Neutrosophic Closed Set and Neutrosophic Continuous Functions

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Abstract

In this paper, we introduce and study the concept of "neutrosophic closed set" and "neutrosophic continuous function". Possible application to GIS topology rules are touched upon.

Keywords: Neutrosophic Closed Set, Neutrosophic Set; Neutrosophic Topology; Neutrosophic Continuous Function.

1 INTRODUCTION

The idea of "neutrosophic set" was first given by Smarandache [11, 12]. Neutrosophic operations have been investigated by Salama et al. [1-10]. Neutrosophy has laid the foundation for a whole family of new mathematical theories, generalizing both their crisp and fuzzy counterparts [9, 13]. Here we shall present the neutrosophic crisp version of these concepts. In this paper, we introduce and study the concept of "neutrosophic closed set" and "neutrosophic continuous function".

2 TERMINOLOGIES

We recollect some relevant basic preliminaries, and in particular the work of Smarandache in [11, 12], and Salama et al. [1-10].

2.1 Definition [5]

A neutrosophic topology (NT for short) an a non empty set \( X \) is a family \( \tau \) of neutrosophic subsets in \( X \) satisfying the following axioms

\( (NT_1) \ O_x, I_x \in \tau \),
\( (NT_2) \ G_1 \cap G_2 \in \tau \ for \ any \ G_1, G_2 \in \tau \),
\( (NT_3) \ \bigcup G_i \in \tau \ \forall \{G_i : i \in J\} \subseteq \tau \)

In this case the pair \( (X, \tau) \) is called a neutrosophic topological space (NTS for short) and any neutrosophic set in \( \tau \) is known as neutrosophic open set (NOS for short) in \( X \). The elements of \( \tau \) are called open neutrosophic sets, A neutrosophic set \( F \) is closed if and only if \( C(F) \) is neutrosophic open.

2.1 Definition [5]

The complement of \( C(A) \) for short) of \( A \) is called a neutrosophic closed set (NCS for short) in \( A \). NOSA NCS X.

3 Neutrosophic Closed Set.

3.1 Definition

Let \( (X, \tau) \) be a neutrosophic topological space. A neutrosophic set \( A \) in \( (X, \tau) \) is said to be neutrosophic closed (in shortly N-closed). If \( Ncl(A) \subseteq G \ whenever A \subseteq G \) and \( G \) is neutrosophic open; the complement of neutrosophic closed set is Neutrosophic open.

3.1 Proposition

If \( A \) and \( B \) are neutrosophic closed sets then \( A \cup B \) is Neutrosophic closed set.

3.1 Remark

The intersection of two neutrosophic closed (N-closed for short) sets need not be neutrosophic closed set.

3.1 Example

Let \( X = \{a, b, c\} \) and
If \( B \subseteq \mu_B, \sigma_B, \nu_B \) is a NS in \( Y \), then the preimage of \( B \) under \( f \) denoted by \( f^{-1}(B) \) is a NS in \( X \) defined by
\[
 f^{-1}(B) = \left\{ x \in X \mid \mu_X(x) = \mu_B(f(x)) \right\}.
\]
ii) If \( A = \mu_A, \sigma_A, \nu_A \) is a NS in \( X \), then the image of \( A \) under \( f \), denoted by \( f(A) \), is a NS in \( Y \) defined by
\[
 f(A) = \left\{ y \in Y \mid \nu_Y(y) = \nu_A(f^{-1}(y)) \right\}.
\]

Here we introduce the properties of images and preimages some of which we shall frequently use in the following sections.

4.1 Corollary
Let \( A, \ \{ A_i : i \in \mathbb{J} \} \) be NSs in \( X \), and \( B_j : j \in \mathbb{K} \) NSs in \( Y \), and \( f : X \rightarrow Y \) a function. Then
(a) \( A_i \subseteq A \Leftrightarrow f(A_i) \subseteq f(A) \).
(b) \( A \subseteq f^{-1}(f(A)) \) and if \( f \) is injective, then
\[
 A = f^{-1}(f(A)).
\]
(c) \( f^{-1}(f(B)) \subseteq B \) and if \( f \) is surjective, then
\[
 f^{-1}(f(B)) = B.
\]
(d) \( \bigcap_{B_j} f^{-1}(B_j) = f^{-1} \left( \bigcap_{B_j} B_j \right) = \bigcap_{B_j} f^{-1}(B_j) \).
(e) \( f(\bigcap A_i) = \bigcap f(A_i) \); \( f(\bigcap A_i) \subseteq \bigcap f(A_i) \); and if \( f \) is injective, then
\[
 f(\bigcap A_i) = \bigcap f(A_i).
\]
(f) \( f^{-1}(0_N) = 1_N \) and \( f(0_N) = 0_Y \).
(g) \( f(1_N) = 1_Y \) if \( f \) is subjectively

Proof
Obvious.

4.2 Definition
Let \( (X, \tau_1) \) and \( (Y, \tau_2) \) be two NTs, and let \( f : X \rightarrow Y \) be a function. Then \( f \) is said to be continuous iff the preimage of each NCS in \( \tau_2 \) is a NCS in \( \tau_1 \).

4.3 Definition
Let \( (X, \tau_1) \) and \( (Y, \tau_2) \) be two NTs, and let \( f : X \rightarrow Y \) be a function. Then \( f \) is said to be open iff the image of each NS in \( \tau_1 \) is a NS in \( \tau_2 \).

4.4 Example
Let \( (X, \tau_1) \) and \( (Y, \tau_2) \) be two NTs, and let \( f : X \rightarrow Y \) be a function. Then \( f \) is said to be open iff the image of each NS in \( \tau_1 \) is a NS in \( \tau_2 \).

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Let \( f : (X, \Gamma_1) \rightarrow (Y, \Gamma_2) \) be a function. If \( f \) is neutrosophic continuous, then the preimage of each NS (neutrosophic closed set) in \( \Gamma_2 \) is a NS in \( \Gamma_1 \).

4.2 Proposition
The following are equivalent to each other:
(a) \( f : (X, \Gamma_1) \rightarrow (Y, \Gamma_2) \) is neutrosophic continuous.
(b) \( f^{-1}(N\text{Int}(B)) \subseteq N\text{Int}(f^{-1}(B)) \) for each CNS \( B \) in \( Y \).
(c) \( N\text{cl}(f^{-1}(B)) \subseteq f^{-1}(N\text{cl}(B)) \) for each NCB in \( Y \).

4.3 Example
Let \( \{Y, \Gamma_2\} \) be a NTS and \( f : X \rightarrow Y \) be a function. In this case \( \Gamma_1 = f^{-1}(H) : H \in \Gamma_2 \) is a NT on \( X \). Indeed, it is the coarsest NT on \( X \) which makes the function \( f : X \rightarrow Y \) continuous. One may call it the initial neutrosophic crisp topology with respect to \( f \).

4.4 Definition
Let \( (X,T) \) and \( (Y,S) \) be two neutrosophic topological spaces, then
(a) A map \( f : (X,T) \rightarrow (Y,S) \) is called N-continuous (in short N-continuous) if the inverse image of every closed set in \( (Y,S) \) is Neutrosophic closed in \( (X,T) \).
(b) A map \( f : (X,T) \rightarrow (Y,S) \) is called neutrosophic-ge irresolute if the inverse image of every Neutrosophic closed set in \( (Y,S) \) is Neutrosophic closed in \( (X,T) \). Equivalently if the inverse image of every Neutrosophic open set in \( (Y,S) \) is Neutrosophic open in \( (X,T) \).
(c) A map \( f : (X,T) \rightarrow (Y,S) \) is said to be strongly neutrosophic continuous if \( f^{-1}(A) \) is both neutrosophic open and neutrosophic closed in \( (X,T) \) for each neutrosophic set \( A \) in \( (Y,S) \).
(d) A map \( f : (X,T) \rightarrow (Y,S) \) is said to be perfectly neutrosophic continuous if \( f^{-1}(A) \) is both neutrosophic open and neutrosophic closed in \( (X,T) \) for each neutrosophic open set \( A \) in \( (Y,S) \).
(e) A map \( f : (X,T) \rightarrow (Y,S) \) is said to be strongly N-continuous if the inverse image of every Neutrosophic open set in \( (Y,S) \) is neutrosophic open in \( (X,T) \).

(F) A map \( f : (X,T) \rightarrow (Y,S) \) is said to be perfectly N-continuous if the inverse image of every Neutrosophic open set in \( (Y,S) \) is both neutrosophic open and neutrosophic closed in \( (X,T) \).

4.3 Proposition
Let \( (X,T) \) and \( (Y,S) \) be any two neutrosophic topological spaces. Let \( f : (X,T) \rightarrow (Y,S) \) be generalized neutrosophic continuous. Then for every neutrosophic set \( A \) in \( X \), \( f(N\text{cl}(A)) \subseteq N\text{cl}(f(A)) \).

4.4 Proposition
Let \( (X,T) \) and \( (Y,S) \) be any two neutrosophic topological spaces. Let \( f : (X,T) \rightarrow (Y,S) \) be generalized neutrosophic continuous. Then for every neutrosophic set \( A \) in \( Y \), \( N\text{cl}(f^{-1}(A)) \subseteq f^{-1}(N\text{cl}(A)) \).

4.5 Proposition
Let \( (X,T) \) and \( (Y,S) \) be any two neutrosophic topological spaces. If \( A \) is a Neutrosophic closedset in \( (X,T) \) and if \( f : (X,T) \rightarrow (Y,S) \) is neutrosophic continuous and neutrosophic-closed then \( f(A) \) is Neutrosophic closed in \( (Y,S) \).

Proof.
Let \( G \) be a neutrosophic-open in \( (Y,S) \). If \( f(A) \subseteq G \), then \( A \subseteq f^{-1}(G) \). Since \( A \) is neutrosophic closed, \( f^{-1}(G) \) is neutrosophic open in \( (X,T) \). Hence, \( f(A) \) is neutrosophic closed in \( (Y,S) \).

4.6 Proposition
Let \( (X,T) \) and \( (Y,S) \) be any two neutrosophic topological spaces. If \( f : (X,T) \rightarrow (Y,S) \) is neutrosophic continuous then it is N-continuous.

The converse of proposition 4.5 need not be true. See Example 4.3.

4.3 Example
Let \( X = \{a, b, c\} \) and \( Y = \{a, b, c\} \). Define neutrosophic sets \( A \) and \( B \) as follows:
\[ A = \{0.4, 0.4, 0.5, 0.0, 0.4, 0.5, 0.0, 0.4\} \]
\[ B = \{0.4, 0.4, 0.5, 0.0, 0.4, 0.5, 0.0, 0.4\} \]
Then the family \( \mathcal{T} = \{0.0, 1.0\} \) is a neutrosophic topology on \( X \) and \( S = \{0.0, 1.0\} \) is a neutrosophic topology on \( Y \). Thus \( (X,T) \) and \( (Y,S) \) are neutrosophic topological spaces. Define \( f : (X,T) \rightarrow (Y,S) \) as \( f(a) = b \), \( f(b) = a \), \( f(c) = c \). Clearly \( f \) is N-continuous. Note \( f \) is not neutrosophic continuous, since \( f^{-1}(B) \notin T \) for \( B \in S \).

4.4 Example
Let \( X = \{a, b, c\} \). Define the neutrosophic sets \( A \) and \( B \) as follows:
\[ A = \{0.4, 0.5, 0.0, 0.4, 0.5, 0.0, 0.4\} \]
\[ B = \{0.4, 0.5, 0.0, 0.4, 0.5, 0.0, 0.4\} \]
Let \( (X,T) \) and \( (Y,S) \) be any neutrosophic topological spaces. If \( f : (X,T) \to (Y,S) \) is strongly Neutrosophic continuous then \( f \) is strongly \( N \)-continuous.

The converse of proposition 3.23 is not true. See Example 4.7.

**Example 4.7**

Let \( X = \{a,b,c\} \) and Define the neutrosophic sets \( A \) and \( B \) as follows.

\[
A = \{(0.9,0.9,0.9),(0.1,0.1,0.1),(0.9,0.9,0.9)\}
\]

\[
B = \{(0.9,0.9,0.9),(0.1,0.1,0.1),(0.9,0.1,0.1),(0.9,0.9,0.9)\}
\]

\[
T = \{0_N, I_N, A, B\}
\]

\[
S = \{0_N, I_N, C\}
\]

Thus \( (X,T) \) and \( (X,S) \) are neutrosophic topological spaces. Let \( f : (X,T) \to (X,S) \) as follows: \( f(a) = a, f(b) = b, f(c) = b \). Clearly \( f \) is strongly \( N \)-continuous but \( f \) is not strongly \( N \)-continuous. Hence the proposition 3.23 is not true.
Then the family $T = \{0_N, 1_N, A\}$, $S = \{0_N, 1_N, B\}$ and $R = \{0_N, 1_N, C\}$ are neutrosophic topologies on $X$. Thus $(X,T),(X,S)$ and $(X,R)$ are neutrosophic topological spaces. Also define $f : (X,T) \to (X,S)$ as $f(a) = b$, $f(b) = a$, $f(c) = c$ and $g : (X,S) \to (X,R)$ as $g(a) = b$, $g(b) = c$, $g(c) = b$. Clearly $f$ and $g$ are $N$-continuous function. But $g \circ f$ is not $N$-continuous. For $1 - C$ is neutrosophic closed in $(X,R)$. $f^{-1}(g^{-1}(1-C))$ is not $N$ closed in $(X,T)$. $g \circ f$ is not $N$-continuous.

References


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