ISBN 1-931233-89-6
52995>
9 781931 233897

SOME SMARANDACHE PROBLEMS

```
                .
                .
                .
                1
              1 a 1
            1 a b a 1
          1 a b c b a 1
        1 a b c d c b a 1
      1 a b c d e d c b a 1
    1 a b c d e f e d c b a 1
. . .1 a b c d e f g f e d c b a
    1 a b c d e f e d c b a 1
      1 a b c d e d c b a 1
        1 a b c d c b a 1
          1 a b c b a 1
            1 a b a 1
              1 a 1
                1
                .
                .
                .
```

Mladen Vassilev-Missana
Krassimir Atanassov

HEXIS
Phoenix, 2004

SOME SMARANDACHE PROBLEMS

Mladen V. Vassilev–Missana[1] and Krassimir T. Atanassov[2]

[1] V. Hugo Str. 5, Sofia–1124, Bulgaria
e-mail: missana@abv.bg

[2] CLBME - Bulgarian Academy of Sciences
Bulgaria, Sofia-1113, P.O.Box 12
e-mail: krat@bas.bg

HEXIS
Phoenix, USA
2004

Mladen V. Vassilev–Missana and Krassimir T. Atanassov

SOME SMARANDACHE PROBLEMS

HEXIS
Phoenix, USA
2004

ISBN 1-931233-89-6
Standard Address Number 297-5092
Printed in the United States of America

Preface

During the five years since publishing [2], we have obtained many new results related to the Smarandache problems. We are happy to have the opportunity to present them in this book for the enjoyment of a wider audience of readers.

The problems in Chapter two have also been solved and published separately by the authors, but it makes sense to collate them here so that they can be better seen in perspective as a whole, particularly in relation to the problems elucidated in Chapter one.

Many of the problems, and more especially the techniques employed in their solution, have wider applicability than just the Smarandache problems, and so they should be of more general interest to other mathematicians, particularly both professional and amateur number theorists.

Mladen V. Vassilev-Missana
Krassimir T. Atanassov

Contents

Chapter 1

On Some Smarandache's problems

In the text below the following notations are used.
$\mathcal{N}$ - the set of all natural numbers (i.e., the set of all positive integers);
$[x]$ – "floor function" (or also so called "bracket function") – the greatest integer which is not greater than the real non-negative number x;
ζ – Riemann's Zeta-function;
Γ – Euler's Gamma-function;
φ – Euler's (totient) function;
ψ – Dedekind's function;
σ – the sum of all divisors of the positive integer argument.

In particular: $\varphi(1) = \psi(1) = \sigma(1) = 1$ and if $n > 1$ and

$$n = \prod_{i=1}^{k} p_i^{\alpha_i}$$

is a prime number factorization of n, then

$$\varphi(n) = n.\prod_{i=1}^{k}(1 - \frac{1}{p_i});$$

$$\psi(n) = n.\prod_{i=1}^{k}(1 + \frac{1}{p_i});$$

$$\sigma(n) = \prod_{i=1}^{k} \frac{p_i^{\alpha_i+1} - 1}{p_i - 1};$$

π – the prime counting function, i.e., $\pi(n)$ denotes the number of primes p such that $p \leq n$;
$\pi_2(k)$ – the twin primes counting function, i.e., $\pi_2(n)$ denotes the number of primes p such that $p \leq n$ and $p + 2$ is also a prime;
$p_2(n)$ – n-th term of the twin primes sequence, i.e.,

$$p_2(1) = 3, p_2(2) = 5, p_2(3) = 7, p_2(4) = 11, p_2(5) = 13, p_2(6) = 17,$$

$$p_2(7) = 19, p_2(8) = 29, p_2(9) = 31, ...$$

1. ON THE 2-ND SMARANDACHE'S PROBLEM[1]

The second problem from [13] (see also 16-th problem from [24]) is the following:

Smarandache circular sequence:

$$\underbrace{1}_{1}, \underbrace{12, 21}_{2}, \underbrace{123, 231, 312}_{3}, \underbrace{1234, 2341, 3412, 4123}_{4},$$

$$\underbrace{12345, 23451, 34512, 45123, 51234}_{5},$$

$$\underbrace{123456, 234561, 345612, 456123, 561234, 612345}_{6}, ...$$

Let $]x[$ be the largest natural number strongly smaller than the real (positive) number x. For instance, $]7.1[= 7$, but $]7[= 6$.

Let $f(n)$ be the n-th member of the above sequence. We shall prove the following

Theorem. For each natural number n:

$$f(n) = \overline{s(s+1)...k12...(s-1)}, \qquad (1)$$

where

$$k \equiv k(n) =]\frac{\sqrt{8n+1}-1}{2}[\qquad (2)$$

and

$$s \equiv s(n) = n - \frac{k(k+1)}{2}. \qquad (3)$$

Proof. If $n = 1$, then from (1) and (2) it follows that $k = 0$, $s = 1$ and from (3) – that $f(1) = 1$. Let us assume that the assertion is valid for some natural number n. Then for $n + 1$ we have the

[1]The results in this section are taken from [9]

following two cases:

1. $k(n+1) = k(n)$, i.e., k is the same as above. Then

$$s(n+1) = n+1 - \frac{k(n+1)(k(n+1)+1)}{2} = n+1 - \frac{k(n)(k(n)+1)}{2}$$

$$= s(n) + 1,$$

i.e.,

$$f(n+1) = \overline{(s+1)...k12...s}.$$

2. $k(n+1) = k(n) + 1$. Then

$$s(n+1) = n+1 - \frac{k(n+1)(k(n+1)+1)}{2}. \qquad (4)$$

On the other hand, it is easy to see, that in (2) the number

$$\frac{\sqrt{8n+1}-1}{2} \text{ is an integer if and only if } n = \frac{m(m+1)}{2}.$$

Also, for any natural numbers n and $m \geq 1$ such that

$$\frac{(m-1)m}{2} < n < \frac{m(m+1)}{2} \qquad (5)$$

it will be valid that

$$]\frac{\sqrt{8n+1}-1}{2}[\; = \;]\frac{\sqrt{\frac{m(m+1)}{2}+1}-1}{2}[\; = m.$$

Therefore, if $k(n+1) = k(n) + 1$, then

$$n = \frac{m(m+1)}{2} + 1$$

and from (4) we obtain:

$$s(n+1) = 1,$$

i.e.,

$$f(n+1) = \overline{12...(n+1)}.$$

Therefore, the assertion is valid.

Let

$$S(n) = \sum_{i=1}^{n} f(i).$$

Then, we shall use again formulae (2) and (3). Therefore,

$$S(n) = \sum_{i=1}^{p} f(i) + \sum_{i=p+1}^{n} f(i),$$

where

$$p = \frac{m(m+1)}{2}.$$

It can be seen directly, that

$$\sum_{i=1}^{p} f(i) = \sum_{i=1}^{m} \overline{12...i} + \overline{23...i1} + \overline{i12...(i-1)} = \sum_{i=1}^{m} \frac{i(i+1)}{2}.\underbrace{11...1}_{i}$$

On the other hand, if $s = n - p$, then

$$\sum_{i=p+1}^{n} f(i)$$

$$= \overline{12...(m+1)} + \overline{23...(m+1)1} + \overline{s(s+1)...m(m+1)12...(s-1)}$$

$$= \sum_{i=0}^{m+1} (\frac{(s+i)(s+i+1)}{2} - \frac{i(i+1)}{2}).10^{m-i}.$$

2. ON THE 8-TH, THE 9-TH, THE 10-TH, THE 11-TH AND THE 103-TH SMARANDACHE'S PROBLEMS[2]

The eight problem from [13] (see also 16-th problem from [24]) is the following:

Smarandache mobile periodicals (I):

...	0	0	0	0	0	0	*1*	0	0	0	0	0	0	...
...	0	0	0	0	0	*1*	*1*	*1*	0	0	0	0	0	...
...	0	0	0	0	*1*	*1*	0	*1*	*1*	0	0	0	0	...
...	0	0	0	0	0	*1*	*1*	*1*	0	0	0	0	0	...
...	0	0	0	0	0	0	*1*	0	0	0	0	0	0	...
...	0	0	0	0	0	*1*	*1*	*1*	0	0	0	0	0	...
...	0	0	0	0	*1*	*1*	0	*1*	*1*	0	0	0	0	...
...	0	0	0	*1*	*1*	0	0	0	*1*	*1*	0	0	0	...
...	0	0	0	0	*1*	*1*	0	*1*	*1*	0	0	0	0	...
...	0	0	0	0	0	*1*	*1*	*1*	0	0	0	0	0	...
...	0	0	0	0	0	0	*1*	0	0	0	0	0	0	...
...	0	0	0	0	0	*1*	*1*	*1*	0	0	0	0	0	...
...	0	0	0	0	*1*	*1*	0	*1*	*1*	0	0	0	0	...
...	0	0	0	*1*	*1*	0	0	0	*1*	*1*	0	0	0	...
...	0	0	*1*	*1*	0	0	0	0	0	*1*	*1*	0	0	...
...	0	0	0	*1*	*1*	0	0	0	*1*	*1*	0	0	0	...
...	0	0	0	0	*1*	*1*	0	*1*	*1*	0	0	0	0	...
...	0	0	0	0	0	*1*	*1*	*1*	0	0	0	0	0	...
...	0	0	0	0	0	0	*1*	0	0	0	0	0	0	...
...	0	0	0	0	0	*1*	*1*	*1*	0	0	0	0	0	...
...	0	0	0	0	*1*	*1*	0	*1*	*1*	0	0	0	0	...
...	0	0	0	*1*	*1*	0	0	0	*1*	*1*	0	0	0	...
...	0	0	*1*	*1*	0	0	0	0	0	*1*	*1*	0	0	...
...	0	*1*	*1*	0	0	0	0	0	0	0	*1*	*1*	0	...
...	0	0	*1*	*1*	0	0	0	0	0	*1*	*1*	0	0	...
			.				.				.			
			.				.				.			

[2]The results in this section are taken from [38]

This sequence has the form

$$\underbrace{1, 111, 11011, 111, 1}_{5}, \underbrace{1, 111, 11011, 1100011, 11011, 111, 1}_{7},$$

$$\underbrace{1, 111, 11011, 1100011, 110000011, , 1100011, 11011, 111, 1}_{9}, \ldots$$

All digits from the above table generate an infinite matrix A. We shall describe the elements of A.

Let us take a Cartesian coordinate system C with origin in the point containing element "1" in the topmost (i.e., the first) row of A. We assume that this row belongs to the ordinate axis of C (see Fig. 1) and that the points to the right of the origin have positive ordinates.

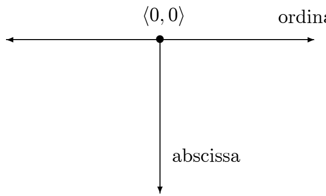

Fig. 1.

The above digits generate an infinite sequence of squares, located in the half-plane (determined by C) where the abscisses of the points are nonnegative. Their diameters have the form

"110...011".

Exactly one of the diameters of each of considered squares lies on the abscissa of C. It can be seen (and proved, e.g., by induction) that the s-th square, denoted by G_s $(s = 0, 1, 2, ...)$ has a diameter with lenght $2s + 4$ and the same square has a highest vertex with

coordinates $\langle s^2 + 3s, 0\rangle$ in C and a lowest vertex with coordinates $\langle s^2 + 5s + 4, 0\rangle$ in C.

Let us denote by $a_{k,i}$ an element of A with coordinates $\langle k, i\rangle$ in C.

First, we determine the minimal nonnegative s for which the inequality

$$s^2 + 5s + 4 \geq k$$

holds. We denote it by $s(k)$. Directly it is seen the following

Lemma. The number $s(k)$ admits the explicit representation:

$$s(k) = \begin{cases} 0, & \text{if } 0 \leq k \leq 4 \\ [\frac{\sqrt{4k+9}-5}{2}], & \text{if } k \geq 5 \text{ and } 4k+9 \text{ is a square of an integer} \\ [\frac{\sqrt{4k+9}-5}{2}] + 1, & \text{if } k \geq 5 \text{ and } 4k+9 \text{ is not a square of an integer} \end{cases} \quad (1)$$

and the inequalities

$$(s(k))^2 + 3s(k) \leq k \leq (s(k))^2 + 5s(k) + 4 \quad (2)$$

hold.

Second, we introduce the integeres $\delta(k)$ and $\varepsilon(k)$ by

$$\delta(k) \equiv k - (s(k))^2 - 3s(k), \quad (3)$$

$$\varepsilon(k) \equiv (s(k))^2 + 5s(k) + 4 - k. \quad (4)$$

From (2) we have $\delta(k) \geq 0$ and $\varepsilon(k) \geq 0$. Let P_k be the infinite strip orthogonal to the abscissa of C and lying between the straight lines passing through those vertices of the square $G_{s(k)}$ lying on the abscissa of C. Then $\delta(k)$ and $\varepsilon(k)$ characterize the location of point with coordinates $\langle k, i\rangle$ in C in strip P_k. Namely, the following assertion is true.

Proposition 1. The elements $a_{k,i}$ of the infinite matrix A are described as follows:
if $k \le (s(k))^2 + 4s(k) + 2$, then

$$a_{k,i} = \begin{cases} 0, & \text{if } \delta(k) < |i| \text{ or } \delta(k) \ge |i| + 2, \\ 1, & \text{if } |i| \le \delta(k) \le |i| + 1 \end{cases}; \qquad (5)$$

if $k > (s(k))^2 + 4s(k) + 2$, then

$$a_{k,i} = \begin{cases} 0, & \text{if } \varepsilon(k) < |i| \text{ or } \varepsilon(k) \ge |i| + 2, \\ 1, & \text{if } |i| \le \varepsilon(k) \le |i| + 1 \end{cases}, \qquad (6)$$

where here and below $s(k)$ is given by (1), $\delta(k)$ and $\varepsilon(k)$ are given by (3) and (4), respectively.

Omitting the obvious proof (it can be done, e.g., by induction), we note that (5) gives a description of $a_{k,i}$ for the case when $\langle k, i\rangle$ belongs to the strip that is orthogonal to the abscissa of C and lying between the straight lines through the points in C with coordinates $\langle (s(k))^2 + 3s(k), 0\rangle$ and $\langle (s(k))^2 + 4s(k) + 2, 0\rangle$ (involving these straight lines), while (6) gives a description of $a_{k,i}$ for the case when $\langle k, i\rangle$ belongs to the strip that is also orthogonal to the abscissa of C, but lying between the straight lines through the points in C with coordinates $\langle (s(k))^2 + 4s(k) + 2, 0\rangle$ and $\langle (s(k))^2 + 5s(k) + 4, 0\rangle$ (without involving the straight line passing through the point in C with coordinates $\langle (s(k))^2 + 4s(k) + 2, 0\rangle$).

Below, we propose another description of elements of A, which can be proved (e.g., by induction) using the same considerations.

$$a_{k,i} = \begin{cases} 1, & \text{if } \langle k, i\rangle \in \\ & \{\langle (s(k))^2 + 3s(k), 0\rangle, \langle (s(k))^2 + 5s(k) + 4, 0\rangle\} \\ & \cup\{\langle (s(k))^2 + 3s(k) + j, -j\rangle, \\ & \langle (s(k))^2 + 3s(k) + j, -j + 1\rangle, \\ & \langle (s(k))^2 + 3s(k) + j, j - 1\rangle, \\ & \langle (s(k))^2 + 3s(k) + j, j\rangle : 1 \le j \le s(k) + 2\} \\ & \langle (s(k))^2 + 5s(k) + 4 - j, -j\rangle, \\ & \langle (s(k))^2 + 5s(k) + 4 - j, -j + 1\rangle, \\ & \langle (s(k))^2 + 5s(k) + 4 - j, j - 1\rangle, \\ & \langle (s(k))^2 + 5s(k) + 4 - j, j\rangle : \\ & 1 \le j \le s(k) + 1\} \\ 0, & \text{otherwise} \end{cases} \qquad (7)$$

Similar representations are possible for all of the next problems.
Let us denote by $\overline{u_1 u_2 ... u_s}$ an s-digit number.
For Smarandache's sequence from Problem 8

$$1, 111, 11011, 111, 1, 111, 11011, 1100011, 11011, 111, 1, ...,$$

that is given above, if we denote it by $\{b_k\}_{k=0}^{\infty}$, then we obtain the representation

$$b_k = \begin{cases} \overline{a_{k,\delta(k)} a_{k,\delta(k)-1} ... a_{k,0} a_{k,1} ... a_{k,\delta(k)-1} a_{k,\delta(k)}}, \\ \qquad \text{if } k \le (s(k))^2 + 4s(k) + 2 \\ \overline{a_{k,\varepsilon(k)} a_{k,\varepsilon(k)-1} ... a_{k,0} a_{k,1} ... a_{k,\varepsilon(k)-1} a_{k,\varepsilon(k)}}, \\ \qquad \text{if } k > (s(k))^2 + 4s(k) + 2 \end{cases}$$

where $a_{k,i}$ are given in an explicit form by (5).

The 10-th Smarandache problem is dual to the above one:
Smarandache infinite numbers (I):

...	1	1	1	1	1	1	*0*	1	1	1	1	1	1	...
...	1	1	1	1	1	*0*	*0*	*0*	1	1	1	1	1	...
...	1	1	1	1	*0*	*0*	1	*0*	*0*	1	1	1	1	...
...	1	1	1	1	1	*0*	*0*	*0*	1	1	1	1	1	...
...	1	1	1	1	1	1	*0*	1	1	1	1	1	1	...
...	1	1	1	1	1	*0*	*0*	*0*	1	1	1	1	1	...
...	1	1	1	1	*0*	*0*	1	*0*	*0*	1	1	1	1	...
...	1	1	1	*0*	*0*	1	1	1	*0*	*0*	1	1	1	...
...	1	1	1	1	*0*	*0*	1	*0*	*0*	1	1	1	1	...
...	1	1	1	1	1	*0*	*0*	*0*	1	1	1	1	1	...
...	1	1	1	1	1	1	*0*	1	1	1	1	1	1	...
...	1	1	1	1	1	*0*	*0*	*0*	1	1	1	1	1	...
...	1	1	1	1	*0*	*0*	1	*0*	*0*	1	1	1	1	...
...	1	1	1	*0*	*0*	1	1	1	*0*	*0*	1	1	1	...
...	1	1	*0*	*0*	1	1	1	1	1	*0*	*0*	1	1	...
...	1	1	1	*0*	*0*	1	1	1	*0*	*0*	1	1	1	...
...	1	1	1	1	*0*	*0*	1	*0*	*0*	1	1	1	1	...
...	1	1	1	1	1	*0*	*0*	*0*	1	1	1	1	1	...
...	1	1	1	1	1	1	*0*	1	1	1	1	1	1	...
...	1	1	1	1	1	*0*	*0*	*0*	1	1	1	1	1	...
...	1	1	1	1	*0*	*0*	1	*0*	*0*	1	1	1	1	...
...	1	1	1	*0*	*0*	1	1	1	*0*	*0*	1	1	1	...
...	1	1	*0*	*0*	1	1	1	1	1	*0*	*0*	1	1	...
...	1	*0*	*0*	1	1	1	1	1	1	1	*0*	*0*	1	...
...	1	1	*0*	*0*	1	1	1	1	1	*0*	*0*	1	1	...
			.				.				.			
			.				.				.			

Further, we will keep the notations: A (for the matrix) and $a_{k,i}$ (for its elements) from the 8-th Smarandache's problem, for each one of the next problems in this section.

Proposition 2. The elements $a_{k,i}$ of the infinite matrix A are described as follows
if $k \leq (s(k))^2 + 4s(k) + 2$, then

$$a_{k,i} = \begin{cases} 0, & \text{if } |i| \leq \delta(k) \leq |i| + 1 \\ \\ 1, & \text{if } \delta(k) < |i| \text{ or } \delta(k) \geq |i| + 2 \end{cases};$$

if $k > (s(k))^2 + 4s(k) + 2$, then

$$a_{k,i} = \begin{cases} 0, & \text{if } |i| \leq \varepsilon(k) \leq |i| + 1 \\ \\ 1, & \text{if } \varepsilon(k) < |i| \text{ or } \varepsilon(k) \geq |i| + 2 \end{cases}.$$

The 9-th Smarandache problem is a modification and extension of the 8-th problem:

Smarandache mobile periodicals (II):

...	0	0	0	0	0	0	1	0	0	0	0	0	0	...
...	0	0	0	0	0	1	1	1	0	0	0	0	0	...
...	0	0	0	0	1	1	2	1	1	0	0	0	0	...
...	0	0	0	0	0	1	1	1	0	0	0	0	0	...
...	0	0	0	0	0	0	1	0	0	0	0	0	0	...
...	0	0	0	0	0	1	1	1	0	0	0	0	0	...
...	0	0	0	0	1	1	2	1	1	0	0	0	0	...
...	0	0	0	1	1	2	3	2	1	1	0	0	0	...
...	0	0	0	0	1	1	2	1	1	0	0	0	0	...
...	0	0	0	0	0	1	1	1	0	0	0	0	0	...
...	0	0	0	0	0	0	1	0	0	0	0	0	0	...
...	0	0	0	0	0	1	1	1	0	0	0	0	0	...
...	0	0	0	0	1	1	2	1	1	0	0	0	0	...
...	0	0	0	1	1	2	3	2	1	1	0	0	0	...
...	0	0	1	1	2	3	4	3	2	1	1	0	0	...
...	0	0	0	1	1	2	3	2	1	1	0	0	0	...
...	0	0	0	0	1	1	2	1	1	0	0	0	0	...
...	0	0	0	0	0	1	1	1	0	0	0	0	0	...
...	0	0	0	0	0	0	1	0	0	0	0	0	0	...
...	0	0	0	0	0	1	1	1	0	0	0	0	0	...
...	0	0	0	0	1	1	2	1	1	0	0	0	0	...
...	0	0	0	1	1	2	3	2	1	1	0	0	0	...
...	0	0	1	1	2	3	4	3	2	1	1	0	0	...
...	0	1	1	2	3	4	5	4	3	2	1	1	0	...
...	0	0	1	1	2	3	4	3	2	1	1	0	0	...
			.				.				.			
			.				.				.			

This sequence has the form

$$\underbrace{1,111,11211,111,1}_{5},\underbrace{1,111,11211,1123211,11211,111,1}_{7},$$

$$\underbrace{1,111,11211,1123211,112343211,,1123211,11211,111,1}_{9},\dots$$

Proposition 3. The elements $a_{k,i}$ of infinite matrix A are described as follow:
if $k \le (s(k))^2 + 4s(k) + 2$, then

$$a_{k,i} = \begin{cases} 0, & \text{if } \delta(k) < |i| \\ 1, & \text{if } \delta(k) = |i| \\ \delta(k) - |i|, & \text{if } \delta(k) > |i| \end{cases}; \tag{8}$$

if $k > (s(k))^2 + 4s(k) + 2$, then

$$a_{k,i} = \begin{cases} 0, & \text{if } \varepsilon(k) < |i| \\ 1, & \text{if } \varepsilon(k) = |i| \\ \varepsilon(k) - |i|, & \text{if } \varepsilon(k) > |i| \end{cases}, \tag{9}$$

For the above sequence

$$1,111,11211,111,1,111,11211,1123211,11211,111,1,...$$

if we denote it by $\{c_k\}_{k=0}^{\infty}$, then we obtain the representation

$$c_k = \begin{cases} \overline{\overline{\overline{a_{k,\delta(k)}a_{k,\delta(k)-1}}\ldots\overline{a_{k,0}a_{k,1}}\ldots\overline{a_{k,\delta(k)-1}a_{k,\delta(k)}}}}, \\ \qquad \text{if } k \le (s(k))^2 + 4s(k) + 2 \\ \overline{\overline{\overline{a_{k,\varepsilon(k)}a_{k,\varepsilon(k)-1}}\ldots\overline{a_{k,0}a_{k,1}}\ldots\overline{a_{k,\varepsilon(k)-1}a_{k,\varepsilon(k)}}}}, \\ \qquad \text{if } k > (s(k))^2 + 4s(k) + 2 \end{cases}$$

where $a_{k,i}$ are given in an explicit form by (8).

The 11-th Smarandache problem is a modification of the 9-th problem:

Smarandache infinite numbers (II):

...	1	1	1	1	1	1	2	1	1	1	1	1	1	...
...	1	1	1	1	1	2	2	2	1	1	1	1	1	...
...	1	1	1	1	2	2	3	2	2	1	1	1	1	...
...	1	1	1	1	1	2	2	2	1	1	1	1	1	...
...	1	1	1	1	1	1	2	1	1	1	1	1	1	...
...	1	1	1	1	1	2	2	2	1	1	1	1	1	...
...	1	1	1	1	2	2	3	2	2	1	1	1	1	...
...	1	1	1	2	2	3	4	3	2	2	1	1	1	...
...	1	1	1	1	2	2	3	2	2	1	1	1	1	...
...	1	1	1	1	1	2	2	2	1	1	1	1	1	...
...	1	1	1	1	1	1	2	1	1	1	1	1	1	...
...	1	1	1	1	1	2	2	2	1	1	1	1	1	...
...	1	1	1	1	2	2	3	2	2	1	1	1	1	...
...	1	1	1	2	2	3	4	3	2	2	1	1	1	...
...	1	1	2	2	3	4	5	4	3	2	2	1	1	...
...	1	1	1	2	2	3	4	3	2	2	1	1	1	...
...	1	1	1	1	2	2	3	2	2	1	1	1	1	...
...	1	1	1	1	1	2	2	2	1	1	1	1	1	...
...	1	1	1	1	1	1	2	1	1	1	1	1	1	...
...	1	1	1	1	1	2	2	2	1	1	1	1	1	...
...	1	1	1	1	2	2	3	2	2	1	1	1	1	...
...	1	1	1	2	2	3	4	3	2	2	1	1	1	...
...	1	1	2	2	3	4	5	4	3	2	2	1	1	...
...	1	2	2	3	4	5	6	5	4	3	2	2	1	...
...	1	1	2	2	3	4	5	4	3	2	2	1	1	...
			⋮				⋮				⋮			

Proposition 4. The elements $a_{k,i}$ of the infinite matrix A are described as follows:

if $k \leq (s(k))^2 + 4s(k) + 2$, then

$$a_{k,i} = \begin{cases} 1, & \text{if } \delta(k) < |i| \\ 2, & \text{if } |i| \leq \delta(k) \leq |i| + 1 \\ \delta(k) - |i| + 1, & \text{if } \delta(k) \geq |i| + 2 \end{cases};$$

if $k > (s(k))^2 + 4s(k) + 2$, then

$$a_{k,i} = \begin{cases} 1, & \text{if } \varepsilon(k) < |i| \\ 2, & \text{if } |i| \leq \varepsilon(k) = |i| + 1 \\ \varepsilon(k) - |i| + 1, & \text{if } \varepsilon(k) \geq |i| + 2 \end{cases},$$

Now, we introduce modifications of the above problems, giving formulae for their (k,i)-th members $a_{k,i}$.

We modify the first of the above problems, now – with a simple countours of the squares in the matrix:

$$
\begin{array}{ccccccccccccccc}
\dots & 0 & 0 & 0 & 0 & 0 & 0 & \mathit{1} & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & \mathit{1} & 0 & \mathit{1} & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & \mathit{1} & 0 & 0 & 0 & \mathit{1} & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & \mathit{1} & 0 & \mathit{1} & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 0 & \mathit{1} & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & \mathit{1} & 0 & \mathit{1} & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & \mathit{1} & 0 & 0 & 0 & \mathit{1} & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & \mathit{1} & 0 & 0 & 0 & 0 & 0 & \mathit{1} & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & \mathit{1} & 0 & 0 & 0 & \mathit{1} & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & \mathit{1} & 0 & \mathit{1} & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 0 & \mathit{1} & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & \mathit{1} & 0 & \mathit{1} & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & \mathit{1} & 0 & 0 & 0 & \mathit{1} & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & \mathit{1} & 0 & 0 & 0 & 0 & 0 & \mathit{1} & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & \mathit{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathit{1} & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & \mathit{1} & 0 & 0 & 0 & 0 & 0 & \mathit{1} & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & \mathit{1} & 0 & 0 & 0 & \mathit{1} & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & \mathit{1} & 0 & \mathit{1} & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 0 & \mathit{1} & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & \mathit{1} & 0 & \mathit{1} & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & \mathit{1} & 0 & 0 & 0 & \mathit{1} & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & \mathit{1} & 0 & 0 & 0 & 0 & 0 & \mathit{1} & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & \mathit{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathit{1} & 0 & 0 & \dots \\
\dots & 0 & \mathit{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathit{1} & 0 & \dots \\
\dots & 0 & 0 & \mathit{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathit{1} & 0 & 0 & \dots \\
 & & & . & & & & . & & & & . & & & \\
 & & & . & & & & . & & & & . & & &
\end{array}
$$

Proposition 5. The elements $a_{k,i}$ of the infinite matrix A are described as follows:

if $k \leq (s(k))^2 + 4s(k) + 2$, then

$$a_{k,i} = \begin{cases} 0, & \text{if } \delta(k) < |i| \text{ or } \delta(k) > |i| \\ 1, & \text{if } \delta(k) = |i| \end{cases} ; \qquad (10)$$

if $k > (s(k))^2 + 4s(k) + 2$, then

$$a_{k,i} = \begin{cases} 0, & \text{if } \varepsilon(k) < |i| \text{ or } \varepsilon(k) > |i| \\ 1, & \text{if } \varepsilon(k) = |i| \end{cases} , \qquad (11)$$

Next, we will modify the third of the above problems, again with a simple countour of the squares:

...	0	0	0	0	0	0	1	0	0	0	0	0	0	...
...	0	0	0	0	0	1	2	1	0	0	0	0	0	...
...	0	0	0	0	1	2	3	2	1	0	0	0	0	...
...	0	0	0	0	0	1	2	1	0	0	0	0	0	...
...	0	0	0	0	0	0	1	0	0	0	0	0	0	...
...	0	0	0	0	0	1	2	1	0	0	0	0	0	...
...	0	0	0	0	1	2	3	2	1	0	0	0	0	...
...	0	0	0	1	2	3	4	3	2	1	0	0	0	...
...	0	0	0	0	1	2	3	2	1	0	0	0	0	...
...	0	0	0	0	0	1	2	1	0	0	0	0	0	...
...	0	0	0	0	0	0	1	0	0	0	0	0	0	...
...	0	0	0	0	0	1	2	1	0	0	0	0	0	...
...	0	0	0	0	1	2	3	2	1	0	0	0	0	...
...	0	0	0	1	2	3	4	3	2	1	0	0	0	...
...	0	0	1	2	3	4	5	4	3	2	1	0	0	...
...	0	0	0	1	2	3	4	3	2	1	0	0	0	...
...	0	0	0	0	1	2	3	2	1	0	0	0	0	...
...	0	0	0	0	0	1	2	1	0	0	0	0	0	...
...	0	0	0	0	0	0	1	0	0	0	0	0	0	...
...	0	0	0	0	0	1	2	1	0	0	0	0	0	...
...	0	0	0	0	1	2	3	2	1	0	0	0	0	...
...	0	0	0	1	2	3	4	3	2	1	0	0	0	...
...	0	0	1	2	3	4	5	4	3	2	1	0	0	...
...	0	1	2	3	4	5	6	5	4	3	2	1	0	...
...	0	0	1	2	3	4	5	4	3	2	1	0	0	...
			⋮				⋮				⋮			

Proposition 6. The elements $a_{k,i}$ of the infinite matrix A are described as follow:

if $k \leq (s(k))^2 + 4s(k) + 2$, then

$$a_{k,i} = \begin{cases} 0, & \text{if } \delta(k) < |i| \\ \delta(k) - |i| + 1, & \text{if } \delta(k) \geq |i| \end{cases} ; \tag{12}$$

if $k > (s(k))^2 + 4s(k) + 2$, then

$$a_{k,i} = \begin{cases} 0, & \text{if } \varepsilon(k) < |i| \\ \varepsilon(k) - |i| + 1, & \text{if } \varepsilon(k) \geq |i| \end{cases} , \tag{13}$$

Third, we will fill the interior of the squares with Fibonacci numbers

$$
\begin{array}{ccccccccccccccc}
\dots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 1 & 1 & 2 & 3 & 2 & 1 & 1 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 1 & 1 & 2 & 3 & 2 & 1 & 1 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 1 & 1 & 2 & 3 & 5 & 3 & 2 & 1 & 1 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 1 & 1 & 2 & 3 & 2 & 1 & 1 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 1 & 1 & 2 & 3 & 2 & 1 & 1 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 1 & 1 & 2 & 3 & 5 & 3 & 2 & 1 & 1 & 0 & 0 & \dots \\
\dots & 0 & 1 & 1 & 2 & 3 & 5 & 8 & 5 & 3 & 2 & 1 & 1 & 0 & \dots \\
\dots & 0 & 0 & 1 & 1 & 2 & 3 & 5 & 3 & 2 & 1 & 1 & 0 & 0 & \dots \\
 & & & \vdots & & & & \vdots & & & & \vdots & & &
\end{array}
$$

Proposition 7. The elements $a_{k,i}$ of the infinite matrix A are described as follows:

if $k \leq (s(k))^2 + 4s(k) + 2$, then

$$a_{k,i} = \begin{cases} 0, & \text{if } \delta(k) < |i| \\ F_{\delta(k)-|i|}, & \text{if } \delta(k) \geq |i| \end{cases}; \tag{14}$$

if $k > (s(k))^2 + 4s(k) + 2$, then

$$a_{k,i} = \begin{cases} 0, & \text{if } \varepsilon(k) < |i| \\ F_{\varepsilon(k)-|i|}, & \text{if } \varepsilon(k) \geq |i| \end{cases}, \tag{15}$$

where F_m $(m = 0, 1, 2, \ldots)$ is the m-th Fibonacci number.

Fourth, we will fill the interior of the squares with powers of 2:

...	0	0	0	0	0	0	1	0	0	0	0	0	0	...
...	0	0	0	0	0	1	2	1	0	0	0	0	0	...
...	0	0	0	0	1	2	4	2	1	0	0	0	0	...
...	0	0	0	0	0	1	2	1	0	0	0	0	0	...
...	0	0	0	0	0	0	1	0	0	0	0	0	0	...
...	0	0	0	0	0	1	2	1	0	0	0	0	0	...
...	0	0	0	0	1	2	4	2	1	0	0	0	0	...
...	0	0	0	1	2	4	8	4	2	1	0	0	0	...
...	0	0	0	0	1	2	4	2	1	0	0	0	0	...
...	0	0	0	0	0	1	2	1	0	0	0	0	0	...
...	0	0	0	0	0	0	1	0	0	0	0	0	0	...
...	0	0	0	0	0	1	2	1	0	0	0	0	0	...
...	0	0	0	0	1	2	4	2	1	0	0	0	0	...
...	0	0	0	1	2	4	8	4	2	1	0	0	0	...
...	0	0	1	2	4	8	16	8	4	2	1	0	0	...
...	0	0	0	1	2	4	8	4	2	1	0	0	0	...
...	0	0	0	0	1	2	4	2	1	0	0	0	0	...
...	0	0	0	0	0	1	2	1	0	0	0	0	0	...
...	0	0	0	0	0	0	1	0	0	0	0	0	0	...
...	0	0	0	0	0	1	2	1	0	0	0	0	0	...
...	0	0	0	0	1	2	4	2	1	0	0	0	0	...
...	0	0	0	1	2	4	8	4	2	1	0	0	0	...
...	0	0	1	2	4	8	16	8	4	2	1	0	0	...
...	0	1	2	4	8	16	32	16	8	4	2	1	0	...
...	0	0	1	2	4	8	16	8	4	2	1	0	0	...
			.				.				.			
			.				.				.			

Proposition 8. The elements $a_{k,i}$ of the infinite matrix A are described as follows:

if $k \leq (s(k))^2 + 4s(k) + 2$, then

$$a_{k,i} = \begin{cases} 0, & \text{if } \delta(k) < |i| \\ 2^{\delta(k)-|i|}, & \text{if } \delta(k) \geq |i| \end{cases}; \tag{16}$$

if $k > (s(k))^2 + 4s(k) + 2$, then

$$a_{k,i} = \begin{cases} 0, & \text{if } \varepsilon(k) < |i| \\ 2^{\varepsilon(k)-|i|}, & \text{if } \varepsilon(k) \geq |i| \end{cases}, \tag{17}$$

Fifth, we will fill the interior of the squares with values 1 and -1 as in the next table:

...	0	0	0	0	0	0	1	0	0	0	0	0	...
...	0	0	0	0	0	1	-1	1	0	0	0	0	...
...	0	0	0	0	1	-1	1	-1	1	0	0	0	...
...	0	0	0	0	0	1	-1	1	0	0	0	0	...
...	0	0	0	0	0	0	1	0	0	0	0	0	...
...	0	0	0	0	0	1	-1	1	0	0	0	0	...
...	0	0	0	0	1	-1	1	-1	1	0	0	0	...
...	0	0	0	1	-1	1	-1	1	-1	1	0	0	...
...	0	0	0	0	1	-1	1	-1	1	0	0	0	...
...	0	0	0	0	0	1	-1	1	0	0	0	0	...
...	0	0	0	0	0	0	1	0	0	0	0	0	...
...	0	0	0	0	0	1	-1	1	0	0	0	0	...
...	0	0	0	0	1	-1	1	-1	1	0	0	0	...
...	0	0	0	1	-1	1	-1	1	-1	1	0	0	...
...	0	0	1	-1	1	-1	1	-1	1	-1	1	0	...
...	0	0	0	1	-1	1	-1	1	-1	1	0	0	...
...	0	0	0	0	1	-1	1	-1	1	0	0	0	...
...	0	0	0	0	0	1	-1	1	0	0	0	0	...
...	0	0	0	0	0	0	1	0	0	0	0	0	...
...	0	0	0	0	0	1	-1	1	0	0	0	0	...
...	0	0	0	0	1	-1	1	-1	1	0	0	0	...
...	0	0	0	1	-1	1	-1	1	-1	1	0	0	...
...	0	0	1	-1	1	-1	1	-1	1	-1	1	0	...
...	0	1	-1	1	-1	1	-1	1	-1	1	-1	1	...
...	0	0	1	-1	1	-1	1	-1	1	-1	1	0	...
...	0	0	0	1	-1	1	-1	1	-1	1	0	0	...
			.				.				.		
			.				.				.		

Proposition 9. The elements $a_{k,i}$ of the infinite matrix A are described as follows:

if $k \leq (s(k))^2 + 4s(k) + 2$, then

$$a_{k,i} = \begin{cases} 0, & \text{if } \delta(k) < |i| \\ (-1)^{\delta(k)-|i|}, & \text{if } \delta(k) \geq |i| \end{cases} ; \tag{18}$$

if $k > (s(k))^2 + 4s(k) + 2$, then

$$a_{k,i} = \begin{cases} 0, & \text{if } \varepsilon(k) < |i| \\ (-1)^{\varepsilon(k)-|i|}, & \text{if } \varepsilon(k) \geq |i| \end{cases} , \tag{19}$$

The following infinite matrix A is a generalization of all previous schemes:

$$
\begin{array}{ccccccccccc}
\ldots & 0 & 0 & 0 & 0 & F(0) & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & F(0) & F(1) & F(0) & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & F(0) & F(1) & F(2) & F(1) & F(0) & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & F(0) & F(1) & F(0) & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & F(0) & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & F(0) & F(1) & F(0) & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & F(0) & F(1) & F(2) & F(1) & F(0) & 0 & 0 & \ldots \\
\ldots & 0 & F(0) & F(1) & F(2) & F(3) & F(2) & F(1) & F(0) & 0 & \ldots \\
\ldots & 0 & 0 & F(0) & F(1) & F(2) & F(1) & F(0) & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & F(0) & F(1) & F(0) & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & F(0) & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & F(0) & F(1) & F(0) & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & F(0) & F(1) & F(2) & F(1) & F(0) & 0 & 0 & \ldots \\
\ldots & 0 & F(0) & F(1) & F(2) & F(3) & F(2) & F(1) & F(0) & 0 & \ldots \\
\ldots & F(0) & F(1) & F(2) & F(3) & F(4) & F(3) & F(2) & F(1) & F(0) & \ldots \\
\ldots & 0 & F(0) & F(1) & F(2) & F(3) & F(2) & F(1) & F(0) & 0 & \ldots \\
\ldots & 0 & 0 & F(0) & F(1) & F(2) & F(1) & F(0) & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & F(0) & F(1) & F(0) & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & F(0) & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & F(0) & F(1) & F(0) & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & F(0) & F(1) & F(2) & F(1) & F(0) & 0 & 0 & \ldots \\
\ldots & 0 & F(0) & F(1) & F(2) & F(3) & F(2) & F(1) & F(0) & 0 & \ldots \\
\ldots & F(0) & F(1) & F(2) & F(3) & F(4) & F(3) & F(2) & F(1) & F(0) & \ldots \\
\ldots & F(1) & F(2) & F(3) & F(4) & F(5) & F(4) & F(3) & F(2) & F(1) & \ldots \\
\ldots & F(0) & F(1) & F(2) & F(3) & F(4) & F(3) & F(2) & F(1) & F(0) & \ldots \\
\ldots & 0 & F(0) & F(1) & F(2) & F(3) & F(2) & F(1) & F(0) & 0 & \ldots \\
 & & \cdot & & & \cdot & & & \cdot & & \\
 & & \cdot & & & \cdot & & & \cdot & &
\end{array}
$$

where F is an arbitrary arithmetic function such that the number $F(0)$ is defined.

Proposition 10. The elements $a_{k,i}$ of the infinite matrix A are described as follows:

if $k \leq (s(k))^2 + 4s(k) + 2$, then

$$a_{k,i} = \begin{cases} 0, & \text{if } \delta(k) < |i| \\ F(\delta(k) - |i|), & \text{if } \delta(k) \geq |i| \end{cases}; \tag{20}$$

if $k > (s(k))^2 + 4s(k) + 2$, then

$$a_{k,i} = \begin{cases} 0, & \text{if } \varepsilon(k) < |i| \\ F(\varepsilon(k) - |i|), & \text{if } \varepsilon(k) \geq |i| \end{cases}, \tag{21}$$

Let we put

$$F(n) = G(H(n)), \ n = 0, 1, 2, ... \tag{22}$$

where $H : \mathcal{N} \cup \{0\} \rightarrow E$ and $G : E \rightarrow \mathcal{N} \cup \{0\}$, are arbitrary functions and E is a fixed set, for example, $E = \mathcal{N} \cup \{0\}$. Then many applications are possible. For example, if

$$G(n) = \psi(n),$$

where function ψ is described in **A6** and $H(n) = 2^n$, we obtain the infinite matrix as given below

$$\begin{array}{ccccccccccc}
\ldots & 0 & 0 & 0 & 0 & \psi(1) & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & \psi(1) & \psi(2) & \psi(1) & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & \psi(1) & \psi(2) & \psi(4) & \psi(2) & \psi(1) & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & \psi(1) & \psi(2) & \psi(1) & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & \psi(1) & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & \psi(1) & \psi(2) & \psi(1) & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & \psi(1) & \psi(2) & \psi(4) & \psi(2) & \psi(1) & 0 & 0 & \ldots \\
\ldots & 0 & \psi(1) & \psi(2) & \psi(4) & \psi(8) & \psi(4) & \psi(2) & \psi(1) & 0 & \ldots \\
\ldots & 0 & 0 & \psi(1) & \psi(2) & \psi(4) & \psi(2) & \psi(1) & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & \psi(1) & \psi(2) & \psi(1) & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & \psi(1) & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & \psi(1) & \psi(2) & \psi(1) & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & \psi(1) & \psi(2) & \psi(4) & \psi(2) & \psi(1) & 0 & 0 & \ldots \\
\ldots & 0 & \psi(1) & \psi(2) & \psi(4) & \psi(8) & \psi(4) & \psi(2) & \psi(1) & 0 & \ldots \\
\ldots & \psi(1) & \psi(2) & \psi(4) & \psi(8) & \psi(16) & \psi(8) & \psi(4) & \psi(2) & \psi(1) & \ldots \\
\ldots & 0 & \psi(1) & \psi(2) & \psi(4) & \psi(8) & \psi(4) & \psi(2) & \psi(1) & 0 & \ldots \\
\ldots & 0 & 0 & \psi(1) & \psi(2) & \psi(4) & \psi(2) & \psi(1) & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & \psi(1) & \psi(2) & \psi(1) & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & \psi(1) & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & \psi(1) & \psi(2) & \psi(1) & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & \psi(1) & \psi(2) & \psi(4) & \psi(2) & \psi(1) & 0 & 0 & \ldots \\
\ldots & 0 & \psi(1) & \psi(2) & \psi(4) & \psi(8) & \psi(4) & \psi(2) & \psi(1) & 0 & \ldots \\
\ldots & \psi(1) & \psi(2) & \psi(4) & \psi(8) & \psi(16) & \psi(8) & \psi(4) & \psi(2) & \psi(1) & \ldots \\
\ldots & \psi(2) & \psi(4) & \psi(8) & \psi(16) & \psi(32) & \psi(16) & \psi(8) & \psi(4) & \psi(2) & \ldots \\
\ldots & \psi(1) & \psi(2) & \psi(4) & \psi(8) & \psi(16) & \psi(8) & \psi(4) & \psi(2) & \psi(1) & \ldots \\
\ldots & 0 & \psi(1) & \psi(2) & \psi(4) & \psi(8) & \psi(4) & \psi(2) & \psi(1) & 0 & \ldots \\
 & \vdots & & & \vdots & & & \vdots & & &
\end{array}$$

or in calculation form:

$$\begin{array}{ccccccccccccccc}
\ldots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 1 & 2 & 4 & 2 & 1 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 1 & 2 & 4 & 2 & 1 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 1 & 2 & 4 & 8 & 4 & 2 & 1 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 1 & 2 & 4 & 2 & 1 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 1 & 2 & 4 & 2 & 1 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 1 & 2 & 4 & 8 & 4 & 2 & 1 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 1 & 2 & 4 & 8 & 7 & 8 & 4 & 2 & 1 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 1 & 2 & 4 & 8 & 4 & 2 & 1 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 1 & 2 & 4 & 2 & 1 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 1 & 2 & 4 & 2 & 1 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 1 & 2 & 4 & 8 & 4 & 2 & 1 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 1 & 2 & 4 & 8 & 7 & 8 & 4 & 2 & 1 & 0 & 0 & \ldots \\
\ldots & 0 & 1 & 2 & 4 & 8 & 7 & 5 & 7 & 8 & 4 & 2 & 1 & 0 & \ldots \\
\ldots & 0 & 0 & 1 & 2 & 4 & 8 & 7 & 8 & 4 & 2 & 1 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 0 & 1 & 2 & 4 & 8 & 4 & 2 & 1 & 0 & 0 & 0 & \ldots \\
 & & & \vdots & & & & \vdots & & & & \vdots & & &
\end{array}$$

The elements of this matrix are described by (20) and (21), if we take

$$F(n) = \psi(2^n), n = 0, 1, 2, ...$$

The elements of the following matrix A has alphabetical form:

...	0	0	0	0	0	0	*a*	0	0	0	0	0	0	...
...	0	0	0	0	0	*a*	*b*	*a*	0	0	0	0	0	...
...	0	0	0	0	*a*	*b*	*c*	*b*	*a*	0	0	0	0	...
...	0	0	0	0	0	*a*	*b*	*a*	0	0	0	0	0	...
...	0	0	0	0	0	0	*a*	0	0	0	0	0	0	...
...	0	0	0	0	0	*a*	*b*	*a*	0	0	0	0	0	...
...	0	0	0	0	*a*	*b*	*c*	*b*	*a*	0	0	0	0	...
...	0	0	0	*a*	*b*	*c*	*d*	*c*	*b*	*a*	0	0	0	...
...	0	0	0	0	*a*	*b*	*c*	*b*	*a*	0	0	0	0	...
...	0	0	0	0	0	*a*	*b*	*a*	0	0	0	0	0	...
...	0	0	0	0	0	0	*a*	0	0	0	0	0	0	...
...	0	0	0	0	0	*a*	*b*	*a*	0	0	0	0	0	...
...	0	0	0	0	*a*	*b*	*c*	*b*	*a*	0	0	0	0	...
...	0	0	0	*a*	*b*	*c*	*d*	*c*	*b*	*a*	0	0	0	...
...	0	0	*a*	*b*	*c*	*d*	*e*	*d*	*c*	*b*	*a*	0	0	...
...	0	0	0	*a*	*b*	*c*	*d*	*c*	*b*	*a*	0	0	0	...
...	0	0	0	0	*a*	*b*	*c*	*b*	*a*	0	0	0	0	...
...	0	0	0	0	0	*a*	*b*	*a*	0	0	0	0	0	...
...	0	0	0	0	0	0	*a*	0	0	0	0	0	0	...
...	0	0	0	0	0	*a*	*b*	*a*	0	0	0	0	0	...
...	0	0	0	0	*a*	*b*	*c*	*b*	*a*	0	0	0	0	...
...	0	0	0	*a*	*b*	*c*	*d*	*c*	*b*	*a*	0	0	0	...
...	0	0	*a*	*b*	*c*	*d*	*e*	*d*	*c*	*b*	*a*	0	0	...
...	0	*a*	*b*	*c*	*d*	*e*	*f*	*e*	*d*	*c*	*b*	*a*	0	...
...	0	0	*a*	*b*	*c*	*d*	*e*	*d*	*c*	*b*	*a*	0	0	...
			.				.				.			
			.				.				.			

and they are described using (20) and (21), because we may put:

$$F(0) = 1, F(1) = b, F(2) = c, F(3) = d, F(4) = e$$

etc.

Of course, by analogy, we can construct different schemes, e.g., the schemes of Problems 12, 13 and 14 from [13], but the benefit of these schemes is not clear.

Essentially more interesting is Problem 103 from [13]:
Smarandache numerical carpet:
has the general form

							.							
							.							
							.							
							1							
						1	*a*	*1*						
					1	*a*	*b*	*a*	*1*					
				1	*a*	*b*	*c*	*b*	*a*	*1*				
			1	*a*	*b*	*c*	*d*	*c*	*b*	*a*	*1*			
		1	*a*	*b*	*c*	*d*	*e*	*d*	*c*	*b*	*a*	*1*		
	1	*a*	*b*	*c*	*d*	*e*	*f*	*e*	*d*	*c*	*b*	*a*	*1*	
1	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*f*	*e*	*d*	*c*	*b*	*a*	*1*
	1	*a*	*b*	*c*	*d*	*e*	*f*	*e*	*d*	*c*	*b*	*a*	*1*	
		1	*a*	*b*	*c*	*d*	*e*	*d*	*c*	*b*	*a*	*1*		
			1	*a*	*b*	*c*	*d*	*c*	*b*	*a*	*1*			
				1	*a*	*b*	*c*	*b*	*a*	*1*				
					1	*a*	*b*	*a*	*1*					
						1	*a*	*1*						
							1							
							.							
							.							
							.							

On the border of level 0, the elements are equal to "1";
they form a rhomb.
Next, on the border of level 1, the elements are equal to "a";
where "a" is the sum of all elements of the previous border;
the "a"s form a rhomb too inside the previous one.
Next again, on the border of level 2, the elements are equal to "b";
where "b" is the sum of all elements of the previous border;
the "b"s form a rhomb too inside the previous one.
And so on ...

The above square, that Smarandache named *"rhomb"*, corresponds to the square from our construction for the case of $s = 6$, if we begin to count from $s = 1$, instead of $s = 0$. In [13] a particular solution of the Problem 103 is given, but there a complete solution is not introduced. We will give a solution below firstly for the case of Problem 103 and then for a more general case.

It can be easily seen that the number of the elements of the s-th square side is $s + 2$ and therefore the number of the elements from the countour of this square is just equal to $4s + 4$.

The s-th square can be represented as a set of sub-squares, each one included in the next. Let us number them inwards, so that the outmost (boundary) square is the first one. As it is written in Problem 103, all of its elements are equal to 1. Hence, the values of the elements of the subsequent (second) square will be (using also the notation from Problem 103):

$$a_1 = a = (s + 2) + (s + 1) + (s + 1) + s = 4(s + 1);$$

the values of the elements of the third square will be

$$a_2 = b = a(4(s - 1) + 4 + 1) = 4(s + 1)(4s + 1);$$

the values of the elements of the fourth square will be

$$a_3 = c = b(4(s - 2) + 4 + 1) = 4(s + 1)(4s + 1)(4s - 3);$$

the values of the elements of the fifth square will be

$$a_4 = d = c(4(s - 3) + 4 + 1) = 4(s + 1)(4s + 1)(4s - 3)(4s - 7);$$

etc., where the square, corresponding to the initial square *(rhomb)*, from Problem 103 has the form

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & . & . & . & & & & \\
 & & 1 & a_1 & . & . & . & a_1 & 1 & & \\
 & 1 & a_1 & a_2 & . & . & . & a_2 & a_1 & 1 & \\
1 & a_1 & a_2 & a_3 & . & . & . & a_3 & a_2 & a_1 & 1 \\
 & 1 & a_1 & a_2 & . & . & . & a_2 & a_1 & 1 & \\
 & & 1 & a_1 & . & . & . & a_1 & 1 & & \\
 & & & & . & . & . & & & & \\
 & & & & & 1 & & & & &
\end{array}$$

It can be proved by induction that the elements of this square that stay on t-th place are given by the formula

$$a_t = 4(s + 1)\prod_{i=0}^{t-2}(4s + 1 - 4i).$$

If we would like to generalize the above problem, we can construct, e.g., the following extension:

$$\begin{array}{ccccccccccc}
 & & & & & x & & & & & \\
 & & & & . & . & . & & & & \\
 & & x & a_1 & . & . & . & a_1 & x & & \\
 & x & a_1 & a_2 & . & . & . & a_2 & a_1 & x & \\
x & a_1 & a_2 & a_3 & . & . & . & a_3 & a_2 & a_1 & x \\
 & x & a_1 & a_2 & . & . & . & a_2 & a_1 & x & \\
 & & x & a_1 & . & . & . & a_1 & x & & \\
 & & & & . & . & . & & & & \\
 & & & & & x & & & & &
\end{array}$$

where x is a given number. Then we obtain

$$a_1 = 4(s + 1)x;$$

$$a_2 = 4(s + 1)(4s + 1)x;$$

$$a_3 = 4(s + 1)(4s + 1)(4s - 3)x;$$

$$a_4 = 4(s+1)(4s+1)(4s-3)(4s-7)x;$$

etc. and for $t \geq 1$

$$a_t = 4(s+1)(\prod_{i=0}^{t-2}(4s+1-4i))x,$$

where it is assumed that

$$\prod_{i=0}^{-1} \bullet = 1.$$

3. ON THE 15-TH SMARANDACHE'S PROBLEM[3]

The 15-th Smarandache's problem from [13] is the following: "*Smarandache's simple numbers:*

$$2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 17, 19, 21, 22, 23, 25, 26, 27,$$

$$29, 31, 33, ...$$

A number n is called "Smarandache's simple number" if the product of its proper divisors is less than or equal to n. Generally speaking, n has the form $n = p$, or $n = p^2$, or $n = p^3$, or $n = pq$, where p and q are distinct primes".

Let us denote: by S - the sequence of all Smarandache's simple numbers and by s_n - the n-th term of S; by $\mathcal{P}$ - the sequence of all primes and by p_n - the n-th term of $\mathcal{P}$; by $\mathcal{P}^2$ - the sequence $\{p_n^2\}_{n=1}^{\infty}$; by $\mathcal{P}^3$ - the sequence $\{p_n^3\}_{n=1}^{\infty}$; by $\mathcal{PQ}$ - the sequence $\{p.q\}_{p,q\in\mathcal{P}}$, where $p < q$.

For an abitrary increasing sequence of natural numbers $C \equiv \{c_n\}_{n=1}^{\infty}$ we denote by $\pi_C(n)$ the number of terms of C, which are not greater that n. When $n < c_1$ we put $\pi_C(n) = 0$.

In the present section we find $\pi_S(n)$ in an explicit form and using this, we find the n-th term of S in explicit form, too.

First, we note that instead of $\pi_P(n)$ we use the well-known notation $\pi(n)$.

Hence

$$\pi_{\mathcal{P}^2}(n) = \pi(\sqrt{n}), \quad \pi_{\mathcal{P}^3}(n) = \pi(\sqrt[3]{n}).$$

Thus, using the definition of S, we get

$$\pi_S(n) = \pi(n) + \pi(\sqrt{n}) + \pi(\sqrt[3]{n}) + \pi_{\mathcal{PQ}}(n). \quad (1)$$

Our first aim is to express $\pi_S(n)$ in an explicit form. For $\pi(n)$ some explicit formulae are proposed in **A2**. Other explicit formulae

[3] The results in this section are taken from [34]

for $\pi(n)$ are given in [18]. One of them is known as Mináč's formula. It is given below

$$\pi(n) = \sum_{k=2}^{n}[\frac{(k-1)!+1}{k} - [\frac{(k-1)!}{k}]]. \tag{2}$$

Therefore, the problem for finding of explicit formulae for functions $\pi(n), \pi(\sqrt{n}), \pi(\sqrt[3]{n})$ is solved successfully. It remains only to express $\pi_{\mathcal{PQ}}(n)$ in an explicit form.

Let $k \in \{1, 2, ..., \pi(\sqrt{n})\}$ be fixed. We consider all numbers of the kind $p_k.q$, with $q \in \mathcal{P}, q > p_k$ for which $p_k.q \leq n$. The quantity of these numbers is $\pi(\frac{n}{p_k}) - \pi(p_k)$, or which is the same

$$\pi(\frac{n}{p_k}) - k. \tag{3}$$

When $k = 1, 2, ..., \pi(\sqrt{n})$, the numbers $p_k.q$, as defined above, describe all numbers of the kind $p.q$, with $p, q \in \mathcal{P}, p < q, p.q \leq n$. But the quantity of the last numbers is equal to $\pi_{\mathcal{PQ}}(n)$. Hence

$$\pi_{\mathcal{PQ}}(n) = \sum_{k=1}^{\pi(\sqrt{n})}(\pi(\frac{n}{p_k}) - k), \tag{4}$$

because of (3). The equality (4), after a simple computation yields the formula

$$\pi_{\mathcal{PQ}}(n) = \sum_{k=1}^{\pi(\sqrt{n})}\pi(\frac{n}{p_k}) - \frac{\pi(\sqrt{n}).(\pi(\sqrt{n})+1)}{2}. \tag{5}$$

In **A5** the identity

$$\sum_{k=1}^{\pi(b)}\pi(\frac{n}{p_k}) = \pi(\frac{n}{b}).\pi(b) + \sum_{k=1}^{\pi(\frac{n}{2})-\pi(\frac{n}{b})}\pi(\frac{n}{p_{\pi(\frac{n}{b})+k}}) \tag{6}$$

is proved, under the condition $b \geq 2$ (b is a real number). When $\pi(\frac{n}{2}) = \pi(\frac{n}{b})$, the right hand-side of (6) is reduced to $\pi(\frac{n}{b}).\pi(b)$. In

the case $b = \sqrt{n}$ and $n \geq 4$ equality (6) yields

$$\sum_{k=1}^{\pi(\sqrt{n})}\pi(\frac{n}{p_k}) = (\pi(\sqrt{n}))^2 + \sum_{k=1}^{\pi(\frac{n}{2})-\pi(\sqrt{n})}\pi(\frac{n}{p_{\pi(\sqrt{n})+k}}). \tag{7}$$

If we compare (5) with (7) we obtain for $n \geq 4$

$$\pi_{\mathcal{PQ}}(n) = \frac{\pi(\sqrt{n}).(\pi(\sqrt{n})-1)}{2} + \sum_{k=1}^{\pi(\frac{n}{2})-\pi(\sqrt{n})}\pi(\frac{n}{p_{\pi(\sqrt{n})+k}}). \tag{8}$$

Thus, we have two different explicit representations for $\pi_{\mathcal{PQ}}(n)$. These are formulae (5) and (8). We note that the right hand-side of (8) reduces to $\frac{\pi(\sqrt{n}).(\pi(\sqrt{n})-1)}{2}$, when $\pi(\frac{n}{2}) = \pi(\sqrt{n})$.

Finally, we observe that (1) gives an explicit representation for $\pi_S(n)$, since we may use formula (2) for $\pi(n)$ (or other explicit formulae for $\pi(n)$) and (5), or (8) for $\pi_{\mathcal{PQ}}(n)$.

The following assertion solves the problem for finding of the explicit representation of s_n.

Theorem. The n-th term s_n of S admits the following three different explicit representations:

$$s_n = \sum_{k=0}^{\theta(n)}[\frac{1}{1+[\frac{\pi_S(k)}{n}]}]; \tag{9}$$

$$s_n = -2\sum_{k=0}^{\theta(n)}\zeta(-2[\frac{\pi_S(k)}{n}]); \tag{10}$$

$$s_n = \sum_{k=0}^{\theta(n)}\frac{1}{\Gamma(1-[\frac{\pi_S(k)}{n}])}, \tag{11}$$

where

$$\theta(n) \equiv [\frac{n^2+3n+4}{4}],\ n = 1, 2, ... \tag{12}$$

Remark. We note that (9)–(11) are representations using, respectively, "floor function", "Riemann's Zeta-function and Euler's Gamma-function. Also, we note that in (9)–(11) $\pi_S(k)$ is given by (1), $\pi(k)$ is given by (2) (or by others formulae like (2)) and $\pi_{\mathcal{PQ}}(n)$ is given by (5), or by (8). Therefore, formulae (9)–(11) are explicit.
Proof of the Theorem. In **A2** the following three universal formulae are proposed, using $\pi_C(k)$ $(k = 0, 1, ...)$, each one of them could apply to represent c_n. They are the following

$$c_n = \sum_{k=0}^{\infty}[\frac{1}{1+[\frac{\pi_C(k)}{n}]}]; \tag{13}$$

$$c_n = -2\sum_{k=0}^{\infty}\zeta(-2[\frac{\pi_C(k)}{n}]); \tag{14}$$

$$c_n = \sum_{k=0}^{\infty}\frac{1}{\Gamma(1-[\frac{\pi_C(k)}{n}])}. \tag{15}$$

In [16] is shown that the inequality

$$p_n \le \theta(n),\ n = 1, 2, ..., \tag{16}$$

holds. Hence

$$s_n \le \theta(n), n = 1, 2, ..., \tag{17}$$

since we have obviously

$$s_n \le p_n,\ n = 1, 2, ... \tag{18}$$

Then, to prove the Theorem it remains only to apply (13)–(15) in the case $C = S$, i.e., for $c_n = s_n$, putting there $\pi_S(k)$ instead of $\pi_C(k)$ and $\theta(n)$ instead of ∞.

4. ON THE 17-TH SMARANDACHE'S PROBLEM[4]

The 17-th problem from [13] (see also the 22-nd problem from [24]) is the following:

Smarandache's digital products:

$$\underbrace{0,1,2,3,4,5,6,7,8,9}, \underbrace{0,1,2,3,4,5,6,7,8,9},$$

$$\underbrace{0,2,4,6,8,19,12,14,16,18}, \underbrace{0,3,6,9,12,15,18,21,24,27},$$

$$\underbrace{0,4,8,12,16,20,24,28,32,36}, \underbrace{0,5,10,15,20,25...}$$

($d_p(n)$ is the product of digits.)

Let the fixed natural number n have the form $n = \overline{a_1a_2...a_k}$, where $a_1, a_2, ..., a_k \in \{0, 1, ..., 9\}$ and $a_1 \ge 1$. Therefore,

$$n = \sum_{i=1}^{k} a_i 10^{i-1}.$$

Hence, $k = [\log_{10} n] + 1$ and

$$a_1(n) \equiv a_1 = [\frac{n}{10^{k-1}}],$$

$$a_2(n) \equiv a_2 = [\frac{n - a_1 10^{k-1}}{10^{k-2}}],$$

$$a_3(n) \equiv a_3 = [\frac{n - a_1 10^{k-1} - a_2 10^{k-2}}{10^{k-3}}],$$

$$\cdots$$

$$a_{[\log_{10} n]}(n) \equiv a_{k-1} = [\frac{n - a_1 10^{k-1} - ... - a_{k-2}10^2}{10}],$$

[4]The results in this section are taken from [7]

$$a_{[\log_{10} n]+1}(n) \equiv a_k = n - a_1 10^{k-1} - ... - a_{k-1}10.$$

Obviously, $k, a_1, a_2, ..., a_k$ are functions only of n. Therefore,

$$d_p(n) = \prod_{i=1}^{[\log_{10} n]+1} a_i(n).$$

5. ON THE 20-TH AND THE 21-ST SMARANDACHE'S PROBLEMS[5]

The 20-th problem from [13] is the following (see also Problem 25 from [24]):
Smarandache divisor products:

$$1, 2, 3, 8, 5, 36, 7, 64, 27, 100, 11, 1728, 13, 196, 225, 1024, 17, 5832, 19,$$

$$8000, 441, 484, 23, 331776, 125, 676, 729, 21952, 29, 810000, 31, 32768,$$

$$1089, 1156, 1225, 10077696, 37, 1444, 1521, 2560000, 41, ...$$

($P_d(n)$ is the product of all positive divisors of n.)

The 21-st problem from [13] is the following (see also Problem 26 from [24]:
Smarandache proper divisor products:

$$1, 1, 1, 2, 1, 6, 1, 8, 3, 10, 1, 144, 1, 14, 15, 64, 1, 324, 1, 400, 21, 22, 1,$$

$$13824, 5, 26, 27, 784, 1, 27000, 1, 1024, 33, 34, 35, 279936, 1, 38, 39,$$

$$64000, 1, ...$$

($p_d(n)$ is the product of all positive divisors of n but n.)

Let us denote by $\tau(n)$ the number of all divisors of n. It is well-known (see, e.g., [17]) that

$$P_d(n) = \sqrt{n^{\tau(n)}} \tag{1}$$

and of course, we have

$$p_d(n) = \frac{P_d(n)}{n}. \tag{2}$$

But (1) is not a good formula for $P_d(n)$, because it depends on function τ and to express $\tau(n)$ we need the prime number factorization of n.

[5] The results in this section are taken from [5, 40]. In [5] there are some misprints.

Below, we give other representations of $P_d(n)$ and $p_d(n)$, which do not use the prime number factorization of n.
Proposition 1. For $n \geq 1$ representation

$$P_d(n) = \prod_{k=1}^{n} k^{[\frac{n}{k}]-[\frac{n-1}{k}]} \tag{3}$$

holds.
Proof. We have

$$\theta(n,k) \equiv [\frac{n}{k}] - [\frac{n-1}{k}]$$

$$= \begin{cases} 1, & \text{if } k \text{ is a divisor of } n \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

Therefore,

$$\prod_{k=1}^{n} k^{[\frac{n}{k}]-[\frac{n-1}{k}]} = \prod_{k/n} k \equiv P_d(n)$$

and Proposition 1 is proved.

Here and further the symbols

$$\prod_{k/n} \bullet \text{ and } \sum_{k/n} \bullet$$

mean the product and the sum, respectively, of all divisors of n.

The following assertion is obtained as a corollary of (2) and (3).
Proposition 2. For $n \geq 1$ representation

$$p_d(n) = \prod_{k=1}^{n-1} k^{[\frac{n}{k}]-[\frac{n-1}{k}]} \tag{5}$$

holds.

For $n = 1$ we have

$$p_d(1) = 1.$$

Proposition 3. For $n \geq 1$ representation

$$P_d(n) = \prod_{k=1}^{n} \frac{[\frac{n}{k}]!}{[\frac{n-1}{k}]!} \tag{6}$$

holds, where here and further we assume that $0! = 1$.
Proof. Obviously, we have

$$\frac{[\frac{n}{k}]!}{[\frac{n-1}{k}]!} = \begin{cases} \frac{n}{k}, & \text{if } k \text{ is a divisor of } n \\ 1, & \text{otherwise} \end{cases}.$$

Hence

$$\prod_{k=1}^{n} \frac{[\frac{n}{k}]!}{[\frac{n-1}{k}]!} = \prod_{k/n} \frac{n}{k} = \prod_{k/n} k \equiv P_d(n),$$

since, if k describes all divisors of n, then $\frac{n}{k}$ describes them, too.

Now (2) and (6) yield.
Proposition 4. For $n \geq 2$ representation

$$p_d(n) = \prod_{k=2}^{n} \frac{[\frac{n}{k}]!}{[\frac{n-1}{k}]!} \tag{7}$$

holds.

Another type of representation of $p_d(n)$ is the following
Proposition 5. For $n \geq 3$ representation

$$p_d(n) = \prod_{k=1}^{n-2} (k!)^{\theta(n,k)-\theta(n,k+1)}, \tag{8}$$

where $\theta(n,k)$ is given by (4).
Proof. Let

$$r(n,k) = \theta(n,k) - \theta(n,k+1).$$

The assertion holds from the fact, that

$$r(n,k) = \begin{cases} 1, & \text{if } k \text{ is a divisor of } n \text{ and} \\ & \quad k+1 \text{ is not a divisor of } n \\ -1, & \text{if } k \text{ is not a divisor of } n \text{ and} \\ & \quad k+1 \text{ is a divisor of } n \\ 0, & \text{otherwise} \end{cases}$$

Further, we need the following

Theorem.[6] For $n \geq 2$ the identity

$$\prod_{k=2}^{n} [\frac{n}{k}]! = \prod_{k=1}^{n-1} (k!)^{[\frac{n}{k}]-[\frac{n}{k+1}]} \tag{9}$$

holds.

Proof. By induction. For $n = 2$ (9) is true. Let us assume, that (9) holds for some $n \geq 2$. Then we must prove that

$$\prod_{k=2}^{n+1} [\frac{n+1}{k}]! = \prod_{k=1}^{n} (k!)^{[\frac{n+1}{k}]-[\frac{n+1}{k+1}]} \tag{10}$$

holds, too.

Dividing (10) by (9) we obtain

$$\prod_{k=2}^{n} \frac{[\frac{n+1}{k}]!}{[\frac{n}{k}]!} = \prod_{k=1}^{n-1} (k!)^{r(n+1,k)}. \tag{11}$$

Sinse, for $k = n + 1$

$$[\frac{n+1}{k}]! = 1$$

and for k = n

$$[\frac{n+1}{k}] - [\frac{n+1}{k+1}] = 0.$$

[6]The Theorem is published in [35]

Then (10) is true, if and only if (11) is true. Therefore, we must prove (11) for proving of the Theorem.

From (7), the left hand-side of (11) is equal to $p_d(n+1)$. From (8), the right side of (11) is equal to $p_d(n+1)$, too. Thetrefore, (11) is true.

Now, we shall deduce some formulae for

$$\prod_{k=1}^{n} P_d(k) \text{ and } \prod_{k=1}^{n} p_d(k).$$

Proposition 6. Let f be an arbitrary arithmetic function. Then the identity

$$\prod_{k=1}^{n} (P_d(k))^{f(k)} = \prod_{k=1}^{n} k^{\rho(n,k)} \tag{12}$$

holds, where

$$\rho(n,k) = \sum_{s=1}^{[\frac{n}{k}]} f(ks).$$

Proof. We use a well-known Dirichlet's identity

$$\sum_{k \leq n} f(k). \sum_{t/k} g(t) = \sum_{k \leq n} g(k). \sum_{s \leq \frac{n}{k}} f(ks),$$

where g is also arbitrary arithmetic function. Putting there $g(x) = \ln x$ for every real positive number x, we obtain (12), since

$$P_d(k) = \prod_{t/k} t.$$

When $f(x) \equiv 1$, as a corollary from (12) we obtain

Proposition 7. For $n \geq 1$ the identity

$$\prod_{k=1}^{n} P_d(k) = \prod_{k=1}^{n} k^{[\frac{n}{k}]} \tag{13}$$

holds.

Now, we need the following

Lemma. For $n \geq 1$ the identity

$$\prod_{k=1}^{n} [\frac{n}{k}]! = \prod_{k=1}^{n} k^{[\frac{n}{k}]} \tag{14}$$

holds.

Proof. In the identity

$$\sum_{k\leq n} f(k). \sum_{s\leq \frac{n}{k}} g(s) = \sum_{k\leq n} g(k). \sum_{s\leq \frac{n}{k}} f(s),$$

that is valid for arbitrary two arithmetic functions f and g, we put:

$$g(x) \equiv 1,$$

$$f(x) = \ln x$$

for any positive real number x and (14) is proved.

From (13) and (14) we obtain

Proposition 8. For $n \geq 1$ the identity

$$\prod_{k=1}^{n} P_d(k) = \prod_{k=1}^{n} [\frac{n}{k}]! \tag{15}$$

holds.

As a corollary from (2) and (15), we also obtain

Proposition 9. For $n \geq 2$ the identity

$$\prod_{k=1}^{n} p_d(k) = \prod_{k=2}^{n} [\frac{n}{k}]! \tag{16}$$

holds.

Fom (9) and (16), we obtain

Proposition 10. For $n \geq 2$ the identity

$$\prod_{k=1}^{n} p_d(k) = \prod_{k=1}^{n-1} (k!)^{[\frac{n}{k}]-[\frac{n}{k+1}]} \tag{17}$$

holds.

As a corollary from (17) we obtain, because of (2)

Proposition 11. For $n \geq 1$ the identity

$$\prod_{k=1}^{n} P_d(k) = \prod_{k=1}^{n} (k!)^{[\frac{n}{k}]-[\frac{n}{k+1}]} \tag{18}$$

holds.

Now, we return to (12) and suppose that

$$f(k) > 0 \ (k = 1, 2, ...).$$

Then after some simple computations we obtain

Proposition 12. For $n \geq 1$ representation

$$P_d(n) = \prod_{k=1}^{n} k^{\sigma(n,k)} \tag{19}$$

holds, where

$$\sigma(n,k) = \frac{\sum\limits_{s=1}^{[\frac{n}{k}]} f(ks) - \sum\limits_{s=1}^{[\frac{n-1}{k}]} f(ks)}{f(n)}.$$

For $n \geq 2$ representation

$$p_d(n) = \prod_{k=1}^{n-1} k^{\sigma(n,k)} \tag{20}$$

holds.

Note that although f is an arbitrary arithmetic function, the situation with (19) and (20) is like the case $f(x) \equiv 1$, because

$$\frac{\sum\limits_{s=1}^{[\frac{n}{k}]} f(ks) - \sum\limits_{s=1}^{[\frac{n-1}{k}]} f(ks)}{f(n)} = \begin{cases} 1, & \text{if } k \text{ is a divisor of } n \\ 0, & \text{otherwise} \end{cases}$$

Finally, we use (12) to obtain some new inequalities, involving $P_d(k)$ and $p_d(k)$ for $k = 1, 2, ..., n$.

Putting

$$F(n) = \sum_{k=1}^{n} f(k)$$

we rewrite (12) as

$$\prod_{k=1}^{n} (P_d(k))^{\frac{f(k)}{F(n)}} = \prod_{k=1}^{n} k^{(\sum_{s=1}^{[\frac{n}{k}]} f(ks))/(F(n))}. \qquad (21)$$

Then we use the well-known Jensen's inequality

$$\sum_{k=1}^{n} \alpha_k x_k \geq \prod_{k=1}^{n} x_k^{\alpha_k},$$

that is valid for arbitrary positive numbers x_k, α_k $(k = 1, 2, ..., n)$ such that

$$\sum_{k=1}^{n} \alpha_k = 1,$$

for the case:

$$x_k = P_d(k),$$

$$\alpha_k = \frac{f(k)}{F(n)}.$$

Thus we obtain from (21) inequality

$$\sum_{k=1}^{n} f(k).P_d(k) \geq (\sum_{k=1}^{n} f(k)). \prod_{k=1}^{n} k^{(\sum_{s=1}^{[\frac{n}{k}]} f(ks))/(\sum_{s=1}^{n} f(s))}. \qquad (22)$$

If $f(x) \equiv 1$ then (22) yields the inequality

$$\frac{1}{n} \sum_{k=1}^{n} P_d(k) \geq \prod_{k=1}^{n} (\sqrt[n]{k})^{[\frac{n}{k}]}. \qquad (23)$$

If we put in (22)

$$f(k) = \frac{g(k)}{k}$$

for $k = 1, 2, ..., n$, then we obtain

$$\sum_{k=1}^{n} g(k).p_d(k) \geq (\sum_{k=1}^{n} \frac{g(k)}{k}). \prod_{k=1}^{n} (\sqrt[k]{k})^{(\sum_{s=1}^{[\frac{n}{k}]} \frac{g(ks)}{s})/(\sum_{s=1}^{n} \frac{g(s)}{s})}, \qquad (24)$$

because of (2).

Let $g(x) \equiv 1$. Then (24) yields the very interesting inequality

$$(\frac{1}{H_n} \sum_{k=1}^{n} p_d(k))^{H_n} \geq \prod_{k=1}^{n} (\sqrt[k]{k})^{H_{[\frac{n}{k}]}},$$

where H_m denotes the m-th partial sum of the harmonic series, i.e.,

$$H_m = \frac{1}{1} + \frac{1}{2} + ... + \frac{1}{m}.$$

All of the above inequalities become equalities if and only if $n = 1$.

6. ON THE 25-TH AND THE 26-TH SMARANDACHE'S PROBLEMS[7]

The 25-th and the 26-th problems from [13] (see also the 30-th and the 31-st problems from [24]) are the following:

Smarandache's cube free sieve:

$$2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15, 17, 18, 19, 20, 21, 22, 23, 25, 26,$$

$$28, 29, 30, 31, 33, 34, 35, 36, 37, 38, 39, 41, 42, 43, 44, 45, 46, 47, 49, 50,$$

$$51, 52, 53, 55, 57, 58, 59, 60, 61, 62, 63, 65, 66, 67, 68, 69, 70, 71, 73, ...$$

Definition: from the set of natural numbers (except 0 and 1):
– take off all multiples of 2^3 (i.e. 8,16,24,32,40,...)
– take off all multiples of 3^3
– take off all multiples of 5^3
... and so on (take off all multiples of all cubic primes).
(One obtains all cube free numbers.)

Smarandache's m-power free sieve:
Definition: from the set of natural numbers (except 0 and 1) take off all multiples of 2^m, afterwards all multiples of 3^m ... and so on (take off all multiples of all m-power primes, $m \geq 2$).
(One obtains all m-power free numbers.)

Here we introduce the solutions for both of these problems.

For every natural number m we denote the increasing sequence $a_1^{(m)}, a_2^{(m)}, a_3^{(m)}, ...$ of all m-power free numbers by $\overline{m}$. Then we have

$$\emptyset \equiv \overline{1} \subset \overline{2} ... \subset \overline{(m-1)} \subset \overline{m} \subset \overline{(m+1)} \subset ...$$

[7]The results in this section are taken from [41]

Also, for $m \geq 2$ we have

$$\overline{m} = \bigcup_{k=1}^{m-1} (\overline{2})^k$$

where

$$(\overline{2})^k = \{x \mid (\exists x_1, ..., x_k \in \overline{2})(x = x_1.x_2. \ldots .x_k)\}$$

for each natural number $k \geq 1$.

Let us consider $\overline{m}$ as an infinite sequence for $m = 2, 3, ...$ Then $\overline{2}$ is a subsequence of $\overline{m}$. Therefore, the inequality

$$a_n^{(m)} \leq a_n^{(2)}$$

holds for $n = 1, 2, 3, ...$

Let $p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7, ...$ be the sequence of all primes. It is obvious that this sequence is a subsequence of $\overline{2}$. Hence, the inequality

$$a_n^{(2)} \leq p_n$$

holds for $n = 1, 2, 3, ...$ But it is well-known that

$$p_n \leq \theta(n) \equiv [\frac{n^2 + 3n + 4}{4}] \qquad (1)$$

(see [16]). Therefore, for any $m \geq 2$ and $n = 1, 2, 3, ...$ we have

$$a_n^{(m)} \leq a_n^{(2)} \leq \theta(n).$$

Hence, there exists $\lambda(n)$ such that $\lambda(n) \leq \theta(n)$ and inequality:

$$a_n^{(m)} \leq a_n^{(2)} \leq \lambda(n) \qquad (2)$$

holds. In particular, it is possible to use $\theta(n)$ instead of $\lambda(n)$.

Further, we will find an explicit formula for $a_n^{(m)}$ when $m \geq 2$ is fixed.

Let for any real x

$$sg(x) = \begin{cases} 1, & x > 0 \\ 0, & x \leq 0 \end{cases}.$$

We define

$$\varepsilon_m(k) = \begin{cases} 1, & k \in \overline{m} \\ 0, & k \notin \overline{m} \end{cases}.$$

Hence,

$$\pi_{\overline{m}}(n) = \sum_{k=2}^{n} \varepsilon_m(k), \qquad (3)$$

where $\pi_{\overline{m}}(n)$ is the number of terms of set $\overline{m}$, which are not greater than n. Using the relation

$$\varepsilon_m(k) = sg\Big(\prod_{\substack{p|k \\ p \text{ is prime}}} \Big[\frac{m-1}{ord_p k}\Big]\Big)$$

we rewrite (3) in the explicit form

$$\pi_{\overline{m}}(n) = \sum_{k=2}^{n} sg\Big(\prod_{\substack{p|k \\ p \text{ is prime}}} \Big[\frac{m-1}{ord_p k}\Big]\Big). \qquad (4)$$

Then, using formulae (1')–(3') from **A4** (that are the universal formulae for the n-th term of an arbitrary increasing sequence of natural numbers) and (2), with $\lambda(n)$ from (2), we obtain

$$a_n^{(m)} = \sum_{k=0}^{\lambda(n)} \Big[\frac{1}{1 + [\frac{\pi_{\overline{m}}(k)}{n}]}\Big]; \qquad (5)$$

(a representation using "floor function"),

$$a_n^{(m)} = -2 \sum_{k=0}^{\lambda(n)} \zeta\Big(-2\Big[\frac{\pi_{\overline{m}}(k)}{n}\Big]\Big); \qquad (6)$$

(a representation using Riemann's Zeta-function),

$$a_n^{(m)} = \sum_{k=0}^{\lambda(n)} \frac{1}{\Gamma(1 - [\frac{\pi_{\overline{m}}(k)}{n}])}, \qquad (7)$$

(a representation using Euler's Gamma-function).

Note that (5)–(7) are explicit formulae, because of (4) and these formulae are valid, too, if we put $\theta(n)$ instead of $\lambda(n)$.

Thus, the 26-th Smarandache's problem is solved and for $m = 3$ the 25-th Smarandache's problem is solved, too.

For $m = 2$ we have the representation

$$\varepsilon_2(k) = |\mu(k)|$$

(here μ is the Möbius function);

$$|\mu(k)| = \Big[\frac{2^{\omega(k)}}{\tau(k)}\Big],$$

where $\omega(k)$ denotes the number of all different prime divisors of k and

$$\tau(k) = \sum_{d|k} 1.$$

Hence,

$$\pi_{\overline{2}}(n) = \sum_{k=2}^{n} |\mu(k)| = \sum_{k=2}^{n} \Big[\frac{2^{\omega(k)}}{\tau(k)}\Big].$$

The following problems are interesting.

Problem 1. Does there exist a constant $C > 1$, such that $\lambda(n) \leq C.n$?

Problem 2. Is $C \leq 2$?

* * *

Below we give the main explicit representation of function $\pi_{\overline{m}}(n)$, that takes part in formulae (5)–(7). In this way we find the main explicit representation for $a_n^{(m)}$, that is based on formulae (5)–(7), too.

Theorem. Function $\pi_{\overline{m}}(n)$ allows representation

$$\pi_{\overline{m}}(n) = n - 1 + \sum_{s \in \overline{2} \cap \{2,3,\dots,[\sqrt[m]{n}]\}} (-1)^{\omega(s)}.[\frac{n}{s^m}]. \qquad (8)$$

Proof. First, we shall note that the variable s from the sum in the right hand-side of (8) is element of the set of only these natural numbers, smaller than $[\sqrt[m]{n}]$, such that $s \in \overline{2}$, i.e., the natural numbers s such that $\mu(s) \neq 0$.

Let $\{b_n^{(m)}\}_{m=1}^{\infty}$ be the sequence defined by

$$b_1^{(m)} = 1, \; b_n^{(m)} = a_{n-1}^{(m)} \text{ for } n \geq 2. \qquad (9)$$

We denote this sequence by m^*.

Let $\pi_{m^*}(n)$ denote the number of terms of m^*, that are not greater than n. Then we have the relation

$$\pi_{\overline{m}}(n) = \pi_{m^*}(n) - 1, \qquad (10)$$

because of (9).

Let $g^{(m)}(k)$ be the function given by

$$g^{(m)}(k) = \begin{cases} 1, & k \in m^* \\ 0, & k \notin m^* \end{cases}. \qquad (11)$$

Then $g^{(m)}(k)$ is a multiplicative function with respect to k, i.e., $g^{(m)}(1) = 1$ and for every two natural numbers a and b, such that $(a, b) = 1$, the relation

$$g^{(m)}(a.b) = g^{(m)}(a).g^{(m)}(b)$$

holds.

Let function $f^{(m)}(k)$ be introduced by

$$f^{(m)}(k) = \sum_{d/k} \mu(\frac{k}{d}) g^{(m)}(d). \qquad (12)$$

Using (12) for $k = p^\alpha$, where p is an arbitrary prime and α is an arbitrary natural number, we obtain

$$f^{(m)}(p^\alpha) = g^{(m)}(p^\alpha) - g^{(m)}(p^{\alpha-1}).$$

Hence,

$$f^{(m)}(p^\alpha) = \begin{cases} 0, & \alpha < m \\ -1, & \alpha = m \\ 0, & \alpha > m \end{cases},$$

because of (11).

Therefore, $f^{(m)}(1) = 1$ and for $k \geq 2$ we have

$$f^{(m)}(k) = \begin{cases} (-1)^{\omega(s)}, & \text{if } k = s^m \text{ and } s \in \overline{2} \\ 0, & \text{otherwise} \end{cases}, \qquad (13)$$

since $f^{(m)}(k)$ is a multiplicative function with respect to k, because of (12).

Using the Möbius inversion formula, equality (12) yields

$$g^{(m)}(k) = \sum_{d/k} f^{(m)}(d). \qquad (14)$$

Now, we use (14) and the representation

$$\pi_{m^*}(n) = \sum_{k=1}^{n} g^{(m)}(k) \qquad (15)$$

in order to obtain

$$\pi_{m^*}(n) = \sum_{k=1}^{n} \sum_{d/k} f^{(m)}(d). \qquad (16)$$

Then both (16) and the identity

$$\sum_{k=1}^{n} \sum_{d/k} f^{(m)}(d) = \sum_{k=1}^{n} f^{(m)}(k).[\frac{n}{k}] \qquad (17)$$

both yield

$$\pi_{m^*}(n) = \sum_{k=1}^{n} f^{(m)}(k).[\frac{n}{k}]. \qquad (18)$$

From (13) and (18) we obtain (8), because of (10) and the fact that $f^{(m)}(1) = 1$. The Theorem is proved.

Finally, we note that some of authors call function $(-1)^{\omega(s)}$ unitary analogue of the Möbius function $\mu(s)$ and denote this function by $\mu^*(s)$ (see [11, 19]). So, if we agree to use the last notation, we may rewrite formula (8) in the form

$$\pi_{\overline{m}}(n) = n - 1 + \sum_{s \in \overline{2} \cap \{2,3,\ldots,[\sqrt[m]{n}]\}} \mu^*(s).[\frac{n}{s^m}].$$

7. ON THE 28-TH SMARANDACHE'S PROBLEM[8]

The 28-th problem from [13] (see also the 94-th problem from [24]) is the following:

Smarandache odd sieve:

$$7, 13, 19, 23, 25, 31, 33, 37, 43, 47, 49, 53, 55, 61, 63, 67, 73, 75, 83,$$

$$85, 91, 93, 97, \ldots$$

(All odd numbers that are not equal to the difference of two primes).
A sieve is used to get this sequence:
- substract 2 from all prime numbers and obtain a temporary sequence;
- choose all odd numbers that do not belong to the temporary one.

We find an explicit form of the n-th term of the above sequence, that will be denoted by $C = \{C_n\}_{n=1}^{\infty}$ below. Let $\pi_C(n)$ be the number of 'he terms of C which are not greater than n. In particular, $\pi_C(0) = 0$.

Firstly, we shall note that the above definition of C can be interpreted to the following equivalent form as follows, having in mind that every odd number is a difference of two prime numbers if and only if it is a difference of a prime number and 2:

Smarandache's odd sieve contains exaclty these odd numbers that cannot be represented as a difference of a prime number and 2.

We can rewrite the last definition to the following equivalent form, too:

Smarandache's odd sieve contains exaclty these odd numbers that are represented as a difference of a composite odd number and 2.

We shall find an explicit form of the n-th term of the above sequence, using the third definition of it. Initially, we shall prove the

[8] The results in this section are taken from [37]

following two Lemmas.

Lemma 1. For every natural number $n \geq 1$, C_{n+1} is exactly one of the numbers: $u \equiv C_n + 2$, $v \equiv C_n + 4$ or $w \equiv C_n + 6$.

Proof. Let us assume that none of the numbers u, v, w coincides with C_{n+1}. Having in mind the third form of the above definition, number u is composite and by assumption u is not a member of sequence C. Therefore v, according to the third form of the definition is a prime number and by assumption it is not a member of sequence C. Finally, w, according to the third form of the definition is a prime number and by assumption it is not a member of sequence C. Therefore, according to the third form of the definition number $w + 2$ is prime.

Hence, from our assumptions we obtained that all of the numbers v, w and $w+2$ are prime, which is impossible, because these numbers are consecutive odd numbers and having in mind that $v = C_n + 4$ and $C_1 = 7$, the smallest of them satisfies the inequality $v \geq 11$.

Corollary. For every natural number $n \geq 1$:

$$C_{n+1} \leq C_n + 6. \tag{1}$$

Lemma 2. For every natural number $n \geq 1$:

$$C_n \leq 6n + 1. \tag{2}$$

Proof. We use induction. For $n = 1$ obviously we have the equality. Let us assume that (2) holds for some n. We shall prove that

$$C_{n+1} \leq 6(n+1) + 1. \tag{3}$$

By (1) and the induction assumption it follows that

$$C_{n+1} \leq C_n + 6 \leq (6n+1) + 6 = 6(n+1) + 1,$$

which proves (3).

Now, we return to the Smarandache's problem.

Let $\pi_C(N)$ be the number of the members of the sequence $\{C_n\}_{n=1}^{\infty}$ that are not greater than N. In particular, $\pi_C(0) = 0$.

In **A2** the following three universal explicit formulae are introduced, using numbers $\pi_C(k)$ $(k = 0, 1, 2, ...)$, that can be used to represent numbers C_n:

$$C_n = \sum_{k=0}^{\infty} [\frac{1}{1 + [\frac{\pi_C(k)}{n}]}], \tag{4}$$

$$C_n = -2. \sum_{k=0}^{\infty} \zeta(-2.[\frac{\pi_C(k)}{n}]), \tag{5}$$

$$C_n = \sum_{k=0}^{\infty} \frac{1}{\Gamma(1 - [\frac{\pi_C(k)}{n}])}. \tag{6}$$

For the present case, having in mind (2), we substitute symbol ∞ with $6n + 1$ in sum $\sum\limits_{k=0}^{\infty}$ for C_n and we obtain the following sums:

$$C_n = \sum_{k=0}^{6n+1} [\frac{1}{1 + [\frac{\pi_C(k)}{n}]}], \tag{7}$$

$$C_n = -2. \sum_{k=0}^{6n+1} \zeta(-2.[\frac{\pi_C(k)}{n}]), \tag{8}$$

$$C_n = \sum_{k=0}^{6n+1} \frac{1}{\Gamma(1 - [\frac{\pi_C(k)}{n}])}. \tag{9}$$

We must show why $\pi_C(n)$ $(n = 1, 2, 3, ...)$ is represented in an explicit form. It can be directly seen that the number of the odd numbers, that are not bigger than n, is exactly equal to

$$\alpha(n) = n - [\frac{n}{2}], \tag{10}$$

because the number of the even numbers that are not greater that n is exactly equal to $[\frac{n}{2}]$.

Let us denote by $\beta(n)$ the number of all odd numbers not bigger that n, that can be represented as a difference of two primes. According the second form of the above given definition, $\beta(n)$ coincides with the number of all odd numbers m such that $m \leq n$ and m has the form $m = p - 2$, where p is an odd prime number. Therefore, we must study all odd prime numbers, because of the inequality $m \leq n$. The number of these prime numbers is exactly $\pi(n+2) - 1$. Therefore,

$$\beta(n) = \pi(n+2) - 1. \tag{11}$$

Omitting from the number of all odd numbers that are not greater than n the quantity of those numbers that are a difference of two primes, we find exactly the quantity of these odd numbers that are not greater than n and that are not a difference of two prime numbers, i.e., $\pi_C(n)$. Therefore, the equality

$$\pi_C(n) = \alpha(n) - \beta(n)$$

holds and from (10) and (11) we obtain:

$$\pi_C(n) = (n - [\frac{n}{2}]) - (\pi(n+2) - 1) = n + 1 - [\frac{n}{2}]) - \pi(n+2),$$

where $\pi(m)$ is the number of primes p such that $p \leq m$. But $\pi(n+2)$ can be represented in an explicit form, e.g., by Mináč's formula (see **A2**):

$$\pi(n+2) = \sum_{k=2}^{n+2}[\frac{(k-1)!+1}{k} - [\frac{(k-1)!}{k}]],$$

and therefore, we obtain that the explicit form of $\pi_C(N)$ is

$$\pi_C(N) = N + 1 - [\frac{N}{2}] - \sum_{k=2}^{N+2}[\frac{(k-1)!+1}{k} - [\frac{(k-1)!}{k}]], \tag{12}$$

where $N \geq 1$ is a fixed natural number.

It is possible to put $[\frac{N+3}{2}]$ instead of $N + 1 - [\frac{N}{2}]$ into (12).

Now, using each of the formulae (7) – (9), we obtain C_n in an explicit form, using (12).

It can be checked directly that

$$C_1 = 7,\ C_2 = 13,\ C_3 = 19,\ C_4 = 23,\ C_5 = 25,\ C_6 = 31,$$

$$C_7 = 33, \ldots$$

and

$$\pi_C(0) = \pi_C(1) = \pi_C(2) = \pi_C(3) = \pi_C(4) = \pi_C(5) = \pi_C(6) = 0.$$

Therefore from (7)–(9) we have the following explicit formulae for C_n

$$C_n = 7 + \sum_{k=7}^{6n+1}[\frac{1}{1 + [\frac{\pi_C(k)}{n}]}],$$

$$C_n = 7 - 2. \sum_{k=7}^{6n+1}\zeta(-2.[\frac{\pi_C(k)}{n}]),$$

$$C_n = 7 + \sum_{k=7}^{6n+1}\frac{1}{\Gamma(1 - [\frac{\pi_C(k)}{n}])},$$

where $\pi_C(k)$ is given by (12).

8. ON THE 46-TH SMARANDACHE'S PROBLEM[9]

The 46-th Smarandache's problem from [13] is the following:

Smarandache's prime additive complements:

$$1, 0, 0, 1, 0, 1, 0, 3, 2, 1, 0, 1, 0, 3, 2, 1, 0, 1, 0, 3, 2, 1, 0, 1, 0, 5, 4, 3, 2, 1,$$

$$0, 1, 0, 5, 4, 3, 2, 1, 0, 3, 2, 1, 0, 5, 4, 3, 2, 1, 0, ...$$

(For each n to find the smallest k such that $n + k$ is prime.)

Remark: Smarandache asked if it is possible to get as large as we want but finite decreasing k, $k-1$, $k-2, ..., 2, 1, 0$ (odd k) sequence included in the previous sequence – i.e., for any even integer are there two primes those difference is equal to it? He conjectured the answer is negative.

Obviously, the members of the above sequence are differences between first prime number that is greater or equal to the current natural number n and the same n. It is well-known that the number of primes smaller than or equal to n is $\pi(n)$. Therefore, the prime number smaller than or equal to n is $p_{\pi(n)}$. Hence, the prime number that is greater than or equal to n is the next prime number, i.e., $p_{\pi(n)+1}$. Finally, the n-th member of the above sequence will be equal to

$$\begin{cases} p_{\pi(n)+1} - n, & \text{if } n \text{ is not a prime number} \\ 0, & \text{otherwise} \end{cases}$$

We shall note that in [4] the following new formula p_n for every natural number n is given:

$$p_n = \sum_{i=0}^{\theta(n)} sg(n - \pi(i)),$$

[9] The results in this section are taken from [8, 39]

where $\theta(n) = [\frac{n^2 + 3n + 4}{4}]$ (for $\theta(n)$ see **A2**) and

$$sg(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ 1, & \text{if } x > 0 \end{cases},$$

Let us denote by a_n the n-th term of the above sequence. Next, we propose a way for obtaining an explicit formula for a_n ($n = 1, 2, 3, ...$). Extending the below results, we give an answer to the Smarandache's question from his own Remark in [13]. At the end, we propose a generalization of Problem 46 and present a proof of an assertion related to Smarandache's conjecture for Problem 46.

Proposition 1. a_n admits the representation

$$a_n = p_{\pi(n-1)+1} - n, \tag{1}$$

where $n = 1, 2, 3, ...$, π is the prime counting function and p_k is the k-th term of prime number sequence.

The proof is a matter of direct check.

It is clear that (1) gives an explicit representation for a_n since several explicit formulae for $\pi(k)$ and p_k are known (see, e.g. [18]).

Let us define

$$n(m) = m! + 2.$$

Then all numbers

$$n(m), n(m) + 1, n(m) + 2, ..., n(m) + m - 2$$

are composite. Hence

$$a_{n(m)} \geq m - 1.$$

This proves Smarandache's conjecture, since m may grow up to infinity. Therefore $\{a_n\}_{n=1}^{\infty}$ is unbounded sequence.

Now, we shall generalize Problem 46.

Let

$$\mathbf{c} \equiv c_1, c_2, c_3, ...$$

be a strictly increasing sequence of positive integers.

Definition. Sequence

$$\mathbf{b} \equiv b_1, b_2, b_3, ...$$

is called c-additive complement of $\mathbf{c}$ if and only if b_n is the smallest non-negative integer, such that $n + b_n$ is a term of $\mathbf{c}$.

The following assertion generalizes Proposition 1.

Proposition 2. b_n admits the representation

$$b_n = c_{\pi_c(n-1)+1} - n, \tag{2}$$

where $n = 1, 2, 3, ...,$ $\pi_c(n)$ is the counting function of $\mathbf{c}$, i.e., $\pi_c(n)$ equals to the quantity of $c_m, m = 1, 2, 3, ...,$ such that $c_m \leq n$.

We omit the proof since it is again a matter of direct check.

Let

$$d_n \equiv c_{n+1} - c_n \ (n = 1, 2, 3, ...).$$

The following assertion is related to Smarandache's conjecture from Problem 46.

Proposition 3. If $\{d_n\}_{n=1}^{\infty}$ is unbounded sequence, then $\{b_n\}_{n=1}^{\infty}$ is unbounded sequence, too.

Proof. Let $\{d_n\}_{n=1}^{\infty}$ be unbounded sequence. Then there exists a strictly increasing sequence of natural numbers $\{n_k\}_{k=1}^{\infty}$, such that sequence $\{d_{n_k}\}_{k=1}^{\infty}$ is strictly increasing, too.. Hence $\{d_n\}_{n=1}^{\infty}$ is unbounded sequence, since it contains a strictly increasing sequence of positive integers.

Open Problem. Formulate necessary conditions for the sequence $\{b_n\}_{n=1}^{\infty}$ to be unbounded.

9. ON THE 78-TH SMARANDACHE'S PROBLEM[10]

Solving of the Diophantine equation

$$2x^2 - 3y^2 = 5 \tag{1}$$

i.e.,

$$2x^2 - 3y^2 - 5 = 0$$

was put as an open Problem 78 by F. Smarandache in [24]. Below this problem is solved completely. Also, we consider here the Diophantive equation

$$l^2 - 6m^2 = -5, \tag{2}$$

i.e.,

$$l^2 - 6m^2 + 5 = 0$$

and the Pellian equation

$$u^2 - 6v^2 = 1, \tag{3}$$

i.e.,

$$u^2 - 6v^2 - 1 = 0.$$

Here we use variables x and y only for equation (1) and l, m for equation (2).

If

$$F(t, w) = 0$$

is an Diophantive equation, then:

(a_1) we use the notation $< t, w >$ if and only if t and w are integers which satisfy this equation.

(a_2) we use the denotation $< t, w > \in \mathcal{N}^2$ if and only if t and w are positive integers;

$K(t, w)$ denotes the set of all $< t, w >$;

[10]The results in this section are taken from [36]

$K^o(t,w)$ denotes the set of all $< t,w > \in \mathcal{N}^2$;

$K'(t,w) = K^o(t,w) - \{< 2,1 >\}$.

Lemma 1. If $< t,w > \in \mathcal{N}^2$ and $< x,y > \neq < 2,1 >$, then there exists $< l,m >$, such that $< l,m > \in \mathcal{N}^2$ and the equalities

$$x = l + 3m \text{ and } y = l + 2m \tag{4}$$

hold.

Lemma 2. Let $< l,m > \in \mathcal{N}^2$. If x and y are given by (1), then x and y satisfy (4) and $< x,y > \in \mathcal{N}^2$.

Note that Lemmas 1 and 2 show that the map $\varphi : K^0(l,m) \to K'(x,y)$ given by (4) is a bijection.

Proof of Lemma 1. Let $< x,y > \in \mathcal{N}^2$ be chosen arbitrarily, but $< x,y > \neq < 2,1 >$. Then $y \geq 2$ and $x > y$. Therefore,

$$x = y + m \tag{5}$$

and m is a positive integer. Substituting (5) into (1), we obtain

$$y^2 - 4my + 5 - 2m^2 = 0. \tag{6}$$

Hence

$$y = y_{1,2} = 2m \pm \sqrt{6m^2 - 5}. \tag{7}$$

For $m = 1$ (7) yields only

$$y = y_1 = 3.$$

indeed

$$1 = y = y_2 < 2$$

contradicts to $y \geq 2$.

Let $m > 1$. Then

$$2m - \sqrt{6m^2 - 5} < 0.$$

Therefore $y = y_2$ is impossible again. Thus we always have

$$y = y_1 = 2m + \sqrt{6m^2 - 5}. \tag{8}$$

Hence

$$y - 2m = \sqrt{6m^2 - 5}. \tag{9}$$

The left hand-side of (9) is a positive integer. Therefore, there exists a positive integer l such that

$$6m^2 - 5 = l^2.$$

Hence l and m satisfy (2) and $< l,m > \in \mathcal{N}^2$.

The equalities (4) hold because of (5) and (8).

Proof of Lemma 2. Let $< l,m > \in \mathcal{N}^2$. Then we check the equality

$$2(l + 3m)^2 - 3(l + 2m)^2 = 5,$$

under the assumption of validity of (2) and the Lemma is proved.

Theorem 108 a, Theorem 109 and Theorem 110 from [17] imply the following

Theorem 1. There exist sets $K_i(l,m)$ such that

$$K_i(l,m) \subset K(l,m) \quad (i = 1,2),$$

$$K_1(l,m) \cap K_2(l,m) = \emptyset,$$

and $K(l,m)$ admits the representation

$$K(l,m) = K_1(l,m) \cup K_2(l,m).$$

The fundamental solution of $K_1(l,m)$ is $< -1,1 >$ and the fundamental solution of $K_2(l,m)$ is $< 1,1 >$.

Moreover, if $< u,v >$ runs $K(u,v)$, then:

(b_1) $< l,m >$ runs $K_1(l,m)$ if and only if the equality

$$l + m\sqrt{6} = (-1 + \sqrt{6})(u + v\sqrt{6}) \tag{10}$$

holds;

(b_2) $< l,m >$ runs $K_2(l,m)$ if and only if the equality

$$l + m\sqrt{6} = (1 + \sqrt{6})(u + v\sqrt{6}) \tag{11}$$

holds.

Note that the fundamental solution of (3) is $< 5, 2 >$. Let u_n and v_n be given by

$$u_n + v_n\sqrt{6} = (5 + 2\sqrt{6})^n \quad (n \in \mathcal{N}. \tag{12}$$

Then u_n and v_n satisfy (11) and $< u_n, v_n > \in \mathcal{N}^2$. Moreover, if n runs $\mathcal{N}$, then $< u_n, v_n >$ runs $K^o(u, v)$.

Let the sets $K_i^o(l, m)$ $(i = 1, 2)$ be introduced by

$$K_i^o(l, m) = K_i(l, m) \cap \mathcal{N}^2. \tag{13}$$

From the above remark and Theorem 1 we obtain

Theorem 2. The set $K^o(l, m)$ may be represented as

$$K^o(l, m) = K_1^o(l, m) \cup K_2^o(l, m), \tag{14}$$

where

$$K_1^o(l, m) \cap K_2^o(l, m) = \emptyset. \tag{15}$$

Moreover:

(c$_1$) If n runs $\mathcal{N}$ and the integers l_n and m_n are defined by

$$l_n + m_n\sqrt{6} = (-1 + \sqrt{6})(5 + 2\sqrt{6})^n, \tag{16}$$

then l_n and m_n satisfy (2) and $< l_n, m_n >$ runs $K_1^o(l, m)$;

(c$_2$) If n runs $\mathcal{N} \cup \{0\}$ and the integers l_n and m_n are defined by

$$l_n + m_n\sqrt{6} = (1 + \sqrt{6})(5 + 2\sqrt{6})^n, \tag{17}$$

then l_n and m_n satisfy (2) and $< l_n, m_n >$ runs $K_2^o(l, m)$.

Let φ be the above mentioned bijection. The sets $K_i'^o(x, y)$ $(i = 1, 2)$ are introduced by

$$K_i'^o(x, y) = \varphi(K_i^o(l, m)). \tag{18}$$

From Theorem 2, and especially from (14), (15), and (18) we obtain the next result.

Theorem 3. The set $K'^o(x, y)$ admits the representation

$$K'^o(x, y) = K_1^o(x, y) \cup K_2^o(x, y), \tag{19}$$

where

$$K_1^o(x, y) \cap K_2^o(x, y) = \emptyset. \tag{20}$$

Moreover:

(d$_1$) If n runs $\mathcal{N}$ and the integers x_n and y_n are defined by

$$x_n = l_n + 3m_n \text{ and } y_n = l_n + 2m_n, \tag{21}$$

where l_n and m_n are introduced by (16), then x_n and y_n satisfy (1) and $< x_n, y_n >$ runs $K_1^o(x, y)$;

(d$_2$) If n runs $\mathcal{N} \cup \{0\}$ and the integers x_n and y_n are defined again by (21), but l_n and m_n now are introduced by (17), then x_n and y_n satisfy (1) and $< x_n, y_n >$ runs $K_2^o(x, y)$.

Theorem 3 completely solves F. Smarandache's Problem 78 from [24], because l_n and m_n could be expressed in explicit form using (16) or (17) as well.

*

* *

Below we introduce a generalization of Smarandache's problem 78 from [24].

If we consider the Diophantine equation

$$2x^2 - 3y^2 = p, \tag{22}$$

where $p \neq 2$ is a prime number, then using [17], Chapter VII, exercize 2 and the same method as in the case of (1), we obtain the following result.

Theorem 4. (1) The necessary and sufficient condition for solvability of (22) is:

$$p \equiv 5 \pmod{24} \quad \text{or} \quad p \equiv 23 \pmod{24} \qquad (23);$$

(2) If (23) is valid, then there exists exactly one solution $< x, y > \in \mathcal{N}^2$ of (22) such that the inequalities

$$x < \sqrt{\frac{3}{2}.p}$$

and

$$y < \sqrt{\frac{2}{3}.p}$$

hold. Every other solution $< x, y > \in \mathcal{N}^2$ of (22) has the form:

$$x = l + 3m$$

$$y = l + 2m,$$

where $< l, m > \in \mathcal{N}^2$ is a solution of the Diophantine equation

$$l^2 - 6m^2 = -p.$$

The problem how to solve the Diophantine equation, a special case of which is the above one, is considered in Theorem 110 from [17].

10. ON FOUR SMARANDACHE'S PROBLEMS[11]

In [21, 25] F. Smarandache formulates the following four problems:

Problem 1. *Let p be an integer ≥ 3. Then:*

p is prime if and only if

$$(p-3)! \textit{ is congruent to } \frac{p-1}{2} (\text{mod } p). \tag{1}$$

Problem 2. *Let p be an integer ≥ 4. Then:*

p is prime if and only if

$$(p-4)! \textit{ is congruent to } (-1)^{\lceil \frac{p}{3} \rceil + 1} \lceil \frac{p+1}{6} \rceil (\text{mod } p). \tag{2}$$

Problem 3. *Let p be an integer ≥ 5. Then:*

p is prime if and only if

$$(p-5)! \textit{ is congruent to } rh + \frac{r^2-1}{24} (\text{mod } p), \tag{3}$$

with $h = \lceil \frac{p}{24} \rceil$ and $r = p - 24h$.

Problem 4. *Let $p = (k-1)!h + 1$ be a positive integer $k > 5$, h natural number. Then:*

p is prime if and only if

$$(p-k)! \textit{ is congruent to } (-1)^t h (\text{mod } p), \tag{4}$$

with $t = h + \lceil \frac{p}{h} \rceil + 1$.

Everywhere above $\lceil x \rceil$ means the inferior integer part of x, i.e., the smallest integer greater than or equal to x.

[11] The results in this section are taken from [10]

Here we shall discuss these four problems.

Problem 1. admits the following representation:
Let $p \geq 3$ be an odd number. Then:

$$p \text{ is prime if and only if } (p-3)! \equiv \frac{p-1}{2} (\text{mod } p). \qquad (1')$$

First, we assume that p is a composite number. Therefore, $p \geq 9$. For p there are two possibilities:

(a) $p = \prod_{i=1}^{s} p_i^{a_i}$, where p_i are different prime numbers and $a_i \geq 1$ are natural numbers ($1 \leq i \leq s$);
(b) $p = q^k$, where q is a prime number and $k \geq 2$ is a natural number.

Let (a) hold. Then there exist odd numbers a and b such that

$$2 < a < b < \frac{p}{2};\ (a,b) = 1;\ a.b = p.$$

The case when $a = 2$ and $b = \frac{p}{2}$ is impossible, because p is an odd number. Hence a and b are two different multipliers of $(p-3)!$ because $\frac{p}{2} < p-3$. Therefore, the number $a.b = p$ divides $(p-3)!$, i.e.,

$$(p-3)! \equiv 0 (\text{mod } p).$$

Hence in case (a) the congruence in the right hand-side of (1') is impossible.

Let (b) hold. Then $q \geq 3$ and we have to consider only two different cases:
(b_1) $k \geq 3$;
(b_2) $k = 2$.

Let (b_1) hold. Then

$$3 \leq q < q^{k-1} < q^k - 3 = p - 3.$$

Hence q and q^{k-1} are two different multipliers of $(p-3)!$. Therefore, the number $q.q^{k-1} = q^k = p$ divides $(p-3)!$, i.e.,

$$(p-3)! \equiv 0 (\text{mod } p).$$

Hence in case (b_1) the congruence in the right hand-side of (1') is impossible.

Let (b_2) hold. Then

$$p - 3 = q^2 - 3 \geq 2q.$$

Hence q and $2q$ are two different multipliers of $(p-3)!$. Therefore, the number $q^2 = p$ divides $(p-3)!$, i.e.,

$$(p-3)! \equiv 0 (\text{mod } p).$$

Hence in case (b_2) the congruence in the right hand-side of (1') is also impossible.

Thus we conclude that if $p > 1$ is an odd composite number, then the congruence

$$(p-3)! \equiv \frac{p-1}{2} (\text{mod } p)$$

is impossible.

Let $p \geq 3$ be prime. In this case we shall prove the above congruence using the well-known Wilson's Theorem (see, e.g. [17]):

$$p \text{ is prime if and only if } (p-1)! \equiv -1 (\text{mod } p). \qquad (5)$$

If we rewrite the congruence from (5) in the form

$$(p-1)(p-2)(p-3)! \equiv p - 1 (\text{mod } p)$$

and using that

$$(p-2) \equiv -2 (\text{mod } p)$$

and

$$(p-1) \equiv -1 (\text{mod } p)$$

we obtain

$$2(p-3)! \equiv p - 1 (\text{mod } p).$$

Hence the congruence

$$(p-3)! \equiv \frac{p-1}{2} (\text{mod } p)$$

is proved, i.e., Problem 1 is solved.

Problem 2. is false, because, for example, if $p = 7$, then (2) obtains the form

$$6 \equiv (-1)^4 2 (\text{mod } 7),$$

where

$$6 = (7-4)!$$

and

$$(-1)^4 2 = (-1)^{\lceil \frac{7}{3} \rceil + 1} \lceil \frac{8}{6} \rceil,$$

i.e.,

$$6 \equiv 2 (\text{mod } 7),$$

which is impossible.

Problem 3. can be modified, having in mind that from $r = p - 24h$ it follows:

$$rh + \frac{r^2-1}{24} = (p-24h).h + \frac{p^2 - 48ph + 24^2h^2 - 1}{24}$$

$$= ph - 24h^2 + \frac{p^2-1}{24} - 2ph + 24h^2 = \frac{p^2-1}{24} - ph,$$

i.e., (3) has the form

p is prime if and only if

$$(p-5)! \text{ is congruent to } \frac{p^2-1}{24} (\text{mod } p), \qquad (3')$$

Let $p \geq 5$ be prime. It is easy to see that $\frac{p^2-1}{24}$ is an integer (because every prime number p has one of the two forms $6k+1$ or $6k+5$ for some natural number k).

From Wilson's Theorem (see, e.g. [2]) and from

$$p^2 \equiv 0 (\text{mod } p)$$

we may write

$$(p-5)!.(p-4).(p-3).(p-2).(p-1) \equiv p^2 - 1 (\text{mod } p).$$

Since

$$(p-i) \equiv -i (\text{mod } p),$$

for $i = 1, 2, 3, 4$, we finally obtain

$$24(p-5)! \equiv p^2 - 1 (\text{mod } p).$$

Hence, the congruence

$$(p-5)! \equiv \frac{p^2-1}{24} (\text{mod } p)$$

is proved.

When p is a composite number and the number $\frac{p^2-1}{24}$ is not integer, the congruence

$$(p-5)! \equiv \frac{p^2-1}{24} (\text{mod } p)$$

is impossible. That is why we consider below only the composite odd numbers $p \geq 5$ for which $\frac{p^2-1}{24}$ is an integer.

Like in the proof of Problem 1, for p we have only the two possibilities (a) and (b).

Let (a) hold. Then $p \geq 15$ and there exist odd numbers a and b such that

$$2 < a < b < \frac{p}{2}; \ (a,b) = 1; \ a.b = p.$$

Hence a and b are two different multipliers of $(p-5)!$ since $\frac{p}{2} < p-5$. Therefore, the number $a.b = p$ divides $(p-5)!$, i.e.,

$$(p-5)! \equiv 0 (\text{mod } p).$$

If we suppose that the congruence from (3') holds too, then we obtain that

$$\frac{p^2-1}{24} \equiv 0 (\text{mod } p),$$

i.e.,

$$p^2 - 1 \equiv 0 (\text{mod } p),$$

i.e.,

$$-1 \equiv 0 (\text{mod } p),$$

which is impossible. Therefore, the congruence in the right hand-side of (3') is impossible.

Let (b) hold. As in the proof of Problem 1, here we have to consider two different cases (b_1) and (b_2).

Let (b_1) hold. Then

$$3 \leq q < q^{k-1} < q^k - 5 = p - 5.$$

Hence q and q^{k-1} are two different multipliers of $(p-5)!$. Therefore, the number $q.q^{k-1} = q^k = p$ divides $(p-5)!$, i.e.,

$$(p-5)! \equiv 0 (\text{mod } p).$$

Therefore, just as in the case (a) we conclude that the congruence in the right hand-side of (3') is impossible.

Let (b_2) hold. If $q \geq 7$, then we have

$$p - 5 = q^2 - 5 \geq 2q.$$

Hence q and $2q$ are two different multipliers of $(p-5)!$. Hence, the number $q^2 = p$ divides $(p-5)!$, i.e.,

$$(p-5)! \equiv 0 (\text{mod } p).$$

Just as in case (a) we conclude that the congruence in the right hand-side of (3') is impossible.

It remains only to consider the cases:

$$p = 3^2 = 9, \;\; p = 5^2 = 25$$

and to finish with (b_2).

If $p = 9$, then $\dfrac{p^2-1}{24}$ is not an integer and as we noted before, the congruence in the right hand of (3') fails.

When $p = 25$ the above congruence yields

$$20! \equiv 26 (\text{mod } 25),$$

i.e.,

$$20! \equiv 1 (\text{mod } 25).$$

On the other hand, 25 divides 20! and therefore,

$$20! \equiv 0 (\text{mod } 25).$$

Hence, the congruence in the right hand of (3') is impossible in the case $p = 25$, too.

Thus the same congruence is impossible for the case (b).

Finally we proved

If $p > 1$ is an odd composite number, then the congruence

$$(p-5)! \equiv \frac{p^2-1}{24} (\text{mod } p)$$

is impossible and Problem 3 is completely solved.

Problem 4. also can be simplified, because

$$t = h + \lceil \frac{p}{h} \rceil + 1$$

$$= h + \lceil \frac{(k-1)!h + 1}{h} \rceil + 1$$

$$= h + (k-1)! + 1 + 1 = h + (k-1)! + 2,$$

i.e.,

$$(-1)^t = (-1)^h,$$

because for $k > 2 : (k-1)! + 2$ is an even number. Therefore, (4) obtains the form

p is prime if and only if

$$(p-k)! \text{ is congruent to } (-1)^h h (\text{mod } p). \qquad (4')$$

Let us assume that (4') is valid. We use again the congruences

$$(p-1) \equiv -1 (\text{mod } p)$$
$$(p-2) \equiv -2 (\text{mod } p)$$
$$\cdot \quad \cdot \quad \cdot$$
$$(p-(k-1)) \equiv -(k-1) (\text{mod } p)$$

and obtain the next form of (4')

p is prime if and only if

$$(p-1)! \equiv (-1)^h.(-1)^{k-1}.(k-1)!.h (\text{mod } p)$$

or

p is prime if and only if

$$(p-1)! \equiv (-1)^{h+k-1}.(p-1) (\text{mod } p).$$

But the last congruence is not valid, because, e.g., for $k = 5, h = 3, p = 73 = (5-1)!3! + 1$ holds

$$72! \equiv (-1)^9.72 (\text{mod } 73),$$

i.e.,

$$72! \equiv 1 (\text{mod } 73),$$

while from Wilson's Theorem it follows that

$$72! \equiv -1 (\text{mod } 73).$$

11. ON FOUR PRIME AND COPRIME FUNCTIONS[12]

In [25] F. Smarandache discussed the following particular cases of the well-known characteristic functions (see, e.g., [14, 42]).

1) Prime function: $P : N \to \{0, 1\}$, with

$$P(n) = \begin{cases} 0, & \text{if } n \text{ is prime} \\ 1, & \text{otherwise} \end{cases}$$

More generally: $P_k : N^k \to \{0, 1\}$, where $k \geq 2$ is an integer, and

$$P_k(n_1, n_2, ..., n_k) = \begin{cases} 0, & \text{if } n_1, n_2, ..., n_k \text{ are all prime numbers} \\ 1, & \text{otherwise} \end{cases}$$

2) Coprime function is defined similarly: $C_k : N^k \to \{0, 1\}$, where $k \geq 2$ is an integer, and

$$C_k(n_1, n_2, ..., n_k) = \begin{cases} 0, & \text{if } n_1, n_2, ..., n_k \text{ are coprime numbers} \\ 1, & \text{otherwise} \end{cases}$$

Here we shall formulate and prove four assertions related to these functions.

Proposition 1. For each $k, n_1, n_2, ..., n_k$ natural numbers:

$$P_k(n_1, ..., n_k) = 1 - \prod_{i=1}^{k} (1 - P(n_i)).$$

Proof. Let the given natural numbers $n_1, n_2, ..., n_k$ be prime. Then, by definition

$$P_k(n_1, ..., n_k) = 0.$$

In this case, for each i $(1 \leq i \leq k)$:

$$P(n_i) = 0,$$

[12]The results in this section are taken from [6]

i.e.,

$$1 - P(n_i) = 1.$$

Therefore

$$\prod_{i=1}^{k}(1 - P(n_i)) = 1,$$

i.e.,

$$1 - \prod_{i=1}^{k}(1 - P(n_i)) = 0 = P_k(n_1, ..., n_k). \qquad (1)$$

If at least one of the natural numbers $n_1, n_2, ..., n_k$ is not prime, then, by definition

$$P_k(n_1, ..., n_k) = 1.$$

In this case, there exists at least one i $(1 \leq i \leq k)$ for which:

$$P(n_i) = 1,$$

i.e.,

$$1 - P(n_i) = 0.$$

Therefore

$$\prod_{i=1}^{k}(1 - P(n_i)) = 0,$$

i.e.,

$$1 - \prod_{i=1}^{k}(1 - P(n_i)) = 1 = P_k(n_1, ..., n_k). \qquad (2)$$

The validity of the assertion follows from (1) and (2).

Similarly it can be proved

Proposition 2. For each $k, n_1, n_2, ..., n_k$ natural numbers:

$$C_k(n_1, ..., n_k) = 1 - \prod_{i=1}^{k}\prod_{j=i+1}^{k}(1 - C_2(n_i, n_j)).$$

Let $p_1, p_2, p_3, ...$ be the sequence of the prime numbers ($p_1 = 2, p_2 = 3, .p_3 = 5, ...$).

Let $\pi(n)$ be the number of the primes that are less than or equal to n.

Proposition 3. For each natural number n:

$$C_{\pi(n)+P(n)}(p_1, p_2, ..., p_{\pi(n)+P(n)-1}, n) = P(n).$$

Proof. Let n be a prime number. Then

$$P(n) = 0$$

and

$$p_{\pi(n)} = n.$$

Therefore

$$C_{\pi(n)+P(n)}(p_1, p_2, ..., p_{\pi(n)+P(n)-1}, n)$$

$$= C_{\pi(n)}(p_1, p_2, ..., p_{\pi(n)-1}, p_{\pi(n)}) = 0,$$

because the primes $p_1, p_2, ..., p_{\pi(n)-1}, p_{\pi(n)}$ are also coprimes.

Let n be a composite number. Then

$$P(n) = 1$$

and

$$p_{\pi(n)} < n.$$

Therefore

$$C_{\pi(n)+P(n)}(p_1, p_2, ..., p_{\pi(n)+P(n)-1}, n)$$

$$= C_{\pi(n)+1}(p_1, p_2, ..., p_{\pi(n)-1}, n) = 1,$$

because, if n is a composite number, then it is divided by at least one of the prime numbers $p_1, p_2, ..., p_{\pi(n)-1}$.

With this the proposition is proved.

The following statement can be proved by analogy

Proposition 4. For each natural number n:

$$P(n) = 1 - \prod_{i=1}^{\pi(n)+P(n)-1}(1 - C_2(p_i, n)).$$

Corollary. For each natural numbers $k, n_1, n_2, ..., n_k$:

$$P_k(n_1, ..., n_k) = 1 - \prod_{i=1}^{k} \prod_{j=1}^{\pi(n_i)+P(n_i)-1} (1 - C_2(p_j, n_i)).$$

These propositions show the connections between the prime and coprime functions.

Chapter 2

Some other results of the authors

In this chapter we present some of the authors' results, that have been already published in various journals on number theory. These results are used in first Chapter and they have independent sense, but admit applications in the solutions of the Smarandache's problems discussed above.

A1. SOME NEW FORMULAE FOR THE TWIN PRIMES COUNTING FUNCTION $\pi_2(n)$[1]

Some different explicit formulae for the twin primes counting function π_2 are given below.

1. A bracket function formula for $\pi_2(n)$ using factorial

$$\pi_2(n) = 1 + \sum_{k=1}^{[\frac{n+1}{6}]} [\frac{2(6k-2)!+(6k)!+2}{36k^2-1} - [\frac{2(6k-2)!+(6k)!}{36k^2-1}]]. \quad (1)$$

Here, and furthermore, $n \geq 5$ and

$$\pi_2(0) = \pi_2(1) = \pi_2(2) = 0; \pi_2(3) = 1.$$

2. Formulae for $\pi_2(n)$ using Riemann's zeta function

$$\pi_2(n) = 1 - 2. \sum_{k=1}^{[\frac{n+1}{6}]} \zeta(\varphi(6k-1) + \varphi(6k+1) - 12k + 2); \quad (2)$$

$$\pi_2(n) = 1 - 2. \sum_{k=1}^{[\frac{n+1}{6}]} \zeta(12k + 2 - \psi(6k-1) - \psi(6k+1)); \quad (3)$$

$$\pi_2(n) = 1 - 2. \sum_{k=1}^{[\frac{n+1}{6}]} \zeta(24k + 4 - 2\sigma(6k-1) - 2\sigma(6k+1)). \quad (4)$$

[1]The results in this section are taken from [29]

3. Bracket function formulae for $\pi_2(n)$ using Euler's function φ

$$\pi_2(n) = 1 + \sum_{k=1}^{[\frac{n+1}{6}]} [\frac{\varphi(36k^2-1)}{36k^2-12k}]; \tag{5}$$

$$\pi_2(n) = 1 + \sum_{k=1}^{[\frac{n+1}{6}]} [\frac{1}{2}.\sqrt{\frac{\varphi(36k^2-1)}{3k(3k-1)}}]; \tag{6}$$

$$\pi_2(n) = 1 + \sum_{k=1}^{[\frac{n+1}{6}]} [\frac{\varphi(6k-1)+\varphi(6k+1)}{12k-2}]; \tag{7}$$

$$\pi_2(n) = 1 + \sum_{k=1}^{[\frac{n+1}{6}]} [\frac{\varphi(6k-1)}{12k-4} + \frac{\varphi(6k+1)}{12k}]; \tag{8}$$

$$\pi_2(n) = 1 + \sum_{k=1}^{[\frac{n+1}{6}]} [\frac{1}{6k - \frac{\varphi(6k-1)+\varphi(6k+1)}{2}}]. \tag{9}$$

4. Bracket function formula for $\pi_2(n)$ using Dedekind's function ψ

$$\pi_2(n) = 1 + \sum_{k=1}^{[\frac{n+1}{6}]} [\frac{36k^2+12k}{\psi(36k^2-1)}]; \tag{10}$$

$$\pi_2(n) = 1 + \sum_{k=1}^{[\frac{n+1}{6}]} [2.\sqrt{\frac{3k(3k+1)}{\psi(36k^2-1)}}]; \tag{11}$$

$$\pi_2(n) = 1 + \sum_{k=1}^{[\frac{n+1}{6}]} [\frac{12k+2}{\psi(6k-1)+\psi(6k+1)}]; \tag{12}$$

$$\pi_2(n) = 1 + \sum_{k=1}^{[\frac{n+1}{6}]} [\frac{3k}{\psi(6k-1)} + \frac{3k+1}{\psi(6k+1)}]; \tag{13}$$

$$\pi_2(n) = 1 + \sum_{k=1}^{[\frac{n+1}{6}]} [\frac{1}{\frac{\psi(6k-1)+\psi(6k+1)}{2} - 6k}]. \tag{14}$$

Remark. The formulae from section 4 are still true if we put $\sigma(n)$ instead of $\psi(n)$.

5. Proofs of the formulae

In order to prove all above formulae we need the arithmetic function

$$\delta(n) = \begin{cases} 1, & \text{if } k \text{ and } k+2 \text{ are twin primes} \\ 0, & \text{otherwise} \end{cases} \tag{15}$$

Since $p = 6k-1$ if p and $p+2$ are twin primes, we obtain for $n \geq 5$:

$$\pi_2(n) = 1 + \sum_{k=1}^{[\frac{n+1}{6}]} \delta(6k-1). \tag{16}$$

First, let us prove (1). It is enough to prove that for $k \geq 5$ the equality

$$\delta(k) = [\frac{2(k-1)!+(k+1)!+2}{k(k+2)} - [\frac{2(k-1)!+(k+1)!}{k(k+2)}]] \tag{17}$$

holds.

We rewrite (17) in the form

$$\delta(k) = [\frac{(k-1)!+1}{k} + \frac{k!-1}{k+2} - [\frac{(k-1)!}{k} + \frac{k!}{k+2}]]. \tag{18}$$

Further, we use a variant of Wilson's Theorem given by Coblyn in 1913 (see [18]): *"The integer $m \geq 2$ is a prime if and only if m divides each of the numbers $(r-1)!(m-r)! + (-1)^{r-1}$ for $r = 1, 2, ..., m-1$."* The cases $r = 1$ and $r = 2$ are called Wilson's and Leibnitz Theorem respectively [20]. We denote by $g(k)$ the right hand-side of (18).

($\mathbf{a_1}$) Let k and $k+2$ be twin primes. Therefore, $(k-1)! + 1 = k.x$ $(x \in \mathcal{N})$ from the Wilson's Theorem and $k! - 1 = ((k+2)-2)! - 1 = (k+2).y$ $(y \in \mathcal{N})$ from the Leibnitz's Theorem. Hence:

$$g(k) = [\frac{kx}{k} + \frac{(k+2)y}{k+2} - [\frac{kx-1}{k} + \frac{(k+2)y+1}{k+2}]]$$

$$= [x + y - [x + y - (\frac{1}{k} - \frac{1}{k+2})]]$$

$$= [x + y - (x + y - 1)] = 1.$$

($\mathbf{a_2}$) Let k be prime and $k+2$ be composite. Therefore, $k > 6$. Now, it is easy to see that $k! = (k+2).y$ $(y \in \mathcal{N})$. The Wilson's Theorem yields $(k-1)! + 1 = k.x$ $(x \in \mathcal{N})$. Hence:

$$g(k) = [\frac{kx}{k} + \frac{(k+2)y-1}{k+2} - [\frac{kx-1}{k} + \frac{(k+2)y}{k+2}]]$$

$$= [x + y - \frac{1}{k+2} - [x + y - \frac{1}{k}]]$$

$$= [x + y - \frac{1}{k+2} - (x + y - 1)]$$

$$= [1 - \frac{1}{k+2}] = 0.$$

($\mathbf{a_3}$) Let k be composite and $k+2$ be prime. Therefore, $k > 6$. Now, it is easy to see that $(k-1)! = k.x$ $(x \in \mathcal{N})$. The Leibnitz's Theorem yields $k! - 1 = (k+2).y$ $(y \in \mathcal{N})$. Hence:

$$g(k) = [\frac{kx+1}{k} + \frac{(k+2)y}{k+2} - [\frac{kx}{k} + \frac{(k+2)y+1}{k+2}]]$$

$$= [x + \frac{1}{k} + y - [x + y + \frac{1}{k+2}]]$$

$$= [x + y + \frac{1}{k} - (x + y)]$$

$$= [\frac{1}{k}] = 0.$$

($\mathbf{a_4}$) Let k and $k+2$ be composite. Therefore, $k \geq 6$. Now, it is easy to see that $(k-1)! = k.x$ $(x \in \mathcal{N})$ and $k! = (k+2).y$ $(y \in \mathcal{N})$. Hence:

$$g(k) = [\frac{kx+1}{k} + \frac{(k+2)y-1}{k+2} - [\frac{kx}{k} + \frac{(k+2)y}{k+2}]]$$

$$= [x + y + \frac{1}{k} - \frac{1}{k+2} - (x + y)]$$

$$= [\frac{1}{k} - \frac{1}{k+2}] = 0.$$

From (a_1) – (a_4) it follows that $g(k) = \delta(k)$ for $k \geq 5$ and the proof of (1) is finished.

Second, let us prove the formulae from section 2. We need the well-known fact that $\zeta(0) = -\frac{1}{2}$ and $\zeta(-2m) = 0$ for $m \in \mathcal{N}$ (see [12]). Since numbers $\varphi(6k-1), \varphi(6k+1), \psi(6k-1), \psi(6k+1), 2\sigma(6k-1), 2\sigma(6k+1)$ are even, and the following inequalities

$$\varphi(6k-1) + \varphi(6k+1) \leq 12k - 2$$

$$\psi(6k-1) + \psi(6k+1) \geq 12k + 2$$

$$\sigma(6k-1) + \sigma(6k+1) \geq 12k + 2$$

are valid, and the fact that the last inequalities become equalities simultaneously if and only if $6k-1$ and $6k+1$ are twin primes, we conclude that the argument of the function ζ in (2) – (4) is everywhere nonpositive even number. Moreover, this argument equals to zero if and only if $6k-1$ and $6k+1$ are twin primes. Therefore, we have

$$\delta(6k-1) = -2\zeta(\varphi(6k-1) + \varphi(6k+1) - 12k + 2)$$

$$= -2\zeta(12k + 2 - \psi(6k - 1) - \psi(6k + 1))$$
$$= -2\zeta(24k + 4 - 2\sigma(6k - 1) - 2\sigma(6k + 1)).$$

Hence, (2) – (4) are proved because of (16).

It remains only to prove the formulae from sections 3 and 4.

First, we use that

$$\varphi(36k^2 - 1) = \varphi(6k - 1).\varphi(6k + 1)$$

and

$$\psi(36k^2 - 1) = \psi(6k - 1).\psi(6k + 1),$$

since, the functions φ and ψ are multiplicative.

Second, we use that inequalities $\varphi(6k - 1) \leq 6k - 2$ and $\varphi(6k + 1) \leq 6k$ (just like inequalities $\psi(6k-1) \geq 6k$ and $\psi(6k+1) \geq 6k+2$) become equalities simultaneously if and only if the numbers $6k - 1$ and $6k + 1$ are twin primes.

Then it is easy to verify that each one of the expressions behind the sum $\sum_{k=1}^{[\frac{n+1}{6}]}$ in (5) – (14) equals to $\delta(6k - 1)$. Hence, the proof of the formulae from sections 3 and 4 falls from (15).

A2. THREE FORMULAE FOR n-th PRIME AND SIX FORMULAE FOR n-th TERM OF TWIN PRIMES[2]

Let $C = \{C_n\}_{n\geq 1}$ be an arbitrary increasing sequence of natural numbers. By $\pi_C(n)$ we denote the number of the terms of C being not greater than n (we agree that $\pi_C(0) = 0$). In the first part of the section we propose six different formulae for C_n $(n = 1, 2, ...)$, which depend on the numbers $\pi_C(k)$ $(k = 0, 1, 2, ...)$. Using these formulae, in the second part of the section we obtain three different explicit formulae for the n-th prime p_n. In the third part of the section, using the formulae from the first part, we propose six explicit formulae for the n-th term of the sequence of twin primes: 3,5,7,11,13,17,19,... The last three of these formulae, related to function π_2, are the main ones for the twin primes.

Part 1: Universal formulae for the n-th term of an arbitrary increasing sequence of natural numbers

1. A bracket function formula for C_n:

$$C_n = \sum_{k=0}^{\infty}[\frac{1}{1 + [\frac{\pi_C(k)}{n}]}]. \tag{1}$$

2. A formula using Riemann's function ζ:

$$C_n = -2.\sum_{k=0}^{\infty}\zeta(-2.[\frac{\pi_C(k)}{n}]). \tag{2}$$

[2]The results in this section are taken from [30]

3. A formula using Euler's function Γ:

$$C_n = \sum_{k=0}^{\infty} \frac{1}{\Gamma(1-[\frac{\pi_C(k)}{n}])}. \qquad (3)$$

Proof of the formulae (1)–(3). First, we represent (2) in the form

$$C_n = \sum_{k=0}^{\infty}(-2).\zeta(-2.[\frac{\pi_C(k)}{n}]). \qquad (2')$$

After that for each one of (1), (2′), (3) we use that

$$\sum_{k=0}^{\infty}\bullet = \sum_{k=0}^{C_n-1}\bullet + \sum_{k=C_n}^{\infty}\bullet.$$

Let $k = 0, 1, ..., C_n - 1$. Then we have

$$\pi_C(k) \leq \pi_C(C_n - 1) < \pi_C(C_n) = n.$$

Hence

$$[\frac{\pi_C(k)}{n}] = 0$$

for $k = 0, 1, ..., C_n - 1$. Therefore, for (1) we have

$$\sum_{k=0}^{C_n-1}[\frac{1}{1+[\frac{\pi_C(k)}{n}]}] = \sum_{k=0}^{C_n-1} 1 = C_n.$$

In the same manner, for (2′) we have

$$\sum_{k=0}^{C_n-1}(-2)\zeta(-2.[\frac{\pi_C(k)}{n}]) = \sum_{k=0}^{C_n-1}(-2)\zeta(0) = \sum_{k=0}^{C_n-1} 1 = C_n,$$

since it is known that $\zeta(0) = -\frac{1}{2}$ (see [12]).

For (3) we have

$$\sum_{k=0}^{C_n-1}\frac{1}{\Gamma(1-[\frac{\pi_C(k)}{n}])} = \sum_{k=0}^{C_n-1}\frac{1}{\Gamma(1)} = \sum_{k=0}^{C_n-1} 1 = C_n.$$

Let $k = C_n, C_n+1, C_n+2, ...$ Then we have $n = \pi_C(C_n) \leq \pi(k)$. Therefore, $[\frac{\pi_C(k)}{n}] \geq 1$ for $k = C_n, C_n+1, C_n+2, ...$ Hence:

$$[\frac{1}{1+[\frac{\pi_C(k)}{n}]}] = 0$$

for $k = C_n, C_n+1, C_n+2, ...$ Therefore, for (1) $\sum_{k=C_n}^{\infty}$ vanishes. This proves (1).

To prove (2′) (i.e., (2)) it remains to show that $\sum_{k=C_n}^{\infty}$ vanishes as in the previous case. But this is obvious from the fact that for $k = C_n, C_n+1, C_n+2, ...$

$$n_k \equiv [\frac{\pi_C(k)}{n}]$$

is a natural number and therefore

$$\zeta(-2n_k) = 0,$$

since, the negative even numbers are trivial zeros of Riemann's Zeta–function (see [12]).

We also have

$$\frac{1}{\Gamma(1-n_k)} = 0$$

for $k = C_n, C_n+1, C_n+2, ...$, since, it is known that the nonpositive integers are poles of Euler's function gamma. Therefore, for (3) the sum $\sum_{k=C_n}^{\infty}$ vanishes too, which proves (3).

4. Three other formulae for $\mathbf{C_n}$:

$$C_n = \sum_{k=0}^{\infty}[\frac{1}{1+[\frac{\pi_C(k)+n}{2n}]}]. \qquad (1^*)$$

$$C_n = -2.\sum_{k=0}^{\infty}\zeta(-2.[\frac{\pi_C(k)+n}{2n}]). \qquad (2^*)$$

$$C_n = \sum_{k=0}^{\infty}\frac{1}{\Gamma(1-[\frac{\pi_C(k)+n}{2n}])}. \qquad (3^*)$$

The validity of these formulae is checked in the same manner.

Part 2: Formulae for n-th prime $\mathbf{p_n}$

Here, as a Corollary from Part 1, we propose three finite formulae for p_n.

Let

$$\theta(n) = [\frac{n^2+3n+4}{4}].$$

It is known (see [16]) that

$$p_n \le \theta(n)$$

for $n = 1, 2, ...$ Hence

$$p_n < n^2$$

for $n > 1$. Then, if we put

$$C_n = p_n$$

for $n = 1, 2, ...$ and using that

$$\pi_C(n) = \pi(n),$$

we obtain the following formulae from (1), (2) and (3):

$$p_n = \sum_{k=0}^{\theta(n)}[\frac{1}{1+[\frac{\pi(k)}{n}]}]; \qquad (4)$$

$$p_n = -2.\sum_{k=0}^{\theta(n)}\zeta(-2.[\frac{\pi(k)}{n}]); \qquad (5)$$

$$p_n = \sum_{k=0}^{\theta(n)}\frac{1}{\Gamma(1-[\frac{\pi(k)}{n}])}. \qquad (6)$$

The above formulae stay valid if we change $\theta(n)$ with n^2. These formulae are explicit ones, because $\pi(k)$ has explicit representations (see [18, 4]).

One may compare (4) with the formula of Willans (see [18]):

$$p_n = 1 + \sum_{k=1}^{2^n}[[\frac{n}{1+\pi(k)}]^{\frac{1}{n}}].$$

Part 3: Formulae for $\mathbf{p_2(n)}$

Let $C_n = p_2(n)$. In this case we have

$$\pi_C(0) = \pi_C(1) = \pi_C(2) = 0;\ \pi_C(3) = \pi_C(4) = 1. \qquad (*)$$

When $k \ge 5$ it is easy to see that

$$\pi_C(k) = \begin{cases} 2\pi_2(k) - 2, & \text{if } k-1 \text{ and } k+1, \text{ or } k \text{ and } k+2 \text{ are twin primes} \\ 2\pi_2(k) - 1, & \text{otherwise} \end{cases}, \qquad (6')$$

or in an explicit form

$$\pi_C(k) = 2\pi_2(k) - 1 - \delta(k-1) - \delta(k), \qquad (6'')$$

where

$$\delta(k) = \begin{cases} 1, & \text{if } k \text{ and } k+2 \text{ are twin primes} \\ 0, & \text{otherwise} \end{cases}.$$

It is easy to give an explicit representation of $\delta(k)$:

$$\delta(k) = [\frac{2(k-1)!+(k+1)!+2}{k(k+2)} - [\frac{2(k-1)!+(k+1)!}{k(k+2)}]]. \qquad (6''')$$

Other criteria for simultaneous primality and coprimality of two numbers are discussed in [22, 23, 25, 26, 27].

Instead of $(6'')$, it is possibe to use the representation:

$$\pi_C(k) = \pi_2(k) + \pi_2(k-2) - 1,$$

since

$$\pi_2(k) = \sum_{j=3}^{k} \delta(j).$$

Therefore, from (1) – (3) we obtain the corresponding formulae for $p_2(n)$:

$$p_2(n) = \sum_{k=0}^{\infty} [\frac{1}{1+[\frac{\pi_C(k)}{n}]}]; \qquad (7)$$

$$p_2(n) = -2.\sum_{k=0}^{\infty} \zeta(-2.[\frac{\pi_C(k)}{n}]); \qquad (8)$$

$$p_2(n) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(1-[\frac{\pi_C(k)}{n}])}, \qquad (9)$$

where $\pi_C(k)$ is given by (*) for $k = 0, 1, 2, 3, 4$, and by $(6'')$ for $k \geq 5$ with $\delta(k)$ is given by $(6''')$.

Three new explicit formulae for $p_2(n)$ for even $n > 2$ are given below, while $p_2(2) = 5$. They correspond to (1^*) – (3^*) and use $(6')$:

$$p_2(n) = 5 + \sum_{k=5}^{\infty} [\frac{1}{1+[\frac{\pi_2(k)-1+\frac{n}{2}}{n}]}]; \qquad (7^*)$$

$$p_2(n) = 5 - 2.\sum_{k=5}^{\infty} \zeta(-2.[\frac{\pi_2(k)-1+\frac{n}{2}}{n}]); \qquad (8^*)$$

$$p_2(n) = 5 + \sum_{k=5}^{\infty} \frac{1}{\Gamma(1-[\frac{\pi_2(k)-1+\frac{n}{2}}{n}])}, \qquad (9^*)$$

They follow from the identity

$$[\frac{\pi_C(k)+n}{2n}] = [\frac{\pi_2(k)-1+\frac{n}{2}}{n}],$$

since for $k \geq 5$ $\pi_C(k)$ is given by $(6')$ and for even $n > 2$ we have

$$[\frac{n-1}{2}] = \frac{n}{2} - 1.$$

Obviously, $p_2(1) = 3$, $p_2(3) = 7$ and for odd $n \geq 5$ we have

$$p_2(n) = p_2(n-1) + 2$$

and we may apply the formulae (7^*) – (9^*) for $p_2(n-1)$ since $n-1$ is an even number.

The last three formulae are main ones for the twin primes.

All formulae for $p_2(n)$ are explicit, because in **A1** some explicit formulae for $\pi_2(n)$ are proposed. One of them is valid for $n \geq 5$:

$$\pi_2(n) = 1 + \sum_{k=1}^{[\frac{n+1}{6}]} [\frac{2(6k-2)!+(6k)!+2}{36k^2-1} - [\frac{2(6k-2)!+(6k)!}{36k^2-1}]].$$

For $\pi(n)$ one may use Mináč's formula (see [18]):

$$\pi(n) = \sum_{k=2}^{n} [\frac{(k-1)!+1}{k} - [\frac{(k-1)!}{k}]],$$

or any of the following formulae, proposed here:

$$\pi(n) = -2.\sum_{k=2}^{n} \zeta(-2.(k-1-\varphi(k)); \qquad (10)$$

$$\pi(n) = -2.\sum_{k=2}^{n} \zeta(-2.(\sigma(k)-k-1); \qquad (11)$$

$$\pi(n) = \sum_{k=2}^{n} [\frac{\varphi(k)}{k-1}]; \quad (12)$$

$$\pi(n) = \sum_{k=2}^{n} [\frac{k+1}{\sigma(k)}]; \quad (13)$$

$$\pi(n) = \sum_{k=2}^{n} [\frac{1}{k-\varphi(k)}]; \quad (14)$$

$$\pi(n) = \sum_{k=2}^{n} [\frac{1}{\sigma(k)-k}]. \quad (15)$$

Remark. In (11), (13), (15) one may prefer to put $\psi(k)$ instead of $\sigma(k)$ and then the formulae will remain valid.

In [4] are published following results:

$$\pi(n) = \sum_{k=2}^{n} \overline{sg}(k-1-\varphi(k));$$

$$\pi(n) = \sum_{k=2}^{n} \overline{sg}(\sigma(k)-k-1);$$

$$\pi(n) = \sum_{k=2}^{n} fr(\frac{k}{(k-1)!}),$$

$$p_n = \sum_{i=0}^{2^n} sg(n-\pi(i)),$$

where:

$$sg(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ 1, & \text{if } x > 0 \end{cases},$$

$$\overline{sg}(x) = \begin{cases} 0, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases},$$

where x is a real number and

$$fr(\frac{p}{q}) = \begin{cases} 0, & \text{if } p = 1 \\ 1, & \text{if } p \neq 1 \end{cases},$$

where p and q are natural numbers, such that $(p, q) = 1$.

Finally, we shall mention that F. Smarandache introduced another formula for $\pi(x)$(see [28]): if x is an integer ≥ 4, then

$$\pi(x) = -1 + \sum_{k=2}^{x} \lceil \frac{S(k)}{k} \rceil,$$

where $S(k)$ is the Smarandache function (the smallest integer m such that $m!$ is divisible by k) and for symbol $\lceil \bullet \rceil$ see page 78.

A3. EXPLICIT FORMULAE FOR THE n-TH TERM OF THE TWIN PRIME SEQUENCE[3]

Three different explicit formulae for the n-th term of the twin prime sequence are proposed and proved, when n is even. They depend on function π_2. The investigation continues **A2**.

We need the following result from **A2** that here formulate for readers' convenience as

Theorem 1. Let $n \geq 4$ be even. Then $p_2(n)$ has each one of the following three representations:

$$p_2(n) = 5 + \sum_{k=5}^{\infty}[\frac{1}{1+H(k;n)}]; \tag{1}$$

$$p_2(n) = 5 - 2.\sum_{k=5}^{\infty}\zeta(-2.H(k;n)); \tag{2}$$

$$p_2(n) = 5 + \sum_{k=5}^{\infty}\frac{1}{\Gamma(1-H(k;n))}, \tag{3}$$

where

$$H(k;n) = [\frac{\pi_2(k) - 1 + \frac{n}{2}}{n}]. \tag{4}$$

Below, we shall prove the following

Theorem 2. Let $n \geq 4$ be integer. Then $p_2(n)$ has each one of the following three representations:

$$p_2(n) = 6 + (-1)^{n-1} + \sum_{k=5}^{\infty}[\frac{1}{1+r(k;n)}]; \tag{1*}$$

$$p_2(n) = 6 + (-1)^{n-1} - 2.\sum_{k=5}^{\infty}\zeta(-2.r(k;n)); \tag{2*}$$

$$p_2(n) = 6 + (-1)^{n-1} + \sum_{k=5}^{\infty}\frac{1}{\Gamma(1-r(k;n))}, \tag{3*}$$

[3]The results in this section are taken from [32]

where

$$r(k;n) = [\frac{\pi_2(k) - 1 + [\frac{n}{2}]}{2.[\frac{n}{2}]}]. \tag{4*}$$

Proof. Let $n \geq 4$ be even. Then $r(k;n) = H(k;n)$ and also $6 + (-1)^{n-1} = 5$. Therefore (1*) coincides with (1), (2*) coincides with (2), and (3*) coincides with (3), which proves Theorem 2 in this case.

Let $n > 4$ be odd. Then

$$r(k;n) = H(k;n-1), \tag{5}$$

since $[\frac{n}{2}] = \frac{n-1}{2}$ and $2.[\frac{n}{2}] = n - 1$.

We have also the relation

$$p_2(n) = 2 + p_2(n-1), \tag{6}$$

since $p_2(n-1)$ and $p_2(n)$ are twin primes. But $n-1$ is even and $n - 1 \geq 4$. Then we apply Theorem 1 with $n-1$ instead of n and from (5) and (6) the proof of Theorem 2 falls, because of the equality $6 + (-1)^{n-1} = 2 + 5$.

Finally, we observe that formulae (1*)–(3*) are explicit, because in **A1** we propose some different explicit formulae for $\pi_2(n)$ when $n \geq 5$. One of these formulae is given below:

$$\pi_2(n) = 1 + \sum_{k=1}^{[\frac{n+1}{6}]}[\frac{2(6k-2)!+(6k)!+2}{36k^2-1} - [\frac{2(6k-2)!+(6k)!}{36k^2-1}]].$$

Of course, all formulae for $p_2(n)$ in **A3** (just like in **A2**) are finite, because it is possible to put

$$\sum_{k=5}^{p_2(n)} \bullet$$

instead of

$$\sum_{k=5}^{\infty} \bullet.$$

But to receive "good" finite formulae for $p_2(n)$ we need something more, namely, the inequality

$$p_2(n) \leq \lambda(n), \tag{7}$$

where $\lambda(n)$ is a function that has an explicit expression. Then, we may put

$$\sum_{k=5}^{\lambda(n)} \bullet$$

instead of

$$\sum_{k=5}^{\infty} \bullet.$$

However, (7) is not found, yet.

A4. SOME EXPLICIT FORMULAE FOR THE COMPOSITE NUMBERS[4]

Explicit formulae for n-th term of the sequence of all composite numbers and for the sequence of all odd composite numbers are proposed.

In **A2** three different formulae are proposed for n-th term C_n of an arbitrary increasing sequence $C = \{c_i\}_{i=1}^{\infty}$ of natural numbers. They are based on the numbers $\pi_C(k)$ ($k = 0, 1, 2, ...$), where $\pi_C(0) = 0$, and for $k \geq 1$ $\pi_C(k)$ denotes the number of terms of C, which are not greater than k. These formulae are given again below:

$$C_n = \sum_{k=0}^{\infty} [\frac{1}{1 + [\frac{\pi_C(k)}{n}]}]. \tag{1}$$

$$C_n = -2. \sum_{k=0}^{\infty} \zeta(-2.[\frac{\pi_C(k)}{n}]). \tag{2}$$

$$C_n = \sum_{k=0}^{\infty} \frac{1}{\Gamma(1 - [\frac{\pi_C(k)}{n}])}. \tag{3}$$

If the inequality

$$C_n \leq \lambda(n)$$

holds for every $n \geq 1$, where the numbers $\lambda(n)$ ($n = 1, 2, 3, ...$) are a priori known, then formulae (1) – (3) take the forms, respectively:

$$C_n = \sum_{k=0}^{\lambda(n)} [\frac{1}{1 + [\frac{\pi_C(k)}{n}]}]. \tag{1'}$$

$$C_n = -2. \sum_{k=0}^{\lambda(n)} \zeta(-2.[\frac{\pi_C(k)}{n}]). \tag{2'}$$

[4] The results in this section are taken from [31]

$$C_n = \sum_{k=0}^{\lambda(n)} \frac{1}{\Gamma(1 - [\frac{\pi_C(k)}{n}])}. \qquad (3')$$

Three different explicit representations for n-th prime number p_n with $\lambda(n) = n^2$, or with

$$\lambda(n) = [\frac{n^2 + 3n + 4}{4}]$$

(by choice) are given in **A2**, using a modification of (1) – (3), with the help of function π_2.

In **A2, A3** three different explicit representations for $p_2(n)$, where $p_2(n)$ means n-th term of the sequence of twin primes are given, using $(1')$ – $(3')$. For example:

$$p_2(1) = 3,\ p_2(2) = 5,\ p_2(3) = 7,\ p_2(4) = 11,\ p_2(5) = 13,$$

$$p_2(6) = 17,\ p_2(7) = 19, ...$$

Let C be the sequence of all composite numbers including 1 (because 1 is not included in the sequence of the prime numbers), i.e.:

$$c_1 = 1,\ c_2 = 4,\ c_3 = 6,\ c_4 = 8,\ c_5 = 9,\ c_6 = 10,\ c_7 = 12,\ c_8 = 14,$$

$$c_9 = 15,\ c_{10} = 16, ...$$

It is trivial to see that for $k \geq 0$:

$$\pi_C(k) = k - \pi(k),$$

where $\pi(k)$ as usually means the number of the prime numbers that are not greater than k. Also, for $n \geq 1$ we have obviously:

$$C_n \leq \lambda(n)$$

with $\lambda(n) = 2n$.

Therefore, applying formulae $(1')$ – $(3')$, we obtain:

$$C_n = \sum_{k=0}^{2n} [\frac{1}{1 + [\frac{\pi_C(k)}{n}]}]. \qquad (5)$$

$$C_n = -2. \sum_{k=0}^{2n} \zeta(-2.[\frac{\pi_C(k)}{n}]). \qquad (6)$$

$$C_n = \sum_{k=0}^{2n} \frac{1}{\Gamma(1 - [\frac{\pi_C(k)}{n}])}. \qquad (7)$$

Let C be the sequence of all odd composite numbers including 1, i.e.:

$$c_1 = 1,\ c_2 = 9,\ c_3 = 15,\ c_4 = 21,\ c_5 = 25, ...$$

It is clear that

$$\pi_C(0) = 0,\ \pi_C(1) = 1 \qquad (8)$$

and for $k \geq 2$:

$$\pi_C(k) = k + 1 - [\frac{k}{2}] - \pi(k). \qquad (9)$$

Also, for $n \geq 1$ the inequality

$$C_n \leq \lambda(n)$$

holds for

$$\lambda(n) = 3(2n - 1) = 6n - 3. \qquad (10)$$

Therefore, applying formulae $(1')$ – $(3')$ and using (8) – (10), we obtain for $n \geq 2$:

$$C_n = 2 + \sum_{k=2}^{6n-3} [\frac{1}{1 + [\frac{k + 1 - [\frac{k}{2}] - \pi(k)}{n}]}],$$

$$C_n = 2 - 2. \sum_{k=2}^{6n-3} \zeta(-2.[\frac{k + 1 - [\frac{k}{2}] - \pi(k)}{n}]),$$

$$C_n = 2 + \sum_{k=2}^{6n-3} \frac{1}{\Gamma(1 - [\frac{k+1-[\frac{k}{2}]-\pi(k)}{n}])}.$$

It is possible to put $[\frac{k+3}{2}]$ instead of $k+1-[\frac{k}{2}]$ in the above formulae.

A5. ON ONE REMARKABLE IDENTITY RELATED TO FUNCTION $\pi(\mathbf{x})$[5]

By $\mathcal{R}_+$ we denote the set of all positive real numbers and $\mathcal{N} = \{1, 2, ...\}$.

Let

$$g = \{g_n\}_{n=1}^{\infty}$$

be sequence such that:

$$g_n \in \mathcal{R}_+, \qquad (a_1)$$

$$(\forall n \in \mathcal{N})(g_n < g_{n+1}), \qquad (a_2)$$

$$g \text{ is unbounded.} \qquad (a_3)$$

For any $x \in \mathcal{R}_+$ we denote by $\pi(x)$ the number of all terms of g, that are not greater than x.

When x satisfies the inequality

$$0 \le x < g_1$$

we put

$$\pi(x) = 0.$$

Remark 1. The condition (a_3) shows that the number $\pi(x)$ is always finite for a fixed x.

The main result here is the following

Theorem. Let $a, b \in \mathcal{R}_+$ and $b \ge g_1$. Then the identity

$$\sum_{i=1}^{\pi(b)} \pi(\frac{a}{g_i}) = \pi(\frac{a}{b}).\pi(b) + \sum_{j=1}^{\pi(\frac{a}{g_1})-\pi(\frac{a}{b})} \pi(\frac{a}{g_{\pi(\frac{a}{b})+j}}) \qquad (1)$$

holds.

Remark 2. When

$$\pi(\frac{a}{g_1}) = \pi(\frac{a}{b})$$

[5] The results in this section are taken from [33]

we put in (1) $\sum_{j=1}^{\pi(\frac{a}{g_1})-\pi(\frac{a}{b})} \bullet$ to be zero, i.e., the right hand-side of (1) reduces to $\pi(\frac{a}{b}).\pi(b)$. Thus, under the conditions of the above Theorem, the identity

$$\sum_{i=1}^{\pi(b)} \pi(\frac{a}{g_i})$$

$$= \begin{cases} \pi(\frac{a}{b}).\pi(b), & \text{if } \pi(\frac{a}{g_1}) = \pi(\frac{a}{b}) \\ \pi(\frac{a}{b}).\pi(b) + \sum_{j=1}^{\pi(\frac{a}{g_1})-\pi(\frac{a}{b})} \pi(\frac{a}{g_{\pi(\frac{a}{b})+j}}), & \text{if } \pi(\frac{a}{g_1}) > \pi(\frac{a}{b}) \end{cases} \qquad (2)$$

holds.

Proof of the Theorem. First, we note that if $a = 0$, then (1), i.e., (2) holds, since

$$\pi(\frac{a}{g_1}) = \pi(\frac{a}{b}) = \pi(0) = 0$$

and therefore, we may use Remark 2.

For that reason, further we assume that $a > 0$.

First, let us prove (1), i.e., (2) for case $b = g_1$.

Now, we have

$$\pi(b) = \pi(g_1) = 1$$

and

$$\pi(\frac{a}{g_1}) = \pi(\frac{a}{b}).$$

Hence

$$\pi(\frac{a}{b}).\pi(b) = \pi(\frac{a}{g_1}).\pi(g_1) = \pi(\frac{a}{g_1})$$

and (1), resp. (2), is proved, since the left hand-side of (2) coincides with $\pi(\frac{a}{g_1})$. Then it remains only to consider the case

$$g_1 < b \qquad (3)$$

and the proof of the Theorem will be completed.

Let (3) hold. We must consider the alternatives

$$b \leq \frac{a}{b} \qquad (e1)$$

and

$$b > \frac{a}{b}. \qquad (e2)$$

Let (e_1) hold. We shall prove (1) in this case. Inequality (3) implies that interval

$$\alpha \equiv [g_1, b] \qquad (4)$$

is well-defined. Also, (3) and (e_1) yield

$$\frac{a}{b} < \frac{a}{g_1}. \qquad (5)$$

Then (5) implies that the interval

$$\beta \equiv (\frac{a}{b}, \frac{a}{g_1}] \qquad (6)$$

is well-defined, too. Obviously, $\alpha \cap \beta = \emptyset$ and moreoevr, β lies to the right side of α on the real axis.

Let $g_i, g_j \in g$ $(i \neq j)$ be arbitrary. We introduce $\tau_{i,j}$ putting

$$\tau_{i,j} = g_i.g_j. \qquad (7)$$

We denote by $\mathcal{P}$ the set of these $\tau_{i,j}$ defined by (7), for which

$$g_i \in g \cap \alpha, \ \ g_j \in g \cap \beta$$

and inequality

$$\tau_{i,j} \leq a \qquad (8)$$

holds. Then we consider the alternatives:

$$\mathcal{P} = \emptyset \qquad (u_1)$$

and

$$\mathcal{P} \neq \emptyset. \qquad (u_2)$$

Let (u_1) holds. Then $g \cap \beta = \emptyset$.

Indeed, if we assume that there exists $g_j \in g \cap \beta$, then we obtain

$$\tau_{1,j} = g_1.g_j \leq g_1.\frac{a}{g_1} = a,$$

i.e., $\tau_{1,j}$ satisfies (8). Therefore, $\tau_{i,j} \in \mathcal{P}$, since $g_1 \in g \cap \alpha$. Hence

$$\mathcal{P} \neq \emptyset.$$

But the last contradicts to (u_1).

Now, $g \cap \beta = \emptyset$ implies

$$\pi(\frac{a}{g_1}) = \pi(\frac{a}{b}). \tag{9}$$

Moreover, the equality

$$\pi(x) = \pi(\frac{a}{b}) \tag{10}$$

holds for each $x \in \beta$.

Let $x_i = \dfrac{a}{g_i}$ for $i = 1, 2, ..., \pi(b)$. Then

$$g_1 \leq g_i \leq b$$

and therefore for $i = 1, 2, ..., \pi(b)$:

$$x_i \in [\frac{a}{b}, \frac{a}{g_1}]. \tag{11}$$

Now, (10) and (11) yield

$$\pi(\frac{a}{g_i}) = \pi(\frac{a}{b}). \tag{12}$$

for each $i = 1, 2, ..., \pi(b)$.

But (12) implies

$$\sum_{i=1}^{\pi(b)} \pi(\frac{a}{g_i}) = \pi(\frac{a}{b}).\pi(b), \tag{13}$$

which proves (1), because of Remark 2.

The case (u_1) is finished.

let (u_2) hold. Then the inequality

$$\pi(\frac{a}{g_1}) > \pi(\frac{a}{b}) \tag{14}$$

holds.

Indeed, the assumption that (9) holds, implies

$$g \cap \beta = \emptyset.$$

Hence $\mathcal{P} = \emptyset$. But the last equality contradicts to (u_2).

Now, (14) implies that

$$g \cap \beta \neq \emptyset$$

and that

$$g_{\pi(\frac{a}{b})+k} \in g \cap \beta$$

at least for $k = 1$. Therefore, the sum $\sum\limits_{j=1}^{\pi(\frac{a}{g_1})-\pi(\frac{a}{b})} \bullet$ from the right hand-side of (1) is well-defined.

We use the following approach to prove (1) in the case (u_2). First, we denote by $\theta(\alpha, \beta)$ the number of all elements of the set $\mathcal{P}$. Second, we calculate $\theta(\alpha, \beta)$ using two different ways. Third, we compare the results of these two different calculations and as a result we establish (1).

First way of calculation

Let

$$E \equiv \{1, 2, ..., \pi(b)\}.$$

If i describes E, then g_i describes $g \cap \alpha$.

Let $E_1 \subset E$ be the set of those $i \in E$ for which there exists at least one j, such that $g_j \in g \cap \beta$ and $\tau_{i,j} \in \mathcal{P}$. For each $i \in E_1$ we denote by δ_i the number of those $g_j \in g \cap \beta$, for which $\tau_{i,j} \in \mathcal{P}$. Then, equality

$$\theta(\alpha, \beta) = \sum_{i \in E_1} \delta_i \tag{15}$$

holds.

On the other hand, from the definition of these g_j it follows that they belong to interval $(\frac{a}{b}, \frac{a}{g_j}]$. Hence, for $i \in E_1$

$$\delta_i = \pi(\frac{a}{g_i}) - \pi(\frac{a}{b}). \tag{16}$$

Remark 3. From the definitions of δ_i and E_1 it follows that $\delta_i > 0$.

Let $i \in E_2$, where

$$E_2 \equiv E - E_1.$$

Then

$$g \cap (\frac{a}{b}, \frac{a}{g_i}] = \emptyset,$$

because in the opposite case we will obtain that $i \in E_1$, that is impossible, since $E_1 \cap E_2 = \emptyset$.

Hence for $i \in E_2$

$$\pi(\frac{a}{g_i}) = \pi(\frac{a}{b}),$$

i.e., for $i \in E_2$

$$\pi(\frac{a}{g_i}) - \pi(\frac{a}{b}) = 0. \tag{17}$$

Now, (15), (16), and (17) imply

$$\theta(\alpha, \beta) = \sum_{i \in E} (\pi(\frac{a}{g_i}) - \pi(\frac{a}{b})),$$

i.e.,

$$\theta(\alpha, \beta) = \sum_{i=1}^{\pi(b)} \pi(\frac{a}{g_i}) - \pi(\frac{a}{b}).\pi(b). \tag{18}$$

Second way of calculation

Let

$$W \equiv \{\pi(\frac{a}{b}) + k \mid k = 1, 2, ..., \pi(\frac{a}{g_1}) - \pi(\frac{a}{b})\}.$$

Of course, we have $W \neq \emptyset$, since (u_2), i.e., (14), is true.

When j describes W, g_j describes $g \cap \beta$. For every such j it is fulfilled

$$g_1 \leq \frac{a}{g_j} < b. \tag{19}$$

Therefore, there exist exactly $\pi(\frac{a}{g_j})$ in number $g_i \in g \cap \beta$, for which $\tau_{i,j} \in \mathcal{P}$. Hence

$$\theta(\alpha, \beta) = \sum_{j \in W} \pi(\frac{a}{g_j}).$$

Thus, using the definition of W, we finally get

$$\theta(\alpha, \beta) = \sum_{j=1}^{\pi(\frac{a}{g_1}) - \pi(\frac{a}{b})} \pi(\frac{a}{g_{\pi(\frac{a}{b})+j}}). \tag{20}$$

If we compare (18) and (20), we prove (1) in case (u_2).

Up to now, we have established that (1) (and (2)) holds, when

$$g_1 \leq b \leq \frac{a}{b} \tag{21}$$

and case (u_2) is finished too.

Now, let (e_2) hold. To prove (2) (and (1)) in this case we consider the alternatives

$$\frac{a}{b} < g_1 \tag{e_{21}}$$

and

$$\frac{a}{b} \geq g_1. \tag{e_{22}}$$

Let (e_{21}) hold. Then $\pi(\frac{a}{b}) = 0$ and (1) takes the form

$$\sum_{i=1}^{\pi(b)} \pi(\frac{a}{g_i}) = \sum_{j=1}^{\pi(\frac{a}{g_1})} \pi(\frac{a}{g_j}). \tag{22}$$

Let us note, that (21) implies $b > \frac{a}{g_1}$. Then (22) will be proved, if we prove that for all $k \in \mathcal{N}$

$$\pi(\frac{a}{g_{\pi(\frac{a}{g_1})+k}}) = 0. \tag{23}$$

But $g_{\pi(\frac{a}{g_1})+k} \notin W$. Then we have that $g_{\pi(\frac{a}{g_1})+k} > \frac{a}{g_1}$. Hence, for all $k \in \mathcal{N}$

$$\frac{a}{g_{\pi(\frac{a}{g_1})+k}} < g_1.$$

The last inequalities prove (23), since $\pi(g_1) = 1$ and for $0 \leq x < g_1$ it is fulfilled $\pi(x) = 0$.

Therefore, (22) is proved, too, and the case (e_{21}) is finished.

Let (e_{22}) hold. Then

$$g_1 \leq \frac{a}{b} < b \tag{24}$$

is valid.

We introduce the number b_1 putting

$$b_1 = \frac{a}{b}. \tag{25}$$

Then, we find

$$b = \frac{a}{b_1}. \tag{26}$$

From (24), (25), and (26) it follows immediatelly

$$g_1 \leq b_1 < \frac{a}{b_1}. \tag{27}$$

Obviously, (27) looks like (21) (only b from (21) is changed with b_1 in (27)). But we proved that (21) implies (1). Therefore, (27) implies (1), but with b_1 instead of b. Hence, the identity

$$\sum_{j=1}^{\pi(b_1)} \pi(\frac{a}{g_j}) = \pi(\frac{a}{b_1}).\pi(b_1) + \sum_{j=1}^{\pi(\frac{a}{g_1})-\pi(\frac{a}{b_1})} \pi(\frac{a}{g_{\pi(\frac{a}{b_1})+j}}) \tag{28}$$

holds and Remark 2 remains also valid substituting b by b_1.

Using (25) we rewrite (28) in the form

$$\sum_{i=1}^{\pi(\frac{a}{b})} \pi(\frac{a}{g_i}) = \pi(\frac{a}{b}).\pi(b) + \sum_{j=1}^{\pi(\frac{a}{g_1})-\pi(b)} \pi(\frac{a}{g_{\pi(b)+j}}). \tag{29}$$

First, let $\pi(b) = \pi(\frac{a}{b})$. In this case (29) coincides with (1) and (1) is proved, since (29) is true.

Second, let $\pi(\frac{a}{b}) < \pi(b)$. Then we add to the two hand-sides of (29) the sum

$$\sum_{j=1}^{\pi(b)-\pi(\frac{a}{b})} \pi(\frac{a}{g_{\pi(\frac{a}{b})+j}})$$

and obtain again (1). This completes the proof of (1) in this case, too, because (29) is true.

Since, we have no other possibilities (the inequality $\pi(b) < \pi(\frac{a}{b})$ is impossible, because of (e_2)), we finish with the case (e_{22}). Hence, the case (e_2) is finished too.

The Theorem is proved.

Further, we use some well-known functions (see, e.g., [15]):

$$\text{ch}x \equiv \frac{e^x + e^{-x}}{2}, \ \text{sh}x \equiv \frac{e^x - e^{-x}}{2}, \ \text{th}x \equiv \frac{\text{sh}x}{\text{ch}x}, \ \text{cth}x \equiv \frac{\text{ch}x}{\text{sh}x}.$$

Corollary 1. Let $a = \text{ch}x, b = \text{sh}x$, where $x \in \mathcal{R}_+$ and $\text{sh}x \geq g_1$. Then, the identity

$$\sum_{i=1}^{\pi(\mathrm{sh}x)} \pi(\frac{\mathrm{ch}x}{g_i})$$

$$= \begin{cases} \pi(\mathrm{sh}x).\pi(\mathrm{cth}x), & \text{if } \pi(\frac{\mathrm{ch}x}{g_1}) = \pi(\mathrm{cth}x) \\ \pi(\mathrm{sh}x).\pi(\mathrm{cth}x) + & \\ \sum_{j=1}^{\pi(\frac{\mathrm{ch}x}{g_1})-\pi(\mathrm{cth}x)} \pi(\frac{\mathrm{ch}x}{g_{\pi(\mathrm{cth}x)+j}}), & \text{if } \pi(\frac{\mathrm{ch}x}{g_1}) > \pi(\mathrm{cth}x) \end{cases} \quad (30)$$

holds.

The same way, putting: $a = \mathrm{sh}x$, $b = \mathrm{ch}x$, where $x \in \mathcal{R}_+$ and $\mathrm{ch}x \geq g_1$, as a corollary of the Theorem, we obtain another identity, that we do not write here since one may get it putting in (30) $\mathrm{ch}x, \mathrm{sh}x, \mathrm{th}x$ instead of $\mathrm{sh}x, \mathrm{ch}x, \mathrm{cth}x$, respectively.

Now, let g be the sequence of all primes, i.e.,

$$g = 2, 3, 5, 7, 11, 13, ...$$

Then the function $\pi(x)$ coincides with the famous function π of the prime number distribution. Thus, from our Theorem we obtain

Corollary 2. Let $a, b \in \mathcal{R}_+$, $b \geq 2$ and $\{p_n\}_{n=1}^{\infty}$ be the sequence of all primes. Then the identity

$$\pi(\frac{a}{p_1}) + \pi(\frac{a}{p_2}) + ... + \pi(\frac{a}{p_{\pi(b)}})$$

$$= \pi(\frac{a}{b}).\pi(b) + \pi(\frac{a}{p_{\pi(\frac{a}{b})+1}}) + \pi(\frac{a}{p_{\pi(\frac{a}{b})+2}}) + ...\pi(\frac{a}{p_{\pi(\frac{a}{2})}}) \quad (31)$$

holds.

In (31) $\pi(x)$ denotes (as usually) the number of primes, that are not greater than x. Also, the right hand-side of (31) reduces to $\pi(\frac{a}{b}).\pi(b)$ if and only if $\pi(\frac{a}{b}) = \pi(\frac{a}{2})$.

Identities (1) and (2) were discovered in 2001 in the Bulgarian village on Black Sea Sinemoretz.

A6. AN ARITHMETIC FUNCTION[6]

For

$$n = \sum_{i=1}^{m} a_i.10^{m-i} \equiv \overline{a_1 a_2 ... a_m},$$

where a_i is a natural number and $0 \leq a_i \leq 9$ $(1 \leq i \leq m)$ let:

$$\varphi(n) = \begin{cases} 0 & , \text{ if } n = 0 \\ \sum_{i=1}^{m} a_i & , \text{ otherwise} \end{cases}$$

and for the sequence of functions $\varphi_0, \varphi_1, \varphi_2, ...$, where ($l$ is a natural number)

$$\varphi_0(n) = n,$$

$$\varphi_{l+1} = \varphi(\varphi_l(n)),$$

let the function ψ be defined by

$$\psi(n) = \varphi_l(n),$$

in which

$$\varphi_{l+1}(n) = \varphi_l(n).$$

This function has the following (and other) properties:

$$\psi(m + n) = \psi(\psi(m) + \psi(n)),$$

$$\psi(m.n) = \psi(\psi(m).\psi(n)) = \psi(m.\psi(n)) = \psi(\psi(m).n),$$

$$\psi(m^n) = \psi(\psi(m)^n),$$

$$\psi(n + 9) = \psi(n),$$

$$\psi(9n) = 9.$$

[6] The results in this section are taken from [1, 2]

Let the sequence $a_1, a_2, ...$ with members – natural numbers, be given and let

$$c_i = \psi(a_i) \ (i = 1, 2, ...).$$

Hence, we deduce the sequence $c_1, c_2, ...$ from the former sequence. If k and $l \geq 0$ exist such that

$$c_{i+l} = c_{k+i+l} = c_{2k+i+l} = ...$$

for $1 \leq i \leq k$, then we say that

$$[c_{l+1}, c_{l+2}, ..., c_{l+k}]$$

is a base of the sequence $c_1, c_2, ...$ with a length k and with respect to function ψ.

For example, the Fibonacci sequence $\{F_i\}_{i=0}^{\infty}$, for which

$$F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n \ (n \geq 0)$$

has a base with a length of 24 with respect to the function ψ and it is the following:

$$[1, 1, 2, 3, 5, 8, 4, 3, 7, 1, 8, 9, 8, 8, 7, 6, 4, 1, 5, 6, 2, 8, 1, 9];$$

the Lucas sequence $\{L_i\}_{i=0}^{\infty}$, for which

$$L_2 = 2, L_1 = 1, L_{n+2} = L_{n+1} + L_n \ (n \geq 0)$$

also has a base with a length of 24 with respect to the function ψ and it is the following:

$$[2, 1, 3, 4, 7, 2, 9, 2, 2, 4, 6, 1, 7, 8, 6, 5, 2, 7, 9, 7, 7, 5, 3, 8];$$

even the Lucas-Lehmer sequence $\{l_i\}_{i=0}^{\infty}$, for which

$$l_1 = 4, l_{n+1} = l_n^2 - 2 \ (n \geq 0)$$

has a base with a length of 1 with respect to the function ψ and it is [5].

The $k-th$ triangular number t_k is defined by the formula

$$t_k = \frac{k(k+1)}{2}$$

and it has a base with a length of 9 with the form

$$[1, 3, 6, 1, 5, 3, 1, 9, 9].$$

It is directly checked that the bases of the sequences $\{n^k\}_{k=1}^{\infty}$ for $n = 1, 2, ..., 9$ are those introduced in the following table.

n	a base of a sequence $\{n^k\}_{i=1}^{\infty}$	a length of the base
1	1	1
2	2,4,8,7,5,1	6
3	9	1
4	4,7,1	3
5	5,7,8,4,2,1	6
6	9	1
7	7,4,1	3
8	8,1	2
9	9	1

On the other hand, the sequence $\{n^n\}_{n=1}^{\infty}$ has a base (with a length of 9) with the form

$$[1, 4, 9, 1, 2, 9, 7, 1, 9],$$

and the sequence $\{k^{n!}\}_{n=1}^{\infty}$ has a base with a length of 9 with the form

$$\begin{cases} [1] & \text{, if } k \neq 3m \text{ some some natural number } m \\ [9] & \text{, if } k = 3m \text{ some some natural number } m \end{cases}$$

Bibliography

[1] Atanassov, K. An arithmetic function and some of its applications. *Bull. of Number Theory and Related Topics,* Vol. IX (1985), No. 1, 18-27.

[2] Atanassov, K. *On Some of the Smarandache's Problems.* American Research Press, Lupton, 1999.

[3] Atanassov K., Remarks on some of the Smarandache's problems. Part 1. *Smarandache Notions Journal,* Vol. 12, No. 1-2-3, Spring 2001, 82-98

[4] Atanassov, K. A new formula for the n-th prime number. *Comptes Rendus de l'Academie Bulgare des Sciences,* Vol. 54, 2001, No. 7, 5-6.

[5] Atanassov K. On the 20-th and the 21-st Smarandache's problems. *Smarandache Notions Journal,* Vol. 12, No. 1-2-3, Spring 2001, 111-113.

[6] Atanassov K. On four prime and coprime functions. *Smarandache Notions Journal,* Vol. 12, No. 1-2-3, Spring 2001, 122-125.

[7] Atanassov, K. On the 17-th Smarandache's problem. *Smarandache Notions Journal,* Vol. 13, No. 1-2-3, 2002, 124-125.

[8] Atanassov, K. On the 46-th Smarandache's problem. *Smarandache Notions Journal,* Vol. 13, No. 1-2-3, 2002, 126-127.

[9] Atanassov K. On the second Smarandache's problem. *Notes on Number Theory and Discrete Mathematics,* Vol. 9, 2003, No. 2, 46-48.

[10] Atanassov K. On four Smarandache's problems. *Notes on Number Theory and Discrete Mathematics,* Vol. 11, 2005, No. 1, 1-6.

[11] Bege, A. A generalization of von Mangoldt's function. *Bulletin of Number Theory and Related Topics,* Vol. XIV, 1990, 73-78.

[12] Davenport, H. *Multiplicative Numebr Theory.* Markham Publ. Co., Chicago, 1967.

[13] Dumitrescu C., V. Seleacu, *Some Sotions and Questions in Number Theory,* Erhus Univ. Press, Glendale, 1994.

[14] Grauert H., Lieb I., Fischer W, *Differential- und Integralrechnung,* Springer-Verlag, Berlin, 1967.

[15] *Handbook of Mathematical Functions,* M. Abramowitz and I. Stegun (Eds.), National Bureau of Standards, Applied Mathematics Series, Vol. 55, 1964.

[16] Mitrinović, D., M. Popadić. *Inequalities in Number Theory.* Niś, Univ. of Niś, 1978.

[17] Nagell T., *Introduction to Number Theory.* John Wiley & Sons, Inc., New York, 1950.

[18] Ribenboim, P. *The New Book of Prime Number Records.* Springer, New York, 1995.

[19] Sandor J., A. Bege, The Mobius function: generalizations and extensions. *Advanced Studies on Contemporary Mathematics,* Vol. 6, 2002, No. 2, 77-128.

[20] Sierpinski, W. *Co wiemy, a czego nie wiemy o liczbach pierwszych.* Panstwowe Zaklady Wydawnictw Szkolnych Warszawa, 1961 (in Polish).

[21] Smarandache F., Criteria for a number to be prime. *Gazeta Matematica,* Vol. XXXVI, No. 2, 49-52, Bucharest, 1981 (in Romanian).

[22] Smarandache, F., Proposed Problem # 328 ("Prime Pairs ans Wilson's Theorem"), <College Mathematics Journal>, USA, March 1988, pp. 191-192.

[23] Smarandache, Florentin, "Characterization of n Prime Numbers Simultaneously", <Libertas Mathematica>, University of Texas at Arlington, Vol. XI, 1991, pp. 151-155.

[24] Smarandache F., *Only Problems, Not Solutions!.* Xiquan Publ. House, Chicago, 1993.

[25] Smarandache, F., *Collected Papers,* Vol. II, Kishinev University Press, Kishinev, 1997.

[26] http://www.gallup.unm.edu/∼smarandache/SIM-PRIM.TXT

[27] http://www.gallup.unm.edu/∼smarandache/COPRIME.TXT

[28] http://www.gallup.unm.edu/∼smarandache/FORMULA.TXT

[29] Vassilev–Missana, M. Some new formulae for the twin primes counting function $\pi_2(n)$. *Notes on Number Theory and Discrete Mathematics,* Vol. 7, 2001, No. 1, 10-14.

[30] Vassilev–Missana, M. Three formulae for **n**-th prime and six for **n**-th term of twin primes. *Notes on Number Theory and Discrete Mathematics,* Vol. 7, 2001, No. 1, 15-20.

[31] Vassilev–Missana, M. Some explicit formulae for the composite numbers. *Notes on Number Theory and Discrete Mathematics,* Vol. 7, 2001, No. 2, 29-31.

[32] Vassilev–Missana, M. Explicit formulae for the n-th term of the twin prime sequence. *Notes on Number Theory and Discrete Mathematics,* Vol. 7, 2001, No. 3, 103-104.

[33] Vassilev–Missana, M., On one remarkable identity related to function $\pi(x)$, *Notes on Number Theory and Discrete Mathematics,* Vol. 8 (2002), No. 4, 129-136.

[34] Vassilev–Missana, M. On 15-th Smarandache's problem. *Notes on Number Theory and Discrete Mathematics,* Vol. 9, 2003, No. 2, 42-45.

[35] Vassilev–Missana, M. Some representations concerning the product of divisors of n. *Notes on Number Theory and Discrete Mathematics,* Vol. 10, 2004, No. 2, 54-56.

[36] Vassilev M., Atanassov K., Note on the Diophantine equation $2x^2 - 3y^2 = p$, *Smarandache Notions Journal,* Vol. 11, No. 1-2-3, 2000, 64-68.

[37] Vassilev–Missana, M., K. Atanassov. On 28-th Smarandache's problem. *Notes on Number Theory and Discrete Mathematics,* Vol. 7 (2001), No. 2, 61-64.

[38] Vassilev - Missana M., K. Atanassov. On five Smarandache's problems. *Notes on Number Theory and Discrete Mathematics,* Vol. 10, 2004, No. 2, 34-53.

[39] Vassilev - Missana M., K. Atanassov. Remarks on the 46-th Smarandache's problem.*Notes on Number Theory and Discrete Mathematics,* Vol. 10, 2004, No. 3, 84-88.

[40] Vassilev - Missana M., K. Atanassov. On two Smarandache's problems. *Notes on Number Theory and Discrete Mathematics,* Vol. 10, 2004, No. 4, 106-112.

[41] Vassilev P., M. Vassilev–Missana, K. Atanassov. On 25-th and 26-th Smarandache's problems. *Notes on Number Theory and Discrete Mathematics,* Vol. 9, 2003, No. 4, 99-104.

[42] Yosida K., *Functional Analysis,* Springer-Verlag, Berlin, 1965.

Referees:

S. Bhattacharya, Alaska Pacific University, USA

A. Shannon, KvB Institute of Technology and University of New South Wales, Australia