Sequential Adaptive Combination of Unreliable Sources of Evidence

Zhun-ga Liu
Quan Pan
Yong-mei Cheng
Jean Dezert

Originally published as Liu Z., Pan Q., Cheng Y., Dezert J., *Sequential adaptive combination of unreliable sources of evidence*, In proc. of International Workshop on Belief Functions, Brest, France, April 2-4, 2010, and reprinted with permission.

***Abstract*-In theories of evidence, several methods have been proposed to combine a group of basic belief assignments altogether at a given time. However, in some applications in defense or in robotics the evidences from different sources are acquired only sequentially and must be processed in real-time and the combination result needs to be updated the most recent information. An approach for combining sequentially unreliable sources of evidence is presented in this paper. The sources of evidence are not considered as equi-reliable in the combination process, and no prior knowledge on their reliability is required. The reliability of each source is evaluated on the fly by a distance measure, which characterizes the variation between one source of evidence with respect to the others. If the source is considered as unreliable, then its evidence is discounted before entering in the fusion process. Dempster's rule of combination and its main alternatives including Yager's rule, Dubois and Prade rule, and PCR5 are adapted to work under different conditions. In this paper, we propose to select the most adapted combination rule according to the value of conflicting belief before combining the evidence. The last part of this paper is devoted to a numerical example to illustrate the interest of this approach.**

Keywords: evidence theory, combination rule, evidence distance, conflicting belief.

I. Introduction

Evidence theories[1] are widely applied in the field of information fusion. A particular attention has been focused on how to efficiently combine sources of evidence altogether at the same time (static approach), and many rules aside Dempster's rule have been proposed [1], [2], [6], [9]. In many applications however, the evidences from different sources are acquired sequentially by different sensors or human experts and the belief updating and decision-making need to be taken in real-time which requires a sequential/dynamic approach rather than a static approach of the fusion problem.

Usually the evidences arising from different independent sources are often considered equally reliable in the combination process, when the prior knowledge about the reliability of each source is unknown. However, all the sources of evidence to be combined can have different reliabilities in real applications. If the sources of evidence are considered as equi-reliable, the unreliable ones may bring a very bad influence on combination result, and even leads to inconsistent results and wrong decisions. Thus, the reliability of each source must be taken account in the fusion process as best as possible to provide a useful and unbiased result. In this work, we propose to evaluate on the fly the relability of the sources to combine based on an evidential distance/reliability measure. From this reliability measure, one can discount accordingly the unreliable sources before applying a rule of combination of basic belief assignments (bba's).

Many rules, like Dempster's rule [7] and its alternatives can be used to combine sources of evidences expressed by bba's and they all have their drawbacks and advantages (see [8], Vol. 1, for a detailed presentation). Dempster's rule, is usually considered well adapted for combining the evidences in low conflict situations and it requires acceptable complexity when the dimension of the frame of discernment is not too large. Dempster's rule however provides counter-intuitive behaviors when the sources evidences become highly conflicting. To palliate this drawback, several interesting alternatives have been proposed when Dempster's rule doesn't work well, mainly: Yager's rule [9], Dubois and Prade rule (DP rule) [2], and PCR5 (proportional conflict redistribution rule no 5) [8] developed in DSmT framework. The difference among Dempster's rule and its main alternatives mainly lies in the distribution of the conflicting belief $m_{\oplus}(\emptyset)$ which is generally used to characterize the total amount of conflict [4] between sources. In this paper, we propose to select the proper rule of combination based on the value of the total degree of conflict $m_{\oplus}(\emptyset)$. The last part of this paper presents a numerical example to show how the approach of sequential adaptive combination of unreliable sources of evidence works.

II. Preliminaries

A. Basics of Dempster-Shafer theory (DST)

DST [7] is developed in Shafer's model. In this model, a fixed set $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$ is called the frame of discernment of fusion problem. All the elements in Θ are mutually exclusive and exhaustive. The set of all subsets of

[1] DST (Dempster-Shafer Theory) [7] or DSmT (Dezert-Smarandache Theory) [8].

Θ is called the power set of Θ, and it is denoted 2^Θ. For instance, if $\Theta = \{\theta_1, \theta_2, \theta_3\}$, then $2^\Theta = \{\emptyset, \theta_1, \theta_2, \theta_3, \theta_1 \cup \theta_2, \theta_1 \cup \theta_3, \theta_2 \cup \theta_3, \theta_1 \cup \theta_2 \cup \theta_3\}$. A basic belief assignment (bba), also called mass of belief, is a mapping $m : 2^\Theta \to [0,1]$ associated to a given body of evidence $\mathcal{B}$ such that $m(\emptyset) = 0$ and $\sum_{A \in 2^\Theta} m(A) = 1$. The credibility (also called belief) of $A \subseteq \Theta$ is defined by $Bel(A) = \sum_{B \in 2^\Theta, B \subseteq A} m(B)$. The commonality function $q(.)$ and the plausibility function $Pl(.)$ are also defined by Shafer in [7]. The functions $m(.)$, $Bel(.)$, $q(.)$ and $Pl(.)$ are in one-to-one correspondence.

Let $m_1(.)$ and $m_2(.)$ be two bba's provided by two independent bodies of evidence $\mathcal{B}_1$ and $\mathcal{B}_2$ over the frame of discernment Θ. The fusion/combination of $m_1(.)$ with $m_2(.)$, denoted $m(.) = [m_1 \oplus m_2](.)$ is obtained in DST with Dempster's rule of combination as follows:

$$\begin{cases} m(\emptyset) = 0 \\ m(A) = \frac{\sum\limits_{X_1 \cap X_2 = A} m_1(X_1) m_2(X_2)}{\sum\limits_{X_1 \cap X_2 \neq \emptyset} m_1(X_1) m_2(X_2)} \quad \forall A \neq \emptyset, A \in 2^\Theta \end{cases} \tag{1}$$

The degree of conflict between the bodies of evidence $\mathcal{B}_1$ and $\mathcal{B}_2$ is defined by

$$m_\oplus(\emptyset) = \sum_{\substack{X_1, X_2 \in 2^\Theta \\ X_1 \cap X_2 = \emptyset}} m_1(X_1) m_2(X_2) \tag{2}$$

Dempster's rule can be directly extended to the combination of S independent and equally reliable sources. It is a commutative and associative rule of combination and it preserves the neutral impact of the vacuous belief assignment defined by $m_{vba}(\Theta) = 1$.

B. Main alternatives to Dempster's rule

Dempster's rule yields counterintuitive results when the evidences highly conflict because of its way of assigning the mass of conflicting belief $m_\oplus(\emptyset)$. Thus, a lot of alternatives to Dempster's rule have been proposed for overcoming limitations of Dempster's rule. The main alternative rules including Yager's rule [9], DP rule [2] and PCR5 [8] are briefly recalled.

- **Yager's rule**: Yager admits the conflicting belief is not reliable. So $m_\oplus(\emptyset)$ is transferred to the total ignorance in Yager's rule. It is given by $m(\emptyset) = 0$ and for $A \neq \emptyset$, $A \in 2^\Theta$ by

$$\begin{cases} m(A) = \sum_{\substack{X, Y \in 2^\Theta \\ X \cap Y = A}} m_1(X) m_2(Y), \text{ for } A \neq \Theta \\ m(\Theta) = m_1(\Theta) m_2(\Theta) + \sum_{\substack{X, Y \in 2^\Theta \\ X \cap Y = \emptyset}} m_1(X) m_2(Y) \end{cases} \tag{3}$$

- **Dubois & Prade rule**: This rule assumes that if two sources of evidence are in conflict, one of them is right but we don't know which one. Thus, if $X \cap Y = \emptyset$, then the mass committed to the set $X \cap Y$ by the conjunctive operator should be transferred to $X \cup Y$. According to this principle, DP rule is defined by $m(\emptyset) = 0$ and for $A \neq \emptyset$ and $A \in 2^\Theta$ by

$$m(A) = \sum_{\substack{X, Y \in 2^\Theta \\ X \cap Y = A}} m_1(X) m_2(Y) + \sum_{\substack{X, Y \in 2^\Theta \\ X \cap Y = \emptyset \\ X \cup Y = A}} m_1(X) m_2(Y) \tag{4}$$

- **PCR5 rule**: PCR5 transfers the partial conflicting mass to the elements involved in the conflict, and it is considered as the most mathematically exact redistribution of conflicting mass to nonempty sets following the logic of the conjunctive rule. PCR5 is defined by $m(\emptyset) = 0$ and $\forall A \neq \emptyset$, $A \in 2^\Theta$ by

$$m(A) = \sum_{\substack{X_1, X_2 \in 2^\Theta \\ X_1 \cap X_2 = A}} m_1(X_1) m_2(X_2) + \sum_{\substack{X_2 \in 2^\Theta \\ X_2 \cap A = \emptyset}} [\frac{m_1(A)^2 m_2(X_2)}{m_1(A) + m_2(X_2)} + \frac{m_2(A)^2 m_1(X_2)}{m_2(A) + m_1(X_2)}] \tag{5}$$

The details, examples and the extension of PCR5 formula (5) for $S > 2$ sources are given in [8].

C. Discounting source of evidence

When the sources of evidences are not considered equally reliable, it is reasonable to discount each unreliable source s_i, $i = 1, 2, \ldots, S$ by a reliability factor $\alpha_i \in [0,1]$. Following the classical discounting method [7], a new discounted bba $m'(.)$ is obtained from the initial bba $m(.)$ provided by the unreliable source s_i as follows

$$\begin{cases} m'(A) = \alpha_i \cdot m(A), \quad A \neq \Theta \\ m'(\Theta) = 1 - \sum_{\substack{A \in 2^\Theta \\ A \neq \Theta}} m'(A) \end{cases} \tag{6}$$

$\alpha_i = 1$ means that the total confidence in the source s_i, and the original bba doesn't need to be discounted. $\alpha_i = 0$ means that the source is s_i is totally unreliable and its bba is revised as a vacouous bba $m'(\Theta) = 1$, which will have a neutral impact in the fusion process. In practice, the discounting method can be used efficiently if one has a good estimation of the reliability factor of each source. We show in the next section how one can evaluate the relability of a source.

III. Evaluating the reliability of each source

Without prior knowledge on the reliability of the sources of evidence, we propose to evaluate the reliability factors of each source based on the distance between the bba from a given source s_i with respect to the others. If the bba of the given source, say s_i varies too much with respect to the others, this source of evidence is considered not reliable and it will be discounted before to be combined. We will show further how the discounting/reliability factor can be estimated. We implicitly assume here that the following principle "Truth is reflected by the majority of opinions" holds.

In [3], Jousselme et al. have proposed the following distance measure $d_J(\mathbf{m}_1, \mathbf{m}_2)$ between two bba's[2] $\mathbf{m}_1 \triangleq m_1(.)$ and $\mathbf{m}_2 \triangleq m_2(.)$ defined on the same power set 2^Θ:

$$d_J(\mathbf{m}_1, \mathbf{m}_2) = \sqrt{\frac{1}{2}(\mathbf{m}_1 - \mathbf{m}_2)^T \mathbf{D}(\mathbf{m}_1 - \mathbf{m}_2)} \quad (7)$$

where $\mathbf{D}$ is a $2^{|\Theta|} \times 2^{|\Theta|}$ positive matrix whose elements are defined as $D_{ij} \triangleq \frac{|A_i \cap B_j|}{|A_i \cup B_j|}$ where A_i and B_j are elements of the power set 2^Θ. $d_J(\mathbf{m}_1, \mathbf{m}_2) \in [0, 1]$ is a distance which measures the similarity between $\mathbf{m}_1$ and $\mathbf{m}_2$ considering both the values and the relative specificity of focal elements of each bba.

The total degree of conflict $m_\oplus(\emptyset)$ obtained from all focal elements which are incompatible doesn't actually capture the similarity between bba's as shown by Martin et al. in [5].

If N pieces of evidence $\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_N$ are combined sequentially, two approaches similarly with [5] could be used to measure the variation between $\mathbf{m}_j$ and the others. One considers the average value d_J between $\mathbf{m}_j$ and the others which is given by

$$d_1^{j-1}(\mathbf{m}_j) = \frac{1}{j-1} \sum_{i=1}^{j-1} d_J(\mathbf{m}_j, \mathbf{m}_{j-i}) \quad (8)$$

The other one is simply defined as

$$d_2^{j-1}(\mathbf{m}_j) = d_J(\mathbf{m}_j, \mathbf{m}_1^{j-1}) \quad (9)$$

where $\mathbf{m}_1^{j-1} \triangleq m_1^{j-1}(.)$ is obtained by the sequential combinination of the bba's $m_1(.)$, $m_2(.)$,..., $m_{j-1}(.)$, i.e. $m_1^{j-1}(.) = (((m_1 \oplus m_2) \oplus m_3) \cdots \oplus m_{j-1})(.)$ with a fusion rule such as Dempster's rule, Yager's rule, DP rule, PCR5, etc. The second measure, $d_2^{j-1}(\mathbf{m}_j)$, reflects only the difference between $\mathbf{m}_j$ and the combined bba $\mathbf{m}_1^{j-1}$ and thus cannot precisely measure the similarity between m_j and the other individual evidences $m_1(.)$, $m_2(.)$, ..., $m_{j-1}(.)$ because some information on specificities of these individual bba's has been lost forever through the fusion process. The following examples will show the distinction between these two methods.

Example 1: Let's consider the frame of discernment $\Theta = \{A, B, C\}$, Shafer's model and the same following bba's

$$\begin{aligned}
m_1(.): \quad & m_1(A) = 0.5, m_1(B) = 0.2 \\
& m_1(A \cup B) = m_1(C) = m_1(\Theta) = 0.1 \\
m_2(.): \quad & m_2(A) = 0.5, m_2(B) = 0.2 \\
& m_2(A \cup B) = m_2(C) = m_2(\Theta) = 0.1 \\
& \vdots \\
m_j(.): \quad & m_j(A) = 0.5, m_j(B) = 0.2 \\
& m_j(A \cup B) = m_j(C) = m_j(\Theta) = 0.1
\end{aligned}$$

The difference between $m_j(.)$, for $j \geq 2$, and all the bba's $m_i(.)$, for $i < j$ according to formula (8) gives $d_1^{j-1}(\mathbf{m}_j) = 0$, which shows correctly that $m_j(.)$ is identical to the other bba's $m_i(.)$, for $i < j$. If one uses the measure $d_2^{j-1}(\mathbf{m}_j)$ defined in (9), one gets the results plotted in Fig. 1.

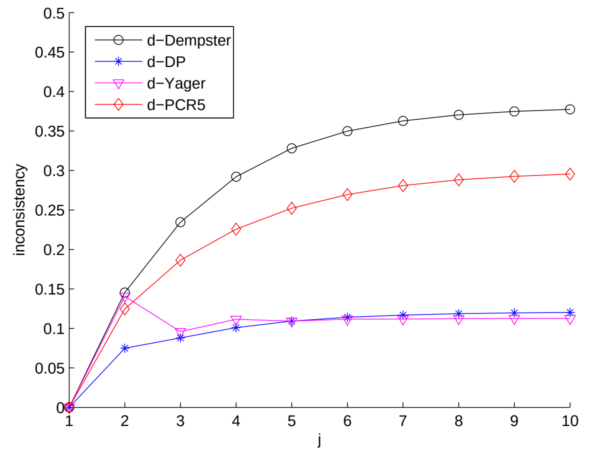

Fig. 1: Variation of the similarity measure $d_2^{j-1}(\mathbf{m}_j)$ based on different fusion rules.

One sees that there exists a variation of the similarity measure using all different fusion rules with a trend to certain values when j increases. This is a bad behavior since we know that $\mathbf{m}_j$ equals to the others bba's and we would expect to get $d_2^{j-1}(\mathbf{m}_j) = 0$ which unfortunately is not the case. That is the main reason why we abondon the use of $d_2^{j-1}(\mathbf{m}_j)$ measure in the sequel of this work.

Example 2: Let's consider the frame of discernment $\Theta = \{A, B, C, D, E\}$, Shafer's model, and the following bba's

$$\begin{aligned}
m_1(.): \quad & m_1(A) = 0.6, m_1(B) = m_1(C) = 0.1 \\
& m_1(D) = m_1(E) = 0.1 \\
m_2(.): \quad & m_2(\Theta) = 1 \\
m_3(.): \quad & m_3(\Theta) = 1 \\
& \vdots \\
m_{j-1}(.): \quad & m_{j-1}(\Theta) = 1 \\
m_j(.): \quad & m_j(A) = 0.6, m_j(B) = m_j(C) = 1, \\
& m_j(D) = m_j(E) = 0.1
\end{aligned}$$

In this example, $m_j(.) = m_1(.)$, but $m_j(.)$ is quite different from the others bba's $m_i(.)$, $i \neq 1$. The similarity measure $d_1^{j-1}(\mathbf{m}_j)$ between $m_j(.)$, for $j \geq 3$ and the bba's $m_i(.)$, $i < j$ is $d_1^{j-1}(\mathbf{m}_j) = \frac{0.2(j-2)}{j-1}$ which shows a trend to 0.2 when j increases. However in such case, one always gets using Dempster's rule, Yager's rule, DP rule or PCR5 rule $d_2^{j-1}(\mathbf{m}_j) = 0$. From such very simple example, one sees that one cannot detect the dissimilarity of $\mathbf{m}_j(.)$ with a majority of quite distinct bba's when $d_2^{j-1}(\mathbf{m}_j)$ measure is used. This shows again that $d_2^{j-1}(\mathbf{m}_j)$ is actually not very appropriate for measuring the similarity between a given bba $\mathbf{m}_j(.)$ and a set of bba's. Therefore we will only consider the measure of similarity $d_1^{j-1}(\mathbf{m}_j)$ in the sequel.

[2]Here for notation convenience, we use the usual vectorial notation $\mathbf{m}_1$ and $\mathbf{m}_2$ (with boldfaced letter) for representing the entire bba's usually denoted $m_1(.)$ and $m_2(.)$. $\mathbf{m}_1$ and $\mathbf{m}_2$ are vectors of dimension $2^{|\Theta|} \times 1$. We assume that the bba's vectors are both ordered using the same order for their components.

For managing the computational burden in applications, a parameter $n \leq j-1$ is introduced in the measure $d_1^{j-1}(\mathbf{m}_j)$ and we define the new measure:

$$d_n^{j-1}(\mathbf{m}_j) = \frac{1}{\min(n, j-1)} \sum_{i=1}^{\min(n,j-1)} d_J(\mathbf{m}_j, \mathbf{m}_{j-i}) \quad (10)$$

The accuracy and the computational complexity of this similarity measure increases when n tends to $j-1$.

Let 's consider a given discounting tolerance threshold ω_d in $[0,1]$. If $d_n^{j-1}(\mathbf{m}_j) \geq \omega_d$, it indicates that the bba $m_j(.)$ will be considered as not similar enough with respect to other bba's and therefore the source s_j is considered as unreliable and must be discounted before entering in the fusion process. The unreliability of the source s_j may be caused by a fault of the sensor or unexpected noises, condition changes, etc. In such case, the bba $m_j(.)$ needs to be discounted by formula (6). As proposed by Martin et al. in [5], the reliability factor of the source s_j is chosen as $\alpha_j = (1 - d_n^{j-1}(\mathbf{m}_j)^{\lambda})^{1/\lambda}$ where the parameter λ is defined in the easiest way with $\lambda = 1$. The larger dissimilarity leads to the less reliability factor. If $d_n^{j-1}(\mathbf{m}_j) < w_d$, it means that the dissimilarity between $m_j(.)$ with other bba's is acceptable, and there is no need to revise/discount $m_j(.)$ in such case.

IV. Selection of combination rules

After evaluating the reliability of the sources, we have to select a suitable combination rule. Dempster's rule is known to offer pretty good performances when the combined bba's are not in too high conflict, otherwise when the conflict becomes too large it is generally considered safer to use alternative rules like Yager's rule, DP rule, and PCR5 rule. The following examples show the difference between the different approaches for the fusion of sources of evidences.

Example 3: This is Zadeh's example [10]. Let's consider $\Theta = \{A, B, C\}$ with Shafer's model and the following bba's

$$\begin{aligned} m_1(.) &: \quad m_1(A) = 0.9, m_1(B) = 0.1 \\ m_2(.) &: \quad m_2(B) = 0.1, m_2(C) = 0.9 \end{aligned}$$

One sees that the two sources are in very high conflict because the total conflict is $m_{\oplus}^{1,2}(\emptyset) = 0.99$. Using Dempster's rule, one gets surprisingly $m(B) = 1$ which is somehow conterintuitive since m_1 and m_2 both believe in B with a little chance, but the fusion result states that B is the only possible solution with certainty, which seems unreasonable[3]. If we use Yager's rule, DP rule, and PCR5, one gets:

- Yager's rule: $m(B) = 0.01$, $m(\Theta) = 0.99$
- DP rule: $m(B) = 0.01$, $m(A \cup B) = 0.09$, $m(A \cup C) = 0.81$, $m(B \cup C) = 0.09$
- PCR5: $m(A) = 0.486$, $m(B) = 0.028$, $m(C) = 0.4860$

These results are more reasonable in some sense, but they are not the same. Yager's rule transfers all the conflicting mass to total ignorance and produces the least specific result in the three rules. DP rule distributes the conflicting mass to the union of the involved sets, which makes the uncertainty of the result still very large. DP rule produces a less specific result than PCR5 but DP is a bit more specific than Yager's rule. PCR5 provides the most specific result since A and C share the same bba whereas B keeps a very low belief assignment.

Therefore, in order to avoid to get counterintuitive results, it is reasonable to use Yager's rule, DP rule, or PCR5 than Dempster's rule as soon as the level of conflict becomes large. The choice among Yager's rule, DP rule, and PCR5 depends on the application and the computational resource one has. PCR5 is very appropriate if a decision has to be made because it provides the most specific solution, but PCR5 requires the most computational burden. Sometimes it better to get less specific result if we don't need to take a clear/precise decision in case of high conflict between sources. In such case, Yager's rule and/or DP rule can be used instead. When the level of conflict between two bba's is low Dempster's rule can be used since it offers a good compromise between computational complexity and the specificity of the result.

Example 4: Let's consider $\Theta = \{A, B, C\}$ and

$$\begin{aligned} m_1(.) &: \quad m_1(A) = 0.35, m_1(B) = 0.3, m_1(A \cup B) = 0.15, \\ &\quad\; m_1(C) = 0.2 \\ m_2(.) &: \quad m_2(A) = 0.35, m_2(B) = 0.3, m_2(A \cup C) = 0.05, \\ &\quad\; m_2(A \cup B) = m_2(C) = m_2(\Theta) = 0.1 \\ m_3(.) &: \quad m_3(A) = 0.3, m_3(B) = 0.3, m_3(A \cup B) = 0.2, \\ &\quad\; m_3(C) = m_3(A \cup C) = 0.1 \end{aligned}$$

The conflicts between each pair of bba's are given by $m_{\oplus}^{1,2}(\emptyset) = 0.455, m_{\oplus}^{1,3}(\emptyset) = 0.395, m_{\oplus}^{2,3}(\emptyset) = 0.395$. The levels of these conflicts are not too large according and the sequential combination $m(.) = [[m_1 \oplus m_2] \oplus m_3](.)$ using the different rules yields

- Dempster's rule: $m(A) = 0.5868$, $m(B) = 0.3592$, $m(A \cup B) = 0.0202$, $m(C) = 0.0338$
- Yager's rule: $m(A) = 0.3105$, $m(B) = 0.243$, $m(C) = 0.0555$, $m(A \cup B) = 0.097$, $m(A \cup C) = 0.0455$, $m(\Theta) = 0.2485$
- DP rule: $m(A) = 0.3255$, $m(B) = 0.2295$, $m(C) = 0.0435$, $m(A \cup B) = 0.1975$, $m(A \cup C) = 0.0575$, $m(B \cup C) = 0.0345$, $m(\Theta) = 0.112$
- PCR5: $m(A) = 0.4889$, $m(B) = 0.3941$, $m(C) = 0.0819$, $m(A \cup B) = 0.0268$, $m(A \cup C) = 0.0083$

All the rules provide reasonable results with assigning the largest belief to A, but Dempster's rule produces the most specific result with a less computational effort. Dempster's rule is thus well appropriate when $m_{\oplus}(\emptyset)$ is not too large.

V. Adaptive combination of sequential evidence

Here we are concerned with the real-time decision-making problem from the sequential acquisition of bba's $m_1(.)$, $m_2(.)$,..., $m_N(.)$ defined on a same frame Θ without any

[3]More generally, one can show that Dempster's rule can become insensitive to the variation of input bba's to combine - see [8], Vol. 1, Chap. 5, p. 114 for example.

prior knowledge about reliability of each source. We start with $m_1(.)$. When $m_2(.)$ is available, one combines it with $m_1(.)$ by a suitable rule according to the value of $m_{\oplus}^{1,2}(\emptyset)$ without evaluating the reliability of the two sources. When $m_j(.)$, for $j \geq 3$ becomes available at the time j, the reliability of the source s_j is evaluated and $m_j(.)$ is discounted (if necessary) by the approach presented in section III. Before combining the discounted bba $m'_j(.)$ (or $m_j(.)$ when no discounting occurs) with the last updated bba $m_1^{j-1}(.)$, the combination rule is selected according to the value of the conflict between $m_j(.)$ and $m_1^{j-1}(.)$. We use a threshold $\omega_{\emptyset}$. If $m_{\oplus}(\emptyset) < \omega_{\emptyset}$, Dempster's rule is selected because it offers a good compromise between complexity and specificity. Otherwise, Yager's rule, DP rule, or PCR5, are selected upon the actual application to avoid to get counterintuitive results.

The tuning of thresholds ω_d and $\omega_{\emptyset}$ is not easy in general. If the thresholds are too large, one takes the risk to get counterintuitive results, whereas if they are set to too low values the non specificity of the result will become large and even will lead to decision-making under big uncertainty. Therefore, both thresholds ω_d and $\omega_{\emptyset}$ need to be determined by accumulated experience depending on the actual application .

VI. Numerical example

Let us suppose a multisensor-based target identification system. From five independent sensors, the system collects five pieces of evidence sequentially (actually we consider here 2 possible bba's $m_{5A}(.)$ and $m_{5B}(.)$ for the fifth source). For decision-making in real-time, the combination result needs to be updated right after the new evidence arrives. The bba's defined on the power set of $\Theta = \{A, B, C\}$ are as follows

$$\begin{aligned}
m_1(.): \quad & m_1(A) = 0.8, m_1(B) = 0.1, m_1(\Theta) = 0.1 \\
m_2(.): \quad & m_2(A) = 0.4, m_2(B) = 0.25, \\
& m_2(C) = 0.2, m_2(B \cup C) = 0.15, \\
m_3(.): \quad & m_3(B) = 0.9, m_3(C) = 0.1, \\
m_4(.): \quad & m_4(B) = 0.45, m_4(C) = 0.45, m_4(B \cup C) = 0.1, \\
m_{5A}(.): \quad & m_{5A}(A) = 0.5, m_{5A}(A \cup B) = 0.25, \\
& m_{5A}(C) = 0.1, m_{5A}(A \cup C) = 0.15, \\
m_{5B}(.): \quad & m_{5B}(B) = 0.5, m_{5B}(A \cup B) = 0.25, \\
& m_{5B}(C) = 0.1, m_{5B}(B \cup C) = 0.15.
\end{aligned}$$

The five pieces of evidence are combined sequentially, and the results are presented in Table 1. The chosen thresholds are $\omega_d = 0.6, \omega_{\emptyset} = 0.6$ and $n = 5$.

All the rules provide reasonable results when combining consistent bba's $m_1(.)$ and $m_2(.)$. The bba $m_3(.)$ is highly conflicting with $m_1(.)$ and $m_2(.)$. If there is no prior information about the reliability of the sources, we evaluate the reliability of each source according to its variation with respect to the others. The average similarity distance between $\mathbf{m}_3$ and $\mathbf{m}_1$, $\mathbf{m}_2$ is so large that $d_n^{j-1}(\mathbf{m}_3) > \omega_d$. Thus, $m_3(.)$ is considered unreliable. If we combine directly (without discounting) $m_3(.)$ with $m_1^2(.)$ using Dempster, Yager, DP or PCR5, one gets a high belief in B with all the rules. With the adaptive rule, the bba of $m_3(.)$ is discounted with the reliability factor $\alpha = 1 - d_n^{j-1}(\mathbf{m}_3)$ to get $m'_3(.)$. The combination of $m'_3(.)$ with $m_1^2(.)$ assigns now the highest belief in A. This adaptive method is helpful to deal with the high conflicts caused by the unreliability of the sources. The difference between $m_4(.)$ and $m_1(.)$, $m_2(.)$, $m_3(.)$ is below the tolerance threshold, but the value of $m_{\oplus}(\emptyset)$ between $m_4(.)$ and $m_1^3(.)$ is very large, and $m_{\oplus}(\emptyset) = 0.8334 > \omega_{\emptyset}$. The result of Dempster's rule indicates that the most credible hypothesis is B, whereas A is not possible to happen, which is not reasonable. The results produced by Yager's rule and DP rule selected in adaptive rule is full of uncertainty, and we even can't make a clear decision from them because of their ways of distributing the mass of conflicting belief. We can get the specific output that most belief focuses on hypothesis A only if PCR5 is selected in the adaptive rule. As we can see, $m_1(.)$ and $m_2(.)$ strongly support the hypothesis A, whereas $m_3(.)$ and $m_4(.)$ strongly support B. It is not easy to be sure what is the true hypothesis. The adaptive rule tends to preserve the earlier decision, since it assumes that $m_1(.)$ and $m_2(.)$ where totally reliable, and then $m_3(.)$ is considered unreliable and thus discounted. When $m_5(.)$ is available, if $m_5(.)$ strongly supports A as with $m_{5A}(.)$, the combination results of all the adaptive rules commit their highest belief in A. If $m_5(.)$ strongly supports B as with $m_{5B}(.)$, the combination results will change and assign the highest belief in B. The results produced by the adaptive rule with selecting combination rules between Dempster and PCR5 are always most specific, which is very useful and helpful for decision-making in real-time. The good performance of adaptive rules lies in the method of evaluating the reliability of sources and the way for automatically selecting suitable combination rules.

VII. Conclusions

An approach for adaptive combination of unreliable sources of evidence has been proposed in this paper for combining sequentially the sources without prior knowledge on their reliabilities. The reliability of each source is evaluated according to its similarity with respect to the others which is measured by an average distance of similarity. When a source is not reliable enough, its bba is discounted to diminish its influence in the fusion process and on decision-making. Before the fusion of the sources, the suitable combination rule is selected depending on the mass of conflicting belief $m_{\oplus}(\emptyset)$ and the compromise between the computational burden and the specificity of the result one wants to deal with. Whenever $m_{\oplus}(\emptyset)$ is below the tolerance threshold, Dempster's rule can be chosen as a good rule of combination for such a compromise. Otherwise, Yager's rule, DP rule, or PCR5 must be selected to avoid to get counterintuitive results. The choice among these three rules depends on the application and the acceptable risk in decision-making errors. PCR5 rule is very appropriate to use in general for decision-making because it provides the most specific fusion results, but it requires more computational resources than other rules. If we want to keep uncertain results and don't necessarily need a very specific decision in case of high conflict between sources, Yager's rule or DP rule can be selected instead. Our numerical example shows the interest of the proposed approach. The main difficulty however lies in the tuning of the thresholds ω_d, $\omega_\emptyset$ and the parameter n involved in its implementation. These parameters must be selected by experience depending on the application. This approach was based on Shafer's model, but could be extended to other models proposed in DSmT.

Acknowledgement

This work was supported by a grant from the National Natural Science Foundation of China(NO.60634030,60775012).

References

[1] Y. Deng, W.K. Shi, Z.F. Zhu, Q. Liu, *Combining belief functions based on distance of evidence*, Decision Support Systems, Vol. 38, No. 3, pp. 489–493, 2004.

[2] D. Dubois, H. Prade, *On the unicity of Dempster's rule of combination*, International Journal of Intelligent Systems, Vol. 1, No. 2, pp. 133–142, 1986.

[3] A.L. Jousselme, D. Grenier, E. Bossé, *A new distance between two bodies of evidence*, Information Fusion, Vol. 2, No. 1, pp. 91–101, 2001.

[4] W. Liu, *Analyzing the degree of conflict among belief functions*, Artificial Intelligence, Vol. 170, No. 11, pp. 909–924, 2006.

[5] A. Martin, A.L. Jousselme, C. Osswald, *Conflict measure for the discounting operation on belief functions*, in Proc. of Fusion 2008 Conference, Cologne, Germany, July 2008.

[6] C.K. Murphy, *Combining belief functions when evidence conflicts*, Decision Support Systems, Vol. 29, No. 1, pp. 1–9, 2000.

[7] G. Shafer, *A Mathematical Theory of Evidence*, Princeton Univ. Press, 1976.

[8] F. Smarandache, J. Dezert (Editors), *Advances and Applications of DSmT for Information Fusion*, American Research Press, Rehoboth, Vol.1-3, 2004-2009.

[9] R.R. Yager, *Hedging in the combination of evidence*, Journal of Information & Optimization Sciences, Vol. 4, No. 1, pp. 73–81, 1983.

[10] L.A. Zadeh, *Review of books: A mathematical theory of evidence*, AI Magazine, Vol. 5, No. 3, pp. 81–83, 1984.

TABLE I
COMBINATION RESULTS BY DIFFERENT RULES

	$\mathbf{m}_1^2$	$\mathbf{m}_1^3$	$\mathbf{m}_1^4$	$\mathbf{m}_1^{5A}$	$\mathbf{m}_1^{5B}$
Dempster's rule	$m(A)=0.7826$ $m(B)=0.1413$ $m(C)=0.0435$ $m(B\cup C)=0.0326$	$m(B)=0.9536$ $m(C)=0.0464$	$m(B)=0.9536$ $m(C)=0.0464$	$m(B)=0.9536$ $m(C)=0.0464$	$m(B)=0.9867$ $m(C)=0.0133$
Yager's rule	$m(A)=0.36$ $m(B)=0.065$ $m(C)=0.02$ $m(B\cup C)=0.015$ $m(\Theta)=0.54$	$m(B)=0.558$ $m(C)=0.0575$ $m(\Theta)=0.3845$	$m(B)=0.4799$ $m(C)=0.2046$ $m(B\cup C)=0.0385$ $m(\Theta)=0.2770$	$m(A)=0.1385$ $m(B)=0.1296$ $m(A\cup B)=0.0692$ $m(C)=0.0885$ $m(A\cup C)=0.0415$ $m(\Theta)=0.5327$	$m(B)=0.5993$ $m(A\cup B)=0.0692$ $m(C)=0.0827$ $m(B\cup C)=0.0473$ $m(\Theta)=0.2015$
DP rule	$m(A)=0.36$ $m(B)=0.065$ $m(A\cup B)=0.24$ $m(C)=0.02$ $m(A\cup C)=0.16$ $m(B\cup C)=0.035$ $m(\Theta)=0.12$	$m(B)=0.414$ $m(A\cup B)=0.324$ $m(C)=0.0335$ $m(A\cup C)=0.036$ $m(B\cup C)=0.0245$ $m(\Theta)=0.168$	$m(B)=0.4925$ $m(C)=0.1249$ $m(B\cup C)=0.2206$ $m(\Theta)=0.1620$	$m(A)=0.081$ $m(B)=0.1783$ $m(A\cup B)=0.2868$ $m(C)=0.1026$ $m(A\cup C)=0.0867$ $m(B\cup C)=0.0493$ $m(\Theta)=0.2153$	$m(B)=0.6897$ $m(A\cup B)=0.0405$ $m(C)=0.0695$ $m(B\cup C)=0.1691$ $m(\Theta)=0.0312$
PCR5 rule	$m(A)=0.7734$ $m(B)=0.1273$ $m(C)=0.0653$ $m(B\cup C)=0.0340$	$m(A)=0.3902$ $m(B)=0.5814$ $m(C)=0.0284$	$m(A)=0.1942$ $m(B)=0.5733$ $m(C)=0.2245$ $m(B\cup C)=0.008$	$m(A)=0.4025$ $m(B)=0.4154$ $m(A\cup B)=0.0296$ $m(C)=0.1346$ $m(A\cup C)=0.0178$ $m(B\cup C)=0.0001$	$m(A)=0.1049$ $m(B)=0.7182$ $m(A\cup B)=0.0296$ $m(C)=0.1334$ $m(B\cup C)=0.0139$
$d_n^{j-1}(\mathbf{m}_j)(n=5)$	0.3813	0.6601	0.4994	0.4115	0.4137
Dempster's rule with discounting	$m(A)=0.7826$ $m(B)=0.1413$ $m(C)=0.0435$ $m(B\cup C)=0.0326$	$m(A)=0.7216$ $m(B)=0.2046$ $m(C)=0.0437$ $m(B\cup C)=0.0301$	$m(B)=0.7565$ $m(C)=0.2255$ $m(B\cup C)=0.018$	$m(B)=0.7608$ $m(C)=0.2392$	$m(B)=0.9194$ $m(C)=0.0770$ $m(B\cup C)=0.0036$
Adaptive rule (Dempster&Yager)	$m(A)=0.7826$ $m(B)=0.1413$ $m(C)=0.0435$ $m(B\cup C)=0.0326$	$m(A)=0.7216$ $m(B)=0.2046$ $m(C)=0.0437$ $m(B\cup C)=0.0301$	$m(B)=0.1261$ $m(C)=0.0376$ $m(B\cup C)=0.0030$ $m(\Theta)=0.8334$	$m(A)=0.4758$ $m(B)=0.0368$ $m(A\cup B)=0.2379$ $m(C)=0.1067$ $m(A\cup C)=0.1427$	$m(B)=0.5550$ $m(A\cup B)=0.2172$ $m(C)=0.0970$ $m(B\cup C)=0.1308$
Adaptive rule (Dempster&DP)	$m(A)=0.7826$ $m(B)=0.1413$ $m(C)=0.0435$ $m(B\cup C)=0.0326$	$m(A)=0.7216$ $m(B)=0.2046$ $m(C)=0.0437$ $m(B\cup C)=0.0301$	$m(B)=0.1261$ $m(A\cup B)=0.3247$ $m(C)=0.0376$ $m(A\cup C)=0.3247$ $m(B\cup C)=0.1147$ $m(\Theta)=0.0722$	$m(A)=0.6232$ $m(B)=0.0765$ $m(A\cup B)=0.126$ $m(C)=0.0988$ $m(A\cup C)=0.0755$	$m(A)=0.1062$ $m(B)=0.5844$ $m(A\cup B)=0.1298$ $m(C)=0.1429$ $m(B\cup C)=0.0367$
Adaptive rule (Dempster&PCR5)	$m(A)=0.7826$ $m(B)=0.1413$ $m(C)=0.0435$ $m(B\cup C)=0.0326$	$m(A)=0.7216$ $m(B)=0.2046$ $m(C)=0.0437$ $m(B\cup C)=0.0301$	$m(A)=0.4634$ $m(B)=0.2975$ $m(C)=0.2273$ $m(B\cup C)=0.0118$	$m(A)=0.7526$ $m(B)=0.1395$ $m(C)=0.1079$	$m(A)=0.2562$ $m(B)=0.6116$ $m(C)=0.1283$ $m(B\cup C)=0.0039$