Neutrosophic Closed Set and Neutrosophic Continuous Functions

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Abstract
In this paper, we introduce and study the concept of "neutrosophic closed set "and "neutrosophic continuous function". Possible application to GIS topology rules are touched upon.

Keywords: Neutrosophic Closed Set, Neutrosophic Set; Neutrosophic Topology; Neutrosophic Continuous Function.

1 INTRODUCTION
The idea of "neutrosophic set" was first given by Smarandache [11, 12]. Neutrosophic operations have been investigated by Salama at el. [1-10]. Neutrosophy has laid the foundation for a whole family of new mathematical theories, generalizing both their crisp and fuzzy counterparts [9, 13]. Here we shall present the neutrosophic crisp version of these concepts. In this paper, we introduce and study the concept of "neutrosophic closed set " and "neutrosophic continuous function".

2 TERMINOLOGIES
We recollect some relevant basic preliminaries, and in particular the work of Smarandache in [11, 12], and Salama at el. [1-10].

2.1 Definition [5]
A neutrosophic topology (NT for short) an a non empty set $X$ is a family $\tau$ of neutrosophic subsets in $X$ satisfying the following axioms

(NT$_1$) $O_x, l_y \in \tau$ ,
(NT$_2$) $G_i \cap G_z \in \tau$ for any $G_i, G_z \in \tau$ ,
(NT$_3$) $\bigcup G_i \in \tau \ \forall \left\{ G_i : i \in J \right\} \subseteq \tau$

In this case the pair $(X, \tau)$ is called a neutrosophic topological space (NTS for short) and any neutrosophic set in $\tau$ is known as neutrosophic open set (NOS for short) in $X$. The elements of $\tau$ are called open neutrosophic sets, A neutrosophic set $F$ is closed if and only if it $C(F)$ is neutrosophic open.

2.1 Definition [5]
The complement of $(C(A)$ for short) of is called a neutrosophic closed set (NCS for short) in $X$. NOSA NCS X.

3 Neutrosophic Closed Set

3.1 Definition
Let $(X, \tau)$ be a neutrosophic topological space. A neutrosophic set $A$ in $(X, \tau)$ is said to be neutrosophic closed (in shortly N-closed). If $\text{Ncl}(A) \subseteq G$ whenever $A \subseteq G$ and $G$ is neutrosophic open; the complement of neutrosophic closed set is Neutrosophic open.

3.1 Proposition
If $A$ and $B$ are neutrosophic closed sets then $A \cup B$ is Neutrosophic closed set.

3.1 Remark
The intersection of two neutrosophic closed (N-closed for short) sets need not be neutrosophic closed set.

3.1 Example
Let $X = \{a, b, c\}$ and
A = \langle (0.5,0.5,0.5), (0.4,0.5,0.5), (0.4,0.5,0.5) \rangle

B = \langle (0.3,0.4,0.4), (0.7,0.5,0.5), (0.3,0.4,0.4) \rangle

Then T = \{ 0_\mathbb{N}, 1_\mathbb{N}, A, B \} is a neutrosophic topology on X. Define the two neutrosophic sets A_1 and A_2 as follows,

A_1 = \langle (0.5,0.5,0.5), (0.6,0.5,0.5), (0.6,0.5,0.5) \rangle

A_2 = \langle (0.7,0.6,0.6), (0.3,0.5,0.5), (0.7,0.6,0.6) \rangle

A_1 and A_2 are neutrosophic closed set but A_1 \cap A_2 is not a neutrosophic closed set.

3.2 Proposition

Let (X, \tau) be a neutrosophic topological space. If B is neutrosophic closed set and A \subseteq A \subseteq \text{Ncl}(B), then A is N-closed.

3.4 Proposition

In a neutrosophic topological space (X,\tau), T=\mathbb{N} (the family of all neutrosophic closed sets) if every neutrosophic subset of (X,\tau) is a neutrosophic closed set.

Proof. Suppose that every neutrosophic set A of (X,\tau) is N-closed. Let A \subseteq A \subseteq A \subseteq A and B \in \text{Ncl}(A). Hence, \text{Ncl}(A) = A, thus, A \subseteq \mathbb{N}. Therefore, B \in \mathbb{N} and hence B \in \mathbb{T}. Therefore \mathbb{T}=\mathbb{N} conversely, suppose that A be a neutrosophic set in (X,\tau). Let B be a neutrosophic open set in (X,\tau). Such that B \subseteq A. By hypothesis, B is neutrosophic N-closed. By definition of neutrosophic closure, \text{Ncl}(A) \subseteq B. Therefore A is N-closed.

3.5 Proposition

Let (X,\tau) be a neutrosophic topological space. A neutrosophic set A is neutrosophic open iff B \subseteq \text{Nint}(A), whenever B is neutrosophic closed and B \subseteq A.

Proof. Let A a neutrosophic open set and B a N-closed, such that B \subseteq A. Now, B \subseteq A \Rightarrow 1-B \subseteq 1-B and 1-A is a neutrosophic closed set \Rightarrow \text{Ncl}(1-A) \subseteq 1-B. That is, B=1-(1-B) \subseteq 1-Ncl(1-A). But 1-Ncl(1-A) = Nint(1-A). Thus, B \subseteq Nint(1-A). Conversely, suppose that B be a neutrosophic set, such that B \subseteq Nint(1-A) whenever B is neutrosophic closed and B \subseteq A. Let 1-A \subseteq B \Rightarrow 1-B \subseteq A. Hence by assumption 1-B \subseteq Nint(1-A), that is, 1-Nint(1-A) \subseteq B. But 1-Nint(1-A) =Ncl(1-A), Hence \text{Ncl}(1-A) \subseteq B. That is 1-A is neutrosophic open set. Therefore, A is neutrosophic open set.

3.6 Proposition

If Nint(A) \subseteq B \subseteq A and if A is neutrosophic open set then B is also neutrosophic open set.

4 Neutrosophic Continuous Functions

4.1 Definition

i) If \( B = \langle \mu_B, \sigma_B, \nu_B \rangle \) is a NS in Y, then the preimage of B under f denoted by \( f^{-1}(B) \) is a NS in X defined by

\[ f^{-1}(B) = \langle f^{-1}(\mu_B), f^{-1}(\sigma_B), f^{-1}(\nu_B) \rangle. \]

ii) If \( A = \langle \mu_A, \sigma_A, \nu_A \rangle \) is a NS in X, then the image of A under f, denoted by \( f(A) \), is the a NS in Y defined by

\[ f(A) = \langle f(\mu_A), f(\sigma_A), f(\nu_A) \rangle. \]

Here we introduce the properties of images and preimages some of which we shall frequently use in the following sections.

4.2 Definition

Let \( \langle X, \mathcal{T}_1 \rangle \) and \( \langle Y, \mathcal{T}_2 \rangle \) be two NTs, and let \( f : X \to Y \) be a function. Then \( f^{'} \) is said to be continuous iff the preimage of each NCS in \( \mathcal{T}_2 \) is a NS in \( \mathcal{T}_1 \).

4.3 Definition

Let \( \langle X, \mathcal{T}_1 \rangle \) and \( \langle Y, \mathcal{T}_2 \rangle \) be two NTs, and let \( f : X \to Y \) be a function. Then \( f^{'} \) is said to be open iff the image of each NS in \( \mathcal{T}_1 \) is a NS in \( \mathcal{T}_2 \).

4.1 Example

Let \( \langle X, \mathcal{T}_0 \rangle \) and \( \langle Y, \psi_{\mathcal{T}_0} \rangle \) be two NTs

(a) If \( f : X \to Y \) is continuous in the usual sense, then in this case, \( f^{'} \) is continuous in the sense of Definition 5.1 too. Here we consider the NTs on X and Y, respectively, as follows : \( \mathcal{T}_1 = \left\{ \langle \mu_G, 0, \mu_G \rangle : G \in \mathcal{T}_0 \right\} \) and
In this case we have, for each \( \mu_H, 0, \mu_H \in \Gamma_2 \),
\( H \in \mathcal{P}_Y \),
\( f^{-1}(\mu_H, 0, \mu_H) = f^{-1}(0, f^{-1}(\mu_H)) \in \Gamma_1 \).

(b) If \( f : X \to Y \) is neutrosophic open in the usual sense, then in this case, \( f \) is neutrosophic open in the sense of Definition 3.2.

Now we obtain some characterizations of neutrosophic continuity:

4.1 Proposition
Let \( f : (X, \Gamma_1) \to (Y, \Gamma_2) \).
\( f \) is neutrosophic continuous iff the preimage of each NS (neutrosophic closed set) in \( \Gamma_2 \) is a NS in \( \Gamma_2 \).

4.2 Proposition
The following are equivalent to each other:
(a) \( f : (X, \Gamma_1) \to (Y, \Gamma_2) \) is neutrosophic continuous.
(b) \( f^{-1}(N_{\text{Int}}(B)) \subseteq N_{\text{Int}}(f^{-1}(B)) \) for each CNS \( B \) in \( Y \).
(c) \( N_{\text{cl}}(f^{-1}(B)) \subseteq f^{-1}(N_{\text{cl}}(B)) \) for each NCB in \( Y \).

4.3 Example
Let \( \{Y, \Gamma_2\} \) be a NTS and \( f : X \to Y \) be a function. In this case \( \Gamma_1 = f^{-1}(H) : H \in \Gamma_2 \) is a NT on \( X \). Indeed, it is the coarsest NT on \( X \) which makes the function \( f : X \to Y \) continuous. One may call it the initial neutrosophic crisp topology with respect to \( f \).

4.4 Definition
Let \( (X, T) \) and \( (Y, S) \) be two neutrosophic topological spaces, then
(a) A map \( f : (X, T) \to (Y, S) \) is called N-continuous (in short N-continuous) if the inverse image of every closed set in \( Y \) is Neutrosophic closed in \( X \).
(b) A map \( f : (X, T) \to (Y, S) \) is called neutrosophic-gc irresolute if the inverse image of every Neutrosophic closed set in \( (Y, S) \) is Neutrosophic closed in \( (X, T) \).
(c) A map \( f : (X, T) \to (Y, S) \) is said to be strongly neutrosophic continuous if \( f^{-1}(A) \) is both neutrosophic open and neutrosophic closed in \( (X, T) \) for each neutrosophic open set \( A \) in \( (Y, S) \).
(d) A map \( f : (X, T) \to (Y, S) \) is said to be perfectly neutrosophic continuous if \( f^{-1}(A) \) is both neutrosophic open and neutrosophic closed in \( (X, T) \) for each neutrosophic open set \( A \) in \( (Y, S) \).
(e) A map \( f : (X, T) \to (Y, S) \) is said to be strongly N-continuous if the inverse image of every Neutrosophic open set in \( Y \) is neutrosophic open in \( X \).

(F) A map \( f : (X, T) \to (Y, S) \) is said to be perfectly N-continuous if the inverse image of every Neutrosophic open set in \( Y \) is both neutrosophic open and neutrosophic closed in \( X, T \).

4.3 Proposition
Let \( (X, T) \) and \( (Y, S) \) be any two neutrosophic topological spaces. Let \( f : (X, T) \to (Y, S) \) be generalized neutrosophic continuous. Then for every neutrosophic set \( A \) in \( X \), \( f(N_{\text{cl}}(A)) \subseteq N_{\text{cl}}(f(A)) \).

4.4 Proposition
Let \( (X, T) \) and \( (Y, S) \) be any two neutrosophic topological spaces. Let \( f : (X, T) \to (Y, S) \) be generalized neutrosophic continuous. Then for every neutrosophic set \( A \) in \( Y \), \( Ncl(f^{-1}(A)) \subseteq f^{-1}(Ncl(A)) \).

4.5 Proposition
Let \( (X, T) \) and \( (Y, S) \) be any two neutrosophic topological spaces. If \( A \) is a Neutrosophic closed set in \( (X, T) \) and if \( f : (X, T) \to (Y, S) \) is neutrosophic continuous and neutrosophic-closed then \( f(A) \) is Neutrosophic closed in \( (Y, S) \).

Proof.
Let \( G \) be a neutrosophic-open in \( (Y, S) \). If \( f(A) \subseteq G \), then \( A \subseteq f^{-1}(G) \) in \( (X, T) \). Since \( A \) is neutrosophic closed and \( f^{-1}(G) \) is neutrosophic open in \( (X, T) \), \( Ncl(A) \subseteq f^{-1}(G) \) (i.e. \( f(Ncl(A)) \subseteq G \)). Now by assumption, \( f(Ncl(A)) \) is neutrosophic closed and \( Ncl(f(Ncl(A))) = f(Ncl(A)) \subseteq G \). Hence, \( f(A) \) is N-closed.

4.5 Proposition
Let \( (X, T) \) and \( (Y, S) \) be any two neutrosophic topological spaces. If \( f : (X, T) \to (Y, S) \) is neutrosophic continuous then it is N-continuous.

The converse of proposition 4.5 need not be true. See Example 4.3.

4.3 Example
Let \( X = \{a, b, c\} \) and \( Y = \{a, b, c\} \). Define neutrosophic sets \( A \) and \( B \) as follows \( A = \{(0.4, 0.4, 0.5), (0.2, 0.4, 0.3), (0.4, 0.4, 0.5)\} \)
\[ B = \{(0.4, 0.5, 0.6), (0.3, 0.2, 0.3), (0.4, 0.5, 0.6)\} \]
Then the family \( T = \{0_b, 1_N, A\} \) is a neutrosophic topology on \( X \) and \( S = \{0_b, 1_N, B\} \) is a neutrosophic topology on \( Y \). Thus \( (X, T) \) and \( (Y, S) \) are neutrosophic topological spaces. Define \( f : (X, T) \to (Y, S) \) as \( f(a) = b, f(b) = a, f(c) = c \). Clearly \( f \) is N-continuous. Now \( f \) is not neutrosophic continuous, since \( f^{-1}(B) \notin T \) for \( B \in S \).

4.4 Example
Let \( X = \{a, b, c\} \). Define the neutrosophic sets \( A \) and \( B \) as follows,
\[ A = \{(0.4, 0.5, 0.4), (0.5, 0.5, 0.5), (0.4, 0.5, 0.4)\} \]
Let $(X,T)$ and $(Y,S)$ be any two neutrosophic topological spaces. If $f : (X,T) \rightarrow (Y,S)$ is strongly N-continuous then $f$ is strongly N-continuous.

The converse of proposition 3.23 is not true. See Example 4.7.

4.7 Example
Let $X = \{a,b,c\}$ and Define the neutrosophic sets $A$ and $B$ as follows.

$$A = \{(0.9,0.9,0.9), (0.1,0.1,0.1), (0.9,0.9,0.9)\}$$
$$B = \{(0.9,0.9,0.9), (0.1,0.1,0.1), (0.9,0.1,0.8)\}$$

and $S = \{0, 1\}$ are neutrosophic topologies on $X$. Thus $(X,T)$ and $(X,S)$ are neutrosophic topological spaces. Also define $f : (X,T) \rightarrow (X,S)$ as follows.

$$f(a) = a, f(b) = c, f(c) = b.$$ Clearly $f$ is neutrosophic continuous. Since $D = \{(0.9,0.9,0.9), (0.1,0.01,0.01), (0.9,0.9,0.99)\}$ is an Neutrosophic open set in $(X,S)$, $f^{-1}(D)$ is not neutrosophic open in $(X,T)$.

4.8 Proposition
Let $(X,T)$ and $(Y,S)$ be any two neutrosophic topological spaces. If $f : (X,T) \rightarrow (Y,S)$ is strongly neutrosophic continuous then $f$ is strongly N-continuous.

The converse of proposition 3.23 is not true. See Example 4.7.

4.9 Proposition
Let $(X,T),(Y,S)$ and $(Z,R)$ be any three neutrosophic topological spaces. Suppose $f : (X,T) \rightarrow (Y,S), g : (Y,S) \rightarrow (Z,R)$ be maps. Assume $f$ is neutrosophic g-c irresolute and $g$ is N-continuous then $g \circ f$ is N-continuous.

4.10 Proposition
Let $(X,T)$, $(Y,S)$ and $(Z,R)$ be any two neutrosophic topological spaces. Let $f : (X,T) \rightarrow (Y,S), g : (Y,S) \rightarrow (Z,R)$ be maps, such that $f$ is strongly N-continuous and $g$ is N-continuous. Then the composition $g \circ f$ is neutrosophic continuous.

4.5 Definition
A neutrosophic topological space $(X,T)$ is said to be neutrosophic $T_{1\beta}$ if every Neutrosophic closed set in $(X,T)$ is neutrosophic closed in $(X,T)$.

4.11 Proposition
Let $(X,T),(Y,S)$ and $(Z,R)$ be any neutrosophic topological spaces. Let $f : (X,T) \rightarrow (Y,S)$ and $g : (Y,S) \rightarrow (Z,R)$ be mapping and $(Y,S)$ be neutrosophic $T_{1\beta}$ if $f$ and $g$ are N-continuous then the composition $g \circ f$ is N-continuous.

The proposition 4.11 is not valid if $(Y,S)$ is not neutrosophic $T_{1\beta}$.

4.8 Example
Let $X = \{a,b,c\}$ and Define the closed sets $A,B$ and $C$ as follows.

$$A = \{0.4,0.4,0.6,0.5\}, \{0.4,0.4,0.3\}$$
$$B = \{0.4,0.5,0.6,0.5\}, \{0.3,0.4,0.3\}$$
$$C = \{0.5,0.3,0.5\}, \{0.5,0.3,0.4\}$$

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Then the family \( T = \{ 0_N, 1_N, A \} \), \( S = \{ 0_N, 1_N, B \} \) and \( R = \{ 0_N, 1_N, C \} \) are neutrosophic topologies on \( X \). Thus \((X,T),(X,S)\) and \((X,R)\) are neutrosophic topological spaces. Also define \( f : (X,T) \rightarrow (X,S) \) as \( f(a) = b, f(b) = a, f(c) = c \) and \( g : (X,S) \rightarrow (X,R) \) as \( g(a) = b, g(b) = c, g(c) = b \). Clearly \( f \) and \( g \) are \( N \)-continuous function. But \( g \circ f \) is not \( N \)-continuous. For \( 1 - C \) is neutrosophic closed in \((X,R)\), \( f^{-1}(g^{-1}(1-C)) \) is not \( N \) closed in \((X,T)\), \( g \circ f \) is not \( N \)-continuous.

References


