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Differential Sandwich-Type results for meromorphic starlike functions

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Abstract In the present paper, we obtain a generalized criterion for meromorphic starlike functions. Using the dual concept of differential subordination and superordination, we find some sandwich-type results regarding meromorphic starlike functions.

Keywords Meromorphic Starlike Function, Differential Subordination, Differential Superordination.
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§1. Introduction and preliminaries

Let $\mathcal{H}$ denote the class of functions analytic in open unit disk $\mathbb{E} = \{z : |z| < 1\}$ and let $\mathcal{H}[a, n]$ denote the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \cdots$ where $a \in \mathbb{C}, n \in \mathbb{N}$. The class of analytic functions f, normalized by the conditions $f(0) = f'(0) - 1 = 0$ is denoted by $\mathcal{A}$.

Let Σ be the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{0}^{\infty} a_n z^n,$$

which are analytic in the punctured unit disc $\mathbb{E}_0 = \mathbb{E} \setminus \{0\}$, where $\mathbb{E} = \{z : |z| < 1\}$.
A function $f \in \Sigma$ is said to be meromorphic starlike of order α if $f(z) \neq 0$ for $z \in \mathbb{E}_0$ and

$$-\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \qquad (0 \leq \alpha < 1; z \in \mathbb{E}).$$

The class of such functions is denoted by $\mathcal{MS}^*(\alpha)$ and write $\mathcal{MS}^* = \mathcal{MS}^*(0)$ - the class of meromorphic starlike functions.
If f is analytic and g is analytic univalent in open unit disk $\mathbb{E}$, we say that $f(z)$ is subordinate to $g(z)$ in $\mathbb{E}$ and written as $f(z) \prec g(z)$ if $f(0) = g(0)$ and $f(\mathbb{E}) \subset g(\mathbb{E})$. To derive certain sandwich-type results, we use the dual concept of differential subordination and superordination.

Let $\Phi : \mathbb{C}^2 \times \mathbb{E} \longrightarrow \mathbb{C}$ ($\mathbb{C}$ is the complex plane) and h be univalent in $\mathbb{E}$. If p is analytic in $\mathbb{E}$ and satisfies the differential subordination

$$\Phi(p(z), zp'(z); z) \prec h(z), \Phi(p(0), 0; 0) = h(0), \tag{1}$$

then p is called a solution of the differential subordination (1). The univalent function q is called a dominant of differential subordination (1) if $p \prec q$ for all p satisfying (1). A dominant $\tilde{q} \prec q$ for all dominants q of (1), is said to be the best dominant of (1).

Let $\Psi : \mathbb{C}^2 \times \mathbb{E} \longrightarrow \mathbb{C}$ ($\mathbb{C}$ is the complex plane) be analytic and univalent in domain $\mathbb{C}^2 \times \mathbb{E}, h$ be analytic in $\mathbb{E}$, p is analytic and univalent in $\mathbb{E}$, with $(p(z), zp'(z); z) \in \mathbb{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$. Then p is called a solution of first order differential superordination if it satisfies

$$h(z) \prec \Psi(p(z), zp'(z); z), h(0) = \Psi(p(0), 0; 0). \tag{2}$$

An analytic function q is called a subordinant of differential superordination (2) if $q \prec p$ for all p satisfying (2). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants q for (2), is said to be the best subordinant of (2).

In the literature of meromorphic functions, many authors have been successfully used the technique of differential subordination to obtain the results involving meromorphic functions. Nunokawa and Ahuja [2] proved the following result.

Theorem 1.1. *Let $\alpha < 0$ and $\gamma \geq 0$. If*

$$f \in \Sigma^*_\gamma \left(\frac{2\alpha - 2\alpha^2 + \gamma\alpha}{2(1-\alpha)} \right),$$

then $f \in \mathcal{MS}^*(\alpha)$

Ravichandaran et al. [6] proved the following results.

Theorem 1.2. *Let $q(z)$ be univalent and $q(z) \neq 0$ in $\mathbb{E}$ and*

(i) $\dfrac{zq'(z)}{q(z)}$ is starlike univalent in $\mathbb{E}$, and

(ii) $\Re \left[1 + \dfrac{zq''(z)}{q'(z)} - \dfrac{zq'(z)}{q(z)} - \dfrac{q(z)}{\gamma} \right] > 0$ for $z \in \mathbb{E}$, $\gamma \neq 0$.

If $f(z) \in \Sigma$ and

$$-\left[(1-\gamma)\frac{zf'(z)}{f(z)} + \gamma \left(1 + \frac{zf''(z)}{f'(z)} \right) \right] \prec q(z) - \gamma \frac{zq'(z)}{q(z)},$$

then

$$-\frac{zf'(z)}{f(z)} \prec q(z)$$

and $q(z)$ is the best dominant.

Theorem 1.3. *Let $\alpha < 0$, $\gamma \neq 0$. If $f(z) \in \Sigma$ and*

$$-\left[(1-\gamma)\frac{zf'(z)}{f(z)} + \gamma \left(1 + \frac{zf''(z)}{f'(z)} \right) \right] \prec \frac{1 + 2[1 - \gamma + (\alpha - 1)\gamma]z + (1-2\alpha)^2 z^2}{1 - 2\alpha z - (1-2\alpha)z^2},$$

then $-\Re \dfrac{zf'(z)}{f(z)} > \alpha$.

Roshian and Ravichandaran [4] proved the following results.

Theorem 1.4. *Let $q(z)$ be univalent in $\mathbb{E}$ and $\dfrac{zq'(z)}{q(z)}$ be starlike in $\mathbb{E}$. If $f \in \Sigma_p$ satisfies*

$$\alpha\frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)} \prec 1 + (1-\alpha)p - \frac{zq'(z)}{q(z)},$$

then

$$-\frac{z^{1+(1-\alpha)p}f'(z)}{pf^{\alpha}(z)} \prec q(z),$$

and $q(z)$ is the best dominant.

Theorem 1.5. *Let $-1 \leq B < A \leq 1$. If $f \in \Sigma$ satisfies*

$$\alpha\frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)} \prec 2 - \alpha - \frac{(A-B)z}{(1+Az)(1+Bz)},$$

then

$$-\frac{z^{2-\alpha}f'(z)}{f^{\alpha}(z)} \prec \frac{(1+Az)}{(1+Bz)}.$$

The main objective of this paper is to generalize the results of above nature and obtain certain sandwich-type results for starlike functions.

We shall use the following lemmas to prove our results.

Lemma 1.1. *([5])Let q be univalent in $\mathbb{E}$ and let θ and ϕ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{E})$, with $\phi(w) \neq 0$, when $w \in q(\mathbb{E})$. Set $Q_1(z) = zq'(z)\phi[q(z)]$, $h(z) = \theta[q(z)] + Q_1(z)$ and suppose that either*

(i) h is convex, or

(ii) Q_1 is starlike.

In addition, assume that

(iii) $\Re\left(\dfrac{zh'(z)}{Q_1(z)}\right) > 0$ for all z in $\mathbb{E}$.

If p is analytic in $\mathbb{E}$, with $p(0) = q(0)$, $p(\mathbb{E}) \subset \mathbb{D}$ and

$$\theta[p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)],\ z \in \mathbb{E},$$

then $p(z) \prec q(z)$ and q is the best dominant.

Definition 1.1. *We denote by Q the set of functions p that are analytic and injective on $\overline{\mathbb{E}} \setminus \mathbb{B}(p)$, where*

$$\mathbb{B}(p) = \left\{\zeta \in \partial\mathbb{E} : \lim_{z\to\zeta} p(z) = \infty\right\},$$

and are such that $p'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{E} \setminus \mathbb{B}(p)$.

Lemma 1.2. *([1]) Let q be the univalent in $\mathbb{E}$ and let θ and ϕ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{E})$. Set $Q_1(z) = zq'(z)\phi[q(z)]$, $h(z) = \theta[q(z)] + Q_1(z)$ and suppose that*

(i)$Q_1(z)$ is starlike in $\mathbb{E}$; and

(ii)$\Re\left(\dfrac{\theta'(q(z))}{\phi(q(z))}\right) > 0$, for $z \in \mathbb{E}$.

if $p \in \mathcal{H}[q(0), 1] \cap Q$, with $p(\mathbb{E}) \subset \mathbb{D}$ and $\theta[p(z)] + zp'(z)\phi[p(z)]$ is univalent in $\mathbb{E}$ and

$$\theta[q(z)] + zq'(z)\phi[q(z)] \prec \theta[p(z)] + zp'(z)\phi[p(z)], z \in \mathbb{E},$$

then $q(z) \prec p(z)$ and q is the best subordinant.

§2. Main Results

Theorem 2.1. *Let q, $q(z) \neq 0$, be a univalent function in $\mathbb{E}$ satisfying therein the condition*
(i) $\Re\left(1 + \frac{zq''(z)}{q'(z)} + (\gamma - 1)\frac{zq'(z)}{q(z)}\right) > 0,$
(ii) $\Re\left(1 + \frac{zq''(z)}{q'(z)} + (\gamma - 1)\frac{zq'(z)}{q(z)} + (\gamma + 1)q(z) + \frac{(1-\alpha)\gamma}{\alpha}\right) > 0$ *for all* $z \in \mathbb{E}$.
If $f \in \Sigma$, $\frac{zf'(z)}{f(z)} \neq 0$, $z \in \mathbb{E}$, *satisfies the differential subordination*

$$\left(-\frac{zf'(z)}{f(z)}\right)^{\gamma}\left(1 + \alpha\left(\frac{zf''(z)}{f'(z)} - 2\frac{zf'(z)}{f(z)}\right)\right)$$
$$\prec \alpha zq'(z)q^{\gamma-1}(z) + \alpha q^{\gamma+1}(z) + (1-\alpha)q^{\gamma}(z), \tag{3}$$

where α, γ *are complex numbers with* $\alpha \neq 0$, *then* $-\frac{zf'(z)}{f(z)} \prec q(z)$ *and* q *is the best dominant.*

Proof. On writing $p(z) = -\frac{zf'(z)}{f(z)}$, the subordination (3) becomes:

$$\alpha zp'(z)p^{\gamma-1}(z) + \alpha p^{\gamma+1}(z) + (1-\alpha)p^{\gamma}(z) \prec \alpha zq'(z)q^{\gamma-1}(z) + \alpha q^{\gamma+1}(z) + (1-\alpha)q^{\gamma}(z). \tag{4}$$

Let us define the functions θ and ϕ as follows:

$$\theta(w) = \alpha w^{\gamma+1} + (1-\alpha)w^{\gamma} \quad \textit{and} \quad \phi(w) = \alpha w^{\gamma-1}$$

Clearly, the functions θ and ϕ are analytic in domain $\mathbb{D} = \mathbb{C} \setminus \{0\}$ and $\phi(w) \neq 0$ in $\mathbb{D}$. Now, define the function h as follows:

$$h(z) = \alpha zq'(z)q^{\gamma-1}(z) + \alpha q^{\gamma+1}(z) + (1-\alpha)q^{\gamma}(z). \tag{5}$$

Differentiate (5) and simplifying a little, we get

$$\frac{zh'(z)}{Q_1(z)} = 1 + \frac{zq''(z)}{q'(z)} + (\gamma - 1)\frac{zq'(z)}{q(z)} + (\gamma + 1)q(z) + \frac{(1-\alpha)\gamma}{\alpha}$$

where $Q_1(z) = \alpha zq'(z)q^{\gamma-1}(z)$. In view of given conditions (i) and (ii), we have Q_1 is starlike in $\mathbb{E}$ and

$$\Re\left(\frac{zh'(z)}{Q_1(z)}\right) > 0, \ z \in \mathbb{E}.$$

In view of lemma 1.2 and subordination (3), we have

$$\theta[p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)].$$

This completes the proof of our theorem. □

Theorem 2.2. *Let q, $q(z) \neq 0$, be a univalent function in $\mathbb{E}$ such that*
(i) $\Re\left(1 + \frac{zq''(z)}{q'(z)} + (\gamma - 1)\frac{zq'(z)}{q(z)}\right) > 0,$
(ii) $\Re\left((\gamma + 1)q(z) + \frac{(1-\alpha)\gamma}{\alpha}\right) > 0$ *for all* $z \in \mathbb{E}$.

If $f \in \Sigma, -\frac{zf'(z)}{f(z)} \in \mathcal{H}[q(0),1] \cap Q$ with $\frac{zf'(z)}{f(z)} \neq 0,\ z \in \mathbb{E}$, satisfies the differential superordination

$$\alpha zq'(z)q^{\gamma-1}(z) + \alpha q^{\gamma+1}(z) + (1-\alpha)q^{\gamma}(z) \prec \left(-\frac{zf'(z)}{f(z)}\right)^{\gamma}\left(1+\alpha\left(\frac{zf''(z)}{f'(z)} - 2\frac{zf'(z)}{f(z)}\right)\right) = h(z), \quad (6)$$

where α, γ are complex numbers with $\alpha \neq 0$, then $q(z) \prec -\frac{zf'(z)}{f(z)}$ and q is the best subordinant.

Proof. Setting $p(z) = -\frac{zf'(z)}{f(z)}$, the superordination (6) becomes:

$$\alpha zq'(z)q^{\gamma-1}(z) + \alpha q^{\gamma+1}(z) + (1-\alpha)q^{\gamma}(z) \prec \alpha zp'(z)p^{\gamma-1}(z) + \alpha p^{\gamma+1}(z) + (1-\alpha)p^{\gamma}(z). \quad (7)$$

By defining the functions θ and ϕ and Q_1 same as in case of Theorem 2.1 and observing that

$$\frac{\theta'(q(z))}{\phi(q(z))} = (\gamma+1)q(z) + \frac{(1-\alpha)\gamma}{\alpha}.$$

The use of lemma 1.2 along with (7) completes the proof on the same lines as in case of Theorem 2.1.

□

On combining Theorem 2.1 and Theorem 2.2, we obtain the following sandwich-type theorem.

Theorem 2.3. *Suppose α, γ are complex numbers with $\alpha \neq 0$ and suppose that $q_1, q_2 (q_1 \neq 0, q_2 \neq 0,\ z \in \mathbb{E})$ are univalent functions in $\mathbb{E}$ such that q_1 satisfies the conditions (i) and (ii) of Theorem 2.2 and q_2 follows the conditions (i) and (ii) of Theorem 2.1. If $f \in \Sigma, -\frac{zf'(z)}{f(z)} \in \mathcal{H}[q_1(0),1] \cap Q$ with $\frac{zf'(z)}{f(z)} \neq 0,\ z \in \mathbb{E}$, satisfies the differential sandwich-type condition*

$$\begin{aligned}\alpha zq_1'(z)q_1^{\gamma-1}(z) + \alpha q_1^{\gamma+1}(z) + (1-\alpha)q_1^{\gamma}(z) &\prec h(z)\\ &= \left(-\frac{zf'(z)}{f(z)}\right)^{\gamma}\left\{1+\alpha\left(\frac{zf''(z)}{f'(z)} - 2\frac{zf'(z)}{f(z)}\right)\right\}\\ &\prec \alpha zq_2'(z)q_2^{\gamma-1}(z) + \alpha q_2^{\gamma+1}(z) + (1-\alpha)q_2^{\gamma}(z),\end{aligned}$$

where h is univalent in $\mathbb{E}$, then $q_1(z) \prec -\frac{zf'(z)}{f(z)} \prec q_2(z)$. Moreover q_1 and q_2 are, respectively, the best subordinant and the best dominant.

Deductions

If we consider the dominant $q(z) = \frac{1+(1-2\lambda)z}{1-z}, 0 \leq \lambda < 1$, a little calculation yields that this dominant satisfies the conditions of Theorem 2.1 in the following particular cases. Select $\gamma = 1$ in Theorem 2.1, we get the following result.

Corollary 2.1. *Suppose that $\alpha (0 < \alpha \leq 1)$ is a real number and if $f \in \Sigma, \frac{zf'(z)}{f(z)} \neq 0$ in $\mathbb{E}$, satisfies*

$$\left(-\frac{zf'(z)}{f(z)}\right)\left\{1+\alpha\left(\frac{zf''(z)}{f'(z)} - 2\frac{zf'(z)}{f(z)}\right)\right\}$$

$$\prec \frac{2\alpha(1-2\lambda)z}{(1-z)^2} + \alpha\left(\frac{1+(1-2\lambda)z}{1-z}\right)^2 + (1-\alpha)\frac{1+(1-2\lambda)z}{1-z},$$

then

$$-\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\lambda)z}{1-z}, \qquad i.e.\ f \in \mathcal{MS}^*(\lambda),\ 0 \le \lambda < 1.$$

Taking $\alpha = \dfrac{1}{2}, \lambda = 0$ in above corollary, we get:

Corollary 2.2. *If* $f \in \Sigma, \dfrac{zf'(z)}{f(z)} \neq 0$ *in* $\mathbb{E}$*, satisfies*

$$\left(-\frac{zf'(z)}{f(z)}\right)\left(1 + \frac{1}{2}\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) \prec \frac{1+2z}{(1-z)^2}$$

then $f \in \mathcal{MS}^*$.

Selecting $\gamma = -1$ in Theorem 2.1, we have:

Corollary 2.3. *Suppose that* α *be a real number such that* $\alpha \in (-\infty, 0) \cup [1, \infty]$ *and let* $f \in \Sigma, \dfrac{zf'(z)}{f(z)} \neq 0$ *in* $\mathbb{E}$*, satisfy*

$$\frac{1 + \alpha\left(\dfrac{zf''(z)}{f'(z)} - 2\dfrac{zf'(z)}{f(z)}\right)}{-\dfrac{zf'(z)}{f(z)}} \prec \alpha + \frac{(1-\alpha)(1-z)}{1+(1-2\lambda)z} + \frac{2\alpha(1-\lambda)z}{(1+(1-2\lambda)z)^2},$$

then

$$-\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\lambda)z}{1-z}, \qquad i.e.\ f \in \mathcal{MS}^*(\lambda),\ 0 \le \lambda < 1.$$

Taking $\gamma = 0$ in Theorem 2.1, we have:

Corollary 2.4. *If* $f \in \Sigma, \dfrac{zf'(z)}{f(z)} \neq 0$ *in* $\mathbb{E}$*, satisfies*

$$1 + \alpha\left(\frac{zf''(z)}{f'(z)} - 2\frac{zf'(z)}{f(z)}\right) \prec 1 - \alpha + \frac{\alpha(1+(1-2\lambda)z)}{1-z} + \frac{2\alpha(1-\lambda)z}{(1-z)(1+(1-2\lambda)z)},$$

where α *is a non-zero complex number. Then*

$$-\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\lambda)z}{1-z},$$

i.e. $f \in \mathcal{MS}^*(\lambda),\ 0 \le \lambda < 1$.

Selecting $\alpha = 1, \lambda = \frac{1}{2}$ in above corollary, we get the following result:

Corollary 2.5. *If* $f \in \Sigma, \dfrac{zf'(z)}{f(z)} \neq 0$ *in* $\mathbb{E}$*, satisfies*

$$1 + \frac{zf''(z)}{f'(z)} - 2\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z},$$

then

$$-\frac{zf'(z)}{f(z)} \prec \frac{1}{1-z},$$

i.e. $f \in \mathcal{MS}^*(\frac{1}{2}),\ z \in \mathbb{E}$.

When we consider the dominant $q(z) = \dfrac{1+az}{1-z},\ -1 < a \leq 1$, a little calculation yields that it satisfies the conditions of Theorem 2.1 in following special cases and consequently we obtain the next results.

Setting $\gamma = 1$ in Theorem 2.1, we have the following corollary.

Corollary 2.6. *Suppose that* $\alpha (0 < \alpha \leq 1)$ *is a real number and if* $f \in \Sigma, \dfrac{zf'(z)}{f(z)} \neq 0$ *in* $\mathbb{E}$*, satisfies*

$$\left(-\frac{zf'(z)}{f(z)}\right)\left\{1 + \alpha\left(\frac{zf''(z)}{f'(z)} - 2\frac{zf'(z)}{f(z)}\right)\right\}$$
$$\prec \frac{\alpha(1+a)z}{(1-z)^2} + \alpha\left(\frac{1+az}{1-z}\right)^2 + (1-\alpha)\frac{1+az}{1-z},$$

then

$$-\frac{zf'(z)}{f(z)} \prec \frac{1+az}{1-z},\ z \in \mathbb{E},\ \ -1 < a \leq 1.$$

Selecting $\gamma = -1$ in Theorem 2.1, we have:

Corollary 2.7. *Suppose that* α *be a real number such that* $\alpha \in (-\infty, 0) \cup [1, \infty)$ *and let* $f \in \Sigma, \dfrac{zf'(z)}{f(z)} \neq 0$ *in* $\mathbb{E}$*, satisfy*

$$\frac{1 + \alpha\left(\dfrac{zf''(z)}{f'(z)} - 2\dfrac{zf'(z)}{f(z)}\right)}{-\dfrac{zf'(z)}{f(z)}} \prec \alpha + \frac{(1-\alpha)(1-z)}{1+az} + \frac{\alpha(1+a)z}{(1+az)^2},$$

then

$$-\frac{zf'(z)}{f(z)} \prec \frac{1+az}{1-z},\ z \in \mathbb{E},\ \ -1 < a \leq 1.$$

Taking $\gamma = 0$ in Theorem 2.1, we have:

Corollary 2.8. *If $f \in \Sigma$, $\dfrac{zf'(z)}{f(z)} \neq 0$ in $\mathbb{E}$, satisfies*

$$1+\alpha\left(\frac{zf''(z)}{f'(z)}-2\frac{zf'(z)}{f(z)}\right) \prec 1-\alpha+\frac{\alpha(1+az)}{1-z}+\frac{\alpha(1+a)z}{(1-z)(1+az)},$$

then

$$-\frac{zf'(z)}{f(z)} \prec \frac{1+az}{1-z},\ z\in\mathbb{E},\ \ -1<a\leq 1.$$

§3. Sandwich-type Results

In this section, we apply Theorem 2.3 to find certain sandwich-type results which give the best subordinant and the best dominant for $-\dfrac{zf'(z)}{f(z)}$. By selecting the subordinant $q_1(z) = 1+az$ and the dominant $q_2(z) = 1+bz, 0<a<b$, in Theorem 2.3, we deduce, below, some criteria for starlike functions. Keeping $\gamma = 1$ in Theorem 2.3, we obtain:

Corollary 3.1. *Suppose α, a, b are real numbers such that $0<\alpha\leq 1, 0<a<b<1$. If $f \in \Sigma$ is such that $-\dfrac{zf'(z)}{f(z)} \in \mathcal{H}[1,1]\cap Q$, with $-\dfrac{zf'(z)}{f(z)} \neq 0$ and satisfies the condition*

$$1+(1+2\alpha)az+\alpha a^2z^2 \prec -\frac{zf'(z)}{f(z)}\left(1+\alpha\left(\frac{zf''(z)}{f'(z)}-2\frac{zf'(z)}{f(z)}\right)\right)$$
$$\prec 1+(1+2\alpha)bz+\alpha b^2z^2,\ z\in\mathbb{E},$$

where $-\dfrac{zf'(z)}{f(z)}\left(1+\alpha\left(\dfrac{zf''(z)}{f'(z)}-2\dfrac{zf'(z)}{f(z)}\right)\right)$ is univalent in $\mathbb{E}$, then

$$1+az \prec -\frac{zf'(z)}{f(z)} \prec 1+bz.$$

Example 3.1. *For $\alpha = 1, a = 1/10, b = 9/10$ and f same as in above corollary, we obtain:*

$$1+\frac{3}{10}z+\frac{1}{100}z^2 \prec -\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-2\frac{zf'(z)}{f(z)}\right) \prec 1+\frac{27}{10}z+\frac{81}{100}z^2 \quad (8)$$

then

$$1+\frac{1}{10}z \prec -\frac{zf'(z)}{f(z)} \prec 1+\frac{9}{10}z. \quad (9)$$

Writing $\gamma = -1$ in Theorem 2.3, we obtain:

Corollary 3.2. *Let α, a, b be real numbers such that $\alpha \in (-\infty, 0) \cup (1, \infty), 0 < a < b < 1$. If $f \in \Sigma$ is such that $-\dfrac{zf'(z)}{f(z)} \in \mathcal{H}[1,1] \cap Q$, with $\dfrac{zf'(z)}{f(z)} \neq 0$ and $\dfrac{1+\alpha\left(\dfrac{zf''(z)}{f'(z)} - 2\dfrac{zf'(z)}{f(z)}\right)}{-\dfrac{zf'(z)}{f(z)}}$ is univalent in $\mathbb{E}$, then*

$$\alpha + \frac{1-\alpha}{1+az} + \frac{\alpha az}{(1+az)^2} \prec \frac{1+\left(\alpha\dfrac{zf''(z)}{f'(z)} - 2\dfrac{zf'(z)}{f(z)}\right)}{-\dfrac{zf'(z)}{f(z)}} \prec \alpha + \frac{1-\alpha}{1+bz} + \frac{\alpha bz}{(1+az)^2},\ z \in \mathbb{E},$$

implies

$$1 + az \prec -\frac{zf'(z)}{f(z)} \prec 1 + bz,\ z \in \mathbb{E}.$$

Writing $\gamma = 0$ in Theorem 2.3, we obtain:

Corollary 3.3. *Suppose α is a non-zero complex number and a, b are real numbers such that $\alpha \in (-\infty, 0) \cup (1, \infty), 0 < a < b < 1$. If $f \in \Sigma$ is such that $-\dfrac{zf'(z)}{f(z)} \in \mathcal{H}[1,1] \cap Q$, with $\dfrac{zf'(z)}{f(z)} \neq 0$ and $1 + \left(\alpha\dfrac{zf''(z)}{f'(z)} - 2\dfrac{zf'(z)}{f(z)}\right)$ is univalent in $\mathbb{E}$, then*

$$1 + \alpha az + \frac{\alpha az}{(1+az)} \prec 1 + \left(\alpha\frac{zf''(z)}{f'(z)} - 2\frac{zf'(z)}{f(z)}\right) \prec 1 + \alpha bz + \frac{\alpha bz}{(1+az)},\ z \in \mathbb{E},$$

implies

$$1 + az \prec -\frac{zf'(z)}{f(z)} \prec 1 + bz,\ z \in \mathbb{E}.$$

Example 3.2. *For $\alpha = 1, a = 1/4, b = 3/4$ and f same as in above corollary, we obtain:*

$$1 + \frac{1}{4}z + \frac{z}{4+z} \prec -\frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)} - 2\frac{zf'(z)}{f(z)}\right) \prec 1 + \frac{3}{4}z + \frac{3z}{4+3z} \qquad (10)$$

then

$$1 + \frac{1}{4}z \prec -\frac{zf'(z)}{f(z)} \prec 1 + \frac{3}{4}z. \qquad (11)$$

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Smarandache nuclei of second Smarandache Bol loops

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Abstract A loop $(G_H, \cdot)$ is called a special loop of the loop $(G, \cdot)$ if the pair $(H, \cdot)$ is an arbitrary non-trivial subloop of $(G, \cdot)$. In particular, $(G_H, \cdot)$ is called a second Smarandache Bol loop (S_{2nd}BL) if it obeys the second Smarandache Bol loop identity $(xs \cdot y)s = x(sy \cdot s)$ for all $x, y \in G$ and $s \in H$. This paper present some characterizations of Smarandache nuclei of second Smarandache Bol loops and its second Smarandache Moufang part $(S_{2nd}M(G_H))$. Some results that holds in classical Bol loops were investigated and generalised. The algebraic connections between right(left) and middle Smarandache nuclei of S_{2nd}BL and its $(S_{2nd}M(G_H))$ via Smarandache autotopism nuclei were shown

Keywords Special loops, Smarandache Nuclei Bol Loops, Smarandache Moufang loops.
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§1. Introduction

A groupoid $(Q, \cdot)$ is a non- empty set Q with a binary operation " $\cdot$ " on Q such that $x \cdot y \in Q$ for all $x, y \in Q$. If the equations: $a \cdot x = b$ and $y \cdot a = b$ have unique solutions $x, y \in Q$ for all $a, b \in Q$, then $(Q, \cdot)$ is called a quasigroup. Let $(Q, \cdot)$ be a quasigroup and there exist a unique element $e \in Q$ called the identity element such that for all $x \in Q, x \cdot e = e \cdot x = x$, then $(Q, \cdot)$ is called a loop. At times, we shall write xy instead of $x \cdot y$ and stipulate that $\cdot$ has lower priority than juxtaposition among factors to be multiplied. Let $(Q, \cdot)$ be a groupoid and a be a fixed element in Q, then the left and right translations L_a and R_a of a are respectively defined by $xL_a = a \cdot x$ and $xR_a = x \cdot a$ for all $x \in Q$. It can now be seen that a groupoid $(Q, \cdot)$ is a quasigroup if its left and right translation mappings are permutations. Since the left and right translation mappings of a quasigroup are bijective, then the inverse mappings L_x^{-1} and R_x^{-1} exist.

Let

$$a \backslash b = bL_a^{-1} \qquad \text{and} \qquad a/b = aR_b^{-1}$$

and note that

$$a \backslash b = c \Longleftrightarrow a \cdot c = b \qquad \text{and} \qquad a/b = c \Longleftrightarrow c \cdot b = a.$$

Thus, for any quasigroup $(Q, \cdot)$, there exist another two new binary operations; right division $(/)$ and left division $(\backslash)$ for any fixed $a \in Q$. Consequently, $(Q, \backslash)$ and $(Q, /)$ are also quasigroups. Using the operations $(\backslash)$ and $(/)$, the definition of a loop can be restated as follows.

A loop $(Q, \cdot, /, \backslash, e)$ is a set G together with three binary operations $(\cdot)$, $(/)$, $(\backslash)$ and one nullary operation e such that

1. $a \cdot (a\backslash b) = b$, $(b/a) \cdot a = b$ for all $a, b \in Q$,
2. $a\backslash(a \cdot b) = b$, $(b \cdot a)/a = b$ for all $a, b \in Q$ and
3. $a\backslash a = b/b$ or $e \cdot a = a$ for all $a, b \in Q$.

We also stipulate that $(/)$ and $(\backslash)$ have higher priority than $(\cdot)$ among factors to be multiplied. For instance, $a \cdot b/z$ and $a \cdot b\backslash z$ stand for $a(b/z)$ and $a(b\backslash z)$ respectively.

In a loop $(Q, \cdot)$ with identity element e, the *left inverse element* of $a \in Q$ is the element $xJ_\lambda = a^\lambda \in Q$ such that $a^\lambda \cdot a = e$ while the *right inverse element* of $a \in G$ is the element $xJ_\rho = a^\rho \in G$ such that $a \cdot a^\rho = e$. For more on quasigroups and loops, check Jaiyéọlá [13], Pflugfelder [5] and Shcherbacov [3]

Smarandache concept in groupoid was first introduced and studied by (W . B Vasantha Kandasamy [18], 2002). In her first paper [20] and her book on Smarandache concept in the study of loops [19], where she initially defined Smarandache loop (S-loop) as a loop with at least a subloop which forms a subgroup under the binary operation of the loop, have started gaining more attractive attention of researchers.

(Muktibodh, [24, 25].2005, 2006) defined Smarandache quasigroup as a non trivial subset H of a quasigroup $(G, \cdot)$ such that $(H, \cdot)$ is an associative subquasigroup of the quasigroup $(G, \cdot)$.

Immediately after the work of Muktibodh, (Jaiyéọlá. [6], 2006) presented a study on holomorphic structures of loop under Smarandache concept. The paper revealed that a loop is a Smarandache loop if and only if its holomorph is a Smarandache loop and he furthered announced that the statement is also true for some weak Smarandache loops such as inverse property, weak inverse property but false for others (conjugacy closed, Bol, central, extra, Burn, A- homogeneous except if their holomorphs are nuclear or central.

In (Jaiyéọlá . [8, 9], 2006) carried out a comprehensive study on parastrophic invariants of Smarandache quasigroups and presented a ground view of the studies of universality of some Smarandache loops of Bol -Moufang type. It was shown in [9] that Smarandache quasigroup (loop) is universal if all its $f, g-$principal isotopes are Smarandache $f, g-$ principal isotopes.

(Jaiyéọlá . [10–12, 14, 15], 2008) in furtherance to his exploit, presented more characterizations of Smarandache concept in the study of quasigroup(loop) structures. In particular, Smarandache isotopic quasigroup and holomorphic study of Smarandache automorphic and cross inverse property loops was investigated in the same manner as the isotopy theory was carried out for groupoids, quasigroups and loops. The same author in [15] introduced the study of double cryptography using the concept of Smarandache Keedwell Cross inverse quasigroup.

In [16, 17], the author furthered his exploration of Smarandache quasigroups(loops) theory by classifying the structures into first Smarandache quasigroup(loop) and second Smarandache quasigroup(loop). The author announced that the most comprehensive study in the Bol Moufang-type identities called Bol loop falls into the second class of Smarandache loops. Hence, the second Smarandache loop is a particular case of the first Smarandache loops and a second Smarandache Bol loop is a generalised form of Bol loops.

In (Osoba et al. [21,22] , 2018) studied relationship of multiplication groups and isostrophic quasigroups and some algebraic characterisations of middle Bol loops while the algebraic connections between left and middle Bol loops and their cores were presented in [23]. More results on the algebraic properties of middle Bol loops were further investigated by (Oyebo and Osoba, [26]).

Furtherance to earlier studies, this research is therefore presented to establish the connections between right left(middle) Smarandache nuclei. Characterisation of Smarandache nuclei of second Smarandache Bol loop were introduced. Some results in this study agreed with or generalised the results in classical cases. In particular, this paper revealed that if $(G_H, \cdot)$ is a second Smarandache Bol loop, then $SN_\rho(G_H) = SN_\mu(G_H)$, $SN_\rho(G_H) \subseteq SN_\lambda(G_H)$ and $SN_\rho(G_H) = SZ(G_H)$. In addition, if $s \in S_{2nd}M(G_H)$ then $(L_s, R_s, R_sL_s) \in S_{1st}\mathrm{AUT}(G_H, \cdot)$. It was shown that if the special loop is S_{2nd}BL, then $SN_\lambda(G_H) \cap SN_\rho(G_H) \subseteq S_{2nd}M(G_H)$, $SN_\rho(G_H) \cap S_{2nd}M(G_H) \subseteq SN_\lambda(G_H)$ and $SN_\mu(G_H) \cap S_{2nd}M(G_H) \subseteq SN_\rho(G_H)$

§2. Preliminaries

Theorem 2.1. [Jaiyeola, [16]] *Let the special loop $(G_H, \cdot)$ be a $S_{2nd}BL$. Then, $S_{2nd}BL$ satisfies $S_{2nd}RIPL$ and $S_{2nd}RAPL$*

Definition 2.2. *A special loop $(G_H, \cdot)$ is called a second Smarandache automorphic inverse loop ($S_{2nd}AIPL$) if $(s \cdot x)^{-1} = s^{-1} \cdot x^{-1}$ for all $s \in H$ and $x \in G$*

Lemma 2.3.[Jaiyeola, [16]] *Let $(G_H, \cdot)$ be a special loop.*

1. *if $(A, B, C) \in S_{2nd}RAUT(G_H, \cdot)$ and G_H has the $S_{2nd}RIP$, then $(C, J'_\rho B J'_\rho, A) \in S_{2nd}RAUT(G_H, \cdot)$*
2. *if $(A, B, C) \in S_{2nd}RAUT(G_H, \cdot)$ and G_H has the $S_{2nd}LIP$, then $(J'_\lambda A J'_\lambda, C, B) \in S_{2nd}LAUT(G_H, \cdot)$*

Theorem 2.4. [Jaiyeola, [16]] *Let $(G_H, \cdot)$ be a special loop. $(G_H, \cdot)$ is a $S_{2nd}BL$ if and only if $(R_s^{-1}, L_sR_s, R_s) \in S_{1st}AUT(G_H, \cdot)$.*

Definition 2.5. *Let $(G_H, \cdot)$ be special groupoid(quasigroup). The Smarandache left, right and middle nucleus are denoted by $SN_\rho(G_H), SN_\rho(G_H)$ and $SN_\mu(G_H)$ respectively.*

1. *The Smarandache left nucleus of G_H is define as $SN_\lambda(G_H, \cdot) = SN_\lambda(G) \cap H = \{s \in H : s \cdot xy = sx \cdot y \text{ for all } x, y \in G\}$*
2. *The Smarandache right nucleus of G_H is define as $SN_\rho(G_H, \cdot) = SN_\rho(G) \cap H = \{s \in H : xy \cdot s = xy \cdot s \text{ for all } x, y \in G\}$*
3. *The Smarandache middle nucleus of G_H is define as $SN_\mu(G_H, \cdot) = SN_\mu(G) \cap H = \{s \in H : x \cdot sy = xs \cdot y \text{ for all } x, y \in G\}$*
4. *The Smarandache nucleus of (G_H) is define as $SC(G_H) = SN(G_H, \cdot) \cap SN_\rho(G_H, \cdot) \cap SN_\mu(G_H, \cdot)$*
5. *The Smarandache center SZ of G_H is define as $SZ(G_H, \cdot) = \{s \in H : s \cdot x = x \cdot s \text{ for all } x \in G\}$, where SN is the Smarandache nucleus of $(G_H, \cdot)$*

Definition 2.6. *Let $(G_H, \cdot)$ be a special quasigroup(loop). A mapping $\phi \in SSYM(G_H)$ is called a*

1. *second Smarandache semi-automorphism(S_{2nd}Semi-automorphism) if and only if $e\phi = e$ and $(sy \cdot s)\phi = (s\phi \cdot y\phi)s\phi$ for all $y \in G$ and all $s \in H$*

2. *second right Smarandache pseudo-automorphism (S_{2nd} right pseudo -automorphism) of G_H if and only if there exists $c \in H$ such that $(\phi, \phi R_c, \phi R_c) \in S_{2nd}RAUM(G_H \cdot)$ where c is the second Smarandache companion (S_{2nd} companion) of ϕ. The set of such ϕs is denoted by $S_{2nd}RAUM(G_H, \cdot) = S_{2nd}RAUM(G_H)$.*

Definition 2.7. *Let the special loop $(G_H, \cdot)$ be a $S_{2nd}BL$. The second Smarandache Moufang part ($S_{2nd}M(G_H)$) of $(G_H, \cdot)$ is define as*
$S_{2nd}M(G_H)$= {for all $s \in H : (s \cdot ys)x = s(y \cdot sx)$ for all $x, y \in G$}

§3. Main Results

Lemma 3.1. *Let $(G_H, \cdot)$ be a special loop. The following hold.*

1. *if $s \in SN_\rho(G_H)$, then $(I, R_s, R_s) \in S_{1st}AUT(G_H, \cdot)$ for all $s \in H$*

2. *if $s \in SN_\mu(G_H)$, then $(R_s^{-1}, L_s, I) \in S_{1st}AUT(G_H, \cdot)$ for all $s \in H$*

3. *if $s \in SN_\lambda(G_H)$, then $(L_s, I, L_s) \in S_{1st}AUT(G_H, \cdot)$ for all $s \in H$*

Proof. 1. $s \in SN_\rho(G_H) \Leftrightarrow xz \cdot s = x \cdot zs \Leftrightarrow x \cdot zR_s = (xz)R_s \Leftrightarrow (I, R_s, R_s) \in S_{1st}AUT(G_H, \cdot)$ for all $s \in H$ and $x, z \in G$.

2. Let $s \in SN_\mu(G_H) \Leftrightarrow xs \cdot z = x \cdot sz \Leftrightarrow xR_s \cdot z = x \cdot zL_s \Leftrightarrow xz = xR_s^{-1} \cdot zL_s \Leftrightarrow (R_s^{-1}, L_s, I) \in S_{1st}AUT(G_H, \cdot)$ for all $s \in H$ and $x, z \in G$.

3. Let $s \in SN_\lambda(G_H) \Leftrightarrow sx \cdot z = s \cdot xz \Leftrightarrow xL_s \cdot z = (xz)L_s \Leftrightarrow (I, L_s, L_s) \in S_{1st}AUT(G_H, \cdot)$ for all $s \in H$ and $x, z \in G$.

□

Lemma 3.2. *Let ϕ be a second right Smarandache pseudo- automorphism of a special loop $(G_H, \cdot)$ with second Smarandache companion c. Then, ϕ^{-1} is also a second right Smarandache pseudo- automorphism.*

Proof. If ϕ is a second right Smarandache pseudo- automorphism with second Smarandache companion c, then

$$(\phi, \phi R_c, \phi R_c) \in S_{2nd}RAUM(G_H, \cdot)$$

So, the inverse $(\phi, \phi R_c, \phi R_c)^{-1} = (\phi^{-1}, R_c^{-1}\phi^{-1}, R_c^{-1}\phi^{-1}) \in S_{2nd}RAUM(G_H \cdot)$ Let

$$(\phi^{-1}, R_c^{-1}\phi^{-1}, R_c^{-1}\phi^{-1}) = (\phi^{-1}, \phi^{-1}R_c, \phi^{-1}R_c) \in S_{2nd}RAUM(G_H, \cdot) \qquad (1)$$

To complete the proof, we only need to show that (1) hold. That is, $R_c^{-1}\phi^{-1} = \phi^{-1}R_c$ for all $c \in H$. Let $t \in G$, then $tR_c^{-1}\phi^{-1} = t\phi^{-1}R_c$. Setting $t = tc$, give

$$(tc)R_c^{-1}\phi^{-1} = (tc)\phi^{-1}R_c \Rightarrow t\phi^{-1} = t\phi^{-1} \cdot c\phi^{-1}R_c \Rightarrow a\phi^{-1}R_c = e,$$

where e is Smarandache identity element. From (1), we have $(\phi^{-1}, \phi^{-1}R_c, \phi^{-1}R_c) \in S_{2nd}RAUM(G_H, \cdot)$ for all $c \in H$

□

Lemma 3.3. *Let $(G_H, \cdot)$ be a special loop.*

1. *if $(G_H, \cdot)$ is a $S_{2nd}RIP$ of $(G, \cdot)$, then $x^{\rho^2} = x$ and $x^\rho = x^\lambda$ for all $x \in G$*
2. *if $(G_H, \cdot)$ is a $S_{2nd}LIP$ of $(G, \cdot)$, then $x^{\lambda^2} = x$ and $x^\rho = x^\lambda$ for all $x \in G$*
3. *$(G_H, \cdot)$ has $S_{2nd}RIP \Leftrightarrow R_{s^{-1}} = R_s^{-1}$*
4. *$(G_H, \cdot)$ has $S_{2nd}LIP \Leftrightarrow L_{s^{-1}} = L_s^{-1}$*
5. *if $(G_H, \cdot)$ is a $S_{2nd}RIP$, then $J_\lambda R_s J_\rho = L_{s^{-1}}$ for all $s \in H$*
6. *if $(G_H, \cdot)$ is a $S_{2nd}LIP$, then $J_\lambda L_s J_\rho = R_{s^{-1}}$ for all $s \in H$*

Proof. 1. Consider the expression $(sx \cdot x^\rho)(x^\rho)^\rho$, then

$$(sx \cdot x^\rho)(x^\rho)^\rho \underbrace{=}_{2_{nd}RIP} s(x^\rho)^\rho = sx \Rightarrow x^{\rho^2} = x \Rightarrow J_\rho^2 = I \Rightarrow J_\rho^{-1} = J_\rho \Rightarrow J_\lambda = J_\rho$$

2. Consider the expression $(x^\lambda \cdot xs)(x^\lambda)^\lambda$, then

$$(x^\lambda \cdot xs)(x^\lambda)^\lambda \underbrace{=}_{2_{nd}LIP} (x^\lambda)^\lambda s = xs \Rightarrow x^{\lambda^2} = x \Rightarrow J_\lambda^2 = I \Rightarrow J_\lambda{-}1 = J_\rho \Rightarrow J_\lambda = J_\rho$$

3. $ys \cdot s^{-1} = y \Leftrightarrow yR_sR_{s^{-1}} = y \Leftrightarrow R_sR_{s^{-1}} = I \Leftrightarrow R_s^{-1} = R_{s^{-1}}$
4. $s^\lambda \cdot sx = x \Leftrightarrow xL_sL_{s^{-1}} = x \Leftrightarrow L_sL_{s^{-1}} = I \Leftrightarrow L_s^{-1} = L_{s^{-1}}$
5. $xJ_\lambda R_s J_\rho = (x^\lambda s)^\rho \underbrace{\Rightarrow}_{S_{2nd}RIP} s^{-1}(x^{-1})^{-1} = s^{-1}x = xL_{s^{-1}}$
6. $xJ_\lambda L_s J_\rho = (sx^\lambda)^\rho \underbrace{\Rightarrow}_{S_{2nd}LIP} (x^{-1})^{-1}s^{-1} = xs^{-1} = xR_{s^{-1}}$

□

Theorem 3.4. *Let the special loop $(G_H, \cdot)$ be a $S_{2nd}BL$. Then $SN_\rho(G_H) = SN_\mu(G_H)$*

Proof. Let $(G_H, \cdot)$ be a $S_{2nd}BL$. Suppose that $s \in SN_\rho(G_H)$, then

$$(xs \cdot z)s = x(sz \cdot s) = (x \cdot sz)s$$

for all $s \in H$, and $x, z \in G$. So, $xs \cdot z = x \cdot sz$. Thus $s \in SN_\mu(G_H)$ for all $s \in H$.

Conversely, Suppose that $s \in SN_\mu(G_H)$, then

$$x(sz \cdot s) = (xs \cdot z)s = (x \cdot sz)s$$

Thus, $s \in SN_\rho(G_H)$ for all $s \in H$. $\square$

Theorem 3.5. *Let the special loop $(G_H, \cdot)$ be a $S_{2nd}BL$ with a $S_{2nd}AIPL$. Then,*

1. $SN_\rho(G_H) \subseteq SN_\lambda(G_H)$
2. $SN_\rho(G_H) = SZ(G_H)$

Proof. Let $J'_\rho : x \mapsto x^{-1}$, where $x \in G$. Using Lemma 3.3 and S_{2nd}AIP,

$$xJ'_\rho R_s J'_\rho = (x^{-1} \cdot s)^{-1} = x \cdot s^{-1} = xR_{s^{-1}}$$

Since in Theorem 3.4 $(G_H, \cdot)$ is a S_{2nd}RIPL, it give

$$J'_\rho R_s J'_\rho = R_s^{-1} \tag{2}$$

for all $s \in H$ and $x \in G$.

Let $s \in SN_\mu(G_H)$, recalling Lemma 3.1, $(R_s^{-1}, L_s, I) \in S_{1st}AUT(G_H, \cdot)$. Use Lemma 2.3 on Lemma 3.1 since $(G_H, \cdot)$ is a S_{2nd}RIPL, give
$A = (I, J'_\rho L_s^{-1} J'_\rho, R_s^{-1}) \in S_{2nd}RAUT(G_H, \cdot)$. Since in Theorem ,

$$SN_\mu(G_H) = SN_\rho(G_H),$$

then $s \in SN_\rho(G_H)$. So B$= (I, R_s, R_s) \in S_{1st}RAUT(G_H, \cdot)$ for all $s \in SN_\rho(G_H)$.

The product, $AB = (I, J'_\rho L_s^{-1} J'_\rho, R_s^{-1})(I, R_s, R_s) = (I, J'_\rho L_s^{-1} J'_\rho R_s, R_s^{-1} R_s) = (I, J'_\rho L_s^{-1} J'_\rho R_s, I)$ is also first Smarandache autotopism of G_H. So,

$$J'_\rho L_s J'_\rho = R_s^{-1} \tag{3}$$

From (2) and (3), we have $L_s = R_s$ for all $s \in SN_\rho(G_H)$. In addition, suppose that $SC(G_H) = \{s \in H : sx = xs \text{ for all } x \in G\}$,
then

$$s \in SN_\rho(G_H) \subseteq SC(G_H)$$

Analogously, let $s \in SN_\rho(G_H)$. Using Theorem 3.4 and $s \in SN_\rho(G_H) \subseteq SC(G_H)$, we have

$$s \cdot xz = xz \cdot s = x \cdot zs = x \cdot sz = xs \cdot z = sx \cdot z$$

for all $s \in H$ and $x, z \in G$. So, $s \in SN_\lambda(G_H)$. Thus,

$$SN_\rho(G_H) \subseteq SC(G_H) \cap SN_\lambda(G_H)$$

Hence,

$$SN_\rho(G_H) \subseteq SC(G_H) \cap SN_\lambda(G_H) \cap SN_\rho(G_H) = SZ(G_H)$$

But, $SZ(G_H) \subseteq SN_\rho(G_H)$. Thus, $SN_\rho(G_H) = SZ(G_H)$ $\square$

Remark 3.6. *In detains, under the hypothesis of Theorem 3.2, $SN_\rho(G_H) \subseteq SN_\lambda(G_H) \cap SC(G_H)$. It does not apparent seems to be true that*

$$SN_\rho(G_H) = SN_\lambda(G_H) \cap SC(G_H)$$

Let $s \in SN_\lambda(G_H) \cap SC(G_H)$, then $R_s = L_s$ and

$$(L_s, I, L_s) = (R_s, I, R_s) \in S_{1st}AUT(G_H, \cdot) \tag{4}$$

Since $(G_H, \cdot)$ is a $S_{2nd}(G_H)BL$, by Theorem 2.4,

$$(R_s^{-1}, L_sR_s, R_s) = (R_s^{-1}, R_s^2, R_s) \in S_{1st}AUT(G_H, \cdot) \tag{5}$$

Using (4) in (5), give $(I, R_s^2, R_s^2) = (I, R_{s^2}, R_{s^2}) \in S_{1st}AUT(G_H, \cdot)$ for all $s \in H$. Thus, $s^2 \in SN_\rho(G_H)$

Theorem 3.7. *Let special loop $(G_H, \cdot)$ be a $S_{2nd}BL$ with second Smarandache Moufang part $S_{2nd}M(G_H)$ of $(G_H, \cdot)$. Then, the following are equivalent:*

1. *$s \in S_{2nd}M(G_H)$.*
2. *$(L_s, R_s, R_sL_s) \in S_{1st}AUT(G_H, \cdot)$*

Proof. Suppose that $s \in S_{2nd}M(G_H)$ for all $s \in H$, we have

$$(s \cdot ys)x = s(y \cdot sx) \tag{6}$$

Put $y = s^{-1}$ in (6), we get $sx = s(s^{-1} \cdot sx) \Leftrightarrow x = s^{-1} \cdot sx \Leftrightarrow xL_sL_{s^{-1}} = x \Leftrightarrow L_sL_{s^{-1}} = e \Leftrightarrow L_s^{-1} = L_{s^{-1}}$.
Consequently, $L_s^{-1} = L_{s^{-1}}$ and $L_{s^{-1}}L_s = e$, that is

$$s \cdot s^{-1}x = x \tag{7}$$

for all $s \in H$, and $x \in G$.

Replacing x by $s^{-1}x$ in (6) and use (7), give $(s \cdot ys)(s^{-1}x) = s \cdot yx$ for all $s \in H$ and $x, y \in G$. So,

$$yR_sL_s \cdot xL_{s^{-1}} = (yx)L_s \Rightarrow$$

A= $(R_sL_s, L_{s^{-1}}, L_s) \in S_{1st}AUT(G_H, \cdot)$. Since $(G_H, \cdot)$ is a second Smarandache Bol loop, by Theorem 2.4, $(R_s^{-1}, L_sR_s, R_s) \in S_{1st}AUT(G_H, \cdot)$.

Let B=

$$(R_sL_s, L_{s^{-1}}, L_s)(R_s^{-1}, L_sR_s, R_s) = (L_s, L_sR_sL_{s^{-1}}, R_sL_s) \in S_{1st}AUT(G_H, \cdot)$$

for all $s \in H$. So, for all $a, b \in G$, we have

$$aL_s \cdot bL_sR_sL_{s^{-1}} = (ab)R_sL_s \tag{8}$$

Set $a = e$ in (8), give

$$bL_sR_sL_{s^{-1}}L_s = bR_sL_s \Leftrightarrow L_sR_s = R_sL_s$$

So, use the last equality in (8), give a new form of B as B= $(L_s, R_s, R_sL_s) \in S_{1st}AUT(G_H, \cdot)$ for all $s \in H$. Thus, 2 hold.

Now, suppose that 2 hold, for all $a, b \in G$ we have

$$aL_s \cdot bR_s = (ab)R_sL_s$$

for all $s \in H$. Set $b = e$, we get $L_sR_s = R_sL_s$.
Let C= $(L_s, L_sR_sL_{s^{-1}}, R_sL_s) \in S_{1st}AUT(G_H, \cdot)$ for all $s \in H$.
Using Theorem 2.4, the product

$$(R_s^{-1}, L_sR_s, R_s)^{-1}(L_s, L_sR_sL_{s^{-1}}, R_sL_s) = (R_sL_s, L_{s^{-1}}, L_s) \in S_{1st}AUT(G_H, \cdot)$$

for all $s \in H$. That is, for all $a, b, \in G$, we have

$$(s \cdot as)(s^{-1}b) = s \cdot ab \tag{9}$$

Let $b \mapsto (sb)$ in 9, give

$$(s \cdot as)b = s(a \cdot sb)$$

for all $s \in H$. Thus, 1 hold □

Theorem 3.8. *Let the special loop $G_H, \cdot)$ be a $S_{2nd}BL$. Then*

1. $SN_\lambda(G_H) \cap SN_\rho(G_H) \subseteq S_{2nd}M(G_H)$
2. $SN_\rho(G_H) \cap S_{2nd}M(G_H) \subseteq SN_\lambda(G_H)$
3. $SN_\mu(G_H) \cap S_{2nd}M(G_H) \subseteq SN_\rho(G_H)$

Proof. 1. Using Lemma 3.1. Let $A = (I, R_s, R_s)$, $B = (L_s, I, L_s)$ and $C = (L_s, R_s, R_sL_s)$ where A, B and C are all first Smarandache autotopisms of G_H for all $s \in H$. Under the componentwise multiplication, we have $AB = C$, that is

$$(I, R_s, R_s)(L_s, I, L_s) = (L_s, R_s, R_sL_s) \in S_{1st}AUT(G_H, \cdot)$$

Thus, $SN_\lambda(G_H) \cap SN_\rho(G_H) \subseteq S_{2nd}M(G_H)$

2. Under the componentwise multiplication, give

$$(I, R_s, R_s)(L_s, R_s, R_sL_s) = (L_s, R_s^2, R_s^2L_s) = (L_s, I, L_S)(I, R_s^2, R_s^2)$$

Thus, $SN_\rho(G_H) \cap S_{2nd}M(G_H) \subseteq SN_\lambda(G_H)$

3. Under the componentwise multiplication, we have

$$(R_s^{-1}, L_s, I)(L_s, R_s, R_sL_s) = (R_s^{-1}L_s, L_sR_s, R_sL_s) = (I, R_S, R_S)(R_s^{-1}L_s, L_s, L_s)$$

Thus, $SN_\mu(G_H) \cap S_{2nd}M(G_H) \subseteq SN_\rho(G_H)$

□

Remark 3.9. *Theorem 3.8 revealed that under the componentwise multiplication, whenever any two of A, B and C are first Smarandache autotopisms of the special loop $(G_H, \cdot)$, then the third is also.*

Theorem 3.10. *Let the special loop $G_H, \cdot)$ be a $S_{2nd}BL$ and ϕ be a second Smarandache right pseudo-automorphism of G_H with second Smarandache companion c. Then*

1. $SN_\lambda(G_H)\phi = SN_\lambda(G_H)$ *and ϕ is a first Smarandache automorphism of* $SN_\lambda(G_H)$
2. $SN_\rho(G_H)\phi = SN_\rho(G_H)$ *and ϕ is a first Smarandache automorphism of* $SN_\rho(G_H)$

Proof. Let ϕ be a second Smarandache right pseudo-automorphism of G_H with second Smarandache companion c ($S_{2nd} - companion$). Let $a \in SN_\rho(G_H)$.

$$\begin{aligned} a\phi \cdot (x\phi \cdot (y\phi \cdot c)) &= a\phi \cdot ((xy)\phi \cdot c) = (a \cdot xy)\phi \cdot c \\ &= (ax \cdot y)\phi \cdot c = (ax)\phi \cdot (y\phi \cdot c) \end{aligned}$$

Thus,

$$a\phi \cdot (x\phi \cdot (y\phi \cdot c) = (ax)\phi \cdot (y\phi \cdot c) \tag{10}$$

for all $c \in H$ and $x, y \in G$. Put $y = c^{-1}\phi^{-1}$ in (10), give

$$a\phi \cdot x\phi = (ax)\phi \tag{11}$$

Thus, ϕ is first Smarandache automorphism of $G_H, \cdot)$ for all $a \in SN_\lambda(G) \cap H$ and $x \in G$. Putting (12) in (10), give

$$a\phi \cdot (x\phi \cdot (y\phi \cdot c)) = (a\phi \cdot x\phi) \cdot (y\phi \cdot c)$$

Thus, $SN_\lambda(G_H)\phi \subseteq SN_\lambda(G_H)$. Since in Lemma 3.2, ϕ^{-1} is also second right Smarandache pseudo-automorphism of $G_H, \cdot)$, we have $SN_\lambda(G_H)\phi = SN_\lambda(G_H)$

Similarly, one can also obtain that $SN_\rho(G_H)\phi = SN_\rho(G_H)$ and ϕ is a first Smarandache automorphism of $SN_\rho(G_H)$ for all $s \in H$ and $x, y \in G$.

□

Theorem 3.11. *Let the special loop $G_H, \cdot)$ be a $S_{2nd}BL$ and ϕ be a second Smarandache right pseudo-automorphism of G_H with second Smarandache companion c. Then*

1. *ϕ is a second Smarandache semi-automorphism.*
2. $S_{2nd}M(G_H)\phi = S_{2nd}M(G_H)$ *if ϕ^{-1} is a second Smarandache right pseudo-automorphism of $(G_H \cdot)$*

Proof. Let $s \in S_{2nd}M(G_H)$ for all $s \in H$. Then

$$\begin{aligned} s\phi \cdot \big[y\phi(s\phi \cdot (x\phi \cdot c))\big] &= s\phi \cdot \big[y\phi((sx)\phi \cdot c)\big] \\ = s\phi \cdot \big[(y \cdot sx)\phi \cdot c)\big] &= \big[s(y \cdot sx)\big]\phi \cdot c = \big[(s \cdot ys)x\big]\phi \cdot c = (s \cdot ys)\phi \cdot (x\phi \cdot c) \end{aligned}$$

for all $s \in H$ and $x, y \in G$.

Thus, $s\phi \cdot \big[y\phi(s\phi \cdot (x\phi \cdot c))\big] = (s \cdot ys)\phi \cdot (x\phi \cdot c)$. Let $a = x\phi \cdot c$ in the last equality, give

$$s\phi \cdot \big[y\phi(s\phi \cdot a\big] = (s \cdot ys)\phi \cdot a \tag{12}$$

for all $s \in H$ and $x, y \in G$.

Let $a = e$ in (12), we get

$$s\phi \cdot (y\phi \cdot s\phi) = (s \cdot ys)\phi \tag{13}$$

Thus, ϕ is a second Smarandache semi-automorphism. Put (13) in (12), get

$$s\phi \cdot \big[y\phi(s\phi \cdot a\big] = [s\phi \cdot (y\phi \cdot s\phi)] \cdot a$$

For all $s \in H$. Then, $s\phi \in S_{2nd}M(G_H)$ whenever $s \in S_{2nd}M(G_H)$.
For the fact that ϕ^{-1} is also second Smarandache right pseudo-automorphism of $(G_H\cdot)$, give

$$S_{2nd}M(G_H)\phi = S_{2nd}M(G_H)$$

for all $s \in H$ and $x, y \in G$. □

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Error estimates of BBP- and Ramanujan-type series

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Abstract The aim of this paper is to establish some asymptotic expansions and inequalities related to BBP- and Ramanujan-type approximation series of the constant π.

Keywords Constant π, asymptotic expansion, inequality.
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§1. Introduction

With the aid of the powerful PSLQ algorithm [3,4], Bailey, Borwein and Plouffe [2] established an amazing series for π:

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right). \tag{1}$$

The formula is significant as it permits the computation of the nth hexadecimal digit of π without calculation the preceding $n-1$ digits. Also in [2], these authors presented the following series:

$$\pi^2 = \frac{9}{8} \sum_{k=0}^{\infty} \frac{1}{64^k} \left(\frac{16}{(6k+1)^2} - \frac{24}{(6k+2)^2} - \frac{8}{(6k+3)^2} - \frac{6}{(6k+4)^2} + \frac{1}{(6k+5)^2} \right). \tag{2}$$

Since the publication of [2], formulas of similar form have been discovered and have become known as BBP-type series. A base-b BBP-type formula is a convergent series formula of the type

$$C = \sum_{k=0}^{\infty} \frac{p(k)}{b^k q(k)},$$

where $p(k)$ and $q(k)$ are integer polynomials in k (see [1] and [4, pp. 54 and 128-129]). Bailey [1] and Borwein and Bailey [4, 128-129] gave a collection of such series.

Ramanujan [6] gave 17 series for $1/\pi$ which are of the following form:

$$\frac{1}{\pi} = \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k (s)_k (1-s)_k}{(1)_k^3} (a+bk) x^k, \qquad (s)_k = \frac{\Gamma(s+k)}{\Gamma(s)}, \tag{3}$$

where $s \in \{1/2, 1/4, 1/3, 1/6\}$ and the parameters x, a, b are algebraic numbers. One example is

$$\frac{1}{\pi} = \frac{1}{16}\sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k^3}{(1)_k^3} \frac{42k+5}{2^{6k}} = \frac{1}{16}\sum_{k=0}^{\infty} \binom{2k}{k}^3 \frac{42k+5}{2^{12k}} = \frac{1}{16}\sum_{k=0}^{\infty} \frac{((2k)!)^3}{(k!)^6} \frac{42k+5}{4096^k}. \tag{4}$$

For $n \geq 1$, let

$$\beta_n = \sum_{k=0}^{n} \frac{1}{16^k}\left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6}\right), \tag{5}$$

$$\gamma_n = \frac{9}{8}\sum_{k=0}^{n} \frac{1}{64^k}\left(\frac{16}{(6k+1)^2} - \frac{24}{(6k+2)^2} - \frac{8}{(6k+3)^2} - \frac{6}{(6k+4)^2} + \frac{1}{(6k+5)^2}\right) \tag{6}$$

and

$$\rho_n = \frac{1}{16}\sum_{k=0}^{n} \frac{((2k)!)^3}{(k!)^6} \frac{42k+5}{4096^k}. \tag{7}$$

Mortici [5] considered the error estimates and proved that for all integers $n \geq 1$,

$$\frac{1}{\left(64n^2 + 184n + 114 + \frac{21}{n+1}\right)16^n} < \pi - \beta_n < \frac{1}{\left(64n^2 + 184n + 114\right)16^n}, \tag{8}$$

$$\frac{9}{8\left(108n^2 + 348n + \frac{883}{3} + \frac{644}{27(n+1)}\right)64^n} < \gamma_n - \pi^2 < \frac{9}{8\left(108n^2 + 348n + \frac{883}{3}\right)64^n} \tag{9}$$

and

$$\rho_n^{(1)} < \frac{1}{\pi} - \rho_n < \rho_n^{(2)}, \tag{10}$$

where

$$\rho_n^{(1)} = \frac{((2n+2)!)^3}{16((n+1)!)^6 4096^{n+1}}\left(\frac{128}{3}n + \frac{1280}{27} + \frac{32}{9(9n+19)}\right)$$

and

$$\rho_n^{(2)} = \frac{((2n+2)!)^3}{16((n+1)!)^6 4096^{n+1}}\left(\frac{128}{3}n + \frac{1280}{27} + \frac{128(n+1)}{324n^2 + 1008n + 677}\right).$$

Using the Maple software, from (8), (9) and (10) we find, as $n \to \infty$,

$$\pi - \beta_n = \frac{1}{16^n}\left\{\frac{1}{64n^2} - \frac{23}{512n^3} + \frac{415}{4096n^4} + O\left(\frac{1}{n^5}\right)\right\}, \tag{11}$$

$$\gamma_n - \pi^2 = \frac{1}{64^n}\left\{\frac{1}{96n^2} - \frac{29}{864n^3} + \frac{827}{10368n^4} + O\left(\frac{1}{n^5}\right)\right\} \tag{12}$$

and

$$\frac{1}{\pi} - \rho_n = \frac{((2n)!)^3 n}{16(n!)^6 4096^n}\left\{\frac{2}{3} - \frac{7}{27n} + \frac{32}{81n^2} - \frac{320}{729n^3} + O\left(\frac{1}{n^4}\right)\right\}. \tag{13}$$

Using the Maple software, formula (13) is given in the appendix.

In this paper, we develop the formulas (11), (12) and (13) to produce complete asymptotic expansions, and then establish the asymptotic inequalities for $\pi - \beta_n$, $\gamma_n - \pi^2$ and $\frac{1}{\pi} - \rho_n$.

The numerical values given in this paper have been calculated via the computer program MAPLE 13.

§2. BBP-type series

Theorem 2.1. *As $n \to \infty$, we have*

$$\begin{aligned}\pi - \beta_n &\sim \frac{1}{16^n}\sum_{k=2}^{\infty}\frac{\lambda_k}{n^k}\\ &= \frac{1}{16^n}\left(\frac{1}{64n^2} - \frac{23}{512n^3} + \frac{415}{4096n^4} - \frac{7091}{32768n^5} + \frac{123871}{262144n^6} - \frac{2321003}{2097152n^7} + \dots\right),\end{aligned} \tag{14}$$

where the coefficients λ_k $(k \geq 2)$ are given by the recurrence relation

$$\begin{aligned}\lambda_2 &= \frac{1}{64} \quad \textit{and}\\ \lambda_k &= \frac{1}{15}\left\{\sum_{j=2}^{k-1}\lambda_j(-1)^{k-j}\binom{k-1}{k-j}\right.\\ &\qquad \left. + (-1)^k\left(\frac{1}{8}\left(\frac{7}{4}\right)^{k-1} - \frac{1}{2}\left(\frac{9}{8}\right)^{k-1} + \frac{1}{8}\left(\frac{13}{8}\right)^{k-1} + \frac{1}{4}\left(\frac{3}{2}\right)^{k-1}\right)\right\}\end{aligned} \tag{15}$$

for $k \geq 3$.

Proof. In view of (11), we can let

$$\pi - \beta_n \sim \frac{1}{16^n}\sum_{k=2}^{\infty}\frac{\lambda_k}{n^k} \quad \text{as} \quad n \to \infty,$$

where λ_k $(k \geq 2)$ are real numbers to be determined. Denote

$$X_n = \pi - \beta_n \quad \text{and} \quad Y_n = \frac{1}{16^n}\sum_{k=2}^{\infty}\frac{\lambda_k}{n^k}.$$

We can let $X_n \sim Y_n$ and

$$\Delta X_n := X_{n+1} - X_n \sim Y_{n+1} - Y_n =: \Delta Y_n \quad \text{as} \quad n \to \infty.$$

We have

$$\begin{aligned}\Delta X_n = \beta_n - \beta_{n+1} &= -\frac{1}{16^{n+1}}\frac{120n^2 + 391n + 318}{(4n+7)(8n+9)(8n+13)(2n+3)}\\ &= -\frac{1}{16^{n+1}}\left(-\frac{1}{8n(1+\frac{7}{4n})} + \frac{1}{2n(1+\frac{9}{8n})} - \frac{1}{8n(1+\frac{13}{8n})} - \frac{1}{4n(1+\frac{3}{2n})}\right)\\ &= \frac{1}{16^{n+1}}\sum_{k=2}^{\infty}(-1)^{k-1}\left(\frac{1}{8}\left(\frac{7}{4}\right)^{k-1} - \frac{1}{2}\left(\frac{9}{8}\right)^{k-1} + \frac{1}{8}\left(\frac{13}{8}\right)^{k-1} + \frac{1}{4}\left(\frac{3}{2}\right)^{k-1}\right)\frac{1}{n^k}.\end{aligned} \tag{16}$$

Direct computation yields

$$\begin{aligned}\sum_{k=2}^{\infty}\frac{\lambda_k}{(n+1)^k} &= \sum_{k=2}^{\infty}\frac{\lambda_k}{n^k}\left(1+\frac{1}{n}\right)^{-k} = \sum_{k=2}^{\infty}\frac{\lambda_k}{n^k}\sum_{j=0}^{\infty}\binom{-k}{j}\frac{1}{n^j}\\ &= \sum_{k=2}^{\infty}\frac{\lambda_k}{n^k}\sum_{j=0}^{\infty}(-1)^j\binom{k+j-1}{j}\frac{1}{n^j} = \sum_{k=2}^{\infty}\sum_{j=2}^{k}\lambda_j(-1)^{k-j}\binom{k-1}{k-j}\frac{1}{n^k}.\end{aligned} \tag{17}$$

We then obtain

$$\Delta Y_n = \frac{1}{16^{n+1}} \sum_{k=2}^{\infty} \left\{ \sum_{j=2}^{k} \lambda_j (-1)^{k-j} \binom{k-1}{k-j} - 16\lambda_k \right\} \frac{1}{n^k}. \tag{18}$$

Equating coefficients of the term n^{-k} on the right-hand sides of (16) and (18) yields

$$(-1)^{k-1}\left(\frac{1}{8}\left(\frac{7}{4}\right)^{k-1} - \frac{1}{2}\left(\frac{9}{8}\right)^{k-1} + \frac{1}{8}\left(\frac{13}{8}\right)^{k-1} + \frac{1}{4}\left(\frac{3}{2}\right)^{k-1}\right)$$
$$= \sum_{j=2}^{k} \lambda_j (-1)^{k-j} \binom{k-1}{k-j} - 16\lambda_k \quad \text{for} \quad k \geq 2.$$

For $k = 2$ we obtain $\lambda_2 = \frac{1}{64}$, and for $k \geq 3$ we have

$$(-1)^{k-1}\left(\frac{1}{8}\left(\frac{7}{4}\right)^{k-1} - \frac{1}{2}\left(\frac{9}{8}\right)^{k-1} + \frac{1}{8}\left(\frac{13}{8}\right)^{k-1} + \frac{1}{4}\left(\frac{3}{2}\right)^{k-1}\right)$$
$$= \sum_{j=2}^{k-1} \lambda_j (-1)^{k-j} \binom{k-1}{k-j} - 15\lambda_k,$$

which gives the desired formula (15). The proof of Theorem 2.1 is complete. □

Theorem 2.2. For all integers $n \geq 1$, we have

$$\beta_n^{(1)} < \pi - \beta_n < \beta_n^{(2)}, \tag{19}$$

where

$$\beta_n^{(1)} = \frac{1}{16^n}\left(\frac{1}{64n^2} - \frac{23}{512n^3} + \frac{415}{4096n^4} - \frac{7091}{32768n^5}\right)$$

and

$$\beta_n^{(2)} = \frac{1}{16^n}\left(\frac{1}{64n^2} - \frac{23}{512n^3} + \frac{415}{4096n^4} - \frac{7091}{32768n^5} + \frac{123871}{262144n^6}\right).$$

Proof. For $n \geq 1$, let

$$x_n = \pi - \beta_n - \beta_n^{(1)} \quad \text{and} \quad y_n = \pi - \beta_n - \beta_n^{(2)}.$$

By (14), we have

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0.$$

In order to prove (19), it suffices to show that the sequence $\{x_n\}$ is strictly decreasing and $\{y_n\}$ is strictly increasing for $n \geq 1$. Direct computation yields

$$x_{n+1} - x_n = \beta_n - \beta_{n+1} + \beta_n^{(1)} - \beta_{n+1}^{(1)}$$
$$= -\frac{P_8(n)}{524288n^5(8n+9)(2n+3)(8n+13)(4n+7)(n+1)^5 16^n}$$

and

$$\begin{aligned}y_{n+1}-y_n &= \beta_n-\beta_{n+1}+\beta_n^{(2)}-\beta_{n+1}^{(2)}\\ &= \frac{P_9(n)}{4194304n^6(8n+9)(2n+3)(8n+13)(4n+7)(n+1)^6 16^n},\end{aligned}$$

where

$$\begin{aligned}P_8(n) = {}& 118916160n^8+1077123864n^7+4233436082n^6+9449307021n^5\\ &+13127042608n^4+11639027744n^3+6435166400n^2\\ &+2027757648n+278761392\end{aligned}$$

and

$$\begin{aligned}P_9(n) = {}& 2228162880n^9+22006726904n^8+96209251962n^7+244893891841n^6\\ &+400582591808n^5+436904487312n^4+317620640864n^3\\ &+148305960784n^2+40341894144n+4869616752.\end{aligned}$$

Hence, we have, for $n \geq 1$,

$$x_{n+1} < x_n \quad \text{and} \quad y_{n+1} > y_n.$$

The proof of Theorem 2.2 is complete. □

Remark 2.1. For all integers $n \geq 1$, we have

$$\beta_n^{(3)} < \pi - \beta_n < \beta_n^{(2)}, \tag{20}$$

where

$$\beta_n^{(3)} = \frac{1}{16^n}\left(\frac{1}{64n^2}-\frac{23}{512n^3}+\frac{415}{4096n^4}-\frac{7091}{32768n^5}+\frac{123871}{262144n^6}-\frac{2321003}{2097152n^7}\right).$$

Following the same method as was used in the proof of Theorem 2.2, we can prove the left-hand side of (20). Here, we omit the proof. For $n \geq 20$, the inequalities (20) are sharper than the inequalities (8). Write (20) as

$$r_n < \pi < s_n, \tag{21}$$

where

$$r_n = \beta_n + \beta_n^{(3)} \quad \text{and} \quad s_n = \beta_n + \beta_n^{(2)}.$$

For $n = 10$ in (21), we have

$$\begin{aligned} r_{10} &= 3.1415926535897932384\ldots,\\ s_{10} &= 3.1415926535897932385\ldots. \end{aligned}$$

We then get the approximate value of π,

$$\pi \approx 3.141592653589793238.$$

The choice $n = 100$ in (21) yields the approximate value of π,

$$\begin{aligned}\pi \approx\, & 3.14159265358979323846264338327950288419716939937510\\ & 58209749445923078164062862089986280348253421170679\\ & 8214808651328230664709384460955058.\end{aligned}$$

Theorem 2.3. As $n \to \infty$, we have

$$\begin{aligned}\gamma_n - \pi^2 &\sim \frac{1}{64^n}\sum_{k=2}^{\infty}\frac{\mu_k}{n^k}\\ &= \frac{1}{64^n}\left(\frac{1}{96n^2} - \frac{29}{864n^3} + \frac{827}{10368n^4} - \frac{7831}{46656n^5} + \frac{1121965}{3359232n^6} - \frac{6580343}{10077696n^7} + \dots\right),\end{aligned} \tag{22}$$

with the coefficients μ_k ($k \ge 2$) given by the recurrence relation

$$\mu_2 = \frac{1}{96}, \quad \mu_k = \frac{1}{63}\left\{64q_k + \sum_{j=2}^{k-1}\mu_j(-1)^{k-j}\binom{k-1}{k-j}\right\}, \tag{23}$$

where

$$\begin{aligned}q_k = \frac{(-1)^k(k-1)}{5464166400}\Bigg(& -31363200\left(\frac{7}{6}\right)^k + 36018675\left(\frac{4}{3}\right)^k\\ & + 9486400\left(\frac{3}{2}\right)^k + 5762988\left(\frac{5}{3}\right)^k - 793800\left(\frac{11}{6}\right)^k\Bigg)\end{aligned}$$

for $k \ge 3$.

Proof. In view of (12), we can let

$$\gamma_n - \pi^2 \sim \frac{1}{64^n}\sum_{k=2}^{\infty}\frac{\mu_k}{n^k} \quad \text{as} \quad n \to \infty,$$

where μ_k ($k \ge 2$) are real numbers to be determined. Denote

$$I_n = \gamma_n - \pi^2 \quad \text{and} \quad J_n = \frac{1}{64^n}\sum_{k=2}^{\infty}\frac{\mu_k}{n^k}.$$

We can let $I_n \sim J_n$ and

$$\Delta I_n := I_{n+1} - I_n \sim J_{n+1} - J_n =: \Delta J_n \quad \text{as} \quad n \to \infty.$$

Direct computation yields

$$\Delta I_n = \gamma_{n+1} - \gamma_n = -\frac{1}{64^n}\frac{Q_8(n)}{1204(6n+7)^2(3n+4)^2(2n+3)^2(3n+5)^2(6n+11)^2},$$

where

$$Q_8(n) = 76444252 + 446923440n + 1124695053n^2 + 1595519856n^3 + 1398337506n^4$$

$$+ 776449152n^5 + 267058944n^6 + 52068096n^7 + 4408992n^8.$$

We find

$$\begin{aligned}
&\frac{Q_8(n)}{1204(6n+7)^2(3n+4)^2(2n+3)^2(3n+5)^2(6n+11)^2}\\
&= \frac{-\frac{9}{32}}{(6n+7)^2} + \frac{\frac{27}{256}}{(3n+4)^2} + \frac{\frac{1}{64}}{(2n+3)^2} + \frac{\frac{27}{1024}}{(3n+5)^2} + \frac{-\frac{9}{512}}{(6n+11)^2}\\
&= \frac{-\frac{9}{32}}{(6n)^2(1+\frac{7}{6n})^2} + \frac{\frac{27}{256}}{(3n)^2(1+\frac{4}{3n})^2} + \frac{\frac{1}{64}}{(2n)^2(1+\frac{3}{2n})^2}\\
&\quad + \frac{\frac{27}{1024}}{(3n)^2(1+\frac{5}{3n})^2} + \frac{-\frac{9}{512}}{(6n)^2(1+\frac{11}{6n})^2}\\
&= \sum_{k=2}^{\infty} \frac{q_k}{n^k},
\end{aligned}$$

where

$$q_k = \frac{(-1)^k(k-1)}{5464166400}\left\{ -31363200\left(\frac{7}{6}\right)^k + 36018675\left(\frac{4}{3}\right)^k + 9486400\left(\frac{3}{2}\right)^k + 5762988\left(\frac{5}{3}\right)^k - 793800\left(\frac{11}{6}\right)^k \right\}.$$

We then obtain

$$64^{n+1}\Delta I_n = -\sum_{k=2}^{\infty} \frac{64q_k}{n^k}. \tag{24}$$

Using (17), we find

$$64^{n+1}\Delta J_n = \sum_{k=2}^{\infty} \left\{ \sum_{j=2}^{k} \mu_j(-1)^{k-j}\binom{k-1}{k-j} - 64\mu_k \right\} \frac{1}{n^k}. \tag{25}$$

Equating coefficients of the term n^{-k} on the right-hand sides of (24) and (25) yields

$$-64q_k = \sum_{j=2}^{k} \mu_j(-1)^{k-j}\binom{k-1}{k-j} - 64\mu_k \quad \text{for} \quad k \geq 2.$$

For $k = 2$ we obtain $\mu_2 = \frac{1}{96}$, and for $k \geq 3$ we have

$$-64q_k = \sum_{j=2}^{k-1} \mu_j(-1)^{k-j}\binom{k-1}{k-j} - 63\mu_k,$$

which gives the desired formula (23). The proof of Theorem 2.3 is complete. □

Following the same method as was used in the proof of Theorem 2.2, we can prove Theorem 2.4 below. Here, we omit the proof.

Theorem 2.4. For all integers $n \geq 1$, we have

$$\gamma_n^{(1)} < \gamma_n - \pi^2 < \gamma_n^{(2)}, \tag{26}$$

where

$$\gamma_n^{(1)} = \frac{1}{64^n}\left(\frac{1}{96n^2} - \frac{29}{864n^3} + \frac{827}{10368n^4} - \frac{7831}{46656n^5} + \frac{1121965}{3359232n^6} - \frac{6580343}{10077696n^7}\right)$$

and

$$\gamma_n^{(2)} = \frac{1}{64^n}\left(\frac{1}{96n^2} - \frac{29}{864n^3} + \frac{827}{10368n^4} - \frac{7831}{46656n^5} + \frac{1121965}{3359232n^6}\right).$$

Remark 2.2. For $n \geq 44$, the inequalities (26) are sharper than the inequalities (9).

§3. Ramanujan-type series for $1/\pi$

Theorem 3.1. As $n \to \infty$, we have

$$\begin{aligned}\frac{1}{\pi} - \rho_n &\sim \frac{((2n)!)^3 n}{16(n!)^6 4096^n}\sum_{k=0}^{\infty}\frac{\nu_k}{n^k} \\ &= \frac{((2n)!)^3 n}{16(n!)^6 4096^n}\left\{\frac{2}{3} - \frac{7}{27n} + \frac{32}{81n^2} - \frac{320}{729n^3} + \frac{3256}{6561n^4} - \frac{3896}{6561n^5}\right. \\ &\qquad \left. + \frac{46730}{59049n^6} - \frac{652264}{531441n^7} + \frac{7137977}{3188646n^8} - \cdots\right\}, \end{aligned} \tag{27}$$

where the coefficients ν_k $(k \geq 0)$ are given by the recurrence relation

$$\begin{aligned} \nu_0 &= \frac{2}{3}, \quad \nu_1 = -\frac{7}{27}, \\ \nu_k &= \frac{64}{63}\left\{(-1)^k\left(\frac{153}{512} + \frac{49}{1024}k - \frac{5}{1024}k^2\right)\right. \\ &\qquad + \sum_{j=0}^{k-1}\nu_j(-1)^{k-j}\left[\frac{1}{64}\binom{k-1}{k-j} + \frac{1}{128}\binom{k-2}{k-j-1}\right] \\ &\qquad \left. + \sum_{j=2}^{k}\sum_{\ell=0}^{k-j}(-1)^{k-\ell}\frac{(j+4)\nu_\ell}{512}\binom{k-j-1}{k-j-\ell}\right\} \end{aligned} \tag{28}$$

for $k \geq 2$.

Proof. In view of (13), we can let

$$\frac{1}{\pi} - \rho_n \sim \frac{((2n)!)^3 n}{16(n!)^6 4096^n}\sum_{k=0}^{\infty}\frac{\nu_k}{n^k} \quad \text{as} \quad n \to \infty,$$

where ν_k $(k \geq 0)$ are real numbers to be determined. Denote

$$U_n = \frac{1}{\pi} - \rho_n \quad \text{and} \quad V_n = \frac{((2n)!)^3 n}{16(n!)^6 4096^n}\sum_{k=0}^{\infty}\frac{\nu_k}{n^k}.$$

We can let $U_n \sim V_n$ and

$$\Delta U_n := U_{n+1} - U_n \sim V_{n+1} - V_n =: \Delta V_n \quad \text{as} \quad n \to \infty.$$

We have

$$\begin{aligned}\Delta U_n &= \rho_n - \rho_{n+1} = -\frac{((2n+2)!)^3}{16\cdot((n+1)!)^6}\frac{42(n+1)+5}{4096^{n+1}}\\ &= -\frac{((2n)!)^3 n}{16\cdot(n!)^6 4096^n}\frac{(42n+47)(2n+1)^3}{512n(n+1)^3}\\ &= -\frac{((2n)!)^3 n}{16\cdot(n!)^6 4096^n}\left\{\frac{21}{32}+\frac{1}{512}\left(\frac{47}{n}-\frac{175}{n+1}+\frac{17}{(n+1)^2}+\frac{5}{(n+1)^3}\right)\right\}. \end{aligned}\tag{29}$$

Direct computation yields

$$\frac{47}{n}-\frac{175}{n+1}+\frac{17}{(n+1)^2}+\frac{5}{(n+1)^3} = -\frac{128}{n}+\sum_{k=2}^{\infty}(-1)^k\left(153+\frac{49}{2}k-\frac{5}{2}k^2\right)\frac{1}{n^k}.\tag{30}$$

Substituting (30) into (29) yields, as $n\to\infty$,

$$\frac{16\cdot(n!)^6 4096^n}{((2n)!)^3 n}\Delta U_n \sim -\frac{21}{32}+\frac{1}{4n}+\sum_{k=2}^{\infty}\frac{p_k}{n^k},\tag{31}$$

where

$$p_k = (-1)^{k-1}\left(\frac{153}{512}+\frac{49}{1024}k-\frac{5}{1024}k^2\right),\qquad k\ge 2.$$

We have

$$\begin{aligned}\Delta V_n &= \frac{((2n+2)!)^3(n+1)}{16((n+1)!)^6 4096^{n+1}}\sum_{k=0}^{\infty}\frac{\nu_k}{(n+1)^k} - \frac{((2n)!)^3 n}{16(n!)^6 4096^n}\sum_{k=0}^{\infty}\frac{\nu_k}{n^k}\\ &= \frac{((2n)!)^3 n}{16(n!)^6 4096^n}\left\{\frac{(2n+1)^3}{512n(n+1)^2}\sum_{k=0}^{\infty}\frac{\nu_k}{(n+1)^k}-\sum_{k=0}^{\infty}\frac{\nu_k}{n^k}\right\}.\end{aligned}$$

It is easy to see that

$$\begin{aligned}\frac{(2n+1)^3}{512n(n+1)^2} &= \frac{1}{512}\left(8+\frac{1}{n}-\frac{5}{n+1}+\frac{1}{(n+1)^2}\right)\\ &= \frac{1}{512}\left(8-\frac{4}{n}+\sum_{k=2}^{\infty}\frac{(-1)^k(k+4)}{n^k}\right)=\sum_{k=0}^{\infty}\frac{a_k}{n^k},\end{aligned}$$

where

$$a_0=\frac{1}{64},\quad a_1=-\frac{1}{128}\quad\text{and}\quad a_k=\frac{(-1)^k(k+4)}{512}\quad\text{for}\quad k\ge 2.$$

Direct computation yields

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{\nu_k}{(n+1)^k} &= \sum_{k=0}^{\infty}\frac{\nu_k}{n^k}\left(1+\frac{1}{n}\right)^{-k} = \sum_{k=0}^{\infty}\frac{\nu_k}{n^k}\sum_{j=0}^{\infty}\binom{-k}{j}\frac{1}{n^j}\\ &= \sum_{k=0}^{\infty}\frac{\nu_k}{n^k}\sum_{j=0}^{\infty}(-1)^j\binom{k+j-1}{j}\frac{1}{n^j} = \sum_{k=0}^{\infty}\frac{b_k}{n^k},\end{aligned}$$

where

$$b_k=\sum_{\ell=0}^{k}\nu_\ell(-1)^{k-\ell}\binom{k-1}{k-\ell}.$$

We then obtain, as $n\to\infty$,

$$\begin{aligned}\frac{16(n!)^6 4096^n}{((2n)!)^3 n}\Delta V_n &\sim \frac{(2n+1)^3}{512n(n+1)^2}\sum_{k=0}^{\infty}\frac{\nu_k}{(n+1)^k}-\sum_{k=0}^{\infty}\frac{\nu_k}{n^k}\\ &=\sum_{k=0}^{\infty}\frac{a_k}{n^k}\sum_{k=0}^{\infty}\frac{b_k}{n^k}-\sum_{k=0}^{\infty}\frac{\nu_k}{n^k}=\sum_{k=0}^{\infty}\left\{\sum_{j=0}^{k}a_jb_{k-j}-\nu_k\right\}\frac{1}{n^k}.\end{aligned}\tag{32}$$

Equating coefficients of the term n^{-k} on the right-hand sides of (31) and (32), we obtain

$$-\frac{21}{32}=a_0b_0-\nu_0=\frac{1}{64}\nu_0-\nu_0\Longrightarrow\nu_0=\frac{2}{3},$$

$$\frac{1}{4}=a_0b_1+a_1b_0-\nu_1=\frac{1}{64}\nu_1+\left(-\frac{1}{128}\right)\nu_0-\nu_1\Longrightarrow\nu_1=-\frac{7}{27},$$

and for $k\ge 2$,

$$p_k=\sum_{j=0}^{k}a_jb_{k-j}-\nu_k=a_0b_k+a_1b_{k-1}+\sum_{j=2}^{k}a_jb_{k-j}-\nu_k,$$

$$\nu_k=-p_k+\frac{1}{64}\sum_{j=0}^{k}\nu_j(-1)^{k-j}\binom{k-1}{k-j}-\frac{1}{128}b_{k-1}+\sum_{j=2}^{k}a_jb_{k-j},$$

$$\nu_k=-p_k+\frac{1}{64}\sum_{j=0}^{k-1}\nu_j(-1)^{k-j}\binom{k-1}{k-j}+\frac{1}{64}\nu_k-\frac{1}{128}b_{k-1}+\sum_{j=2}^{k}a_jb_{k-j},$$

$$\nu_k=\frac{64}{63}\left\{-p_k+\frac{1}{64}\sum_{j=0}^{k-1}\nu_j(-1)^{k-j}\binom{k-1}{k-j}-\frac{1}{128}b_{k-1}+\sum_{j=2}^{k}a_jb_{k-j}\right\},$$

which gives the desired formula (28). The proof of Theorem 3.1 is complete. □

By using the formula (28), we now show how easily we can determine ν_k's in (27). We obtain the first few coefficients λ_k as follows:

$$\begin{aligned}&\nu_0=\frac{2}{3},\quad \nu_1=-\frac{7}{27},\\ &\nu_2=\frac{8}{21}+\frac{1}{84}\nu_0-\frac{1}{42}\nu_1=\frac{32}{81},\\ &\nu_3=-\frac{17}{42}-\frac{1}{72}\nu_0+\frac{1}{28}\nu_1-\frac{5}{126}\nu_2=-\frac{320}{729},\end{aligned}$$

$$\begin{aligned}
\nu_4 &= \frac{211}{504} + \frac{1}{63}\nu_0 - \frac{25}{504}\nu_1 + \frac{19}{252}\nu_2 - \frac{1}{18}\nu_3 = \frac{3256}{6561},\\
\nu_5 &= -\frac{71}{168} - \frac{1}{56}\nu_0 + \frac{11}{168}\nu_1 - \frac{1}{8}\nu_2 + \frac{11}{84}\nu_3 - \frac{1}{14}\nu_4 = -\frac{3896}{6561},\\
\nu_6 &= \frac{5}{12} + \frac{5}{252}\nu_0 - \frac{1}{12}\nu_1 + \frac{4}{21}\nu_2 - \frac{43}{168}\nu_3 + \frac{17}{84}\nu_4 - \frac{11}{126}\nu_5 = \frac{46730}{59049},\\
\nu_7 &= -\frac{101}{252} - \frac{11}{504}\nu_0 + \frac{13}{126}\nu_1 - \frac{23}{84}\nu_2 + \frac{25}{56}\nu_3 - \frac{11}{24}\nu_4 + \frac{73}{252}\nu_5 - \frac{13}{126}\nu_6 = -\frac{652264}{531441},\\
\nu_8 &= \frac{3}{8} + \frac{1}{42}\nu_0 - \frac{1}{8}\nu_1 + \frac{95}{252}\nu_2 - \frac{121}{168}\nu_3 + \frac{19}{21}\nu_4 - \frac{377}{504}\nu_5 + \frac{11}{28}\nu_6 - \frac{5}{42}\nu_7 = \frac{7137977}{3188646}.
\end{aligned}$$

We note that the values of ν_k (for $k = 0, 1, 2, 3$) here are equal to the coefficients of $1/n^k$ (for $k = 0, 1, 2, 3$) in (13), respectively.

Theorem 3.2. For all integers $n \geq 1$, we have

$$\rho_n^{(3)} < \frac{1}{\pi} - \rho_n < \rho_n^{(4)} \tag{33}$$

and

$$\rho_n^{(5)} < \frac{1}{\pi} - \rho_n < \rho_n^{(6)}, \tag{34}$$

where

$$\rho_n^{(3)} = \frac{((2n)!)^3 n}{16(n!)^6 4096^n}\left(\frac{2}{3} - \frac{7}{27n} + \frac{32}{81n^2} - \frac{320}{729n^3} + \frac{3256}{6561n^4} - \frac{3896}{6561n^5}\right),$$

$$\rho_n^{(4)} = \frac{((2n)!)^3 n}{16(n!)^6 4096^n}\left(\frac{2}{3} - \frac{7}{27n} + \frac{32}{81n^2} - \frac{320}{729n^3} + \frac{3256}{6561n^4} - \frac{3896}{6561n^5} + \frac{46730}{59049n^6}\right),$$

$$\rho_n^{(5)} = \frac{((2n)!)^3 n}{16(n!)^6 4096^n}\left(\frac{2}{3} - \frac{7}{27n} + \frac{32}{81n^2} - \frac{320}{729n^3} + \frac{3256}{6561n^4} - \frac{3896}{6561n^5} + \frac{46730}{59049n^6} - \frac{652264}{531441n^7}\right)$$

and

$$\begin{aligned}
\rho_n^{(6)} = \frac{((2n)!)^3 n}{16(n!)^6 4096^n}\biggl(&\frac{2}{3} - \frac{7}{27n} + \frac{32}{81n^2} - \frac{320}{729n^3} + \frac{3256}{6561n^4}\\
&- \frac{3896}{6561n^5} + \frac{46730}{59049n^6} - \frac{652264}{531441n^7} + \frac{7137977}{3188646n^8}\biggr).
\end{aligned}$$

Proof. We only prove inequality (33). The proof of (34) is analogous. For $n \geq 1$, let

$$\theta_n = \frac{1}{\pi} - \rho_n - \rho_n^{(3)} \quad \text{and} \quad \vartheta_n = \frac{1}{\pi} - \rho_n - \rho_n^{(4)}.$$

We have

$$\lim_{n\to\infty} \theta_n = \lim_{n\to\infty} \vartheta_n = 0.$$

In order to prove (33), it suffices to show that the sequence $\{\theta_n\}$ is strictly decreasing and $\{\vartheta_n\}$ is strictly increasing for $n \geq 1$. Direct computation yields

$$\theta_{n+1} - \theta_n = \rho_n - \rho_{n+1} + \rho_n^{(3)} - \rho_{n+1}^{(3)}$$

$$
\begin{aligned}
&= -\frac{((2n)!)^3 n}{16\cdot(n!)^6 4096^n}\frac{(42n+47)(2n+1)^3}{512n(n+1)^3}\\
&\quad+\frac{((2n)!)^3 n}{16(n!)^6 4096^n}\left(\frac{2}{3}-\frac{7}{27n}+\frac{32}{81n^2}-\frac{320}{729n^3}+\frac{3256}{6561n^4}-\frac{3896}{6561n^5}\right)\\
&\quad-\frac{((2n)!)^3 n}{16(n!)^6 4096^n}\frac{((2n+2)(2n+1))^3}{4096(n+1)^6}\left(1+\frac{1}{n}\right)\\
&\quad\times\left(\frac{2}{3}-\frac{7}{27(n+1)}+\frac{32}{81(n+1)^2}-\frac{320}{729(n+1)^3}+\frac{3256}{6561(n+1)^4}-\frac{3896}{6561(n+1)^5}\right)\\
&= -\frac{((2n)!)^3 n}{16\cdot(n!)^6 4096^n}\frac{P_6(n)}{419904n^5(n+1)^7}
\end{aligned}
$$

and

$$
\begin{aligned}
\vartheta_{n+1}-\vartheta_n &= \rho_n-\rho_{n+1}+\rho_n^{(4)}-\rho_{n+1}^{(4)}\\
&= -\frac{((2n)!)^3 n}{16\cdot(n!)^6 4096^n}\frac{(42n+47)(2n+1)^3}{512n(n+1)^3}\\
&\quad+\frac{((2n)!)^3 n}{16(n!)^6 4096^n}\left(\frac{2}{3}-\frac{7}{27n}+\frac{32}{81n^2}-\frac{320}{729n^3}+\frac{3256}{6561n^4}-\frac{3896}{6561n^5}+\frac{46730}{59049n^6}\right)\\
&\quad-\frac{((2n)!)^3 n}{16(n!)^6 4096^n}\frac{((2n+2)(2n+1))^3}{4096(n+1)^6}\left(1+\frac{1}{n}\right)\\
&\quad\times\left(\frac{2}{3}-\frac{7}{27(n+1)}+\frac{32}{81(n+1)^2}-\frac{320}{729(n+1)^3}+\frac{3256}{6561(n+1)^4}\right.\\
&\qquad\left.-\frac{3896}{6561(n+1)^5}+\frac{46730}{59049(n+1)^6}\right)\\
&= \frac{((2n)!)^3 n}{16\cdot(n!)^6 4096^n}\frac{P_7(n)}{15116544n^6(n+1)^8},
\end{aligned}
$$

where

$$
P_6(n) = 249344+1537024n+3961856n^2+5475328n^3+4290732n^4+1816203n^5+327110n^6
$$

and

$$
\begin{aligned}
P_7(n) &= 11962880+86726656n+270651392n^2+471961600n^3+497662976n^4\\
&\quad+318319755n^5+114970790n^6+18263392n^7.
\end{aligned}
$$

Hence, we have, for $n \geq 1$,

$$
\theta_{n+1} < \theta_n \quad \text{and} \quad \vartheta_{n+1} > \vartheta_n.
$$

The proof of Theorem 3.2 is complete. □

Remark 3.1. For $n \geq 14$, the inequalities (34) are sharper than the inequalities (10). Write (34) as

$$
u_n < \pi < v_n, \tag{35}
$$

where

$$u_n=\frac{1}{\rho_n+\rho_n^{(6)}}\quad\text{and}\quad v_n=\frac{1}{\rho_n+\rho_n^{(5)}}.$$

For $n=10$ in (35), we have

$$u_{10}=3.141592653589793238462643383 1\ldots,$$
$$v_{10}=3.141592653589793238462643383 8\ldots.$$

We then get the approximate value of π,

$$\pi\approx 3.14159265358979323846264338 3.$$

The choice $n=100$ in (35) yields the approximate value of π,

$$\begin{aligned}\pi\approx&3.14159265358979323846264338327950288419716939937510\\&58209749445923078164062862089986280348253421170679\\&82148086513282306647093844609550582231725359408128\\&48111745028410270193852110555964462294895493038.\end{aligned}$$

Appendix A. A derivation of formula (13)

Using the Maple software, we find, as $n\to\infty$,

$$\begin{aligned}\rho_n^{(1)}&=\frac{((2n)!)^3}{16(n!)^6 4096^n}\frac{((2n+2)(2n+1))^3}{(n+1)^6 4096}\left(\frac{128}{3}n+\frac{1280}{27}+\frac{32}{9(9n+19)}\right)\\&\sim\frac{((2n)!)^3 n}{16(n!)^6 4096^n}\left\{\frac{2}{3}-\frac{7}{27n}+\frac{32}{81n^2}-\frac{320}{729n^3}+\frac{26041}{52488n^4}-\ldots\right\}\end{aligned}$$

and

$$\begin{aligned}\rho_n^{(2)}&=\frac{((2n)!)^3}{16(n!)^6 4096^n}\frac{((2n+2)(2n+1))^3}{(n+1)^6 4096}\left(\frac{128}{3}n+\frac{1280}{27}+\frac{128(n+1)}{324n^2+1008n+677}\right)\\&\sim\frac{((2n)!)^3 n}{16(n!)^6 4096^n}\left\{\frac{2}{3}-\frac{7}{27n}+\frac{32}{81n^2}-\frac{320}{729n^3}+\frac{3256}{6561n^4}-\ldots\right\}.\end{aligned}$$

We then obtain the following asymptotic formula for $\frac{1}{\pi}-\rho_n$:

$$\frac{1}{\pi}-\rho_n=\frac{((2n)!)^3 n}{16(n!)^6 4096^n}\left\{\frac{2}{3}-\frac{7}{27n}+\frac{32}{81n^2}-\frac{320}{729n^3}+O\left(\frac{1}{n^4}\right)\right\},\qquad n\to\infty.$$

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On general divisor sums involving the coefficients of triple product L-functions

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Abstract In this paper, we consider general divisor problems involving the coefficients of triple product L-function $L(f \times g \times h, s)$ attached to holomorphic cusp forms $f(z), g(z)$ and $h(z)$ of even integral weight k for the full modular group $SL(2, \mathbb{Z})$.

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§1. Introduction and Main results

Throughout this paper, $\Gamma = SL(2, \mathbb{Z})$ is the full modular group. Let H_k^* be the set of primitive holomorphic cusp forms of even integral weight k for Γ. Let $\lambda_f(n), \lambda_g(n)$ and $\lambda_h(n)$ be the normalized Fourier coefficients of holomorphic Hecke cusp forms $f(z) \in H_{k_1}^*, g(z) \in H_{k_2}^*$ and $h(z) \in H_{k_3}^*$, respectively. $f(z)$ has the following Fourier expansions at the cusp ∞,

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{k-1}{2}} e^{2\pi i z}.$$

By the Hecke operators theory, for integers $m \geq 1$ and $n \geq 1$, $\lambda_f(n)$ is real and multiplicative. In 1974, Deligne proved that

$$|\ \lambda_f(n)\ | \leq d(n), \tag{1}$$

where $d(n)$ is the divisor function.

For $\Re e(s) > 1$, the Hecke L-function associated with $f(z)$ is defined by

$$L(f, s) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p \left(1 - \frac{\alpha_f(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta_f(p)}{p^s}\right)^{-1},$$

where $\alpha_f(p)$ and $\beta_f(p)$ satisfy

$$\lambda_f(p) = \alpha_f(p) + \beta_f(p), \qquad \alpha_f(p)\beta_f(p) = |\alpha_f(p)| = |\beta_f(p)| = 1.$$

For any fixed integer $j \geq 1$, we define

$$L(f,s)^j = \sum_{n=1}^{\infty} \frac{\lambda_{j,f}(n)}{n^s}.$$

It's easy to find that

$$\lambda_{j,f}(n) = \sum_{n=n_1 n_2 \cdots n_j} \lambda_f(n_1)\lambda_f(n_2)\cdots\lambda_f(n_j).$$

The sums $\sum\limits_{n\leq x} \lambda_{j,f}(n)$ has been studied by many researchers. Hecke [2] firstly gave the result

$$\sum_{n\leq x} \lambda_f(n) \ll x^{\frac{1}{2}+\varepsilon}.$$

Furthermore, Wu [13] proved that

$$\sum_{n\leq x} \lambda_f(n) \ll x^{\frac{1}{3}}(\log x)^{\delta},$$

where $\delta = -0.118....$ Moreover, Landau's classical results implied that

$$\sum_{n\leq x} \lambda_{j,f}(n) \ll x^{\frac{2j-1}{2j+1}+\varepsilon}.$$

Recently, Lü [6] showed that

$$\sum_{n\leq x} \lambda_{j,f}(n) \ll x^{\frac{3}{5}+\varepsilon}, \quad j=3,$$

and

$$\sum_{n\leq x} \lambda_{j,f}(n) \ll x^{\frac{2j-3}{2j}+\varepsilon}, \quad j\geq 4.$$

For $f(z) \in H_{k_1}^*, g(z) \in H_{k_2}^*$ and $h(z) \in H_{k_3}^*$, the associated triple product L-function is given by

$$\begin{aligned} L(f\times g\times h,s) &= \sum_{n=1}^{\infty} \frac{\lambda_{f\times g\times h}(n)}{n^s} \\ &= \prod_p \prod_{m=0}^{1}\prod_{u=0}^{1}\prod_{v=0}^{1}\left(1-\frac{\alpha_f(p)^{1-m}\beta_f(p)^m\alpha_g(p)^{1-u}\beta_g(p)^u\alpha_h(p)^{1-v}\beta_h(p)^v}{p^s}\right)^{-1}, \end{aligned}$$

where $\Re e(s) > 1$.

Then

$$L(f\times g\times h,s)^j = \sum_{n=1}^{\infty} \frac{\lambda_{j,f\times g\times h}(n)}{n^s}$$

for $\Re e(s) > 1$. We can show that

$$\lambda_{j,f\times g\times h}(n) = \sum_{n=n_1 n_2\cdots n_j} \lambda_{f\times g\times h}(n_1)\lambda_{f\times g\times h}(n_2)\cdots\lambda_{f\times g\times h}(n_j).$$

For $f(z)=g(z)=h(z)$, Liu [5] established

$$\sum_{n\le x}\lambda_{j,f\times f\times f}(n)\ll x^{1-\frac{3}{10j}+\varepsilon}.$$

In this paper, our first aim is to investigate the sums

$$S_j(x)=\sum_{n\le x}\lambda_{j,f\times g\times h}(n), \tag{2}$$

where $f(z),g(z)$ and $h(z)$ are different.

Theorem 1.1. *For any $\varepsilon>0$ and $j\ge 1$, we have*

$$S_j(x)\ll x^{\frac{4j-1}{4j}+\varepsilon}.$$

Some scholars studied the shifted convolution sums which plays an important role in many aspects, such as subconvexity and unique ergodicity. The shifted convolution sums of $GL(3)$ Fourier coefficients was first investigated by Pitt [12]. Based on the work of [8], Lü and Wang [9] recently gave the upper bounds for shifted convolution sums of coefficients of $L(f,s)^j$,

$$\sum_{l\le H}\sum_{N<n\le 2N}\lambda_{2,f}(n)\lambda_{2,f}(n+l)\ll N^{\frac{6}{5}+\varepsilon}H^{\frac{2}{5}},\quad N^{\frac{1}{3}+\varepsilon}\le H\le N^{1-\varepsilon},$$

$$\sum_{l\le H}\sum_{N<n\le 2N}\lambda_{3,f}(n)\lambda_{3,f}(n+l)\ll N^{\frac{4}{3}+\varepsilon}H^{\frac{1}{3}},\quad N^{\frac{1}{2}+\varepsilon}\le H\le N^{1-\varepsilon},$$

and for $j\ge 4$

$$\sum_{l\le H}\sum_{N<n\le 2N}\lambda_{j,f}(n)\lambda_{j,f}(n+l)\ll N^{\frac{4j+1}{2j+4}+v_j+\varepsilon}H^{\frac{2}{j+2}},\quad N^{\frac{2j-3}{2j}+\varepsilon}\le H\le N^{1-\varepsilon}.$$

In fact, the proof of [9] would apply to any L-function for which analogous subconvexity and moment estimates are known. The argument is structured in such a way that improved moment estimates would immediately yield an better result. Motivated by this, our second aim is to study the averages of shifted convolution sums of triple product L-function,

$$S_j(N,H):=\sum_{l\le H}\sum_{N<n\le 2N}\lambda_{j,f\times g\times h}(n)\lambda_{j,f\times g\times h}(n+l), \tag{3}$$

where $f(z),g(z)$ and $h(z)$ are different. Then we have the following result.

Theorem 1.2. *Suppose $1\le H\le N$. For any $\varepsilon>0$ and $j\ge 1$, we have*

$$S_j(N,H)\ll N^{\frac{8j}{4j+1}}H^{\frac{1}{4j+1}},$$

where $1\le H\le N^{\frac{1}{4j}}$.

§2. Preliminaries

In this section, we will give some lemmas for the proof of Theorem 1.1 and Theorem 1.2.

A series of works [1,3,11] established the automorphy and cuspidality of the triple product $\pi_1 \times \pi_2 \times \pi_3$, where π_1, π_2 and π_3 are unitary automorphic cuspidal representations of $GL_2(Q_A)$ with Fourier coefficients $\lambda_{\pi_i}(n)$ $(i = 1, 2, 3)$. Hence we know $L(f \times g \times h, s)$ is automorphic and it's also a general L-function (see [10]). We can get the following averaged and individual convexity bounds of $L(f \times g \times h, s)$ with degree 8 by Lemma 2.5 of [7].

Lemma 2.1. *Suppose $f(z) \in H^*_{k_1}, g(z) \in H^*_{k_2}$ and $h(z) \in H^*_{k_3}$, we have*

$$\int_T^{2T} \mid L\left(f \times g \times h, \sigma + it\right) \mid^2 dt \ll T^{\max\{8(1-\sigma),1\}+\varepsilon}, \tag{4}$$

$$L(f \times g \times h, \sigma + it) \ll (1 + \mid t \mid)^{4(1-\sigma)+\varepsilon} \tag{5}$$

uniformly for $\frac{1}{2} \leq \sigma \leq 1 + \varepsilon$, $T \geq 1$ and $\mid t \mid \geq 1$.

Lemma 2.2. *Suppose that the series $f(s) = \sum\limits_{n\geq 1} a_n n^{-s}$ converges absolutely for $\sigma > 1$ and $|a_n| \leq A(n)$, where $A(n)$ is a positive monotonously increasing function of n and*

$$\sum_{n\geq 1} |a_n| n^{-\sigma} = O\left((\sigma - 1)^{-\alpha}\right), \qquad \sigma \to 1^+$$

for some $\alpha > 0$. The formula

$$\sum_{n\leq x} a_n = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} f(s) \frac{x^s}{s} ds + O\left(\frac{x^b}{T(b-1)^\alpha}\right) + O\left(\frac{xA(2x)\ln x}{T}\right)$$

holds truly for $1 < b \leq b_0$, $T \geq 2$ and $x = N + \frac{1}{2}$ (the constants in O-symbols depend on b_0).

Proof. See Karatsuba and Voronin [4] pp. 334-336. □

§3. Proof of Theorem 1.1.

Suppose $f(z) \in H^*_{k_1}, g(z) \in H^*_{k_2}$ and $h(z) \in H^*_{k_3}$ are different. Since $L(f \times g \times h, s)$ is automorphic, there is no pole for $\Re e(s) > 1$. By Lemma 2.2, we can express (2) in the form

$$S_j(x) = \frac{1}{2\pi i} \int_{1+\varepsilon-iT}^{1+\varepsilon+iT} L(f \times g \times h, s)^j \frac{x^s}{s} ds + O\left(\frac{x^{1+\varepsilon}}{T}\right),$$

where T is a parameter to be chosen later.

The line of integration can be shifted to the parallel segment $\Re e(s) = \frac{1}{2} + \varepsilon$ by Cauchy Residue Theorem ,

$$\begin{aligned} S_j(x) =&\frac{1}{2\pi i}\left(\int_{\frac{1}{2}+\varepsilon-iT}^{\frac{1}{2}+\varepsilon+iT} + \int_{\frac{1}{2}+\varepsilon+iT}^{1+\varepsilon+iT} + \int_{1+\varepsilon-iT}^{\frac{1}{2}+\varepsilon-iT}\right) L(f \times g \times h, s)^j \frac{x^s}{s} ds + O\left(\frac{x^{1+\varepsilon}}{T}\right) \\ :=&J_1 + J_2 + J_3 + O\left(\frac{x^{1+\varepsilon}}{T}\right). \end{aligned} \tag{6}$$

By (4) and (5), J_1 becomes

$$\begin{aligned}J_1 \ll& x^{\frac{1}{2}+\varepsilon}\int_1^T \left|L(f\times g\times h,\frac{1}{2}+\varepsilon+it)\right|^j t^{-1}dt + x^{\frac{1}{2}+\varepsilon}\\ \ll& x^{\frac{1}{2}+\varepsilon}\log T \max_{T_1\le T}\frac{1}{T_1}\max_{\frac{T_1}{2}\le t\le T_1}\Big\{L(f\times g\times h,\frac{1}{2}+\varepsilon+it)^{j-2}\\ &\times\int_{\frac{T_1}{2}}^{T_1}\left|L(f\times g\times h,\frac{1}{2}+\varepsilon+it)\right|^2 dt\Big\}+x^{\frac{1}{2}+\varepsilon}\\ \ll& x^{\frac{1}{2}+\varepsilon}T^{2j-1}+x^{\frac{1}{2}+\varepsilon}.\end{aligned}$$

For the integral over the horizontal segments, we also use (5) to get

$$\begin{aligned}J_2+J_3 &\ll \int_{\frac{1}{2}+\varepsilon}^{1+\varepsilon} x^{\sigma}\mid L(f\times g\times h,\sigma+iT)\mid^j T^{-1}d\sigma\\ &\ll \max_{\frac{1}{2}+\varepsilon\le\sigma\le 1+\varepsilon} x^{\sigma}T^{4j(1-\sigma)+\varepsilon}T^{-1}\\ &\ll \frac{x^{1+\varepsilon}}{T}+x^{\frac{1}{2}+\varepsilon}T^{2j-1+\varepsilon}.\end{aligned}$$

Taking $T=x^{\frac{1}{4j}}$, we obtain that

$$J_1+J_2+J_3 \ll x^{\frac{4j-1}{4j}+\varepsilon}. \tag{7}$$

Now inserting (7) into (6), we can get the result of Theorem 1.

§4. Proof of Theorem 1.2.

For $j\ge 1$, we change the order of summations of (3) to get

$$\begin{aligned}S_j(N,H) &= \sum_{l\le H}\sum_{N<n\le 2N}\lambda_{j,f\times g\times h}(n)\lambda_{j,f\times g\times h}(n+l)\\ &=\sum_{N<n\le 2N}\lambda_{j,f\times g\times h}(n)\sum_{l\le H}\lambda_{j,f\times g\times h}(n+l).\end{aligned} \tag{8}$$

By Lemma 2.2 and Cauchy Residue Theorem, it follows that

$$\begin{aligned}\sum_{l\le H}\lambda_{j,f\times g\times h}(n+l) =& \sum_{m\le n+H}\lambda_{j,f\times g\times h}(m)-\sum_{m\le n}\lambda_{j,f\times g\times h}(m)\\ =&\frac{1}{2\pi i}\int_{1+\varepsilon-iT}^{1+\varepsilon+iT} L(f\times g\times h,s)^j\Big((n+H)^s-n^s\Big)\frac{ds}{s}+O\left(\frac{N^{1+\varepsilon}}{T}\right)\\ =&\frac{1}{2\pi i}\left(\int_{\frac{1}{2}+\varepsilon-iT}^{\frac{1}{2}+\varepsilon+iT}+\int_{\frac{1}{2}+\varepsilon+iT}^{1+\varepsilon+iT}+\int_{1+\varepsilon-iT}^{\frac{1}{2}+\varepsilon-iT}\right)L(f\times g\times h,s)^j\\ &\times\Big((n+H)^s-n^s\Big)\frac{ds}{s}+O\left(\frac{N^{1+\varepsilon}}{T}\right)\\ :=&I_1+I_2+I_3+O\left(\frac{N^{1+\varepsilon}}{T}\right).\end{aligned} \tag{9}$$

We can estimate I_2+I_3 by (5) to get

$$I_2+I_3 \ll \frac{N^{1+\varepsilon}}{T}+N^{\frac{1}{2}+\varepsilon}T^{2j-1+\varepsilon} \ll \frac{N^{1+\varepsilon}}{T}, \tag{10}$$

where $T \leq N^{\frac{1}{4j}}$.

Inserting (10) into (9), we have

$$\sum_{l\leq H}\lambda_{j,f\times g\times h}(n+l)=\frac{1}{2\pi i}\int_{\frac{1}{2}+\varepsilon-iT}^{\frac{1}{2}+\varepsilon+iT}L(f\times g\times h,s)^j\Big((n+H)^s-n^s\Big)\frac{ds}{s}+O\left(\frac{N^{1+\varepsilon}}{T}\right). \tag{11}$$

Taking (11) into the second equation of (8), it can be easily seen that

$$\begin{aligned} S_j(N,H) \ll &\sum_{N<n\leq 2N}\lambda_{j,f\times g\times h}(n)\left\{\frac{1}{2\pi i}\int_{\frac{1}{2}+\varepsilon-iT}^{\frac{1}{2}+\varepsilon+iT}L(f\times g\times h,s)^j\Big((n+H)^s-n^s\Big)\frac{ds}{s}\right\}\\ &+\frac{N^{1+\varepsilon}}{T}\sum_{N<n\leq 2N}\mid \lambda_{j,f\times g\times h}(n)\mid \\ :=&G_1(N,H)+G_2(N,H). \end{aligned} \tag{12}$$

The Deligne's bound (1) and multiplicative property of $\lambda_{f\times g\times h}(n)$ imply that

$$G_2(N,H) \ll \frac{N^{2+\varepsilon}}{T}. \tag{13}$$

For $G_1(N,H)$, we change the order of integration and summation to get

$$G_1(N,H)=\frac{1}{2\pi i}\int_{\frac{1}{2}+\varepsilon-iT}^{\frac{1}{2}+\varepsilon+iT}L(f\times g\times h,s)^j\sum_{N<n\leq 2N}\lambda_{j,f\times g\times h}(n)\Big((n+H)^s-n^s\Big)\frac{ds}{s}. \tag{14}$$

Using Abel transformation to the sum over n, then

$$\begin{aligned} &\sum_{N<n\leq 2N}\lambda_{j,f\times g\times h}(n)\Big((n+H)^s-n^s\Big)\\ =&\sum_{N<n\leq 2N}\left(\left(1+\frac{H}{n}\right)^s-1\right)\lambda_{j,f\times g\times h}(n)n^s\\ =&\left(\left(1+\frac{H}{2N}\right)^s-1\right)\sum_{N<n\leq 2N}\lambda_{j,f\times g\times h}(n)n^s\\ &+sH\int_N^{2N}\left(1+\frac{H}{x}\right)^{s-1}\left(\sum_{N<n\leq x}\lambda_{j,f\times g\times h}(n)n^s\right)\frac{dx}{x^2}. \end{aligned} \tag{15}$$

Inserting (15) into (14), we have

$$\begin{aligned} G_1(N,H)=&\frac{1}{2\pi i}\int_{\frac{1}{2}+\varepsilon-iT}^{\frac{1}{2}+\varepsilon+iT}L(f\times g\times h,s)^j\left(\left(1+\frac{H}{2N}\right)^s-1\right)\left(\sum_{N<n\leq 2N}\lambda_{j,f\times g\times h}(n)n^s\right)\frac{ds}{s}\\ &+\frac{1}{2\pi i}\int_{\frac{1}{2}+\varepsilon-iT}^{\frac{1}{2}+\varepsilon+iT}\int_N^{2N}sH\cdot L(f\times g\times h,s)^j\left(1+\frac{H}{x}\right)^{s-1}\\ &\times\left(\sum_{N<n\leq x}\lambda_{j,f\times g\times h}(n)n^s\right)\frac{dx}{x^2}\frac{ds}{s}\\ :=&G_1^1(N,H)+G_1^2(N,H). \end{aligned} \tag{16}$$

By Newton–Leibniz formula, we get

$$\begin{aligned} G_1^1(N,H) =& \frac{1}{2\pi i}\int_0^H \int_{\frac{1}{2}+\varepsilon-iT}^{\frac{1}{2}+\varepsilon+iT} L(f\times g\times h,s)^j \\ & \times\left(1+\frac{\theta}{2N}\right)^{s-1}\left(\sum_{N<n\le 2N}\lambda_{j,f\times g\times h}(n)n^s\right)dsd\theta. \end{aligned} \tag{17}$$

Moreover, (16) and (17) imply that

$$G_1(N,H)\ll \frac{H}{N}\max_{N<x\le 2N}\int_{-T}^{T}\left|L\left(f\times g\times h,\frac{1}{2}+\varepsilon+it\right)\right|^j\left|\sum_{N<n\le x}\lambda_{j,f\times g\times h}(n)n^{\frac{1}{2}+\varepsilon+it}\right|dt. \tag{18}$$

Applying Lemma 2.2, the inner summation of (17) can be written as

$$\begin{aligned} \sum_{N<n\le x}\lambda_{j,f\times g\times h}(n)n^{\frac{1}{2}+\varepsilon+it} =& \frac{1}{2\pi i}\int_{\frac{3}{2}+\varepsilon-2iT}^{\frac{3}{2}+\varepsilon+2iT} L\left(f\times g\times h,s_1-\left(\frac{1}{2}+\varepsilon+it\right)\right)^j \\ & \times\left(x^{s_1}-N^{s_1}\right)\frac{ds_1}{s_1}+O\left(\frac{N^{\frac{3}{2}+\varepsilon}}{T}\right), \end{aligned}$$

where $-T\le t\le T$ and $2\le T\le N$ is a parameter to be chosen later.

By Cauchy Residue Theorem, we change the line of integration to the parallel with $\Re e(s)=1$ to obtain

$$\begin{aligned} & \sum_{N<n\le x}\lambda_{j,f\times g\times h}(n)n^{\frac{1}{2}+\varepsilon+it} \\ =& \frac{1}{2\pi i}\left\{\int_{1+\varepsilon-2iT}^{1+\varepsilon+2iT}+\int_{1+\varepsilon+2iT}^{\frac{3}{2}+\varepsilon+2iT}+\int_{\frac{3}{2}+\varepsilon-2iT}^{1+\varepsilon-2iT}\right\}L\left(f\times g\times h,s_1-\left(\frac{1}{2}+\varepsilon+it\right)\right)^j \\ & \times\left(x^{s_1}-N^{s_1}\right)\frac{ds_1}{s_1}+O\left(\frac{N^{\frac{3}{2}+\varepsilon}}{T}\right) \\ :=& H_1+H_2+H_3+O\left(\frac{N^{\frac{3}{2}+\varepsilon}}{T}\right). \end{aligned} \tag{19}$$

Similarly, for H_2+H_3, we use (5) to get

$$H_2+H_3\ll NT^{2j-1}+N^{\frac{3}{2}}T^{-1}\ll N^{\frac{3}{2}}T^{-1}, \tag{20}$$

where we suppose $T\le N^{\frac{1}{4j}}$.

Taking (20) into (19), then

$$\begin{aligned} \sum_{N<n\le x}\lambda_{j,f\times g\times h}(n)n^{\frac{1}{2}+\varepsilon+it} =& \frac{1}{2\pi i}\int_{1+\varepsilon-2iT}^{1+\varepsilon+2iT} L\left(f\times g\times h,s_1-\left(\frac{1}{2}+\varepsilon+it\right)\right)^j \\ & \times\left(x^{s_1}-N^{s_1}\right)\frac{ds_1}{s_1}+O\left(\frac{N^{\frac{3}{2}+\varepsilon}}{T}\right). \end{aligned} \tag{21}$$

(18) and (21) imply that

$$\begin{aligned} G_1(N,H) \ll & \frac{H}{N} \max_{N<N\leq 2N} \int_{-T}^{T} \int_{1+\varepsilon-2iT}^{1+\varepsilon+2iT} \left| L\Big(f\times g\times h, \frac{1}{2}+\varepsilon+it\Big)\right|^j \\ & \times \left| L\Big(f\times g\times h, s_1 - \Big(\frac{1}{2}+\varepsilon+it\Big)\Big)^j \Big(x^{s_1}-N^{s_1}\Big)\right| \frac{ds_1}{s_1} dt \\ & + \frac{H}{N}\int_{-T}^{T} \left| L\Big(f\times g\times h, \frac{1}{2}+\varepsilon+it\Big)\right|^j \frac{N^{\frac{3}{2}+\varepsilon}}{T} dt \\ := & E_1(N,H)+E_2(N,H). \end{aligned} \tag{22}$$

By (5), it follows that

$$\begin{aligned} E_1(N,H) \ll & H\int_{-2T}^{2T} \Big(\int_{-T}^{T} \left| L\Big(f\times g\times h, \frac{1}{2}+\varepsilon+it\Big)^j\right|^2 dt\Big)^{\frac{1}{2}} \\ & \times \Big(\int_{-T}^{T} \left| L\Big(f\times g\times h, \frac{1}{2}+\varepsilon+i(t_1-t)\Big)^j\right|^2 dt\Big)^{\frac{1}{2}} \frac{dt_1}{1+|t_1|} \\ \ll & HT^{4j}. \end{aligned} \tag{23}$$

For $E_2(N,H)$, we have

$$E_2(N,H) \ll \frac{H}{N} N^{\frac{3}{2}+\varepsilon} T^{2j-1} = HN^{\frac{1}{2}+\varepsilon} T^{2j-1}. \tag{24}$$

(22), (23) and (24) imply that

$$G_1(N,H) \ll HT^{4j} + HN^{\frac{1}{2}+\varepsilon} T^{2j-1}. \tag{25}$$

Inserting (13) and (25) into (12) to get

$$S_j(N,H) \ll HT^{4j} + HN^{\frac{1}{2}+\varepsilon} T^{2j-1} + N^{2+\varepsilon}T^{-1}.$$

Noting $T \leq N^{\frac{1}{4j}}$, hence we deduce that

$$S_j(N,H) \ll N^{\frac{8j}{4j+1}} H^{\frac{1}{4j+1}},$$

where we take $T = N^{\frac{2}{4j+1}+\varepsilon} H^{\frac{-1}{4j+1}+\varepsilon}$. This completes the proof of Theorem 1.2.

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Dirichlet density on sets of primes for linear combinations of Hecke eigenvalues

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Abstract Let f and g be two holomorphic cusp forms for the full modular group $\mathrm{SL}(2,\mathbb{Z})$. Denote by $\lambda_f(n)$ and $\lambda_g(n)$ the Hecke eigenvalue of f and g, respectively. In this paper, we are interested in Dirichlet density on sets of primes for linear combinations of $\lambda_f(n)$ and $\lambda_g(n)$.

Keywords Dirichlet density, linear combination, Hecke eigenvalues, holomorphic cusp form.
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§1. Introduction and Main results

Let H_k^* be the set of Hecke primitive eigencuspforms of even integral weight $k \geq 2$ for the full modular group $\mathrm{SL}(2,\mathbb{Z})$. Suppose that $f \in H_k^*$ is an eigenfunction of Hecke operators,

$$T_n f = \lambda_f(n) f,$$

where the Hecke operators is defined by

$$(T_n f)(z) = \frac{1}{\sqrt{n}} \sum_{ad=n} \Big(\frac{a}{d}\Big)^k \sum_{b \ mod \ d} f\Big(\frac{az+b}{d}\Big).$$

Then the primitive cusp form f has the following Fourier expansion at the cusp ∞

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{(k-1)}{2}} e^{2\pi niz},$$

here $\lambda_f(n) \in \mathbb{R}$ and $\lambda_f(1) = 1$. From the theory of Hecke operators, for all integers $m, n \geq 1$, $\lambda_f(n)$ is real and satisfies the Hecke multiplicity:

$$\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \lambda_f\left(\frac{mn}{d^2}\right), \tag{1}$$

Hence $\lambda_f(n)$ is not only the n-th normalized Fourier coefficient of f but also the normalized eigenvalue of T_n, abbreviated Hecke eigenvalues in this paper.

Now we recall the definition of Dirichlet density. A set E of primes has Dirichlet density (or analytic density) $\gamma > 0$ if and only if

$$\sum_{p\in \mathrm{E}} \frac{1}{p^s} \sim \gamma \sum_{p} \frac{1}{p^s}, \qquad s \to 1^+.$$

In this paper, we write $\delta(\mathrm{E}) = \gamma$.

Kowalski et al. [4] firstly showed if the sign of $\lambda_f(p)$ is the same as $\lambda_g(p)$ for any primes up to the exceptional set of Dirichlet density at least $\frac{1}{32}$, then $f = g$. Motivated by [4], Chiriac [1] successfully proved that

$$\delta(\{p|\lambda_f(p) < \lambda_g(p)\}) \geq \frac{1}{16},$$
$$\delta(\{p|\lambda_f^2(p) < \lambda_g^2(p)\}) \geq \frac{1}{16}$$

for $p \in \mathrm{P}$, where P is the set of all primes and $f, g \in H_k^*$ are different.

Very recently, based on the work of [1], Lao [5] considered the Hecke eigenvalues at prime powers, i.e., for $p \in \mathrm{P}$,

$$\delta(\{p|\lambda_f(p^j) < \lambda_g(p^j)\}) \geq \frac{1}{16[\frac{j+1}{2}]^2}, \qquad 1 \leq j \leq 8,$$

$$\delta(\{p|\lambda_f^2(p^j) < \lambda_g^2(p^j)\}) \geq \frac{1}{4j(j+1)^2}, \qquad 1 \leq j \leq 4,$$

where $f, g \in H_k^*$ are different.

Chiriac and Jorza [2] were able to prove the density bound for the set $\{v|a < \lambda_1 a_v(\pi_1) + \lambda_2 a_v(\pi_2) < b\}$ in the context of unitary cuspidal representations that satisfy the Ramanujan conjecture. Recently, by using ideas of Chiriac [1], Gao [3] obtained the Dirichlet densities for the set $\{p|\lambda_f(p) + m\lambda_g(p) + n < 0\}$ and $\{p|\lambda_f^2(p) + m\lambda_g^2(p) + n < 0\}$. In this paper, We further consider the Hecke eigenvalues at prime powers and generalize the results of Gao [3].

Theorem 1.1. *Let $f, g \in H_k^*$ be two different cusp forms. Then*

$$\delta(\{p|\lambda_f(p^j) + m\lambda_g(p^j) + n < 0\}) \geq \frac{1 + m^2 - n(j+1)(1+|m|)}{2(j+1)(j+1+(j+1)|m| - n)(1+|m|)},$$

where $m \in \mathbb{R}$ and $-(j+1)(1+|m|) < n < \frac{1+m^2}{(j+1)(1+|m|)}$.

Theorem 1.2. *Let $f, g \in H_k^*$ be two different cusp forms.*

(i) For $m \geq 0$ and $-(j+1)^2(1+m) < n < -\frac{(j^2+j)m^2+(2j^2+4j)m+j^2+j}{(j^2+2j)(1+m)}$,

$$\begin{aligned}&\delta(\{p|\lambda_f^2(p^j) + m\lambda_g^2(p^j) + n < 0\}) \\ \geq &\frac{(j^2+j)m^2 + (2j^2+4j)m + (j^2+2j)n + (j^2+2j)mn + j^2 + j}{n(j+1)^2(1+m)}.\end{aligned}$$

(ii) For $m < 0$ and $-(j+1)^2 < n < \frac{(j+1)m^2+(1-j^2-2j)m-j^2-j}{j^2+2j-m}$,

$$\begin{aligned}&\delta(\{p|\lambda_f^2(p^j) + m\lambda_g^2(p^j) + n < 0\}) \\ \geq &\frac{(j+1)m^2 + (1-j^2-2j)m + (-j^2-2j)n + mn - j^2 - j}{(j+1)^2(-(j+1)^2 m - n)(1-m)}.\end{aligned}$$

§2. Preliminaries

In this section, we will recall and establish some preliminary results which are used to prove Theorems 1.1 – 1.2.

Let $f \in H_k^*$ be a cusp form. The i-th symmetric power L-function attached to f is defined by

$$L(\mathrm{sym}^i f, s) = \prod_p \prod_{m=0}^{i} \left(1 - \frac{\alpha_f(p)^{i-m}\beta_f(p)^m}{p^s}\right)^{-1}, \qquad \Re s > 1, \tag{2}$$

where $\alpha_f(p)$ and $\beta_f(p)$ are two complex numbers satisfying

$$\alpha_f(p)\beta_f(p) = |\alpha_f(p)| = |\beta_f(p)| = 1, \qquad \lambda_f(p) = \alpha_f(p) + \beta_f(p). \tag{3}$$

We express it as a Dirichlet series:

$$L(\mathrm{sym}^i f, s) = \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^i f}(n)}{n^s} = \prod_p \left(1 + \sum_{v=1}^{\infty} \frac{\lambda_{\mathrm{sym}^i f}(p^v)}{p^{vs}}\right), \qquad \Re s > 1,$$

where $\lambda_{\mathrm{sym}^i f}(n)$ is a real multiplicative function, and

$$\lambda_{\mathrm{sym}^i f}(p) = \sum_{m=0}^{i} \alpha_f(p)^{i-m}\beta_f(p)^m = \lambda_f(p^i). \tag{4}$$

Let $f, g \in H_k^*$ be two different cusp forms, Rankin–Selberg L-function attached to $\mathrm{sym}^i f$ and $\mathrm{sym}^j g$ is defined by

$$L(\mathrm{sym}^i f \times \mathrm{sym}^j g, s) = \prod_p \prod_{m=0}^{i} \prod_{m'=0}^{j} \left(1 - \frac{\alpha_f(p)^{i-m}\beta_f(p)^m \alpha_g(p)^{j-m'}\beta_g(p)^{m'}}{p^s}\right)^{-1}, \quad \Re s > 1. \tag{5}$$

For $\Re s > 1$, we expand it into a Dirichlet series

$$L(\mathrm{sym}^i f \times \mathrm{sym}^j g, s) = \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j g}(n)}{n^s} = \prod_p \left(1 + \sum_{v=1}^{\infty} \frac{\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j g}(p^v)}{p^{vs}}\right), \tag{6}$$

where $\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j g}(n)$ is real and multiplicative. By the multiplicative property and the definitions of (3) and (4), it's easily seen that

$$\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j g}(p) = \sum_{m=0}^{i} \sum_{m'=0}^{j} \alpha_f(p)^{i-m}\beta_f(p)^m \alpha_g(p)^{j-m'}\beta_g(p)^{m'} = \lambda_{\mathrm{sym}^i f}(p)\lambda_{\mathrm{sym}^j g}(p), \tag{7}$$

where i, j are positive integers.

Lemma 2.1. *Let $f \in H_k^*$ be a Hecke cusp form. Then for $j \geq 1$, we have*

$$|\lambda_f(p^j)| \leq j + 1,$$

and

$$|\lambda_f^2(p^j)| \leq (j+1)^2.$$

Proof. Using (3) and (4), we can easily obtain the conclusion. □

Lemma 2.2. *Let $f \in H_k^*$ be a Hecke cusp form. Then for $j \geq 1$, $L(\mathrm{sym}^j f, s)$ has an analytic continuation as an entire function in the whole complex plane $\mathbb{C}$ (see Lemma 1 of [6]).*

According to Lemma 2.2 and a series of definitions of (4) – (7), it is not hard to deduce that for $j \geq 1$,

$$\sum_p \frac{\lambda_f(p^j)}{p^s} = O(1), \qquad s \to 1^+. \tag{8}$$

Lemma 2.3. *Let $f, g \in H_k^*$ be two different cusp forms. Then for $i, j \geq 1$, $L(\mathrm{sym}^i f \times \mathrm{sym}^j g, s)$ has an analytic continuation as an entire function in the whole complex plane $\mathbb{C}$. In particular, when $f = g$, $L(\mathrm{sym}^i f \times \mathrm{sym}^j g, s)$ has simple poles at $s = 0, 1$ (see Lemma 2 of [6]).*

Hence, when $f = g$, we have

$$\sum_p \frac{\lambda_f(p^j)\lambda_g(p^j)}{p^s} = \sum_p \frac{1}{p^s} + O(1), \qquad s \to 1^+. \tag{9}$$

In other cases, we get

$$\sum_p \frac{\lambda_f(p^j)\lambda_g(p^j)}{p^s} = O(1), \qquad s \to 1^+. \tag{10}$$

The next lemma follows plainly from [6, Lemma 7].

Lemma 2.4. *Let $f, g \in H_k^*$ be two cusp forms. Then*

1) when $f = g$, we have

$$\sum_p \frac{\lambda_f^2(p^j)\lambda_g^2(p^j)}{p^s} = (j+1)\sum_p \frac{1}{p^s} + O(1), \qquad s \to 1^+, \tag{11}$$

2) when $f \neq g$, we have

$$\sum_p \frac{\lambda_f^2(p^j)\lambda_g^2(p^j)}{p^s} = \sum_p \frac{1}{p^s} + O(1), \qquad s \to 1^+. \tag{12}$$

Lemma 2.5. *Let $f, g \in H_k^*$ be two different cusp forms. Then for $j \geq 1$, we have*

$$\sum_p \frac{(\lambda_f(p^j) + m\lambda_g(p^j) + n)^2}{p^s} = (1 + m^2 + n^2)\sum_p \frac{1}{p^s} + O(1), \qquad s \to 1^+,$$

$$\sum_p \frac{(\lambda_f^2(p^j) + m\lambda_g^2(p^j) + n)^2}{p^s} = H(m,n)\sum_p \frac{1}{p^s} + O(1), \qquad s \to 1^+,$$

where

$$H(m,n) = (j+1)m^2 + n^2 + 2m + 2n + 2mn + (j+1).$$

Proof. We have

$$\begin{aligned}&\sum_p \frac{(\lambda_f(p^j) + m\lambda_g(p^j) + n)^2}{p^s} \\ =&\sum_p \frac{\lambda_f^2(p^j) + m^2\lambda_g^2(p^j) + 2m\lambda_f(p^j)\lambda_g(p^j) + 2n\lambda_f(p^j) + 2mn\lambda_g(p^j) + n^2}{p^s}.\end{aligned}$$

According to Lemma 2.4 and a series of results (8) – (10), we obtain

$$\sum_p \frac{(\lambda_f(p^j)+m\lambda_g(p^j)+n)^2}{p^s} = (1+m^2+n^2)\sum_p \frac{1}{p^s} + O(1), \qquad s \to 1^+.$$

Similarly, we get that as $s \to 1^+$,

$$\begin{aligned}
&\sum_p \frac{(\lambda_f^2(p^j)+m\lambda_g^2(p^j)+n)^2}{p^s}\\
=&\sum_p \frac{\lambda_f^4(p^j)+m^2\lambda_g^4(p^j)+2m\lambda_f^2(p^j)\lambda_g^2(p^j)+2n\lambda_f^2(p^j)+2mn\lambda_g^2(p^j)+n^2}{p^s}\\
=&\Big((j+1)m^2+n^2+2m+2n+2mn+(j+1)\Big)\sum_p \frac{1}{p^s} + O(1).
\end{aligned}$$

□

§3. Proof of Theorem 1.1

Proof. Let

$$A = \{p | \lambda_f(p^j) + m\lambda_g(p^j) + n < 0\}$$

for any fixed $m, n \in \mathbb{R}$ and define

$$A(f,g,p) = \lambda_f(p^j) + m\lambda_g(p^j) + n.$$

Let $\delta(A)$ be the Dirichlet density of A, we know that $0 < \delta(A) < 1$. Then we have the following inequations

$$\begin{aligned}
&\min_{p\in \mathrm{P}} A(f,g,p) < 0,\\
&\max_{p\in \mathrm{P}} A(f,g,p) > 0.
\end{aligned}$$

Then by Lemma 2.1, we have

$$\begin{aligned}
-(j+1)-(j+1)|m|+n &< 0,\\
j+1+(j+1)|m|+n &> 0.
\end{aligned}$$

By a straightforward calculation, we get

$$-(j+1)(1+|m|) < n < (j+1)(1+|m|). \tag{13}$$

On the one hand, for $p \in A$, we have

$$A(f,g,p) < 0.$$

Then we obtain

$$|A(f,g,p)| = -\lambda_f(p^j) - m\lambda_g(p^j) - n \le (j+1)(1+|m|) - n. \tag{14}$$

Hence

$$\sum_{p\in A}\frac{A^2(f,g,p)}{p^s}\le\big((j+1)(1+|m|)-n\big)^2\sum_{p\in A}\frac{1}{p^s}. \tag{15}$$

On the other hand, for $p\notin A$, we konw that

$$A(f,g,p)\ge 0.$$

It follows from Lemma 2.1 that

$$|A(f,g,p)|=\lambda_f(p^j)+m\lambda_g(p^j)+n\le(j+1)(1+|m|)+n. \tag{16}$$

Using (16), we deduce that as $s\to 1^+$,

$$\begin{aligned}\sum_{p\notin A}\frac{A^2(f,g,p)}{p^s}\le&\big((j+1)(1+|m|)+n\big)\sum_{p\notin A}\frac{A(f,g,p)}{p^s}\\ =&\big((j+1)(1+|m|)+n\big)\Bigg(\sum_p\frac{A(f,g,p)}{p^s}-\sum_{p\in A}\frac{A(f,g,p)}{p^s}\Bigg)\\ =&\big((j+1)(1+|m|)+n\big)\Bigg(\sum_p\frac{A(f,g,p)}{p^s}+\sum_{p\in A}\frac{|A(f,g,p)|}{p^s}\Bigg).\end{aligned}$$

From (8), we have

$$\sum_p\frac{A(f,g,p)}{p^s}=\sum_p\frac{\lambda_f(p^j)+m\lambda_g(p^j)+n}{p^s}=n\sum_p\frac{1}{p^s}+O(1),\qquad s\to 1^+, \tag{17}$$

Combining (14) with (17) yields

$$\sum_{p\notin A}\frac{A^2(f,g,p)}{p^s}\le n\big((j+1)(1+|m|)+n\big)\sum_p\frac{1}{p^s}+H_1(m,n)\sum_{p\in A}\frac{1}{p^s}+O(1),\qquad s\to 1^+, \tag{18}$$

where

$$H_1(m,n)=(j+1)^2(1+|m|)^2-n^2.$$

By (15) and (18), it is shown that as $s\to 1^+$

$$\sum_p\frac{A^2(f,g,p)}{p^s}\le n\big((j+1)(1+|m|)+n\big)\sum_p\frac{1}{p^s}+H_2(m,n)\sum_{p\in A}\frac{1}{p^s}+O(1), \tag{19}$$

where

$$H_2(m,n)=2(j+1)(1+|m|)\big((j+1)(1+|m|)-n\big).$$

Applying Lemma 2.5 on the left-hand side in (19), we can see that

$$\sum_{p\in A}\frac{1}{p^s}\ge\frac{1+m^2-n(j+1)(1+|m|)}{2(j+1)(1+|m|)\big((j+1)(1+|m|)-n\big)}\sum_p\frac{1}{p^s}+O(1),\qquad s\to 1^+.$$

Hence, we get

$$\delta(A)\ge\frac{1+m^2-n(j+1)(1+|m|)}{2(j+1)(1+|m|)\big((j+1)(1+|m|)-n\big)}.$$

Then, in order to $\delta(A) > 0$, we need

$$\frac{1+m^2-n(j+1)(1+|m|)}{2(j+1)(1+|m|)\big((j+1)(1+|m|)-n\big)} > 0.$$

According to the above inequality and (13), we have

$$-(j+1)(1+|m|) < n < \frac{1+m^2}{(j+1)(1+|m|)}.$$

This completes the proof of Theorem 1.1. $\square$

§4. Proof of Theorem 1.2

Proof. For fixed $m, n \in \mathbb{R}$, the set B is defined by

$$B = \{p | \lambda_f^2(p^j) + m\lambda_g^2(p^j) + n < 0\}$$

and let

$$B(f,g,p) = \lambda_f^2(p^j) + m\lambda_g^2(p^j) + n.$$

We only consider the case $m \geq 0$, since the case $m < 0$ is similar. Let $\delta(B)$ be the analytic density of B, we need to limit $\delta(B)$, such that $0 < \delta(B) < 1$. Then

$$\min_{p\in \mathrm{P}} B(f,g,p) < 0,$$

$$\max_{p\in \mathrm{P}} B(f,g,p) > 0,$$

By Lemma 2.1 and a simple calculation, we have

$$-(j+1)^2(1+m) < n < 0. \tag{20}$$

Firstly, for $p \in B$, we get

$$B(f,g,p) < 0.$$

It is easy to see that

$$|B(f,g,p)| = -\lambda_f^2(p^j) - m\lambda_g^2(p^j) - n \leq -n. \tag{21}$$

Hence, we obtain

$$\sum_{p\in B} \frac{B^2(f,g,p)}{p^s} \leq n^2 \sum_{p\in B} \frac{1}{p^s}. \tag{22}$$

Then, for $p \notin B$, we have

$$B(f,g,p) \geq 0.$$

Therefore, one can find that

$$|B(f,g,p)| = \lambda_f^2(p^j) + m\lambda_g^2(p^j) + n \leq (j+1)^2(1+m) + n. \tag{23}$$

From (23), we show that as $s \to 1^+$,

$$\begin{aligned}\sum_{p \notin B} \frac{B^2(f,g,p)}{p^s} \le & \big((j+1)^2(1+m)+n\big) \sum_{p \notin B} \frac{B(f,g,p)}{p^s} \\ = & \big((j+1)^2(1+m)+n\big) \bigg(\sum_{p} \frac{B(f,g,p)}{p^s} - \sum_{p \in B} \frac{B(f,g,p)}{p^s} \bigg) \\ = & \big((j+1)^2(1+m)+n\big) \bigg(\sum_{p} \frac{B(f,g,p)}{p^s} + \sum_{p \in B} \frac{|B(f,g,p)|}{p^s} \bigg).\end{aligned}$$

By (9) and (19), we deduce that

$$\sum_{p \notin B} \frac{B^2(f,g,p)}{p^s} \le H_3(m,n) \sum_{p} \frac{1}{p^s} + H_4(m,n) \sum_{p \in B} \frac{1}{p^s} + O(1), \qquad s \to 1^+, \tag{24}$$

where

$$\begin{aligned} H_3(m,n) = & \big((j+1)^2(1+m)+n\big)(1+m+n), \\ H_4(m,n) = & -n\big((j+1)^2(1+m)+n\big). \end{aligned}$$

Combining (22) with (24) implies

$$\sum_{p} \frac{B^2(f,g,p)}{p^s} \le H_3(m,n) \sum_{p} \frac{1}{p^s} + H_5(m,n) \sum_{p \in B} \frac{1}{p^s} + O(1), \qquad s \to 1^+, \tag{25}$$

where

$$H_5(m,n) = -n(j+1)^2(1+m).$$

By Lemma 2.5, we obtain

$$\delta(B) \ge \frac{(j^2+j)m^2 + (2j^2+4j)m + (j^2+2j)n + (j^2+2j)mn + j^2 + j}{n(j+1)^2(1+m)}.$$

To ensure $\delta(B) > 0$ it suffices that

$$\frac{(j^2+j)m^2 + (2j^2+4j)m + (j^2+2j)n + (j^2+2j)mn + j^2 + j}{n(j+1)^2(1+m)} > 0. \tag{26}$$

From (20) and (26), we have

$$-(j+1)^2 < n < \frac{(j+1)m^2 + (1-j^2-2j)m - j^2 - j}{j^2+2j-m}.$$

This finishes the proof of Theorem 1.2. $\square$

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Intuitionistic fuzzy subalgebras, ideals and positive implicative ideals of BCI-algebras under norms

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Abstract In this paper, as using norms(T and C), we introduce the concepts of intuitionistic fuzzy subalgebras, intuitionistic fuzzy ideals and intuitionistic fuzzy positive implicative ideals of BCI-algebras and present some fundamental properties of them. Also we investigatem them under intersections, direct products, image and preimage of homomorphisms in BCI-algebras.

Keywords Algebra and orders, theory of fuzzy sets, intuitionistic fuzzy sets, norms, products and intersections, homomorphisms.
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§1. Introduction

In 1965, Zadeh[31] introduced the notion of a fuzzy subset of a set as a method for representing uncertainty in real physical world. The idea of intuitionistic fuzzy set was first published by Atanassov[2, 3] as a generalization of the notion of fuzzy sets. Imai and Iseki introduced the notion of BCI-algebra[5]. Touqeer and Aslam Malik[30] considerd the intuitionistic fuzzification of the concept of BCI-positive implicative ideals in BCI-algebras and investigated some of their properties. The author by using norms, investigated some properties of fuzzy algebraic structures[8-29]. In this work, we define the concepts of intuitionistic fuzzy subalgebras, intuitionistic fuzzy ideals and intuitionistic fuzzy positive implicative ideals of BCI-algebras under t-norm T and t-conorm C and obtaine some basic properties of them. Next we show the characterization properties of them with subalgebras, ideals and implicative ideals of BCI-algebras. Finally, we consider them under intersections, direct products and BCI-homomorphisms(image and preimage) and prove some basic properties of them.

§2. Preliminaries

In this section we cite the fundamental definitions and results that will be used in the sequel.

Definition 2.1.([30]) an algebra $(X, *, 0)$ of type $(2, 0)$ is called a BCI-algebra if it satisfies the following conditions:

(1) $((x * y) * (x * z)) * (z * y) = 0$
(2) $(x * (x * y)) * y = 0$
(3) $x * x = 0$
(4) $x * y = 0$ and $y * x = 0$ imply $x = y$
(5) $(x * y) * z = (x * z) * y$
(6) $x * 0 = x$
(7) $0 * (x * y) = (0 * x) * (0 * y)$
(8) $0 * (0 * (x * y)) = 0 * (y * x)$
for all $x, y, z \in X$.
In a BCI-algebra, we can define a partial ordering " $\leq$ " by $x \leq y$ if and only if $x * y = 0$.
(9) $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$
(10) $(x * z) * (y * z) \leq x * y$
for all $x, y, z \in X$.

Definition 2.2.([6]) A non-empty subset I of a BCI-algebra X is called an ideal of X if
(1) $0 \in I$,
(2) $x * y \in I$ and $y \in I$ imply that $x \in I$ for all $x, y \in X$.

Definition 2.3.([6]) A non-empty subset I of a BCI-algebra X is said to be a positive implicative ideal of X if it satisfies:
(1) $0 \in I$,
(2) $((x * z) * z) * (y * z) \in I$ and $y \in I$ imply $x * z \in I$,
for all $x, y, z \in X$. 3

Definition 2.4.([30]) A non-empty subset I of a BCI-algebra X is called subalgebra of X if $x * y \in I$ for all $x, y \in I$.

Definition 2.5.([30]) A mapping $f : X \to Y$ of BCI-algebras is called a homomorphism if $f(x * y) = f(x) * f(y)$, for all $x, y \in X$.

Definition 2.6.([7]) Let X be an arbitrary set. A fuzzy subset of X, we mean a function from X into $[0, 1]$. The set of all fuzzy subsets of X is called the $[0, 1]$-power set of X and is denoted $[0, 1]^X$. For a fixed $s \in [0, 1]$, the set $\mu_s = \{x \in X : \mu(x) \geq s\}$ is called an upper level of μ and the set $\mu_t = \{x \in X : \mu(x) \leq t\}$ is called a lower level of μ.

Definition 2.7.([2, 3]) Let X be a nonempty set. A complex mapping $A = (\mu_A, \nu_A) : X \to [0, 1] \times [0, 1]$ is called an intuitionistic fuzzy set (in short, IFS) in X if $\mu_A + \nu_A \leq 1$ where the mappings $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) for each $x \in X$ to A, respectively. In particular $\emptyset_X$ and U_X denote the intuitionistic fuzzy empty set and intuitionistic fuzzy whole set in X defined by $\emptyset_X(x) = (0, 1) \sim 0$ and $U_X(x) = (1, 0) \sim 1$, respectively. We will denote the set of all $IFSs$ in X as $IFS(X)$.

Definition 2.8.([7]) Let φ be a function from set X into set Y such that $A = (\mu_A, \nu_A) \in IFS(X)$ and $B = (\mu_B, \nu_B) \in IFS(Y)$. For all $x \in X, y \in Y$, we define

$$\begin{aligned} \varphi(A)(y) &= (\varphi(\mu_A)(y), \varphi(\nu_A)(y)) \\ &= \begin{cases} (\sup\{\mu_A(x) \mid x \in X, \varphi(x) = y\}, \inf\{\nu_A(x) \mid x \in X, \varphi(x) = y\}) & \text{if } \varphi^{-1}(y) \neq \emptyset \\ (0, 1) & \text{if } \varphi^{-1}(y) = \emptyset \end{cases} \end{aligned}$$

Also $\varphi^{-1}(B)(x) = (\varphi^{-1}(\mu_B)(x), \varphi^{-1}(\nu_B)(x)) = (\mu_B(\varphi(x)), \nu_B(\varphi(x)))$.

Definition 2.9.([4]) A t-norm T is a function $T : [0,1] \times [0,1] \to [0,1]$ having the following four properties:

(T1) $T(x, 1) = x$ (neutral element),
(T2) $T(x, y) \leq T(x, z)$ if $y \leq z$ (monotonicity),
(T3) $T(x, y) = T(y, x)$ (commutativity),
(T4) $T(x, T(y, z)) = T(T(x, y), z)$ (associativity),
for all $x, y, z \in [0, 1]$.

It is clear that if $x_1 \geq x_2$ and $y_1 \geq y_2$, then $T(x_1, y_1) \geq T(x_2, y_2)$.

Example 2.1. (1) Standard intersection t-norm $T_m(x, y) = \min\{x, y\}$.
(2) Bounded sum t-norm $T_b(x, y) = \max\{0, x + y - 1\}$.
(3) algebraic product t-norm $T_p(x, y) = xy$.
(4) Drastic t-norm

$$T_D(x,y) = \begin{cases} y & \text{if } x = 1 \\ x & \text{if } y = 1 \\ 0 & \text{otherwise.} \end{cases}$$

(5) Nilpotent minimum t-norm

$$T_{nM}(x,y) = \begin{cases} \min\{x,y\} & \text{if } x + y > 1 \\ 0 & \text{otherwise.} \end{cases}$$

(6) Hamacher product T-norm

$$T_{H_0}(x,y) = \begin{cases} 0 & \text{if } x = y = 0 \\ \frac{xy}{x+y-xy} & \text{otherwise.} \end{cases}$$

The drastic t-norm is the pointwise smallest t-norm and the minimum is the pointwise largest t-norm: $T_D(x, y) \leq T(x, y) \leq T_{\min}(x, y)$ for all $x, y \in [0, 1]$.

Definition 2.10.([4]) A t-norm C is a function $C : [0,1] \times [0,1] \to [0,1]$ having the following four properties:

(1) $C(x, 0) = x$,
(2) $C(x, y) \leq C(x, z)$ if $y \leq z$,
(3) $C(x, y) = C(y, x)$,
(4) $C(x, C(y, z)) = C(C(x, y), z)$,
for all $x, y, z \in [0, 1]$.

We say that T and C be idempotent if for all $x \in [0, 1]$ we have $T(x, x) = x$ and $C(x, x) = x$.

Example 2.2. The basic t-conorms are

$$C_m(x, y) = \max\{x, y\},$$

$$C_b(x, y) = \min\{1, x + y\}$$

and

$$C_p(x, y) = x + y - xy$$

for all $x, y \in [0,1]$.
S_m is standard union, C_b is bounded sum, C_p is algebraic sum.

Definition 2.11.([8]) Let $A = (\mu_A, \nu_A) \in IFS(X)$ and $B = (\mu_B, \nu_B) \in IFS(X)$. Define

$$A \cap B = (\mu_{A\cap B}, \nu_{A\cap B}) : X \to [0,1]$$

as $\mu_{A\cap B}(x) = T(\mu_A(x), \mu_B(x))$ and $\nu_{A\cap B}(x) = C(\nu_A(x), \nu_B(x))$ for all $x \in X$.

Definition 2.12.([8]) Let $A = (\mu_A, \nu_A) \in IFS(X)$ and $B = (\mu_B, \nu_B) \in IFS(Y)$. The cartesian product of A and B is denoted by $A \times B : X \times Y \to [0,1]$ is defined by

$$(A \times B)(x, y) = ((\mu_A, \nu_A) \times (\mu_B, \nu_B))(x, y) = (\mu_{A\times B}, \nu_{A\times B})(x, y)$$

$$= (\mu_{A\times B}(x, y), \nu_{A\times B}(x, y)) = (T(\mu_A(x), \mu_B(y)), C(\nu_A(x), \nu_B(y)))$$

for all $(x, y) \in X \times Y$.

Lemma 2.1.([1]) Let C be a t-conorm and T be a t-norm. Then

$$C(C(x, y), C(w, z)) = C(C(x, w), C(y, z)),$$

and

$$T(T(x, y), T(w, z)) = T(T(x, w), T(y, z)),$$

for all $x, y, w, z \in [0,1]$.

§3. $(T,C)IFS(X), (T,C)IFI(X), (T,C)IFPII(X)$

Definition 3.1. Let $A = (\mu_A, \nu_A) \in IFS(X)$ then A is called an intuitionistic fuzzy subalgebra of BCI-algebra X under norms(t-norm T and t-conorm C) if
(1) $\mu_A(x * y) \geq T(\mu_A(x), \mu_A(y))$,
(2) $\nu_A(x * y) \leq C(\nu_A(x), \nu_A(y))$
for all $x, y \in X$.
Denote by $(T,C)IFS(X)$, the set of all intuitionistic fuzzy subalgebras of BCI-algebra X under norms (t-norm T and t-conorm C).

Example 3.1. Let $X = \{0, a, b, c\}$ be a set given by the following Cayley table:

*	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Then $(X, *, 0)$ is a BCI-algebra.
Define fuzzy subset $\mu_A : (X, *, 0) \to [0,1]$ as

$$\mu_A(x) = \begin{cases} 0.55 & \text{if } x = 0, a, c \\ 0.25 & \text{if } x = b \end{cases}$$

and

$$\nu_A(x) = \begin{cases} 0.15 & \text{if } x = 0, a, c \\ 0.45 & \text{if } x = b. \end{cases}$$

Let $T(a,b) = T_p(a,b) = ab$ and $C(a,b) = C_p(a,b) = a + b - ab$ for all $a, b \in [0,1]$ then $A = (\mu_A, \nu_A) \in (T,C)IFS(X)$.

Proposition 3.1. Let $A = (\mu_A, \nu_A) \in IFS(X)$ and T, C be idempotent. Then $A = (\mu_A, \nu_A) \in (T,C)IFS(X)$ if and only if the set

$$A_{s,t} = \{x \in X : A(x) \supseteq (s,t)\}$$

be either empty or a subalgebra of X for every $s, t \in [0,1]$.

Proof. Let $A = (\mu_A, \nu_A) \in IFSN(X)$ and $x, y \in A_{s,t}$. Then

$$\mu_A(x * y) \geq T(\mu_A(x), \mu_A(y)) \geq T(s,s) = s$$

and

$$\nu_A(x * y) \leq C(\nu_A(x), \nu_A(y)) \leq C(t,t) = t$$

so

$$A(x * y) = (\mu_A(x * y), \nu_A(x * y)) \supseteq A(s,t)$$

thus $x * y \in A_{s,t}$ and so $A_{s,t}$ will be a subalgebra of X for every $s, t \in [0,1]$.
Conversely, let $A_{s,t}$ is either empty or a subalgebra of X for every $s, t \in [0,1]$. Let $s = T(\mu_A(x), \mu_A(y))$ and $t = C(\nu_A(x), \nu_A(y))$ and $x, y \in A_{s,t}$. As $A_{s,t}$ is a subalgebra of X so $x * y \in A_{s,t}$ and thus

$$\mu_A(x * y) \geq s = T(\mu_A(x), \mu_A(y))$$

and

$$\nu_A(x * y) \leq t = C(\nu_A(x), \nu_A(y))$$

so $A = (\mu_A, \nu_A) \in (T,C)IFS(X)$. □

Proposition 3.2. Let $A = (\mu_A, \nu_A) \in (T,C)IFS(X)$ and T, C be idempotent. Then $A(0) \supseteq A(x)$ for all $x \in X$.

Proof. Let $x \in X$. Then

$$\mu_A(0) = \mu_A(x * x) \geq T(\mu_A(x), \mu_A(x)) = \mu_A(x)$$

and

$$\nu_A(0) = \nu_A(x * x) \leq C(\nu_A(x), \nu_A(x)) = \nu_A(x)$$

thus

$$A(0) = (\mu_A(0), \nu_A(0)) \supseteq (\mu_A(x), \nu_A(x)) = A(x).$$

□

Definition 3.2. Define $A = (\mu_A, \nu_A) \in IFS(X)$ is an intuitionistic fuzzy ideal of BCI-algebra X under norms(t-norm T and t-conorm C) if it satisfies the following inequalities:
(1) $\mu_A(0) \geq \mu_A(x)$,
(2) $\mu_A(x) \geq T(\mu_A(x * y), \mu_A(y))$,
(3) $\nu_A(0) \leq \nu_A(x)$,
(4) $\nu_A(x) \leq C(\nu_A(x * y), \nu_A(y))$,
for all $x, y \in X$.
Denote by $(T, C)IFI(X)$, the set of all intuitionistic fuzzy ideals of X under norms(t-norm T and t-conorm C).

Example 3.2. Let $X = \{0, a, 1, 2, 3\}$ be a set given by the following Cayley table:

*	0	a	1	2	3
0	0	0	3	2	1
a	a	0	3	2	1
1	1	1	0	3	2
2	3	2	1	0	3
3	3	3	2	1	0

Then $(X, *, 0)$ is a BCI-algebra. Define $A = (\mu_A, \nu_A) \in IFS(X)$ as

$$\mu_A(x) = \begin{cases} t_0 & \text{if } x = 0, \\ t_1 & \text{if } x = a, \\ t_2 & \text{if } x = 1, 2, 3, \end{cases}$$

and

$$\nu_A(x) = \begin{cases} s_0 & \text{if } x = 0, \\ s_1 & \text{if } x = a, \\ s_2 & \text{if } x = 1, 2, 3, \end{cases}$$

with $t_0 > t_1 > t_2$ and $s_0 < s_1 < s_2$ such that $0 < t_i + s_i < 1$ and $t_i, s_i \in [0, 1]$.

Let $T(a, b) = T_p(a, b) = ab$ and $C(a, b) = C_p(a, b) = a + b - ab$ for all $a, b \in [0, 1]$ then $A = (\mu_A, \nu_A) \in (T, C)IFI(X)$.

Proposition 3.3. Let $A = (\mu_A, \nu_A) \in IFS(X)$ and T, C be idempotent. Then $A = (\mu_A, \nu_A) \in (T, C)IFI(X)$ if and only if the

$$A_{s,t} = \{x \in X : A(x) \supseteq (s, t)\}$$

be either empty or an ideal of BCI-algebra X for every $s, t \in [0, 1]$.

Proof. Let $A = (\mu_A, \nu_A) \in (T, C)IFI(X)$ and $x, y \in X$. Then $\mu_A(0) \geq \mu_A(x) \geq s$ and $\nu_A(0) \leq \nu_A(x) \leq t$ so $A(0) = (\mu_A(0), \nu_A(0)) \supseteq (s, t)$ and then $0 \in A_{s,t}$. Also let $x * y \in A_{s,t}$ and $y \in A_{s,t}$. Then

$$\mu_A(x) \geq T(\mu_A(x * y), \mu_A(y)) \geq T(s, s) = s$$

and

$$\nu_A(x) \leq C(\nu_A(x*y), \nu_A(y)) \leq C(t,t) = t$$

then $A(x) = (\mu_A(x), \nu_A(x)) \supseteq (s,t)$ thus $x \in A_{s,t}$. Then $A_{s,t}$ will be an ideal of BCI-algebra X for every $t \in [0,1]$.
Conversely, let $A_{s,t}$ be either empty or an ideal of BCI-algebra X for every $s,t \in [0,1]$. Let $s = T(\mu_A(x*y), \mu_A(y))$ and $t = C(\nu_A(x*y), \nu_A(y))$ with $x*y \in A_{s,t}$ and $y \in A_{s,t}$. Then $x \in A_{s,t}$ thus

$$\mu_A(x) \geq s = T(\mu_A(x*y), \mu_A(y))$$

and

$$\nu_A(x) \leq t = C(\nu_A(x*y), \nu_A(y))$$

so $A = (\mu_A, \nu_A) \in (T,C)IFI(X)$. □

Proposition 3.4. Let $A = (\mu_A, \nu_A) \in (T,C)IFI(X)$ and $x*y \leq z$. Then $\mu_A(x) \geq T(\mu_A(y), \mu_A(z))$ and $\nu_A(x) \leq C(\nu_A(y), \nu_A(z))$ for all $x,y,z \in X$.

Proof. As $x*y \leq z$ so $(x*y)*z = 0$ for all $x,y,z \in X$. Then

$$\begin{aligned}
\mu_A(x) &\geq T(\mu_A(x*y), \mu_A(y)) \\
&\geq T(T(\mu_A((x*y)*z), \mu_A(z)), \mu_A(y)) \\
&= T(T(\mu_A(0), \mu_A(z)), \mu_A(y)) \\
&= T(\mu_A(z), \mu_A(y)) \\
&= T(\mu_A(y), \mu_A(z))
\end{aligned}$$

thus $\mu_A(x) \geq T(\mu_A(y), \mu_A(z))$. Also

$$\begin{aligned}
\nu_A(x) &\leq C(\nu_A(x*y), \nu_A(y)) \\
&\leq C(C(\nu_A((x*y)*z), \nu_A(z)), \nu_A(y)) \\
&= C(C(\nu_A(0), \nu_A(z)), \nu_A(y)) \\
&= C(\nu_A(z), \nu_A(y)) \\
&= C(\nu_A(y), \nu_A(z))
\end{aligned}$$

so $\nu_A(x) \leq C(\nu_A(y), \nu_A(z))$. □

Proposition 3.5. Let $A = (\mu_A, \nu_A) \in (T,C)IFI(X)$ and $x \leq y$ for all $x,y \in X$. Then $A(x) \supseteq A(y)$.

Proof. Since $x \leq y$ so $x*y = 0$ for all $x,y \in X$. Then

$$\mu_A(x) \geq T(\mu_A(x*y), \mu_A(y)) = T(\mu_A(0), \mu_A(y)) = \mu_A(y)$$

and

$$\nu_A(x) \leq C(\nu_A(x*y), \nu_A(y)) = C(\nu_A(0), \nu_A(y)) = \nu_A(y)$$

therefore

$$A(x) = (\mu_A(x), \nu_A(x)) \supseteq (\mu_A(y), \nu_A(y)) = A(y).$$

$\square$

Following proposition provides that every $(T, C)IFI(X)$ is $(T, C)IFS(X)$.

Proposition 3.6. Let $A = (\mu_A, \nu_A) \in (T, C)IFI(X)$. Then $A = (\mu_A, \nu_A) \in (T, C)IFS(X)$.

Proof. We know that $x * y \leq x$ and from Proposition 3.2 we get that $A(x * y) \supseteq A(x)$. Now

$$\mu_A(x * y) \geq \mu_A(x) \geq T(\mu_A(x * y), \mu_A(y)) \geq T(\mu_A(x), \mu_A(y))$$

and

$$\nu_A(x * y) \leq \nu_A(x) \leq C(\nu_A(x * y), \nu_A(y)) \leq C(\nu_A(x), \nu_A(y))$$

and then $A = (\mu_A, \nu_A) \in (T, C)IFS(X)$. $\square$

Remark 3.1. The converse of Proposition 3.6 may not be true. For example in Example 3.1we have that $A = (\mu_A, \nu_A) \in (T, C)IFS(X)$ but since $\mu_A(b) = 0.25 \ngeq T(\mu_A(b * a), \mu_A(a)) = T(\mu_A(a), \mu_A(a)) = \mu_A(a) = 0.55$ so $A = (\mu_A, \nu_A) \notin (T, C)IFI(X)$.

As under a condition every $(T, C)IFS(X)$ is $(T, C)IFI(X)$.

Proposition 3.7. Let $A = (\mu_A, \nu_A) \in (T, C)IFS(X)$. If $\mu_A(x) \geq T(\mu_A(y), \mu_A(z))$ and $\nu_A(x) \leq C(\nu_A(y), \nu_A(z))$ and $x * y \leq z$ for all $x, y, z \in X$, then $A = (\mu_A, \nu_A) \in (T, C)IFI(X)$.

Proof. As Proposition 3.4 we get that $\mu_A(0) \geq \mu_A(x)$ and $\nu_A(0) \leq \nu_A(x)$. As $x * (x * y) \leq y$ so $\mu_A(x) \geq T(\mu_A(x * y), \mu_A(y))$ and $\nu_A(x) \leq C(\nu_A(x * y), \nu_A(y))$. (From the hypothesis)
Then $A = (\mu_A, \nu_A) \in (T, C)IFI(X)$. $\square$

Proposition 3.8. Let $A = (\mu_A, \nu_A) \in IFS(X)$. Then

$$A = (\mu_A, \nu_A) \in (T, C)IFI(X) \Longleftrightarrow \triangle A = (\mu_A, \bar{\mu_A}) \in (T, C)IFI(X)$$

$$\textit{and } \bigtriangledown A = (\bar{\nu_A}, \nu_A) \in (T, C)IFI(X)$$

such that $\bar{\mu_A} = 1 - \mu_A$ and $\bar{\nu_A} = 1 - \nu_A$.

Proof. Let $x, y \in X$. Let $A = (\mu_A, \nu_A) \in (T, C)IFI(X)$ then

$$\mu_A(0) \geq \mu_A(x) \qquad \textit{(1)}$$

and

$$\mu_A(x) \geq T(\mu_A(x * y), \mu_A(y)) \qquad \textit{(2)}$$

also

$$\bar{\mu_A}(0) = 1 - \mu_A(0) \leq 1 - \mu_A(x) = \bar{\mu_A}(x) \qquad \textit{(3)}$$

and

$$\bar{\mu_A}(x) = 1 - \mu_A(x) \leq 1 - T(\mu_A(x * y), \mu_A(y)) = C(\bar{\mu_A}(x * y), \bar{\mu_A}(y)) \qquad \textit{(4)}$$

thus $\triangle A = (\mu_A, \bar{\mu_A}) \in IFIN(X)$.
Also

$$\nu_A(0) \leq \nu_A(x) \qquad (1)$$

and

$$\nu_A(x) \leq C(\nu_A(x * y), \nu_A(y)) \qquad (2)$$

and

$$\bar{\nu_A}(0) = 1 - \nu_A(0) \geq 1 - \nu_A(x) = \bar{\nu_A}(x) \qquad (3)$$

and

$$\bar{\nu_A}(x) = 1 - \nu_A(x) \geq 1 - C(\nu_A(x * y), \nu_A(y)) = T(\bar{\nu_A}(x * y), \bar{\nu_A}(y)) \qquad (4)$$

thus $\bigtriangledown A = (\bar{\nu_A}, \nu_A) \in IFIN(X)$.
Conversely, let $\triangle A = (\mu_A, \bar{\mu_A}) \in (T, C)IFI(X)$ and $\bigtriangledown A = (\bar{\nu_A}, \nu_A) \in IFIN(X)$ then we will have that $A = (\mu_A, \nu_A) \in (T, C)IFI(X)$. □

Definition 3.3. We say that $A = (\mu_A, \nu_A) \in IFS(X)$ is an intuitionistic fuzzy positive implicative ideal of BCI-algebra X under norms(t-norm T and t-conorm C) if it satisfies the following inequalities:
(1) $\mu_A(0) \geq \mu_A(x)$,
(2) $\mu_A(x * z) \geq T(\mu_A(((x * z) * z) * (y * z)), \mu_A(y))$,
(3) $\nu_A(0) \leq \nu_A(x)$,
(4) $\nu_A(x * z) \leq C(\nu_A(((x * z) * z) * (y * z)), \nu_A(y))$,
for all $x, y, z \in X$.
Denote by $A = (\mu_A, \nu_A) \in (T, C)IFPII(X)$, the set of all intuitionistic fuzzy positive implicative ideals of BCI-algebra X under norms(t-norm T and t-conorm C).

Example 3.3. Let $X = \{0, 1, 2, 3\}$ be a set given by the following Cayley table:

*	0	1	2	3
0	0	0	0	3
1	1	0	0	3
2	2	2	0	3
3	3	3	3	0

Then $(X, *, 0)$ is a BCI-algebra. Define $A = (\mu_A, \nu_A) \in IFS(X)$ as

$$\mu_A(x) = \begin{cases} 1 & \text{if } x = 0, 3 \\ t & \text{if } x = 1, 2 \end{cases}$$

and

$$\nu_A(x) = \begin{cases} 0 & \text{if } x = 0, 3 \\ s & \text{if } x = 1, 2 \end{cases}$$

such that $0 < t + s \leq 1$ and $t, s \in (0, 1)$. Let $T(a, b) = T_p(a, b) = ab$ and $C(a, b) = C_p(a, b) = a + b - ab$ for $all a, b \in [0, 1]$ then $A = (\mu_A, \nu_A) \in (T, C)IFPII(X)$.

Proposition 3.9. Let $A = (\mu_A, \nu_A) \in IFS(X)$ and T, C be idempotent. Then $A = (\mu_A, \nu_A) \in (T, C)IFPII(X)$ if and only if the set

$$A_{s,t} = \{x \in X : A(x) \supseteq (s, t)\}$$

be either empty or a positive implicative ideal of BCI-algebra X for every $s, t \in [0, 1]$.

Proof. Let $A = (\mu_A, \nu_A) \in (T, C)IFPII(X)$ and $A_{s,t} = \{x \in X : A(x) \supseteq (s, t)\}$ be not empty then for any $x \in A_{s,t}$ we have $\mu_A(x) \geq s$ and $\nu_A(x) \leq t$ so $\mu_A(0) \geq \mu_A(x) \geq s$ and $\nu_A(0) \leq \nu_A(x) \leq t$ thus $A(0) \supseteq (s, t)$ which means that $0 \in A_{s,t}$.
Also let $((x * z) * z) * (y * z) \in A_{s,t}$ and $y \in A_{s,t}$. Then

$$\mu_A(x * z) \geq T(\mu_A(((x * z) * z) * (y * z)), \mu_A(y)) \geq T(s, s) = s$$

and

$$\nu_A(x * z) \leq C(\nu_A(((x * z) * z) * (y * z)), \nu_A(y)) \leq C(t, t) = t$$

thus $x * z \in A_{s,t}$. Then $A_{s,t}$ is a posive implicative ideal of X for every $s, t \in [0, 1]$.
Conversely, let $A_{s,t}$ be not empty and be a positive implicative ideal of X for every $s, t \in [0, 1]$. Then for any $x \in A_{s,t}$ we have $A(0) \supseteq (s, t)$ then $A(0) \supseteq (s, t)$ and so $\mu_A(x) \geq s$ and $\nu_A(x) \leq t$. Let $s = T(\mu_A(((x * z) * z) * (y * z)), \mu_A(y))$ and $t = C(\nu_A(((x * z) * z) * (y * z)), \nu_A(y))$ with $((x * z) * z) * (y * z) \in A_{s,t}$ and $y \in A_{s,t}$. Thus $x * z \in A_{s,t}$. Therefore

$$\mu_A(x * z) \geq s = T(\mu_A(((x * z) * z) * (y * z)), \mu_A(y))$$

and

$$\nu_A(x * z) \leq t = C(\nu_A(((x * z) * z) * (y * z)), \nu_A(y))$$

so $A = (\mu_A, \nu_A) \in (T, C)IFPII(X)$. □

We prove that every $(T, C)IFPII(X)$ will be $(T, C)IFI(X)$ as following proposition. Whereas the converse of this proposition may not be true. For this consider the following example.

Proposition 3.10. If $A = (\mu_A, \nu_A) \in (T, C)IFPII(X)$, then $A = (\mu_A, \nu_A) \in (T, C)IFI(X)$.

Proof. Let $x, y, z \in X$ and $A = (\mu_A, \nu_A) \in (T, C)IFPII(X)$. Then
(1) $\mu_A(0) \geq \mu_A(x)$,
(2) $\mu_A(x * z) \geq T(\mu_A(((x * z) * z) * (y * z)), \mu_A(y))$,
(3) $\nu_A(0) \leq \nu_A(x)$,
(4) $\nu_A(x * z) \leq C(\nu_A(((x * z) * z) * (y * z)), \nu_A(y))$.
Now in (2) and (4) let $z = 0$ then

$$\mu_A(x * 0) \geq T(\mu_A(((x * 0) * 0) * (y * 0)), \mu_A(y))$$

and

$$\nu_A(x * 0) \leq C(\nu_A(((x * 0) * 0) * (y * 0)), \nu_A(y))$$

which mean that

$$\mu_A(x) \geq T(\mu_A(x * y), \mu_A(y))$$

and

$$\nu_A(x) \leq C(\nu_A(x * y, \nu_A(y)).$$

Therefore $A = (\mu_A, \nu_A) \in (T, C)IFI(X)$. □

Example 3.4. Consider the BCI-algebra $X = \{0, 1, 2, 3, 4\}$ with the following caley table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	3	4	1	0

Then $(X, *, 0)$ is a BCI-algebra. Define $A = (\mu_A, \nu_A) \in IFS(X)$ as

$$\mu_A(x) = \begin{cases} 1 & \text{if } x = 0, 2 \\ t & \text{if } x = 1, 3, 4 \end{cases}$$

and

$$\nu_A(x) = \begin{cases} 0 & \text{if } x = 0, 2 \\ s & \text{if } x = 1, 3, 4 \end{cases}$$

such that $0 < t + s \leq 1$ and $t, s \in (0, 1)$. Let $T(a, b) = T_p(a, b) = ab$ and $C(a, b) = C_p(a, b) = a + b - ab$ for all $a, b \in [0, 1]$ then $A = (\mu_A, \nu_A) \in (T, C)IFI(X)$ but $A = (\mu_A, \nu_A) \notin (T, C)IFPII(X)$ because: as we let $x = 4, z = 3, y = 2$ so from $\mu_A(x * z) \geq T(\mu_A(((x * z) * z) * (y * z)), \mu_A(y))$ we get that $t \geq 1$ and this is a contradition with $t, s \in (0, 1)$.

Proposition 3.11. If $A = (\mu_A, \nu_A) \in (T, C)IFPII(X)$, then $\triangle A = (\mu_A, \bar{\mu_A}) \in (T, C)IFPII(X)$.

Proof. Let $x, y, z \in X$. As $A = (\mu_A, \nu_A) \in (T, C)IFPII(X)$ so

$$\mu_A(0) \geq \mu_A(x) \qquad (1)$$

and then $1 - \mu_A(0) \leq 1 - \mu_A(x)$ then

$$\bar{\mu_A}(0) \leq \bar{\mu_A}(x). \qquad (2)$$

Also

$$\mu_A(x * z) \geq T(\mu_A(((x * z) * z) * (y * z)), \mu_A(y)), \qquad (3)$$

thus

$$1 - \mu_A(x * z) \leq 1 - T(\mu_A(((x * z) * z) * (y * z)), \mu_A(y))$$

then

$$\bar{\mu_A}(x * z) \leq C(1 - \mu_A(((x * z) * z) * (y * z)), 1 - \mu_A(y))$$

then

$$\bar{\mu_A}(x * z) \leq C(\bar{\mu_A}(((x * z) * z) * (y * z)), \bar{\mu_A}(y)). \qquad (4)$$

Now (1)-(4) give us that $\triangle A = (\mu_A, \bar{\mu_A}) \in (T, C)IFPII(X)$. □

Proposition 3.12. If $A = (\mu_A, \nu_A) \in (T,C)IFPII(X)$, then $\triangledown A = (\bar{\nu_A}, \nu_A) \in (T,C)IFPII(X)$.

Proof. Let $x, y, z \in X$. As $A = (\mu_A, \nu_A) \in (T,C)IFPII(X)$ so

$$\nu_A(0) \leq \nu_A(x) \qquad (1)$$

and then $1 - \nu_A(0) \geq 1 - \nu_A(x)$ then

$$\bar{\nu_A}(0) \geq \bar{\nu_A}(x). \qquad (2)$$

Also

$$\nu_A(x * z) \leq C(\nu_A(((x * z) * z) * (y * z)), \nu_A(y)) \qquad (3)$$

thus

$$1 - \nu_A(x * z) \geq 1 - C(\nu_A(((x * z) * z) * (y * z)), \nu_A(y))$$

then

$$\bar{\nu_A}(x * z) \geq T(1 - \nu_A(((x * z) * z) * (y * z)), 1 - \nu_A(y)))$$

then

$$\bar{\nu_A}(x * z) \geq T(\bar{\nu_A}(((x * z) * z) * (y * z)), \bar{\nu_A}(y)). \qquad (4)$$

As (1)-(4) so $\triangledown A = (\mu_A, \bar{\mu_A}) \in IFPIIN(X)$. □

Proposition 3.13. $A = (\mu_A, \nu_A) \in (T,C)IFPII(X)$ if and only if $\triangle A = (\mu_A, \bar{\mu_A}) \in (T,C)IFPII(X)$ and $\triangledown A = (\bar{\nu_A}, \nu_A) \in (T,C)IFPII(X)$.

Proof. Use Proposition 3.11 and Proposition 3.12. □

§4. $(T,C)IFS(X), (T,C)IFI(X), (T,C)IFPII(X)$ under intersections, cartesian products and homomorphisms

Proposition 4.1. Let $A = (\mu_A, \nu_A) \in (T,C)IFS(X)$ and $B = (\mu_B, \nu_B) \in (T,C)IFS(X)$. Then $A \cap B \in (T,C)IFS(X)$.

Proof. Let $x, y \in X$. Then

$$\begin{aligned}
\mu_{A\cap B}(x * y) &= T(\mu_A(x * y), \mu_B(x * y)) \\
&\geq T(T(\mu_A(x), \mu_A(y)), T(\mu_B(x), \mu_B(y))) \\
&= T(T(\mu_A(x), \mu_B(x)), T(\mu_A(y), \mu_B(y))) \\
&= T(\mu_{A\cap B}(x), \mu_{A\cap B}(y))
\end{aligned}$$

thus

$$\mu_{A\cap B}(x * y) \geq T(\mu_{A\cap B}(x), \mu_{A\cap B}(y)).$$

Also

$$\begin{aligned}\nu_{A\cap B}(x*y) &= C(\nu_A(x*y),\nu_B(x*y))\\ &\leq C(C(\nu_A(x),\nu_A(y)),C(\nu_B(x),\nu_B(y)))\\ &= C(C(\nu_A(x),\nu_B(x)),C(\nu_A(y),\nu_B(y)))\\ &= C(\nu_{A\cap B}(x),\nu_{A\cap B}(y))\end{aligned}$$

then

$$\nu_{A\cap B}(x*y)\leq C(\nu_{A\cap B}(x),\nu_{A\cap B}(y)).$$

Thus $A\cap B=(\mu_{A\cap B},\nu_{A\cap B})\in(T,C)IFS(X)$. □

Proposition 4.2. Let $A=(\mu_A,\nu_A)\in(T,C)IFI(X)$ and $B=(\mu_B,\nu_B)\in(T,C)IFI(X)$. Then $A\cap B\in(T,C)IFI(X)$.

Proof. Let $x,y\in X$. Then

(1)

$$\mu_{A\cap B}(0)=T(\mu_A(0),\mu_B(0))\geq T(\mu_A(x),\mu_B(x))=\mu_{A\cap B}(x)$$

thus

$$\mu_{A\cap B}(0)\geq\mu_{A\cap B}(x).$$

(2)

$$\begin{aligned}\mu_{A\cap B}(x) &= T(\mu_A(x),\mu_B(x))\\ &\geq T(T(\mu_A(x*y),\mu_A(y)),T(\mu_B(x*y),\mu_B(y)))\\ &= T(T(\mu_A(x*y),\mu_B(x*y)),T(\mu_A(y),\mu_B(y)))\\ &= T(\mu_{A\cap B}(x*y),\mu_{A\cap B}(y))\end{aligned}$$

so

$$\mu_{A\cap B}(x)\geq T(\mu_{A\cap B}(x*y),\mu_{A\cap B}(y)).$$

(3)

$$\nu_{A\cap B}(0)=C(\nu_A(0),\mu_B(0))\leq C(\nu_A(x),\nu_B(x))=\nu_{A\cap B}(x)$$

so

$$\nu_{A\cap B}(0)\leq\nu_{A\cap B}(x).$$

(4)

$$\begin{aligned}\nu_{A\cap B}(x) &= C(\nu_A(x),\mu_B(x))\\ &\leq C(C(\nu_A(x*y),\nu_A(y)),C(\nu_B(x*y),\nu_B(y)))\\ &= C(C(\nu_A(x*y),\nu_B(x*y)),C(\nu_A(y),\nu_B(y)))\\ &= C(\nu_{A\cap B}(x*y),\nu_{A\cap B}(y))\end{aligned}$$

then

$$\nu_{A\cap B}(x) \leq C(\nu_{A\cap B}(x * y), \nu_{A\cap B}(y)).$$

Now (1)-(4) give us that $A \cap B = (\mu_{A\cap B}, \nu_{A\cap B}) \in (T, C)IFI(X)$. $\square$

Proposition 4.3. Let $A = (\mu_A, \nu_A) \in (T, C)IFPII(X)$ and $B = (\mu_B, \nu_B) \in (T, C)IFPII(X)$. Then $A \cap B \in (T, C)IFPII(X)$.

Proof. Let $x, y, z \in X$. Then
(1)

$$\mu_{A\cap B}(0) = T(\mu_A(0), \mu_B(0)) \geq T(\mu_A(x), \mu_B(x)) = \mu_{A\cap B}(x)$$

thus

$$\mu_{A\cap B}(0) \geq \mu_{A\cap B}(x).$$

(2)

$$\begin{aligned}\mu_{A\cap B}(x * z) &= T(\mu_A(x * z), \mu_B(x * z))\\ &\geq T(T(\mu_A(((x * z) * z) * (y * z)), \mu_A(y)), T(\mu_B(((x * z) * z) * (y * z)), \mu_B(y)))\\ &= T(T(\mu_A(((x * z) * z) * (y * z)), \mu_B(((x * z) * z) * (y * z))), T(\mu_A(y), \mu_B(y)))\\ &= T(\mu_{A\cap B}(((x * z) * z) * (y * z))), \mu_{A\cap B}(y))\end{aligned}$$

so

$$\mu_{A\cap B}(x * z) \geq T(\mu_{A\cap B}(((x * z) * z) * (y * z))), \mu_{A\cap B}(y)).$$

(3)

$$\nu_{A\cap B}(0) = C(\nu_A(0), \mu_B(0)) \leq C(\nu_A(x), \nu_B(x)) = \nu_{A\cap B}(x)$$

so

$$\nu_{A\cap B}(0) \leq \nu_{A\cap B}(x).$$

(4)

$$\begin{aligned}\nu_{A\cap B}(x * z) &= C(\nu_A(x * z), \nu_B(x * z))\\ &\leq C(C(\nu_A(((x * z) * z) * (y * z)), \nu_A(y)), C(\nu_B(((x * z) * z) * (y * z)), \nu_B(y)))\\ &= C(C(\nu_A(((x * z) * z) * (y * z)), \nu_B(((x * z) * z) * (y * z))), C(\nu_A(y), \nu_B(y)))\\ &= C(\nu_{A\cap B}(((x * z) * z) * (y * z))), \nu_{A\cap B}(y))\end{aligned}$$

so

$$\nu_{A\cap B}(x * z) \leq C(\nu_{A\cap B}(((x * z) * z) * (y * z))), \nu_{A\cap B}(y)).$$

Now (1)-(4) give us that $A \cap B = (\mu_{A\cap B}, \nu_{A\cap B}) \in (T, C)IFPII(X)$. $\square$

Proposition 4.4. Let $A = (\mu_A, \nu_A) \in (T, C)IFS(X)$ and $B = (\mu_B, \nu_B) \in (T, C)IFS(Y)$. Then $A \times B \in (T, C)IFS(X \times Y)$.

Proof. Let $(x_1, y_1), (x_2, y_2) \in X \times Y$. Then

$$\begin{aligned}(\mu_{A\times B})((x_1, y_1) * (x_2, y_2)) &= (\mu_{A\times B})(x_1 * x_2, y_1 * y_2)\\ &= T(\mu_A(x_1 * x_2), \mu_B(y_1 * y_2))\\ &\geq T(T(\mu_A(x_1), \mu_A(x_2)), T(\mu_B(y_1), \mu_B(y_2)))\\ &= T(T(\mu_A(x_1), \mu_B(y_1)), T(\mu_A(x_2), \mu_B(y_2)))\\ &= T(\mu_{A\times B}(x_1, y_1), \mu_{A\times B}(x_2, y_2))\end{aligned}$$

thus

$$\mu_{A\times B}((x_1, y_1) * (x_2, y_2)) \geq T(\mu_{A\times B}(x_1, y_1), \mu_{A\times B}(x_2, y_2)).$$

Also

$$\begin{aligned}(\nu_{A\times B})((x_1, y_1) * (x_2, y_2)) &= (\nu_{A\times B})(x_1 * x_2, y_1 * y_2)\\ &= C(\nu_A(x_1 * x_2), \nu_B(y_1 * y_2))\\ &\leq C(C(\nu_A(x_1), \nu_A(x_2)), C(\nu_B(y_1), \nu_B(y_2)))\\ &= C(C(\nu_A(x_1), \nu_B(y_1)), C(\nu_A(x_2), \nu_B(y_2)))\\ &= C(\nu_{A\times B}(x_1, y_1), \nu_{A\times B}(x_2, y_2))\end{aligned}$$

then

$$\nu_{A\times B}((x_1, y_1) * (x_2, y_2)) \leq C(\nu_{A\times B}(x_1, y_1), \nu_{A\times B}(x_2, y_2)).$$

Therefore

$$A \times B = (\mu_{A\times B}, \nu_{A\times B}) \in (T, C)IFS(X \times Y).$$

□

Proposition 4.5. Let $A = (\mu_A, \nu_A) \in (T, C)IFI(X)$ and $B = (\mu_B, \nu_B) \in (T, C)IFI(Y)$. Then $A \times B \in (T, C)IFI(X \times Y)$.

Proof. Let $(x, y) \in X \times Y$. Then

$$\mu_{A\times B}(0, 0) = T(\mu_A(0), \mu_B(0)) \geq T(\mu_A(x), \mu_B(y)) = \mu_{A\times B}(x, y)$$

and

$$\nu_{A\times B}(0, 0) = C(\nu_A(0), \nu_B(0)) \leq C(\nu_A(x), \nu_B(y)) = \nu_{A\times B}(x, y).$$

Also let $x_i \in X$ and $y_i \in Y$ for $i = 1, 2$. Now

$$\begin{aligned}\mu_{A\times B}(x_1, y_1) &= T(\mu_A(x_1), \mu_B(y_1))\\ &\geq T(T(\mu_A(x_1 * x_2), \mu_A(x_2)), T(\mu_B(y_1 * y_2), \mu_B(y_2)))\\ &= T(T(\mu_A(x_1 * x_2), \mu_B(y_1 * y_2)), T(\mu_A(x_2), \mu_B(y_2)))\\ &= T(\mu_{A\times B}(x_1 * x_2, y_1 * y_2), \mu_{A\times B}(x_2, y_2))\end{aligned}$$

$$= T(\mu_{A\times B}((x_1, y_1) * (x_2, y_2)), \mu_{A\times B}(x_2, y_2))$$

thus

$$\mu_{A\times B}(x_1, y_1) \geq T(\mu_{A\times B}((x_1, y_1) * (x_2, y_2)), \mu_{A\times B}(x_2, y_2)).$$

Also

$$\begin{aligned}
\nu_{A\times B}(x_1, y_1) &= C(\nu_A(x_1), \nu_B(y_1)) \\
&\leq C(C(\nu_A(x_1 * x_2), \nu_A(x_2)), C(\nu_B(y_1 * y_2), \nu_B(y_2))) \\
&= C(C(\nu_A(x_1 * x_2), \nu_B(y_1 * y_2)), C(\nu_A(x_2), \nu_B(y_2))) \\
&= C(\nu_{A\times B}(x_1 * x_2, y_1 * y_2), \nu_{A\times B}(x_2, y_2)) \\
&= C(\nu_{A\times B}((x_1, y_1) * (x_2, y_2)), \nu_{A\times B}(x_2, y_2))
\end{aligned}$$

thus

$$\nu_{A\times B}(x_1, y_1) \leq C(\nu_{A\times B}((x_1, y_1) * (x_2, y_2)), \nu_{A\times B}(x_2, y_2)).$$

Therefore

$$A \times B = (\mu_{A\times B}, \nu_{A\times B}) \in (T, C)IFI(X \times Y).$$

□

Proposition 4.6. Let $A = (\mu_A, \nu_A) \in (T, C)IFPII(X)$ and $B = (\mu_B, \nu_B) \in (T, C)IFPII(Y)$. Then $A \times B \in (T, C)IFPII(X \times Y)$.

Proof. Let $(x, y) \in X \times Y$. Then

$$\mu_{A\times B}(0, 0) = T(\mu_A(0), \mu_B(0)) \geq T(\mu_A(x), \mu_B(y)) = \mu_{A\times B}(x, y)$$

and

$$\nu_{A\times B}(0, 0) = C(\nu_A(0), \nu_B(0)) \leq C(\nu_A(x), \nu_B(y)) = \nu_{A\times B}(x, y).$$

Also let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times Y$. Then

$$\mu_{A\times B}((x_1, x_2) * (z_1, z_2)) = \mu_{A\times B}(x_1 * z_1, x_2 * z_2) = T(\mu_A(x_1 * z_1), \mu_B(x_2 * z_2))$$

$$\geq T(T(\mu_A(((x_1 * z_1) * z_1) * (y_1 * z_1)), \mu_A(y_1)), T(\mu_B(((x_2 * z_2) * z_2) * (y_2 * z_2)), \mu_B(y_2)))$$

$$= T(T(\mu_A(((x_1 * z_1) * z_1) * (y_1 * z_1)), \mu_B(((x_2 * z_2) * z_2) * (y_2 * z_2))), T(\mu_A(y_1), \mu_B(y_2)))$$

$$= T(\mu_{A\times B}(((x_1 * z_1) * z_1) * (y_1 * z_1), ((x_2 * z_2) * z_2) * (y_2 * z_2)), \mu_{A\times B}(y_1, y_2))$$

$$= T(\mu_{A\times B}((((x_1, x_2) * (z_1, z_2)) * (z_1, z_2)) * ((y_1, y_2) * (z_1, z_2))), \mu_{A\times B}(y_1, y_2))$$

therefore

$$\mu_{A\times B}((x_1, x_2)*(z_1, z_2)) \geq T(\mu_{A\times B}((((x_1, x_2)*(z_1, z_2))*(z_1, z_2))*((y_1, y_2)*(z_1, z_2))), \mu_{A\times B}(y_1, y_2)).$$

Also

$$\nu_{A\times B}((x_1, x_2) * (z_1, z_2)) = \nu_{A\times B}(x_1 * z_1, x_2 * z_2) = C(\nu_A(x_1 * z_1), \nu_B(x_2 * z_2))$$

$$\leq C(C(\nu_A(((x_1 * z_1) * z_1) * (y_1 * z_1)), \nu_A(y_1)), C(\nu_B(((x_2 * z_2) * z_2) * (y_2 * z_2)), \nu_B(y_2)))$$

$$= C(C(\nu_A(((x_1 * z_1) * z_1) * (y_1 * z_1)), \nu_B(((x_2 * z_2) * z_2) * (y_2 * z_2))), C(\nu_A(y_1), \nu_B(y_2)))$$

$$= C(\nu_{A\times B}(((x_1 * z_1) * z_1) * (y_1 * z_1), ((x_2 * z_2) * z_2) * (y_2 * z_2)), \nu_{A\times B}(y_1, y_2))$$

$$= C(\nu_{A\times B}((((x_1, x_2) * (z_1, z_2)) * (z_1, z_2)) * ((y_1, y_2) * (z_1, z_2))), \nu_{A\times B}(y_1, y_2))$$

therefore

$$\nu_{A\times B}((x_1, x_2)*(z_1, z_2)) \leq C(\nu_{A\times B}((((x_1, x_2)*(z_1, z_2))*(z_1, z_2))*((y_1, y_2)*(z_1, z_2))), \nu_{A\times B}(y_1, y_2)).$$

Therefore

$$A \times B = (\mu_{A\times B}, \nu_{A\times B}) \in (T, C)IFPII(X \times Y).$$

□

Proposition 4.7. If $A = (\mu_A, \nu_A) \in (T, C)IFS(X)$ and $\varphi : X \to Y$ be a homomorphism of BCI-algebras, then $\varphi(A) \in (T, C)IFS(Y)$.

Proof. Let $y_1, y_2 \in Y$ and $x_1, x_2 \in X$ such that $\varphi(x_1) = y_1$ and $\varphi(x_2) = y_2$. Then

$$\varphi(\mu_A)(y_1 * y_2) = \sup\{\mu_A(x_1 * x_2) \mid x_1, x_2 \in X, \varphi(x_1) = y_1, \varphi(x_2) = y_2\}$$

$$\geq \sup\{T(\mu_A(x_1), \mu_A(x_2) \mid x_1, x_2 \in X, \varphi(x_1) = y_1, \varphi(x_2) = y_2\}$$

$$= T(\sup\{\mu_A(x_1) \mid x_1 \in X, \varphi(x_1) = y_1\}, \sup\{\mu_A(x_2) \mid x_2 \in X, \varphi(x_2) = y_2\})$$

$$= T(\varphi(\mu_A)(y_1), \varphi(\mu_A)(y_2))$$

thus

$$\varphi(\mu_A)(y_1 * y_2) \geq T(\varphi(\mu_A)(y_1), \varphi(\mu_A)(y_2)).$$

Also

$$\varphi(\nu_A)(y_1 * y_2) = \inf\{\nu_A(x_1 * x_2) \mid x_1, x_2 \in X, \varphi(x_1) = y_1, \varphi(x_2) = y_2\}$$

$$\leq \inf\{C(\nu_A(x_1), \nu_A(x_2) \mid x_1, x_2 \in X, \varphi(x_1) = y_1, \varphi(x_2) = y_2\}$$

$$= C(\inf\{\mu_A(x_1) \mid x_1 \in X, \varphi(x_1) = y_1\}, \inf\{\nu_A(x_2) \mid x_2 \in X, \varphi(x_2) = y_2\})$$

$$= C(\varphi(\nu_A)(y_1), \varphi(\mu_A)(y_2))$$

so

$$\varphi(\nu_A)(y_1 * y_2) \leq C(\varphi(\nu_A)(y_1), \varphi(\nu_A)(y_2)).$$

Then $\varphi(A) = (\varphi(\mu_A), \varphi(\nu_A)) \in (T, C)IFS(Y)$. □

Proposition 4.8. If $B = (\mu_B, \nu_B) \in (T, C)IFS(Y)$ and $\varphi : X \to Y$ be a homomorphism of BCI-algebras, then $\varphi^{-1}(B) \in (T, C)IFS(X)$.

Proof. Let $x_1, x_2 \in X$. Then

$$\begin{aligned}\varphi^{-1}(\mu_B)(x_1 * x_2) &= \mu_B(\varphi(x_1 * x_2))\\ &= \mu_B(\varphi(x_1) * \varphi(x_2))\\ &\geq T(\mu_B(\varphi(x_1)), \mu_B(\varphi(x_2)))\\ &= T(\varphi^{-1}(\mu_B)(x_1), \varphi^{-1}(\mu_B)(x_2))\end{aligned}$$

thus

$$\varphi^{-1}(\mu_B)(x_1 * x_2) \geq T(\varphi^{-1}(\mu_B)(x_1), \varphi^{-1}(\mu_B)(x_2)).$$

Also

$$\begin{aligned}\varphi^{-1}(\nu_B)(x_1 * x_2) &= \nu_B(\varphi(x_1 * x_2))\\ &= \nu_B(\varphi(x_1) * \varphi(x_2))\\ &\leq C(\nu_B(\varphi(x_1)), \nu_B(\varphi(x_2)))\\ &= C(\varphi^{-1}(\nu_B)(x_1), \varphi^{-1}(\nu_B)(x_2))\end{aligned}$$

then

$$\varphi^{-1}(\nu_B)(x_1 * x_2) \leq C(\varphi^{-1}(\nu_B)(x_1), \varphi^{-1}(\nu_B)(x_2)).$$

Therefore $\varphi^{-1}(B) = (\varphi^{-1}(\mu_B), \varphi^{-1}(\nu_B)) \in (T, C)IFS(X)$. □

Proposition 4.9. If $A = (\mu_A, \nu_A) \in (T, C)IFI(X)$ and $\varphi : X \to Y$ be a homomorphism of BCI-algebras, then $\varphi(A) \in (T, C)IFI(Y)$.

Let $x \in X$ and $y \in Y$ with $\varphi(x) = y$. Now

$$\varphi(\mu_A)(0) = \sup\{\mu_A(0) \mid 0 \in X, \varphi(0) = 0\} \geq \sup\{\mu_A(x) \mid x \in X, \varphi(x) = y\} = \varphi(\mu_A)(y)$$

thus

$$\varphi(\mu_A)(0) \geq \varphi(\mu_A)(y)$$

and

$$\varphi(\nu_A)(0) = \inf\{\nu_A(0) \mid 0 \in X, \varphi(0) = 0\} \leq \inf\{\nu_A(x) \mid x \in X, \varphi(x) = y\} = \varphi(\nu_A)(y)$$

then

$$\varphi(\nu_A)(0) \leq \varphi(\nu_A)(y).$$

Also let $x, x_1 \in X$ such that $\varphi(x) = y, \varphi(x_1) = y_1$. Then

$$\varphi(\mu_A)(y) = \sup\{\mu_A(x) \mid x \in X, \varphi(x) = y\}$$

$$\geq \sup\{T(\mu_A(x * x_1), \mu_A(x_1)) \mid x, x_1 \in X, \varphi(x) = y, \varphi(x_1) = y_1\}$$

$$= T(\sup\{\mu_A(x * x_1) \mid x, x_1 \in X, \varphi(x) = y, \varphi(x_1) = y_1\}, \sup\{\mu_A(x_1) \mid x_1 \in X, \varphi(x_1) = y_1\})$$

$$= T(\sup\{\mu_A(x * x_1) \mid x, x_1 \in X, \varphi(x * x_1) = y * y_1\}, \sup\{\mu_A(x_1) \mid x_1 \in X, \varphi(x_1) = y_1\}$$

$= T(\varphi(\mu_A)(y * y_1), \varphi(\mu_A)(y_1))$

therefore

$$\varphi(\mu_A)(y) \geq T(\varphi(\mu_A)(y * y_1), \varphi(\mu_A)(y_1)).$$

And

$\varphi(\nu_A)(y) = \inf\{\nu_A(x) \mid x \in X, \varphi(x) = y\}$

$\leq \inf\{C(\nu_A(x * x_1), \nu_A(x_1)) \mid x, x_1 \in X, \varphi(x) = y, \varphi(x_1) = y_1\}$

$= C(\inf\{\nu_A(x * x_1) \mid x, x_1 \in X, \varphi(x) = y, \varphi(x_1) = y_1\}, \inf\{\nu_A(x_1) \mid x_1 \in X, \varphi(x_1) = y_1\})$

$= C(\inf\{\nu_A(x * x_1) \mid x, x_1 \in X, \varphi(x * x_1) = y * y_1\}, \inf\{\nu_A(x_1) \mid x_1 \in X, \varphi(x_1) = y_1\}$

$= C(\varphi(\nu_A)(y * y_1), \varphi(\nu_A)(y_1))$

thus

$$\varphi(\nu_A)(y) \leq C(\varphi(\nu_A)(y * y_1), \varphi(\nu_A)(y_1)).$$

Therefore $\varphi(A) = (\varphi(\mu_A), \varphi(\nu_A)) \in (T, C)IFI(Y)$.

Proposition 4.10. If $B = (\mu_B, \nu_B) \in (T, C)IFI(Y)$ and $\varphi : X \to Y$ be a homomorphism of BCI-algebras, then $\varphi^{-1}(B) \in (T, C)IFI(X)$.

Proof. Let $x \in X$. Then

$$\varphi^{-1}(\mu_B)(0) = \mu_B(\varphi(0)) \geq \mu_B(\varphi(x)) = \varphi^{-1}(\mu_B)(x)$$

and

$$\varphi^{-1}(\nu_B)(0) = \nu_B(\varphi(0)) \leq \nu_B(\varphi(x)) = \varphi^{-1}(\nu_B)(x).$$

Let $x, x_1 \in X$. As

$$\begin{aligned} \varphi^{-1}(\mu_B)(x) &= \mu_B(\varphi(x)) \\ &\geq T(\mu_B(\varphi(x) * \varphi(x_1)), \mu_B(\varphi(x_1))) \\ &= T(\mu_B(\varphi(x * x_1)), \mu_B(\varphi(x_1))) \\ &= T(\varphi^{-1}(\mu_B)(x * x_1), \varphi^{-1}(\mu_B)(x_1)) \end{aligned}$$

so

$$\varphi^{-1}(\mu_B)(x) \geq T(\varphi^{-1}(\mu_B)(x * x_1), \varphi^{-1}(\mu_B)(x_1))$$

and

$$\begin{aligned} \varphi^{-1}(\nu_B)(x) &= \nu_B(\varphi(x)) \\ &\leq C(\nu_B(\varphi(x) * \varphi(x_1)), \nu_B(\varphi(x_1))) \\ &= C(\nu_B(\varphi(x * x_1)), \nu_B(\varphi(x_1))) \\ &= C(\varphi^{-1}(\nu_B)(x * x_1), \varphi^{-1}(\nu_B)(x_1)) \end{aligned}$$

thus

$$\varphi^{-1}(\nu_B)(x) \leq C(\varphi^{-1}(\nu_B)(x * x_1), \varphi^{-1}(\nu_B)(x_1)).$$

Therefore $\varphi^{-1}(B) = (\varphi^{-1}(\mu_B), \varphi^{-1}(\nu_B)) \in (T, C)IFI(X)$. □

Proposition 4.11. If $A = (\mu_A, \nu_A) \in (T, C)IFPII(X)$ and $\varphi : X \to Y$ be a homomorphism of BCI-algebras, then $\varphi(A) \in (T, C)IFPII(Y)$.

Let $x \in X$ and $y \in Y$ with $\varphi(x) = y$. Now

$$\varphi(\mu_A)(0) = \sup\{\mu_A(0) \mid 0 \in X, \varphi(0) = 0\} \geq \sup\{\mu_A(x) \mid x \in X, \varphi(x) = y\} = \varphi(\mu_A)(y)$$

thus

$$\varphi(\mu_A)(0) \geq \varphi(\mu_A)(y)$$

and

$$\varphi(\nu_A)(0) = \inf\{\nu_A(0) \mid 0 \in X, \varphi(0) = 0\} \leq \inf\{\nu_A(x) \mid x \in X, \varphi(x) = y\} = \varphi(\nu_A)(y)$$

then

$$\varphi(\nu_A)(0) \leq \varphi(\nu_A)(y).$$

Also let $x_i \in X$ such that $\varphi(x_i) = y_i$ and $i = 1, 2, 3$. Then

$$\varphi(\mu_A)(y_1 * y_2) = \sup\{\mu_A(x_1 * x_2) \mid x_i \in X, \varphi(x_i) = y_i\}$$

$$\geq \sup\{T(\mu_A(((x_1 * x_2) * x_2) * (x_3 * x_2)), \mu_A(x_3)) \mid x_i \in X, \varphi(x_i) = y_i\}$$

$$= T(\sup\{\mu_A(((x_1 * x_2) * x_2) * (x_3 * x_2)) \mid x_i \in X, \varphi(x_i) = y_i\}, \sup\{\mu_A(x_3) \mid x_3 \in X, \varphi(x_3) = y_3\})$$

$$= T(\varphi(\mu_A)(((y_1 * y_2) * y_2) * (y_3 * y_2)), \varphi(\mu_A)(y_3))$$

therefore

$$\varphi(\mu_A)(y_1 * y_2) \geq T(\varphi(\mu_A)(((y_1 * y_2) * y_2) * (y_3 * y_2)), \varphi(\mu_A)(y_3)).$$

And

$$\varphi(\nu_A)(y_1 * y_2) = \inf\{\nu_A(x_1 * x_2) \mid x_i \in X, \varphi(x_i) = y_i\}$$

$$\leq \inf\{C(\nu_A(((x_1 * x_2) * x_2) * (x_3 * x_2)), \mu_A(x_3)) \mid x_i \in X, \varphi(x_i) = y_i\}$$

$$= C(\inf\{\nu_A(((x_1 * x_2) * x_2) * (x_3 * x_2)) \mid x_i \in X, \varphi(x_i) = y_i\}, \inf\{\nu_A(x_3) \mid x_3 \in X, \varphi(x_3) = y_3\})$$

$$= C(\varphi(\nu_A)(((y_1 * y_2) * y_2) * (y_3 * y_2)), \varphi(\nu_A)(y_3))$$

therefore

$$\varphi(\nu_A)(y_1 * y_2) \leq C(\varphi(\nu_A)(((y_1 * y_2) * y_2) * (y_3 * y_2)), \varphi(\nu_A)(y_3)).$$

Therefore $\varphi(A) = (\varphi(\mu_A), \varphi(\nu_A)) \in (T, C)IFPII(Y)$.

Proposition 4.12. If $B = (\mu_B, \nu_B) \in (T, C)IFPII(Y)$ and $\varphi : X \to Y$ be a homomorphism of BCI-algebras, then $\varphi^{-1}(B) \in (T, C)IFPII(X)$.

Proof. Let $x \in X$. Then

$$\varphi^{-1}(\mu_B)(0) = \mu_B(\varphi(0)) \geq \mu_B(\varphi(x)) = \varphi^{-1}(\mu_B)(x)$$

and

$$\varphi^{-1}(\nu_B)(0) = \nu_B(\varphi(0)) \leq \nu_B(\varphi(x)) = \varphi^{-1}(\nu_B)(x).$$

Let $x_1, x_2, x_3 \in X$. As

$$\begin{aligned}
\varphi^{-1}(\mu_B)(x_1 * x_2) &= \mu_B(\varphi(x_1 * x_2)) \\
&= \mu_B(\varphi(x_1) * \varphi(x_2)) \\
&\geq T(\mu_B(((\varphi(x_1) * \varphi(x_2)) * \varphi(x_2)) * (\varphi(x_3) * \varphi(x_2)))), \mu_B(\varphi(x_3))) \\
&= T(\mu_B(\varphi)(((x_1 * x_2) * x_2) * (x_3 * x_2)), \mu_B(\varphi(x_3))) \\
&= T(\varphi^{-1}(\mu_B)(((x_1 * x_2) * x_2) * (x_3 * x_2)), \varphi^{-1}(\mu_B)(x_3))
\end{aligned}$$

so

$$\varphi^{-1}(\mu_B)(x_1 * x_2) \geq T(\varphi^{-1}(\mu_B)(((x_1 * x_2) * x_2) * (x_3 * x_2)), \varphi^{-1}(\mu_B)(x_3)).$$

Also

$$\begin{aligned}
\varphi^{-1}(\nu_B)(x_1 * x_2) &= \nu_B(\varphi(x_1 * x_2)) \\
&= \nu_B(\varphi(x_1) * \varphi(x_2)) \\
&\leq C(\nu_B(((\varphi(x_1) * \varphi(x_2)) * \varphi(x_2)) * (\varphi(x_3) * \varphi(x_2)))), \nu_B(\varphi(x_3))) \\
&= C(\nu_B(\varphi)(((x_1 * x_2) * x_2) * (x_3 * x_2)), \nu_B(\varphi(x_3))) \\
&= C(\varphi^{-1}(\nu_B)(((x_1 * x_2) * x_2) * (x_3 * x_2)), \varphi^{-1}(\nu_B)(x_3))
\end{aligned}$$

so

$$\varphi^{-1}(\nu_B)(x_1 * x_2) \leq C(\varphi^{-1}(\nu_B)(((x_1 * x_2) * x_2) * (x_3 * x_2)), \varphi^{-1}(\nu_B)(x_3)).$$

Therefore $\varphi^{-1}(B) = (\varphi^{-1}(\mu_B), \varphi^{-1}(\nu_B)) \in (T, C)IFPII(X)$. □

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Leverrier-Takeno and Faddeev-Sominsky algorithms

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Abstract We exhibit representations of the coefficients of the characteristic polynomial of any matrix $A_{n\times n}$, especially in terms of the (exponential) complete Bell polynomials. Besides, we use the Faddeev-Sominsky method to obtain the Lanczos formula for the resolvent of a matrix. We indicate that the Newton's recurrence formula can be solved via the inversion of a triangular matrix.

Keywords Faddeev-Sominsky's algorithm, Cayley-Hamilton- Frobenius theorem, Leverrier-Takeno's procedure, (Exponential) complete Bell polynomials.
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§1. Introduction

For an arbitrary matrix $A_{n\times n} = (A^i{}_j)$ its characteristic polynomial [1–3]:

$$p(\lambda) \equiv \lambda^n + a_1\lambda^{n-1} + ... + a_{n-1}\lambda + a_n, \tag{1}$$

can be obtained, through several procedures [1,4–8], directly from the condition $det\ (A^i{}_j - \lambda\delta^i_j) = 0$. The method of Leverrier-Takeno [4, 9–13] is a simple and interesting technique to construct (1) based in the traces of the powers A^r, $r = 1, ..., n$.

On the other hand, it is well known that an arbitrary matrix satisfies its characteristic equation, which is the Cayley-Hamilton-Frobenius identity [1–3, 14]:

$$A^n + a_1A^{n-1} + ...a_{n-1}A + a_nI = 0. \tag{2}$$

If A is non-singular (that is, $det\ A \neq 0$), then from (2) we obtain its inverse matrix:

$$A^{-1} = -\frac{1}{a_n}\left(A^{n-1} + a_1A^{n-2} + ... + a_{n-1}I\right), \tag{3}$$

where $a_n \neq 0$ because $a_n = (-1)^n \; det \; A$.

Faddeev-Sominsky [15–24] proposed an algorithm to determine A^{-1} in terms of A^r and their traces, which is equivalent [23] to the Cayley-Hamilton-Frobenius theorem (2) plus the Leverrier-Takeno's method to construct the characteristic polynomial of a matrix A. In Sec. 2, we use the Faddeev-Sominsky's procedure to deduce the Lanczos expression [25] for the resolvent of A [20, 21, 26, 27], that is, the Laplace transform of $exp\,(tA)$ [28].

§2. Leverrier-Takeno technique

If we define the quantities:

$$a_0 = 1, \quad s_k = tr\; A^k \quad k = 1, 2, ..., n \tag{4}$$

then the approach of Leverrier-Takeno [4, 9–13] implies (1) wherein the a_i are determined with the Newton's recurrence relation:

$$r a_r + s_1 a_{r-1} + s_2 a_{r-2} + ... + s_{r-1} a_1 + s_r = 0, \quad r = 1, 2, ..., n \tag{5}$$

therefore:

$$\begin{gathered} a_1 = -s_1, \quad 2!\; a_2 = (s_1)^2 - s_2, \quad 3!\; a_3 = -(s_1)^3 + 3s_1 s_2 - 2s_3, \\ 4!\; a_4 = (s_1)^4 - 6(s_1)^2 s_2 + 8 s_1 s_3 + 3(s_2)^2 - 6 s_4, \\ 5!\; a_5 = -24 s_5 - (s_1)^5 + 10(s_1)^3 s_2 - 20(s_1)^2 s_3 - 15(s_2)^2 s_1 + 30 s_1 s_4 + 20 s_2 s_3, \; etc, \end{gathered} \tag{6}$$

in particular, $det\; A = (-1)^n a_n$, that is, the determinant of any matrix only depends on the traces s_r, which means that A and its transpose have the same determinant. In [29–31] we find the general expression:

$$a_k = \frac{(-1)^k}{k!} \begin{vmatrix} s_1 & k-1 & 0 & ... & 0 \\ s_2 & s_1 & k-2 & ... & 0 \\ ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... \\ s_{k-1} & s_{k-2} & ... & ... & 1 \\ s_k & s_{k-1} & ... & ... & s_1 \end{vmatrix}, \qquad k = 1, ..., n, \tag{7}$$

which allows reproduce the values (6).

We can exhibit a relation to determine the coefficients a_j via the complete Bell polynomials [8, 32–40], in fact, we have the following representation [8]:

$$m!\; a_m = Y_m\Big(-0!s_1, -1!s_2, -2!s_3, -3!s_4, ..., -(m-2)!s_{m-1}, -(m-1)!s_m\Big). \tag{8}$$

such that [37, 41]:

$$
Y_m(x_1,x_2,...,x_m)=\begin{vmatrix}
\binom{m-1}{0}x_1 & \binom{m-1}{1}x_2 & ... & \binom{m-1}{m-2}x_{m-1} & \binom{m-1}{m-1}x_m \\
-1 & \binom{m-2}{0}x_1 & ... & \binom{m-2}{m-3}x_{m-2} & \binom{m-2}{m-2}x_{m-1} \\
0 & -1 & ... & \binom{m-3}{m-4}x_{m-3} & \binom{m-3}{m-3}x_{m-2} \\
... & ... & ... & ... & ... \\
... & ... & ... & ... & ... \\
0 & 0 & ... & \binom{1}{0}x_1 & \binom{1}{1}x_2 \\
0 & 0 & ... & -1 & \binom{0}{0}x_1
\end{vmatrix}, \tag{9}
$$

therefore:

$$
\begin{gathered}
Y_0=1,\ \ Y_1=x_1,\ \ Y_2=x_1^2+x_2,\\
Y_3=x_1^3+3\,x_1x_3+x_3,\ \ Y_4=x_1^4+6\,x_1^2x_2+4\,x_1x_3+3\,x_2^2+x_4,\\
Y_5=x_1^5+10\,x_1^3x_2+10\,x_1^2x_3+15\,x_1x_2^2+5\,x_1x_4+10\,x_2x_3+x_5,\ \ ...
\end{gathered} \tag{10}
$$

We see that (8) and (10) imply (6) if we employ $x_1=-s_1$, $x_2=-s_2$, $x_3=-2s_3$, $x_4=-6s_4$, $x_5=-24s_5$, ... It is simple to prove that (9) with $x_k=-(k-1)!\,s_k$ gives (7), thus the coefficients of the characteristic polynomial (1) are generated by the (exponential) complete Bell polynomials.

In the Newtons formula (5) the quantities s_r are known, and the a_j are solutions of the triangular linear system [42–44]:

$$
A_{n\times n}\,(a_j)_{n\times 1}\equiv
\begin{pmatrix}
1 & 0 & 0 & ... & ... & 0\\
s_1 & 2 & 0 & ... & ... & 0\\
s_2 & s_1 & 3 & ... & ... & ...\\
... & ... & ... & ... & ... & ...\\
... & ... & ... & ... & ... & ...\\
... & ... & ... & ... & ... & 0\\
s_{n-1} & s_{n-2} & s_{n-3} & ... & s_1 & n
\end{pmatrix}
\begin{pmatrix} a_1\\ a_2\\ a_3\\ ...\\ ...\\ ...\\ a_n\end{pmatrix}
=-\begin{pmatrix} s_1\\ s_2\\ s_3\\ ...\\ ...\\ ...\\ s_n\end{pmatrix}, \tag{11}
$$

then:

$$
\begin{pmatrix} a_1\\ ...\\ ...\\ a_n\end{pmatrix}=-A^{-1}\begin{pmatrix} s_1\\ ...\\ ...\\ s_n\end{pmatrix}, \tag{12}
$$

which gives the opportunity to invert a triangular matrix via interesting algorithms applying the Faddeev-Sominsky method [15–24], matrix multiplication [45, 46] or binomial series [47].

§3. Faddeev-Sominsky procedure

The Faddeev-Sominsky's technique to obtain A^{-1} is a sequence of algebraic computations on the powers A^r and their traces, in fact, this algorithm is given via the following instructions:

$$\begin{array}{ccc} B_1 = A, & q_1 = \mathrm{tr}\ B_1, & C_1 = B_1 - q_1 I, \\ B_2 = C_1\ A, & q_2 = \frac{1}{2}\ \mathrm{tr}\ B_2, & C_2 = B_2 - q_2 I, \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ B_{n-1} = C_{n-2}\ A, & q_{n-1} = \frac{1}{n-1}\ \mathrm{tr}\ B_{n-1}, & C_{n-1} = B_{n-1} - q_{n-1} I, \\ B_n = C_{n-1}\ A, & q_n = \frac{1}{n}\ \mathrm{tr}\ B_n, & \end{array} \tag{13}$$

then:

$$A^{-1} = \frac{1}{q_n}\ C_{n-1}. \tag{14}$$

For example, if we apply (13) for $n = 4$, then it is easy to see that the corresponding q_r imply (6) with $q_j = -a_j$, and besides (14) reproduces (3). By mathematical induction one can prove that (13) and (14) are equivalent to (3), (4) and (5), showing [23] thus that the Faddeev-Sominsky's technique has its origin in the Leverrier-Takeno method plus the Cayley-Hamilton-Frobenius theorem.

From (13) we can see that [26]:

$$C_k = A^k + a_1 A^{k-1} + a_2 A^{k-2} + \ldots + a_{k-1} A + a_k I, \quad k = 1, 2, \ldots, n-1, \quad C_n = B_n - q_n I = 0, \tag{15}$$

and for $k = n - 1$:

$$C_{n-1} = A^{n-1} + a_1 A^{n-2} + a_2 A^{n-3} + \ldots + a_{n-2} A + a_{n-1} I \overset{(3)}{=} -a_n A^{-1}$$

in harmony with (14) because $a_n = -q_n$. The property $C_n = 0$ is equivalent to (2); if A is singular, the process (13) gives the adjoint matrix of A [2,3,16], in fact, $Adj\ A = (-1)^{n+1} C_{n-1}$.

If the roots of (1) have distinct values, then the Faddeev-Sominsky's algorithm allows obtain the corresponding eigenvectors of A [6]:

$$A\vec{u}_k = \lambda_k \vec{u}_k, \qquad k = 1, 2, \ldots, n, \tag{16}$$

because for a given value of k, each column of:

$$Q_k \equiv \lambda_k^{n-1} I + \lambda_k^{n-2} C_1 + \ldots + C_{n-1}. \tag{17}$$

satisfies (16) [16, 18, 27], and therefore all columns of Q_k are proportional to each other, that is, $rank\ Q_k = 1$ [18]; we note that $Q_k = Q(\lambda_k)$ with the participation of the matrix:

$$Q(z) \equiv z^{n-1}I + z^{n-2}C_1 + z^{n-3}C_2 + ... + zC_{n-2} + C_{n-1}. \tag{18}$$

By synthetic division of two polynomials [1]:

$$\frac{p(z)}{z-\lambda} = \sum_{r=0}^{n-1} \Big(\lambda^r + a_1\lambda^{r-1} + a_2\lambda^{r-2} + ... + a_{r-1}\lambda + a_r \Big) z^{n-1-r},$$

then under the change $\lambda \to A$ we obtain the Lanczos expression [25] for the resolvent of a matrix [20, 21, 26, 27]:

$$\frac{1}{zI - A} = \frac{1}{p(z)} \sum_{r=0}^{n-1} z^{n-1-r}\ C_r = \frac{Q(a)}{p(z)}, \tag{19}$$

if A is non-singular, then [19] for $z = 0$ implies (14). McCarthy [48] used (19) and the Cauchy's integral theorem in complex variable to show the Cayley-Hamilton-Frobenius identity indicated in (2); the relation (19) is the Laplace transform of $exp\ (tA)$ [28].

On the other hand, Sylvester [49–52] obtained the following interpolating definition of $f(A)$:

$$f(A) = \sum_{j=1}^{n} f(\lambda_j) \prod_{k \neq j} \frac{A - \lambda_k I}{\lambda_j - \lambda_k}, \tag{20}$$

which is valid if all eigenvalues are different from each other. Buchheim [53] generalized (20) to multiple proper values using Hermite interpolation, thereby giving the first completely general definition of a matrix function. From (19) and (20) for $f(s) = 1/(z-s)$ we deduce the properties:

$$Q(z) = \sum_{j=1}^{n} \prod_{k=1, k\neq j}^{n} \frac{z - \lambda_k}{\lambda_j - \lambda_k} \Big(A - \lambda_k I \Big), \qquad Q_j = \prod_{k=1, k\neq j}^{n} \Big(A - \lambda_k I \Big)$$

$$Q_j\ \vec{u}_j = \prod_{k=1, k\neq j}^{n} \Big(\lambda_j - \lambda_k \Big) \vec{u}_j, \tag{21}$$

hence the eigenvectors of A showed in (16) also are proper vectors of the matrices Q_j. Besides, from (16) and (21):

$$A\ Q_j\ \vec{u}_j = \prod_{k=1, k\neq j}^{n} \Big(\lambda_j - \lambda_k \Big) \lambda_j \vec{u}_j = \lambda_j\ Q_j\ \vec{u}_j, \qquad A\ Q_j = \lambda_j\ Q_j \tag{22}$$

that is, each column of Q_j is eigenvector of A with proper value λ_j. The resolvent (19) implies the relation $(A - zI)Q(z) = -p(z)I$, then $(A - \lambda_k I)\ Q(\lambda_k) = -p(\lambda_k)I = 0$ in according with (22).

From the Sylvester's formula (20) with $f(z) = p(z)$ we obtain $p(A) = 0$, which is the Cayley-Hamilton-Frobenius theorem indicated in (2). If $f(z) = e^{tz}$, then (20) allows to construct $exp\ (tA)$ that, in particular, is valuable to determine the motion of classical charged particles into a homogeneous electromagnetic field, and to integrate the Frenet-Serret equations with constant curvatures [54]. In [51, 55] we find that the book of Frazer-Duncan-Collar [56] emphasizes the important role of the matrix exponential in solving differential equations and was the first to employ matrices as an engineering tool, and indeed the first book to treat matrices as a branch of applied mathematics.

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A note on the Smarandache LCM function

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Abstract For any integer $k \geq 1$, denoting by $[1, 2, \cdots, k]$ the least common multiple (LCM) of the integers 1 through k, the Smarandache LCM function, denoted by $SL(n)$, is defined as the minimum k such that n divides $[1, 2, ..., k]$. Also, the pseudo Smarandache function, denoted by $Z(n)$, is defined as the minimum m such that n divides $\frac{m(m+1)}{2}$. This paper considers two Diophantine equations involving the functions $SL(n)$ and $Z(n)$, namely, the equations $Z(n) = SL(n)$ and $Z(n) + SL(n) = n$.

Keywords Smarandache LCM function, Pseudo Smarandache function, Diophantine equation.
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§1. Introduction and preliminaries

The pseudo Smarandache function $Z(n)$, introduced by Kashihara [1], is defined as follows:

$$Z(n) = \min \{m : n \text{ divides } \tfrac{m(m+1)}{2}\}.$$

Some elementary properties of $Z(n)$ have been studied by Kashihara [1], Ibstedt [2], Ashbacher [3] and Majumdar [4, 5]. For a recent review of the pseudo Smarandache function, we refer the reader to Liu [6].

The Smarandache LCM function, denoted by $SL(n)$, is defined as

$$SL(n) = \min \{k \geq 1 : n \text{ divides } [1, 2, \cdots, k]\},$$

where $[1, 2, \cdots, k]$ is the least common multiple (LCM) of the integers $1, 2, \cdots, k$ for any $k \geq 1$.

The reader is referred to Xiaolin Chen [7] for a brief survey on the Smarandache LCM function. An explicit expression of $SL(n)$, due to Murthy [8], is given in the following lemma.

Lemma 1.1. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ be the (unique) representation of the integer n in terms of its r prime factors $p_1, p_2, \cdots, p_r$. Then,

$$SL(n) = \max \{p_1^{\alpha_1}, p_2^{\alpha_2}, \cdots, p_r^{\alpha_r}\}.$$

Clearly, $SL(p) = p$ for any prime $p \geq 2$. Using Lemma 1.1, we may derive the following values of $SL(n)$:

$$SL(1) = 1, SL(2) = 2, SL(3) = 3, SL(4) = 4, SL(5) = 5, SL(6) = 3,$$
$$SL(7) = 7, SL(8) = 8, SL(9) = 9, SL(10) = 5, SL(11) = 11, SL(12) = 4,$$
$$SL(13) = 13, SL(14) = 7, SL(15) = 5, SL(16) = 16, SL(17) = 17.$$

Murthy [8] proposed the solution of the Diophantine equation $S(n) = SL(n)$, where $S(n)$ is the Smarandache function. The complete solution of the equation has been given by Maohua [9]. This paper considers the following two Diophantine equations:

$$Z(n) = SL(n), \tag{1.1}$$

$$Z(n) + SL(n) = n. \tag{1.2}$$

In connection with the equation (1.1), we would need the following results, proofs of which are given in Majumdar [4].

Lemma 1.2. Let $p(\geq 3)$ be a prime and let $n(\geq 1)$ be an integer such that $2n$ divides $p + 1$. Then, $Z(np) = p$.

Lemma 1.3. Let $p(\geq 3)$ be a prime and let $n(\geq 1)$ be an integer such that $2n$ divides $p^2 + 1$. Then, $Z(np^2) = p^2$.

Xin Xu [10] considers the Diophantine equation (1.2). This paper follows a simple approach to the solution of the equation. To do so, we would need the following result. Theorem 4.2.2 in Majumdar [4] gives a method of finding $Z(pq)$, where p and $q(> p)$ are distinct primes. We follow the same method to find $Z(2^k p^\alpha)$ for some special cases in the lemma below.

Lemma 1.4. Let the integer n be of the form $n = 2^k p^\alpha$, where $p \geq 3$ is a prime, and $k \geq 1$ and $\alpha \geq 1$ are integers. Then,

(i) $Z(n) = p^\alpha(2^k - 1)$, if 2^k divides $(p^\alpha - 1)$ but 2^{k+1} does not divide $(p^\alpha - 1)$,

(ii) $Z(n) = 2^k(p^\alpha - 1)$, if p^α divides $(2^k - 1)$,

(iii) $Z(n) = p^\alpha - 1$, if 2^{k+1} divides $(p^\alpha - 1)$.

Proof. Since

$$Z(n) = Z(2^k p^\alpha) = \min\left\{m : 2^k p^\alpha \text{ divides } \tfrac{m(m+1)}{2}\right\},$$

the following two possibilities may arise :

Case 1. When 2^{k+1} divides m, p^α divides $(m + 1)$.

In this case, there are integers $x \geq 1$ and $y \geq 1$ such that

$$m = 2^{k+1}x, m + 1 = p^\alpha y,$$

which lead to the combined Diophantine equation :

$$p^\alpha y - 2^{k+1}x = 1. \tag{1}$$

Case 2. When 2^{k+1} divides $(m + 1)$, p^α divides m.

Then, there are integers $x \geq 1$ and $y \geq 1$ such that

$$m + 1 = 2^{k+1}x, m = p^\alpha y,$$

so that the combined Diophantine equation is

$$2^{k+1}x - p^\alpha y = 1. \tag{2}$$

Now, we consider the following three possible cases.

(i) Let 2^k divide $(p^\alpha - 1)$ but 2^{k+1} does not divide $(p^\alpha - 1)$. Then,

$$p^\alpha - 1 = 2^k b \text{ for some integer } b \geq 1, b \neq 2.$$

Therefore, the equations (1) and (2) can be recast as

$$(2^k b + 1)y - 2^{k+1}x = 1, 2^{k+1}x - (2^k b + 1)y = 1,$$

that is,

$$2^k(by - 2x) + y = 1, \tag{3}$$

$$2^k(2x - by) - y = 1. \tag{4}$$

In this case, the minimum solution, obtained from (4), is as follows :

$$2x - by = 1, y = 2^k - 1,$$

Thus, the minimum m is $m = p^\alpha y = p^\alpha(2^k - 1)$.

(ii) Let p^α divide $(2^k - 1)$. Here,

$$2^k - 1 = p^\alpha c \text{ for some integer } c \geq 1.$$

Then, the equations (1) and (2) become

$$p^\alpha y - 2(p^\alpha c + 1)x = 1, 2(p^\alpha c + 1)x - p^\alpha y = 1,$$

that is,

$$p^\alpha(y - 2cx) - 2x = 1, \tag{5}$$

$$p^\alpha(2cx - y) + 2x = 1. \tag{6}$$

The minimum solution, obtained from (5), is

$$y - 2cx = 1, 2x = p^\alpha - 1.$$

Consequently, the minimum m is $m = 2^{k+1}x = 2^k(p^\alpha - 1)$.

(iii) Let 2^{k+1} divide $(p^\alpha - 1)$. Then,

$$p^\alpha - 1 = 2^{k+1}a \text{ for some integer } a \geq 1.$$

Then, from the equations (1) and (2), we get

$$(2^{k+1}a + 1)y - 2^{k+1}x = 1, 2^{k+1}x - (2^{k+1}a + 1)y = 1,$$

that is,

$$2^{k+1}(ay - x) + y = 1, \tag{7}$$

$$2^{k+1}(x - ay) - y = 1. \tag{8}$$

Therefore, the minimum solution is obtained from (7) with

$$y = 1, ay - x = 0.$$

Hence, the minimum m is given by $m = p^\alpha - 1$.

All these complete the proof of the lemma.

Lemma 1.5. For any integer $a \geq 1$, **(1)** 3 divides $2^{2a} - 1$, **(2)** 3 divides $2^{2a-1} + 1$.

Proof. The proof is by induction on a. The results are true for $a = 1$. So, we assume that the results hold true for some integer a. Now, writing

$$2^{2(a+1)} - 1 = 4(2^{2a} - 1) + 3,$$

it follows, by the induction hypothesis, that 3 divides $2^{2(a+1)} - 1$. Again, since

$$2^{2a+1} + 1 = 4(2^{2a-1} + 1) - 3,$$

we see that $2^{2a+1} + 1$ is divisible by 3. All these complete the proof by induction.

Lemma 1.6. For any integer $a \geq 1$, **(1)** 2^2 divides $5^a - 1$, **(2)** 2^3 divides $5^{2a} - 1$.

Proof. The proof is by induction on a. The result is true for $a = 1$. So, we assume that the result is true for some integer a. Now, since

$$5^{a+1} - 1 = 5(5^a - 1) + 4,$$

it follows, by the induction hypothesis, that 2^2 divides $5^{a+1} - 1$. Again, since

$$5^{2(a+1)} - 1 = 25(5^{2a} - 1) + 24,$$

we see that $5^{2(a+1)} - 1$ is divisible by 2^3. All these complete the proof.

Example 1.1. Since 2^3 divides $3^2 - 1$, by part (i) of Lemma 1.4, $Z(2^3.3^2) = 3^2(2^3 - 1) = 63$. Also, by part (iii) of Lemma 1.4, $Z(2^2.3^2) = 3^2 - 1 = 8$. Again, since 3 divides $2^2 - 1$, using Lemma 1.5, we get, by part (ii) of Lemma 1.4, $Z(2^2.3) = 8, Z(2^4.3) = 32$.

Example 1.2. Since 2^2 divides $5 - 1$, by part (i) of Lemma 1.4, $Z(2^2.5) = 15$. Again, since 5 divides 2^4-1, by part (ii) of Lemma 1.4, $Z(2^4.5) = 64$. Finally, noting that 2^3 divides 5^2-1, and

2^4 divides $5^4 - 1$, we get $Z(2^2.5^2) = 24, Z(2^3.5^2) = 175, Z(2^3.5^4) = 624$ and $Z(2^4.5^4) = 9375$.

§2. Main Results

This section gives the solutions of the Diophantine equations (1.1) and (1.2), posed in Section 1. First, we consider the equation (1.1). The following theorem shows that the equation possesses infinite number of solutions.

Theorem 2.1. $n = kp$ is a solution of the Diophantine equation $Z(n) = SL(n)$, where $p(\geq 3)$ is a prime and $k(\geq 1)$ is an integer such that $2k$ divides $(p + 1)$.

Proof. Let $n = kp$, where $p(\geq 3)$ is a prime and $2k$ divides $(p + 1)$. Then, clearly, $p > n$, so that, by Lemma 1.1, $SL(n) = \max\{k, p\} = p$. Now, appealing to Lemma 1.2, the theorem is proved.

From Theorem 2.1, note that, in particular,

$$Z(2p) = SL(2p) = p \text{ for any prime } p(\geq 3) \text{ such that 4 divides } (p + 1).$$

Note that, any p satisfying the condition above must be of the form $p = 4a + 3(a \geq 1)$. A few examples are given below.

$$Z(6) = SL(6) = 3, Z(14) = SL(14) = 7, Z(22) = SL(22) = 11.$$

It may be mentioned here that, if $p(\geq 3)$ is a prime and $k(\geq 1)$ is an integer such that $2k$ divides $(p^2 + 1)$, then by virtue of Lemma 1.3 and Lemma 1.1,

$$Z(kp^2) = p^2 = SL(kp^2).$$

Thus, for example, $Z(5.3^2) = SL(5.3^2) = 3^2, Z(5.7^2) = SL(5.7^2) = 7^2, Z(13.5^2) = SL(13.5^2) = 5^2$.

Theorem 2.2. The Diophantine equation $Z(n) + SL(n) = n$ has the solution

$$n = 2^k p^\alpha,$$

where 2^k and p^α satisfy one of the two conditions (i) and (ii) of Lemma 1.4.

Proof. With $n = 2^k p^\alpha$, by Lemma 1.1, $SL(n) = \max\{2^k, p^\alpha\}$. Now, if the condition (i) of Lemma 2.1 is satisfied, then $p^\alpha > 2^k$, while $2^k > p^\alpha$ if the condition (ii) of Lemma 2.1 holds. In either case, the given equation is satisfied.

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A note on the pseudo Smarandache square-free function

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Abstract The pseudo Smarandache square-free function, denoted by $Z_w(n)$, is a Smarandache-type arithmetic function, and is defined as the minimum integer $m(\geq 1)$ such that n divides m^n. This paper derives some properties of $Z_w(n)$. Some relationships involving the functions $Z_w(n)$ and the Smarandache function $S(n)$ as well as the pseudo Smarandache function $Z(n)$ have been established.

Keywords Pseudo Smarandache spuare-free function, Smarandache function, Pseudo Smarandache function, Diophantine equation.
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§1. Introduction

The pseudo Smarandache square-free function, denoted by $Z_w(n)$, is defined as the smallest integer $m(\geq 1)$ such that n divides m^n (see, Russo [1]). That is

$$Z_w(n) = \min m : n \text{ divides } m^n,\ n \geq 1.$$

An alternative definition, due to Smarandache [2] is that, $Z_w(n)$ is the largest square-free integer that divides n (that is, the square-free kernel of n). Recall that an integer $N(> 0)$ is called square free if for some prime p, p divides N but p^2 does not divide N.

Some of the properties of $Z_w(n)$ have been studied by Russo [1], who also posed several open problems, some of which were later addressed by Maohua [3], Guan [4] and Li [5]. Russo [1] also provides a table of values of $Z_w(n)$ for $1 \leq n \leq 100$. The first few values of $Z_w(n)$ are listed below.

$Z_w(1) = 1, Z_w(2) = 2, Z_w(3) = 3, Z_w(4) = 2, Z_w(5) = 5, Z_w(6) = 6,$
$Z_w(7) = 7, Z_w(8) = 2, Z_w(9) = 3, Z_w(10) = 10, Z_w(11) = 11, Z_w(12) = 6,$
$Z_w(13) = 13, Z_w(14) = 14, Z_w(15) = 15, Z_w(16) = 2, Z_w(17) = 17.$

Since n divides n^n for any integer $n(\geq 1)$, it follows that

$$Z_w(n) \leq n \text{ for all } n \geq 1. \tag{1.1}$$

An explicit expression of $Z_w(n)$ is given in the theorem below (see Russo [1] and Maohua [3]).

Theorem 1.1. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ be the (unique) representation of the integer n in terms of its r prime factors $p_1, p_2, \cdots, p_r(\alpha_1, \alpha_2, \cdots, \alpha_r$ being non-negative integers). Then,

$$Z_w(n) = p_1 p_2 \cdots p_r.$$

From Theorem 1.1, we see immediately that the range of the function $Z_w(n)$ is the set of all square-free integers. Using Theorem 1.1, we may derive the following result.

Lemma 1.1. $Z_w(n)$ is even if and only if n is even.

Proof. Note that, in Theorem 1.1, $Z_w(n)$ is even if and only if one of the r prime factors is 2, and in such a case, n is also even.

An immediate consequence of Lemma 1.1 is that the equation $Z_w(n+1) = Z_w(n)$ has no solution. Some of the properties of the function $Z_w(n)$ are given in the following corollary.

Corollary 1.1. The following properties hold :

(1) $Z_w(n)$=$p_1p_2...p_r$ for some distinct primes $p_1, p_2, ..., p_r$ if and only if n is of the form

$n = p_1^{\alpha_1} p_2^{\alpha_2} \ ... \ p_r^{\alpha_r}$ for some non-negative integers $\alpha_1, \alpha_2, ..., \alpha_r$,

(2) $Z_w(n) \geq 1$ for all $n \geq 1$ and $Z_w(n) = 1$ if and only if $n = 1$,

(3) $0 < \frac{Z_w(n)}{n} \leq 1$ for any $n \geq 1$,

(4) $Z_w(p) = p$ for any prime $p \geq 2$,

(5) $Z_w(n) = n$ if and only if n is square-free,

(6) $Z_w(.)$ is multiplicative, that is, for any pair of positive integers n and m with $(n, m) = 1$,

$$Z_w(nm) = Z_w(n)Z_w(m),$$

(7) $Z_w(.)$ is not additive, that is, it is not true that

$Z_w(n + m) = Z_w(n) + Z_w(m)$ for any positive integers n and m.

Proof. See Russo [1].

Recall that an arithmetic function $f(n)$ is called completely multiplicative if

$f(nm) = f(n)f(m)$ for all integers $n(\geq 1)$ and $m(\geq 1)$.

The pseudo Smarandache square-free function $Z_w(n)$ is one of the few Smarandache-type functions that is multiplicative; however, note that $Z_w(n)$ is not completely multiplicative. For example,

$$Z_w(4) = 2 \neq Z_w(2)Z_w(2).$$

The Smarandache function, denoted by $S(n)$, is defined as follows :

$$S(n) = \min \{m : n \text{ divides } m!\}.$$

The function $S(n)$ has been studied to some extent by Ashbacher [6] and Sandor [7]. Some more results may be found by Majumdar [8, 9], which also summarizes the significant contributions of other researchers as well. For a recent review of the pseudo Smarandache function may be found in Liu [10].

The pseudo Smarandache function $Z(n)$, introduced by Kashihara [11], is defined as follows :

$$Z(n) = \min \{m : n \text{ divides } \tfrac{m(m+1)}{2}\}.$$

Some of the properties of the function $Z(n)$ have been studied by Kashihara [11], Ibstedt [12], Ashbacher [13] and Majumdar [8, 9]. A recent review of the pseudo Smarandache function may be found in Liu [14].

This paper studies some interesting properties of the pseudo Smarandache Square-free function $Z_w(n)$. This is done in Section 3, where some relationships between the function $Z_w(n)$ and each of the functions $S(n)$ and $Z(n)$ are established. Some of the problems addressed here were listed as open problems in Russo [1]. Some background materials are given in Section 2.

We conclude the paper with some remarks and open problems in the final Section 4.

§2. Background Material

In this section, we give some background material that would be needed later. We start with the lemma below, whose proof is simple and is omitted here.

Lemma 2.1. For any integer $n \geq 1$,

$$gcd(n, n+1) = 1, gcd(2n+1, 2n+3) = 1, gcd(2n, 2(n+1)) = 2.$$

Lemma 2.2. For any integer $k \geq 1$, 3 divides $2^{2k} - 1$.

Proof. The proof is by induction on k. The result is clearly true for $k = 1$. So we assume that the result is true for some k. Now, writing $2^{2(k+1)} - 1$ as follows

$$2^{2(k+1)} - 1 = 4(2^{2k} - 1) + 3,$$

we see that 3 divides $2^{2(k+1)} - 1$, so that the result is true for $k + 1$ as well.

Corollary 2.1. For any integer $k \geq 1$, 3 does not divides $2^{2k+1} - 1$.

Proof. Since $2^{2k+1} - 1 = 2(2^{2k} - 1) + 1$, the result follows by virtue of Lemma 2.2.

The two lemmas below gives the forms of $S(n)$ is some particular cases of n. The proofs of the lemmas may be found in Majumdar [8].

Lemma 2.3. Let $n = p_1^{\alpha_1} p_2^{\alpha_2}$ ... $p_r^{\alpha_r}$ be the (unique) factorization of n in terms of its r primes factors $p_1, p_2,$, ..., p_r (where α_1, α_2, ..., α_r are non-negative integers). Then,

$$S(n) = max\{S(p_1^{\alpha_1}), S(p_2^{\alpha_2}), ..., S(p_r^{\alpha_r})\}.$$

Corollary 2.2. $S(p_1, p_2, ..., p_r) = max\{p_1, p_2, ..., p_r\}$ for any distinct primes $p_1, p_2, ..., p_r$.

Lemma 2.4. For any prime $p(\geq 2), S(p) = p, S(p^2) = 2p, S(p^p) = p^2 = S(p^{p+1})$.

The following lemma gives the expressions of $Z(2p)$ and $Z(3p)$ for some particular cases of the prime p. For proof of the lemma, we refer the reader to Majumdar [8].

Lemma 2.5. The following results hold :

(1) $Z(2p) = p$ for any prime $p(\geq 3)$ such that 4 divides $(p + 1)$,

(2) $Z(3p) = p$ for any prime $p(\geq 5)$ such that 3 divides $(p + 1)$.

The two lemmas below, due to Majumdar [8], give explicit expressions of $Z(2^k)$, $Z(p^k)(p \geq 3), Z(3.2^{2k}), Z(5.2^{4k}), Z(5.2^{4k+1})$ and $Z(7.2^{3k})$.

Lemma 2.6. For any integer $k \geq 1$,

(1) $Z(2^k) = 2^{k+1} - 1$, (2) $Z(p^k) = p^k - 1$ for any prime $p \geq 3$.

Lemma 2.7. For any integer $k \geq 1$,

(1) $Z(3.2^{2k}) = 2^{2k+1}$, (2) $Z(5.2^{4k}) = 2^{4k+2}, Z(5.2^{4k+1}) = 2^{4k+2}$, (3) $Z(7.2^{3k}) = 3.2^{3k+1}$.

Given a multiplicative function $f(n)$, we can form another multiplicative function using the lemma below.

Lemma 2.8. Let $f(n)$ be a multiplicative function. Let

$$F(n) = \sum_{d \ divides \ n} f(d).$$

Then, the function $F(n)$ is multiplicative.

Proof. See, for example, Gioia [15] or Hardy and Wright [16, Theorem 265].

Example 2.1. Given the multiplicative function $Z_w(n)$, using Lemma 2.8 above, we can form the following new multiplicative function $G(n)$:

$$G(n) = \sum_{d \text{ divides } n} Z_w(d).$$

Thus, for example, since for any prime $p(\geq 2)$ has just two divisors, namely, 1 and p, with $Z_w(1) = 1$ and $Z_w(p) = p$, it follows that

$$G(p) = 1 + p \text{ for any prime } p(\geq 2).$$

Similarly, to find $G(pq)$, where p and q are distinct primes, note that the only divisors of pq are $1, p, q$ and pq, so that

$$G(pq) = 1 + p + q + pq \text{ for any two distinct primes } p \text{ and } q.$$

Again, since the only divisors of p^k (where $p(\geq 2)$ is a prime and k is any positive integer) are $1, p, p^2, ..., p^k$, it follows that

$$G(p^k) = 1 + kp \text{ for any prime } p(\geq 2).$$

Thus, for example, $G(1) = 1, G(2) = 3, G(3) = 4, G(4) = 5, G(5) = 6$ and $G(6) = 12$. Note that, as expected, $G(2.3) = 12 = G(2)G(3) = 3 \times 4$.

The main results of the paper are given in the next section.

§3. Main Results

This section gives some interesting properties satisfied by the function $Z_w(n)$. By equation (1.1), $Z_w(2n) \leq 2n$ for any integer $n \geq 1$. However, we can prove the result below.

Lemma 3.1. If $n(\geq 2)$ is an even integer, then $Z_w(2n) \leq n$.

Proof. Let

$$n = 2^{\alpha} p_1^{\alpha_1} p_2^{\alpha_2} \ ... \ p_r^{\alpha_r} \text{ with } \alpha \geq 1; \alpha_1, \alpha_2, ..., \alpha_r \geq 0,$$

where $p_1, p_2, ..., p_r$ are r odd (distinct) prime factors of n. Then, by Theorem 1.1,

$$Z_w(2n) = 2p_1p_2...p_r \leq n.$$

From the proof of Lemma 3.1, we see that

$$Z_w(2n) \leq 2n \text{ for any odd integer } n(\geq 3).$$

Another immediate consequence of Lemma 3.1 is the following.

Corollary 3.1. $Z_w(2(p-1)) \leq p - 1$ for any prime $p \geq 3$.

The following result is mentioned in Russo [1]. We give here a more general proof.

Lemma 3.2. $k(\geq 1)$ being an integer, neither of the equations

$$\frac{Z_w(n)}{Z_w(n+1)} = k \quad \text{or} \quad \frac{Z_w(n+1)}{Z_w(n)} = k$$

has a solution.

Proof. Since $(n, n+1) = 1$, let

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \ ... \ p_r^{\alpha_r}, \ n + 1 = q_1^{\beta_1} q_2^{\beta_2} \ ... \ q_s^{\beta_s},$$

where $p_1, p_2, ..., p_r, q_1, q_2, ..., q_s$ are $(r + s)$ number of distinct primes, and $\alpha_1, \alpha_2, ..., \alpha_r \geq 0$ (not all of which are zero) and $\beta_1, \beta_2, ..., \beta_s \geq 0$ (not all zero) are non-negative integers. Then,

$$\frac{Z_w(n)}{Z_w(n+1)} = \frac{p_1p_2...p_r}{q_1q_2...q_s},$$

which shows that the ratio on the R.H.S. cannot be an integer (since, for example, the prime q_1 does not divide any of the r primes in the numerator). By similar reasoning, $\frac{Z_w(n+1)}{Z_w(n)}$ cannot be an integer.

Lemma 3.3. Each of the inequalities $\frac{Z_w(n)}{Z_w(n+1)} < 1$ and $\frac{Z_w(n)}{Z_w(n+1)} > 1$ is satisfied for an infinite number of integers n.

Proof. Let $n = 2^{\alpha}$ for $\alpha \geq 1$. Then, $Z_w(n) = 2$. Now, since any prime factor of $n+1$ is greater than or equal to 3, it follows that $Z_w(n+1) \geq 3$. Therefore, with this n,

$$\frac{Z_w(n)}{Z_w(n+1)} = \frac{2}{Z_w(2^\alpha+1)} \leq 1 \text{ for any } \alpha \geq 1.$$

Next, let $n+1 = 2^\beta$ for $\beta \geq 1$. Then, $Z_w(n+1) = 2, Z_w(n) = Z_w(2^\beta - 1) \geq 3$. Then, with this n,

$$\frac{Z_w(n)}{Z_w(n+1)} = \frac{Z_w(2^\beta-1)}{2} > 1 \text{ for any } \beta \geq 1.$$

The first few terms of the sequence $\{\frac{2}{Z_w(2^\alpha+1)}\}_{\alpha=1}^{\infty}$ are

$$\frac{2}{3}, \frac{2}{5}, \frac{2}{3}, \frac{2}{17}, \frac{2}{33}, \frac{2}{65}, \frac{2}{129}, \cdots,$$

and the first few leading terms of the sequence $\{\frac{Z_w(2^\beta-1)}{2}\}_{\beta=1}^{\infty}$ are

$$\frac{3}{2}, \frac{7}{2}, \frac{15}{2}, \frac{31}{2}, \frac{21}{2}, \frac{127}{2}, \cdots,$$

while the first few terms of the sequence $\{\frac{Z_w(n)}{Z_w(n+1)}\}_n^{\infty} = 1$ are

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{2}, \frac{2}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{2}, \frac{2}{3}, \frac{3}{10}, \frac{10}{11}, \frac{11}{6}, \frac{6}{13}, \cdots.$$

Lemma 3.4. For any $\epsilon > 0$, however small, there is an integer $n(> 1)$ such that $\frac{Z_w(n)}{n}$.

Proof. Let $n = 2^k$ for some integer $k(\geq 1)$. Then, $\frac{Z_w(n)}{n} = \frac{2}{2^k} = \frac{1}{2^{k-1}}$. Therefore, given real number $\epsilon > 0$, choose $k > -\frac{ln\epsilon}{ln2} + 1$. With this n and k,

$$\frac{Z_w(n)}{n} = \frac{1}{2^{k-1}} < \epsilon.$$

For example, if $\epsilon = 0.001$, then choose $n = 2^{11}$ so that $\frac{Z_w(n)}{n} = \frac{1}{2^{10}} < 0.001$.

Lemma 3.5. $|Z_w(n+1) - Z_w(n)|$ is unbounded.

Proof. Let n be a prime of the form $n = 4m+3, m \geq 1$, so that (by Corollary 1.1) $Z_w(n) = 4m+3$. Now, by Lemma 3.1, $Z_w(n+1) \geq \sqrt{2(m+1)}$. Therefore,

$$|Z_w(n+1) - Z_w(n)| = Z_w(n) - Z_w(n+1) \geq 4m+3 - \sqrt{2(m+1)},$$

which may be made arbitrarily large by the proper choice of m such that $n = 4m+3$ is prime.

Lemma 3.6. The equation $Z_w(mn) = m^k Z_w(n)$ has an infinite number of solutions.

Proof. Clearly, the equation is trivially satisfied when $m = 1$. So, we assume that $m \geq 2$. To find a solution of the equation

$$Z_w(mn) = m^k Z_w(n) \tag{3.1}$$

we proceed as follows : Let $gcd(m,n) = 1$, so that $Z_w(mn) = Z_w(m)Z_w(n)$. Then, the equation (3.1) takes the form $Z_w(m)Z_w(n) = m^k Z_w(n)$, so that we must have $Z_w(m) = m^k$. Then, by Theorem 1.1, $k = 1$; moreover, m must be square-free. Thus,

$$k = 1,\ m \text{ is a square-free integer},\ n \text{ is any integer with } gcd(m,n) = 1$$

is a solution of the equation (3.1). In particular, $k = 1, m = 2, n = p$ ($p \geq 3$ being a prime) is a solution of the equation (3.1), which shows that the equation possesses an infinite number of solutions.

Lemma 3.7. None of the equations below has a solution :

(1) $Z_w(n).Z_w(n+1) = Z_w(n+2)$,

(2) $Z_w(n) = Z_w(n+1).Z_w(n+2)$,

(3) $Z_w(n).Z_w(n+1) = Z_w(n+2).Z_w(n+3)$.

Proof. To prove part (1), we assume that the equation is satisfied for some n, and then we would reach to a contradiction. Now, since $gcd(n+1, n+2) = 1$, let

$$n+1 = q_1^{\beta_1} q_2^{\beta_2} ... q_s^{\beta_s},\ n+2 = r_1^{\gamma_1} r_2^{\gamma_2} ... r_t^{\gamma_t},$$

so that the equation takes the form

$$Z_w(n) q_1 q_2 ... q_s = r_1 r_2 ... r_t,$$

and we arrive at a contradiction (since neither of the s primes $q_1, q_2, ..., q_s$ divides any of the t

primes $r_1, r_2, ..., r_t$).

To prove part (2), let the equation be satisfied for some n. Now, letting

$$n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_r^{\alpha_r}, \; n+1 = q_1^{\beta_1} q_2^{\beta_2} ... q_s^{\beta_s}$$

the equation becomes

$$p_1 p_2 ... p_r = q_1 q_2 ... q_s Z_w(n+2),$$

and we reach to a contradiction.

To prove part (3), let the equation be satisfied for some integer n. Then, letting

$$n+1 = q_1^{\beta_1} q_2^{\beta_2} ... q_s^{\beta_s}, \; n+2 = r_1^{\gamma_1} r_2^{\gamma_2} ... r_t^{\gamma_t},$$

we get

$$Z_w(n) q_1 q_2 ... q_s = r_1 r_2 ... r_t Z_w(n+3).$$

Now, by Lemma 2.1, if n is odd, so also is $n+2$ with $(n, n+2) = 1$; on the other hand, if n is even, so also is $n+2$ with $gcd(n, n+2) = 2$. In either case, none of the t primes $r_1, r_2, ..., r_t$ divides $Z_w(n)$, and we arrive at a contradiction.

Lemma 3.8. The equation $[Z_w(n)]^k = kZ_w(kn)$ has an infinite number of solutions.

Proof. There is nothing to prove if $k = 1$. So, let $k \geq 2$. Now, we prove that $gcd(k, n) \neq 1$. For otherwise, the equation, after simplification (since $Z_w(n) \neq 0$), becomes

$$[Z_w(n)]^{k-1} = kZ_w(k).$$

Since in the equation above, only square-free prime factors of the integers n and k appear, we must have $k = 2$, so that

$$Z_w(n) = 2Z_w(2) = 4.$$

But there is no integer n satisfying the above condition. Hence, $gcd(k, n) \neq 1$.

Now, let p be a prime factor of k. Since $Z_w(kn)$ is square-free, the prime factor of k appears on the R.H.S. of the equation as (at most) second power. Thus, $k \leq 2$. With $k = 2$, the equation reads as

$$[Z_w(n)]^2 = 2Z_w(2n),$$

where n must be even. Then, n cannot have any prime factor greater than 2. Letting $n = 2^\alpha, \alpha \geq 1$, it is easy to verify that the above equation is satisfied.

The proof above shows that, any (non-trivial) solution of the equation $[Z_w(n)]^k = kZ_w(kn)$ must be of the form $k = 2, n = 2^\alpha, \alpha \geq 1$ being an integer.

As has been mentioned in Russo [1], the equation $Z_w(n) = n$ has an infinite number of solutions, namely, $n = p$, where $p(\geq 2)$ is a prime. However, we have the following result.

Lemma 3.9. If $k \geq 2$, then the following equation has no solution :

$$[Z_w(n)]^k + [Z_w(n)]^{k-1} + ... + Z_w(n) = n.$$

Proof. If possible, let the equation have a solution of the form

$$n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_r^{\alpha_r}$$

for some integer $k(\geq 2)$, where $p_1, p_2, ..., p_r$ are the distinct prime factors of n, and $\alpha_1, \alpha_2, ..., \alpha_r$ are all non-negative integers. Then, substituting in the equation, we get, after canceling out the common factor $p_1 p_2 ... p_r$ on both sides,

$$(p_1 p_2 ... p_r)^{k-1} + (p_1 p_2 ... p_r)^{k-2} + ... + 1 = p_1^{\alpha_1 - 1} p_2^{\alpha_2 - 1} ... p_r^{\alpha_r - 1}.$$

Then, we must have $\alpha_1 - 1 = 0 = \alpha_2 - 1 = ... = \alpha_r - 1$. But then we have

$$(p_1 p_2 ... p_r)^{k-1} + (p_1 p_2 ... p_r)^{k-2} + ... + p_1 p_2 ... p_r = 0,$$

which is absurd.

Since $S(p) = p = Z_w(p)$ for any prime $p(\geq 2)$, it follows that the Diophantine equation $S(n) = Z_w(n)$ has an infinite number of solutions. Russo [1] asks for composite n satisfying the $S(n) = Z_w(n)$. The lemma below answers his question.

Lemma 3.10. Let $n = 2p^2$ where $p(\geq 3)$ is a prime. Then, $S(n) = Z_w(n)$.

Proof. By Theorem 1.1, $Z(n) = 2p$. Now, by Lemma 2.3 and Lemma 2.4,

$$S(n) = max\{S(2), S(p^2)\} = S(p^2) = 2p \text{ for all } p \geq 3.$$

Hence, $S(n) = Z_w(n)$.

A more general solution to the equation $S(n) = Z_w(n)$ has been given by Guan [4]. The following five lemmas gives more relationships involving $S(n)$ and $Z_w(n)$. The first two lemmas show that each of the inequalities $Z_w(n) > S(n)$ and $Z_w(n) < S(n)$ is satisfied for an infinite number of integers n.

Lemma 3.11. There is an infinite number of integers n satisfying the inequality $Z_w(n) > S(n)$.

Proof. Let p and q be two primes with $q > p \geq 2$. Then, using Corollary 2.2,

$$Z_w(pq) = pq > q = S(pq).$$

Thus, with such p and $q, n = pq$ satisfies the given inequality.

Note that, we can find other n satisfying the inequality $Z_w(n) > S(n)$. For example, $n = pqr$ also satisfies the inequality, where p, q and r are distinct primes with $r > q > p \geq 2$.

Lemma 3.12. There is an infinite number of n satisfying the inequality $S(n) > Z_w(n)$.

Proof. We have, using Lemma 2.4,

$$S(p^p) = p^2 > p = Z_w(p^p).$$

Therefore, $n = p^p$ satisfies the given inequality.

Since $S(p^{p+1}) = p^2 > p = Z_w(p^{p+1})$, we see that $n = p^{p+1}$ also satisfies the inequality $S(n) > Z_w(n)$.

Lemma 3.13. There is an infinite number of integers n such that $S(Z_w(n)) = Z_w(S(n))$.

Proof. Let p and q be two primes with $q > p \geq 2$. Then,

$$S(Z_w(pq)) = S(pq) = q, Z_w(S(pq)) = Z_w(q) = q,$$

so that $S(Z_w(pq)) = Z_w(S(pq))$.

Lemma 3.14. The inequality $Z_w(S(n)) > S(Z_w(n))$ is satisfied by an infinite number of n.

Proof. Let $p(\geq 3)$ be any prime. Then, by Lemma 2.4,

$$Z_w(S(p^2)) = Z^w(2p) = 2p, S(Z_w(p^2)) = S(p) = p.$$

Thus, with $n = p^2$ ($p \geq 3$ being a prime), the inequality $Z_w(S(n)) > S(Z_w(n))$ is satisfied.

Lemma 3.15. The inequality $Z_w(S(n)) < S(Z_w(n))$ is satisfied by an infinite number of n.

Proof. Let p and q be two primes with $q > p(\geq 3)$ such that $p^2 > 2q$. Then, by Lemma 2.3 and Lemma 2.4,

$$S(p^pq^2) = max\{S(p^p), S(q^2)\} = max\{p^2, 2q\} = p^2,$$

so that

$$Z_w(S(p^pq^2)) = Z_w(p^2) = p, S(Z_w(p^pq^2)) = S(pq) = q.$$

Thus, with this $n = p^pq^2$, the inequality $Z_w(S(n)) < S(Z_w(n))$ is satisfied.

It may be mentioned here that, given any prime $p(\geq 3)$, there always exists another prime $q(> p)$ such that $p^2 > 2q$. To see this, it is sufficient to consider the case when p and $q = p + 2$ are the twin primes. Now,

$$p^2 > 2(p + 2) \text{ if and only if } p \geq 3.$$

The following five lemmas involve the two functions $Z(n)$ and $Z_w(n)$.

Lemma 3.16. The equation $Z_w(n) = Z(n)$ has no solution.

Proof. If possible, let $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_r^{\alpha_r}$ be a solution of the given equation, where, without any loss of generality the r prime factors of n satisfies the inequality $p_1 < p_2 < ... < p_r$, and $\alpha_1, \alpha_2, ..., \alpha_r$ are non-negative integers. Then,

$$Z(n) = p_1 p_2 ... p_r,$$

so that, by definition, $p_1 p_2 ... p_r$ is the minimum integer such that n divides

Hence, we must have $\alpha_1 = 1 = \alpha_2 = ... = \alpha_r$, so that n has the form $n = p_1 p_2 ... p_r$. If $p_1 = 2$, then $n = 2p_2 ... p_r$ cannot divide $p_2 ... p_r(p_1 p_2 ... p_r + 1)$. On the other hand, if all the prime factors of n are odd, then $n = p_1 p_2 ... p_r$ divides $\frac{p_1 p_2 ... p_r(p_1 p_2 ... p_r - 1)}{2}$ so that $Z(n) \leq p_1 p_2 ... p_r - 1$, and we reach to a contradiction.

Though the equation $Z_w(n) = Z(n)$ has no solution, the two lemmas below prove that each of the inequalities $Z_w(n) > Z(n)$ and $Z_w(n) < Z(n)$ is satisfied by an infinite number of n.

Lemma 3.17. There is an infinite number of integers n satisfying $Z_w(n) > Z(n)$.

Proof. Let $p \geq 3$ be a prime such that 4 divides $(p + 1)$. Then, using Lemma 2.5,

$$Z_w(2p) = 2p > p = Z(2p).$$

Thus, with such $p, n = 2p$ satisfies the given inequality.

Lemma 3.16 finds the integer n such that $Z_w(n) > Z(n)$. We can find more such n. For example, let $n = 3p$, where $p(\geq 5)$ is a prime such that 3 divides $(p + 1)$. Then, using Lemma 2.5, we get

$$Z_w(3p) = 3p > p = Z(3p).$$

Lemma 3.18. There is an infinite number of n satisfying the inequality $Z(n) > Z_w(n)$.

Proof. Using Lemma 2.6, we get

$$Z(2^k) = 2^{k+1} - 1 > 2 = Z_w(2^k) \text{ for any } k \geq 1.$$

Let $p(\geq 3)$ be a prime. Then, using Lemma 2.6, we get

$$Z(p^k) = p^k - 1 > p = Z_w(p^k) \text{ for any } k \geq 2.$$

This provides a second example of n such that $Z(n) > Z_w(n)$. Note that

$$Z_w(Z(2)) = Z_w(3) = 3 = Z(2) = Z(Z_w(2)), Z_w(Z(3)) = Z_w(2) = 2 = Z(3) = Z(Z_w(3)).$$

The following lemma proves more.

Lemma 3.19. The equation $Z_w(Z(n)) = Z(Z_w(n))$ has an infinite number of solution n.

Proof. Let p be an odd prime such that 4 divides $(p + 1)$. Then, using Lemma 2.5,

$$Z_w(Z(2p)) = Z_w(p) = p, Z(Z_w(2p)) = Z(2p) = p,$$

so that for such p, $Z_w(Z(2p)) = Z(Z_w(2p))$.

It is possible to construct other examples of n such that $Z_w(Z(n)) = Z(Z_w(n))$. For example, let $p(\geq 5)$ be a prime such that 3 divides $(p + 1)$. Then, using Lemma 2.5,

$$Z_w(Z(3p)) = Z_w(p) = p = Z(Z_w(3p)) = Z(3p).$$

Lemma 3.20. There is an infinite number of integers n satisfying $Z(Z_w(n)) > Z_w(Z(n))$.

Proof. By Lemma 2.7, for any $k \geq 1$,

$$Z(Z_w(3.2^{2k})) = Z(3.2) = 3, Z_w(Z(3.2^{2k})) = Z_w(2^{2k+1}) = 2.$$

Therefore, $Z(Z_w(3.2^{2k})) > Z_w(Z(3.2^{2k}))$ for all $k \geq 1$.

To find other n such that $Z(Z_w(n)) > Z_w(Z(n))$, note that, by Lemma 2.7, for any $k \geq 1$,

$$Z(Z_w(5.2^{4k})) = Z(5.2) = 4, Z_w(Z(5.2^{4k})) = Z_w(2^{4k+2}) = 2,$$

so that $Z(Z_w(5.2^{4k})) > Z_w(Z(5.2^{4k}))$. Again, since

$$Z(Z_w(5.2^{4k+1})) = Z(5.2) = 4, Z_w(Z(5.2^{4k+1})) = Z_w(2^{4k+2}) = 2,$$

we see that $Z(Z_w(5.2^{4k+1})) > Z_w(Z(5.2^{4k}))$. Finally, since for any $k \geq 1$,

$$Z(Z_w(7.2^{3k})) = Z(7.2) = 7, Z_w(Z(7.2^{3k})) = Zw(3.2^{3k+1}) = 3.2,$$

it follows that $Z(Z_w(7.2^{3k})) > Z_w(Z(7.2^{3k}))$.

Lemma 3.21. There is an infinite number of integers n such that $Z(Z_w(n)) < Z_w(Z(n))$.

Proof. By Lemma 2.6, $k \geq 1$,

$$Z(Z_w(2^{2k})) = Z(2) = 3, Z_w(Z(2^{2k})) = Z_w(2^{2k+1} - 1).$$

But, by Corollary 2.1, any prime factor of $2^{2k+1} - 1$ is greater than 3, so that $Z_w(2^{2k+1} - 1) > 5$ for all $k \geq 1$. Thus,

$$Z(Z_w(2^{2k})) < Z_w(Z(2^{2k})) \text{ for all } k \geq 1.$$

§4. Some Remarks

This paper gives some new results related to the pseudo Smarandache Square-free function $Z_w(n)$. It addresses some open problems posed by Russo [1]. In several cases, we found multiple solutions. For example, two examples are mentioned to illustrate that each of the inequalities $Z_w(n) > S(n)$ and $Z_w(n) < S(n)$ has an infinite number of solutions. Thus, we may set the following problems.

Problem 4.1. Find all values of n such that $Z_w(n) > S(n)$.

Problem 4.2. Find all values of n such that $Z_w(n) < S(n)$.

Problem 4.3. Find all values of n such that $S(Z_w(n)) = Z_w(S(n))$.

Problem 4.4. Find all values of n such that $S(Z_w(n)) > Z_w(S(n))$.

Problem 4.5. Find all values of n such that $S(Z_w(n)) < Z_w(S(n))$.

Lemma 3.16 shows that the equation $Z_w(n) = Z(n)$ has no solution; however, it has been proved that each of the inequalities $Z_w(n) > Z(n)$ and $Z_w(n) < Z(n)$ has an infinite number of solutions, and in each case, two examples are given. This tempts us to pose the following problems.

Problem 4.6. Find all values of n such that $Z_w(n) > Z(n)$.

Problem 4.7. Find all values of n such that $Z_w(n) < Z(n)$.

Lemmas 3.19-3.21 involve the functions $Z(Z_w(n))$ and $Z_w(Z(n))$. Two examples are provided to demonstrate that $Z(Z_w(n)) = Z_w(Z(n))$, while in support of the inequality $Z(Z_w(n)) > Z_w(Z(n))$, four examples have been found. We thus reiterate the following problems.

Problem 4.8. Find all values of n such that $Z(Z_w(n)) = Z_w(Z(n))$.

Problem 4.9. Find all values of n such that $Z(Z_w(n)) > Z_w(Z(n))$.

Problem 4.10. Find all values of n such that $Z(Z_w(n)) < Z_w(Z(n))$.

Russo [1] asks the following question.

Problem 4.11. For what values of $k(\geq 2), m(\geq 2)$ and $n(\geq 2)$ does the functional equation

$$[Z_w(n)]^k + [Z_w(n)]^{k-1} + ... + [Z_w(n)]^2 + Z_w(n) = mn \tag{4.1}$$

has a solution?

We find an answer to Problem 4.11, note that, when $k = 1$, the equation (4.1) becomes $Z(n) = mn$, so that we must have $m = 1$ (since, by (1.1), $mn = Z(n) \leq n$). So, let $k \geq 2$. We now find a partial answer to Problem 4.11 under the assumption that n is a square-free integer, so that $Z(n) = n$. Then, the equation (4.1) takes the form

$$n^k + n^{k-1} + ... + n^2 + n = mn,$$

that is,

$$n^{k-1} + n^{k-2} + ... + n + 1 = m, \tag{4.2}$$

which does not have a solution if $m = 1, k \geq 2$. Now, for any $k \geq 2$ fixed, the equation (4.2) has a solution for any square-free integer $n(\geq 2)$ if and only if $m = \frac{n^k-1}{n-1}$. Thus, for example, when $n = 2$, the equation (4.1) has a solution only if $m = 2^k - 1; k = 1, 2, \cdots$.

The following problem, posed by Russo [1], still remains open.

Open Problem 4.1. Solve the equation

$$Z_w(n) + Z_w(n+1) = Z_w(n+2). \tag{4.3}$$

Clearly, in the equation (4.3) above, all the three integers $n, n+1$ and $n+2$ cannot be square free. Now, the L.H.S. is odd, and hence, $Z(n+2)$ is odd, so that $n+2$ is also odd. Then, $Z(n)$ and n both are odd. Assuming that

$$n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_r^{\alpha_r}, n+1 = 2^{\beta} q_1^{\beta_1} q_2^{\beta_2} ... q_s^{\beta_s}, n+2 = r_1^{\gamma_1} r_2^{\gamma_2} ... r_t^{\gamma_t}, \tag{4.4}$$

we get

$$p_1 p_2 ... p_r + 2 q_1 q_2 ... q_s = r_1 r_2 ... r_t, \tag{4.5}$$

where the three sets of primes, namely, $\{p_1, p_2, ..., p_r\}, \{q_1, q_2, ..., q_s\}$ and $\{r_1, r_2, ..., r_t\}$ are all odd and distinct with

$$r_1^{\gamma_1} r_2^{\gamma_2} ... r_t^{\gamma_t} = 2^{\beta} q_1^{\beta_1} q_2^{\beta_2} ... q_s^{\beta_s} + 1 = p_1^{\alpha_1} p_2^{\alpha_2} ... p_r^{\alpha_r} + 2. \tag{4.6}$$

Russo [1] reports that, searching for $1 \leq n \leq 1000$ such that $Z_w(n)$ satisfies (4.3), only six solutions are found, which are as follows :

$$Z_w(1) + Z_w(2) = Z_w(3), Z_w(3) + Z_w(4) = Z_w(5),$$
$$Z_w(15) + Z_w(16) = Z_w(17), Z_w(31) + Z_w(32) = Z_w(33),$$
$$Z_w(127) + Z_w(128) = Z_w(129), Z_w(255) + Z_w(256) = Z_w(257).$$

Note that, in all of the above six cases, $n+1$ is of the form $n+1 = 2^{\beta}$ for some integer $\beta \geq 1$. Searching for $1 \leq n \leq 100$, the solution of the inequality $Z_w(n) > Z_w(n+1) + Z_w(n+2)$, the following 8 instances have been observed :

$$Z_w(7) > Z_w(8) + Z_w(9), Z_w(23) > Z_w(24) + Z_w(25),$$
$$Z_w(26) > Z_w(27) + Z_w(28), Z_w(47) > Z_w(48) + Z_w(49),$$
$$Z_w(62) > Z_w(63) + Z_w(64), Z_w(74) > Z_w(75) + Z_w(76),$$
$$Z_w(79) > Z_w(80) + Z_w(81), Z_w(97) > Z_w(98) + Z_w(99).$$

Thus, we have the following problems.

Open Problem 4.2. Find all values n such that $Z_w(n) > Z_w(n+1) + Z_w(n+2)$.

Open Problem 4.3. Solve the inequality : $Z_w(n) < Z_w(n+1) + Z_w(n+2)$.

In addition, we have the following open problem, posed by Russo [1].

Open Problem 4.4. Solve the equation

$$Z_w(n) = Z_w(n+1) + Z_w(n+2). \tag{4.7}$$

Russo [1] reports that, searching for $1 \leq n \leq 1000$, no solution of the equation (4.7) has been found. Note that the R.H.S. of (4.7) is odd, and hence, $Z(n)$ (and n) is odd, so that $Z(n+2)$ (and $n+2$) is also odd. Assuming solution of the form (4.4), we get

$$p_1 p_2 ... p_r = 2 q_1 q_2 ... q_s + r_1 r_2 ... r_t, \tag{4.8}$$

where the primes $p_1, p_2, ..., p_r, q_1, q_2, ..., q_s$ and $r_1, r_2, ..., r_t$ are all odd and distinct with

$$r_1^{\gamma_1} r_2^{\gamma_2} ... r_t^{\gamma_t} = 2^{\beta} q_1^{\beta_1} q_2^{\beta_2} ... q_s^{\beta_s} + 1 = p_1^{\alpha_1} p_2^{\alpha_2} ... p_r^{\alpha_r} + 2.$$

Searching for $1 \le n \le 100$, the solution of $Z(n+1) - Z_w(n) = Z_w(n+2) - Z_w(n+1)$, the following 14 instances have been found :

$$Z(2) - Z_w(1) = Z_w(3) - Z_w(2), \quad Z(6) - Z_w(5) = Z_w(7) - Z_w(6),$$
$$Z(14) - Z_w(13) = Z_w(15) - Z_w(14), \quad Z(22) - Z_w(21) = Z_w(23) - Z_w(22),$$
$$Z(30) - Z_w(29) = Z_w(31) - Z_w(30), \quad Z(34) - Z_w(33) = Z_w(35) - Z_w(34),$$
$$Z(38) - Z_w(37) = Z_w(39) - Z_w(38), \quad Z(42) - Z_w(41) = Z_w(43) - Z_w(42),$$
$$Z(58) - Z_w(57) = Z_w(59) - Z_w(58), \quad Z(66) - Z_w(65) = Z_w(67) - Z_w(66),$$
$$Z(70) - Z_w(69) = Z_w(71) - Z_w(70), \quad Z(78) - Z_w(77) = Z_w(79) - Z_w(78),$$
$$Z(86) - Z_w(85) = Z_w(87) - Z_w(86), \quad Z(94) - Z_w(93) = Z_w(95) - Z_w(94).$$

We now pose the following problem.

Open Problem 4.5. Find all values n such that

$$Z(n+1) - Z_w(n) = Z_w(n+2) - Z_w(n+1). \tag{4.9}$$

The equation (4.9) is satisfied if the integers $n, n+1$ and $n+2$ are all square-free. Assuming a solution of the form

$$n = p_1 p_2 ... p_r, n+1 = q_1 q_2 ... q_s, n+2 = r_1 r_2 ... r_t, \tag{4.10}$$

we get

$$p_1 p_2 ... p_r + r_1 r_2 ... r_t = 2 q_1 q_2 ... q_s, \tag{4.11}$$

where $p_1, p_2, ..., p_r, q_1, q_2, ..., q_s, r_1, r_2, ..., r_t$ are distinct primes with

$$q_1 q_2 ... q_s = p_1 p_2 ... p_r + 1, \tag{4.12}$$

$$r_1 r_2 ... r_t = q_1 q_2 ... q_s + 1. \tag{4.13}$$

Without any loss of generality, we may assume that

$$p_1 < p_2 < ... < p_r; q_1 < q_2 < ... < q_s; r_1 < r_2 < ... < r_t. \tag{4.14}$$

Now, if $q_1 q_2 ... q_s$ is odd, then $p_1 p_2 ... p_r$ is even (with $p_1 = 2$), so that, $q_1 q_2 ... q_s = 2 p_2 ... p_r + 1$. But then from (4.13), $r_1 r_2 ... r_t = 2 p_2 ... p_r + 2$ is even (with $r_1 = 2$), so that

$$2 r_2 r_3 ... r_t = 2 p_2 p_3 ... p_r + 2,$$

that is,

$$r_2 r_3 ... r_t = p_2 p_3 ... p_r + 1,$$

and we arrive at a contradiction. Hence, $q_1 q_2 ... q_s$ must be even, say, $q_1 q_2 ... q_s = 2 q_2 q_3 ... q_s$. In the simplest case, $Z_w(n+1) = 2q$ (q is a prime), and then $Z_w(n) = p = 2q - 1$ and $Z_w(n+2) = r = 2q + 1$ are twin primes. For example, corresponding to $q = 3$, the solution $p = 5, r = 7$ is obtained. Another possibility is that $Z_w(n+1) = 2q, Z_w(n) = p = 2q - 1, Z_w(n+2) = r_1 r_2 = 2q + 1$. Then, corresponding to $q = 7$, we get the solution $(Z_w(n), Z_w(n+1), Z_w(n+2)) = (13, 14, 15)$, while the solution corresponding to $q = 19$ is $(Z_w(n), Z_w(n+1), Z_w(n+2)) = (37, 38, 39)$. The third possibility is that $Z_w(n+1) = 2q, Z_w(n) = p_1 p_2 = 2q - 1, Z_w(n+2) = r = 2q + 1$. In this case, corresponding to $q = 11$, we get the solution $(Z_w(n), Z_w(n+1), Z_w(n+2)) = (21, 22, 23)$, and with $q = 29$, the solution $(Z_w(n), Z_w(n+1), Z_w(n+2)) = (57, 58, 59)$. Considering the fourth possibility that $Z_w(n+1) = 2q_1 q_2, Z_w(n) = p = 2q_1 q_2 - 1, Z_w(n+2) = r = 2q_1 q_2 + 1$, we get the solutions $(Z_w(n), Z_w(n+1), Z_w(n+2)) = (29, 30, 31)$ corresponding to $p = 29$, and the solution (41, 42, 43) corresponding to $p = 41$. The fifth possibility is that $Z_w(n+1) = 2q, Z_w(n) = p_1 p_2 =$

$2q-1, Z_w(n+2) = r_1r_2 = 2q+1$. Here, corresponding to $q = 17, 43, 47$, we get respectively the solutions $(Z_w(n), Z_w(n+1), Z_w(n+2)) = (33, 34, 35), (85, 86, 87), (93, 94, 95)$. The sixth possibility is that $Z_w(n+1) = 2q_1q_2, Z_w(n) = p_1p_2 = 2q_1q_2 - 1, Z_w(n+2) = r = 2q+1$. Examples are the solutions $(Z_w(n), Z_w(n+1), Z_w(n+2)) = (65, 66, 67), (69, 70, 71), (77, 78, 79)$, corresponding to $r = 67, 71, 79$ respectively.

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Some remarks on UP-algebras and KU-algebras

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Abstract We show that UP-algebras are the same as Komori's BCC-algebras and that KU-algebras are reversed BCK-algebras. Moreover, we prove that pseudo-KU algebras are in fact reversed pseudo-BCK algebras.

Keywords UP-/KU-/BCC-/BCK-algebra, pseudo-BCK/pseudo-UP/pseudo-KU algebra.
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§1. Introduction

In 1966, K. Iséki [12] introduced BCK algebras as algebras connected to certain kinds of logics. Hundred of papers and the books [16] and [10] were writen on BCK algebras. To solve some problems on BCK algebras, in 1983, Y. Komori [13, 14] introduced BCC algebras. These algebras (also called BIK$^+$-algebras) are an algebraic model of BIK$^+$-logic and they have been widely investigated in literature (see e.g. [2]– [5]). In 2001, G. Georgescu and A. Iorgulescu [6] defined pseudo-BCK algebras as a non-commutative extension of BCK-algebras. The paper [6] contains basic properties of pseudo-BCK algebras and their connections with some other algebras of logic. Y. B. Jun [11] obtained a characterization of pseudo-BCK algebras. A symplified axiomatization of these algebras was given by A. Walendziak in [19]. A. Iorgulescu defined and studied reversed BCK-algebras (cf. part I of [10]) and reversed pseudo-BCK algebras (cf. part II of [10]), see also [9]. The monograph [15] (the habilitation thesis of J. Kühr) presents many of the most important results on pseudo-BCK algebras. In 2009, C. Prabpayak and U. Leerawat [17] introduced KU-algebras. Later on, in 2017, A. Iampan [8] introduced UP-algebras. Recently, D. A. Romano [18] defined pseudo-UP and pseudo-KU algebras as a natural generalizations of UP-algebras and KU-algebras, respectively.

In this paper, we show that UP-algebras are the same as Komori's BCC-algebras and that KU-algebras are reversed BCK-algebras. Moreover, we prove that pseudo-KU algebras are in fact reversed pseudo-BCK algebras.

§2. Results

We start with the following

Definition 2.1. ([8]) An algebra $(A, \cdot, 0)$ of type $(2, 0)$ is called a *UP-algebra* if it verifies the following axioms:

(UP-1) $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0$,
(UP-2) $0 \cdot x = x$,
(UP-3) $x \cdot 0 = 0$,
(UP-4) $x \cdot y = 0 = y \cdot x \Longrightarrow x = y$.

We consider here the concept of BCC-algebra defined by Y. Komori as follows:

Definition 2.2. ([13,14]) An algebra $(A, \rightarrow, 1)$ of type $(2, 0)$ is called a *BCC-algebra* if it verifies the following axioms:

(BCC-1) $(y \rightarrow z) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z)) = 1$,
(BCC-2) $1 \rightarrow x = x$,
(BCC-3) $x \rightarrow 1 = 1$,
(BCC-4) $x \rightarrow y = 1 = y \rightarrow x \Longrightarrow x = y$,
(BCC-5) $x \rightarrow x = 1$.

Remark 2.3. Note that in the papers [2]– [5] on BCC-algebras, the dual notation with $\cdot$ and 0 is used.

Replacing "$\cdot$" by "$\rightarrow$" and "0" by "1" in (UP-1)–(UP-4), we obtain (BCC-1)–(BCC-4). Observe that (BCC-5) follows from (BCC-1) and (BCC-2). Indeed, putting $x = y = 1$ in (BCC-1), we get $(1 \rightarrow z) \rightarrow ((1 \rightarrow 1) \rightarrow (1 \rightarrow z)) = 1$. Hence, by (BCC-2), we have $z \rightarrow z = 1$, that is, (BCC-5) holds. Conversely, replacing "$\rightarrow$" by "$\cdot$" and "1" by "0" in (BCC-1)–(BCC-4), we obtain (UP-1)–(UP-4). Then, we have the following

Theorem 2.4. *An algebra is a UP-algebra if and only if it is a BCC-algebra.*

The notion of KU-algebra was defined by C. Prabpayak and U. Leerawat as follows:

Definicja 2.5. ([17]) An algebra $(A, \cdot, 0)$ of type $(2, 0)$ is called a *KU-algebra* if it verifies the following axioms: (UP-2)–(UP-4) and

(KU-1) $(y \cdot x) \cdot ((x \cdot z) \cdot (y \cdot z)) = 0$.

Now, we recall the definition of a BCK-algebra.

Definicja 2.6. An algebra $(A, *, 0)$ of type $(2, 0)$ is called a *BCK-algebra* ([12]) if it verifies the following axioms:

(1) $((z * y) * (z * x)) * (x * y) = 0$,
(2) $(x * (x * y)) * y = 0$,
(3) $x * x = 0$,
(4) $x * y = 0 = y * x \Longrightarrow x = y$,
(5) $0 * x = 0$,

or, equivalently, ([7]) verifies the axioms:

(BCK-1) $((z * y) * (z * x)) * (x * y) = 0$,
(BCK-2) $x * 0 = x$,
(BCK-3) $0 * x = 0$
(BCK-4) $x * y = 0 = y * x \Longrightarrow x = y$.

The reversed BCK-algebra is obtained by reversing the operation $*$, that is, by replacing $x * y$ by $y \rightarrow x$ for all x, y. We obtain the following definition:

Definicja 2.7. ([10]) A *reserved BCK-algebra* is an algebra $(A, \rightarrow, 0)$ of type $(2, 0)$ verifying the following axioms:

(BCK-1') $(y \rightarrow x) \cdot ((x \rightarrow z) \rightarrow (y \rightarrow z)) = 0$,

(BCK-2') $0 \to x = x$
(BCK-3') $x \to 0 = 0$,
(BCK-4') $x \to y = 0 = y \to x \Longrightarrow x = y$.

Remark 2.8. R. A. Borzooei and S. Khosravi Shoar [1] defined and investigated dual BCK-algebras. Note that these algebras are the same as reserved BCK-algebras.

Combining Definitions 2.5 and 2.7, we get

Theorem 2.9. *A KU-algebra is in fact a reserved BCK-algebra.*

D. A. Romano introduced the following notion:

Definicja 2.10. ([18]) A *pseudo-KU algebra* is a structure $(A, \leq, \cdot, *, 0)$, where $\leq$ is a binary relation on a set A, $\cdot$ and $*$ are binary operation on A and 0 is an element of A, verifying the following axioms:

(pKU-1) $y \cdot x \leq (x \cdot z) * (y \cdot z)$, $y * x \leq (x * z) \cdot (y * z)$,
(pKU-2) $0 \cdot x = 0 * x = x$,
(pKU-3) $x \leq 0$,
(pKU-4) $(x \leq y$ and $y \leq x) \Longrightarrow x = y$,
(pKU-5) $x \leq y \Longleftrightarrow x \cdot y = 0 \Longleftrightarrow x * y = 0$,

or, equivalently, it is an algebra $(A, \cdot, *, 0)$ of type $(2, 2, 0)$ verifying the axioms:

(pKU-1') $(y \cdot x) * ((x \cdot z) * (y \cdot z)) = 0$, $(y * x) \cdot ((x * z) \cdot (y * z)) = 0$,
(pKU-2') $0 \cdot x = 0 * x = x$,
(pKU-3') $x \cdot 0 = x * 0 = 0$,
(pKU-4') $x \cdot y = 0 = y * x \Longrightarrow x = y$,
(pKU-5') $x \cdot y = 0 \Longleftrightarrow x * y = 0$.

Now we recall the notion of pseudo-BCK algebra introduced by G. Georgescu and A. Iorgulescu as follows:

Definicja 2.11. ([6]) A *pseudo-BCK algebra* is a structure $A = (A, \leq, *, \circ, 0)$, where $\leq$ is a binary relation on A, $*$ and $\circ$ are binary operations on A and 0 is an element of A, verifying the axioms:

(A-1) $(z * y) \circ (z * x) \leq x * y$, $\quad (z \circ y) * (z \circ x) \leq x \circ y$,
(A-2) $0 \leq x$,
(A-3) $(x \leq y$ and $y \leq x) \implies x = y$,
(A-4) $x \leq y \Leftrightarrow x * y = 0 \Longleftrightarrow x \circ y = 0$,
(A-5) $x * (x \circ y) \leq y$, $\quad x \circ (x * y) \leq y$,
(A-6) $x \leq x$.

Lemma 2.12. (Theorem 1.6 (9) of [6]) *A pseudo-BCK algebra satisfies*

(A-7) $x * 0 = x = x \circ 0$.

Proposition 2.13. *A structure* $\mathcal{A} = (A, \leq, *, \circ, 0)$ *is a pseudo-BCK algebra if and only if it satisfies* (A-1)–(A-4) *and* (A-7).

Proof. Let $\mathcal{A}$ satisfy (A-1)–(A-4) and (A-7). Putting $y = 0$ in (A-1), we get $(z*0)\circ(z*x) \leq x * 0$ and $(z \circ 0) * (z \circ x) \leq x \circ 0$. Applying (A-7), we obtain $z \circ (z * x) \leq x$ and $z * (z \circ x) \leq x$, that is, (A-5) holds. To prove (A-6), we first put $y = 0$ in (A-5). Then $x * (x \circ 0) \leq 0$. Hence, using (A-7), we have $x * x \leq 0$. By (A-2), $0 \leq x * x$. From (A-3) we conclude that $x * x = 0$. Applying (A-4), we see that $x \leq x$. Thus $\mathcal{A}$ satisfies (A-6).

Conversely, let $\mathcal{A}$ be a pseudo-BCK algebra. By definition and Lemma 2.12, (A-1)–(A-4) and (A-7) hold in $\mathcal{A}$. $\square$

According to the above proposition, we say that $(A, *, \circ, 0)$ is a pseudo BCK-algebra if it verifies

(pBCK-1) $((z * y) \circ (z * x)) \circ (x * y) = 0$, $((z \circ y) * (z \circ x)) * (x \circ y) = 0$,

(pBCK-2) $x * 0 = x = x \circ 0$,

(pBCK-3) $0 * x = 0 = 0 \circ x$,

(pBCK-4) $x * y = 0 = y \circ x \Longrightarrow x = y$,

(pBCK-5) $x * y = 0 \Longleftrightarrow x \circ y = 0$.

The reversed pseudo-BCK algebra is obtained by reversing the operations $*$ and $\circ$, i.e., by replacing $x * y$ by $y \to x$ and $x \circ y$ by $y \rightsquigarrow x$, for all x, y (see [9] or [10]). We have

Definicja 2.14. A *reversed pseudo-BCK algebra* is an algebra $(A, \to, \rightsquigarrow, 0)$ of type $(2, 2, 0)$ verifying the following the axioms:

(pBCK-1') $(y \to x) \rightsquigarrow ((x \to z) \rightsquigarrow (y \to z)) = 0$

$(y \rightsquigarrow x) \to ((x \rightsquigarrow z) \to (y \rightsquigarrow z)) = 0$,

(pBCK-2') $0 \to x = x = 0 \rightsquigarrow x$,

(pBCK-3') $x \to 0 = 0 = x \rightsquigarrow 0$,

(pBCK-4') $x \to y = 0 = y \rightsquigarrow x \Longrightarrow x = y$,

(pBCK-5') $x \to y = 0 \Longleftrightarrow x \rightsquigarrow y = 0$.

Combaining Definitions 2.10 and 2.14, we obtain

Theorem 2.15. *A pseudo-KU algebra is in fact a reversed pseudo-BCK algebra.*

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T-fuzzy G-submodules

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Abstract In this paper, we define fuzzy G-submodules under t-norms and obtain some results about them. Next, we investigate the relationship between them and submodules. Later, we prove that the intersection, sum and direct sum of them is also T-fuzzy G-module under t-norm. Finally, we investigate them under G-module homomorphisms.

Keywords Theory of modules, groups, homomorphism, fuzzy set theory, norms, direct sums.
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§1. Introduction

In mathematics and abstract algebra, group theory studies the algebraic structures known as groups. The concept of a group is central to abstract algebra: other well-known algebraic structures, such as rings, fields, and vector spaces, can all be seen as groups endowed with additional operations and axioms. Groups recur throughout mathematics, and the methods of group theory have influenced many parts of algebra. Linear algebraic groups and Lie groups are two branches of group theory that have experienced advances and have become subject areas in their own right. Various physical systems, such as crystals and the hydrogen atom, may be modelled by symmetry groups. Thus group theory and the closely related representation theory have many important applications in physics, chemistry, and materials science. Group theory is also central to public key cryptography. In mathematics, a module is one of the fundamental algebraic structures used in abstract algebra. A module over a ring is a generalization of the notion of vector space over a field, wherein the corresponding scalars are the elements of an arbitrary given ring (with identity) and a multiplication (on the left and/or on the right) is defined between elements of the ring and elements of the module. Thus, a module, like a vector space, is an additive abelian group; a product is defined between elements of the ring and elements of the module that is distributive over the addition operation of each parameter and is compatible with the ring multiplication. Modules are very closely related to the representation theory of groups. They are also one of the central notions of commutative algebra and homological algebra, and are used widely in algebraic geometry and algebraic topology. A vector space (also called a linear space) is a collection of objects called vectors, which may be added together and multiplied ("scaled") by numbers, called scalars. Scalars are often taken to be real numbers, but there are also vector spaces with scalar multiplication by complex numbers, rational numbers, or generally any field. The operations of vector addition and scalar multiplication must satisfy certain requirements. Euclidean vectors are an example of a vector space. In mathematics, given a group G, a G-module is an abelian group M on which G acts compatibly with the abelian group structure on M. This widely applicable notion generalizes that of a representation of G. Group (co)homology provides an important set of tools for studying general G-modules. The term G-module is also used for the more general notion of an R-module on which G acts linearly (i.e. as a group of R-module automorphisms). Representation theory (G-module theory) has had its origin in the 20^{th} century. In the 19^{th} century, groups were generally regarded as subsets of some permutation set, or of the set $GL(V)$ of automorphisms of a vector space V, closed under composition and inverse. Only in the 20^{th} century was the notion of an abstract group formed, making it possible to make a distincton between properties of the abstract group and properties of the particular realizatior as a subgroup of the permutation group or $GL(V)$. Most of the problems in economics, engineering, medical science, environments etc. have various uncertainties. We cannot successfully use classical methods to solve these uncertainties because of various uncertainties typical for those problems. Hence some kinds of theories were given like theory of fuzzy sets, i.e., which we can use as mathematical tools for dealing with uncertainties. In 1965, Zadeh [33] introduced the concept of fuzzy subset as a generalization of the notion of characteristic function in classicalset theory. Shery Fernadez introduced and studied the notion of fuzzy G-modules in [4]. The triangular norm, T-norm, originated from the studies of probabilistic metric spaces in which triangular inequalities were extended using the theory of T-norm. Later, Hohle [6], Alsina et al. [2] introduced the T-norm and the S-norm into fuzzy set theory and suggested that the T-norm be used for the intersection and union of fuzzy sets. Since then, many other researchers have presented various types of T-norms for particular purposes [5, 32]. In practice, Zadeh's conventional T-norm, $\bigwedge$ and $\bigvee$, have been used in almost every design for fuzzy logic controllers and even in the modelling of other decision-making processes. However, some theoretical and experimental studies seem to indicate that other types of T-norms may work better in some situations, especially in the context of decisionmaking processes. The author by using norms, investigated some properties of fuzzy algebraic structures [10] - [29]. Here in this paper, we introduced fuzzy G-submodules under t-norms and some related results like intersection, sum and direct sum of them has also been discussed. Also some of their properties has been investigated under G-module homomorphisms.

§2. Preliminaries

Throughout the paper, $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ will always be rational, real and complex numbers, respectively.

Definition 2.1. (See [8]) Let R be a ring. A commutative group $(M, +)$ is called a left R-module or a left module over R with respect to a mapping

$$. : R \times M \to M$$

if for all $r, s \in R$ and $m, n \in M$,
(1) $r.(m + n) = r.m + r.n$,
(2) $r.(s.m) = (rs).m$,

(3) $(r+s).m = r.m + s.m$.
If R has an identity 1 and if $1.m = m$ for all $m \in M$, then M is called a unitary or unital left R-module.
A right R-module can be defined in a similar fashion.

Definition 2.2. (See [8]) Let M be an R-module and N be a nonempty subset of M. Then N is called a submodule of M if N is a subgroup of M and for all $r \in R, a \in N$, we have $ra \in N$.

Definition 2.3. (See [3]) Let G be a finite group. A vector space M over a field K is called a G-module if for every $g \in G$ and $m \in M$, there exist a product (called the action of G on M) $m.g \in M$ satisfyirlg the following axioms:
(1) $m.1_G = m, \forall m \in M$ (1_G being the identity element in G),
(2) $m.(g.h) = (m.g).h$, $\forall m \in M : g, h \in G$ and
(3) $(k_1m_1 + k_2m_2).g = k_1(m_1.g) + k_2(m_2.g)$ $\forall m_1, m_2 \in M : g \in G : k_1, k_2 \in K$.

Example 2.1. Let $G = \{1, -1, i, -i\}$ and $M = \mathbb{C}^n$ with $n \geq 1$. Then M is a vector spacc over $\mathbb{C}$ and under the usual addition and multiplication of complex numbers, we can show that M is a G-module.

Remark 2.1. The operation $(m, g) \to m.g$ defined above may be called a right-action of G on M and M may be said to be a right G-module. In a similar way, we can define left-action and left G-module. We shall consider all G-modules as left G-modules.

Definition 2.4. (See [3]) Let M be a G-module. A vector subspaee N of M is a G-submodule if N is also a G-module under the same action of G. Thus N is G-submodule of G-module M if and only if N is submodule of M and N be a G-module.

Example 2.2. Let $\mathbb{Q}$ be the field of rationals and $G = \{1, -1\}$ and $M = \mathbb{R}$. Then M is a G-module over $\mathbb{Q}$. Now for each $r \notin \mathbb{Q}$ we get that $N = \mathbb{Q}(r)$ is a G-submodule of M.

Definition 2.5. (See [7]) Let M and N be G-modules. A mapping $f : M \to M$ is a G-module homomorphism if
(1) $f(k_1m_1 + k_2m_2) = k_1f(m_1) + k_2f(m_2)$
(2) $f(gm) = gf(m)$
for all $m_1, m_2m \in M$ and $k_1, k_2 \in K$ and $g \in G$.

Definition 2.6. (See [9]) Let X a non-empty sets. A fuzzy subset μ of X is a function $\mu : X \to [0, 1]$. Denote by $[0, 1]^X$, the set of all fuzzy subset of X.

Definition 2.7. (See [31]) A fuzzy set μ of a non-empty set M is a mapping $\mu : M \to [0, 1]$. For any $\alpha \in [0, 1]$, the set $U(\mu, \alpha) = \{x \in M : \mu(x) \geq \alpha\}$ and $L(\mu, \alpha) = \{x \in M : \mu(x) \leq \alpha\}$ are, respectively, called the upper α-level cut and the lower α-level cut of μ.

Definition 2.8. (See [9]) Let f be a mapping from R-module M into R-module N. Let $\mu \in [0, 1]^M$ and $\nu \in [0, 1]^N$. Define $f(\mu) \in [0, 1]^N$ and $f^{-1}(\nu) \in [0, 1]^M$ as $\forall y \in N$, $f(\mu)(y) = \sup\{\mu(x) \mid x \in M, f(x) = y\}$ if $f^{-1}(y) \neq \emptyset$ and $f(\mu)(y) = 0$ if $f^{-1}(y) = \emptyset$. Also $\forall x \in M$, $f^{-1}(\nu)(x) = \nu(f(x))$.

Definition 2.9. (See [1]) A t-norm T is a function $T : [0, 1] \times [0, 1] \to [0, 1]$ having the following four properties:
(T1) $T(x, 1) = x$ (neutral element),
(T2) $T(x, y) \leq T(x, z)$ if $y \leq z$ (monotonicity),

(T3) $T(x, y) = T(y, x)$ (commutativity),
(T4) $T(x, T(y, z)) = T(T(x, y), z)$ (associativity),
for all $x, y, z \in [0, 1]$.
We say that T be idempotent if $T(x, x) = x$ for all $x \in [0, 1]$.

It is clear that if $x_1 \geq x_2$ and $y_1 \geq y_2$, then $T(x_1, y_1) \geq T(x_2, y_2)$.

Example 2.3. (1) Standard intersection T-norm $T_m(x, y) = \min\{x, y\}$.
(2) Bounded sum T-norm $T_b(x, y) = \max\{0, x + y - 1\}$.
(3) algebraic product T-norm $T_p(x, y) = xy$.
(4) Drastic T-norm

$$T_D(x, y) = \begin{cases} y & \text{if } x = 1 \\ x & \text{if } y = 1 \\ 0 & \text{otherwise.} \end{cases}$$

(5) Nilpotent minimum T-norm

$$T_{nM}(x, y) = \begin{cases} \min\{x, y\} & \text{if } x + y > 1 \\ 0 & \text{otherwise.} \end{cases}$$

(6) Hamacher product T-norm

$$T_{H_0}(x, y) = \begin{cases} 0 & \text{if } x = y = 0 \\ \frac{xy}{x+y-xy} & \text{otherwise.} \end{cases}$$

The drastic t-norm is the pointwise smallest t-norm and the minimum is the pointwise largest t-norm: $T_D(x, y) \leq T(x, y) \leq T_{\min}(x, y)$ for all $x, y \in [0, 1]$.

Lemma 2.1. (See [1]) Let T be a t-norm. Then

$$T(T(x, y), T(w, z)) = T(T(x, w), T(y, z)),$$

for all $x, y, w, z \in [0, 1]$.

Definition 2.10. (See [30]) The intersection of fuzzy subsets μ_1 and μ_1 in a set X with respect to a t-norm T we mean the fuzzy subset $\mu = \mu_1 \cap \mu_2$ in the set X such that for any $x \in X$

$$\mu(x) = (\mu_1 \cap \mu_2)(x) = T(\mu_1(x), \mu_2(x)).$$

§3. Main Results

Definition 3.1. Let G be a finite group and M be a G-module over K, which is a subfield of $\mathbb{C}$. Then a fuzzy G-module on M under t-norm T (T-fuzzy G-submodule of M) is a fuzzy subset $\mu : M \to [0, 1]$ such that
(1) $\mu(ax + by) \geq T(\mu(x), \mu(y))$
(2) $\mu(gm) \geq \mu(m)$

for all $a, b \in K : x, y \in M : m \in M$ and $g \in G$.
Denote by $TFG(M)$, the set of all T-fuzzy G-submodules of M.

Example 3.1. Let $G = \{1, -1\}$ and $M = \mathbb{R}^4$ is a vector space over real field $\mathbb{R}$. Then M is a G-module over $\mathbb{R}$. Define $\mu : M \to [0, 1]$ by

$$\mu(x) = \begin{cases} 1 & \text{if } x_i = 0, \forall i \\ 0.60 & \text{if atleast one } x_i = 0 \end{cases}$$

where $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ such that $x_i \in \mathbb{R}$. If T be standard intersection t-norm $T(a, b) = T_m(a, b) = \min\{a, b\}$ for all $a, b \in [0, 1]$, then $\mu \in TFG(M)$.

Example 3.2. Let F be a field, K be an extension field of F and $a \in K$. Let $F(a)$ be the field obtained by adjoining a to F as $F(a) = \{b_0 + b_1 a + b_2 a^2 + ...\}$ with $b_i \in F$. If $G = (a)$, be the cyclic group generated by a, then $M = F(a)$ will be G-module. Define $\mu : M \to [0, 1]$ by

$$\mu(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0.5 & \text{if } x \in F - \{0\} \\ 0.25 & \text{if } x \in F(a) - F. \end{cases}$$

Let T be bounded sum t-norm $T(a, b) = T_b(a, b) = \max\{0, a + b - 1\}$ for all $a, b \in [0, 1]$ then $\mu \in TFG(M)$.

Example 3.3. Consider the G-module $M = \mathbb{C}$ over the field $\mathbb{R}$ where $G = \{\pm 1\}$
Define $\mu : M \to [0, 1]$ by

$$\mu(z) = \begin{cases} 1 & \text{if } z = 0 \\ 0.65 & \text{if } z \in \mathbb{R} - \{0\} \\ 0.45 & \text{if } z \in \mathbb{C} - \mathbb{R}. \end{cases}$$

Let T be algebraic product t-norm $T(x, y) = T_p(x, y) = xy$ for all $x, y \in [0, 1]$ then $\mu \in TFG(M)$.

Proposition 3.1. Let M be a G-module over K and μ be a fuzzy set of M. If $\mu \in TFG(M)$ and T be idempotent t-norm, then $U(\mu, \alpha)$ will be G-submodule of M.

Proof. If $U(\mu, \alpha) = \emptyset$, then nothing to prove. Therefore, suppose that $U(\mu, \alpha) \neq \emptyset$, and let $x, y \in U(\mu, \alpha)$ and $a, b \in K$. Then $\mu(x) \geq \alpha$ and $\mu(y) \geq \alpha$ and as $\mu \in TFG(M)$ so $\mu(ax + by) \geq T(\mu(x), \mu(y)) \geq T(\alpha, \alpha) = \alpha$ and then $\mu(ax + by) \geq \alpha$ so $ax + by \in U(\mu, \alpha)$. Also $\mu(gx) \geq \mu(x) \geq \alpha$ and then $gx \in U(\mu, \alpha)$. Thus $U(\mu, \alpha)$ is G-submodule of M. □

Proposition 3.2. Let $\mu_1, \mu_2 \in TFG(M)$. Then $(\mu_1 \cap \mu_2) \in TFG(M)$.

Proof. Let $x, y \in M$ and $a, b \in K$ and $g \in G$.
(1)

$$\begin{aligned} (\mu_1 \cap \mu_2)(ax + by) &= T(\mu_1(ax + by), \mu_2(ax + by)) \\ &\geq T(T(\mu_1(x), \mu_1(y)), T(\mu_2(x), \mu_2(y))) \end{aligned}$$

$$= T(T(\mu_1(x), \mu_2(x)), T(\mu_1(y), \mu_2(y))) \quad \textit{(from Lemma 2.1)}$$
$$= T((\mu_1 \cap \mu_2)(x), (\mu_1 \cap \mu_2)(y)).$$

(2)
$$(\mu_1 \cap \mu_2)(gx) = T(\mu_1(gx), \mu_2(gx)) \geq T(\mu_1(x), \mu_2(x)) = (\mu_1 \cap \mu_2)(x).$$

Thus $(\mu_1 \cap \mu_2) \in TFG(M)$. □

Corollary 3.1. Let $\{\mu_i \mid i \in I_n = 1, 2, ..., n\} \subseteq TFG(M)$. Then so is $\cap_{i \in I_n} \mu_i$.

Proposition 3.3. Let $f : M \to N$ be a G-module epimorphism. If $\mu \in TFG(M)$, then $f(\mu) \in TFG(N)$.

Proof. Let $y_1, y_2 \in N$ and $a, b \in K$.
(1)

$$\begin{aligned}
f(\mu)(ay_1 + by_2) &= \sup\{\mu(ax_1 + bx_2) \mid x_1, x_2 \in M, f(ax_1) = ay_1, f(bx_2) = by_2\} \\
&= \sup\{\mu(ax_1 + bx_2) \mid x_1, x_2 \in M, af(x_1) = ay_1, bf(x_2) = by_2\} \\
&\geq \sup\{T(\mu(x_1), \mu(x_2)) \mid x_1, x_2 \in M, f(x_1) = y_1, f(x_2) = y_2\} \\
&= T(\sup\{\mu(x_1) \mid f(x_1) = y_1\}, \sup\{\mu(x_2) \mid f(x_2) = y_2\}) \\
&= T(f(\mu)(y_1), f(\mu)(y_2)).
\end{aligned}$$

(2) Let $y \in N$ and $g \in G$.

$$\begin{aligned}
f(\mu)(gy) &= \sup\{\mu(gx) \mid x \in M, f(gx) = gy\} \\
&= \sup\{\mu(gx) \mid x \in M, gf(x) = gy\} \\
&\geq \sup\{\mu(x) \mid x \in M, f(x) = y\} \\
&= f(\mu)(y).
\end{aligned}$$

Therefore $f(\mu) \in TFG(N)$. □

Proposition 3.4. Let $f : M \to N$ be a G-module homomorphism. If $\nu \in TFG(N)$, then $f^{-1}(\nu) \in TFG(M)$.

Proof. Let $x_1, x_2 \in M$ and $a, b \in K$. Then
(1)

$$\begin{aligned}
f^{-1}(\nu)(ax_1 + bx_2) &= \nu(f(ax_1 + bx_2)) \\
&= \nu(f(ax_1) + f(bx_2)) \\
&= \nu(af(x_1) + bf(x_2)) \\
&\geq T(\nu(f(x_1), \nu(f(x_2))) \\
&= T(f^{-1}(\nu)(x_1), f^{-1}(\nu)(x_2)).
\end{aligned}$$

(2) Let $x \in M$ and $g \in G$. Then

$$f^{-1}(\nu)(gx) = \nu(f(gx)) = \nu(gf(x)) \geq \nu(f(x)) = f^{-1}(\nu)(x).$$

Hence $f^{-1}(\nu) \in TFG(M)$. □

Definition 3.2. The sum of two $\mu_1, \mu_2 \in TFG(M)$ is defined as follows:

$$(\mu_1 + \mu_2)(x) = \sup\{T(\mu_1(y), \mu_2(z)) \mid x = y + z \in M\}.$$

Proposition 3.5. Let $\mu_1, \mu_2 \in TFG(M)$. Then $(\mu_1 + \mu_2) \in TFG(M)$.

Proof. (1) Let $x_1, x_2, y_1, y_2, z_1, z_2 \in M$ and $a, b \in K$. Then
$(\mu_1 + \mu_2)(ax_1 + bx_2)$

$$= \sup\{T(\mu_1(ay_1 + by_2), \mu_2(az_1 + bz_2)) \mid ax_1 + bx_2 = ay_1 + by_2 + az_1 + bz_2\}$$

$$\geq \sup\{T(T(\mu_1(y_1), \mu_1(y_2)), T(\mu_2(z_1), \mu_2(z_2))) \mid ax_1 + bx_2 = ay_1 + az_1 + by_2 + bz_2\}$$

$$= \sup\{T(T(\mu_1(y_1), \mu_1(y_2)), T(\mu_2(z_1), \mu_2(z_2))) \mid ax_1 = ay_1 + az_1, bx_2 = by_2 + bz_2\}$$

$$= \sup\{T(T(\mu_1(y_1), \mu_1(y_2)), T(\mu_2(z_1), \mu_2(z_2))) \mid x_1 = y_1 + z_1, x_2 = y_2 + z_2\}$$

$$(\ from\ Lemma\ 2.1\ )$$

$$= \sup\{T(T(\mu_1(y_1), \mu_2(z_1)), T(\mu_1(y_2), \mu_2(z_2))) \mid x_1 + x_2 = y_1 + z_1 + y_2 + z_2\}$$

$$= T(\sup\{T(\mu_1(y_1), \mu_2(z_1)) \mid x_1 = y_1 + z_1)\}, \sup\{T(\mu_1(y_2), \mu_2(z_2)) \mid x_2 = y_2 + z_2\})$$

$$= T((\mu_1 + \mu_2)(x_1), (\mu_1 + \mu_2)(x_2)).$$

(2) Let $x, y, z \in M$ and $g \in G$.

$$\begin{aligned}(\mu_1 + \mu_2)(gx) &= \sup\{T(\mu_1(gy), \mu_2(gz)) \mid gx = gy + gz\} \\ &\geq \sup\{T(\mu_1(y), \mu_2(z)) \mid x = y + z\} \\ &= (\mu_1 + \mu_2)(x).\end{aligned}$$

Therefore $(\mu_1 + \mu_2) \in TFG(M)$. □

Proposition 3.6. Let M be a G-module and N be a subset of M. Let

$$\mu(x) = \begin{cases} 1 & \text{if } x \in N \\ \alpha & \text{if } x \notin N \end{cases}$$

with $\alpha \in [0, 1)$. Then $\mu \in TFG(M)$ if and only if N is a G-submodule of M.

Proof. Let $\mu \in TFG(M)$ and we prove that N is a submodule of M. Let $x, y \in N \subseteq M$ and $a, b \in K$. Now

$$\mu(ax+by) \geq T(\mu(x), \mu(y)) = T(1,1) = 1$$

so $\mu(ax+by) = 1$ and then $ax+by \in N$.
Also let $g \in G$ and then $\mu(gx) \geq \mu(x) = 1$ so $\mu(gx) = 1$ and then $gx \in N$.
Therefore N is a submodule of M and since N be a subset of M so N will be N is a G-submodule of M.
Conversely, let N is a submodule of M and we prove that $\mu \in TFG(M)$. Suppose $x, y \in M$ and $a, b \in K$ and we investigate the following conditions:
(1) If $x, y \in N$, then

$$\mu(ax+by) = 1 \geq 1 = T(1,1) = T(\mu(x), \mu(y)).$$

(2) For any $x \in N$ and $y \notin N$ then $ax+by \notin N$ and so

$$\mu(ax+by) = \alpha \geq 0 = T(1,0) = T(\mu(x), \mu(y)).$$

(3) Let $x \notin N$ and $y \in N$ then $ax+by \notin N$ and then

$$\mu(ax+by) = \alpha \geq 0 = T(0,1) = T(\mu(x), \mu(y)).$$

(4) Finally, if $x, y \notin N$, $ax+by \notin N$ and so

$$\mu(ax+by) = \alpha \geq 0 = T(0,0) = T(\mu(x), \mu(y)).$$

Therefore from (1)-(4) we have that

$$\mu(ax+by) \geq T(\mu(x), \mu(y)).$$

Now let $x \in M$ and $g \in G$. Then we have:
(1) If $x \in N$ then $gx \in N$ and then $\mu(gx) = 1 \geq \mu(x)$.
(2) If $x \notin N$, then $gx \notin N$ and so $\mu(gx) = 0 \geq 0 = \mu(x)$.
Therefore from (1) and (2) we have that $\mu(gx) \geq \mu(x)$.
Hence $\mu \in TFG(M)$. □

Proposition 3.7. Any n-dimensional G-module M has a T-fuzzy G-module μ with $\mid \mu \mid = n+1$ where $\mid \mu \mid$ is called level cardinality of μ.

Proof. Let $B = \{m_1, m_2, ...m_n\}$ hc the basis for M. Then $\mu : M \to [0,1]$ with

$$\mu(c_1m_1+c_2m_2+...+c_nm_n)=\begin{cases} 1 & \text{if } c_1=c_2=...=c_n=0 \\ \frac{1}{2} & \text{if } c_1\neq 0, c_2=c_3=...=c_n=0 \\ \frac{1}{3} & \text{if } c_2\neq 0, c_3=...=c_n=0 \\ \frac{1}{4} & \text{if } c_3\neq 0, c_4=c_5=...=c_n=0 \\ . & \\ . & \\ . & \\ . & \\ \frac{1}{n+1} & \text{if } c_n\neq 0 \end{cases}$$

then $\mu \in TFG(M)$ with $|\ \mu\ |= n+1$. □

Example 3.4. Let $G=\{\pm 1\}$ and $M=\mathbb{R}^4$ be G-module over field $K=\mathbb{R}$. Let $B=\{m_1=(1,0,0,0), m_2=(0,1,0,0), m_3=(0,0,1,0), m_4=(0,0,0,1)\}$ be the standard ordered basis for M. Define $\mu : M \to [0,1]$ by

$$\mu(c_1m_1+c_2m_2+...+c_nm_n)=\begin{cases} 1 & \text{if } c_1=c_2=c_3=c_4=0 \\ \frac{1}{2} & \text{if } c_1\neq 0, c_2=c_3=c_4=0 \\ \frac{1}{3} & \text{if } c_2\neq 0, c_3=c_4=0 \\ \frac{1}{4} & \text{if } c_3\neq 0, c_4=0 \\ \frac{1}{5} & \text{if } c_4\neq 0. \end{cases}$$

Let T be algebraic product t-norm $T(x,y)=T_p(x,y)=xy$ for all $x,y\in[0,1]$ then $\mu\in TFG(M)$ with $|\ \mu\ |=5$.

Remark 3.1. The above construction can be extended to infinite dimensional G-modules also.

Proposition 3.8. Let M be a G-module over K and $M=\oplus_{i=1}^{n}M_i$, where M_i are G-submodules of M. Define $\mu : M=\oplus_{i=1}^{n}M_i \to [0,1]$ by $\mu(m=\sum_i^n m_i)=\bigwedge\{\mu_i(m_i) : i=1,2,3,...,n\}$ where $\bigwedge$ denote minimum [infimum]. If $\mu_i\in TFG(M_i)$, then $\mu\in TFG(M)$.

Proof. Let $x=\sum_i^n m_i$ and $y=\sum_j^n m_j$ and $a,b\in K$ and $g\in G$. Then
(1)

$$\begin{aligned} \mu(ax+by) &= \mu(a\sum_i^n m_i + b\sum_j^n m_j) \\ &= \mu(\sum_i^n am_i + \sum_j^n bm_j) \\ &= \bigwedge\{\mu_i(am_i+bm_j) : i,j=1,2,3,...,n\} \\ &\geq \bigwedge\{T(\mu_i(m_i,m_j) : i,j=1,2,3,...,n\} \qquad (Since \quad \mu_i\in TFG(M_i)) \end{aligned}$$

$$= T(\bigwedge\{\mu_i(m_i) : i = 1, 2, 3, ..., n\}, \bigwedge\{\mu_i(m_j) : j = 1, 2, 3, ..., n\})$$
$$= T(\mu(x), \mu(y)).$$

(2)

$$\begin{aligned}
\mu(gx) &= \mu(g\sum_{i}^{n} m_i) \\
&= \mu(\sum_{i}^{n} gm_i) \\
&= \bigwedge\{\mu_i(gm_i) : i = 1, 2, 3, ..., n\} \\
&\geq \bigwedge\{\mu_i(m_i) : i = 1, 2, 3, ..., n\} \qquad (Since \quad \mu_i \in TFG(M_i)) \\
&= \bigwedge\{\mu_i(m_i) : i = 1, 2, 3, ..., n\} \\
&= \mu(x).
\end{aligned}$$

Therefore $\mu \in TFG(M)$. □

Remark 3.2. In thc above proposition, if $\mu_i(0)$ are all equal then we have $\mu(0) = \bigwedge\{\mu_i(0) : i = 1, 2, 3, ..., n\} = \mu_i(0)$ for all i.

Definition 3.3. The T-fuzzy G-module μ on $M = \oplus_{i=1}^{n} M_i$, in Proposition 3.8 with $\mu(0) = \mu_i(0)$ for all i is called the direct sum of the T-fuzzy G-modules μ_i, and is denoted by $\mu = \oplus_{i=1}^{n} \mu_i$.

Example 3.5. Let $G = \{\pm 1\}$ and $M = \mathbb{C}$ over $\mathbb{R}$. Then M is a G-module. We have $M = M_1 \oplus M_2$, where $M_1 = \mathbb{R}$ and $M_2 = i\mathbb{R}$. Let T be algebraic product t-norm $T(x, y) = T_p(x, y) = xy$ for all $x, y \in [0, 1]$. Define $\mu : M \to [0, 1]$ as

$$\mu(x + iy) = \begin{cases} 1 & \text{if } x = y = 0 \\ \frac{1}{2} & \text{if } x \neq 0, y = 0 \\ \frac{1}{3} & \text{if } y \neq 0, \end{cases}$$

then $\mu \in TFG(M)$. Also define
$\mu_1 : M_1 \to [0, 1]$ as

$$\mu_1(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{2} & \text{if } x \neq 0, \end{cases}$$

and
$\mu_2 : M_2 \to [0, 1]$ as

$$\mu_2(y) = \begin{cases} 1 & \text{if } y = 0 \\ \frac{1}{3} & \text{if } y \neq 0, \end{cases}$$

then $\mu_1 \in TFG(M_1)$ and $\mu_2 \in TFG(M_2)$. Also we obtain that $\mu = \mu_1 \oplus \mu_2$.

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