Abstract - The purpose of this paper is to define the product related neutrosophic topological space and proved some theorems based on this. We introduce the concept of neutrosophic semi-open sets and neutrosophic semi-closed sets in neutrosophic topological spaces and derive some of their characterization. Finally, we analyze neutrosophic semi-interior and neutrosophic semi-closure operators also.

Mathematics Subject Classification : 03E72

Keywords: Neutrosophic semi-open set, Neutrosophic semi-closed set, Neutrosophic semi-interior operator and Neutrosophic semi-closure operator.

INTRODUCTION

Theory of Fuzzy sets [17], Theory of Intuitionistic fuzzy sets [2], Theory of Neutrosophic sets [9] and the theory of Interval Neutrosophic sets [11] can be considered as tools for dealing with uncertainties. However, all of these theories have their own difficulties which are pointed out in [9]. In 1965, Zadeh [17] introduced fuzzy set theory as a mathematical tool for dealing with uncertainties where each element had a degree of membership. The Intuitionistic fuzzy set was introduced by Atanassov [2] in 1983 as a generalization of fuzzy set, where besides the degree of membership and the degree of non-membership of each element. The Neutrosophic set was introduced by Smarandache [9] and explained, neutrosophic set is a generalization of Intuitionistic fuzzy set. In 2012, Salama, Alblowi [15], introduced the concept of Neutrosophic topological spaces. They introduced neutrosophic topological space as a generalization of Intuitionistic fuzzy topological space and a Neutrosophic set besides the degree of membership, the degree of indeterminacy and the degree of non-membership of each element.

This paper consists of six sections. The section I consists of the basic definitions and some properties which are used in the later sections. The section II, we define product related neutrosophic topological space and proved some theorem related to this definition. The section III deals with the definition of neutrosophic semi-open set in neutrosophic topological spaces and its various properties. The section IV deals with the definition of neutrosophic semi-closed set in neutrosophic topological spaces and its various properties. The section V and VI are dealt with the concepts of neutrosophic semi-interior and neutrosophic semi-closure operators.

I. PRELIMINARIES

In this section, we give the basic definitions for neutrosophic sets and its operations.

Definition 1.1 [15] Let X be a non-empty fixed set. A neutrosophic set [ NS for short ] A is an object having the form A = \{ (x, \mu_A(x), \sigma_A(x), \gamma_A(x)) : x \in X \} where \( \mu_A(x) \), \( \sigma_A(x) \) and \( \gamma_A(x) \) which represents the degree of membership function, the degree
indeterminacy and the degree of non-membership function respectively of each element \( x \in X \) to the set \( A \).

**Remark 1.2** [15] A neutrosophic set \( A = \{ \{ x, \mu_A(x), \sigma_A(x), \gamma_A(x) \} : x \in X \} \) can be identified to an ordered triple \( (\mu_A, \sigma_A, \gamma_A) \) in \( 0,1 \ast \) on \( X \).

**Remark 1.3** [15] For the sake of simplicity, we shall use the symbol \( A = ( x, \mu_A, \sigma_A, \gamma_A ) \) for the neutrosophic set \( A = \{ \{ x, \mu_A(x), \sigma_A(x), \gamma_A(x) \} : x \in X \} \). Since our main purpose is to construct the tools for developing neutrosophic set and neutrosophic topology, we must introduce the NS \( 0_N \) and \( 1_N \) in \( X \) as follows:

\( 0_N \) may be defined as :
- \( (0_1) \hspace{1cm} 0_N = \{ x, 0, 0, 0 \} : x \in X \}
- \( (0_2) \hspace{1cm} 0_N = \{ x, 0, 1, 0 \} : x \in X \}
- \( (0_3) \hspace{1cm} 0_N = \{ x, 1, 0, 0 \} : x \in X \}
- \( (0_4) \hspace{1cm} 0_N = \{ x, 0, 0, 0 \} : x \in X \}

\( 1_N \) may be defined as :
- \( (1_1) \hspace{1cm} 1_N = \{ x, 1, 0, 0 \} : x \in X \}
- \( (1_2) \hspace{1cm} 1_N = \{ x, 1, 1, 0 \} : x \in X \}
- \( (1_3) \hspace{1cm} 1_N = \{ x, 1, 1, 1 \} : x \in X \}
- \( (1_4) \hspace{1cm} 1_N = \{ x, 1, 1, 1 \} : x \in X \}

**Definition 1.5** [15] Let \( A = (\mu_A, \sigma_A, \gamma_A) \) be a NS on \( X \), then the complement of the set \( A \) \( C(A) \) for short may be defined as three kinds of complements:
- \( (C_1) \hspace{1cm} C(A) = \{ x, 1 - \mu_A(x), 1 - \sigma_A(x), 1 - \gamma_A(x) \} : x \in X \}
- \( (C_2) \hspace{1cm} C(A) = \{ x, \gamma_A(x), \sigma_A(x), \mu_A(x) \} : x \in X \}
- \( (C_3) \hspace{1cm} C(A) = \{ x, \gamma_A(x), 1 - \sigma_A(x), 1 - \mu_A(x) \} : x \in X \}

One can define several relations and operations between \( NSs \) follows:

**Definition 1.6** [15] Let \( x \) be a non-empty set, and neutrosophic sets \( A \) and \( B \) in the form \( A = \{ x, \mu_A(x), \sigma_A(x), \gamma_A(x) \} : x \in X \} \) and \( B = \{ x, \mu_B(x), \sigma_B(x), \gamma_B(x) \} : x \in X \} \). Then we may consider two possible definitions for subsets \( A \subseteq B \)

\( A \subseteq B \) may be defined as :
1. \( A \subseteq B \) if \( \mu_A(x) \leq \mu_B(x), \sigma_A(x) \leq \sigma_B(x) \) and \( \gamma_A(x) \geq \gamma_B(x) \) for all \( x \in X \}
2. \( A \subseteq B \) if \( \mu_A(x) \leq \mu_B(x), \sigma_A(x) \leq \sigma_B(x) \) and \( \gamma_A(x) \geq \gamma_B(x) \) for all \( x \in X \}

**Proposition 1.7** [15] For any neutrosophic set \( A \), then the following conditions are holds :
1. \( 0_N \subseteq A \), \( 0_N \subseteq 0_N \)
2. \( A \subseteq 1_N \), \( 1_N \subseteq A \)

**Definition 1.8** [15] Let \( X \) be a non-empty set, and \( A = \{ x, \mu_A(x), \sigma_A(x), \gamma_A(x) \} \} \) and \( B = \{ x, \mu_B(x), \sigma_B(x), \gamma_B(x) \} \} \}

\( (I_1) \hspace{1cm} A \cap B = \{ x, \mu_A(x) \land \mu_B(x), \sigma_A(x) \land \sigma_B(x) \land \gamma_A(x) \lor \gamma_B(x) \} \}
\( (I_2) \hspace{1cm} A \cap B = \{ x, \mu_A(x) \land \mu_B(x), \sigma_A(x) \lor \sigma_B(x) \lor \gamma_A(x) \lor \gamma_B(x) \} \}
\( (U_1) \hspace{1cm} A \cup B = \{ x, \mu_A(x) \lor \mu_B(x), \sigma_A(x) \lor \sigma_B(x) \lor \gamma_A(x) \lor \gamma_B(x) \} \}
\( (U_2) \hspace{1cm} A \cup B = \{ x, \mu_A(x) \lor \mu_B(x), \sigma_A(x) \land \sigma_B(x) \land \gamma_A(x) \land \gamma_B(x) \} \}

We can easily generalize the operations of intersection and union in Definition 1.8 to arbitrary family of \( NSs \) as follows:

**Definition 1.9** [15] Let \( \{ A_j : j \in J \} \) be a arbitrary family of \( NSs \) in \( X \), then
\( (1) \hspace{1cm} \bigcap_{j} A_j \) may be defined as:
- \( (i) \hspace{1cm} \bigcap_{j} A_j = \{ x, \bigwedge_{j} \mu_{A_j}(x), \bigwedge_{j} \sigma_{A_j}(x), \bigwedge_{j} \gamma_{A_j}(x) \} \}
- \( (ii) \hspace{1cm} \bigcap_{j} A_j = \{ x, \bigwedge_{j} \mu_{A_j}(x), \bigwedge_{j} \sigma_{A_j}(x), \bigwedge_{j} \gamma_{A_j}(x) \} \}

\( (2) \hspace{1cm} \bigcup_{j} A_j \) may be defined as:
- \( (i) \hspace{1cm} \bigcup_{j} A_j = \{ x, \bigvee_{j} \sigma_{A_j}(x) \land \sigma_{A_j}(x), \bigvee_{j} \sigma_{A_j}(x) \land \gamma_{A_j}(x) \} \}
- \( (ii) \hspace{1cm} \bigcup_{j} A_j = \{ x, \bigvee_{j} \mu_{A_j}(x), \bigvee_{j} \sigma_{A_j}(x) \land \sigma_{A_j}(x), \bigvee_{j} \sigma_{A_j}(x) \land \gamma_{A_j}(x) \} \}

**Proposition 1.10** [15] For all \( A \) and \( B \) are two neutrosophic sets then the following conditions are true :
1. \( C(A \cap B) = C(A) \cup C(B) \)
2. \( C(A \cup B) = C(A) \cap C(B) \).

Here we extend the concepts of fuzzy topological space [5] and Intuitionistic fuzzy topological space [6,7] to the case of neutrosophic sets.

**Definition 1.11** [15] A neutrosophic topology \( NT \) for short is a non-empty set \( X \) is a family \( \tau \) of neutrosophic subsets in \( X \) satisfying the following axioms :
- \( (NT_1) \hspace{1cm} 0_N, 1_N \in \tau \),
- \( (NT_2) \hspace{1cm} G_1 \cap G_2 \in \tau \) for any \( G_1, G_2 \in \tau \),
- \( (NT_3) \hspace{1cm} G_1 \cup G_2 \in \tau \) for every \( G_i : i \in J \) \( \subseteq \tau \)
In this case the pair \((X, \tau)\) is called a neutrosophic topological space \([NTS\ for\ short]\). The elements of \(\tau\) are called neutrosophic open sets \([NOS\ for\ short]\). A neutrosophic set \(F\) is closed if and only if \(C(F)\) is neutrosophic open.

**Example 1.12** [15] Any fuzzy topological space \((X, \tau_0)\) in the sense of Chang is obviously a \(NTS\) in the form \(\tau = \{A : \mu_A \in \tau_0\}\) wherever we identify a fuzzy set in \(X\) whose membership function is \(\mu_A\) with its counterpart.

**Remark 1.13** [15] Neutrosophic topological spaces are very natural generalizations of fuzzy topological spaces allowing more general functions to be members of fuzzy topology.

**Example 1.14** [15] Let \(X = \{x\}\) and \(A = \{(x, 0.5, 0.5, 0.4) : x \in X\}\), \(B = \{(x, 0.4, 0.6, 0.8) : x \in X\}\), \(D = \{(x, 0.5, 0.6, 0.4) : x \in X\}\), and \(C = \{(x, 0.4, 0.5, 0.8) : x \in X\}\). Then the family \(\tau = \{0_N, A, B, C, D, 1_N\}\) of \(NSs\) in \(X\) is neutrosophic topology on \(X\).

**Definition 1.15** [15] The complement of \(A [C(A)\ for\ short]\) of \(NOS\) is called a neutrosophic closed set \([NCS\ for\ short]\) in \(X\).

Now, we define neutrosophic closure and neutrosophic interior operations in neutrosophic topological spaces:

**Definition 1.16** [15] Let \((X, \tau)\) be \(NTS\) and \(A = \langle x, \mu_A(x), \sigma_A(x), \gamma_A(x) \rangle\) be a \(NS\) in \(X\). Then the neutrosophic closure and neutrosophic interior of \(A\) are defined by

\[
NCl(A) = \cap \{K : K \text{ is a } NCS \text{ in } X \text{ and } A \subseteq K\}
\]

\[
NInt(A) = \cup \{G : G \text{ is a } NOS \text{ in } X \text{ and } G \subseteq A\}.
\]

It can also be shown that \(NCl(A)\) is \(NCS\) and \(NInt(A)\) is a \(NOS\) in \(X\).

a) \(A\) is \(NOS\) if and only if \(A = NInt(A)\).

b) \(A\) is \(NCS\) if and only if \(A = NCl(A)\).

**Proposition 1.17** [15] For any neutrosophic set \(A\) in \((X, \tau)\) we have

\(a) \quad NCl(C(A)) = C(NInt(A))\),

\(b) \quad NInt(C(A)) = C(NCl(A))\).

**Proposition 1.18** [15] Let \((X, \tau)\) be a \(NTS\) and \(A, B\) be two neutrosophic sets in \(X\). Then the following properties are holds:

\(a) \quad NInt(A) \subseteq A\),

\(b) \quad A \subseteq NCl(A)\).

\(c) \quad A \subseteq B \Rightarrow NInt(A) \subseteq NInt(B)\),

\(d) \quad A \subseteq B \Rightarrow NCI(A) \subseteq NCI(B)\),

\(e) \quad NInt(NInt(A)) = NInt(A)\),

\(f) \quad NCI(NCI(A)) = NCI(A)\),

\(g) \quad NInt(A \cap B) = NInt(A) \cap NInt(B)\),

\(h) \quad NCI(A \cup B) = NCI(A) \cup NCI(B)\),

\(i) \quad NInt(0_N) = 0_N\),

\(j) \quad NInt(1_N) = 1_N\),

\(k) \quad NCI(0_N) = 0_N\),

\(l) \quad NCI(1_N) = 1_N\),

\(m) \quad A \subseteq B \Rightarrow C(B) \subseteq C(A)\),

\(n) \quad NCI(A \cap B) \subseteq NCI(A) \cap NCI(B)\),

\(o) \quad NInt(A \cup B) \supseteq NInt(A) \cup NInt(B)\).

**II. PRODUCT RELATED NEUTROSOFTOPOLITICAL SPACES**

In this section, we define some basic and important results which are very useful in later sections. In order topology, the product of the closure is equal to the closure of the product and product of the interior is equal to the interior of the product. But this result is not true in neutrosophic topological space. For this reason, we define the product related neutrosophic topological space. Using this definition, we prove the above mentioned result.

**Definition 2.1** A subfamily \(\beta\) of \(NTS\) \((X, \tau)\) is called a base for \(\tau\) if each \(NS\) of \(\tau\) is a union of some members of \(\beta\).

**Definition 2.2** Let \(X, Y\) be nonempty neutrosophic sets and \(A = \langle x, \mu_A(x), \sigma_A(x), \gamma_A(x) \rangle, B = \langle y, \mu_B(y), \sigma_B(y), \gamma_B(y) \rangle\) \(NSs\) of \(X\) and \(Y\) respectively. Then \(A \times B\) is a \(NS\) of \(X \times Y\) is defined by

\[(P_1) \quad (A \times B) \times (x, y) = \langle (x, y), \min (\mu_A(x), \mu_B(y)), \min (\sigma_A(x), \sigma_B(y)), \max (\gamma_A(x), \gamma_B(y)) \rangle\]

\[(P_2) \quad (A \times B) \times (x, y) = \langle (x, y), \min (\mu_A(x), \mu_B(y)), \max (\sigma_A(x), \sigma_B(y)), \max (\gamma_A(x), \gamma_B(y)) \rangle\]

Notice that

\[(CP_1) \quad C((A \times B) \times (x, y)) = \langle (x, y), \max (\mu_A(x), \mu_B(y)), \max (\sigma_A(x), \sigma_B(y)), \min (\gamma_A(x), \gamma_B(y)) \rangle\]

\[(CP_2) \quad C((A \times B) \times (x, y)) = \langle (x, y), \min (\mu_A(x), \mu_B(y)), \min (\sigma_A(x), \sigma_B(y)), \min (\gamma_A(x), \gamma_B(y)) \rangle\]

**Lemma 2.3** If \(A\) is the \(NS\) of \(X\) and \(B\) is the \(NS\) of \(Y\), then

\[(i) \quad (A \times 1_N) \cap (1_N \times B) = A \times B\]

\[(ii) \quad (A \times 1_N) \cup (1_N \times B) = C(C(A) \times C(B))\]

\[(iii) \quad C(A \times B) = (C(A) \times 1_N) \cup (1_N \times C(B))\]
Proof: Let $A = \{ x, \mu_A(x), \sigma_A(x), \gamma_A(x) \}$, $B = \{ y, \mu_B(y), \sigma_B(y), \gamma_B(y) \}$. 

(i) Since $A \times I_N = \{ x, \min (\mu_A, I_N), \min (\sigma_A, I_N), \max (\gamma_A, 0_N) \} = \{ x, \mu_A(x), \sigma_A(x), \gamma_A(x) \} = A$ and similarly $1_N \times B = \{ y, \min (\mu_B, 1_N), \min (\sigma_B, 1_N), \max (\gamma_B, 0_N) \} = B$, we have $(A \times I_N) \cap (1_N \times B) = (A \times B) \cap (1_N \times 1_N) = (A \times 1_N) \cap (1_N \times B) = (A \times B) \cap (1_N \times B) = A \times B$.

(ii) Similarly to (i).

(iii) Obvious by putting $A, B$ instead of $C (A), C (B)$ in (ii).

Definition 2.4 Let $X$ and $Y$ be two nonempty neutrosophic sets and $f : X \rightarrow Y$ be a neutrosophic function. (i) If $B = \{ y, \mu_B(y), \sigma_B(y), \gamma_B(y) \}: y \in Y$ is a NS in $Y$, then the pre image of $B$ under $f$ is denoted and defined by $f^{-1}(B) = \{ x, f^{-1}(\mu_B(x)), f^{-1}(\sigma_B(x)), f^{-1}(\gamma_B(x)) \}: x \in X$.

(ii) If $A = \{ x, \alpha_A(x), \delta_A(x), \lambda_A(x) \}: x \in X$ is a NS in $X$, then the image of $A$ under $f$ is denoted and defined by $f(A) = \{ y, f(\alpha_A(x)), f(\delta_A(x)), f_{-}(\lambda_A(x)) \}: y \in Y$ where $f_{-}(\lambda_A) = C (f (C (A)))$.

In (i), (ii), since $\mu_B, \sigma_B, \gamma_B, \alpha_A, \delta_A, \lambda_A$ are neutrosophic sets, we explain that $f^{-1}(\mu_B(x)) = \mu_B(f(x))$, and $f(\alpha_A(x)) = \{ \sup \alpha_A(x) \text{ if } x \in f^{-1}(y) \}$.

Definition 2.5 Let $(X, \tau)$ and $(Y, \sigma)$ be NTSSs. The neutrosophic product topological space $[NPTS$ for short] of $(X, \tau)$ and $(Y, \sigma)$ is the cartesian product $X \times Y$ of $X$ and $Y$ with the $NT \xi$ of $X \times Y$ which is generated by the family $\{ P_{-1}(A), P_{-1}(B) : A \in \tau, B \in \sigma \}$. We show that $\{ P_{-1}(A), P_{-1}(B) : A \in \tau, B \in \sigma \}$ forms a base for $NPTS \xi$ of $X \times Y$.

Remark 2.6 In the above definition, since $P_{-1}(A) = A \times I_N$ and $P_{-1}(B) = I_N \times B$, the family $\{ A \times B : A \in \tau, B \in \sigma \}$ forms a base for $NPTS \xi$ of $X \times Y$.

Definition 2.7 Let $f_1 : X_1 \rightarrow Y_1$ and $f_2 : X_2 \rightarrow Y_2$ be the two neutrosophic functions. Then the neutrosophic product $f_1 \times f_2 : X_1 \times X_2 \rightarrow Y_1 \times Y_2$ is defined by $(f_1 \times f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2))$ for all $(x_1, x_2) \in X_1 \times X_2$.

Definition 2.8 Let $A, A_i$ (i $\in J$) be NSs in $X$ and $B, B_j$ (j $\in K$) be NSs in $Y$ and $f : X \rightarrow Y$ be the neutrosophic function. Then

(i) $f^{-1}(\cup B_j) = \cup f^{-1}(B_j)$,

(ii) $f^{-1}(\cap B_j) = \cap f^{-1}(B_j)$,

(iii) $f^{-1}(1_N) = 1_N$, $f^{-1}(0_N) = 0_N$.

(iv) $f^{-1}(C (B)) = C (f^{-1}(B))$.

(v) $f (\cup A_i) = \cup f (A_i)$.

Definition 2.9 Let $f : X \rightarrow Y$ be the neutrosophic function. Then the neutrosophic graph $g : X \rightarrow X \times Y$ of $f$ is defined by $g(x) = (x, f(x))$ for all $x \in X$.

Lemma 2.10 Let $f_i : X_i \rightarrow Y_i$ (i = 1, 2) be the neutrosophic functions and $A, B$ be NSs of $Y_1$, $Y_2$ respectively. Then $(f_1 \times f_2)^{-1}(A, B) = (f_1^{-1}(A) \times f_2^{-1}(B))$.

Proof: Let $A = \{ x_1, \mu_A(x_1), \sigma_A(x_1), \gamma_A(x_1) \}$, $B = \{ x_2, \mu_B(x_2), \sigma_B(x_2), \gamma_B(x_2) \}$. For each $(x_1, x_2) \in X_1 \times X_2$, we have $(f_1 \times f_2)^{-1}(A, B) = (A \times B) \cap (f_1 \times f_2)^{-1}(A, B) = (f_1^{-1}(A) \times f_2^{-1}(B))$.

Lemma 2.11 Let $g : X \rightarrow X \times Y$ be the neutrosophic graph of the neutrosophic function $f : X \rightarrow Y$. If $g^{-1}(X \times B) = (A \cap f^{-1}(B))(x)$.

Proof: Let $A = \{ x, \mu_A(x), \sigma_A(x), \gamma_A(x) \}$, $B = \{ x, \mu_B(x), \sigma_B(x), \gamma_B(x) \}$. For each $x \in X$, we have $g^{-1}(X \times B))(x) = (A \times B) \cap (X \times f(x)) = (A, f(x)) \cap (\mu_A(x), \mu_B(x), f(x)) \cap (\sigma_A(x), \sigma_B(f(x)), \gamma_A(x), \gamma_B(f(x)))$.

Lemma 2.12 Let $A, B, C$ and $D$ be NSs in $X$. Then $A \subset C \subset D \Rightarrow A \times C \subset B \times D$.

Proof: Let $A = \{ x, \mu_A(x), \sigma_A(x), \gamma_A(x) \}$, $B = \{ x, \mu_B(x), \sigma_B(x), \gamma_B(x) \}$, $C = \{ x, \mu_C(x), \sigma_C(x), \gamma_C(x) \}$ and $D = \{ x, \mu_D(x), \sigma_D(x), \gamma_D(x) \}$ be NSs. Since $A \subset B \Rightarrow \mu_A \leq \mu_B$, $\sigma_A \leq \sigma_B$, $\gamma_A \leq \gamma_B$ and also $C \subset D \Rightarrow \mu_C \leq \mu_D$, $\sigma_C \leq \sigma_D$, $\gamma_C \leq \gamma_D$, we have $\min (\mu_A, \mu_C) \leq \min (\mu_B, \mu_D)$, $\min (\sigma_A, \sigma_C) \leq \min (\sigma_B, \sigma_D)$ and $\max (\gamma_A, \gamma_C) \geq \max (\gamma_B, \gamma_D)$. Hence the result.

Lemma 2.13 Let $(X, \tau)$ and $(Y, \sigma)$ be any two NTSSs such that $X$ is neutrosophic product relative to $Y$. Let $A$ and $B$ be NCSSs in $NTSSS X$ and $Y$ respectively. Then $A \times B$ is the NCS in the $NPTS$ of $X \times Y$.

Proof: Let $A = \{ x, \mu_A(x), \sigma_A(x), \gamma_A(x) \}$, $B = \{ x, \mu_B(y), \sigma_B(y), \gamma_B(y) \}$. From Lemma 2.3, $C (A \times B)(x, y) = (C (A) \times 1_N) \cup (1_N \times C (B))(x, y)$. Since $C (A) \times 1_N$ and $1_N \times C (B)$ are NSSs in $X$ and $Y$ respectively. Hence $C (A) \times 1_N \cup 1_N \times C (B)$ is $NOS$ of $X \times Y$. Hence $C (A \times B)$ is a $NOS$ of $X \times Y$ and consequently $A \times B$ is the NCS of $X \times Y$. 

ISSN: 2231-5373 http://www.ijmttjournal.org Page 217
Theorem 2.14 If A and B are NSs of NTSs X and Y respectively, then
(i) $NCI (A) \times NCI (B) \supseteq NCI (A \times B)$,
(ii) $NInt (A) \times NInt (B) \subseteq NInt (A \times B)$.
Proof: (i) Since $A \subseteq NCI (A)$ and $B \subseteq NCI (B)$, hence $A \times B \subseteq NCI (A) \times NCI (B)$. This implies that $NCI (A \times B) \subseteq NCI (A \times NCI (B))$ and from Lemma 2.13, $NCI (A \times B) \subseteq NCI (A) \times NCI (B)$.
(ii) follows from (i) and the fact that $NInt (C (A)) = C (NCI (A))$.

Definition 2.15 Let $(X, \tau)$, $(Y, \sigma)$ be NTSs and $A \in \tau$, $B \in \sigma$. We say that $(X, \tau)$ is neutrosophic product related to $(Y, \sigma)$ if for any NSs $C$ of $X$ and $D$ of $Y$, whenever $C (A) \nsubseteq C$ and $C (B) \nsubseteq D \Rightarrow C (A) \times 1_N \subseteq 1_N \times C (B)$ there exist $e_1 \in \tau$, $e_2 \in \sigma$ such that $C (A(e_1)) \subseteq C$ or $C (B(e_2)) \subseteq D$ and $C (A(e_1) \times 1_N \subseteq 1_N \times C (B(e_2))) = C (A) \times 1_N \subseteq 1_N \times C (B)$.

Lemma 2.16 For NSs $A_i$'s and $B_j$'s of NTSs $X$ and $Y$ respectively, we have
(i) $\cap \{ A_i \cup B_j \} = \min (\cap \{ A_i \cup B_j \})$,
$\cup \{ A_i \cup B_j \} = \max (\cup \{ A_i \cup B_j \})$.
(ii) $\cap \{ A_i \times 1_X \} = (\cap A_i) \times 1_X$,
$\cup \{ A_i \times 1_X \} = (\cup A_i) \times 1_X$.
(iii) $\cap \{ 1_X \times B_j \} = 1_X \times (\cap B_j)$,
$\cup \{ 1_X \times B_j \} = 1_X \times (\cup B_j)$.
Proof: Obvious.

Theorem 2.17 Let $(X, \tau)$ and $(Y, \sigma)$ be NTSs such that $X$ is neutrosophic product related to $Y$. Then for $NSs$ $A$ of $X$ and $B$ of $Y$, we have
(i) $NCI (A \times B) = NCI (A) \times NCI (B)$,
(ii) $NInt (A \times B) = NInt (A) \times NInt (B)$.
Proof: (i) Since $NCI (A \times B) \subseteq NCI (A) \times NCI (B)$, then $NCI (A \times B) = C (NCI (A \times B))$, and $NCI (A) \times NCI (B) = C (NCI (A)) \times C (NCI (B))$. Therefore $NCI (A) \times NCI (B) = NCI (A \times B)$.

III. NEUTROSOHIC SEMI-OPEN SETS IN NEUTROSOHIC TOPOLOGICAL SPACES

In this section, the concepts of the neutrosophic semi-open set is introduced and also discussed their characterizations.

Definition 3.1 Let $A$ be NS of a NTS $X$. Then $A$ is said to be neutrosophic semi-open [written $NSO$] set of $X$ if there exists a neutrosophic open set $NO$ such that $NO \subseteq A \subseteq NCI (NO)$.

The following theorem is the characterization of $NSO$ set in $NTS$.

Theorem 3.2 A subset $A$ in a NTS $X$ is NSO set if and only if $A \subseteq NCI (NInt (A))$.
Proof: Sufficiency: Let $A \subseteq NCI (NInt (A))$. Then for $NO = NInt (A)$, we have $NO \subseteq A \subseteq NCI (NO)$. Necessity: Let $A$ be $NSO$ set in $X$. Then $NO \subseteq A \subseteq NCI (NO)$ for some neutrosophic open set $NO$. But $NO \subseteq NInt (A)$ and thus $NCl (NO) \subseteq NCi (NInt (A))$. Hence $A \subseteq NCI (NO) \subseteq NCI (NInt (A))$.

Theorem 3.3 Let $(X, \tau)$ be a NTS. Then union of two NSO sets is a NSO set in the $NTS$.
Proof: Let $A$ and $B$ be NSO sets in $X$. Then $A \subseteq NCI (NInt (A))$ and $B \subseteq NCI (NInt (B))$. Therefore $A \cup B \subseteq NCI (NInt (A)) \cup NCI (NInt (B)) = NCI (NInt (A) \cup NInt (B)) \subseteq NCI (NInt (A \cup B))$. By Proposition 1.18 (a) [ ], Hence $A \cup B$ is NSO set in $X$.

Theorem 3.4 Let $(X, \tau)$ be a NTS. If $\{ A_\alpha \}_{\alpha \in \Delta}$ is a collection of NSO sets in a NTS $X$. Then $\cup_{\alpha \in \Delta} A_\alpha$ is NSO set in $X$.
Proof: For each $\alpha \in \Delta$, we have a neutrosophic open set $NO_\alpha$ such that $NO_\alpha \subseteq A_\alpha \subseteq NCI (NO_\alpha)$. Then $\cup_{\alpha \equiv \Delta} NO_\alpha \subseteq \cup_{\alpha \equiv \Delta} A_\alpha \subseteq \cup_{\alpha \equiv \Delta} NCI (NO_\alpha) \subseteq NCI (\cup_{\alpha \equiv \Delta} NO_\alpha)$. Hence let $NO = \cup_{\alpha \equiv \Delta} NO_\alpha$. 

ISSN: 2231-5373 http://www.jimttjournal.org
Remark 3.5 The intersection of any two NSO sets need not be a NSO set in X as shown by the following example.

Example 3.6 Let $X = \{ a, b \}$ and $A = \langle (0.3, 0.5, 0.4), (0.6, 0.2, 0.5) \rangle$ and $B = \langle (0.2, 0.6, 0.7), (0.5, 0.3, 0.1) \rangle$. Then $\tau = \{ 0, A, B, C, D, 1 \}$ is NTS on X. Now, we define the two NSO sets as follows:

$A_1 = \langle (0.4, 0.6, 0.4), (0.8, 0.3, 0.4) \rangle$ and $A_2 = \langle (1, 0.9, 0.2), (0.5, 0.7, 0) \rangle$. Here $\mathrm{Nhnt}(A_1) = A, \mathrm{NCI}(\mathrm{Nhnt}(A_1)) = I$ and $\mathrm{Nhnt}(A_2) = B, \mathrm{NCI}(\mathrm{Nhnt}(A_2)) = I$. But $A_1 \cap A_2 = \langle (0.4, 0.6, 0.4), (0.5, 0.3, 0.4) \rangle$ is not a NSO set in X.

Theorem 3.7 Let $A$ be NSO set in the NTS X and suppose $A \subseteq B \subseteq \mathrm{NCI}(A)$. Then B is NSO set in X.

Proof: There exists a neutrosophic open set NO such that $\mathrm{NO} \subseteq A \subseteq \mathrm{NCI}(NO)$. Then NO $\subseteq B$. But $\mathrm{NCI}(A) \subseteq \mathrm{NCI}(NO)$ and thus $B \subseteq \mathrm{NCI}(NO)$. Hence $\mathrm{NO} \subseteq B \subseteq \mathrm{NCI}(NO)$ and B is NSO set in X.

Theorem 3.8 Every neutrosophic open set in the NTS X is NSO set in X.

Proof: Let $A$ be neutrosophic open set in NTS X. Then $A = \mathrm{Nhnt}(A)$. Also $\mathrm{Nhnt}(A) \subseteq \mathrm{NCI}(\mathrm{Nhnt}(A))$. This implies that $A \subseteq \mathrm{NCI}(\mathrm{Nhnt}(A))$. Hence by Theorem 3.2, $A$ is NSO set in X.

Remark 3.9 The converse of the above theorem need not be true as shown by the following example.

Example 3.10 Let $X = \{ a, b, c \}$ with $\tau = \{ 0, A, B, I \}$. Some of the NSO sets are

$A = \langle (0.4, 0.5, 0.2), (0.3, 0.2, 0.1), (0.9, 0.6, 0.8) \rangle$

$B = \langle (0.2, 0.4, 0.5), (0.1, 0.1, 0.2), (0.6, 0.5, 0.8) \rangle$

$C = \langle (0.5, 0.6, 0.1), (0.4, 0.3, 0.1), (0.9, 0.8, 0.5) \rangle$

$D = \langle (0.3, 0.5, 0.4), (0.1, 0.6, 0.2), (0.7, 0.5, 0.8) \rangle$

$E = \langle (0.5, 0.6, 0.1), (0.4, 0.6, 0.1), (0.9, 0.8, 0.5) \rangle$

$F = \langle (0.3, 0.5, 0.4), (0.1, 0.3, 0.2), (0.7, 0.5, 0.8) \rangle$

$G = \langle (0.4, 0.5, 0.2), (0.3, 0.6, 0.1), (0.9, 0.6, 0.8) \rangle$

$H = \langle (0.3, 0.5, 0.4), (0.1, 0.2, 0.2), (0.7, 0.5, 0.8) \rangle$

$I = \langle (0.4, 0.5, 0.2), (0.3, 0.3, 0.1), (0.9, 0.6, 0.8) \rangle$

$J = \langle (0.3, 0.5, 0.4), (0.1, 0.2, 0.2), (0.7, 0.5, 0.8) \rangle$.

Here C, D, E, F, G, H, I and J are NSO sets but are not neutrosophic open sets.

Proposition 3.11 If X and Y are NTS such that X is neutrosophic product related to Y. Then the neutrosophic product $A \times B$ of a neutrosophic semi-open set $A$ of X and a neutrosophic semi-open set $B$ of Y is a neutrosophic semi-open set of the neutrosophic product topological space $X \times Y$.

Proof: Let $O_1 \subseteq A \subseteq \mathrm{NCI}(O_1)$ and $O_2 \subseteq B \subseteq \mathrm{NCI}(O_2)$ where $O_1$ and $O_2$ are neutrosophic open sets in X and Y respectively. Then $O_1 \times O_2 \subseteq A \times B \subseteq \mathrm{NCI}(O_1 \times O_2)$. By Theorem 2.17 (i) , $\mathrm{NCI}(O_1 \times O_2) = \mathrm{NCI}(O_1 \times O_2)$. Therefore $O_1 \times O_2 \subseteq A \times B \subseteq \mathrm{NCI}(O_1 \times O_2)$. Hence by Theorem 3.1, $A \times B$ is neutrosophic semi-open set in $X \times Y$.

IV. NEUTROSOPHIC SEMI-CLOSED SETS IN NEUTROSOPHIC TOPOLOGICAL SPACES

In this section, the neutrosophic semi-closed set is introduced and studied their properties.

Definition 4.1 Let $A$ be NS of a NTS X. Then $A$ is said to be neutrosophic semi-closed set of X if there exists a neutrosophic closed set NC such that $\mathrm{Nhnt}(\mathrm{NC}) \subseteq A \subseteq \mathrm{NC}$.

Theorem 4.2 A subset $A$ in a NTS X is NCS if and only if $\mathrm{Nhnt}(\mathrm{NC}(A)) \subseteq A$.

Proof: Sufficiency: Let $\mathrm{Nhnt}(\mathrm{NC}(A)) \subseteq A$. Then for $\mathrm{NC} = \mathrm{NC}(A)$, we have $\mathrm{Nhnt}(\mathrm{NC}) \subseteq A \subseteq \mathrm{NC}$. Necessity: Let $A$ be NSC set in X. Then $\mathrm{Nhnt}(\mathrm{NC}) \subseteq A \subseteq \mathrm{NC}$ for some neutrosophic closed set NC. But $\mathrm{NC}(A) \subseteq \mathrm{NC}$ and thus $\mathrm{Nhnt}(\mathrm{NC}(A)) \subseteq \mathrm{Nhnt}(\mathrm{NC})$. Hence $\mathrm{Nhnt}(\mathrm{NC}(A)) \subseteq A$.

Proposition 4.3 Let (X, $\tau$) be a NTS and A be a neutrosophic subset of X. Then A is NSC set if and only if $C(A)$ is NSO set in X.

Proof: Let A be a neutrosophic semi-closed subset of X. Then by Theorem 4.2, $\mathrm{Nhnt}(\mathrm{NC}(A)) \subseteq A$. Taking complement on both sides, $C(A) \subseteq C(\mathrm{Nhnt}(\mathrm{NC}(A))) = \mathrm{NC}(C(\mathrm{NC}(A)))$. By using Proposition 1.17 (b), $C(A) \subseteq \mathrm{Nhnt}(\mathrm{NC}(C(A)))$. By Theorem 3.2, C(A) is neutrosophic semi-open. Conversely let C(A) is neutrosophic semi-open. By Theorem 3.2,
C (A) ⊆ NCI (NInt (C (A))). Taking complement on both sides, A ⊇ C (NCI (NInt (C (A))) = NInt (C (NInt (C (A)))). By using Proposition 1.17 (b), A ⊇ NInt (NCI (A)). By Theorem 4.2, A is neutrosophic semi-closed set.

**Theorem 4.4** Let (X, τ) be a NTS. Then intersection of two NSC sets is a NSC set in the NTS X.

**Proof**: Let A and B are NSC sets in X. Then NInt (NCI (A)) ⊆ A and NInt (NCI (B)) ⊆ B. Therefore A ∩ B ⊇ NInt (NCI (A)) ∩ NInt (NCI (B)) = NInt (NCI (A) ∩ NCNI (B)) ⊇ NInt (NCI (A)) ⊆ A and B is NSC set in X.

**Theorem 4.5** Let \{Aα\}α∈Δ be a collection of NSC sets in a NTS X. Then \( \cap_{α∈Δ} A_α \) is NSC set in X.

**Proof**: For each α ∈ Δ, we have a neutrosophic closed set NC\(\alpha \) such that NInt (NC\(\alpha \)) ⊆ A\(\alpha \) ⊆ NC\(\alpha \). Then NInt (∩\(\alpha∈Δ\) NC\(\alpha \)) ⊆ ∩\(\alpha∈Δ\) NInt (NC\(\alpha \)) ⊆ ∩\(\alpha∈Δ\) A\(\alpha \). Hence NC = ∩\(\alpha∈Δ\) NC\(\alpha \).

**Remark 4.6** The union of any two NSC sets need not be a NSC set in X as shown by the following example.

**Example 4.7** Let X = \{a\} and \( A = \langle (1, 0.5, 0.7) \rangle \)
A = \( \langle (0, 0.9, 0.2) \rangle \)
B = \( \langle (1, 0.9, 0.2) \rangle \)
D = \( \langle (0, 0.5, 0.7) \rangle \).
Then \( \tau = \{0_N, A, B, C, D, 1_N\} \) is NTS on X. Now, we define the two NSC sets as follows:
A1 = \( \langle (0.4, 0.5, 1) \rangle \)
A2 = \( \langle (0.2, 0.8) \rangle \). Here NC\(\alpha \) (A\(\alpha \)) = \( \langle (0.7, 0.5, 1) \rangle \), NInt (NC\(\alpha \)) = 0\(\alpha \) and NInt (A\(\alpha \)) = \( \langle (0.2, 0.1, 0) \rangle \). NInt (NC\(\alpha \)) = 0\(\alpha \). But A\(\alpha \) ∪ A\(\beta \) = \( \langle (0.4, 0.5, 0.8) \rangle \) is not a NSC set in X.

**Theorem 4.8** Let A be NSC set in the NTS X and suppose NInt (A) ⊆ B ⊆ A. Then B is NSC set in X.

**Proof**: There exists a neutrosophic closed set NC such that NInt (NC) ⊆ A ⊆ NC. Then B ⊆ NC. But NInt (NC) ⊆ NInt (A) and thus NInt (NC) ⊆ B. Hence NInt (NC) ⊆ B ⊆ NC and B is NSC set in X.

**Theorem 4.9** Every neutrosophic closed set in the NTS X is NSC set in X.

**Proof**: Let \( A \) be neutrosophic closed set in NTS X. Then \( A = NCI (A) \). Also NInt (NCI (A)) ⊆ NCI (A). This implies that NInt (NCI (A)) ⊆ A. Hence by Theorem 4.2, A is NSC set in X.

**Remark 4.10** The converse of the above theorem need not be true as shown by the following example.

**Example 4.11** Let X = \{a, b, c\} with \( \tau = \{0_N, A, B, 1_N\} \) and C (τ) = \{1_N, C, D, 0_N\} where \( A = \langle (0.5, 0.6, 0.3), (0.1, 0.7, 0.9), (1, 0.6, 0.4) \rangle \)
B = \( \langle (0.4, 0.7), (0.1, 0.6, 0.9), (0.5, 0.5, 0.8) \rangle \)
C = \( \langle (0.3, 0.4, 0.5), (0.9, 0.3, 0.1), (0.4, 0.4, 1) \rangle \)
D = \( \langle (0.7, 0.6, 0), (0.9, 0.4, 0.1), (0.8, 0.5, 0.5) \rangle \).
E = \( \langle (0.2, 0.4, 0.9), (0, 0.2, 0.9), (0.3, 0.2, 1) \rangle \). Here the NSC sets are C, D and E.

**Proposition 4.12** If X and Y are neutrosophic spaces such that X is neutrosophic product related to Y. Then the neutrosophic product A × B of a neutrosophic semi-closed set A of X and a neutrosophic semi-closed set B of Y is a neutrosophic semi-closed set of the neutrosophic product topological space X × Y.

**Proof**: Let \( NInt (C_1) \subseteq A \subseteq C_1 \) and \( NInt (C_2) \subseteq B \subseteq C_2 \) where C\(\alpha \) and C\(\beta \) are neutrosophic closed sets in X and Y respectively. Then \( NInt (C_1) \times NInt (C_2) \subseteq A \times B \subseteq C_1 \times C_2 \). By Theorem 2.17 (ii), \( NInt (C_1) \times NInt (C_2) = NInt (C_1 \times C_2) \). Therefore \( NInt (C_1 \times C_2) \subseteq A \times B \subseteq C_1 \times C_2 \). Hence by Theorem 4.1, A × B is neutrosophic semi-closed set in X × Y.

**V. NEUTROSOPHIC SEMI-INTERIOR IN NEUTROSOPHIC TOPOLOGICAL SPACES**

In this section, we introduce the neutrosophic semi-interior operator and their properties in neutrosophic topological space.

**Definition 5.1** Let (X, τ) be a NTS. Then for a neutrosophic subset A of X, the neutrosophic semi-interior of A [NS Int (A) for short] is the union of all neutrosophic semi-open sets of X contained in A.

That is, \( NS \ Int (A) = \cup \{G : G is a NSO set in X and G \subseteq A\} \).
Proposition 5.2 Let \((X, \tau)\) be a NTS. Then for any neutrosophic subsets \(A\) and \(B\) of a NTS \(X\) we have
(i) \(NS\ Int (A) \subseteq A\)
(ii) \(A\) is NSO set in \(X \iff NS\ Int (A) = A\)
(iii) \(NS\ Int (NS\ Int (A)) = NS\ Int (A)\)
(iv) If \(A \subseteq B\) then \(NS\ Int (A) \subseteq NS\ Int (B)\)

Proof: (i) follows from Definition 5.1.

Let \(A\) be NSO set in \(X\). Then \(A \subseteq NS\ Int (A)\). By using (i) we get \(A = NS\ Int (A)\). Conversely assume that \(A = NS\ Int (A)\). By using Definition 5.1, \(A\) is NSO set in \(X\). Thus (ii) is proved.

By using (ii), \(NS\ Int (NS\ Int (A)) = NS\ Int (A)\). This proves (iii).

Since \(A \subseteq B\), by using (i), \(NS\ Int (A) \subseteq A \subseteq B\). That is \(NS\ Int (A) \subseteq B\). By (iii), \(NS\ Int (NS\ Int (A)) \subseteq NS\ Int (B)\). Thus \(NS\ Int (A) \subseteq NS\ Int (B)\). This proves (iv).

Theorem 5.3 Let \((X, \tau)\) be a NTS. Then for any neutrosophic subset \(A\) and \(B\) of a NTS, we have
(i) \(NS\ Int (A \cap B) = NS\ Int (A) \cap NS\ Int (B)\)
(ii) \(NS\ Int (A \cup B) = NS\ Int (A) \cup NS\ Int (B)\)

Proof: Since \(A \cap B \subseteq A\) and \(A \cap B \subseteq B\), by using Proposition 5.2 (iv), \(NS\ Int (A \cap B) \subseteq NS\ Int (A)\) and \(NS\ Int (A \cap B) \subseteq NS\ Int (B)\). This implies that \(NS\ Int (A \cap B) \subseteq NS\ Int (A) \cap NS\ Int (B)\).---(1).

By using Proposition 5.2 (i), \(NS\ Int (A) \subseteq A\) and \(NS\ Int (B) \subseteq B\). This implies that \(NS\ Int (A) \cap NS\ Int (B) \subseteq A \cap B\). Now applying Proposition 5.2 (iv), \(NS\ Int ((NS\ Int (A) \cap NS\ Int (B)) \subseteq NS\ Int (A \cap B)\).

By Proposition 5.2 (iii), \(NS\ Int (A) \cap NS\ Int (B) \subseteq NS\ Int (A \cap B)\) ----(2). From (1) and (2), \(NS\ Int (A \cap B) = NS\ Int (A) \cap NS\ Int (B)\). This implies (i).

Since \(A \subseteq A \cup B\) and \(B \subseteq A \cup B\), by using Proposition 5.2 (iv), \(NS\ Int (A) \subseteq NS\ Int (A \cup B)\) and \(NS\ Int (B) \subseteq NS\ Int (A \cup B)\). This implies that \(NS\ Int (A) \cup NS\ Int (B) \subseteq NS\ Int (A \cup B)\). Hence (ii).

The following example shows that the equality need not be hold in Theorem 5.3 (ii).

Example 5.4 Let \(X = \{a, b, c\}\) and \(\tau = \{0_\infty, A, B, C, D, 1_N\}\) where
\[A = \{(0.4, 0.7, 0.1), (0.5, 0.6, 0.2), (0.9, 0.7, 0.3)\},\]
\[B = \{(0.4, 0.6, 0.1), (0.7, 0.7, 0.2), (0.9, 0.5, 0.1)\},\]
\[C = \{(0.4, 0.7, 0.1), (0.7, 0.7, 0.2), (0.9, 0.7, 0.1)\},\]
\[D = \{(0.4, 0.6, 0.1), (0.5, 0.6, 0.2), (0.9, 0.5, 0.3)\}.

Then \((X, \tau)\) is a NTS. Consider the NSs are
\[E = \{(0.7, 0.6, 0.1), (0.7, 0.6, 0.1), (0.9, 0.5, 0)\}\]
and \(F = \{(0.4, 0.6, 0.1), (0.5, 0.7, 0.2), (1, 0.7, 0.1)\}.

Then \(NS\ Int (E) = D\) and \(NS\ Int (F) = D\). This implies that \(NS\ Int (E) \cup NS\ Int (F) = D\). Now, \(E \cup F = \{(0.7, 0.6, 0.1), (0.7, 0.7, 0.1), (1, 0.7, 0)\}\), it follows that \(NS\ Int (E \cup F) = B\). Then \(NS\ Int (E \cup F) \subseteq NS\ Int (E) \cup NS\ Int (F)\).

VI. NEUTROSOPHIC SEMI-CLOSURE IN NEUTROSOPHIC TOPOLOGICAL SPACES

In this section, we introduce the concept of neutrosophic semi-closure operators in a NTS.

Definition 6.1 Let \((X, \tau)\) be a NTS. Then for a neutrosophic subset \(A\) of \(X\), the neutrosophic semi-closure of \(A\) \([NS\ Cl (A)\) for short \(]\) is the intersection of all neutrosophic semi-closed sets of \(X\) contained in \(A\). That is, \(NS\ Cl (A) = \cap \{ K : K\) is a NSC set in \(X\) and \(K \supseteq A\}\).

Proposition 6.2 Let \((X, \tau)\) be a NTS. Then for any neutrosophic subsets \(A\) of \(X\),
(i) \(C (NS\ Int (A)) = NS\ Cl (C (A))\),
(ii) \(NS\ Cl (NS\ Cl (A)) = NS\ Cl (C (A))\).

Proof: By using Definition 5.1, \(NS\ Int (A) \equiv \cup \{ G : G\) is a NSO set in \(X\) and \(G \subseteq A\}\). Taking complement on both sides, \(C (NS\ Int (A)) = C (\cup \{ G : G\) is a NSO set in \(X\) and \(G \subseteq A\}) = \cap \{ C (G) : C (G)\) is a NSC set in \(X\) and \(C (A) \subseteq C (G)\}\). Replacing \(C (G)\) by \(K\), we get \(C (NS\ Int (A)) = \cap \{ K : K\) is a NSC set in \(X\) and \(K \supseteq C (A)\}\). By Definition 6.1, \(C (NS\ Int (A)) = NS\ Cl (C (A))\). This proves (i).

By using (i), \(C (NS\ Cl (C (A))) = NS\ Cl (NS\ Cl (C (A))) = NS\ Cl (A)\). Taking complement on both sides, we get \(NS\ Cl (C (A)) = C (NS\ Cl (A))\). Hence proved (ii).

Proposition 6.3 Let \((X, \tau)\) be a NTS. Then for any neutrosophic subsets \(A\) and \(B\) of a NTS \(X\) we have
(i) \(A \subseteq NS\ Cl (A)\)
(ii) \(A\) is NSC set in \(X \iff NS\ Cl (A) = A\)
(iii) \(NS\ Cl (NS\ Cl (A)) = NS\ Cl (A)\)
(iv) If \(A \subseteq B\) then \(NS\ Cl (A) \subseteq NS\ Cl (B)\)

Proof: (i) follows from Definition 6.1.

Let \(A\) be NSC set in \(X\). By using Proposition 4.3, \(C (A)\) is NSO set in \(X\). By Proposition 6.2 (ii), \(NS\ Int (C (A)) = C (A) \iff C (NS\ Cl (A)) = C (A) \iff NS\ Cl (A) = A\). Thus proved (ii).

By using (ii), \(NS\ Cl (NS\ Cl (A)) = NS\ Cl (A)\). This proves (iii).

Since \(A \subseteq B\), \(C (B) \subseteq C (A)\). By using Proposition 5.2 (iv), \(NS\ Cl (C (B)) \subseteq NS\ Cl (C (A))\). Taking complement on both sides, \(C (NS\ Int (C (B))) \subseteq C (NS\ Int (C (A)))\). By Proposition 6.2 (ii), \(NS\ Cl (A) \subseteq NS\ Cl (B)\). This proves (iv).
Proposition 6.4 Let A be a neutrosophic set in a NTS X. Then NInt(A) ⊆ NS Int(A) ⊆ A ⊆ NS Cl(A) ⊆ NCl(A).

Proof: It follows from the definitions of corresponding operators.

Proposition 6.5 Let (X, τ) be a NTS. Then for a neutrosophic subset A and B of a NTS X, we have
(i) NS Cl(A ∪ B) = NS Cl(A) ∪ NS Cl(B) and
(ii) NS Cl(A ∩ B) ⊆ NS Cl(A) ∩ NS Cl(B).

Proof: Since NS Cl(A ∪ B) = NS Cl(C(C(A ∪ B))), by using Proposition 6.2 (i), NS Cl(A ∪ B) = C(NS Int(C(A) ∩ C(B))). Again using Proposition 5.3 (i), NS Cl(A ∪ B) = C(NS Int(C(A)) ∩ NS Int(C(B))) = C(NS Int(C(A)) ∩ NS Int(C(B))). By using Proposition 6.2 (i), NS Cl(A ∩ B) = NS Cl(C(C(A))) ∩ NS Cl(B). Thus proved (ii).

The following example shows that the equality need not be hold in Proposition 6.5 (ii).

Example 6.6 Let X = {a, b, c} with τ = {0_N, A, B, C, D, 1_N} and C(τ) = {1_N, E, F, G, H, 0_N} where
A = ((0.5, 0.6, 0.1), (0.6, 0.7, 0.1), (0.9, 0.5, 0.2))
B = ((0.4, 0.5, 0.2), (0.8, 0.6, 0.3), (0.9, 0.7, 0.3))
C = ((0.4, 0.5, 0.2), (0.6, 0.6, 0.3), (0.9, 0.5, 0.3))
D = ((0.5, 0.6, 0.1), (0.8, 0.7, 0.1), (0.9, 0.7, 0.2))
E = ((0.1, 0.4, 0.5), (0.1, 0.3, 0.6), (0.2, 0.5, 0.9))
F = ((0.2, 0.5, 0.4), (0.3, 0.4, 0.8), (0.3, 0.3, 0.9))
G = ((0.2, 0.5, 0.4), (0.3, 0.4, 0.6), (0.3, 0.3, 0.9))
H = ((0.1, 0.4, 0.5), (0.1, 0.3, 0.8), (0.2, 0.3, 0.9)).

Then (X, τ) is a NTS. Consider the NSs are
I = ((0.1, 0.2, 0.5), (0.2, 0.3, 0.7), (0.3, 0.3, 1)) and J = ((0.2, 0.4, 0.8), (0.2, 0.4, 0.8), (0.2, 0.5, 0.9)). Then NS Cl(I) = G and NS Cl(J) = H.

This implies that NS Cl(I) ∩ NS Cl(J) = G. Now, I ∩ J = ((0.1, 0.2, 0.8), (0.1, 0.2, 0.8), (0.2, 0.3, 1)), it follows that NS Cl(I ∩ J) = H. Then NS Cl(I) ∩ NS Cl(J) ⊆ NS Cl(I ∩ J).

Theorem 6.7 If A and B are NSs of NTSS X and Y respectively, then
(i) NS Cl(A × X) ⊆ NS Cl(A × X),
(ii) NS Int(A × X) ⊆ NS Int(A × X).

Proof: (i) Since A ⊆ NS Cl(A) and B ⊆ NS Cl(B), hence A × B ⊆ NS Cl(A) × NS Cl(B). This implies that NS Cl(A × B) ⊆ NS Cl(A × B), and from Proposition 4.12, NS Cl(A × B) ⊆ NS Cl(A × B).

(ii) From Proposition 6.8, NS Int(A) = C(NS Cl(A)).

Lemma 6.8 For NSs A_i’s and B_j’s of NTSS X and Y respectively, we have
(i) \( \bigcap \{ A_i, B_j \} = \min ( \bigcap A_i, \bigcap B_j ) \);
(ii) \( \bigcup \{ A_i, B_j \} = \max ( \bigcup A_i, \bigcup B_j ) \);
(iii) \( \bigcup \{ A_i \times B_j \} = \bigcup \{ A_i \} \times \bigcup \{ B_j \} \);
(iv) \( \bigcup \{ A_i \times B_j \} = \bigcap \{ A_i \} \times \bigcup \{ B_j \} \).

Proof: Obvious.

Theorem 6.9 Let (X, τ) and (Y, σ) be NTSS such that X is neutrosophic product related to Y. Then for NSs A of X and B of Y, we have
(i) NS Cl(A × B) ⊆ NS Cl(A) × NS Cl(B),
(ii) NS Int(A × B) = NS Int(A) × NS Int(B).

Proof: (i) Since NS Cl(A × B) ⊆ NS Cl(A) × NS Cl(B) (By Theorem 6.7 (ii)) it is sufficient to show that NS Cl(A × B) ⊆ NS Cl(A) × NS Cl(B). Let \( A_i \in \tau \) and \( B_j \in \sigma \). Then NS Cl(A × B) = (X(x, y), \bigcap C(\{A_i × B_j\}) = \bigcap A_i × B_j, \bigcup C(\{A_i × B_j\}) = \bigcup A_i × B_j, \bigcup (A_i × B_j, A_i × B_j) = \bigcap (A_i × B_j, A_i × B_j). Thus proved (ii).

Example 6.10 Let (X, τ) be a NTS. Then for a neutrosophic subset A and B of a NTS X, we have
(i) NS Cl(A × B) = NS Cl(A) × NS Cl(B),
(ii) NS Int(A × B) = NS Int(A) × NS Int(B).

Proof: (i) Since NS Cl(A × B) ⊆ NS Cl(A) × NS Cl(B) (By Theorem 6.7 (ii)) it is sufficient to show that NS Cl(A × B) ⊆ NS Cl(A) × NS Cl(B). Let \( A_i \in \tau \) and \( B_j \in \sigma \). Then NS Cl(A × B) = (X(x, y), \bigcap C(\{A_i × B_j\}) = \bigcap A_i × B_j, \bigcup C(\{A_i × B_j\}) = \bigcup A_i × B_j, \bigcup (A_i × B_j, A_i × B_j) = \bigcap (A_i × B_j, A_i × B_j). Thus proved (ii).

Theorem 6.11 Let (X, τ) and (Y, σ) be NTSS such that X is neutrosophic product related to Y. Then for NSs A of X and B of Y, we have
(i) NS Cl(A × B) = NS Cl(A) × NS Cl(B),
(ii) NS Int(A × B) = NS Int(A) × NS Int(B).

Proof: (i) Since NS Cl(A × B) ⊆ NS Cl(A) × NS Cl(B) (By Theorem 6.7 (ii)) it is sufficient to show that NS Cl(A × B) ⊆ NS Cl(A) × NS Cl(B). Let \( A_i \in \tau \) and \( B_j \in \sigma \). Then NS Cl(A × B) = (X(x, y), \bigcap C(\{A_i × B_j\}) = \bigcap A_i × B_j, \bigcup C(\{A_i × B_j\}) = \bigcup A_i × B_j, \bigcup (A_i × B_j, A_i × B_j) = \bigcap (A_i × B_j, A_i × B_j). Thus proved (ii).

Theorem 6.12 Let (X, τ) be a NTS. Then for a neutrosophic subset A and B of a NTS X, we have
(i) NS Cl(A × B) = NS Cl(A) × NS Cl(B),
(ii) NS Int(A × B) = NS Int(A) × NS Int(B).

Proof: (i) Since NS Cl(A × B) ⊆ NS Cl(A) × NS Cl(B) (By Theorem 6.7 (ii)) it is sufficient to show that NS Cl(A × B) ⊆ NS Cl(A) × NS Cl(B). Let \( A_i \in \tau \) and \( B_j \in \sigma \). Then NS Cl(A × B) = (X(x, y), \bigcap C(\{A_i × B_j\}) = \bigcap A_i × B_j, \bigcup C(\{A_i × B_j\}) = \bigcup A_i × B_j, \bigcup (A_i × B_j, A_i × B_j) = \bigcap (A_i × B_j, A_i × B_j). Thus proved (ii).
(i) \( \text{NS CI} (A) \supseteq A \cup \text{NS CI} (\text{NS Int} (A)) \),
(ii) \( \text{NS Int} (A) \subseteq A \cap \text{NS Int} (\text{NS CI} (A)) \),
(iii) \( \text{NInt} (\text{NS CI} (A)) \subseteq \text{NInt} (\text{NCl} (A)) \),
(iv) \( \text{NInt} (\text{NS CI} (A)) \supseteq \text{NInt} (\text{NS Cl} (\text{NS Int} (A))) \).

**Proof**: By Proposition 6.3 (i), \( A \subseteq \text{NS CI} (A) \) -----(1). Again using Proposition 5.2 (i), \( \text{NS Int} (A) \subseteq A \). Then \( \text{NS CI} (\text{NS Int} (A)) \subseteq \text{NS CI} (A) \) -----(2). By (1) & (2) we have, \( A \cup \text{NS CI} (\text{NS Int} (A)) \subseteq \text{NS CI} (A) \). This proves (i).

By Proposition 5.2 (i), \( \text{NS Int} (A) \subseteq A \) -----(1). Again using proposition 6.3 (i), \( A \subseteq \text{NS CI} (A) \). Then \( \text{NS Int} (A) \subseteq \text{NS Int} (\text{NS CI} (A)) \) -----(2). From (1) & (2), we have \( \text{NS Int} (A) \subseteq A \cap \text{NS Int} (\text{NS CI} (A)) \). This proves (ii).

By Proposition 6.4, \( \text{NS CI} (A) \subseteq \text{NCl} (A) \). We get \( \text{NInt} (\text{NS CI} (A)) \subseteq \text{NInt} (\text{NCl} (A)) \). Hence (iii).

By (i), \( \text{NS CI} (A) \supseteq A \cup \text{NS CI} (\text{NS Int} (A)) \). We have \( \text{NInt} (\text{NS CI} (A)) \supseteq \text{NInt} (A \cup \text{NS CI} (\text{NS Int} (A))) \).

Since \( \text{NInt} (A \cup B) \supseteq \text{NInt} (A) \cup \text{NInt} (B) \), \( \text{NInt} (\text{NS CI} (A)) \supseteq \text{NInt} (\text{NS CI} (\text{NS Int} (A))) \) \supseteq \text{NInt} (\text{NS CI} (\text{NS Int} (A))). Hence (iv).

**REFERENCES**

ITKRS Session, Sofia ( June 1983 central Sci. and Techn.
 Library, Bulg.Academy of Sciences (1984)).

[2] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and

[3] K. Atanassov, Review and new result on intuitionistic fuzzy

14-32.

24 (1968), 182-190.


[7] Florentin Smarandache, Neutrosophy and Neutrosophic
Logic, First International Conference on Neutrosophy,
Neutrosophic Logic, Set, Probability, and Statistics
University of New Mexico, Gallup, NM 87301, USA (2002),
smarand@unm.edu

[8] Florentin Smarandache, A Unifying Field in Logics :
Neutrosophic Logic. Neutrosophy, Neutrosophic Set,
Neutrosophic Probability. American Research Press,
Rehoboth, NM, 1999.

of Intuitionistic Fuzzy set, Journal of Defense Resourses

[10] I. M. Hanafy, Completely continuous functions in
intuitionistic fuzzy topological spaces, Czechoslovak

Proceedings of 13th WSEAS. J. International conference on
Applied Mathematics (MATH’08) Kybernetes, 38 (2009),
621-624.


[13] Reza Saadati, J. HanPark, On the intuitionistic fuzzy
topological space, Chaos, Solitons and Fractals 27 (2006),
331-344.

[14] A.A. Salama and S.A. Alblowi, Generalized Neutrosophic
Set and Generalized Neutrosophic Topological Spaces,

[15] A.A.Salama and S.A.Alblowi, Neutrosophic set and
neutrosophic topological space, ISOR J. mathematics,

[16] R. Usha Parameswari, K. Bageerathi, On fuzzy \( \gamma \)-semi open
sets and fuzzy \( \gamma \)-semi closed sets in fuzzy topological spaces,

[17] L.A. Zadeh, Fuzzy Sets, Inform and Control 8 (1965), 338-
353.