Interval-valued neutrosophic competition graphs

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Abstract. We first introduce the concept of interval-valued neutrosophic competition graphs. We then discuss certain types, including $k$-competition interval-valued neutrosophic graphs, $p$-competition interval-valued neutrosophic graphs and $m$-step interval-valued neutrosophic competition graphs. Moreover, we present the concept of $m$-step interval-valued neutrosophic neighbourhood graphs.

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1. Introduction

In 1975, Zadeh [35] introduced the notion of interval-valued fuzzy sets as an extension of fuzzy sets [34] in which the values of the membership degrees are intervals of numbers instead of the numbers. Interval-valued fuzzy sets provide a more adequate description of uncertainty than traditional fuzzy sets. It is therefore important to use interval-valued fuzzy sets in applications, such as fuzzy control. One of the computationally most intensive part of fuzzy control is defuzzification [19]. Atanassov [12] proposed the extended form of fuzzy set theory by adding a new component, called, intuitionistic fuzzy sets. Smarandache [26, 27] introduced the concept of neutrosophic sets by combining the non-standard analysis. In neutrosophic set, the membership value is associated with three components: truth-membership ($t$), indeterminacy-membership ($i$) and falsity-membership ($f$), in which each membership value is a real standard or non-standard subset of the non-standard unit interval $[0,1]^*$. and there is no restriction on their sum. Smarandache [28] and Wang et al. [29] presented the notion of single-valued neutrosophic sets to apply neutrosophic sets in real life problems more conveniently. In single-valued neutrosophic sets, three components are independent and their values are taken from the standard unit interval $[0,1]$. Wang et al. [30] presented the concept of interval-valued neutrosophic
sets, which is more precise and more flexible than the single-valued neutrosophic set. An interval-valued neutrosophic set is a generalization of the concept of single-valued neutrosophic set, in which three membership \((t, i, f)\) functions are independent, and their values belong to the unit interval \([0, 1]\).

Kauffman [18] gave the definition of a fuzzy graph. Fuzzy graphs were narrated by Rosenfeld [22]. After that, some remarks on fuzzy graphs were represented by Bhattacharya [13]. He showed that all the concepts on crisp graph theory do not have similarities in fuzzy graphs. Wu [32] discussed fuzzy digraphs. The concept of fuzzy \(k\)-competition graphs and \(p\)-competition fuzzy graphs was first developed by Samanta and Pal in [23], it was further studied in [11, 21, 25]. Samanta et al. [24] introduced the generalization of fuzzy competition graphs, called \(m\)-step fuzzy competition graphs. Samanta et al. [24] also introduced the concepts of fuzzy \(m\)-step neighbourhood graphs, fuzzy economic competition graphs, and \(m\)-step economic competition graphs. The concepts of bipolar fuzzy competition graphs and intuitionistic fuzzy competition graphs are discussed in [21, 25]. Hongmei and Lianhua [16], gave definition of interval-valued fuzzy graphs. Akram et al. [1, 2, 3, 4] have introduced several concepts on interval-valued fuzzy graphs and interval-valued neutrosophic graphs. Akram and Shahzadi [6] introduced the notion of neutrosophic soft graphs with applications. Akram [7] introduced the notion of single-valued neutrosophic planar graphs. Akram and Shahzadi [8] studied properties of single-valued neutrosophic graphs by level graphs. Recently, Akram and Nasir [5] have discussed some concepts of interval-valued neutrosophic graphs. In this paper, we first introduce the concept of interval-valued neutrosophic competition graphs. We then discuss certain types, including \(k\)-competition interval-valued neutrosophic graphs, \(p\)-competition interval-valued neutrosophic graphs and \(m\)-step interval-valued neutrosophic competition graphs. Moreover, we present the concept of \(m\)-step interval-valued neutrosophic neighbourhood graphs.

We have used standard definitions and terminologies in this paper. For other notations, terminologies and applications not mentioned in the paper, the readers are referred to [6, 9, 10, 13, 14, 15, 17, 20, 26, 33, 36].

2. INTERVAL-VALUED NEUTROSOPHIC COMPETITION GRAPHS

**Definition 2.1 ([35])**. The interval-valued fuzzy set \(A\) in \(X\) is defined by

\[
A = \{ s, [t_A^t(s), t_A^i(s)] : s \in X \},
\]

where, \(t_A^t(s)\) and \(t_A^i(s)\) are fuzzy subsets of \(X\) such that \(t_A^t(s) \leq t_A^i(s)\) for all \(x \in X\).

An interval-valued fuzzy relation on \(X\) is an interval-valued fuzzy set \(B\) in \(X \times X\).

**Definition 2.2 ([30, 31])**. The interval-valued neutrosophic set (IVN-set) \(A\) in \(X\) is defined by

\[
A = \{ s, [t_A^t(s), t_A^u(s), t_A^s(s), i_A^t(s), i_A^u(s), i_A^s(s), f_A^t(s), e_A^u(s)] : s \in X \},
\]

where, \(t_A^t(s)\), \(t_A^u(s)\), \(t_A^s(s)\), \(i_A^t(s)\), \(i_A^u(s)\), \(i_A^s(s)\), \(f_A^t(s)\), \(e_A^u(s)\) are neutrosophic subsets of \(X\) such that \(t_A^t(s) \leq t_A^u(s)\), \(i_A^t(s) \leq i_A^u(s)\), \(i_A^s(s) \leq i_A^s(s)\), \(t_A^t(s) \leq t_A^u(s)\) and \(f_A^t(s) \leq f_A^u(s)\) for all \(s \in X\). An interval-valued neutrosophic relation (IVN-relation) on \(X\) is an interval-valued neutrosophic set \(B\) in \(X \times X\).
Definition 2.3 ([5]). An interval-valued neutrosophic digraph (IVN-digraph) on a non-empty set $X$ is a pair $G = (A, \overrightarrow{B})$, (in short, $G$), where $A = ([t_A^l, t_A^u], [i_A^l, i_A^u], [f_A^l, f_A^u])$ is an IVN-set on $X$ and $B = ([t_B^l, t_B^u], [i_B^l, i_B^u], [f_B^l, f_B^u])$ is an IVN-relation on $X$, such that:

(i) $t_B^l(s, w) \leq t_A^l(s) \wedge t_A^l(w)$, \quad $t_B^u(s, w) \leq t_A^u(s) \wedge t_A^u(w)$,

(ii) $i_B^l(s, w) \leq i_A^l(s) \wedge i_A^l(w)$, \quad $i_B^u(s, w) \leq i_A^u(s) \wedge i_A^u(w)$,

(iii) $f_B^l(s, w) \leq f_A^l(s) \wedge f_A^l(w)$, \quad $f_B^u(s, w) \leq f_A^u(s) \wedge f_A^u(w)$, \quad for all $s, w \in X$.

Example 2.4. We construct an IVN-digraph $G = (A, \overrightarrow{B})$ on $X = \{a, b, c\}$ as shown in Fig. 1.

**Figure 1.** IVN-digraph

Definition 2.5. Let $\overrightarrow{G}$ be an IVN-digraph then interval-valued neutrosophic out-neighbourhoods (IVN-out-neighbourhoods) of a vertex $s$ is an IVN-set

$$\mathbb{N}^+(s) = (X_s^+, [t_s^{(l)+}, t_s^{(u)+}],[i_s^{(l)+}, i_s^{(u)+}],[f_s^{(l)+}, f_s^{(u)+}]),$$

where

$X_s^+ = \{w | t_B^l(s, w) > 0, t_B^u(s, w) > 0, i_B^l(s, w) > 0, i_B^u(s, w) > 0, f_B^l(s, w) > 0, f_B^u(s, w) > 0\},$

such that $t_s^{(l)+} : X_s^+ \rightarrow [0, 1]$, defined by $t_s^{(l)+}(w) = t_B^l(s, w)$, $t_s^{(u)+} : X_s^+ \rightarrow [0, 1]$, defined by $t_s^{(u)+}(w) = t_B^u(s, w)$,

$i_s^{(l)+} : X_s^+ \rightarrow [0, 1]$, defined by $i_s^{(l)+}(w) = i_B^l(s, w)$, $i_s^{(u)+} : X_s^+ \rightarrow [0, 1]$, defined by $i_s^{(u)+}(w) = i_B^u(s, w)$,

$f_s^{(l)+} : X_s^+ \rightarrow [0, 1]$, defined by $f_s^{(l)+}(w) = f_B^l(s, w)$, $f_s^{(u)+} : X_s^+ \rightarrow [0, 1]$, defined by $f_s^{(u)+}(w) = f_B^u(s, w)$.

Definition 2.6. Let $\overrightarrow{G}$ be an IVN-digraph then interval-valued neutrosophic in-neighbourhoods (IVN-in-neighbourhoods) of a vertex $s$ is an IVN-set

$$\mathbb{N}^-(s) = (X_s^-, [t_s^{(l)-}, t_s^{(u)-}],[i_s^{(l)-}, i_s^{(u)-}],[f_s^{(l)-}, f_s^{(u)-}]),$$

where

$X_s^- = \{w | t_B^l(s, w) > 0, t_B^u(s, w) > 0, i_B^l(s, w) > 0, i_B^u(s, w) > 0, f_B^l(s, w) > 0, f_B^u(s, w) > 0\},$

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such that \( t_s^{(l)} : X_s^{-} \to [0,1] \), defined by \( t_s^{(l)}(w) = t_B^l(w,s) \), \( t_s^{(u)} : X_s^{-} \to [0,1] \), defined by \( t_s^{(u)}(w) = t_B^u(w,s) \), \( i_s^{(l)} : X_s^{-} \to [0,1] \), defined by \( i_s^{(l)}(w) = i_B^l(w,s) \), \( i_s^{(u)} : X_s^{-} \to [0,1] \), defined by \( i_s^{(u)}(w) = i_B^u(w,s) \), \( f_s^{(l)} : X_s^{-} \to [0,1] \), defined by \( f_s^{(l)}(w) = f_B^l(w,s) \), \( f_s^{(u)} : X_s^{-} \to [0,1] \), defined by \( f_s^{(u)}(w) = f_B^u(w,s) \).

**Example 2.7.** Consider an IVN-digraph \( G = (A, \overrightarrow{B}) \) on \( X = \{a, b, c\} \) as shown in Fig. 2.

![Figure 2. IVN-digraph](image)

We have Table 1 and Table 2 representing interval-valued neutrosophic out and in-neighbourhoods, respectively.

**Table 1.** IVN-out-neighbourhoods

<table>
<thead>
<tr>
<th>( s )</th>
<th>( N^+(s) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>{b, [0.1,0.2],[0.2,0.3],[0.1,0.6]}, {c, [0.1,0.2],[0.1,0.3],[0.2,0.6]}</td>
</tr>
<tr>
<td>b</td>
<td>\Ø</td>
</tr>
<tr>
<td>c</td>
<td>{b, [0.1,0.2],[0.2,0.3],[0.2,0.5]}</td>
</tr>
</tbody>
</table>

**Table 2.** IVN-in-neighbourhoods

<table>
<thead>
<tr>
<th>( s )</th>
<th>( N^-(s) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>\Ø</td>
</tr>
<tr>
<td>b</td>
<td>{a, [0.1,0.2],[0.2,0.3],[0.1,0.6]}, {c, [0.1,0.2],[0.2,0.3],[0.2,0.5]}</td>
</tr>
<tr>
<td>c</td>
<td>{a, [0.1,0.2],[0.1,0.3],[0.2,0.6]}</td>
</tr>
</tbody>
</table>

**Definition 2.8.** The height of IVN-set \( A = (s, [t_A^{i}], [i_A^{u}], [f_A^{i}], [f_A^{u}]) \) in universe of discourse \( X \) is defined as: for all \( s \in X \),

\[
h(A) = ([h_A^1(s), h_A^u(A)], [h_A^2(s), h_A^u(A)], [h_A^3(s), h_A^u(A)]),
\]

\[
= ([\sup_{s \in X} t_A^i(s), \sup_{s \in X} t_A^u(s)], [\inf_{s \in X} i_A^i(s), \inf_{s \in X} i_A^u(s)], [\inf_{s \in X} f_A^i(s), \inf_{s \in X} f_A^u(s)]).
\]
Definition 2.9. An interval-valued neutrosophic competition graph (IVNC-graph) of an interval-valued neutrosophic graph (IVN-graph) $\overrightarrow{G} = (A, \overrightarrow{B})$ is an undirected IVN-graph $\mathbb{C}(\overrightarrow{G}) = (A, W)$ which has the same vertex set as in $\overrightarrow{G}$ and there is an edge between two vertices $s$ and $w$ if and only if $N^+(s) \cap N^+(w) \neq \emptyset$. The truth-membership, indeterminacy-membership and falsity-membership values of the edge $(s, w)$ are defined as: for all $s, w \in X$,

(i) $t^i_W(s, w) = (t^i_A(s) \land t^i_A(w))h^i_1(N^+(s) \cap N^+(w))$,
(ii) $i^i_W(s, w) = (i^i_A(s) \land i^i_A(w))h^i_2(N^+(s) \cap N^+(w))$,
(iii) $f^i_W(s, w) = (f^i_A(s) \land f^i_A(w))h^i_3(N^+(s) \cap N^+(w))$.

Example 2.10. Consider an IVN-digraph $G = (A, \overrightarrow{B})$ on $X = \{a, b, c\}$ as shown in Fig. 3.

![IVN-digraph](image)

**Figure 3. IVN-digraph**

We have Table 3 and Table 4 representing interval-valued neutrosophic out and in-neighbourhoods, respectively.

**Table 3. IVN-out-neighbourhoods**

<table>
<thead>
<tr>
<th>$s$</th>
<th>$N^+(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>${(b, [0.1,0.2],[0.2,0.3],[0.1,0.6]), (c, [0.1,0.2],[0.1,0.3],[0.2,0.6])}$</td>
</tr>
<tr>
<td>b</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>c</td>
<td>${(b, [0.1,0.2],[0.2,0.3],[0.2,0.5])}$</td>
</tr>
</tbody>
</table>

**Table 4. IVN-in-neighbourhoods**

<table>
<thead>
<tr>
<th>$s$</th>
<th>$N^-(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>b</td>
<td>${(a, [0.1,0.2],[0.2,0.3],[0.1,0.6]), (c, [0.1,0.2],[0.2,0.3],[0.2,0.5])}$</td>
</tr>
<tr>
<td>c</td>
<td>${(a, [0.1,0.2],[0.1,0.3],[0.2,0.6])}$</td>
</tr>
</tbody>
</table>
Then IVNC-graph of Fig. 3 is shown in Fig. 4.

**Definition 2.11.** Consider an IVN-graph $G = (A, B)$, where $A = ([A^1_1, A^1_2], [A^2_1, A^2_2], [A^3_1, A^3_2])$ and $B = ([B^1_1, B^1_2], [B^2_1, B^2_2], [B^3_1, B^3_2])$. Then, an edge $(s, w)$, $s, w \in X$ is called independent strong, if

\[
\begin{align*}
\frac{1}{2} [A^1_1(s) \land A^1_2(w)] &< B^1_1(s, w), \\
\frac{1}{2} [A^1_2(s) \land A^2_1(w)] &< B^1_2(s, w), \\
\frac{1}{2} [A^2_1(s) \land A^2_2(w)] &< B^2_1(s, w), \\
\frac{1}{2} [A^2_2(s) \land A^1_1(w)] &< B^2_2(s, w), \\
\frac{1}{2} [A^3_1(s) \land A^3_2(w)] &> B^3_1(s, w), \\
\frac{1}{2} [A^3_2(s) \land A^1_1(w)] &> B^3_2(s, w).
\end{align*}
\]

Otherwise, it is called weak.

We state the following theorems without their proofs.

**Theorem 2.12.** Suppose $\overline{G}$ is an IVN-digraph. If $N^+(s) \cap N^+(w)$ contains only one element of $\overline{G}$, then the edge $(s, w)$ of $\overline{C(G)}$ is independent strong if and only if

\[
\begin{align*}
||N^+(s) \cap N^+(w)||_1 &> 0.5, & ||N^+(s) \cap N^+(w)||_\infty &> 0.5, \\
||N^+(s) \cap N^+(w)||_1 &> 0.5, & ||N^+(s) \cap N^+(w)||_\infty &> 0.5, \\
||N^+(s) \cap N^+(w)||_1 &< 0.5, & ||N^+(s) \cap N^+(w)||_\infty &< 0.5.
\end{align*}
\]

**Theorem 2.13.** If all the edges of an IVN-digraph $\overline{G}$ are independent strong, then

\[
\begin{align*}
B^1_1(s, w) &> 0.5, & B^1_1(s, w) &> 0.5, \\
B^1_2(s, w) &> 0.5, & B^1_2(s, w) &> 0.5, \\
B^2_1(s, w) &> 0.5, & B^2_1(s, w) &> 0.5, \\
B^2_2(s, w) &< 0.5, & B^2_2(s, w) &< 0.5, \\
B^3_1(s, w) &< 0.5, & B^3_1(s, w) &< 0.5, \\
B^3_2(s, w) &< 0.5, & B^3_2(s, w) &< 0.5,
\end{align*}
\]

for all edges $(s, w)$ in $\overline{C(G)}$.

**Definition 2.14.** The interval-valued neutrosophic open-neighbourhood (IVN-open-neighbourhood) of a vertex $s$ of an IVN-graph $G = (A, B)$ is IVN-set $N(s) = (X_s, [t^i_s, t^u_s], [i^i_s, i^u_s], [f^i_s, f^u_s])$, where
\[X_s = \{w|B_1(s, w) > 0, B_2(s, w) > 0, B_3(s, w) > 0, B_4(s, w) > 0, B_5(s, w) > 0\},\]

and \(t_s^\prime : X_s \rightarrow [0, 1]\) defined by \(t_s^\prime(w) = B_1(s, w)\), \(t_s^\prime : X_s \rightarrow [0, 1]\) defined by \(t_s^\prime(w) = B_2(s, w)\), \(t_s^\prime : X_s \rightarrow [0, 1]\) defined by \(t_s^\prime(w) = B_3(s, w)\), \(t_s^\prime : X_s \rightarrow [0, 1]\) defined by \(t_s^\prime(w) = B_4(s, w)\), \(t_s^\prime : X_s \rightarrow [0, 1]\) defined by \(t_s^\prime(w) = B_5(s, w)\), respectively.

**Definition 2.15.** Suppose \(G = (A, B)\) is an IVN-graph. Interval-valued neutrosophic open-neighbourhood graph (IVN-open-neighbourhood graph) of \(G\) is an IVN-graph \(N(G) = (A, B')\) which has the same IVN-set of vertices in \(G\) and has an interval-valued neutrosophic edge between two vertices \(s, w \in X\) in \(N(G)\) if and only if \(N[s] \cap N[w]\) is a non-empty IVN-set in \(G\). The truth-membership, indeterminacy-membership, falsity-membership values of the edge \((s, w)\) are given by:

\[B_1^{\prime\prime}(s, w) = [A_1'(s) \wedge A_1(w)]h_1(N(s) \cap N(w)),\]
\[B_2^{\prime\prime}(s, w) = [A_2'(s) \wedge A_2(w)]h_2(N(s) \cap N(w)),\]
\[B_3^{\prime\prime}(s, w) = [A_3'(s) \wedge A_3(w)]h_3(N(s) \cap N(w)),\]
\[B_4^{\prime\prime}(s, w) = [A_4'(s) \wedge A_4(w)]h_4(N(s) \cap N(w)),\]
\[B_5^{\prime\prime}(s, w) = [A_5'(s) \wedge A_5(w)]h_5(N(s) \cap N(w)),\]

**Definition 2.16.** Suppose \(G = (A, B)\) is an IVN-graph. Interval-valued neutrosophic closed-neighbourhood graph (IVN-closed-neighbourhood graph) of \(G\) is an IVN-graph \(N(G) = (A, B')\) which has the same IVN-set of vertices in \(G\) and has an interval-valued neutrosophic edge between two vertices \(s, w \in X\) in \(N(G)\) if and only if \(N[s] \cap N[w]\) is a non-empty IVN-set in \(G\). The truth-membership, indeterminacy-membership, falsity-membership values of the edge \((s, w)\) are given by:

\[B_1^{\prime\prime}(s, w) = [A_1'(s) \wedge A_1(w)]h_1(N[s] \cap N[w]),\]
\[B_2^{\prime\prime}(s, w) = [A_2'(s) \wedge A_2(w)]h_2(N[s] \cap N[w]),\]
\[B_3^{\prime\prime}(s, w) = [A_3'(s) \wedge A_3(w)]h_3(N[s] \cap N[w]),\]
\[B_4^{\prime\prime}(s, w) = [A_4'(s) \wedge A_4(w)]h_4(N[s] \cap N[w]),\]
\[B_5^{\prime\prime}(s, w) = [A_5'(s) \wedge A_5(w)]h_5(N[s] \cap N[w]),\]

We now discuss the method of construction of interval-valued neutrosophic competition graph of the Cartesian product of IVN-digraph in following theorem which can be proof using similar method as used in [21], hence we omit its proof.
Theorem 2.17. Let $\overrightarrow{G_1} = (A_1, B_1)$ and $\overrightarrow{G_2} = (A_2, B_2)$ be two IVNC-graphs of IVN-digraphs $G_1 = (A_1, L_1)$ and $G_2 = (A_2, L_2)$, respectively. Then $\overrightarrow{G_1 \cup G_2} = G_1 \cup_{(\overrightarrow{G_1} \cap \overrightarrow{G_2})^*}$ is an IVN-graph on the crisp graph $(X_1 \times X_2, E_{\overrightarrow{G_1}} \cap E_{\overrightarrow{G_2}})$, $\overrightarrow{G_1}^*$ and $\overrightarrow{G_2}^*$ are the crisp competition graphs of $\overrightarrow{G_1}$ and $\overrightarrow{G_2}$, respectively. $\overrightarrow{G}$ is an IVN-graph on $(X_1 \times X_2, \overrightarrow{G})$ such that:

1. $E^0 = \{(s_1, s_2)(w_1, w_2) : w_1 \in N^- (s_1), w_2 \in N^+ (s_2)\}$
   \[ E_{\overrightarrow{G_1}} \cap E_{\overrightarrow{G_2}} = \{(s_1, s_2)(w_1, w_2) : s_1 \in X_1, s_2 w_2 \in E_{\overrightarrow{G_2}}\}, \]
   \[ \cup \{(s_1, s_2)(w_1, s_2) : s_2 \in X_2, s_1 w_1 \in E_{\overrightarrow{G_1}}\}. \]

2. $t^1_{A_1 \oplus A_2} = t^1_{A_1}(s_1) \land t^1_{A_2}(s_2), \quad i^1_{A_1 \oplus A_2} = i^1_{A_1}(s_1) \land i^1_{A_2}(s_2), \quad f^1_{A_1 \oplus A_2} = f^1_{A_1}(s_1) \land f^1_{A_2}(s_2),$
   \[ t^u_{A_1 \oplus A_2} = t^u_{A_1}(s_1) \land t^u_{A_2}(s_2), \quad i^u_{A_1 \oplus A_2} = i^u_{A_1}(s_1) \land i^u_{A_2}(s_2), \quad f^u_{A_1 \oplus A_2} = f^u_{A_1}(s_1) \land f^u_{A_2}(s_2). \]

3. $t^l_B((s_1, s_2)(s_1, w_2)) = [t^l_{A_1}(s_1) \land t^l_{A_2}(s_1) \land t^l_{A_2}(w_2)] \land v_{a_2} \{t^l_{A_1}(s_1) \land t^l_{A_2}(w_2) \land t^l_{A_2}(w_2,a_2)\},$
   \[ (s_1, s_2)(s_1, w_2) \in E_{\overrightarrow{G_1}} \cap E_{\overrightarrow{G_2}}, \quad a_2 \in (N^+(s_2) \land N^+(w_2)). \]

4. $i^l_B((s_1, s_2)(s_1, w_2)) = [i^l_{A_1}(s_1) \land i^l_{A_2}(s_2) \land i^l_{A_2}(w_2)] \land v_{a_2} \{i^l_{A_1}(s_1) \land i^l_{A_2}(w_2) \land i^l_{A_2}(w_2,a_2)\},$
   \[ (s_1, s_2)(s_1, w_2) \in E_{\overrightarrow{G_1}} \cap E_{\overrightarrow{G_2}}, \quad a_2 \in (N^+(s_2) \land N^+(w_2)). \]

5. $f^l_B((s_1, s_2)(s_1, w_2)) = [f^l_{A_1}(s_1) \land f^l_{A_2}(s_2) \land f^l_{A_2}(w_2)] \land v_{a_2} \{f^l_{A_1}(s_1) \land f^l_{A_2}(w_2) \land f^l_{A_2}(w_2,a_2)\},$
   \[ (s_1, s_2)(s_1, w_2) \in E_{\overrightarrow{G_1}} \cap E_{\overrightarrow{G_2}}, \quad a_2 \in (N^+(s_2) \land N^+(w_2)). \]

6. $t^u_B((s_1, s_2)(s_1, w_2)) = [t^u_{A_1}(s_1) \land t^u_{A_2}(s_1) \land t^u_{A_2}(w_2)] \land v_{a_2} \{t^u_{A_1}(s_1) \land t^u_{A_2}(w_2) \land t^u_{A_2}(w_2,a_2)\},$
   \[ (s_1, s_2)(s_1, w_2) \in E_{\overrightarrow{G_1}} \cap E_{\overrightarrow{G_2}}, \quad a_2 \in (N^+(s_2) \land N^+(w_2)). \]

7. $i^u_B((s_1, s_2)(s_1, w_2)) = [i^u_{A_1}(s_1) \land i^u_{A_2}(s_2) \land i^u_{A_2}(w_2)] \land v_{a_2} \{i^u_{A_1}(s_1) \land i^u_{A_2}(w_2) \land i^u_{A_2}(w_2,a_2)\},$
   \[ (s_1, s_2)(s_1, w_2) \in E_{\overrightarrow{G_1}} \cap E_{\overrightarrow{G_2}}, \quad a_2 \in (N^+(s_2) \land N^+(w_2)). \]

8. $f^u_B((s_1, s_2)(s_1, w_2)) = [f^u_{A_1}(s_1) \land f^u_{A_2}(s_2) \land f^u_{A_2}(w_2)] \land v_{a_2} \{f^u_{A_1}(s_1) \land f^u_{A_2}(w_2) \land f^u_{A_2}(w_2,a_2)\},$
   \[ (s_1, s_2)(s_1, w_2) \in E_{\overrightarrow{G_1}} \cap E_{\overrightarrow{G_2}}, \quad a_2 \in (N^+(s_2) \land N^+(w_2)). \]

9. $t^l_B((s_1, s_2)(w_1, s_2)) = [t^l_{A_1}(s_1) \land t^l_{A_2}(w_1) \land t^l_{A_2}(s_2)] \land v_{a_1} \{t^l_{A_2}(w_1) \land t^l_{A_2}(s_2) \land t^l_{A_1}(w_1,a_1)\},$
   \[ (s_1, s_2)(w_1, s_2) \in E_{\overrightarrow{G_1}} \cap E_{\overrightarrow{G_2}}, \quad a_1 \in (N^+(s_1) \land N^+(w_1)). \]

10. $i^l_B((s_1, s_2)(w_1, s_2)) = [i^l_{A_1}(s_1) \land i^l_{A_2}(w_1) \land i^l_{A_2}(s_2)] \land v_{a_1} \{i^l_{A_2}(w_1) \land i^l_{A_2}(s_2) \land i^l_{A_1}(w_1,a_1)\},$
    \[ (s_1, s_2)(w_1, s_2) \in E_{\overrightarrow{G_1}} \cap E_{\overrightarrow{G_2}}, \quad a_1 \in (N^+(s_1) \land N^+(w_1)). \]
(11) $f_B^u((s_1, s_2)(w_1, s_2)) = [f_{A_1}^v(s_1) \wedge f_{A_2}^v(s_2)] \times \forall a_1 \{t_{A_2}^u(s_2) \wedge f_{L_1}^u(w_1)(s_1 a_1) \}
\quad \{s_1, s_2(w_1, s_2) \in E_{C(G_1)}, a_1 \in (N^+(s_1) \cap N^+(w_1))^*\}.
(12) t_B^u((s_1, s_2)(w_1, s_2)) = [t_{A_1}^v(s_1) \wedge t_{A_2}^v(s_2)] \times \forall a_1 \{t_{A_2}^u(s_2) \wedge t_{L_1}^u(w_1)(s_1 a_1) \}
\quad \{s_1, s_2(w_1, s_2) \in E_{C(G_1)}, a_1 \in (N^+(s_1) \cap N^+(w_1))^*\}.
(13) i_B^u((s_1, s_2)(w_1, s_2)) = [i_{A_1}^v(s_1) \wedge i_{A_2}^v(s_2)] \times \forall a_1 \{i_{A_2}^u(s_2) \wedge i_{L_1}^u(w_1)(s_1 a_1) \}
\quad \{s_1, s_2(w_1, s_2) \in E_{C(G_1)}, a_1 \in (N^+(s_1) \cap N^+(w_1))^*\}.

A. k-competition interval-valued neutrosophic graphs

We now discuss an extension of IVN-graphs, called k-competition IVN-graphs.

**Definition 2.18.** The cardinality of an IVN-set $A$ is denoted by

$$|A| = \left( |A|_{\mu}, |A|_{\nu}, [|A|_{\mu}, |A|_{\nu}], [|A|_{\mu}, |A|_{\nu}] \right).$$

Where $[|A|_{\mu}, |A|_{\nu}], [|A|_{\mu}, |A|_{\nu}]$ and $[|A|_{\mu}, |A|_{\nu}]$ represent the sum of truth-membership values, indeterminacy-membership values and falsity-membership values, respectively, of all the elements of $A$.
Example 2.19. The cardinality of an IVN-set $A = \{(a, [0.5, 0.7], [0.2, 0.8], [0.1, 0.3]), (b, [0.1, 0.2], [0.1, 0.5], [0.7, 0.9]), (c, [0.3, 0.5], [0.3, 0.8], [0.6, 0.9])\}$ in $X = \{a, b, c\}$ is

$$|A| = \left(\frac{|A|_1}{|A|_1}, \frac{|A|_2}{|A|_2}, \frac{|A|_3}{|A|_4}\right) \cup \left(\frac{|A|_1}{|A|_1}, \frac{|A|_2}{|A|_3}, \frac{|A|_4}{|A|_4}\right) \cup \left(\frac{|A|_1}{|A|_1}, \frac{|A|_3}{|A|_2}, \frac{|A|_4}{|A|_4}\right) = (0.9, 1.4, 0.6, 2.1, 1.4, 2.1)$$

We now discuss $k$-competition IVN-graphs.

Definition 2.20. Let $k$ be a non-negative number. Then $k$-competition IVN-graph $\overrightarrow{C_k(G)}$ of an IVN-digraph $G = (A, B)$ is an undirected IVN-graph $G = (A, B)$ which has same IVN-set of vertices as in $G$ and has an interval-valued neutrosophic edge between two vertices $s, w \in X$ in $\overrightarrow{C_k(G)}$ if and only if $|N^+(s) \cap N^+(w)|_1 > k$, $|N^+(s) \cap N^+(w)|_2 > k$, $|N^+(s) \cap N^+(w)|_3 > k$, and $|N^+(s) \cap N^+(w)|_4 > k$. The interval-valued truth-membership value of edge $(s, w)$ in $\overrightarrow{C_k(G)}$ is $t^k_B(s, w) = \frac{k_3 - k}{k_3} |t^k_A(s) \wedge t^k_A(w)|_1 |N^+(s) \cap N^+(w)|_1$, where $k_3 = |N^+(s) \cap N^+(w)|_1$, and $t^k_B(s, w) = \frac{k_3 - k}{k_3} |t^k_A(s) \wedge t^k_A(w)|_2 |N^+(s) \cap N^+(w)|_2$, where $k_2 = |N^+(s) \cap N^+(w)|_2$, and $t^k_B(s, w) = \frac{k_3 - k}{k_3} |t^k_A(s) \wedge t^k_A(w)|_3 |N^+(s) \cap N^+(w)|_3$, where $k_2 = |N^+(s) \cap N^+(w)|_3$, and $t^k_B(s, w) = \frac{k_3 - k}{k_3} |t^k_A(s) \wedge t^k_A(w)|_4 |N^+(s) \cap N^+(w)|_4$, where $k_2 = |N^+(s) \cap N^+(w)|_4$.

Example 2.21. Consider an IVN-digraph $G = (A, B)$ on $X = \{s, w, a, b, c\}$, such that $A = \{(s, [0.4, 0.5], [0.5, 0.7], [0.8, 0.9]), (w, [0.6, 0.7], [0.4, 0.6], [0.2, 0.3]), (a, [0.2, 0.6], [0.3, 0.6], [0.2, 0.6]), (b, [0.2, 0.6], [0.1, 0.6], [0.2, 0.6]), (c, [0.2, 0.7], [0.3, 0.5], [0.2, 0.6])\}$, and $B = \{(s, a), [0.1, 0.4], [0.3, 0.6], [0.2, 0.6]), (s, b), [0.2, 0.4], [0.1, 0.5], [0.2, 0.6]), (s, c), [0.2, 0.5], [0.3, 0.5], [0.2, 0.6]), (w, a), [0.2, 0.5], [0.2, 0.5], [0.2, 0.3]), ([w, b), [0.2, 0.6], [0.1, 0.6], [0.2, 0.3]), ([w, c), [0.2, 0.7], [0.3, 0.5], [0.2, 0.3])\}$, as shown in Fig. 5.

We calculate $N^+(s) = \{(a, [0.1, 0.4], [0.3, 0.6], [0.2, 0.6]), (b, [0.2, 0.4], [0.1, 0.5], [0.2, 0.6]), (c, [0.2, 0.5], [0.3, 0.5], [0.2, 0.6])\}$ and $N^+(w) = \{(a, [0.2, 0.5], [0.2, 0.5], [0.2, 0.3]), (b, [0.2, 0.6], [0.1, 0.6], [0.2, 0.3]), (c, [0.2, 0.7], [0.3, 0.5], [0.2, 0.3])\}$. Therefore, $N^+(s) \cap N^+(w) = \{(a, [0.1, 0.4], [0.2, 0.5], [0.2, 0.3]), (b, [0.2, 0.4], [0.1, 0.5], [0.2, 0.3]), (c, [0.2, 0.5], [0.3, 0.5], [0.2, 0.3])\}$. So, $k_1 = 0.5$, $k_2 = 1.3$, $k_3 = 0.6$, $k_4 = 1.5$, $k_3 = 0.6$ and $k_4 = 0.9$. Let $k = 0.4$, then, $t^k_B(s, w) = 0.02$, $t^k_B(s, w) = 0.06$, $t^k_B(s, w) = 0.08$, $f^k_B(s, w) = 0.02$ and $f^k_B(s, w) = 0.11$. This graph is depicted in Fig. 6.
Theorem 2.22. Let $\overrightarrow{G} = (A, \overrightarrow{B})$ be an IVN-digraph. If

\[ h^1_1(N^+(s) \cap N^+(w)) = 1, \quad h^2_2(N^+(s) \cap N^+(w)) = 1, \quad h^3_3(N^+(s) \cap N^+(w)) = 1, \]

\[ h^4_4(N^+(s) \cap N^+(w)) = 1, \quad h^5_5(N^+(s) \cap N^+(w)) = 1, \quad h^6_6(N^+(s) \cap N^+(w)) = 1, \]

and

\[ |(N^+(s) \cap N^+(w))|_t > 2k, \quad |(N^+(s) \cap N^+(w))|_i > 2k, \quad |(N^+(s) \cap N^+(w))|_i > 2k, \]

\[ |(N^+(s) \cap N^+(w))|_l > 2k, \quad |(N^+(s) \cap N^+(w))|_l > 2k, \quad |(N^+(s) \cap N^+(w))|_l < 2k, \]

Then the edge $(s, w)$ is independent strong in $C_k(\overrightarrow{G})$.

Proof. Let $\overrightarrow{G} = (A, \overrightarrow{B})$ be an IVN-digraph. Let $C_k(\overrightarrow{G})$ be the corresponding $k$-competition IVN-graph.

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If $h^l_1(N^+(s) \cap N^+(w)) = 1$ and $|(N^+(s) \cap N^+(w))|_{\mu} > 2k$, then $k^l_1 > 2k$. Thus,

$$t^l_B(s, w) = \frac{k^l_1 - k}{k^l_1} [t^l_A(s) \wedge t^l_A(w)]h^l_1(N^+(s) \cap N^+(w))$$

or,

$$t^l_B(s, w) = \frac{k^l_1 - k}{k^l_1} [t^l_A(s) \wedge t^l_A(w)]$$

$$\frac{t^l_B(s, w)}{[t^l_A(s) \wedge t^l_A(w)]} = \frac{k^l_1 - k}{k^l_1} > 0.5.$$ 

If $h^u_1(N^+(s) \cap N^+(w)) = 1$ and $|(N^+(s) \cap N^+(w))|_{\upsilon} > 2k$, then $k^u_1 > 2k$. Thus,

$$t^u_B(s, w) = \frac{k^u_1 - k}{k^u_1} [t^u_A(s) \wedge t^u_A(w)]h^u_1(N^+(s) \cap N^+(w))$$

or,

$$t^u_B(s, w) = \frac{k^u_1 - k}{k^u_1} [t^u_A(s) \wedge t^u_A(w)]$$

$$\frac{t^u_B(s, w)}{[t^u_A(s) \wedge t^u_A(w)]} = \frac{k^u_1 - k}{k^u_1} > 0.5.$$ 

If $h^l_2(N^+(s) \cap N^+(w)) = 1$ and $|(N^+(s) \cap N^+(w))|_{\mu} > 2k$, then $k^l_2 > 2k$. Thus,

$$i^l_B(s, w) = \frac{k^l_2 - k}{k^l_2} [i^l_A(s) \wedge i^l_A(w)]h^l_2(N^+(s) \cap N^+(w))$$

or,

$$i^l_B(s, w) = \frac{k^l_2 - k}{k^l_2} [i^l_A(s) \wedge i^l_A(w)]$$

$$\frac{i^l_B(s, w)}{[i^l_A(s) \wedge i^l_A(w)]} = \frac{k^l_2 - k}{k^l_2} > 0.5.$$ 

If $h^u_2(N^+(s) \cap N^+(w)) = 1$ and $|(N^+(s) \cap N^+(w))|_{\upsilon} > 2k$, then $k^u_2 > 2k$. Thus,

$$i^u_B(s, w) = \frac{k^u_2 - k}{k^u_2} [i^u_A(s) \wedge i^u_A(w)]h^u_2(N^+(s) \cap N^+(w))$$

or,

$$i^u_B(s, w) = \frac{k^u_2 - k}{k^u_2} [i^u_A(s) \wedge i^u_A(w)]$$

$$\frac{i^u_B(s, w)}{[i^u_A(s) \wedge i^u_A(w)]} = \frac{k^u_2 - k}{k^u_2} > 0.5.$$ 

If $h^l_3(N^+(s) \cap N^+(w)) = 1$ and $|(N^+(s) \cap N^+(w))|_{\mu} < 2k$, then $k^l_3 < 2k$. Thus,

$$f^l_B(s, w) = \frac{k^l_3 - k}{k^l_3} [f^l_A(s) \wedge f^l_A(w)]h^l_3(N^+(s) \cap N^+(w))$$

or,

$$f^l_B(s, w) = \frac{k^l_3 - k}{k^l_3} [f^l_A(s) \wedge f^l_A(w)]$$

$$\frac{f^l_B(s, w)}{[f^l_A(s) \wedge f^l_A(w)]} = \frac{k^l_3 - k}{k^l_3} < 0.5.$$
If \( h^u_{3}(N^+(s) \cap N^+(w)) = 1 \) and \( |(N^+(s) \cap N^+(w))|^{f_s} < 2k \), then \( k^u_3 < 2k \). Thus,
\[
f_B^u(s, w) = \frac{k^u_3 - k}{k^u_3} [f_A^u(s) \land f_A^u(w)]h^u_3(N^+(s) \cap N^+(w))
\]
or,
\[
f_B^u(s, w) = \frac{k^u_3 - k}{k^u_3} [f_A^u(s) \land f_A^u(w)]
\]
\[
\frac{f_B^u(s, w)}{[f_A^u(s) \land f_A^u(w)]} = \frac{k^u_3 - k}{k^u_3} < 0.5.
\]
So, the edge \((s, w)\) is independent strong in \(C_k(\overrightarrow{G})\). \(\square\)

**B. \(p\)-competition interval-valued neutrosophic graphs**

We now define another extension of IVNC-graphs, called \(p\)-competition IVN-graphs.

**Definition 2.23.** The support of an IVN-set \(A = (s, [t^a_1, t^a_3], [t^b_1, t^b_3], [f^a_1, f^a_3])\) in \(X\) is the subset of \(X\) defined by
\[
supp(A) = \{s \in X : |t^a_1(s) \neq 0, t^a_3(s) \neq 0, t^b_1(s) \neq 0, t^b_3(s) \neq 0, f^a_1(s) \neq 1, f^a_3(s) \neq 1\}
\]
and \(|supp(A)|\) is the number of elements in the set.

**Example 2.24.** The support of an IVN-set \(A = \{(a, [0.5, 0.7], [0.2, 0.8], [0.1, 0.3]), (b, [0.1, 0.2], [0.1, 0.5], [0.7, 0.9]), (c, [0.3, 0.5], [0.3, 0.8], [0.6, 0.9]), (d, [0, 0], [0, 0], [1, 1])\}\) in \(X = \{a, b, c, d\}\) is \(supp(A) = \{a, b, c\}\) and \(|supp(A)| = 3\).

We now define \(p\)-competition IVN-graphs.

**Definition 2.25.** Let \(p\) be a positive integer. Then \(p\)-competition IVN-graph \(\overrightarrow{C_p}(\overrightarrow{G})\) of the IVN-digraph \(\overrightarrow{G} = (A, \overrightarrow{B})\) is an undirected IVN-graph \(G = (A, B)\) which has same IVN-set of vertices as in \(\overrightarrow{G}\) and has an interval-valued neutrosophic edge between two vertices \(s, w \in X\) in \(\overrightarrow{C_p}(\overrightarrow{G})\) if and only if \(|supp(N^+(s) \cap N^+(w))| \geq p\).

The interval-valued truth-membership value of edge \((s, w)\) in \(\overrightarrow{C_p}(\overrightarrow{G})\) is \(t_B^i(s, w) = \frac{(p-1)i+1}{i}[t^i_1(s) \land t^i_3(w)]h^i_3(N^+(s) \cap N^+(w))\), and \(t_B^i(s, w) = \frac{(p-1)i+1}{i}[t^i_1(s) \land t^i_3(w)]h^i_3(N^+(s) \cap N^+(w))\), the interval-valued indeterminacy-membership value of edge \((s, w)\) in \(\overrightarrow{C_p}(\overrightarrow{G})\) is \(i_B^i(s, w) = \frac{(p-1)i+1}{i}[i^i_1(s) \land i^i_3(w)]h^i_3(N^+(s) \cap N^+(w))\), and \(i_B^i(s, w) = \frac{(p-1)i+1}{i}[i^i_1(s) \land i^i_3(w)]h^i_3(N^+(s) \cap N^+(w))\), the interval-valued falsity-membership value of edge \((s, w)\) in \(\overrightarrow{C_p}(\overrightarrow{G})\) is \(f_B^i(s, w) = \frac{(p-1)i+1}{i}[f^i_1(s) \land f^i_3(w)]h^i_3(N^+(s) \cap N^+(w))\), and \(f_B^i(s, w) = \frac{(p-1)i+1}{i}[f^i_1(s) \land f^i_3(w)]h^i_3(N^+(s) \cap N^+(w))\), where \(i = |supp(N^+(s) \cap N^+(w))|\).

**Example 2.26.** Consider an IVN-digraph \(G = (A, \overrightarrow{B})\) on \(X = \{s, w, a, b, c\}\), such that \(A = \{(s, [0.4, 0.5], [0.5, 0.7], [0.8, 0.9]), (w, [0.6, 0.7], [0.4, 0.6], [0.2, 0.3]), (a, [0.2, 0.6], [0.3, 0.6]), (b, [0.2, 0.6], [0.1, 0.6], [0.2, 0.6]), (c, [0.2, 0.7], [0.3, 0.5], [0.2, 0.6])\}\) and \(B = \{(s, a, [0.1, 0.4], [0.3, 0.6], [0.2, 0.6]), (s, b, [0.2, 0.4], [0.1, 0.5], [0.2, 0.6]), (s, c, [0.2, 0.5], [0.3, 0.5], [0.2, 0.6]), (w, a, [0.2, 0.5], [0.2, 0.5], [0.2, 0.3]), (w, b, [0.2, 0.6], [0.1, 0.6], [0.2, 0.3]), (w, c, [0.2, 0.7], [0.3, 0.5], [0.2, 0.3])\}\), as shown in Fig. 7.
We state the following theorem without its proof.

**Theorem 2.27.** Let $\overrightarrow{G} = (A, \overrightarrow{B})$ be an IVN-digraph. If

\[
\begin{align*}
& h_1^v(N^+(s) \cap N^+(w)) = 1, & h_2^v(N^+(s) \cap N^+(w)) = 1, & h_3^v(N^+(s) \cap N^+(w)) = 0, \\
& h_1^h(N^+(s) \cap N^+(w)) = 1, & h_2^h(N^+(s) \cap N^+(w)) = 1, & h_3^h(N^+(s) \cap N^+(w)) = 0,
\end{align*}
\]

then there exists a path from $s$ to $w$.

We calculate $N^+(s) = \{ (a, [0.1,0.4], [0.3,0.6], [0.2,0.6]), (b, [0.2,0.4], [0.1,0.5], [0.2,0.6]), (c, [0.2,0.5], [0.3,0.5], [0.2,0.6]) \}$ and $N^+(w) = \{ (a, [0.2,0.5], [0.2,0.5], [0.2,0.3]), (b, [0.2,0.6], [0.1,0.6], [0.2,0.3]), (c, [0.2,0.7], [0.3,0.5], [0.2,0.3]) \}$. Therefore, $N^+(s) \cap N^+(w) = \{ (a, [0.1,0.4], [0.2,0.5], [0.2,0.3]), (b, [0.2,0.4], [0.1,0.5], [0.2,0.3]), (c, [0.2,0.5], [0.3,0.5], [0.2,0.3]) \}$. Now, $i = \text{supp}(N^+(s) \cap N^+(w)) = 3$. For $p = 3$, we have, $t_B^i(s, w) = 0.02$, $t_B^w(s, w) = 0.08$, $t_B^h(s, w) = 0.04$, $i_B^h(s, w) = 0.1$, $f_B^i(s, w) = 0.01$ and $f_B^w(s, w) = 0.03$. This graph is depicted in Fig. 8.
We now define another extension of IVNC-graph known as $m$-step IVNC-graph. We will use the following notations:

- $P_{s,w}^m$: An interval-valued neutrosophic path of length $m$ from $s$ to $w$.
- $\overrightarrow{P}_{s,w}^m$: A directed interval-valued neutrosophic path of length $m$ from $s$ to $w$.
- $\mathbb{N}^m_s(s)$: $m$-step interval-valued neutrosophic out-neighbourhood of vertex $s$.
- $\mathbb{N}^m_s(s)$: $m$-step interval-valued neutrosophic in-neighbourhood of vertex $s$.
- $\mathbb{N}_m(G)$: $m$-step interval-valued neutrosophic neighbourhood graph of IVN-graph $G$.
- $\mathbb{G}_m(G)$: $m$-step IVN-graph of the IVN-digraph $\overrightarrow{G}$.

**Definition 2.28.** Suppose $\overrightarrow{G} = (A, \overrightarrow{B})$ is an IVN-digraph. The $m$-step IVN-digraph of $\overrightarrow{G}$ is denoted by $\overrightarrow{G}_m = (A, B)$, where IVN-set of vertices of $\overrightarrow{G}_m$ is same with IVN-set of vertices of $\overrightarrow{G}$ and has an edge between $s$ and $w$ in $\overrightarrow{G}_m$ if and only if there exists an interval-valued neutrosophic directed path $\overrightarrow{P}_{s,w}^m$ in $\overrightarrow{G}$.

**Definition 2.29.** The $m$-step interval-valued neutrosophic out-neighbourhood (IVN-out-neighbourhood) of vertex $s$ in an IVN-digraph $\overrightarrow{G} = (A, \overrightarrow{B})$ is IVN-set

$$\mathbb{N}^+_m(s) = \{X^+_s, [t^+_s, i^+_s], [i^+_s, i^+_s], [f^+_s, f^+_s])\},$$

where $X^+_s = \{w\}$ there exists a directed interval-valued neutrosophic path of length $m$ from $s$ to $w$, $\overrightarrow{P}_{s,w}^m$, $t^+_s(X^+_s) \rightarrow [0, 1]$, $i^+_s(X^+_s) \rightarrow [0, 1]$, $t^+_s(X^+_s) \rightarrow [0, 1]$, $i^+_s(X^+_s) \rightarrow [0, 1]$, $f^+_s(X^+_s) \rightarrow [0, 1]$, $f^+_s(X^+_s) \rightarrow [0, 1]$ are defined by $t^+_s = \min\{t^+\}$, $f^+_s = \min\{f^+\}$, $(s_1, s_2)$ is an edge of $\overrightarrow{P}_{s,w}^m$, $t^+_s = \min\{t^+\}$, $f^+_s = \min\{f^+\}$, $(s_1, s_2)$ is an edge of $\overrightarrow{P}_{s,w}^m$, $t^+_s = \min\{t^+\}$, $f^+_s = \min\{f^+\}$, $(s_1, s_2)$ is an edge of $\overrightarrow{P}_{s,w}^m$, $t^+_s = \min\{t^+\}$, $f^+_s = \min\{f^+\}$, $(s_1, s_2)$ is an edge of $\overrightarrow{P}_{s,w}^m$, respectively.

**Example 2.30.** Consider an IVN-digraph $G = (A, \overrightarrow{B})$ on $X = \{s, w, a, b, c, d\}$, such that $A = \{s, [0.4, 0.5], [0.5, 0.7], [0.8, 0.9], (w, [0.6, 0.7], [0.4, 0.6], [0.2, 0.3]), (a, [0.2, 0.6], [0.3, 0.6], [0.2, 0.6]), (b, [0.2, 0.6], [0.1, 0.6], [0.2, 0.6]), (c, [0.2, 0.7], [0.3, 0.5], [0.2, 0.6]), (d, [0.2, 0.6], [0.3, 0.6], [0.2, 0.6]), (a, c), [0.2, 0.6], [0.3, 0.5], [0.2, 0.6]), ([a, d], [0.2, 0.6], [0.3, 0.5], [0.2, 0.4]), ([a, d], [0.2, 0.6], [0.1, 0.6], [0.2, 0.3]), ([a, d], [0.2, 0.4], [0.1, 0.2], [0.1, 0.3]), ([b, d], [0.1, 0.3], [0.1, 0.2], [0.2, 0.4]), as shown in Fig. 9.
Example 2.32. Consider an IVN-digraph $G = (A, \overrightarrow{B})$ on $X = \{s, w, a, b, c, d\}$, such that $A = \{(s, [0.4, 0.5], [0.5, 0.7], [0.8, 0.9]), (w, [0.6, 0.7], [0.4, 0.6], [0.2, 0.3]), (a, [0.2, 0.6], [0.3, 0.6], [0.2, 0.6]), (b, [0.2, 0.6], [0.1, 0.6], [0.2, 0.6]), (c, [0.2, 0.7], [0.3, 0.5], [0.2, 0.6]), d([0.2, 0.6], [0.3, 0.6], [0.2, 0.6]), \} \), and $B = \{(s, a), (0.1, 0.4), (0.3, 0.6), (0.2, 0.6), (a, c), (0.2, 0.6), (0.3, 0.5), (0.2, 0.6), \} , (b, c), (0.2, 0.4), (0.1, 0.2), (0.1, 0.3), (b, d), (0.1, 0.3), (0.1, 0.2), (0.2, 0.4)) \}$, as shown in Fig. 10.
We calculate 2-step IVN-in-neighbourhoods as, $N^-_s(s) = \{c, [0.1,0.4], [0.3,0.5], [0.2,0.6]\}$, $(d, [0.1,0.4], [0.3,0.5], [0.2,0.4])$ and $N^-_w(w) = \{c, [0.2,0.4], [0.1,0.2], [0.1,0.3]\}$, $(d, [0.1,0.3], [0.1,0.2], [0.2,0.3])$.

**Definition 2.33.** Suppose $\vec{G} = (A, \vec{B})$ is an IVN-digraph. The $m$-step IVNGraph of IVN-digraph $\vec{G}$ is denoted by $C_m(\vec{G}) = (A, B)$ which has same IVN-set of vertices as in $\vec{G}$ and has an edge between two vertices $s, w \in X$ in $C_m(\vec{G})$ if and only if $(N^+_m(s) \cap N^+_m(w))$ is a non-empty IVN-set in $\vec{G}$. The interval-valued truth-membership value of edge $(s, w)$ in $C_m(\vec{G})$ is $t^m_B(s, w) = [t^m_A(s) \land t^m_A(w)]h^m(N^+_m(s) \cap N^+_m(w))$, and $t^m_B(s, w) = [t^m_A(s) \land t^m_A(w)]h^m(N^+_m(s) \cap N^+_m(w))$, the interval-valued indeterminacy-membership value of edge $(s, w)$ in $C_m(\vec{G})$ is $i^m_B(s, w) = [i^m_A(s) \land i^m_A(w)]h^m(N^+_m(s) \cap N^+_m(w))$, and $i^m_B(s, w) = [i^m_A(s) \land i^m_A(w)]h^m(N^+_m(s) \cap N^+_m(w))$, the interval-valued falsity-membership value of edge $(s, w)$ in $C_m(\vec{G})$ is $f^m_B(s, w) = [f^m_A(s) \land f^m_A(w)]h^m(N^+_m(s) \cap N^+_m(w))$, and $f^m_B(s, w) = [f^m_A(s) \land f^m_A(w)]h^m(N^+_m(s) \cap N^+_m(w))$.

The 2-step IVNGraph is illustrated by the following example.

**Example 2.34.** Consider an IVN-digraph $G = (A, \vec{B})$ on $X = \{s, w, a, b, c, d\}$, such that $A = \{(s, [0.4,0.5], [0.5,0.7], [0.8,0.9]), (w, [0.6,0.7], [0.4,0.6], [0.2,0.3]), (a, [0.2,0.6], [0.3,0.6], [0.2,0.6]), (b, [0.2,0.6], [0.1,0.6], [0.2,0.6]), (c, [0.2,0.7], [0.3,0.5], [0.2,0.6]), d([0.2,0.6], [0.3,0.6], [0.2,0.6])\}$, and $B = \{(s, a), [0.1,0.4], [0.3,0.6], [0.2,0.6], (a, c), [0.2,0.6], [0.3,0.5], [0.2,0.6], (a, d), [0.2,0.6], [0.3,0.5], [0.2,0.4], ([w, b], [0.2,0.6], [0.1,0.6], [0.2,0.3]), ([b, c], [0.2,0.4], [0.1,0.2], [0.1,0.3]), ([b, d], [0.1,0.3], [0.1,0.2], [0.2,0.4])\}$, as shown in Fig. 11.
We calculate $N^+_2(s) = \{(c, [0.1, 0.4], [0.3, 0.5], [0.2, 0.6]), (d, [0.1, 0.4], [0.3, 0.5], [0.2, 0.4])\}$ and $N^+_2(w) = \{(c, [0.2, 0.4], [0.1, 0.2], [0.1, 0.3]), (d, [0.1, 0.2], [0.2, 0.6]), (d, [0.1, 0.2], [0.1, 0.2], [0.2, 0.2])\}$. Therefore, $N^+_2(s) \cap N^+_2(w) = \{(c, [0.1, 0.4], [0.1, 0.2], [0.2, 0.6]), (d, [0.1, 0.3], [0.1, 0.2], [0.2, 0.4])\}$. Thus, $t^s_B(s, w) = 0.04$, $t^w_B(s, w) = 0.20$, $i^s_B(s, w) = 0.04$, $i^w_B(s, w) = 0.12$, $f^s_B(s, w) = 0.04$ and $f^w_B(s, w) = 0.12$. This graph is depicted in Fig. 12.

If a predator $s$ attacks one prey $w$, then the linkage is shown by an edge $\overrightarrow{(s, w)}$ in an IVN-digraph. But, if predator needs help of many other mediators $s_1, s_2, \ldots, s_{m-1}$, then linkage among them is shown by interval-valued neutrosophic directed path $\overrightarrow{P}^{m}_{s_1w}$ in an IVN-digraph. So, $m$-step prey in an IVN-digraph is represented by a vertex which is the $m$-step out-neighbourhood of some vertices. Now, the strength of an IVNC-graphs is defined below.

**Definition 2.35.** Let $\overrightarrow{G} = (A, \overrightarrow{B})$ be an IVN-digraph. Let $w$ be a common vertex of $m$-step out-neighbourhoods of vertices $s_1, s_2, \ldots, s_l$. Also, let $\overrightarrow{B}^1(u_1, v_1)$, $\overrightarrow{B}^1(u_2, v_2), \ldots, \overrightarrow{B}^1(u_r, v_r)$ and $\overrightarrow{B}^1(u_1, v_1)$, $\overrightarrow{B}^1(u_2, v_2), \ldots, \overrightarrow{B}^1(u_r, v_r)$ be the minimum interval-valued truth-membership values, $\overrightarrow{B}^2(u_1, v_1)$, $\overrightarrow{B}^2(u_2, v_2), \ldots, \overrightarrow{B}^2(u_r, v_r)$ and $\overrightarrow{B}^2(u_1, v_1)$, $\overrightarrow{B}^2(u_2, v_2), \ldots, \overrightarrow{B}^2(u_r, v_r)$ be the minimum indeterminacy-membership...
Let all preys may not be strong. This can be explained as:

The converse of the above theorem is not true, i.e.,

**Remark 2.38.**

\[ S > (1, u_1, v_1) \] \( \rightarrow \)

**Theorem 2.37.** If a prey \( w \) is strong, then the strength of \( w \),

\[ S(w) = \frac{1}{r} \left\{ \sum_{i=1}^{r} B^1_i(u_i, v_i) + \sum_{i=1}^{r} B^2_i(u_i, v_i) + \sum_{i=1}^{r} B^3_i(u_i, v_i) \right\} \]

**Example 2.36.** Consider an IVN-digraph \( \overrightarrow{G} = (A, \overrightarrow{B}) \) as shown in Fig. 11, the strength of the prey \( c \) is equal to

\[ \frac{(0.2 + 0.2) + (0.6 + 0.4) + (0.1 + 0.1) + (0.6 + 0.2) - (0.2 + 0.1) - (0.3 + 0.3)}{2} = 1.5 > 0.5. \]

Hence, \( c \) is strong 2-step prey.

We state the following theorem without its proof.

**Theorem 2.37.** If a prey \( w \) of \( \overrightarrow{G} = (A, \overrightarrow{B}) \) is strong, then the strength of \( w \), \( S(w) > 0.5 \).

**Remark 2.38.** The converse of the above theorem is not true, i.e., if \( S(w) > 0.5 \), then all preys may not be strong. This can be explained as:

Let \( S(w) > 0.5 \) for a prey \( w \) in \( \overrightarrow{G} \). So,

\[ S(w) = \frac{1}{r} \left\{ \sum_{i=1}^{r} B^1_i(u_i, v_i) + \sum_{i=1}^{r} B^2_i(u_i, v_i) + \sum_{i=1}^{r} B^3_i(u_i, v_i) \right\} \]

Hence,

\[ \left\{ \sum_{i=1}^{r} B^1_i(u_i, v_i) + \sum_{i=1}^{r} B^2_i(u_i, v_i) + \sum_{i=1}^{r} B^3_i(u_i, v_i) \right\} > \frac{r}{2} \]

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This result does not necessarily imply that
\[ \overrightarrow{B}_1^1(u, v) > 0.5, \quad \overrightarrow{B}_2^1(u, v) > 0.5, \quad \overrightarrow{B}_3^1(u, v) < 0.5, \]
\[ \overrightarrow{B}_1^2(u, v) > 0.5, \quad \overrightarrow{B}_2^2(u, v) > 0.5, \quad \overrightarrow{B}_3^2(u, v) < 0.5, \text{ for all } i = 1, 2, \ldots, r. \]

Since, all edges of the directed paths \( \overrightarrow{P}_{s, w}^m, \overrightarrow{P}_{s, w}^{m'}, \ldots, \overrightarrow{P}_{s, w}^{m''} \), are not strong. So, the converse of the above statement is not true i.e., if \( S(w) > 0.5 \), the prey \( w \) of \( \overrightarrow{G} \) may not be strong. Now, \( m \)-step interval-valued neutrosophic neighbourhood graphs are defined below.

**Definition 2.39.** The \( m \)-step IVN-out-neighbourhood of vertex \( s \) of an IVN-digraph \( \overrightarrow{G} = (A, \overrightarrow{B}) \) is IVN-set

\[ N_m(s) = (X_s, \{t^i_s, t^u_s\}, \{i^i_s, i^u_s\}, \{f^i_s, f^u_s\}), \]

where \( X_s = \{ w \mid \text{there exists a directed interval-valued neutrosophic path of length } m \text{ from } s \text{ to } w, \} \). \( \overrightarrow{P}_{s, w}^m \), \( t^i_s : X_s \to [0, 1], \) \( t^u_s : X_s \to [0, 1], \) \( i^i_s : X_s \to [0, 1], \) \( i^u_s : X_s \to [0, 1], \) \( f^i_s : X_s \to [0, 1], \) \( f^u_s : X_s \to [0, 1], \) are defined by \( t^i_s = \min\{t^i(s_1, s_2), (s_1, s_2) \text{ is an edge of } \overrightarrow{P}_{s, w}^m \}, t^u_s = \min\{t^u(s_1, s_2), (s_1, s_2) \text{ is an edge of } \overrightarrow{P}_{s, w}^m \}, i^i_s = \min\{i^i(s_1, s_2), (s_1, s_2) \text{ is an edge of } \overrightarrow{P}_{s, w}^m \}, i^u_s = \min\{i^u(s_1, s_2), (s_1, s_2) \text{ is an edge of } \overrightarrow{P}_{s, w}^m \}, f^i_s = \min\{f^i(s_1, s_2), (s_1, s_2) \text{ is an edge of } \overrightarrow{P}_{s, w}^m \}, f^u_s = \min\{f^u(s_1, s_2), (s_1, s_2) \text{ is an edge of } \overrightarrow{P}_{s, w}^m \}, \) respectively.

**Definition 2.40.** Suppose \( G = (A, B) \) is an IVN-graph. Then \( m \)-step interval-valued neutrosophic neighbourhood graph \( \overrightarrow{N}_m(G) \) is defined by \( \overrightarrow{N}_m(G) = (\overrightarrow{A}, \overrightarrow{B}) \), where \( A = ([A^1_1, A^u_1], [A^2_1, A^u_2], [A^3_1, A^u_3]), B = ([\overrightarrow{A}^1_1, \overrightarrow{A}^u_1], [\overrightarrow{A}^2_1, \overrightarrow{A}^u_2], [\overrightarrow{B}^1_1, \overrightarrow{B}^u_1]), \)

\( \overrightarrow{B}^1_1 : X \times X \to [0, 1], \overrightarrow{B}^u_1 : X \times X \to [0, 1], \overrightarrow{B}^2_1 : X \times X \to [0, 1], \overrightarrow{B}^u_2 : X \times X \to [0, 1], \overrightarrow{B}^3_1 : X \times X \to [0, 1], \) and \( \overrightarrow{B}^u_3 : X \times X \to [0, 1] \) are such that:

\[ \overrightarrow{B}^1_1(s, w) = A^1_1(s) \land A^1_1(w)h^1(N_m(s) \cap N_m(w)), \]
\[ \overrightarrow{B}^2_1(s, w) = A^2_1(s) \land A^2_1(w)h^2(N_m(s) \cap N_m(w)), \]
\[ \overrightarrow{B}^3_1(s, w) = A^3_1(s) \land A^3_1(w)h^3(N_m(s) \cap N_m(w)), \]
\[ \overrightarrow{B}^1_2(s, w) = A^1_2(s) \land A^1_2(w)h^1(N_m(s) \cap N_m(w)), \]
\[ \overrightarrow{B}^2_2(s, w) = A^2_2(s) \land A^2_2(w)h^2(N_m(s) \cap N_m(w)), \]
\[ \overrightarrow{B}^3_2(s, w) = A^3_2(s) \land A^3_2(w)h^3(N_m(s) \cap N_m(w)), \]

We state the following theorems without their proofs.

**Theorem 2.41.** If all preys of \( \overrightarrow{G} = (A, \overrightarrow{B}) \) are strong, then all edges of \( \overrightarrow{C}_m(\overrightarrow{G}) = (A, \overrightarrow{B}) \) are strong.

A relation is established between \( m \)-step IVNC-graph of an IVN-digraph and IVN-digraph of \( m \)-step IVN-digraph.

**Theorem 2.42.** If \( \overrightarrow{G} \) is an IVN-digraph and \( \overrightarrow{G}_m \) is the \( m \)-step IVN-digraph of \( \overrightarrow{G} \), then \( \overrightarrow{C}(\overrightarrow{G}_m) = \overrightarrow{C}(\overrightarrow{G}) \).

**Theorem 2.43.** Let \( \overrightarrow{G} = (A, \overrightarrow{B}) \) be an IVN-digraph. If \( m > \vert X \vert \) then \( \overrightarrow{C}_m(\overrightarrow{G}) = (A, \overrightarrow{B}) \) has no edge.
Theorem 2.44. If all the edges of IVN-digraph $\tilde{G} = (\tilde{A}, \tilde{B})$ are independent strong, then all the edges of $C_m(\tilde{G})$ are independent strong.

3. Conclusions

Graph theory is an enjoyable playground for the research of proof techniques in discrete mathematics. There are many applications of graph theory in different fields. We have introduced IVNC-graphs and $k$-competition IVN-graphs, $p$-competition IVN-graphs and $m$-step IVNC-graphs as the generalized structures of IVNC-graphs. We have described interval-valued neutrosophic open and closed-neighbourhood. Also we have established some results related to them. We aim to extend our research work to (1) Interval-valued fuzzy rough graphs; (2) Interval-valued fuzzy rough hypergraphs, (3) Interval-valued fuzzy rough neutrosophic graphs, and (4) Decision support systems based on IVN-graphs.

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