SCIENTIA MAGNA

An international journal

Edited by School of Mathematics
Northwest University, P.R.China

Vol. 16, No. 1, 2021 ISSN 1556-6706

SCIENTIA MAGNA

An international journal

Edited by

School of Mathematics

Northwest University

Xi'an, Shaanxi, China

Information for Authors

Scientia Magna is a peer-reviewed, open access journal that publishes original research articles in all areas of mathematics and mathematical sciences. However, papers related to Smarandache's problems will be highly preferred.

The submitted manuscripts may be in the format of remarks, conjectures, solved/unsolved or open new proposed problems, notes, articles, miscellaneous, etc. Submission of a manuscript implies that the work described has not been published before, that it is not under consideration for publication elsewhere, and that it will not be submitted elsewhere unless it has been rejected by the editors of Scientia Magna.

Manuscripts should be submitted electronically, preferably by sending a PDF file to ScientiaMagna@hotmail.com.

On acceptance of the paper, the authors will also be asked to transmit the TeX source file. PDF proofs will be e-mailed to the corresponding author.

Contents

Scientia Magna
Vol. 16 (2021), No. 1, 1-34

A survey on Smarandache notions in number theory: proper divisor product, sieve sequences, integer part sequences, pseudo-odd, pseudo-even and related sequences

Yuchan Qi

School of Mathematics, Northwest University
Xi'an 710127, China
E-mail: ycqimath@163.com

Abstract In this paper we give a survey on Smarandache divisor product sequences, sieve sequences, integer part sequences, Smarandache pseudo-odd, pseudo-even and pseudo-multiples sequences, Smarandache kn-digital sequences and related sequences.

Keywords proper divisor product, sieve sequences, integer part, pseudo-odd, pseudo-even, pseudo-multiples, kn-digital.
2010 Mathematics Subject Classification 11B83, 11N25, 11N37.

§1. Introduction and preliminaries

Number Theory is one of the oldest of mathematical disciplines. Problems are often easy to state, but extremely difficult to solve, which is the origin of much of their charm. People have always been interested in the "purity" of the integers. Divisibility is the backbone of number theory, which has a close relationship with prime factorizations, therefore the values of number theoretic functions can often be computed by formulas based on the prime factorization. For example: The Euler phi function $\Phi(n)$ is the number of integers m, where $1 \leq m \leq n$ and m and n are relatively prime. Sum of divisors function σ(n) is the sum of all the positive divisors of n.

In the 1970 's, Florentin Smarandache created a new function in number theory. Let $S(n)$, for $n \in \mathbb{N}^+$ denote the Smarandache function, then $S(n)$ is defined as the smallest $m \in \mathbb{N}^+$, with $n \mid m!$. The consequences of this simple definition encompass many areas of mathematics. This paper is a survey on Smarandache divisor product sequences, Smarandache sieve sequences, integer part sequences, LCM ratio sequences, Smarandache pseudo-odd, pseudo-even and pseudo-multiples sequences, simple sequences, primitive numbers, Smarandache kn-digital sequences.

§2. Smarandache divisor product sequences

F.Smarandache introduced the function $P_d(n) := \prod_{d|n} d$ in Problem 25[1]. For example, $P_d(1) = 1, P_d(2) = 2, P_d(3) = 3\ P_d(4) = 8, \cdots, P_d(p) = p, \cdots$. $q_d(n)$ denotes the product of all proper divisors of n. That is, $q_d(n) = \prod_{d|n, d<n} d$. For example, $q_d(1) = 1, q_d(2) = 1, q_d(3) = 1, q_d(4) = 2, \cdots$. In problem 25 and 26 of [1], Professor F. Smarandache asked people to study the properties of the sequences $\{p_d(n)\}$ and $\{q_d(n)\}$. F. Liang studied the properties of the sequences of $\{p_d(n)\}$ and $\{q_d(n)\}$ for k -th divisor sum function and gave two more general results.

F. Liang [28] *Let $n = p^\alpha, p$ be a prime and α be a positive integer. Then for any fixed positive integer k, we have the inequality*

$$\sigma_k\left(\phi\left(p_d(n)\right)\right) \geq \frac{1}{2^k} p_d^k(n),$$

$$\sigma_k\left(\phi\left(q_d(n)\right)\right) \geq \frac{1}{2^k} q_d^k(n),$$

where $\sigma_k(n) = \sum_{d|n} d^k$ is the k -th divisor sum function.

A similar function $P_{sd}(n)$ denotes the product of all square-free divisors of n, i.e.,

$$P_{sd}(n) := \prod_{\substack{d|n \\ \mu(d)\neq 0}} d.$$

Y. Han [15] *We have the asymptotic formula*

$$\sum_{n\leq x} \log P_{sd}(n) = A_1 x \log^2 x + A_2 x \log x + A_3 x + O\left(x^{\frac{1}{2}} \exp\left(-D(\log x)^{\frac{3}{5}} (\log\log x)^{-\frac{1}{5}}\right)\right.$$

where A_1, A_2, A_3 are constants, $D > 0$ is an absolute constant.

Let $1 \leq l \leq q$ be fixed integers. Define a function $P_{ld}(n) := \prod_{\substack{d|n \\ d\equiv l(\bmod q)}} d.$

X. Ma [46] *For any real number $x > 1$, we have*

$$\sum_{n\leq x} \log P_{ad}(n) = \frac{1}{2q} x \log^2 x + \frac{\gamma - 1}{q} x \log x + cx + O((q^{-1}x)^{\frac{27}{82}+\varepsilon}),$$

where c is a constant which depends on q and l, γ is the Euler constant, ε is a fixed small positive constant.

For any positive integer $n \geq 1$, the Smarandache Superior Prime Part $P_p(n)$ is defined as the smallest prime number greater than or equal to n. For example, the first few values of $P_p(n)$ are $P_p(1) = 2, P_p(2) = 2, P_p(3) = 3, P_p(4) = 5, P_p(5) = 5, P_p(6) = 7, P_p(7) = 7, P_P(8) = 11$ $P_p(9) = 11, P_p(10) = 11, P_p(11) = 11, P_p(12) = 13, P_p(13) = 13, P_p(14) = 17, P_p(15) = 17 \cdots$ For any positive integer $n \geq 2$, define the Smarandache Inferior Prime Part $p_p(n)$ as the largest prime number less than or equal to n. Its first few values are $p_p(2) = 2\ p_p(3) = 3, p_p(4) = 3, p_p(5) = 5, p_p(6) = 5, p_p(7) = 7, p_p(8) = 7, p_p(9) = 7, p_p(10) = 7\ p_p(11) = 11, \cdots$. There is a close relationship between the Smarandache prime part and the prime distribution problem. Define

$$I_n = \left\{p_p(2) + p_p(3) + \cdots + p_p(n)\right\} / n,$$

and

$$S_n = \{P_p(2) + P_p(3) + \cdots + P_p(n)\}/n.$$

There are two problems in problem 10 of reference [22]:
(A). If $\lim_{n\to\infty} (S_n - I_n)$ converges or diverges. If it converges, find the limit.
(B). If $\lim_{n\to\infty} \frac{S_n}{I_n}$ converges or diverges. If it converges, find the limit.

The problem (A) is still open. But for the problem (B), X. Yan got asymptotic properties of $\frac{S_n}{I_n}$, and gave a shaper asymptotic formula for it.

X. Yan [65] *For any positive integer $n > 1$, we have the asymptotic formula*

$$\frac{S_n}{I_n} = 1 + O\left(n^{-\frac{1}{3}}\right).$$

From this theorem it can be deduced that the limit S_n/I_n converges as $n \longrightarrow \infty$, and

$$\lim_{n\to\infty} \frac{S_n}{I_n} = 1.$$

This solved the problem B of reference [22].

For any positive integer n, a new additive function $F(n)$ is defined as $F(0) = 0$ and $F(n) = \alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_k p_k$ for $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$.

S. Gou [10] *Let N be a positive integer. Then for any fixed real number $x > 1$, we have*

$$\sum_{n\in x} F\left(P_d(n)\right) = \sum_{i=1}^{N} d_i \cdot \frac{x^2}{\ln^i x} + O\left(\frac{x^2}{\ln^{N+1} x}\right),$$

where $d_1, \cdots, d_N)$ are computable constants and $u_1 = \frac{\pi^4}{72}$. And we also have

$$\sum_{n\in x} F\left(q_d(n)\right) = \sum_{i=1}^{N} h_i \cdot \frac{x^2}{\ln^i x} + O\left(\frac{x^2}{\ln^{N+1} x}\right),$$

where $h_1, \cdots, h_N)$ are computable constants and $h_1 = \frac{\pi^4}{72} - \frac{\pi^2}{12}$.

T. Zhang [82] *For any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{n\leq x} \ln P_d(n) = \frac{1}{2} x \ln^2 x + (C-1) x \ln x - (C-1)x + O\left(x^{\frac{1}{2}} \ln x\right),$$

where C is the Euler constant.

For any real number $x \geq 1$, we have the asymptotic formula

$$\sum_{n\leq x} \ln q_d(n) = x \ln^2 x + (C-2) x \ln x - (C-2)x + O\left(x^{\frac{1}{2}} \ln x\right).$$

W. Zhu [84] *For any real number $x > 1$, we have the asumptotic formula*

$$\sum_{n\leq x} \frac{1}{P_d(n)} = \ln\ln x + C_1 + O\left(\frac{1}{\ln x}\right),$$

where C_1 is a constant. And we also have

$$\sum_{n\leq x}\frac{1}{q_d(n)}=\pi(x)+(\ln\ln x)^2+B\ln\ln x+C_2+O\left(\frac{\ln\ln x}{\ln x}\right),$$

where $\pi(x)$ is the number of all primes $\leq x, B$ and C_2 are constants.

H. Liu and W. Zhang gave the following results.

H. Liu [33] *For any positive integer n, we have the inequality*

$$\sigma\left(\phi\left(P_d(n)\right)\right)\geq\frac{1}{2}P_d(n),$$

where $\phi(k)$ is the Euler's function and $\sigma(k)$ is the divisor sum function. And we also have the inequality

$$\sigma\left(\phi\left(q_d(n)\right)\right)\geq\frac{1}{2}\eta_d(n).$$

M. Le gave a formulae of $P_d(n)$.

M. Le [27] *Let $n=p_1^{r_1}\ldots p_k^{r_k}$ be the factorization of n and let*

$$r(n)=\begin{cases}1/2\,(r_1+1)\ldots(r_k+1) & \text{if } n \text{ is not a square,}\\ 1/2\,((r_1+1)\ldots(r_k+1)-1) & \text{if } n \text{ is a square.}\end{cases}$$

Then we have $P_d(n)=n^{r(n)}$.

§3. Smarandache sieve sequences

The definition of Smarandache irrational root sieve is: from the set of natural numbers (except 0 and 1): take off all powers of $2^k, k\geq 2$; take off all powers of $3^k, k\geq 2$; take off all powers of $5^k, k\geq 2$; take off all powers of $6^k, k\geq 2$; take off all powers of $7^k, k\geq 2$; take off all powers of $10^k, k\geq 2,\cdots$ (take off all k -powers, $k\geq 2$). For example: $2,3,5,6,7,10,$ $11,12,13,14,15,17,18,19\cdots$ are all irrational root sieve sequence. Let A denotes the set of all the irrational root sieve. X. Zhang studied the mean value of the irrational root sieve sequence and gave an interesting asymptotic formula for it.

X. Zhang and Y. Lou [78] *Let $d(n)$ denote the divisor function. Then for any real mumber $x\geq 1$, we have the asymptotic formula*

$$\begin{aligned}\sum_{n\in A}d(n) &= \left(x-\frac{3}{4\pi^2}\sqrt{x}\ln x+A_1x^{\frac{1}{3}}\ln^2 x+A_2x^{\frac{1}{3}}\ln x+A_3x^{\frac{1}{3}}+A_4\sqrt{x}\right)\ln x\\ &\quad +(2\gamma-1)x+A_5\sqrt{x}+A_6x^{\frac{1}{3}}+O\left(x^{\frac{139}{429}+\epsilon}\right),\end{aligned}$$

where ϵ denotes any fixed positive number, γ is the Euler constant, A_1, A_2, A_3 A_4, A_5, A_6 are the computable constants.

Let $\mathcal{A}$ denote the set of all numbers in the k -th power free sieve sequence.

J. Su [51] *Let $k \geq 2$ be a fixed positive integer, A denotes the set of all power k sieve sequence. Then for any real $x > 1$, we have the asymptotic formula*

$$\sum_{\substack{n \leq x \\ n \in A}} 1 = c(k) \cdot x + O\left(x^{\frac{1}{k}}\right),$$

where $c(k) = \prod_{n=2}^{\infty}\left(1 - \frac{1}{n^k}\right)$ is a constant, and $c(2) = \frac{1}{2}$. For any real number $x > 1$, from the Prime Number Theorem we know that there are at most $O\left(\frac{x}{\ln x}\right)$ primes in the interval $[1, x]$, so from the theorem above one can get that there are an infinity of numbers of the power k sieve sequence which are not prime. Therefore, there is an infinity of numbers of the power k sieve sequence which are not prime.

J. Guo and X. Zhao studied the convergent property of some infinite series involving this sequence and gave some interesting identities.

J. Guo and X. Zhao [14] 1. *Let $k \geq 2$ be any positive integer. For any real number $\alpha > 1$, we have the identity:*

$$\sum_{\substack{n=1 \\ n \in \mathcal{A}}}^{\infty} \frac{1}{n^{\alpha}} = \frac{\zeta(\alpha)}{\zeta(k\alpha)},$$

where $\zeta(s)$ denotes the Riemann-zeta function. Moreover, let $\mathcal{B}$ be the set of all numbers in the square free sieve seaquence $\mathcal{C}$ be the set of all numbers in the cubic free sieve sequence. Then

$$\sum_{\substack{n=1 \\ n \in \mathcal{B}}}^{\infty} \frac{1}{n^2} = \frac{15}{\pi^2} \quad \text{and} \quad \sum_{\substack{n=1 \\ n \in C}}^{\infty} \frac{1}{n^2} = \frac{315}{2\pi^4}.$$

2. *Let $k \geq 2$ be any positive integer. For any real number $\alpha > 1$, we have the identity:*

$$\sum_{\substack{n=1 \\ n \in \mathcal{A}}}^{\infty} \frac{d(n)}{n^{\alpha}} = \frac{\zeta^2(\alpha)}{\zeta(k\alpha)} \prod_{p} \left(1 - \frac{k\left(p^{\alpha} - 1\right)}{p^{(k+1)\alpha} - p^{\alpha}}\right).$$

X. Pan and B. Liu [48] *Let $\mathcal{A}$ denote the set of all elements of the irrational root sieve sequence. Then for any real number $x > 1$, we have the asymptotic formula*

$$\sum_{\substack{a \leq x \\ a \in \mathcal{A}}} 1 = x - \frac{6}{\pi^2}\sqrt{x} - \frac{6}{\pi^2} x^{\frac{1}{3}} + O\left(x^{\frac{1}{4}} \cdot \ln x\right).$$

The odd sieve sequence is the sequence which is composed of all odd number that are not equal to the difference of two primes. For example: $7, 11, 19, 23, 25, \cdots$. Let $\mathcal{A}$ denote the set of the odd sieve numbers. W. Yao used analytic method to study the mean value properties of this sequence and gave two interesting asymptotic formulae.

W. Yao [63] 1. *For any positive number $x > 1$, we have the asymptotic formula*

$$\sum_{\substack{n \leq x \\ n \in \mathcal{A}}} n = \frac{x^2}{4} - \frac{x^2}{2 \ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

2. *For any positive number $x > 1$, we have the asymptotic formula*

$$\sum_{\substack{n\le x\\ n\in\mathcal{A}}} \frac{1}{n} = \frac{1}{2}\ln\frac{x}{2} - \ln\ln(x+2) + \frac{1}{2}\gamma - A + B + O\left(\frac{1}{\ln x}\right),$$

where A, B are computable constants, γ is the Euler's constant.

R. Xie [61] *Let φ be Euler function. For any positive number $x > 1$, we have the asymptotic formula*

$$\sum_{\substack{n\le x\\ n\in\mathcal{A}}} \varphi(n) = \frac{3}{\pi^2} + O\left(x^{\frac{3}{2}+\epsilon}\right),$$

where ϵ is any fixed positive number.

L. Qian [50] *Let $\sigma(n)$ denote the sum of positive factor of n: $\sigma(n) = \sum_{d|n} d$. Then for any real number $x \geqslant 1$ we have*

$$\sum_{\substack{n\le x\\ n\in\mathcal{A}}} \sigma_\alpha(n) = \frac{1}{2}\zeta(\alpha+1)g(\alpha+1)x^2 + O\left(x^{\frac{3}{2}+\varepsilon}\right),$$

where $g(\alpha) = \prod_p (1 + (p-1)p^{-s} + (1-p)p^{\alpha-2^s})$, $\zeta(s)$ is Riemann zeta- function and ε is a fixed positive number.

Let p be a prime, $e_p(n)$ denote the largest exponent of power p which is included n. L. Qi studied the mean value properties of $e_p(n)$ acting on the irrational root sieve sequences and gave an asymptotic formula.

L. Qi [49] *Let A denote the set of all elements of the irrational root sieve sequence. For any real number $x \geqslant 1$, we have*

$$\sum_{\substack{n\leqslant x\\ n\in A}} e_p(n) = \frac{1}{p-1}x - \frac{2}{p-1}x^{\frac{1}{2}} - \frac{3}{p-1}x^{\frac{1}{3}} + O\left(x^{\frac{1}{2}+\varepsilon}\right).$$

§4. The Inferior and Superior parts of integers sequences

For any positive integer n, the Smarandache Superior m-th power part sequence SSMP(n) is defined as the smallest m -th power greater than or equal to n. The Smarandache Inferior m -th power part sequence SIMP(n) is defined as the largest m -th power less than or equal to n. For example, if $m = 2$, then the first few terms of SIMP(n) are: $0, 1, 1, 1, 4, 4, 4, 4, 4, 9, \cdots$. The first few terms of SSMP(n) are $1, 4, 4, 4, 9, 9, 9, 9, 9, .16, 16, \cdots$. Now we let

$$\begin{aligned}
S_n &= (\mathrm{SSMP}(1) + \mathrm{SSMP}(2) + \cdots + \mathrm{SSMP}(n))/n\\
I_n &= (\mathrm{SIMP}(1) + \mathrm{SIMP}(2) + \cdots + \mathrm{SIMP}(n))/n\\
K_n &= \sqrt[n]{\mathrm{SSMP}(1) + \mathrm{SSMP}(2) + \cdots + \mathrm{SSMP}(n)}\\
I_n &= \sqrt[n]{\mathrm{SIMP}(1) + \mathrm{SIMP}(2) + \cdots + \mathrm{SIMP}(n)}.
\end{aligned}$$

Y. Wang [58] 1. *Let $m \geq 2$ be an integer, then for any real number $x > 1$, we have the asymptotic formula*

$$\sum_{n \leq x} \mathrm{SSMP}(n) = \frac{x^2}{2} + O\left(x^{\frac{2m-1}{m}}\right)$$

and

$$\sum_{n \leq x} \mathrm{SIMP}(n) = \frac{x^2}{2} + O\left(x^{\frac{2m-1}{m}}\right).$$

2. *For any fixed positive integer $m > 2$ and any positive integer n we have the asymptotic formula*

$$S_n - I_n = \frac{m(m-1)}{2m-1} n^{1-\frac{1}{m}} + O\left(n^{1-\frac{2}{m}}\right),$$

$$\frac{S_n}{I_n} = 1 + O\left(n^{-\frac{1}{m}}\right),$$

and

$$\frac{K_n}{L_n} = 1 + O\left(\frac{1}{n}\right).$$

Y. Yu [74] *For any positive integer n we have*

$$\lim_{n \longrightarrow \infty} S_n - I_n = \infty.$$

Let n be a positive integer, $u_6(n)$ be the largest hexagon number less than or equal to n and $v_6(n)$ be the smallest hexagn number less than n, $a(n)$ and $b(n)$ are complement numbers of $u_6(n)$ and $v_6(n)$ respectively. Define

$$\begin{aligned} S_6(m) &= [a(1) + a(2) + a(3) + \cdots + a(m)]/m = \tfrac{1}{m} \textstyle\sum_{i=1}^{m} a(i), \\ I_6(m) &= [b(1) + b(2) + b(3) + \cdots + b(m)]/m = \tfrac{1}{m} \textstyle\sum_{i=1}^{m} b(i), \\ K_6(m) &= \sqrt[m]{a(1) + a(2) + a(3) + \cdots + a(m)} = \left(\textstyle\sum_{i=1}^{m} a(i)\right)^{\frac{1}{m}}, \\ L_6(m) &= \sqrt[m]{b(1) + b(2) + b(3) + \cdots + b(m)} = \left(\textstyle\sum_{i=1}^{m} b(i)\right)^{\frac{1}{m}}. \end{aligned}$$

W. Huang and Y. Ma [19] 1. *For any positive integer n we have*

$$\frac{S_6(n)}{I_6(n)} = 1 + O\left(n^{-\frac{1}{2}}\right), \lim_{n \to +\infty} \frac{S_6(n)}{I_6(n)} = 1.$$

2. *For any positive integer n we have*

$$\frac{K_6(n)}{L_6(n)} = 1 + O\left(n^{-\frac{1}{2n}}\right), \lim_{n \to +\infty} \frac{K_6(n)}{L_6(n)} = 1, \lim_{n \to +\infty} \left(K_6(n) - L_6(n)\right) = 0.$$

3. *For any positive integer n we have*

$$S_6(n) - I_6(n) = 4 + \sqrt{2} n^{-\frac{1}{2}} + O(1),$$

$$\lim_{n \to +\infty} \left(I_6(n) - S_6(n)\right) = 4, \lim_{n \to +\infty} \left(I_6(n) - S_6(n)\right)^{\frac{1}{n}} = 1.$$

Let n be a positive integer, $a(n)$ be the smallest m factorial number greater than or equal to n, and $b(n)$ be the largest m factorial number less than or equal to n. for example: $a(4) = 3!, a(5) = 3!, a(6) = 3!, a(7) = 4!, \cdots$. $b(1) = 1!, b(2) = 2!, b(3) = 2!, b(4) = 2!, b(5) = 2!, b(6) = 3!, \cdots$. Define

$$S_n(n) = \frac{1}{n}\sum_{i=1}^{n} \log(a(n)), \quad I_n(n) = \frac{1}{n}\sum_{i=1}^{n} \log(b(n)),$$

$$K_n(n) = (\sum_{i=1}^{n} \log(a(n))^{\frac{1}{n}}, \quad L_n(n) = (\sum_{i=1}^{n} \log(b(n))^{\frac{1}{n}}.$$

W. Huang [20] 1. *For any real number $x > 1$ we have*

$$\sum \log(a(n)) = x\log x + O\left(\frac{x\log x \cdot \log\log\log x}{\log\log x}\right),$$
$$\sum_{n\leqslant s} \log(b(n)) = x\log x + O\left(\frac{x\log x \cdot \log\log\log x}{\log\log x}\right).$$

2. *For any positive integer n we have*

$$\frac{S_n(n)}{I_n(n)} = 1 + O\left(\frac{\log n \cdot \log\log\log n}{\log\log n}\right)^{-1}, \lim_{n\to\infty} \frac{S_n(n)}{I_n(n)} = 1,$$

$$\frac{K_n(n)}{L_n(n)} = 1 + O\left(\frac{\log n \cdot \log\log\log n}{\log\log n}\right)^{-\frac{1}{n}}, \lim_{n\to\infty} \frac{K_n(n)}{L_n(n)} = 1, \lim_{n\to\infty} (K_n(n) - L_n(n)) = 0,$$

$$S_n(n) - I_n(n) = \frac{1}{n}\left(m^2(m-1)!\log\log n + O(\log\log m)\right), \lim_{n\to\infty} S_n(n) - I_n(n) = 0.$$

Define $u_3(n) = \max\{\frac{m(m+1)}{2} : n \geq \frac{m(m+1)}{2},\ m \in \mathbb{N}^+\}$, $v_3(n) = \max\{\frac{m(m+1)}{2} : n \leq \frac{m(m+1)}{2},\ m \in \mathbb{N}^+\}$. let $u_3(n)$ denote the partial sequence of the largest triangular numbers less than or equal to n and $v_3(n)$ denote the partial sequence of the smallest triangular numbers greater than or equal to n.

W. Huang [21] 1. *For any real number $x \geq 1$ we have*

$$\sum_{n\leqslant x} u_3(n) = \tfrac{1}{2}x^2 + O\left(x^{\frac{3}{2}}\right),$$
$$\sum_{n\leqslant x} v_3(n) = \tfrac{1}{2}x^2 + O\left(x^{\frac{3}{2}}\right).$$

2. *For any positive number n we have*

$$\frac{S_3(n)}{I_3(n)} = 1 + O\left(n^{-\frac{1}{2}}\right), \lim_{n\to\infty} \frac{S_3(n)}{I_3(n)} = 1,$$

$$\frac{K_3(n)}{L_3(n)} = 1 + O\left(n^{-\frac{1}{2n}}\right), \lim_{n\to\infty} \frac{K_3(n)}{L_3(n)} = 1, \lim_{n\to\infty} (K_3(n) - L_3(n)) = 0,$$

$$S_3(n) - I_3(n) = 2\sqrt{2}n^{\frac{1}{2}} + O(1),$$
$$\lim_{n\to\infty} \frac{I_3(n) - S_3(n)}{n^{\frac{1}{2}}} = 2\sqrt{2}, \quad \lim_{n\to\infty} (I_3(n) - S_3(n))^{\frac{1}{n}} = 1.$$

For any positive integer n, the Smarandache Superior Square Part $SP(n)$ is the smallest square greater than or equal to n, the Smarandache Inferior Square Part $IP(n)$ is the largest square less than or equal to n. Define

$$S_n = (SP(1) + SP(2) + \cdots + SP(n))/n;$$
$$I_n = (IP(1) + IP(2) + \cdots + IP(n))/n;$$
$$K_n = \sqrt[n]{SP(1) + SP(2) + \cdots SP(n)};\ L_n = \sqrt[n]{IP(1) + IP(2) + \cdots IP(n)}$$

M. Yang [67] 1. *For any real number $x > 1$, we have*

$$\sum_{n\leq x} \Omega(IP(n)) = 2x \ln\ln x + Cx + O\left(\frac{x}{\ln x}\right),$$

where Ω is the function of the number of prime factors.

2. *For any real number $x > 1$, we have*

$$\sum_{n\leq x} \Omega(SP(n)) = 2x \ln\ln x + Cx + O\left(\frac{x}{\ln x}\right),$$

where Ω is the function of the number of prime factors.

F. Li [30] 1. *For any real number $x > 2$, we have*

$$\sum_{n\leqslant x} SP(n) = \frac{x^2}{2} + O\left(x^{\frac{3}{2}}\right); \sum_{n\leqslant x} IP(n) = \frac{x^2}{2} + O\left(x^{\frac{3}{2}}\right).$$

2. *For any integer n, we have*

$$\frac{S_n}{I_n} = 1 + O\left(n^{-\frac{1}{2}}\right), \quad \lim_{n\to\infty} \frac{S_n}{I_n} = 1.$$
$$\frac{K_n}{L_n} = 1 + O\left(n^{-\frac{1}{2}}\right), \quad \lim_{n\to\infty} \frac{K_n}{L_n} = 1, \lim_{n\to\infty} (K_n - L_n) = 0.$$

Let m and k are two fixed positive integers with $k \geq 2$. For any positive integer n, define arithmetical function $b_m(n)$ as the integer part of the m-th root of n. That is, $b_m(n) = \left[n^{\frac{1}{m}}\right]$, where $[x]$ denotes the greatest integer $\leq x$. For example, $b_2(1) = 1, b_2(2) = 1, b_2(3) = 1, b_2(4) = 2, b_2(5) = 2, b_2(6) = 2, b_2(7) = 2, b_2(8) = 2, b_2(9) = 3, \cdots$. Let $\mathcal{A}_k$ denote the set of all k -th power free numbers. Z. Li used the elementary methods to study the mean value properties of $b_m(n)$ over the set $\mathcal{A}_k$.

Z. Li [37] 1. *Let m and k are two fixed positive integers with $k \geq 2$. Then for any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{\substack{n\leq x\\ n\in\mathcal{A}_k}} b_m(n) = \frac{1}{\zeta(k)} \frac{m}{m+1} x^{\frac{m+1}{m}} + O(x),$$

where $\zeta(k)$ is the Riemann zeta-function.

2. *For any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{\substack{n\leq x\\ n\in\mathcal{A}_2}} b_2(n) = \frac{4}{\pi^2} x^{\frac{3}{2}} + O(x) \text{ and } \sum_{\substack{n\leq x\\ n\in\mathcal{A}_2}} b_3(n) = \frac{9}{2\pi^2} x^{\frac{4}{3}} + O(x).$$

3. *For any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{\substack{n\leq x\\ n\in\mathcal{A}_4}} b_4(n) = \frac{64}{\pi^4}x^{\frac{5}{4}} + O(x) \text{ and } \sum_{\substack{n\leq x\\ n\in\mathcal{A}_4}} b_5(n) = \frac{75}{\pi^4}x^{\frac{6}{5}} + O(x).$$

Let n is a positive integer. It is clear that there exists an integer k such that

$$k^m \leq n < (k+1)^m.$$

Define $b_m(n) = k^m$. That is, $b_m(n)$ is the largest m -th power not exceeding n. In problems 40 and 41 of reference [1], Professor F.Smarandache asked people to study the properties of the sequences $\{b_2(n)\}$ and $\{b_3(n)\}$. D. Liu and J. Li studied the convergent properties of the Dirichlet series $f(s) = \sum_{n=1}^{\infty} \frac{1}{b_m^s(n)}$.

D. Liu and J. Li [39] *Let m be a fixed positive integer. Then for any real number $s > 1$, the Dirichlet series $f(s)$ is convergent and $f(s) = C_m^1\zeta(ms-m+1) + C_m^2\zeta(ms-m+2)+\cdots+ C_m^{m-1}\zeta(ms-1)+\zeta(ms)$ where $C_m^n = \frac{m!}{n!(m-n)!}$, and $\zeta(s)$ is the Riemann zeta-function.*

Taking $m = 2$ and $s = 3/2$ or $m = s = 2$ in the above theorem, then

$$\sum_{n=1}^{\infty}\frac{1}{b_2{}^{\frac{3}{2}}(n)} = \frac{\pi^2}{3} + \zeta(3) \text{ and } \sum_{n=1}^{\infty}\frac{1}{b_2^2(n)} = 2\zeta(3) + \frac{\pi^4}{90}.$$

Taking $m = 3$ and $s = 2$ or $m = 2$ and $s = 3$ in the above theorem, then

$$\sum_{n=1}^{\infty}\frac{1}{b_3^2(n)} = \frac{\pi^4}{30} + 3\zeta(5) + \frac{\pi^6}{945} \text{ and } \sum_{n=1}^{\infty}\frac{1}{b_2{}^3(n)} = 2\zeta(5) + \frac{\pi^6}{945}.$$

For any positive integer n, the inferior factorial part denoted by $a(n)$ is the largest factorial less than or equal to n. It is the sequence: $1, 2, 2, 2, 2, 6, 6, 6, 6, 6, 6, 6, 6, 6, \cdots$, On the other hand, the superior factorial part denoted by $b(n)$ is the smallest factorial greater than or equal to n. It is the sequence as follows: $1, 2, 6, 6, 6, 6, 24, 24, 24, 24, 24, 24, 24, 24\cdots$. J,Li studied two infinite series involving $a(n)$ and $b(n)$ as follows:

$$I = \sum_{n=1}^{\infty}\frac{1}{a^{\alpha}(n)}, \quad S = \sum_{n=1}^{\infty}\frac{1}{b^{\alpha}(n)}$$

and gave some sufficient conditions of the convergent property of them.

J. Li [37] *Let α be any positive real number. Then the infinite series I and S are convergent if $\alpha > 1$, divergent if $\alpha \leq 1$. Especially, when $\alpha = 2$, we have the following Corollary. We have the identity*

$$\sum_{n=1}^{\infty}\frac{1}{a^2(n)} = e.$$

X. Zhang [81] *Let $x \geq 1$, Then we have the asymptotic formula*

$$\sum_{\substack{n=1\\ a(n)\leq x}}^{\infty}\frac{1}{a(n)} = \frac{\ln^2 x}{2(\ln\ln x)^2} + O\left(\frac{\ln^2 x\ln\ln\ln x}{(\ln\ln x)^3}\right).$$

For any positive integer n, inferior prime part function $p_p(n)$ is defined as the largest prime number less than or equal to n, and Superior prime part function $P_n(n)$ is denoted as the smallest prime number greater than or equal to n. Y. Lou studied the mean value of $p_p(n)$ and $P_p(n)$, and gave two sharp asymptotic formulae.

Y. Lou [44] 1. *For any real number $x \geq 1$, we have*

$$\sum_{n\leq x} p_p(n) = \frac{x^2}{2} + O\left(x^{\frac{23}{18}+\epsilon}\right),$$

where ϵ is any fixed positive number.

2. *For any real number $x \geq 1$, we have*

$$\sum_{n\leq x} P_p(n) = \frac{x^2}{2} + O\left(x^{\frac{23}{18}+\epsilon}\right).$$

Let $n, k > 1$ are two positive integers. $m \geqslant 0$ is another fixed integer. The general Smarandache Sum mands function $S(n,m,k)$ is defined by $S(n,m,k) = \sum\limits_{i=0}^{[(n-m)/k]} (n-ki)$.

F. Li [29] *Let $k > 1, m \geqslant 0$ be fixed integers. Then for any integer $x > 1$ we have*

$$\sum_{i\leqslant x} S(n,m,k) = \frac{1}{6k}x^3 + \left(\frac{3}{4k} - \frac{1}{4}\right)x^2 + \left(\frac{1}{k} - \frac{3}{4} + \frac{k}{12} + m + \frac{m}{2k} - \frac{m^2}{2k}\right)x + R(x,k),$$

where $|R(x,k)| \leqslant \frac{5}{12}k^2 + \frac{1}{2}km$.

Let $n > 2$ be a positive integer, $a(n)$ denote the integral part of the k -th root sequence. In paper [2], Jozsef Sandor defined the following analogue of the Smarandache function:

$$S_1(x) = \min\{m \in N : x \leq m!\}, x \in (1, \infty),$$

which is defined on a subset of real numbers. W. Yao studied the mean value properties of the additive analogue Smarandache function acting on the floor of the k -th root sequence, and obtained two interesting asymptotic formulae.

W. Yao [63] 1. *For any real number $x \geq 2$ and integer $k \geq 2$, we have the asymptotic formula*

$$\sum_{n\leq x} S_1(a(n)) = \frac{x\log x}{k\log\log x^{\frac{1}{k}}} + O\left(\frac{x(\log x)\left(\log\log\log x^{\frac{1}{k}}\right)}{\left(\log\log x^{\frac{1}{k}}\right)^2}\right).$$

2. *For any real number $x \geq 2$, we have the estimate*

$$\sum_{n\leq x} d(n)S_1(n) = \frac{x\log^2 x}{\log\log x}\left(1 + O\left(\frac{\log\log\log x}{\log\log x}\right)\right),$$

where $d(n)$ be the divisor function.

Define $\Omega(n)$ and $\omega(n)$ as $\Omega(n) = \alpha_1 + \alpha_2 + \ldots + \alpha_r, \omega(n) = r$, if $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$ be the factorization of n into prime powers.

M. Yang and H. Li [66] *For any real number $x > 1$, we have the asymptotic formula*

$$\begin{aligned}\textstyle\sum_{n\le x}\omega(a(n)) &= x\ln\ln x + (A-\ln k)x + O\left(\tfrac{x}{\ln x}\right),\\ \textstyle\sum_{n\le x}\Omega(a(n)) &= x\ln\ln x + (B-\ln k)x + O\left(\tfrac{x}{\ln x}\right),\end{aligned}$$

where

$$A = \gamma + \sum_p \left(\ln\left(1-\frac{1}{p}\right)+\frac{1}{p}\right), B = A + \sum_p \frac{1}{p(p-1)}$$

are two constants. Taking $k = 3$ on the above, one can immediately obtain that

$$\begin{aligned}\textstyle\sum_{n\le x}\omega(a(n)) &= x\ln\ln x + (A-\ln 3)x + O\left(\tfrac{x}{\ln x}\right),\\ \textstyle\sum_{n\le x}\Omega(a(n)) &= x\ln\ln x + (B-\ln 3)x + O\left(\tfrac{x}{\ln x}\right).\end{aligned}$$

X. He and J. Guo [16] 1. *Let n be a positive integer, and $a(n) = [\sqrt{n}]$, then*

$$\sum_{n\le x} a(n) = \sum_{n\le x}[\sqrt{n}] = \frac{2}{3}x^{\frac{3}{2}} + \frac{3}{2}x + O\left(x^{\frac{1}{2}}\right).$$

2. *Let n be a positive integer, and $a(n) = \left[n^{\frac{1}{3}}\right]$, then*

$$\sum_{n\le x} a(n) = \sum_{n\le x}\left[n^{\frac{1}{3}}\right] = \frac{3}{4}x^{\frac{4}{3}} + \frac{5}{2}x + \frac{11}{4}x^{\frac{2}{3}} + O\left(x^{\frac{1}{3}}\right).$$

3.*Let n be a positive integer, and $a(n) = \left[n^{\frac{1}{k}}\right]$,then*

$$\sum_{n\le x} a(n) = \sum_{n\le x}\left[n^{\frac{1}{k}}\right] = \frac{k}{k+1}x^{\frac{k+1}{k}} + O(x).$$

4. *Let n be a positive integer, and $b(n) = (a(n))^2 = [\sqrt{n}]^2$, then*

$$\sum_{n\le x} b(n) = \sum_{n\le x}[\sqrt{n}]^2 = \frac{1}{2}x^2 + \frac{4}{3}x^{\frac{4}{3}} + O(x).$$

X. He and J. Guo [17] 1. *Let n be a positive integer,and $a(n) = [\sqrt{n}], d(n)$ be divisor function,then*

$$\sum_{n\le x} d(a(n)) = \sum_{n\le x} d([\sqrt{n}]) = \frac{1}{2}x\log x + \left(2c - \frac{1}{2}\right)x + O\left(x^{\frac{2}{4}}\right)$$

Where c is Euler's constant.

2. *Let n be a positive integer, and $a(n) = \left[n^{\frac{1}{3}}\right], d(n)$ be divisor function, then*

$$\sum_{n\le x} d(a(n)) = \sum_{n\le x} d\left(\left[n^{\frac{1}{3}}\right]\right) = \frac{1}{3}x\log x + \left(2c - \frac{1}{3}\right)x + O\left(x^{\frac{5}{6}}\right),$$

Where c is Euler's constant.

3. *Let n be a positive integer, and $a(n) = \left[n^{\frac{1}{k}}\right], d(n)$ be divisor function, then*

$$\sum_{n\leq x} d(a(n)) = \sum_{n\leq x} d\left(\left[n^{\frac{1}{k}}\right]\right) = \frac{1}{k}x\log x + O(x).$$

W. Yao [68] *Let n be a positive integer, and $a(n) = \left[n^{\frac{1}{k}}\right]$. For any real number $x > 1$ we have*

$$\sum_{n\leq x}\sigma_\alpha(a(n)) = \begin{cases} \frac{k\zeta(\alpha+1)}{\alpha+k}x^{\frac{\alpha+k}{k}} + O\left(x^{\frac{\beta+k-1}{k}}\right) & \text{if } \alpha > 0; \\ \frac{1}{k}x\log x + O(x) & \text{if } \alpha = 0; \\ \zeta(2)x + 0\left(x^{\frac{k+\varepsilon-1}{k}}\right) & \text{if } \alpha = -1; \\ \zeta(1-\alpha)x + O\left(x^{\frac{\delta+k-1}{k}}\right) & \text{if } \alpha < 0 \text{ and } \alpha \neq -1. \end{cases}$$

where $\sigma_\alpha(n) = \sum_{d|n} d^\alpha$ be the divisor function, $\zeta(n)$ be the Riemann Zeta function, $\beta = \max\{1,\alpha\}, \delta = \max\{0, 1+\alpha\}$ and $\varepsilon > 0$ be an arbitrary real number.

X. He and J. Guo [17] 1. *Let n be a positive integer, and $a(n) = [\sqrt{n}], \varphi(n)$ be Euler totient function, then*

$$\sum_{n\leq x}\varphi(a(n)) = \sum_{n\leq x}\varphi([\sqrt{n}]) = \frac{4}{\pi^2}x^{\frac{3}{2}} + O(x\log x).$$

2. *Let n be a positive integer,and $a(n) = \left[n^{\frac{1}{3}}\right], \varphi(n)$ be Euler totient function, then*

$$\sum_{n<x}\varphi(a(n)) = \sum_{n\leq x}\varphi\left(\left[n^{\frac{1}{3}}\right]\right) = \frac{9}{2\pi^2}x^{\frac{4}{3}} + O(x\log x).$$

3. *Let n be a positive integer, and $a(n) = \left[n^{\frac{1}{k}}\right], \varphi(n)$ be Euler totient function, then*

$$\sum_{n<x}\varphi(a(n)) = \sum_{n\leq x}\varphi\left(\left[n^{\frac{1}{k}}\right]\right) = \frac{6k}{(k+1)\pi^2}x^{\frac{k+1}{k}} + O(x\log x).$$

Q. Feng and J. Guo [6] 1. *Let n be a positive integer, $a(n) = \left[n^{\frac{1}{2}}\right]$, we have*

$$\sum_{n\leq x} a(n)^{\frac{1}{3}} = \frac{6}{7}x^{\frac{7}{6}} + \frac{3}{4}x^{\frac{4}{3}} + C\left(x^{\frac{2}{3}}\right).$$

2. *Let n be a positive integer, $a(n) = \left[n^{\frac{1}{3}}\right]$, we have*

$$\sum_{n\leq x} a(n)^{\frac{1}{3}} = \frac{9}{10}x^{\frac{10}{9}} + O\left(x^{\frac{7}{9}}\right).$$

3. *Let n, k be positive integers, $a(n) = \left[n^{\frac{1}{k}}\right]$, then we have*

$$\sum_{n\leqslant x} a(n)^{\frac{1}{2}} = \frac{2k}{2k+1}x^{\frac{2k+1}{2}} + O\left(x^{\frac{2x-1}{2\pi}}\right).$$

4. *Let n, k be positive integers, $a(n) = \left[n^{\frac{1}{2}}\right]$, then we have*

$$\sum_{k \le x} a(n)^{\frac{1}{2}} = \frac{2k}{2k+1} x^{\frac{2k+1}{2k}} + O\left(x^{\frac{k+1}{2k}}\right).$$

5. *Let n, k be positive integers, $a(n) = \left[n^{\frac{1}{3}}\right]$, we have*

$$\sum_{k \le x} a(n)^{\frac{1}{k}} = \frac{3k}{3k+1} x^{\frac{3k+1}{3k}} + O\left(x^{\frac{2k+1}{3k}}\right).$$

6. *Let n, k be positive integers, $a(n) = \left[n^{\frac{1}{k}}\right]$, we have*

$$\sum_{kx} (a(n))^{\frac{1}{k}} = \frac{k^2}{k^2+1} x^{\frac{k^2+1}{k^2}} + O\left(x^{\frac{k^2-k+1}{k^2}}\right).$$

For any fixed positive integer $k > 1$ and any positive integer n, let $a_k(n)$ and $b_k(n)$ denote the inferior and superior k -power part of n respectively. That is, $a_k(n)$ denotes the largest k power less than or equal to n, and $b_k(n)$ denotes the smallest k -power greater than or equal to n For example, let $k = 2$, then $a_2(1) = a_2(2) = a_2(3) = 1, a_2(4) = a_2(5) = a_2(6) = a_2(7) = 4 \ldots$, $b_2(1) = 1, b_2(2) = b_2(3) = b_2(4) = 4, b_2(5) = b_2(6) = b_2(7) = b_2(8) = 8, \cdots$.

F. Li [31] 1. *For any complex number s with* $\mathrm{Re}s > 2$, *we have*

$$\sum_{n=1}^{\infty} \frac{\sigma\left(a_k(n)\right)}{\left(a_k(n)\right)^s} = \sum_{i=0}^{k-1} C_k^i \zeta(ks-i)\zeta(ks-k-i) \prod_p \left[1 + p^{-(ks-i)} \frac{p-p^k}{1-p}\right],$$

where $\zeta(s)$ denotes the Rieman zeta-function, and $\sigma(n)$ denotes the Dirichlet divisor function.

2. *For any complex number s with* $\mathrm{Re}s > 2$, *we have*

$$\sum_{n=1}^{\infty} \frac{\sigma\left(b_k(n)\right)}{\left(b_k(n)\right)^s} = \sum_{i=0}^{k-1} (-1)^{k-i+1} C_k^i \zeta(ks-i)\zeta(ks-k-i) \prod_p \left[1 + p^{-(ks-i)} \frac{p-p^k}{1-p}\right].$$

3. *For any complex number s with* $\mathrm{Re}s > 2$, *we have*

$$\sum_{n=1}^{\infty} \frac{\varphi\left(a_k(n)\right)}{\left(a_k(n)\right)^s} = \sum_{i=0}^{k-1} C_k^i \frac{\zeta(ks-k-i)}{\zeta(ks-k-i+1)},$$

where $\varphi(n)$ denotes the Euler function.

4. *For any complex number s with* $\mathrm{Re}s > 2$, *we have*

$$\sum_{n=1}^{\infty} \frac{\varphi\left(b_k(n)\right)}{\left(b_k(n)\right)^s} = \sum_{i=0}^{k-1} (-1)^{k-i+1} C_k^i \frac{\zeta(ks-k-i)}{\zeta(ks-k-i+1)}.$$

X. Du [4] *Let $a_m(n) = \left[n^{\frac{1}{m}}\right]$. Let m be a fixed positive integer. Then for any real number $s > 1$, the Dirichlet series $f(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{a_m^s(n)}$ is convergent and*

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{a_m^s(n)} = \left(\frac{1}{2^{s-1}} - 1\right) \zeta(s),$$

where $\zeta(s)$ is the Riemann zeta-function.

Taking $s = 2$ and $s = 3$ in the Theorem respectively, then we have the identities

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{a_m^2(n)} = -\frac{\pi^2}{12} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{a_m^3(n)} = -\frac{3}{4}\zeta(3).$$

Let n be a positive integer. It is clear that there exists one and only one integer k such that

$$k^m \leq n < (k+1)^m.$$

Define $b_m(n) = k^m$.

X. Du [4] *Let m be a fixed positive integer. Then for any real number $s > \frac{1}{m}$, the Dirichlet series $g_m(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{b_m^s(n)}$ is convergent and*

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{b_m^s(n)} = \left(\frac{1}{2^{ms-1}} - 1\right)\zeta(ms).$$

For any positive integer s and $m \geq 2$, we have

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{b_m^s(n)} = \sum_{n=1}^{\infty} \frac{(-1)^n}{b_s^m(n)}.$$

For any positive integer m, let $a(m)$ denotes the integer part of the k -th root of m. That is, $a(m) = \left[m^{1/k}\right]$. T. Zhang studied the asymptotic properties of

$$\sigma_{-\alpha}(f(a(m))),$$

where $0 < \alpha \leq 1$ be a fixed real number, $\sigma_{-\alpha}(n) = \sum_{l|n} \frac{1}{l^\alpha}$, $f(x)$ be a polynomial with integer coefficients.

T. Zhang and Y. Ma [83] *Let $0 < \alpha \leq 1$ be a fixed real number, $f(x)$ be a polynomial with integer coefficients. Then for any real number $x > 1$, we have the asymptotic formula*

$$\sum_{m \leq x} \sigma_{-\alpha}(f(a(m))) = C_f(\alpha)x + O\left(x^{1-\alpha/k+\varepsilon}\right),$$

where

$$\sigma_{-\alpha}(n) = \sum_{l|n} \frac{1}{l^\alpha}, \quad C_f(\alpha) = \sum_{d=1}^{\infty} P_f(d) d^{-1-\alpha}, \quad P_f(d) = \sum_{f(n) \equiv 0 (\bmod d), 0 < n \leq d} 1$$

and ε denotes any fixed positive number.

H. Yang and R. Fu [69] *Let $s_k(n)$ denote the integer part of k -th root of n. For any real number $x > 1$, we have the asymptotic formula*

$$\sum_{n < x} \Omega(s_k(n)) = x \ln\ln x + (A - \log k)x + O\left(\frac{x}{\ln x}\right),$$

where $\Omega(n)$ denotes the total number of prime divisors of n, A is a constant.

R. Fu and H. Yang [7] 1. *For any real number $x > 1$ and any fixed positive integer $k > 1$, we have the asymptotic formula*

$$\sum_{n\leq x} \frac{\varphi\left(s_k(n)\right)}{s_k(n)} = \frac{6}{\pi^2}x + O\left(x^{1-\frac{1}{k}-\varepsilon}\right),$$

where ε is any real number.

2. *For any real number $x > 1$ and any fixed positive integer $k > 1$, we have the asymptotic formula*

$$\sum_{n\leq x} \frac{1}{\varphi\left(s_k(n)\right)} = \frac{k\zeta(2)\zeta(3)}{(k-1)\zeta(6)}x^{1-\frac{1}{k}} + A + O\left(x^{1-\frac{2}{k}}\log x\right),$$

where $A = \gamma\sum_{n=1}^{\infty}\frac{\mu^2(n)}{n\varphi(n)} - \sum_{n=1}^{\infty}\frac{\mu^2(n)\log n}{n\varphi(n)}$.

H. Yang and R. Fu [69] *Let m be a fixed positive integer and $\varphi(n)$ be the Euler totient function, then for any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{n\leq x} \varphi((s_k(n), m)) = h(m)x + (k+1)h(m) + O\left(x^{1-\frac{1}{2k}+\varepsilon}\right),$$

where $(s_k(n), m)$ denotes the greatest common divisor of $s_k(n)$ and m, $h(m) = \frac{\varphi(m)}{m}\prod_{p^\alpha || m}\left(1+\alpha-\frac{\alpha-1}{p}\right)$, and ε is any positive number.

For any positive integer n, let $m_q(n) = \left[n^{\frac{1}{k}}\right]$.

N. Gao [9] *m is any fixed positive integer, α is a real number. For any real number $x > 1$, we have the asymptotic formula*

$$\sum_{n\leq x} \sigma_\alpha\left((m_q(n), m)\right) = \frac{(2k-1)\sigma_{1-\alpha}(m)}{m^{1-\alpha}}x + O\left(x^{1-\frac{1}{2k}+\varepsilon}\right),$$

where $\sigma_\alpha(n) = \sum_{d|n} d^\alpha$, ε is any fixed positive number.

For any positive integer n, let $Sg(n)$ denotes the smallest square greater than or equal to n. For example, $Sg(1) = 1, Sg(2) = 4, Sg(3) = 4, Sg(4) = 4$ $Sg(5) = 9, Sg(6) = 9, Sg(7) = 9, \cdots, Sg(9) = 9, Sg(10) = 16\cdots$ Let $Sk(n)$ denote the smallest power k greater than or equal to n, and $Gk(n) = Sk(n) - n$.

L. Ding [1] *Let $x \geq 1$, we have the asymptotic formula*

$$\sum_{n\leq x} Gk(n) = \frac{k^2}{2(2k-1)}x^{\frac{2k-1}{k}} + O\left(x^{\frac{2k-2}{k}}\right).$$

From this theorem. When $k = 2, 3$, it follows that

$$\sum_{n\leq x}(Sg(n) - n) = \frac{2}{3}x^{\frac{3}{2}} + O(x).$$

Let $x \geq 1$ and $S_c(n)$ denotes the smallest cube greater than or equal to n, then

$$\sum_{n\leq x}(S_c(n) - n) = \frac{9}{10}x^{\frac{5}{3}} + O\left(x^{\frac{4}{3}}\right).$$

For a fixed positive integer $k > 1$, and any positive integer n, let $a(n)$ denote the largest k-th power less than or equal to $n, b(n)$ denote the smallest k -th power greater than or equal to n.

H. Zhang [78] 1. *For any real number $x > 1$, we have*

$$\sum_{n\leq x} d(a(n)) = \frac{1}{kk!}\left(\frac{6}{k\pi^2}\right)^{k-1} A_0 x\ln^k x + A_1 x \ln^{k-1} x + \cdots + A_{k-1} x\ln x + A_k x + O\left(x^{1-\frac{1}{2k}+\varepsilon}\right),$$

where $A_0 A_h, \cdots A_k$ are constants, especially when k equals to $2, A_0 = 1; d(n)$ denotes the Dirichlet divisor function, ε is any fixed positive number.

2. *For any real number $x > 1$, we have the asymptotic formula*

$$\sum_{n\leq x} d(b(n)) = \frac{1}{kk!}\left(\frac{6}{k\pi^2}\right)^{k-1} A_0 x\ln^k x + A_1 x \ln^{k-1} x + \cdots + A_{k-1} x\ln x + A_k x + O\left(x^{1-\frac{1}{2k}+\varepsilon}\right).$$

§5. LCM ratio sequences

Let $(x_1, x_2, \ldots, x_t)$ and $[x_1, x_2, \ldots, x_t]$ denote the greatest common divisor and the least common multiple of any positive integers $x_1, x_2, \ldots, x_t$ respectively. Let r be a positive integer with $r > 1$. For any positive integer n, let

$$T(r,n) = \frac{[n, n+1, \ldots, n+r-1]}{[1, 2, \ldots, r]}$$

then the sequences $SLR(r) = T(r,n)_\infty$ is called the F.Samarandache LCM ratio sequences of degree r. M. Le [26] proved that

$$T(3,n) = \begin{cases} \frac{1}{6}n(n+1)(n+2), & \text{if } n \text{ is odd} \\ \frac{1}{12}n(n+1)(n+2), & \text{if } n \text{ is even} \end{cases}$$
$$T(4,n) = \begin{cases} \frac{1}{24}n(n+1)(n+2)(n+3), & \text{if } n \neq 0(\text{mod}3) \\ \frac{1}{72}n(n+1)(n+2)(n+3), & \text{if } n \equiv 0(\text{mod}3) \end{cases}$$

Furthermore, Wang Ting [57] computing the value of $T(5,n)$:

$$T(5,n) = \begin{cases} \frac{1}{1440}n(n+1)(n+2)(n+3)(n+4), & \text{if } n \equiv 0, 8(\text{mod}12) \\ \frac{1}{120}n(n+1)(n+2)(n+3)(n+4), & \text{if } n \equiv 1, 7(\text{mod}12) \\ \frac{1}{720}n(n+1)(n+2)(n+3)(n+4), & \text{if } n \equiv 2, 6(\text{mod}12) \\ \frac{1}{360}n(n+1)(n+2)(n+3)(n+4), & \text{if } n \equiv 3, 5, 9, 11(\text{mod}12), \\ \frac{1}{480}n(n+1)(n+2)(n+3)(n+4), & \text{if } n \equiv 4(\text{mod}12) \\ \frac{1}{240}n(n+1)(n+2)(n+3)(n+4), & \text{if } n \equiv 10(\text{mod}12) \end{cases}$$

R. Ma studied the recurrence relations between $T(r+1,n)$ and $T(r,n)$, and got three recurrence formulas for it.

R. Ma [46] 1. *For any natural number n and r, we have the recurrence formula:*

$$T(r+1,n) = \frac{n+r}{r+1} \cdot \frac{([1, 2, \ldots, r], r+1)}{([n, n+1, \ldots, n+r-1], n+r)} \cdot T(r,n)$$

Especially, if both $r+1$ *and* $n+r$ *are primes, then we can get a simple formula*

$$T(r+1,n)=\frac{n+r}{r+1}\cdot T(r,n).$$

2. *For each natural number* n *and* r*, we also have another recurrence formula:*

$$T(r,n+1)=\frac{n+r}{n}\cdot\frac{(n,[n+1,\ldots,n+r])}{([n,n+1,\ldots,n+r-1],n+r)}\cdot T(r,n).$$

Especially, if both n *and* $n+r$ *are primes with* $r<n$*, then we can also get a simple formula*

$$T(r,n+1)=\frac{n+r}{n}\cdot T(r,n).$$

If both n *and* $n+r$ *are primes with* $r\geq n$*, then we have*

$$T(r,n+1)=(n+r)\cdot T(r,n).$$

3. *For each natural number* n *and* r*, we have*

$$T(r+1,n+1)=\frac{n+r}{n}\cdot\frac{n+r+1}{r+1}\cdot\frac{([1,2,\ldots,r],r+1)}{([n+1,\ldots,n+r],n+r+1)}\cdot\frac{(n,[n+1,\ldots,n+r])}{([n,n+1,\ldots,n+r-1],n+r)}\cdot T(r,n).$$

T. Wang [56] 1. *If* $n\equiv 0,15 \bmod 20$*, then*

$$SLR(6)=\frac{1}{7200}n(n+1)(n+2)(n+3)(n+4)(n+5).$$

If $n\equiv 1,2,6,9,13,14,17,18 \bmod 20$*, then*

$$SLR(6)=\frac{1}{720}n(n+1)(n+2)(n+3)(n+4)(n+5).$$

If $n\equiv 5,10 \bmod 20$*, then*

$$SLR(6)=\frac{1}{3600}n(n+1)(n+2)(n+3)(n+4)(n+5).$$

If $n\equiv 3,4,7,8,11,12,16,19 \bmod 20$*, then*

$$SLR(6)=\frac{1}{1440}n(n+1)(n+2)(n+3)(n+4)(n+5).$$

2. *For any positive integer* n*, we have the following If* $n\equiv 0,24,30,54$ *mod 60 , then*

$$SLR(7)=\frac{1}{302400}n(n+1)(n+2)(n+3)(n+4)(n+5)(n+6).$$

If $n\equiv 1,13,17,37,41,53$ *mod 60 , then*

$$SLR(7)=\frac{1}{5040}n(n+1)(n+2)(n+3)(n+4)(n+5)(n+6).$$

If $n\equiv 2,8,16,22,26,28,32,38,46,52,56,58$ *mod* 60*, then*

$$SLR(7)=\frac{1}{20160}n(n+1)(n+2)(n+3)(n+4)(n+5)(n+6).$$

If $n \equiv 3, 27, 51 \text{ mod } 60,$ *then*

$$SLR(7) = \frac{1}{30240}n(n+1)(n+2)(n+3)(n+4)(n+5)(n+6).$$

If $n \equiv 4, 10, 14, 20, 34, 40, 44, 50\ mod\ 60,$ *then*

$$SLR(7) = \frac{1}{100800}n(n+1)(n+2)(n+3)(n+4)(n+5)(n+6).$$

If $n \equiv 5, 25, 29, 49\ mod\ 60$ *, then*

$$SLR(7) = \frac{1}{25200}n(n+1)(n+2)(n+3)(n+4)(n+5)(n+6).$$

If $n \equiv 6, 12, 18, 36, 42, 48\ mod\ 60,$ *then*

$$SLR(7) = \frac{1}{60480}n(n+1)(n+2)(n+3)(n+4)(n+5)(n+6).$$

If $n \equiv 7, 11, 23, 31, 43, 47 \text{ mod } 60,$ *then*

$$SLR(7) = \frac{1}{10080}n(n+1)(n+2)(n+3)(n+4)(n+5)(n+6).$$

If $n \equiv 9, 45 \text{ mod } 60,$ *then*

$$SLR(7) = \frac{1}{75600}n(n+1)(n+2)(n+3)(n+4)(n+5)(n+6).$$

If $n \equiv 15, 39\ mod\ 60$ *, then*

$$\mathrm{SLR}(7) = \frac{1}{151200}n(n+1)(n+2)(n+3)(n+4)(n+5)(n+6).$$

If $n \equiv 19, 55, 59, 35\ mod\ 60$ *, then*

$$\mathrm{SLR}(7) = \frac{1}{50400}n(n+1)(n+2)(n+3)(n+4)(n+5)(n+6).$$

If $n \equiv 21, 33, 57 \text{ mod } 60,$ *then*

$$SLR(7) = \frac{1}{15120}n(n+1)(n+2)(n+3)(n+4)(n+5)(n+6).$$

The Smarandache LCM ratio function of the second typeis defined as:

$$SL(n,r) = \frac{[n, n-1, n-2, \ldots, n-r+1]}{[1, 2, 3, \ldots, r]}, r \leq n; n, r \in N.$$

A. A. K. Majumdar [24] 1. *For any* $n \geq 1, \mathrm{SL}(n,1) = n.$ *For any* $n \geq 2, SL(n,2) = \frac{n(n-1)}{2}.$

2. *For any* $n \geq 3,$

$$SL(n,3) = \begin{cases} \frac{n(n-1)(n-2)}{6} & \text{if } n \text{ is odd }, \\ \frac{n(n-1)(n-2)}{12} & \text{if } n \text{ is even.} \end{cases}$$

3. *For any* $n \geq 4$,

$$SL(n,4)=\begin{cases}\frac{n(n-1)(n-2)(n-3)}{72} & \textit{if } 3 \textit{ divides } n,\\ \frac{n(n-1)(n-2)(n-3)}{24} & \textit{if } 3 \textit{ does not divide } n.\end{cases}$$

4. *For any* $n \geq 5$,

$$SL(n,5)=\begin{cases}\frac{n(n-1)(n-2)(n-3)(n-4)}{1440} & \textit{if } n=12m,12m+4\\ \frac{n(n-1)(n-2)(n-3)(n-4)}{120} & \textit{if } n=12m+1,12m+3,12m+7,12m+9\\ \frac{n(n-1)(n-2)(n-3)(n-4)}{720} & \textit{if } n=12m+2\\ \frac{n(n-1)(n-2)(n-3)(n-4)}{360} & \textit{if } n=12m+5,12m+11\\ \frac{n(n-1)(n-2)(n-3)(n-4)}{480} & \textit{if } n=12m+6,12m+10\\ \frac{n(n-1)(n-2)(n-3)(n-4)}{240} & \textit{if } n=12m+8\end{cases}$$

S. M. Khairnar, A. W.Vyawahare and J. N. Salunke [23]

$$SL(n,r)-SL(n-1,r)=SL(n-1,r-1), r<n,$$

$$\frac{SL(n,r+1)}{SL(n,r)}=\frac{n-r}{r+1},$$

$$\frac{1}{SL(n,r)}+\frac{1}{SL(n,r+1)}=\frac{n+1}{(r+1)\cdot SL(n,r+1)},$$

$$SL(n,\tau)\cdot SL(n,r+1)=\frac{n^2\cdot(n-1)^2\cdot(n-2)^2\cdot(n-3)^2\cdot\cdots\cdot(n-r+1)^2\cdot(n-r)}{r!\cdot(r+1)!}.$$

X. Pan [48] *For any fixed positive integer n, we have*

$$T(n,n)^{\frac{1}{n}}=e+O\left(exp\left(\frac{-c(\ln n)^{\frac{3}{5}}}{(\ln\ln n)^{\frac{1}{5}}}\right)\right),$$

where c is a positive constant and $exp(y)=\mathrm{e}^y$.

Y. Li [38] *For any fixed positive integer n, we have*

$$T(5,n)=\begin{cases}\frac{1}{120}n(n+1)(n+2)(n+3)(n+4) & n\not\equiv 0(\mathrm{mod}2), n\not\equiv 0(\mathrm{mod}3), n\not\equiv 0(\mathrm{mod}4), n+1\not\equiv 0(\mathrm{mod}3)\\ \frac{1}{240}n(n+1)(n+2)(n+3)(n+4) & n\equiv 0(\mathrm{mod}2), n\not\equiv 0(\mathrm{mod}3), n\not\equiv 0(\mathrm{mod}4), n+1\not\equiv 0(\mathrm{mod}3)\\ \frac{1}{360}n(n+1)(n+2)(n+3)(n+4)\begin{cases} n\equiv 0(\mathrm{mod}2), n\not\equiv 0(\mathrm{mod}2)\\ n\not\equiv 0(\mathrm{mod}2), n\not\equiv 0(\mathrm{mod}3), n\not\equiv 0(\mathrm{mod}4), n+1\not\equiv 0(\mathrm{mod}3)\end{cases}\\ \frac{1}{480}n(n+1)(n+2)(n+3)(n+4), n\not\equiv 0(\mathrm{mod}3), n\equiv 0(\mathrm{mod}4), n+1\not\equiv 0(\mathrm{mod}3)\\ \frac{1}{720}n(n+1)(n+2)(n+3)(n+4)\begin{cases} n\equiv 0(\mathrm{mod}2), n\not\equiv 0(\mathrm{mod}3), n\not\equiv 0(\mathrm{mod}4), n+1\equiv 0(\mathrm{mod}3)\\ n\equiv 0(\mathrm{mod}2), n\equiv 0(\mathrm{mod}3), n\not\equiv 0(\mathrm{mod}4)\end{cases}\\ \frac{1}{1440}n(n+1)(n+2)(n+3)(n+4)\begin{cases} n\equiv 0(\mathrm{mod}3), n\equiv 0(\mathrm{mod}4)\\ n\not\equiv 0(\mathrm{mod}3), n+1\equiv 0(\mathrm{mod}3), n\equiv 0(\mathrm{mod}4)\end{cases}\end{cases}$$

§6. Smarandache pseudo-odd, pseudo-even and pseudo-multiples sequences

A number is called pseudo-even number if some permutation of its digits is an even number, including the identity permutation. For example: $0, 2, 4, 6, 8,\ 10, 12, 14, 16, 18, 20, 21, \cdots$ are pseudo-even numbers. Let A denote the set of all the pseudo-even numbers. Similarly, we can define the pseudo-odd number. That is, a number is called pseudo-odd number if some permutation of its digits is an odd number, such as $1, 3, 5, 7, 9, 10, 11, 12, 13, \cdots$ are pseudo-odd numbers. Let B denote the set of all the pseudo-odd numbers.

X. Zhang and Y. Lou [80] 1. *For any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{\substack{n\in A\\ n\leq x}} f(n) = \sum_{n\leq x} f(n) + O\left(Mx\frac{\ln 5}{\ln 10}\right),$$

where $M = \max_{1<n<x}\{|f(n)|\}$.

2. *For any real number $x \geq 1$, we have*

$$\sum_{\substack{n\in B\\ n\leq x}} f(n) = \sum_{n\leq x} f(n) + O\left(Mx^{\frac{\ln 5}{\ln 10}}\right).$$

Y. Lou [43] 1. *For any real number $x \geq 1$, we have the asymptotic formula*

$$\ln(x - \sum_{\substack{n\in A\\ n\leq x}} 1) = \frac{\ln 5}{\ln 10}\ln x + O(1).$$

2. *For any real number $x \geq 1$, we have the asymptotic formula*

$$\ln(x - \sum_{\substack{n\in B\\ n\leq x}} 1) = \frac{\ln 5}{\ln 10}\ln x + O(1).$$

X. Liu and J. Guo [41] 1. *For any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{\substack{n\in B\\ n\leq x}} \frac{\varphi(x)}{n} = \frac{6}{\pi^2}x + O(\frac{1}{2}(\ln x)^2),$$

where $\varphi(x)$ is Euler function.

2. *For any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{\substack{n\in B\\ n\leq x}} d(n) = \frac{6}{\pi^2}x(\ln x)^3 + ax(\ln x)^2 + bx\ln x + c\ln x + O(x^{\frac{\ln 5}{\ln 10}+\epsilon}),$$

where $d(x)$ is the Dirichlet divisor function and a, b, c are constants.

A number is called the second Smarandache pseudo-odd number if it is an even number, and some permutation of its digits is an odd number. For example: $10, 12, 14, 16, 18, 30, 32, 34, 36, 38, 50, 52, \cdots$ are the second Smarandache pseudo-odd numbers. Let A denote the set of all the second Smarandache pseudo-odd numbers. Similarly, we can define the second Smarandache pseudo-even number. That is, a number is called the second Smarandache pseudo-even number if it is an odd number, and some permutation of its digits is an even number, such as $21, 23, 25, 27, 29, 41, 43, 45, 47, 49, \cdots$ are the second Smarandache pseudo-even numbers. Let B denote the set of all the second Smarandache pseudo-even numbers. Y. Liu studied the mean value properties of these two sequences.

Y. Liu [42] 1. *For any real number $x > 1$, we have the asymptotic formulae*

$$\sum_{\substack{n\in A\\ n\le x}} 1 = \frac{1}{2}x + O\left(x^{\frac{\ln 5}{\ln 10}}\right) \ \ and \ \sum_{\substack{n\in B\\ n\le x}} 1 = \frac{1}{2}x + O\left(x^{\frac{\ln 5}{\ln 10}}\right).$$

2. *For any real number $x \ge 1$, let $d(n)$ denote the Dirichlet divisor function. then we have the asymptotic formulae*

$$\sum_{\substack{n\in A\\ n\le z}} d(n) = \frac{3}{4}x\ln x + \left(\frac{3}{2}\gamma - \frac{\ln 2}{2} - \frac{3}{4}\right)x + O\left(x^{\frac{\ln 5}{\ln 10}+\varepsilon}\right)$$

and

$$\sum_{\substack{n\in B\\ n\le x}} d(n) = \frac{1}{4}x\ln x + \left(\frac{1}{2}\gamma + \frac{\ln 2}{2} - \frac{1}{4}\right)x + O\left(x^{\frac{\ln 5}{\ln 10}+\varepsilon}\right),$$

where γ is the Euler constant, ε is any fixed positive integer.

A positive integer is called pseudo-multiple of 5 if some permutation of its digits is a multiple of 5. Let C denote the set of pseudo-multiple of 5 numbers.

X. Wang and J. Guo [54] *For any real number $x \ge 1$, we have the asymptotic formula*

$$\sum_{n\in X} \frac{\varphi(n)}{n^2} = \frac{6\ln x}{\pi^2} + O(1),$$

where $\varphi(n)$ is Euler function.

Y. Lou [43] *For any real number $x \ge 1$, we have the asymptotic formula*

$$\ln(x - \sum_{\substack{n\in C\\ n\le x}} 1) = \frac{\ln 8}{\ln 10}\ln x + O(1).$$

X. Wang [56] *For any real number $x \ge 1$, we have the asymptotic formula*

$$\sum_{\substack{n\in C\\ n\le x}} f(n) = \sum_{n\le x} f(n) + O\left(Mx^{\frac{\ln 8}{\ln 10}}\right),$$

where $M = \max_{1\le n\le x}\{|f(n)|\}$. Taking $f(n) = d(n), \Omega(n)$ as the Dirichlet divisor function and the function of the number of prime factors respectively, then we have the following: For any real number $x \ge 1$, we have

$$\sum_{\substack{n\in C\\ n\le x}} d(n) = x\ln x + (2\gamma - 1)x + O\left(x^{\frac{\ln 8}{\ln 10}} + \epsilon\right),$$

where γ is the Euler constant, ϵ is any fixed positive number. And

$$\sum_{\substack{n\in C\\ n<x}} \Omega(n) = x\ln\ln x + Bx + O\left(\frac{x}{\ln x}\right),$$

where B is a computable constant.

C. Wu and Q. Yuan [55] 1. *Let A denote the set of pseudo-multiple of 10 numbers. For any real number $x \geq 1$, we have*

$$\begin{array}{l} \sum_{n\in A}\frac{1}{n} = \ln x + \gamma - C + O\left(x^{\frac{2\ln 3}{\ln 10}-1}\right) \\ \sum_{n\in B}\frac{1}{n} = \frac{9}{10}\ln x + \frac{9\gamma}{10} + D + O\left(x^{\frac{2\ln 3}{\ln 10}-1}\right) \end{array},$$

where C, D are constants.

2. *Let A denote the set of pseudo-multiple of 10 numbers. For any real number $x \geq 1$, we have*

$$\sum_{n\in A} f(n) = \sum_{n\leqslant x} f(n) + O\left(Mx^{\frac{2\ln 3}{\ln 0}}\right),$$

where $M = \max\{|f(n)|\}$.

For any real number $x > 1$, let $[x]$ denote the expansion in base 10 of the integer part of x: $[x] = a_m 10^m + a_{m-1}10^{m-1} + \cdots + a_1 10 + a_0$, where $0 \leqslant a_i \leqslant 9 (i = 0, 1, \cdots, m)$ are positive integer and $a_m \neq 0$. Let $k = \max\{i : a_i = 0, i = 0, 1, \cdots, m\}$.

C. Wu and Q. Yuan [55] 1. *For the pseudo-multiple of 10 numbers, we have*

(1) *If $0 \leqslant k < m$, we have $\sum_{\substack{n\in A\\ n\leqslant x}} 1 = [x] - \frac{9}{8}(9^m - 1) - \sum_{i=k+1}^{m}(a_i - 1)9^i$;*

(2) *If k does not exist, we have $\sum_{n\in A} 1 = [x] - \frac{9}{8}(9^m - 1) - \sum_{i=0}^{m}(a_i - 1)9^i - 1$.*

2. *For the second pseudo-multiple of 10 numbers, we have*

(1) *If $0 < k < m$ and $a_0 = 0$, then*

$$\sum_{\substack{n\in B\\ n\leqslant x}} 1 = [x] - \frac{9}{8}(9^m - 1) - \sum_{i=k+1}^{m}(a_i - 1)9^i - \sum_{i=1}^{m} a_i 10^{i-1} - 1.$$

If $0 < k < m$ and $a_0 \neq 0$, then

$$\sum_{\substack{n\in B\\ n\leqslant x}} 1 = [x] - \frac{9}{8}(9^m - 1) - \sum_{i=k+1}^{m}(a_i - 1)9^i - \sum_{i=1}^{m} a_i 10^{i-1}$$

(2) *If k does not exist, we have $\sum_{\substack{n\in B\\ n\leq x}} 1 = [x] - \frac{9}{8}(9^m - 1) - \sum_{i=0}^{m}(a_i - 1)9^i - \sum_{i=1}^{m} a_i 10^{i-1} - 1$.*

N. Wu [59] *Let A be a set of Smarandache pseudo-even number.*

(1) *If $0 \leq k \leq m$, then*

$$\sum_{\substack{n\in A\\ n\leqslant x}} 1 = [x] - \frac{5}{4}(5^m - 1) - \sum_{i=k}^{m}\left[\frac{a_i}{2}\right]5^i,$$

$$\sum_{\substack{n\in B\\ n\leqslant x}} 1 = [x] - \frac{5}{4}(5^m - 1) - \sum_{i=k}^{m}\left[\frac{a_i}{2}\right]5^i - 5\times 10^{m-1} - 5\sum_{i=0}^{m}(a_i - 1)10^{i-1} - [\frac{a_0}{2}] - 1.$$

(2) *If k does not exist , we have*

$$\sum_{\substack{n\in A\\ n\leq x}} 1=[x]-\frac{5}{4}\left(5^m-1\right)-\sum_{i=0}^{m}\left[\frac{a_i}{2}\right]5^i-1.$$

$$\sum_{\substack{n\in B\\ n\leq x}} 1=[x]-\frac{5}{4}\left(5^m-1\right)-\sum_{i=0}^{m}\left[\frac{a_i}{2}\right]5^i-5\times 10^{m-1}-5\sum_{i=0}^{m}(a_i-1)10^{i-1}-[\frac{a_0}{2}]-2.$$

Let $A(n)$ denote the sum of all the digits of the base 10 digits of n. That is,

$$A(n)=\sum_{i=0}^{k}a_i,$$

if $n=a_k10^k+a_{k-1}10^{k-1}+\cdots+a_1 10+a_0$.

Z. Li [35] 1. *For any integer number $x\geq 1$, we have the asymptotic formula*

$$\sum_{\substack{n\in C\\ n\leq x}} A^m(n)=x\left(\frac{9}{2}\log x\right)^m+O\left(x(\log x)^{m-1}\right).$$

2. *Let B denote the set of all Smarandache Pseudo-even numbers. For any integer number $x\geq 1$, we have the asymptotic formula*

$$\sum_{\substack{n\in B\\ n\leq x}} A^m(n)=x\left(\frac{9}{2}\log x\right)^m+O\left(x(\log x)^{m-1}\right).$$

3. *Let C denote the set of all Smarandache Pseudo-odd numbers. For any integer number $x\geq 1$, we have the asymptotic formula*

$$\sum_{\substack{n\in C\\ n\leq x}} A^m(n)=x\left(\frac{9}{2}\log x\right)^m+O\left(x(\log x)^{m-1}\right).$$

A positive integer is called the second class pseudo-multiple of 5 if it is not a multiple of 5, but its some permutation of its digits is a multiple of 5. For convenience, let B denote the set of all second class pseudo-multiple of 5 sequences.

N. Gao [9] *For any real number $x\geq 1$, we have the asymptotic formula*

$$\sum_{\substack{n\in B\\ n\leq x}} d(n)=\frac{16}{25}x\left(\ln x+2\gamma-1+\frac{\ln 5}{2}\right)+O\left(x^{\frac{\ln 8}{\ln 10}+\varepsilon}\right),$$

where $d(n)$ is the Dirichlet divisor function, γ is the Euler constant, and ε denotes any fixed positive number.

X. Wang and J. Guo [54] *For any real number $x\geq 1$, we have the asymptotic formula*

$$\sum_{n\in B}\frac{\varphi(n)}{n^2}=\frac{144\ln x}{25\pi^2}+O(1),$$

where $\varphi(n)$ is the Euler function.

J. Li [36] 1. *For any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{\substack{n\in B\\ n\leq x}} \frac{1}{n} = \frac{4}{5}\ln x + \frac{4\gamma + \ln 5}{5} - c + O(x^{\frac{\ln 8}{\ln 10}-1}),$$

where c is a constant.

2. *Let A denote the set of all the second Smarandache Pseudo-even numbers. For any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{\substack{n\in B\\ n\leq x}} \frac{1}{n} = \frac{1}{2}\ln x + \frac{\gamma + \ln 2}{2} - c + O(x^{-\ln 2}),$$

where c is a constant.

3. *Let D denote the set of all the second Smarandache Pseudo-odd numbers. For any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{\substack{n\in C\\ n\leq x}} \frac{1}{n} = \frac{1}{2}\ln x + \frac{\gamma + \ln 2}{2} - c + O(x^{-\ln 2}),$$

where c is a constant.

§7. Simple sequences

A number n is called simple number if the product of its proper divisors is less than or equal to n. For example: $2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 17, 19, 21, \cdots$. In problem 23 of [1], Professor F.Smarandach asked us to study the properties of the sequence of the simple numbers. Let A is a set of simple numbers, that is, $A = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 17, 19, 21, \cdots\}$. In this paper, we use the elementary methods to study the properties of this sequence, and give several interesting asymptotic formulae. That is, we shall prove the following:

H. Liu and Wenpeng Zhang [32] 1. *For any positive number $x > 1$, we have the asymptotic formula*

$$\sum_{\substack{n\in A\\ n\leq x}} \frac{1}{n} = (\ln\ln x)^2 + B_1 \ln\ln x + B_2 + O\left(\frac{\ln\ln x}{\ln x}\right),$$

where B_1, B_2 are the constants.

2. *For any positive number $x > 1$, we have the asymptotic formula*

$$\sum_{\substack{n\in A\\ n\leq x}} \frac{1}{\phi(n)} = (\ln\ln x)^2 + C_1 \ln\ln x + C_2 + O\left(\frac{\ln\ln x}{\ln x}\right),$$

where C_1, C_2 are the constants, $\phi(n)$ is Euler function.

3. *For any positive number $x > 1$, we have the asymptotic formula*

$$\sum_{\substack{n\in A\\ n\leq x}} \frac{1}{\sigma(n)} = (\ln\ln x)^2 + D_1 \ln\ln x + D_2 + O\left(\frac{\ln\ln x}{\ln x}\right),$$

where D_1, D_2 are the constants, $\sigma(n)$ is divisor function.

For $n > 1$, let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ denote the factorization of n into prime powers. If one of the divisor d of n satisfying $\tau(d) \leq 4$ (where $\tau(n)$ denotes the numbers of all divisors of n), then we call d as a simple number divisor. Define

$$\tau_{sp}(n) = \sum_{\substack{\sum \\ d|n}} 1,$$

which we called the simple function.

Q. Yang [70] *For any real number $x > 1$, we have the asymptotic formula*

$$\sum_{n \leq x} \tau_{sp}(n) = \frac{1}{2} x (\log\log x)^2 + \frac{1}{2}(a+1) x \log\log x + \left(\frac{b+A}{2} + B + C \right) x + O\left(\frac{x \log\log x}{\log x} \right),$$

where a and b are two computable constants, $A = \gamma + \sum_p (\log(1-1/p) + 1/p), \gamma$ is the Euler constant, $B = \sum_p \frac{1}{p^2}$ and $C = \sum_p \frac{1}{p^3}$.

Let A denote the set of all the simple numbers. Generally speaking, n has the form: $n = p$, or p^2, or p^3, or pq, where p and q are distinct primes. In [2], Jason Earls defined sop $fr(n)$ as a new Smarandache sequence as following: Let sopf $r(n)$ denote the sum of primes dividing n (with repetition). That is,

$$\operatorname{sopf} r(n) = \sum_{p|n} p.$$

For example:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
sopf $r(n)$	0	2	3	4	5	5	7	6	6	7	11	7	13	9	8	8	17	8

X. Zhang and M. Yang [78] *For any real number $x \geq 1$, we have*

$$\sum_{n \in A} \operatorname{sopf} r(n) = A_1 \frac{x^2}{\ln x} + A_2 \frac{x^2}{\ln^2 x} + O\left(\frac{x^2}{\ln^3 x} \right),$$

where A_1, A_2 are computable constants.

Let $S_p(x) = \min\{m \in \mathbb{N}^+ : p^x \leq m!!\}$ and $S_p^*(x) = \min\{m \in \mathbb{N}^+ : m!! \leq p^x\}$ where $m!! = 2 \times 4 \times \cdots \times m$ if m is an even and $m!! = 1 \times 3 \times \cdots \times m$ if m is an odd. Then $S_p(x)$ and $S_p^*(x)$ are called the additive analogue of Smarandache simple function.

M. Zhu [85] 1. *For any real number $x \geq 1$, we have*

$$\sum_{n \leq x} d(S_p(n)) = 2x(\ln x - 2 \ln\ln x) + O(x \ln p).$$

2. *For any real number $x \geq 1$, we have*

$$\sum_{n \leq x} d(S_p^*(n)) = 2x(\ln x - 2 \ln\ln x) + O(x \ln p).$$

Let $Sdf(n) = \min\{m \in \mathbb{N}^+ : n \mid m!!\}$ and $a_k(n) = [n^{\frac{1}{k}}]$.

M. Zhu [86] *For any real number $x \geq 1$, we have*

$$\sum_{n \leq x} Sdf(a_k(n)) = \frac{7\pi^2}{12(k+1)} \frac{x^{\frac{k+1}{k}}}{\ln x} + O(\frac{x^{\frac{k+1}{k}}}{\ln^2 x}).$$

§8. The primitive numbers

Let p be a prime, n be any positive integer, $S_p(n)$ denotes the smallest integer such that $S_p(n)!$ is divisible by p^n. For example, $S_3(1) = 3, S_3(2) = 6, S_3(3) = 9, S_3(4) = 9, \cdots$.

Y. Yi and F. Liang [72] *For any real number $x \geq 2$, let p be a prime and n be any positive integer. Then we have the asymptotic formula*

$$\sum_{n\leq x} \frac{1}{p} \left|S_p\left(a_{n+1}\right) - S_p\left(a_n\right)\right| = x^{\frac{1}{k}} \cdot \left(1 - \frac{1}{p}\right) + O_k\left(\frac{\ln x}{\ln p}\right),$$

where O_k denotes the O -constant depending only on parameter k.

L. Ding [3] 1. *Let p be an odd prime, m_i be positive integer. Then we have the triangle inequality*

$$S_p\left(\sum_{i=1}^{k} m_i\right) \leq \sum_{i=1}^{k} S_p\left(m_i\right).$$

2. *There are infinite integers $m_i (i = 1, 2, \cdots, k.)$ satisfying*

$$S_p\left(\sum_{i=1}^{k} m_i\right) = \sum_{i=1}^{k} S_p\left(m_i\right).$$

W. Zhang and D. Liu [77] *For any fixed prime p and any positive integer n, we have the asymptotic formula*

$$S_p(n) = (p-1)n + O\left(\frac{p}{\ln p} \cdot \ln n\right).$$

M. Liu [40] *For any fixed positive integer $k > 1$ and any positive integer n, we have the asymptotic formula*

$$S_k(n) = \alpha(p-1)n + O\left(\frac{p}{\ln p} \ln n\right),$$

where $k = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ be the factorization of k into prime powers, and $\alpha(p-1) = \max_{1\leq i\leq r}\{\alpha_i\,(p_i-$ 1) $\}$.

Z. Xu [62] *For any prime p and complex number s, we have the identity:*

$$\sum_{n=1}^{\infty} \frac{1}{S_p^s(n)} = \frac{\zeta(s)}{p^s - 1},$$

where $\zeta(s)$ is the Riemann zeta-function.

Specially, taking $s = 2, 4$ and $p = 2, 3, 5$, we have

$$\sum_{n=1}^{\infty} \frac{1}{S_2^2(n)} = \frac{\pi^2}{18}; \quad \sum_{n=1}^{\infty} \frac{1}{S_3^2(n)} = \frac{\pi^2}{48}; \quad \sum_{n=1}^{\infty} \frac{1}{S_5^2(n)} = \frac{\pi^2}{144};$$
$$\sum_{n=1}^{\infty} \frac{1}{S_2^4(n)} = \frac{\pi^4}{1350}; \quad \sum_{n=1}^{\infty} \frac{1}{S_3^4(n)} = \frac{\pi^4}{7200}; \quad \sum_{n=1}^{\infty} \frac{1}{S_5^4(n)} = \frac{\pi^4}{56160}.$$

Z. Xu [62] *Let p be any fixed prime. Then for any real number $x \geq 1$, we have the asymptotic formula:*

$$\sum_{\substack{n=1 \\ s_p(n)\leq x}}^{\infty} \frac{1}{S_p(n)} = \frac{1}{p-1}\left(\ln x + \gamma + \frac{p \ln p}{p-1}\right) + O\left(x^{-\frac{1}{j}+\epsilon}\right),$$

where γ is the Euler constant, c denotes any fixed positive number.

Z. Xu [62] *Let k be any positive integer. Then for any prime p and real number $x \geq 1$, we have the asymptotic formula:*

$$\sum_{\substack{n=1 \\ s_p(n)\leq x}}^{\infty} S_p^k(n) = \frac{x^{k+1}}{(k+1)(p-1)} + O\left(x^{k+\frac{1}{2}+\epsilon}\right).$$

W. Zhu [87] *For any prime p, we have the calculating formulas*
(1) $\sum_{d|n} S_p(d) = p\sigma(n)$, *if* $1 \leq n \leq p$;
(2) $\sum_{d|n} S_p(d) = p\sigma(n) - (n-1)p$, *if* $p < n \leq 2p$, *where $\sigma(n)$ denotes the summation over all divisors of n.*

W. Zhu [88] *Let p be a prime. Then for any positive integers n and k with $1 \leq n \leq p$ and $1 < k < p$, we have the identities:*

$$S_p(kn) = kS_p(n), \text{ if } 1 < kn < p,$$

$$S_p(kn) = kS_p(n) - p\left[\frac{kn}{p}\right], \text{ if} p < kn < p^2,$$

where $[x]$ denotes the integer part of x.

Y. Yi [72] *For any real number $x \geq 2$, let p be a prime and n be any positive integer. Then we have the asymptotic formula*

$$\frac{1}{p}\sum_{n<x} |S_p(n+1) - S_p(n)| = x\left(1 - \frac{1}{p}\right) + O\left(\frac{\ln x}{\ln p}\right).$$

L. Ding [2] *For any real number $x \geq 2$, let n be any positive integer. Then we have the asymptotic formula*

$$\sum_{p\leq x} S_p(n) = \frac{nx^2}{2\log x} + \sum_{m=1}^{k-1} \frac{na_m x^2}{\log^{m+1} x} + O\left(\frac{nx^2}{\log^{k+1} x}\right),$$

where $a_m (m = 1, 2, \cdots, k-1)$ are computable constants.

§9. The Smarandache kn-digital sequences

For any positive integer n and any fixed positive integer $k \geq 2$, the Smarandache kn -digital subsequence $\{S_k(n)\}$ is defined as the numbers $S_k(n)$, which can be partitioned into two groups such that the second is k times bigger than the first. For example, The Smarandache $4n$ -digital subsequence are: $S_4(1) = 14, S_4(2) = 28, S_4(3) = 312$ $S_4(4) = 416, S_4(5) = 520, S_4(6) = 624, S_4(7) = 728, S_4(8) = 832, S_4(9) = 936, S_4(10) = 1040$ $S_4(11) = 1144, S_4(12) = 1248, S_4(13) = 1352, S_4(14) = 1456, S_4(15) = 1560, \cdots$. The Smarandache $5n$ -digital subsequence are: $S_5(1) = 15, S_5(2) = 210, S_5(3) = 315,$ $S_5(4) = 420, S_5(5) = 525, S_5(6) = 630, S_5(7) = 735, S_5(8) = 840, S_5(9) = 945, S_5(10) = 1050$ $S_5(11) = 1155, S_5(12) = 1260, S_5(13) = 1365, S_5(14) = 1470, S_5(15) = 1575, \cdots$.

C. Zhang and Y. Liu [75] *Let z be a real number. If $z > \frac{1}{2}$, then the infinite series*

$$f(z,k) = \sum_{n=1}^{+\infty} \frac{1}{S_k^z(n)}$$

is convergent. If $z \leq \frac{1}{2}$, then the infinite series is divergent. In these Smarandache kn -digital subsequences, it is very hard to find a complete square number.

For any positive integer n, the Smarandache Prime- Digital Subsequence (SPDS) is defined as follows:
A positive integer n is an element of $SPDS$, if it satisfies the following properties:
a) m is a prime.
b) All of the digits of m are prime, i.e, they are all elements of the set $\{2,3,5,7\}$.
For example, the first few values of $SPDS$ are: $2, 3, 5, 7, 23, 53, 73, 223, 227, 233, 257, 277, 377, \cdots$.

S. Shang and J. Su [52] *Let $SPDSN(n)$ denote the number of all elements of $SPDS$ that are less than or equal to n, and $\pi(n)$ denote the number of all primes not exceeding n. Then we have the limit*

$$\lim_{n\to\infty} \frac{\text{SPDSN}(n)}{\pi(n)} = 0.$$

H. Le studied the hybrid mean value properties of the Smarandache kn -digital sequence with $SL(n)$ function and divisor function $d(n)$, where $SL(n)$ is defined as the smallest positive integer k such that $n \mid [1, 2, \ldots, k]$, that is $SL(n) = \min\{k : k \in N, n \mid [1, 2, \ldots, k]\}$. And obtained the following results:

H. Le [25] 1. *Let $1 < k < 9$, then for any real number $x > 1$, we have the asymptotic formula*

$$\sum_{n\leq x} \frac{SL(n)}{S_k(n)} = \frac{3\pi^2}{k \cdot 20} \cdot \ln\ln x + O(1).$$

2.*Let $1 < k < 9$, then for any real number $x > 1$, we have the asymptotic formula*

$$\sum_{n\leq x} \frac{d(n) \cdot SL(n)}{S_k(n)} = \frac{\pi^4}{k \cdot 20} \cdot \ln\ln x + O(1).$$

S. Gou [10] *Let $k \in \mathbf{N}$. If either $2 \leqslant k \leqslant 5$ or $6 \leqslant k \leqslant 9$. For any positive number x, we have*

$$\sum_{1\leqslant n\leqslant x} \frac{n}{S_k(n)} = \frac{9}{k \cdot 10 \cdot \ln 10} \ln x + O(x).$$

S. Gou [12] 1. *Let k be an integer and $1 \leqslant k \leqslant 9$, for any positive number x, we have*

$$\sum_{n\leqslant x} \frac{\sigma(n)}{S_k(n)} = \frac{3\pi^2}{k \cdot 20 \cdot \ln 10} \cdot \ln x + O(1).$$

2. *Let p be a prime, k be an integer and $1 \leqslant k \leqslant 9$, for any positive number x, we have*

$$\sum_{p\leq x} \frac{p}{S_k(p)} = \frac{9}{k \cdot 10 \cdot \ln 10} \cdot \ln\ln x + O(1).$$

J. Hu, Y. He and H. Bai [18] *For any positive integer $N > 1$, we have*

$$\sum_{n¡N} \frac{n}{S_3(n)} = \frac{3}{10 \ln 10} \cdot \ln N + O(1).$$

For any positive integer n. the famous Smarandache $3n$ -digital sequence is defined as $\{a_n\} = \{13, 26, 39, 412, 515, 618, 721, 824, \cdots, \}$. That is, the numbers that can be partitioned into two groups such that the second is three times bigger than the first. For example, $a_{10} = 1030, a_{21} = 2163, a_{32} = 3296, a_{100} = 100300, \cdots$. Professor Zhang Wenpeng proposed the following conjecture: There does not exist any complete square number in the Smarandache $3n-$ digital sequence $\{a_n\}$. That is, the equation $a_n = m^2$ has no positive integer solution. N. Wu studied this problem, and prove that the Zhang's conjecture is correct for some special positive integers.

N. Wu [59] 1. *If positive integer n is a square-free number (That is, for any prime p, if $p \mid n$, then $p^2 \nmid n$), then a_n is not a complete square number.*

2. *If positive integer n is a complete square number, then a_n is not a complete square number. If a_n be a complete square number, then we must have*

$$n = 2^{2\alpha_1} \cdot 3^{2\alpha_2} \cdot 5^{2\alpha_3} \cdot 11^{2\alpha_4} \cdot n_1,$$

where $(n_1, 330) = 1$.

S. Gou [13] *For any positive integer $N > 1$, we have*

$$\sum_{n<N} \ln a_n = 2N \ln N + O(N).$$

References

[1] Liping Ding. On the primitive numbers of power P and its triangle inequality. Research on S-marandache problems in number theory (2004), 37-38.

[2] Liping Ding. On the primitive numbers of power P and its mean value properties. Scientia Magna 1 (2005), No. 2, 36-38.

[3] Liping Ding. A new function and its mean value. Scientia Magna 1 (2005), No. 1, 167-169.

[4] Xiaoying Du. On the integer part of the M-th root and the largest M-th power not exceeding N. Scientia Magna 2 (2006), No. 4, 91-94.

[5] M. Farris and P. Mitchell. Bounding the Smarandache function. Smarandache Notions Journal 13 (2002), 37-42.

[6] Qiang Feng and Jinbao GUo. Generalized some asymptotic formulars of F. Smarandache. Journal of Yanan University (Natural Science Edition) 23 (2004), No. 4, 8-9.

[7] Guqin Fu and Hai Yang. On the 82-th problem of F. Smarandache. Scientia Magna 1 (2005), No. 1, 131-133.

[8] Nan Gao. On the 83-th problem of F. Smarandache. Scientia Magna 1 (2005), No. 1, 83-87.

[9] Nan Gao. On the second class pseudo-multiples of 5 sequences. Research on Smarandache problems in number theory II (2005), 83-86.

[10] Su Gou. The Smarandache kn-digital sequence and its mean value properties. Basic Sciences Journal of Textile Universities 24 (2011), No. 2, 250-253. (In Chinese with English abstract).

[11] Su Gou. A new additive function and Smarandache sequences. Journal of Northwest University(Natural Secience Edition) 41 (2011), No. 1, 19-21. (In Chinese with English abstract).

[12] Su Gou. Hybrid mean value of Smarandache kn-digital series with divisor sum function. Journal of Xi'an Shiyou University (Natural Science Edition) 26 (2011), No. 2, 107-110. (In Chinese with English abstract).

[13] Su Gou. The Smarandache $3n$-digital sequence and its some asymptotic properties. Journal of Inner Mongolia Normal University (Natural Science Edition) 39 (2010), No. 6, 563-564. (In Chinese with English abstract).

[14] Jinbao Guo and Xiqing Zhao. On the k-th power free sieve sequence. Research on Smarandache problems in number theory (2004), 155-157.

[15] Yanyan Han. On the product of the square-free divisor of a natural number. Scientia Magna 2 (2006), No.2, 55-59.

[16] Xianlin He and Jinbao Guo. On the 80th problem of F. Smarandache (I). Smarandache Notions Journal 14 (2004), 70-73.

[17] Xianlin He and Jinbao Guo. On the 80th problem of F. Smarandache (II). Smarandache Notions Journal 14 (2004), 74-79.

[18] Jiayuan Hu, Yu He and Hui Bai. The asymptotic formula of numerical sequence containing F. Smarandache $3n$. Journal of Hetao College 13 (2016), No. 1, 76-78. (In Chinese with English abstract).

[19] Wei Huang and Yan Ma. On the inferior and superior series of hexagon number's complement. Journal of Beihua University (Natural Science) 14 (2013), No. 5, 526-529. (In Chinese with English abstract).

[20] Wei Huang. On the Smarandache inferior and superior factorial series. Journal of Gansu Sciences 23 (2011), No. 4, 5-8. (In Chinese with English abstract).

[21] Wei Huang. On inferior and superior partial sequences of the triangular numbers. Journal of Baoji University of Arts and Sciences (Natural Science) 31 (2011), No. 1, 23-25. (In Chinese with English abstract).

[22] Kenichiro Kasgihara. Comments and topics on Smarandache notions and problems. Erhus University Press, USA, 1996.

[23] Khairnar. On Smarandache least common multiple ratio. Scientia Magna 5 (2009), No. 1, 29-36.

[24] A. A. K. Majumdar. On the Smarandache LCM ratio. Scientia Magna 11 (2016), No. 1, 4-11.

[25] Huan Le. On the phybrid mean value of the Smarandache kn-digital sequence with $SL(n)$ function and divisor function $d(n)$. Scientia Magna 8 (2012), No. 1, 31-36.

[26] Maohua Le. Two formulas for Smarandache LCM ratio sequences. Smarandache Notions Journal 14 (2014), 183-185.

[27] Maohua Le. On Smarandache divisor products. Smarandache Notions Journal 10 (1999), 144-145.

[28] Fangchi Liang. Generalization of the divisor products and proper divisor products sequences. Scientia Magna 3 (2005), No. 1, 29-32.

[29] Fanbei Li. On the mean value of the Smarandache summands. Journal of Inner Mongolia Normal University (Natural Science Edition) 39 (2010), No. 6, 555-556. (In Chinese with English abstract).

[30] Fenju Li. On the difference between the mean value of the Smarandache square parts sequences $SP(n)$ and $IP(n)$. Pure and Applied Mathematics 26 (2010), No. 1, 69-72. (In Chinese with English abstract).

[31] Feng Li. Some dirichlet series involving special sequences. Scientia Magna 3 (2007), No. 3, 36-39.

[32] Hongzhi Liu and Wenpeng Zhang. On the simple numbers and the mean value properties. Smarandache Notions Journal 14 (2004), 171-175.

[33] Hongyan Liu and Wenpeng Zhang. On the divisor product and proper divisor products sequences. Smarandache Notions Journal 13 (2002), 128-133.

[34] Zhanhu Li. On the integer part of the m-th root and the k-th power free number. Research on Smarandache problems in number theory II (2005), 41-43.

[35] Zhanhui Li. On the Smarandache pseudo-number sequences. Scientia Magna 1 (2005), No. 2, 111-113.

[36] Jie Li. On the second Smarandache pseudo-multiples of 5 sequence. Journal of Northwest University (Natural Science Edition) 36 (2006), No. 4, 517-518. (In Chinese with English abstract).

[37] Jie Li. On the inferior and superior factorial part sequences. Research on Smarandache problems in number theory (2004), 47-48.

[38] Youcheng Li. Reduction formulas for Smarandache LCM ratio sequences. Jiangxi Science 26 (2008), No. 6, 851-857. (In Chinese with English abstract).

[39] Duansen Liu and Junzhuang Li. On the largest M-th power not exceeding. Research on Smarandache problems in number theory II (2005), 59-61.

[40] Miaohua Liu. On the generalization of the primitive number function. Scientia Magna 3 (2007), No. 1, 18-21.

[41] Xingru Liu and Jinbao Guo. On two asymptotic formulas about Smarandache pseudo-odd sequence and number theory function. Science Technology and Engineering 11 (2011), No. 4, 798-799.

[42] Yanni Liu. On the Smarandache pseudo number sequence. Chinese Quarterly Journal of Mathematics 21 (2006), No. 4, 581-584. (In Chinese with English abstract).

[43] Yuanbing Lou. On the Smarandache pseudo- number. Research on Smarandache problems in number theory (2004), 111-113.

[44] Yanbing Lou. On the inferior and superior prime part. Research on Smarandache problems in number theory (2004), 57-59.

[45] Xinwen Ma. The asymptotic formal of $\sum\limits_{n\leq x} \log P_{ld}(n)$. Scientia Magna 2 (2006), No.2, 72-76.

[46] Rong Ma. On the F. Smarandache LCM ratio sequence. Scientia Magna 2 (2006), No. 2, 44-47.

[47] Xiaowei Pan and Baoli Liu. On the irrational root sieve sequence. Scientia Magna 3 (2007), No. 4, 1-3.

[48] Xiaowei Pan. A limit problem involving the Smarandache LCM ration sequences. Journal of Northwest University (Natural Science Edition) 37 (2007), No. 5, 708-710. (In Chinese with English abstract).

[49] Lan Qi. The properties of an arithmetical function and the Smarandache irrational root sieve. Henan Science 30 (2012), No. 12, 1698-1700. (In Chinese with English abstract).

[50] Longjiang Qian. A mean value formulae on Smarandache odd sieve sequence. Science Technology and Engineering 10 (2010), No. 36, 9033-9035.

[51] Juanli Su. A problem related to the Smarandache n-ary power sieve. Scientia Magna 4 (2008), No. 1, 83-85.

[52] Songye Shang and Juanli Su. On the Smarandache prime-digital subsequence sequences. Scientia Magna 4 (2008), No. 4, 12-14.

[53] Xiaoying Wang. On the Smarandache pseudo-multiples of 5 sequence. Research on Smarandache problems in number theory (2004), 17-19.

[54] Xiangyuan Wang and Jingjie Guo. On two asymptotic fomulas of Smarandache pseudo-multiples of 5 number sequence. Journal of Yanan University (Natural Science Edition) 31 (2012), No. 4, 9-10. (In Chinese with English abstract).

[55] Chaoxia Wu and Quan Yuan. Research on the Smarandache pseudo-multiples of 10 number sequence. Journal of Inner Mongolia Normal University (Natural Science Edition) 43 (2014), No. 6, 669-673. (In Chinese with English abstract).

[56] Ting Wang. Two formulas for Smarandache LCM ratio sequences. Scientia Magna 1 (2005), No. 1, 109-113.

[57] Ting Wang. A formula for Smarandache LCM ratio sequence. Research on Smarandache problems in number theory II (2005), 45-46.

[58] Yiren Wang. On the mean value of $SSMP(n)$ and $SIMP(n)^1$. Scientia Magna 5 (2009), No. 3, 1-5.

[59] Nan Wu. On the Smarandache $3n$-digital sequence and the Zhang wenpeng's conjecture. Scientia Magna 4 (2008), No.4, 120-122.

[60] Nan Wu. On the Smarandache pseudo-even number. Basic Sciences Journal of Textile Universities 21 (2008), No. 3, 378-380. (In Chinese with English abstract).

[61] Riqin Xie. An mean value formulae on Smarandache odd sieve sequence. Science Technology and Engineering 10 (2010), No. 26, 6490-6491.

[62] Zhefeng Xu. Some arithmetical properties of primitive numbers of power p. Scientia Magna 2 (2006), No. 1, 9-12.

[63] Weili Yao. On the odd sieve sequence. Scientia Magna 1 (), No. 1, 137-139.

[64] Weili Yao. The additive analogue of Smarandache function. Research on Smarandache problems in number theory (2004), 99-102.

[65] Xiaoxia Yan. On the Smarandache prime part. Scientia Magna 3 (2007), No. 3, 74-77.

[66] Mingshun Yang and Hailong Li. On the integer part of the K-th root of a positive integer. Research on Smarandache problems in number theory (2004), 143-145.

[67] Mingshun Yang. The mean value of two Smarandache sequences. Jiangxi Science 28 (2010), No. 1, 9-10. (In Chinese with English abstract).

[68] Weili Yao. On the generalization of the floor of the square root sequence. Scientia Magna 1 (2005), No. 1, 183-186.

[69] Hai Yang and Ruiqin Fu. On the integer part of a positive integer's K-th root. Scientia Magna 1 (2014), No. 1, 61-66.

[70] Qianli Yang. On the mean value of the F. Smarandache simple divisor function. Scientia Magna 2 (2006), No. 1, 35-37.

[71] Yuan Yi and Fangchi Liang. On the primitive numbers of power P and K-power roots. Research on Smarandache problems in number theory (2004), 5-8.

[72] Yuan Yi. On the primitive numbers of power P and its asymptotic property. Scientia Magna 1 (2005), No. 1, 175-177.

[73] Yahui Yu and Lixiang Cai. On the Smarandache prime part sequences and its two conjectures. Scientia Magna 4 (2008), No. 3, 33-35.

[74] Yahui Yu and Zhenping Li. Limit problem of Smarandache square sequences. Journal of North University of China (Natural Science Edition) 32 (2011), No. 4, 411-423. (In Chinese with English abstract).

[75] Cuncao Zhang and Yanyan Liu. On the Smarandache kn-digital subsequence. Scientia Magna 4 (2008), No. 4, 4-6.

[76] Jianfeng Zhang. On the inferior and superior k-th power part of a positive integer abd divisor function. Smarandache Notions Journal 14 (2002), 88-91.

[77] Wenpeng zhang and Duansen Liu. On the primitive numbers of power P and its asymptotic property. Smarandache Notions Journal 13 (2002), 173-175.

[78] Xiaobeng Zhang and Mingshun Yang. On a new Smarandache sequence. Research on Smarandache problems in number theory (2004), 159-162.

[79] Xiaobeng Zhang and Yanbing Lou. The Smarandache irrational root sieve sequences. Research on Smarandache problems in number theory (2004), 27-31.

[80] Xiaobeng Zhang and Yanbing Lou. On the Smarandache pseudo-even number sequence. Research on Smarandache problems in number theory (2004), 85-87.

[81] Xiaobeng Zhang. The Smarandache factorial sequence. Scientia Magna 1 (2014), No. 1, 123-124.

[82] Tianping Zhang. Two asymptotic formulae on the divisor product sequences. Research on Smarandache problems in number theory (2004), 81-83.

[83] Tianping Zhang and Yuankui Ma. On the integer part of the K-th root of a positive integer. Scientia Magna 1 (2006), No. 1, 5-8.

[84] Weiyi Zhu. On the divisor product sequences. Smarandache Notions Journal 14 (2004), 144-146.

[85] Minhui Zhu. On a problem of F. Smarandache. Jiangxi Science 27 (2009), No. 3, 337-338. (In Chinese with English abstract).

[86] Minhui Zhu. Hybrid mean value on some Smarandache function. Basic Sciences Journal of Textile Universities 22 (2009), No. 3, 295-198. (In Chinese with English abstract).

[87] Weiyi Zhu. On the primitive numbers of power p. Scientia Magna 2 (2006), No. 3, 92-95.

[88] Weiyi Zhu. The relationship between $S_p(n)$ and $S_p(kn)$. Scientia Magna 2 (2006), No. 4, 87-90.

.

Scientia Magna
Vol. 16 (2021), No. 1, 35-50

A survey on Smarandache notions in number theory: Pseudo-Smarandache-Squarefree function and Smarandache dual function

Zehua Liu

School of Mathematics, Northwest University
Xi'an 710127, China
E-mail: zehua@stumail.nwu.edu.cn

Abstract In this paper we give a survey on recent results on Pseudo-Smarandache-Squarefree function, Smarandache dual function and other related Smarandache functions.

Keywords Pseudo-Smarandache-Squarefree function, Smarandache dual function, mean value.
2010 Mathematics Subject Classification 11B83.

§1. the Pseudo-Smarandache-Squarefree function

For any positive integer n, the Pseudo-Smarandache-Squarefree function $Z\omega(n)$ is defined as the smallest integer m such that m^n is divisible by n. That is,

$$Z\omega(n) = min\{m : n \mid m^n, m \in N\},$$

where N denotes the set of all positive integers.

In reference [28], Felice Russo studied the properties of $Z\omega(n)$, and proposed some new problems and conjectures. Many scholars have studied these issues, have made great achievements, and have solved some of them.

M. Le [17]. *1. If $n > 1$, then $Z\omega(n) = p_1 p_2 ... p_k$, where $p_1 p_2 ... p_k$ are distinct prime divisors of n.*

2. The difference $|Z\omega(n+1) - Z\omega(n)|$ is unbounded.

3. $Z\omega(n)$ is not a Lipschitz function.

4. $P = \bigcap_{n=1}^{\infty} \frac{1}{Z\omega(n)} = 0$

5. For any positive number a,

$$S(a) = \sum_{n=1}^{\infty} \frac{1}{(Z\omega(n))^a} \quad (a \in \mathbb{R}, a > 0)$$

is divergence.

6. The following three equations have no positive integer solutions n.

$$Z\omega(n) = Z\omega(n+1) \cdot Z\omega(n+2);$$

$$Z\omega(n) \cdot Z\omega(n+1) = Z\omega(n+2);$$
$$Z\omega(n) \cdot Z\omega(n+1) = Z\omega(n+2) \cdot Z\omega(n+3).$$

7. The equation $Z\omega(mn) = m^k Z\omega(n)$ *has infinitely many positive integer solutions* (m, n, k). *Moreover, every solution* (m, n, k) *of the equation can be expressed as*

$$m = p_1 p_2 \dots p_r, \ n = t, k = 1,$$

where $p_1 p_2 \dots p_r$ *are distinct primes,* t *is a positive integer with* $gcd(m, t) = 1$.

8. The equation

$$(Z\omega(n))^k = k \cdot Z\omega(kn), \ k, n \in \mathbb{N}, k > 1, n > 1$$

has infinitely many solutions (n, k). *Moreover, every solution* (n, k) *of the equation can be expressed as*

$$n = 2^r, \ k = 2, r \in \mathbb{N}.$$

9. The equation

$$(Z\omega(n))^k + (Z\omega(n))^{k-1} + \dots + Z\omega(n) = n, \ k, n \in \mathbb{N}, k > 1$$

has no isolutions (n, k).

H. Liu and J. Gao [25]. *1. For any real numbers* α, s *with* $s - \alpha > 1$ *and* $\alpha > 0$, *we have*

$$\sum_{n=1}^{\infty} \frac{Z\omega^{\alpha}(n)}{n^s} = \frac{\zeta(s)\zeta(s-\alpha)}{\zeta(2s-2\alpha)} \prod_p \left[1 - \frac{1}{p^s + p^{\alpha}}\right],$$

where $\zeta(s)$ *is the Riemann zeta function,* $\prod_p$ *denotes the product over all prime numbers.*

2. For any real numbers $\alpha > 0$ *and* $x \geq 1$, *we have*

$$\sum_{n \leq x} Z\omega^{\alpha}(n) = \frac{\zeta(\alpha+1)x^{\alpha+1}}{\zeta(2)(\alpha+1)} \prod_p \left[1 - \frac{1}{p^{\alpha}(p+1)}\right] + O\left(x^{\alpha+\frac{1}{2}+\varepsilon}\right).$$

Noting that $\sum_{n \leq x} Z\omega^0(n) = x + O(1)$ *and* $\lim_{\alpha \to 0^+} \alpha\zeta(\alpha+1) = 1$, *so from 2 we immediately have the limit*

$$\lim_{\alpha \to 0^+} \frac{1}{\alpha} \prod_p \left(1 - \frac{1}{p^{\alpha}(p+1)}\right) = \zeta(2).$$

J. Li [22]. *For any positive integer* $k > 1$, *we have the asymptotic formula*

$$\frac{Z\omega(k)}{\theta(k)} = \frac{Z\omega(k)}{\sum_{n \leq k} \ln(Z\omega(n))} = O\left(\frac{1}{\ln k}\right)$$

B. Cheng [1]. *1. Let* $k > 1$ *be an integer, then for any positive integer* $m_1 m_2 \dots m_k$, *we have the inequality*

$$\sqrt[k]{Z\omega\left(\prod_{i=1}^{k} m_i\right)} < \frac{\sum_{i=1}^{k} Z\omega(m_i)}{k} \leq Z\omega\left(\prod_{i=1}^{k} m_i\right),$$

and the equality holds if and only all $m_1 m_2 ... m_k$ have the same prime divisors.

2. For any positive integer $k \geq 1$, the equation

$$\sum_{i=1}^{k} Z\omega(m_i) = Z\omega\left(\sum_{i=1}^{k} m_i\right)$$

has infinity positive integer solutions $(m_1 m_2 ... m_k)$.

W. Xiong [35]. *For any positive integer $n > 1$, we have the asymptotic formula*

$$\frac{1}{n}\sum_{k=2}^{n} \frac{\ln(Z\omega(k))}{\ln k} = 1 + O\left(\frac{1}{\ln n}\right).$$

Noting that from this theorem we immediately have the limit

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=2}^{n} \frac{\ln(Z\omega(k))}{\ln k} = 1.$$

The above scholars studied the mean value of $Z\omega(n)$, and gave a few asymptotic formulae. Then some scholars used the elementary and analytic method to study the hybrid mean value involving the Pseudo-Smarandache-Squarefree function and other number theory functions, and give interesting asymptotic formulae.

Definition 1. *For any positive integer n, the famous Smarandache function $S(n)$ is defined by*

$$S(n) = min\{k : k \in N, n \mid k!\}.$$

W. Guan [9]. *1.The following three equations have no positive integer solution.*

$$Z\omega(n) = Z\omega(n+1) \cdot Z\omega(n+2);$$
$$Z\omega(n) \cdot Z\omega(n+1) = Z\omega(n+2);$$
$$Z\omega(n) \cdot Z\omega(n+1) = Z\omega(n+2) \cdot Z\omega(n+3).$$

2. There exist infinite positive integers n such that the equation $S(n) = Z\omega(n)$.

X. Fan [3]. *1. Let $k \geq 2$ be any fixed positive integer. Then for any real number $x \geq 2$, we have the asymptotic formula*

$$\sum_{n\leq x} Z\omega(S(n)) = \frac{\pi^2}{12} \cdot \frac{x^2}{\ln x} + \sum_{i=2}^{k} \frac{c_i \cdot x^2}{\ln^i x} + O\left(\frac{x^2}{\ln^{k+1} x}\right),$$

where c_i $(i = 1, 2, ..., k)$ are computable constants.

2. Let $k \geq 2$ be any fixed positive integer. Then for any real numbers $x \geq 2$, we have the asymptotic formula

$$\sum_{n\leq x} Z\omega(n) \cdot S(n) = \frac{\zeta(2)\zeta(3)}{3\zeta(4)} \prod_{p} \left(1 - \frac{1}{p+p^3}\right) \cdot \frac{x^3}{\ln x} + \sum_{i=2}^{k} \frac{e_i \cdot x^3}{\ln^i x} + O\left(\frac{x^3}{\ln^{k+1} x}\right),$$

where $\zeta(s)$ is the Riemann zeta function, $\prod_p$ denotes the product over all prime numbers, e_i $(i = 1, 2, ..., k)$ are computable constants.

Second, we define a new Pseudo-Smarandache function $K(n) = m = \frac{n(n+1)}{2} + k$, where k is the smallest natural number such that $n|m$. Then Y. Wang studied the mean value properties of the new composite function $Z\omega\left(K(n) - \frac{n(n+1)}{2}\right)$.

Y. Wang [31]. *For any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{n\leq x} Z\omega\left(K(n) - \frac{n(n+1)}{2}\right) = \frac{9}{40}x^2\prod_p\left(1 - \frac{1}{p+p^2}\right) + O\left(x^{1.5+\varepsilon}\right)$$

where $\prod_p$ denotes the product over all prime numbers, ε is any fixed positive number.

Definition 2. *For any positive integer n, the least prime divisor function and the greatest prime divisor function $V(n)$, $U(n)$ are defined as:*
(1) V(1)=U(1)=1;
(2) $n = p_1^{\alpha_1}p_2^{\alpha_2}...p_r^{\alpha_r}$ is the standard decomposition of n, when $n > 1$. Then

$$V(n) = \min_{1\leq i\leq r}\{\alpha_1 p_1, \alpha_2 p_2, ..., \alpha_r p_r\},$$

$$U(n) = \max_{1\leq i\leq r}\{\alpha_1 p_1, \alpha_2 p_2, ..., \alpha_r p_r\}.$$

X. Zhao, J. Guo, X. Mu and T. He [37]. *1. Let $k \geq 2$ be any fixed positive integer. Then for any real number $x \geq 2$, we have the asymptotic formula*

$$\sum_{n\leq x}\frac{V(n)\cdot U(n)}{Z\omega(n)} = \frac{x^2}{2}\cdot\sum_{i=1}^{k}\frac{a_i}{\ln^i x} + O\left(\frac{x^2}{\ln^{k+1} x}\right),$$

where $a_i = (i-1)!$.

2. Let $k \geq 2$ be any fixed positive integer. Then for any real numbers $x \geq 2$, we have the asymptotic formula

$$\sum_{n\leq x} V(n)\cdot Z\omega(n) = \frac{x^3}{3}\cdot\sum_{i=1}^{k}\frac{a_i}{\ln^i x} + O\left(\frac{x^3}{\ln^{k+1} x}\right),$$

where $a_i = (i-1)!$.

Definition 3. *For any positive integer n, a new Pseudo-Smarandache function $Z(n)$ is defined as the smallest positive integer m such that $n|\frac{m(m+1)}{2}$. That is,*

$$Z(n) = min\{m : n \mid \frac{m(m+1)}{2}, m \in \mathbb{N}\}.$$

X. Wang, L. Gao, G. Li and Y. Xue (HS) [29]. *Let $k \geq 2$ be any fixed positive integer. Then for any real number $x \geq 2$, we have the asymptotic formula*

$$\sum_{n\leq x} Z\omega(n)\cdot Z(n) = \frac{\zeta(2)\zeta(3)}{3\zeta(4)}\cdot\prod_p\left(1 - \frac{1}{p+p^3}\right)\cdot\frac{x^3}{\ln x} + \sum_{i=2}^{k}\frac{d_i\cdot x}{\ln^i x} + O\left(\frac{x^3}{\ln^{k+1} x}\right),$$

where $\zeta(s)$ *is the Riemann zeta function,* $\prod_p$ *denotes the product over all prime numbers,* d_i $(i = 1, 2, ..., k)$ *are computable constants.*

X. Fan and C. Tian [4]. *For any real number* $x \geq 2$*, we have the asymptotic formula*

$$\sum_{n\leq x} Z\omega(Z(n)) = \left(1 + \prod_p \left(1 + \frac{1}{p(p^2-1)}\right)\right) \cdot \frac{4\sqrt{2}}{\pi^2} \cdot x^{\frac{3}{2}} + O\left(x^{\frac{5}{4}}\right),$$

where $\prod_p$ *denotes the product over all prime numbers.*

Last, some scholars studied the hybrid mean value involving the Pseudo-Smarandache-Squarefree function and common number theory functions.

X. Fan and H. Zhou [5]. *1. Let* $k \geq 2$ *be any fixed integer, then for any real numbers* $x \geq 2$ *we have the asymptotic formula*

$$\sum_{n\leq x} p(n) \cdot Z\omega(n) = x^3 \sum_{i=1}^{k} \frac{b_i}{\ln^i x} + O\left(\frac{x^3}{\ln^{k+1} x}\right),$$

where b_i $(i = 1, 2, ..., k)$ *are computable constants with* $b_1 = \frac{1}{3}$ *and* $p(n)$ *represents the smallest prime factor.*

2. Let $k \geq 2$ *be any fixed integer, then for any real numbers* $x \geq 2$ *we have the asymptotic formula*

$$\sum_{n\leq x} \Lambda(n) \cdot Z\omega(n) = x^2 \sum_{i=1}^{k} \frac{c_i}{\ln^{i-1} x} + O\left(\frac{x^2}{\ln^k x}\right),$$

where c_i $(i = 1, 2, ..., k)$ *are computable constants with* $c_1 = \frac{1}{2}$ *and* $\Lambda(n) = \begin{cases} \ln p, & \textit{if } p \textit{ is prime}, \\ 0, & \textit{otherwise}. \end{cases}$

X. Wang, L. Gao, G. Li and Y. Xue (JYU) [30]. *For any real number* $x \geq 1, m \in \mathbb{N}$*, we have the asymptotic formula*

$$\sum_{n\leq x} Z\omega(n) \cdot \phi(n^m) = \frac{6x^{m+1}}{\pi^2(m+1)} \cdot \prod_p \left(1 + \frac{p-3}{1+p}\right) + O\left(x^{m+\frac{1}{2}+\varepsilon}\right),$$

where ε *is any fixed positive number.*

§2. the Smarandache dual function

Definition 4. *For any positive integer* n*, the famous Smarandache function* $S(n)$ *is defined as the smallest positive integer* m *such that* $n \mid m!$*. That is,*

$$S(n) = min\{m : n \mid m!\}.$$

Similarly, we introduce another function as following which have close relationship with the Smarandache function $S(n)$. It is the Smarandache dual function $S^*(n)$ which denotes the greatest positive integer m such that $m! \mid n$, where n denotes any positive integer. That is,

$$S^*(n) = max\{m : m! \mid n\}.$$

J. Li [22]. *For any real number $\alpha \leq 1$, the infinite series*

$$\sum_{n=1}^{\infty} \frac{S^*(n)}{n^\alpha}$$

is divergent, it is convergent if $\alpha > 1$, and

$$\sum_{n=1}^{\infty} \frac{S^*(n)}{n^\alpha} = \zeta(\alpha) \cdot \sum_{n=1}^{\infty} \frac{1}{(n!)^\alpha},$$

where $\zeta(s)$ is the Riemann zeta-function.

S. Xue [33]. *1. For any real number $s > 1$, we have the identities*

$$\sum_{n=1}^{\infty} \frac{(S^*(n))^k}{n^s} = \zeta(s) \cdot \sum_{n=1}^{\infty} \frac{n^k - (n-1)^k}{(n!)^s},$$

and

$$\sum_{n=1}^{\infty} \frac{1}{S^*(n) \cdot n^s} = \zeta(s) \cdot \left(1 - \sum_{n=1}^{\infty} \frac{1}{n(n+1)((n+1)!)^s}\right),$$

where $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ is the Riemann zeta-function.

2. For any real number $x > 1$, we have the asymptotic formula

$$\sum_{n \leq x} S^*(n) = (e-1)x + O\left(\frac{\ln^2 x}{(\ln \ln x)^2}\right).$$

F. Li [18]. *For any real number $x > 1$, we have the asymptotic formula*

$$\sum_{n \leq x} (S^*(n))^k = x \cdot \sum_{m=1}^{\infty} \frac{m^{k+1}}{(m+1)!} + O\left(\frac{\ln^{k+1} x}{(\ln \ln x)^{k+1}}\right).$$

M. Liu [27]. *For any positive integer n, we have the identity about the sum of Smarandache dual function $\sum_{d|n} S^*(d)$*

$$\sum_{d|n} S^*(d) = \begin{cases} (\alpha_1+1)(\alpha_2+1)...(\alpha_k+1), & \text{if } n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}, p_i \text{ is odd prime}; \\ (2\alpha+1)(\alpha_1+1)(\alpha_2+1)...(\alpha_k+1), & \text{if } n = 2^\alpha p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}, p_1 \neq 3; \\ (2\alpha+1+\alpha_1+3\alpha\alpha_1)(\alpha_2+1)...(\alpha_k+1), & \text{if } n = 2^\alpha p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}, p_1 = 3, \alpha = 1, 2. \end{cases}$$

where p_i are different odd primes; α, α_i are positive integers.

Next we introduce another Smarandache dual function $S^{**}(n)$.

Definition 5. *For any positive integer n, the new Smarandache dual function $S^{**}(n)$ is defined as the greatest positive integer $2m-1$ such that $(2m-1)!! \,|\, n$, if n is an odd number. $S^{**}(n)$ is the greatest positive integer $2m$ such that $(2m)!! \,|\, n$, if n is an even number. That is,*

$$S^{**}(n) = \begin{cases} max\{2m : m \in \mathbb{N}^*, (2m)!! \,|\, n\}, & \text{if } 2 \,|\, n; \\ max\{2m-1 : m \in \mathbb{N}^*, (2m-1)!! \,|\, n\}, & \text{if } 2 \nmid n. \end{cases}$$

S. Gou and X. Du [10]. *For any real number $s > 1$, the infinite series*

$$\sum_{n=1}^{\infty} \frac{S^{**}(n)}{n^s}$$

is convergent, and

$$\sum_{n=1}^{\infty} \frac{S^{**}(n)}{n^s} = \zeta(s)\left(1-\frac{1}{2^s}\right)\left(1+\sum_{m=1}^{\infty}\frac{2}{((2m+1)!!)^s}\right)+\zeta(s)\sum_{m=1}^{\infty}\frac{2}{((2m)!!)^s},$$

where $\zeta(s)$ is the Riemann zeta-function.

Q. Zhao and Y. Wang [39]. *For any real number $s > 1$, we have the identity*

$$\sum_{n=1}^{\infty} \frac{S^{**}(n)^2}{n^s} = \zeta(s)\left[1-\frac{1}{2^s}+\left(1-\frac{1}{2^s}\right)\sum_{m=1}^{\infty}\frac{8m}{((2m+1)!!)^s}+\sum_{m=1}^{\infty}\frac{8m-4}{((2m)!!)^s}\right],$$

where $\zeta(s)$ is the Riemann zeta-function.

From this theorem we immediately deduce the following limit formula:

$$\lim_{s\to 1}(s-1)\left(\sum_{n=1}^{\infty}\frac{S^{**}(n)^2}{n^s}\right)=\frac{13}{2}.$$

Then we introduce the Smarandache LCM function $SL(n)$.

Definition 6. *For any positive integer n, the Smarandache LCM function $SL(n)$ is defined as the smallest positive integer k such that $n\,|\,[1,2,...,k]$, where $[1,2,...,k]$ denotes the least common multiple of $1,2,...,k$. That is,*

$$SL(n) = min\{k : k \in \mathbb{N}, n\,|\,[1,2,...,k]\}.$$

Similarly, we introduce another function as following which have close relationship with the Smarandache LCM function $SL(n)$. It is the Smarandache LCM dual function $SL^*(n)$ which denotes the greatest positive integer k such that $[1,2,...,k]\,|\,n$, where n denotes any positive integer. That is,

$$SL^*(n) = max\{k : k \in \mathbb{N}, [1,2,...,k]\,|\,n\}.$$

J. Zhao and W. Duan [38]. *For any real number $x \geq 2$, we have the asymptotic formula*

$$\sum_{n\leq x}\frac{SL^*(n)}{n} = c+\frac{c}{x}+O\left(\frac{\ln^2 x}{x}\right).$$

where $c = \sum_{\alpha=1}^{\infty}\sum_{p}\frac{(p^\alpha-1)(p-1)}{[1,2,...,p]}$ is a constant.

Last, we define the Smarandache double factorial function $Sdf(n)$.

Definition 7. *For any positive integer n, $Sdf(n)$ is defined as the smallest positive integer m such that $n\,|\,m!!$. That is,*

$$Sdf(n) = min\{m : m \in \mathbb{N}, n\,|\,m!!\}.$$

L. Huan [15]. *1. For $n = p_1 p_2 \ldots p_k$ is a square-free number, we have the identity*

$$\prod_{d|n} S(d) = p_1 \cdot p_2^2 \cdot \ldots \cdot p_{k-1}^{2^{k-2}} \cdot p_k^{2^{k-1}}.$$

2. For $n = p_1 p_2 \ldots p_k$ is a square-free number or $n = p^k$ is a power of prime, we have the identity

$$\prod_{d|n} SL(d) = \begin{cases} p_1 \cdot p_2^2 \cdot \ldots \cdot p_{k-1}^{2^{k-2}} \cdot p_k^{2^{k-1}}, & \textit{if } n = p_1 p_2 \ldots p_k; \\ p^{\frac{k(k+1)}{2}}, & \textit{if } n = p^k. \end{cases}$$

3. For $n = p_1 p_2 \ldots p_k$ is a square-free number or $n = p^k$ is a power of prime, we have the identity

$$\prod_{d|n} SL^*(d) = \begin{cases} p_1^{2^{k-1}} \cdot p_2^{2^{k-2}} \cdot p_3^{2^{k-3}} \cdot \ldots \cdot p_{k-1}^2 \cdot p_k, & \textit{if } n = p_1 p_2 \ldots p_k; \\ p^{\frac{k(k+1)}{2}}, & \textit{if } n = p^k. \end{cases}$$

4. For $n = p_1 p_2 \ldots p_k$ is a square-free odd number or $n = 2p_1 p_2 \ldots p_k$ is a square-free even number, we have the identity

$$\prod_{d|n} Sdf(d) = \begin{cases} p_1 \cdot p_2^2 \cdot \ldots \cdot p_{k-1}^{2^{k-2}} \cdot p_k^{2^{k-1}}, & \textit{if } n = p_1 p_2 \ldots p_k; \\ 2^{2^k} \cdot p_1^2 \cdot p_2^{2^2} \cdot p_3 2^3 \cdot \ldots \cdot p_k^{2^k}, & \textit{if } n = 2p_1 p_2 \ldots p_k. \end{cases}$$

5. For $n = p_1 p_2 \ldots p_k$ is a square-free number or $n = p^k$ is a power of prime, we have the identity

$$\prod_{d|n} Z\omega(d) = \begin{cases} (p_1 p_2 \ldots p_k)^{2^{k-1}}, & \textit{if } n = p_1 p_2 \ldots p_k; \\ p^k, & \textit{if } n = p^k. \end{cases}$$

F. Zhang and J. Li [40]. *Let n be any positive integer. Then for any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{n \leq x} Sdf(n) = \frac{x \ln x}{\ln \ln x} + O\left(\frac{x \ln x}{(\ln \ln x)^2}\right)$$

§3. the other related Smarandache functions

In this part, we will introduce some new functions which are generalizations of the famous Smarandache function $S(n)$. Many scholars studied the elementary properties of the functions.

Definition 8. *Let p be a prime, n be any positive integer, $S_p(n)$ denotes the smallest integer m such that $m!$ is divisible by p^n. That is,*

$$S_p(n) = min\{m : m \in \mathbb{N}, p^n \mid m!\}.$$

F. Liang and Y. Yi [24]. *For any real number $x \geq 2$, let p be a prime and n be any positive integer. Then we have the asymptotic formula*

$$\sum_{\substack{n \leq x \\ S_p(n+1)=S_p(n)}} 1 = \frac{x}{p} + O\left(\frac{\ln x}{\ln p}\right).$$

R. Fu [6]. *Let p be a prime and n be any positive integer, then the equation*

$$S_p(1 \times 2) + S_p(2 \times 3) + ... + S_p(n \times (n+1)) = S_p\left(\frac{n(n+1)(n+2)}{3}\right)$$

has finite solutions.

1. If $p = 2, 3, 7$, the equation has only one solution $n = 1$;

2. If $p = 5, 11$, the equation has solutions $n = 1, 2, 3$;

3. If $p \geq 13$, the equation has finite positive integer solutions $n = 1, 2, ..., n_p$, where $n_p \geq 1$ is a positive integer and

$$n_p = \left[\sqrt[3]{\frac{3p}{2} + \sqrt{\frac{9p^2}{4} - \frac{1}{27}}} + \sqrt[3]{\frac{3p}{2} - \sqrt{\frac{9p^2}{4} - \frac{1}{27}}} - 1\right],$$

where $[x]$ denotes the greatest integer not exceeding x.

Definition 9. *For any positive integer n, the functions $Z_*(n)$ and $Z(n)$ will be defined as*

$$Z_*(n) = max\{m : m \in \mathbb{N}, \leq n\}$$

and

$$Z(n) = min\{m : m \in \mathbb{N}, n \leq \frac{m(m+1)}{2}\}.$$

J. Gao [8]. *1. For any complex number s, the infinite series*

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(Z_*(n))^s},$$

and

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(Z(n))^s}$$

are convergent if $s > 0$, and divergent if $s \leq 0$.

2. For any complex number s with Re $s > 2$, we have the identities

$$\sum_{n=1}^{\infty} \frac{1}{(Z_*(n))^s} = \zeta(s-1) + \zeta(s),$$

and

$$\sum_{n=1}^{\infty} \frac{1}{(Z(n))^s} = \zeta(s-1),$$

where $\zeta(s)$ is the Riemann zeta-function.

3. For any positive integer n and complex number s with Re $s > 1$, we have

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(Z_*(n))^s} = \frac{2}{4^s}\zeta(s) - \frac{1}{2^s}\zeta(s),$$

and

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(Z(n))^s} = \left(1 - \frac{1}{2^s}\right)\zeta(s) - \frac{2}{4^s}\zeta\left(s, \frac{1}{4}\right),$$

where $\zeta(s, \alpha)$ is the Hurwitz zeta-function.

Definition 10. *Define the arithmetic function $D(n)$ by*

$$D(n) = \begin{cases} 1, & \text{if } n = 1; \\ \prod_{p^\alpha | n} \alpha p^{\alpha - 1}, & \text{if } n > 1. \end{cases}$$

K. Liu [26]. *There exists an absolute constant $c > 0$, such that the asymptotic formula*

$$\sum_{n \le x} \frac{1}{D(n)} = c_1 x + O\left(x^{\frac{1}{2}} e^{-c\delta(x)}\right)$$

is ture, where

$$c_1 := \frac{6}{\pi^2} \prod_p \left(1 - \frac{p^2}{p+1}\left(\log\left(1 - \frac{1}{p^2}\right) + \frac{1}{p^2}\right)\right),$$

and

$$\delta(x) := \log^{\frac{3}{5}} x (\log \log x)^{-\frac{1}{5}}.$$

Definition 11. *For any positive integer n, we define a new Smarandache function $G(n)$ as the smallest positive integer m such that $\prod_{k=1}^{m} \phi(k)$ is divisible by n. That is,*

$$G(n) = min\{m : m \in \mathbb{N}, n \mid \prod_{k=1}^{m} \phi(k)\},$$

where $\phi(k)$ is the Euler function.

J. Fu and Y. Wang [7]. *1. For any prime p, we have the calculating formulae*

$$G(n) = min\{p^2, q(p, 1)\};$$
$$G(p^2) = q(p, 2),\ if\ q(p, 2) < p^2;\ G(p^2) = p^2,\ if\ q(p, 1) < p^2 < q(p, 2);$$
$$G(p^2) = q(p, 1),\ if\ p^2 < q(p, 1) < 2p^2;\ and\ G(p^2) = 2p^2,\ if\ q(p, 1) > 2p^2,$$

where $q(p, i)$ is the i-th prime in the arithmetical series $\{np + 1\}$.

2. $G(n)$ is a Smarandache multiplicative function, and moreover, the Dirichlet series $\sum_{n=1}^{\infty} \frac{G(n)}{n^2}$ is divergent.

3. Let $k \geq 2$ be a fixed positive integer, then for any positive integer group $(m_1, m_2, ..., m_k)$, we have the inequality

$$G(m_1 m_2 ... m_k) \leq G(m_1) G(m_2) ... G(m_k).$$

Definition 12. *For any positive integer $n \geq 3$, we define a arithmetic function $C(n)$ as the greatest positive integer $m \leq n-2$ such that $n \,|\, C_n^m = \frac{n!}{m! \cdot (n-m)!}$. That is,*

$$C(n) = max\{m : m \leq n-2, n \,|\, C_n^m\},$$

and define $C(1) = C(2) = 1$.

F. Li [19]. *1. Let $k > 2$ be a fixed positive integer, then there exists positive integer n such that*

$$n - C(n) \geq k.$$

2. For any positive integer $n \geq 4$, we have the asymptotic formula

$$C(n) = n + O\left(exp\left(\frac{c_1 \cdot \ln n}{\ln \ln n}\right)\right),$$

where $exp(y) = e^y$, $c_1 > 0$ is a constant.

3. For any real number $N \geq 4$, we have the asymptotic formula

$$\sum_{n \leq N} C(n) = \frac{1}{2} \cdot N^2 + O\left(N \cdot \ln N\right).$$

Definition 13. *For any positive integer n, we define the function $P(n)$ as the smallest prime p such that $n \,|\, p!$. That is,*

$$P(n) = \min\{p : n \,|\, p!, \text{where}\, p\, \text{be a prime}\}$$

H. Li [23]. *1. For any real number $x > 1$, we have the asymptotic formula*

$$\sum_{n \leq x} P(n) = \frac{1}{2} \cdot x^2 + O\left(x^{\frac{19}{12}}\right).$$

2. For any real number $x > 1$, we also have the mean value formula

$$\sum_{n \leq x} (P(n) - \overline{P}(n))^2 = \frac{2}{3} \cdot \zeta\left(\frac{3}{2}\right) \cdot \frac{x^{\frac{3}{2}}}{\ln x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x}\right),$$

where $\overline{P}(n)$ denotes the largest prime divisor of n, and $\zeta(s)$ is the Riemann zeta-function.

Definition 14. *For any positive integer n, the Smarandache reciprocal function $S_c(n)$ is defined as the largest positive integer m such that $y \mid n!$ for all integers $1 \leq y \leq m$, and $m+1 \nmid n!$. That is,*

$$S_c(n) = \max\{m : y \mid n! \,\text{for all}\, 1 \leq y \leq m, \text{and}\, m+1 \nmid n!\}$$

L. Ding [2]. *1. For any real number $x > 1$, we have the asymptotic formula*

$$\sum_{n \leq x} S_c(n) = \frac{1}{2} \cdot x^2 + O\left(x^{\frac{19}{12}}\right).$$

2. For any real number $x > 1$, we have the low bound estimate

$$\frac{1}{x} \sum_{n \leq x} (S_c(n) - n)^2 \geq \frac{1}{3} \cdot \ln^2 x + O\left(x^{-\frac{5}{12} \cdot \ln^2 x}\right).$$

Then F. Li studied the solvability of an equation involving $S_c(n)$ and $P(n)$, and give a new critical method for twin primes. That is, we shall have the following:

F. Li [20]. *For any positive integer $n > 3$, n and $n+2$ are twin primes if and only if n satisfy the equation*

$$S_c(n) = P(n) + 1.$$

Definition 15. *For any positive integer n, the Smarandache prime addictive function $SPAC(n)$ is defined as the smallest non-negative integer k such that $n + k$ is prime.*

Y. Guo and Y. Lu [11]. *For any positive integer n, we have the estimation*

$$A_n = \frac{1}{n} \sum_{a=1}^{n} SPAC(a) \geq \frac{1}{2} \cdot \ln n + O(1).$$

Y. Guo and G. Ren [12]. *For any real number $x > 1$, we have the asymptotic formula*

$$\sum_{n \leq x} (n + SPAC(n)) = \frac{1}{2} \cdot x^2 + O\left(x^{\frac{19}{12}}\right).$$

In section 2 we defined the least prime divisor function and the greatest prime divisor function $V(n)$, $U(n)$, now W. Huang, Y. He and Q. Qi gave many asymptotic formula of the functions.

Y. He [13]. *For any real number x, we have the asymptotic formula*

$$\sum_{n \leq x} V(n) \cdot U(n) = \frac{x^3}{3} \cdot \sum_{i=1}^{k} \frac{a_i}{\ln^i x} + O\left(\frac{x^3}{\ln^{k+1} x}\right),$$

where $a_i = (i-1)!$ is computable constant.

Y. He and Q. Qi [14]. *For any real number x, we have the asymptotic formula*

$$\sum_{n\leq x} V(n)\cdot p(n) = \frac{x^3}{3}\cdot\sum_{i=1}^{k}\frac{a_i}{\ln^i x} + O\left(\frac{x^3}{\ln^{k+1} x}\right),$$

where $p(n)$ represents the smallest prime factor, and $a_i = (i-1)!$ is computable constant.

W. Huang [16]. *1. Let r be any fixed positive integer, for any real number $x > 1$, we have the asymptotic formula*

$$\sum_{n\leq x} U(n) = x^2\cdot\sum_{i=1}^{r}\frac{f_i}{\ln^i x} + O\left(\frac{x^2}{\ln^{r+1} x}\right),$$

where $f_i (i = 1, 2, 3, ..., r)$ are computable constants and $f_1 = \frac{\pi^2}{12}$.

2. Let r, m be any fixed positive integer, for any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\leq x} V^m(n) = \frac{1}{m+1}x^{m+1}\cdot\sum_{i=1}^{r}\frac{a_i}{\ln^i x} + O\left(\frac{x^{m+1}}{\ln^{r+1} x}\right),$$

where $a_i (i = 1, 2, 3, ..., r)$ are computable constants.

Definition 16. *For any positive integer n, a new addictive function $G(n)$ is defined as follows:*
(1) $G(1) = 0$;
(2) if $n > 1$ and $n = p_1^{\alpha_1}p_2^{\alpha_2}...p_k^{\alpha_k}$ denotes the prime power factorization of n, then $G(n) = \frac{\alpha_1}{p_1} + \frac{\alpha_2}{p_2} + ... + \frac{\alpha_k}{p_k}$.

Now we define the Smarandache divisor product sequences $\{p_d(n)\}$ and $\{q_d(n)\}$ as follows: $p_d(n)$ denotes the product of all positive divisors of n; $q_d(n)$ denotes the product of all positive divisors of n but n. That is,

$$p_d(n) = \prod_{d|n} d = n^{\frac{d(n)}{2}}; q_d(n) = \prod_{d|n, d<n} d = n^{\frac{d(n)}{2}-1},$$

where $d(n)$ denotes the Dirichlet divisor function.

W. Yao and T. Cao [36]. *1. For any real number $x > 1$, we have the asymptotic formula*

$$\sum_{n\leq x} G(p_d(n)) = B\cdot x\cdot\ln x + (2\gamma\cdot B - D - B)\cdot x + O\left(\sqrt{x}\ln\ln x\right),$$

where $B = \sum_p \frac{1}{p^2}, D = \sum_p \frac{\ln p}{p^2}$, γ is the Euler constant, and $\sum_p$ denotes the summation over all primes.

2. For any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\leq x} G(q_d(n)) = B\cdot x\cdot\ln x + (2\gamma\cdot B - D - 2B)\cdot x + O\left(\sqrt{x}\ln\ln x\right),$$

where B and D are defined as same as in 1.

Definition 17. *For any positive integer n and two fixed positive integer m and $k(k > 1)$, a generalized power-sum Smarandache function $P(n, m, k)$ is defined as*

$$P(n, m, k) = \sum_{i=0}^{[(\ln m+\ln n)/\ln k]} (n - k^i).$$

Y. Wang and L. Hua [32]. *Let m and $k(k > 1)$ be any fixed positive integer, for any real number $x > 1$, we have the asymptotic formula*

$$\sum_{n\le x} P(n, m, k) = \frac{1}{2\ln k} \cdot x^2 \cdot \ln(mx) + R(n, m, k),$$

where $|R(n, m, k)| \le \left(\frac{1}{4} + \frac{mk}{k-1}\right) x^2 + \frac{x}{k-1} + \frac{2x\ln(mx)+2x+\ln(mx)+x\ln k}{2\ln k}$.

References

[1] Bin Cheng. On the pseudo Smarandache square-free function. Scientia Magna 4 (2008), no. 2, 21 - 24.

[2] Liping Ding. On the Smarandache reciprocal function and its mean value. Scientia Magna 4 (2008), no. 1, 120 - 123.

[3] Xuhui Fan. On the Pseudo-Smarandache-Squarefree function and Smarandache function. Scientia Magna 4 (2008), no. 4, 7 - 11.

[4] Xuhui Fan and Chengliang Tian. On the mean value of the Pseudo-Smarandache-Squarefree function. Scientia Magna 4 (2008), no. 1, 86 - 89.

[5] Xuhui Fan and Hang Zhou. Pseudo-Smarandache-Squarefree function and its value distribution. Jounal of Kunming University of Science and Technology (Science and Technology) 33 (2008), no. 6, 117 - 120. (In Chinese with English abstract).

[6] Ruiqin Fu. An equation involving the Smarandache primitive function and a kind of sum of natural numbers products. Journal of Yanan University (Natural Science Edition) 27 (2008), no. 3, 3 - 7. (In Chinese with English abstract).

[7] Jing Fu and Yu Wang. A new Smarandache function and its elementary properties. Scientia Magna 4 (2008), no. 3, 26 - 28.

[8] Jianli Gao. On the additive analogues of the simple function. Scientia Magna 2 (2016), no. 3, 88 - 91.

[9] Wenji Guan. Four problems related to the Pseudo-Smarandache-Squarefree function. Scientia Magna 3 (2007), no. 4, 45 - 47.

[10] Su Gou and Xiaoying Du. On the Smarandache dual functions. Pure and Applied Mathematics 24 (2008), no. 1, 17 - 21. (In Chinese with English abstract).

[11] Yanchun Guo and Yulin Lu. On the sequence of Smarandache prime additive complement. Basic Sciences Journal of Textile Universities 21 (2008), no. 1, 128 - 131. (In Chinese with English abstract).

[12] Yanchun Guo and Ganglian Ren. On the Smarandache prime additive complement. Journal of Weinan Normal University 27 (2012), no. 2, 14 - 17. (In Chinese with English abstract).

[13] Yanfeng He. One hybrid mean value formula involving two Smarandache functions. Journal of Natural Science of Heilongjiang University 25 (2008), no. 4, 477 - 480. (In Chinese with English abstract).

[14] Yanfeng He and Qiong Qi. One hybrid mean value formula involving Smarandache function and the least prime divisor function. Basic Sciences Journal of Textile Universities 21 (2008), no. 3, 376 - 378. (In Chinese with English abstract).

[15] Le Huan. On the properties of Smarandache function and its related functions. Journal of Southwest University (Natural Science Edition) 35 (2013), no. 4, 67 - 71. (In Chinese with English abstract).

[16] Wei Huang. Two hybrid mean value formulas of involving Smarandache functions. Mathematics in practice and theory 41 (2011), no. 24, 252 - 255. (In Chinese with English abstract).

[17] Maohua Le. On the Pseudo-Smarandache squarefree function. Smarandache Notions Journal 13 (2002), 229 - 236.

[18] Fanbei Li. On the k-th mean value of Smarandache dual function. Journal of Inner Mongolia University of Finance and Economics 11 (2013), no. 5, 134 - 137. (In Chinese with English abstract).

[19] Fanbei Li. A new Smarandache function and its mean value. Pure and Applied Mathematics 24 (2008), no. 4, 682 - 686. (In Chinese with English abstract).

[20] Fanbei Li. A new critical method for twin primes. Scientia Magna 4 (2008), no. 2, 56 - 58.

[21] Jianghua Li. On the Pseudo-Smarandache-Squarefree function. Scientia Magna 3 (2007), no. 3, 93 - 96.

[22] Jie Li. On Smarandache dual function. Scientia Magna 2 (2006), no. 1, 111 - 113.

[23] Hailong Li. A generalization of the Smarandache function. Scientia Magna 4 (2008), no. 1, 31 - 34.

[24] Fangchi Liang and Yuan Yi. The primitive numbers of power p and its asymptotic property. Research on Smarandache problems in number theory, 129 - 131, Hexis, Phoenix

[25] Huaning Liu and Jing Gao. Mean value on the Pseudo-Smarandache-Squarefree function. Research on Smarandache problems in number theory. Vol. I, 9C11, Hexis, Phoenix, AZ, 2004.

[26] Kui Liu. On mean values of an arithmetic function. Scientia Magna 3 (2007), no. 4, 12 - 15.

[27] Miaohua Liu. A study of Smarandache dual function calculation formula. Journal of Air Force Engineering University (Natural Science Edition) 14 (2013), no. 5, 92 - 95. (In Chinese with English abstract).

[28] Felice Russo. A set of new Smarandache function, sequences and conjectures in number theory. Lupton USA, 2000.

[29] Xihan Wang, Li Gao, Guorong Li and Yang Xue. A hybrid mean value of the Pseudo-Smarandache-Squarefree function. Henan Science 34 (2016), no. 9, 1410 - 1414. (In Chinese with English abstract).

[30] Xihan Wang, Li Gao, Guorong Li and Yang Xue. The hybrid mean value of the Pseudo-Smarandache-Squarefree function and Euler function. Journal of Yanan University (Natural Science Edition) 35 (2016), no. 3, 8 - 10. (In Chinese with English abstract).

[31] Yu Wang. On the mean value of the new pseudo F.Smarandache function. Journal of Science of Teachers' College and University 31 (2011), no. 3, 1 - 3.(In Chinese with English abstract).

[32] Yang Wang and Liuqing Hua. On the mean value of generalized power-sum Smarandache function. Journal of Inner Mongolia Normal University (Natural Science Edition) 43 (2014), no. 2, 149 - 152. (In Chinese with English abstract).

[33] Shejiao Xue. On the Smarandache dual function. Scientia Magna 3 (2007), no. 1, 29 - 32.

[34] Shejiao Xue and Chunliang Yan. Near pseudo modification of Smarandache function. Journal of Weinan Teachers College 21 (2006), no. 5, 15 - 19. (In Chinese with English abstract).

[35] Wenjing Xiong. On a problem of Pseudo-Smarandache-Squarefree function. Jounal of Northwest University (Natural Science Edition) 38 (2008), no. 4, 192 - 195. (In Chinese with English abstract).

[36] Weili Yao and Tieming Cao. A new additive function and the Smarandache divisor product sequences. Scientia Magna 5 (2009), no. 2, 1 - 5.

[37] Xinghua Zhao, Jingbao Guo, Xiumei Mu and Tao He. Hybrid mean value involving the Pseudo-Smarandache-Squarefree function. Henan Science 29 (2011), no. 11, 1279 - 1282. (In Chinese with English abstract).

[38] Jiaolian Zhao and Weiguo Duan. Mean value involving the Smarandache LCM dual function. Science Technology and Engineering 8 (2008), no. 5, 1265 - 1268.

[39] Qiuhong Zhao and Yang Wang. A limit problem of the Smarandache dual function $S^{**}(n)$. Scientia Magna 5 (2009), no. 3, 87 - 89.

[40] Fuling Zhang and Jianghua Li. On the mean value of Smarandache function $d_f(n)$. Journal of Northwest University (Natural Science Edition) 38 (2008), no. 4, 531 - 533. (In Chinese with English abstract). , AZ, 2004.

Scientia Magna
Vol. 16 (2021), No. 1, 51-58

Asymmetric Fuglede - Putnam theorem for $p-(\alpha, \beta)$-normal operators

Ahmed Bachir[1],T. Prasad[2] and Nawal Ali Sayyaf [3]

[1]Department of Mathematics, King Khalid University
P.O.Box 9004, Abha, Saudi Arabia
E-mail: abishr@kku.edu.sa
[2]Department of Mathematics, Cochin University of Science and Technology
Cochin 22, India
E-mail: prasadvalapil@gmail.com
[3]Department of Mathematics, Bisha University
Bisha, Saudi arabia
E-mail: nasayyaf@ub.edu.sa

Abstract In this paper, we obtain some properties of $p-(\alpha, \beta)$-normal and we prove the following assertions. (i) If T is $p-(\alpha, \beta)$-normal operator, S is an invertible operator and X is a Hilbert-Schmidt operator such that $TX = XS$, then $T^*X = XS^*$. (ii) If T is totally $p-(\alpha, \beta)$-normal operator, then the range of generalized derivation $\delta_T : B(\mathcal{H}) \ni X \to TX - XT \in B(\mathcal{H})$ is orthogonal to its kernel.

Keywords Fuglede-Putnam Theorem - Hyponormal operator - (α, β)-Normal operator - $p-(\alpha, \beta)$-Normal operator - Orthogonality.
2010 Mathematics Subject Classification 47B20, 47A10, 47A30.

§1. Introduction and preliminaries

Spectral theory has a key important role in the modern functional analysis and its applications in various fields. Basically, it is incorporated with specific inverse operators, their common properties and their dealings with the original operators. Such inverse operators play a major role in solving systems of linear algebraic equations, differential and Sylvester equations.

Throughout this paper, $\mathcal{H}$ denotes an infinite dimensional complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and $\mathcal{B}(\mathcal{H})$ denotes the algebra of all bounded linear operators acting on $\mathcal{H}$. The kernel, the range, the spectrum, point spectrum of an operator T will be denoted by $\ker T$, $\operatorname{ran}(T)$, $\sigma(T)$, $\sigma_p(T)$, respectively. For any operator $T \in \mathcal{B}(\mathcal{H})$, set, as usual $|T| = (T^*T)^{1/2}$, and consider the following standard definitions: normal if $T^*T = TT^*$ and T is hyponormal if $|T^*|^2 \leq |T|^2$. An operator $T \in B(\mathcal{H})$ is said to be $p-(\alpha, \beta)$-normal operator $(0 \leq \alpha \leq 1 \leq \beta)$ for $0 < p \leq 1$ if

$$\alpha^2 (T^*T)^p \leq (TT^*)^p \leq \beta^2 (T^*T)^p.$$

When $p = 1$, the operator T is said (α, β)-normal operator.

S.S. Dragomir and M.S. Moslehian [9] has given various inequalities between the operator norm and the numerical radius of (α, β)-normal operator.

An operator is said to be

(i) p-hyponormal [1] for $0 < p < 1$ if $(TT^*)^p \leq (T^*T)^p$;

(ii) dominant if $\operatorname{ran}(T - \lambda I) \subseteq \operatorname{ran}(T - \lambda I)^*$ for all $\lambda \in \mathbb{C}$, or equivalently

$$\exists M_\lambda \in \mathbb{C} : \|(T - \lambda I)^* x\| \leq M_\lambda \|(T - \lambda I)x\| \text{ for each } x \in \mathcal{H};$$

(iii) M-hyponormal if there exists a constant M such that $M_\lambda \leq M$ for all $\lambda \in \mathbb{C}$.

The weighted shift defined by $We_1 = e_2$, $We_2 = 2e_3$ and $We_i = e_{i+1}$ for $i \geq 2$ is an example of M-hyponormal [25], is not an p-(α, β)-normal, which is either normal nor hyponormal. It is proved in [19] that every p-(α, β)-normal operator is M-hyponormal.

So, we conclude that the class p-(α, β)-normal lies between hyponormal and M-hyponormal operators, and we have the following classes inclusion

$$\begin{aligned} \text{Normal} \quad &\subseteq \quad \text{Hyponormal} \subseteq (\alpha, \beta) - \text{normal} \subseteq p - (\alpha, \beta) - \text{normal} \\ &\subseteq \quad M - \textit{hyponormal} \subseteq \text{Dominant} \end{aligned}$$

The familiar Fuglede-Putnam theorem asserts that if $T \in B(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{H})$ are normal operators and $TX = XS$ for some $X \in B(\mathcal{H})$, then $T^*X = XS^*$ (see [17]). Many authors extended this theorem for several nonnormal classes of operators (see [4–6, 12, 14, 16, 18, 22, 24]). Following [26], we say that an operator $T \in B(\mathcal{H})$ is finite if

$$\|I - (TX - XT)\| \geq 1$$

holds for all $X \in B(\mathcal{H})$. The above inequality is the starting point of the study of commutator approximations, a topic with roots in quantum theory [27]. Let $\mathcal{B}$ denote a Banach algebra. Recall that $b \in \mathcal{B}$ is said to be Brikhoff orthogonal to $a \in \mathcal{B}$, written as $b \perp a$, if the inequality

$$\|a\| \leq \|a + \mu b\|$$

holds for all $\mu \in \mathbb{C}$. The above definition of orthogonality has natural geometric meaning, namely, $b \perp a$ if and only if the line $\{a + \mu b : \mu \in \mathbb{C}\}$ is tangent to the ball of center zero and radius $\|a\|$. If $\mathcal{B} = \mathcal{H}$, then the orthogonality means usual sense $\langle a, b \rangle = 0$.

The generalized derivation $\delta_{S,T} : B(\mathcal{H}) \to B(\mathcal{H})$ for $S, T \in B(\mathcal{H})$ is defined by $\delta_{S,T}(X) = SX - XT$ for $X \in B(\mathcal{H})$, and we set $\delta_{T,T} = \delta_T$. If the following inequality

$$\|S - (TX - XT)\| \geq \|S\|$$

holds for all $S \in \ker \delta_T$ and for all $X \in B(\mathcal{H})$, then we say that the range of δ_T is orthogonal to the kernel of δ_T .

Let $T \in B(\mathcal{H})$ and let $\{e_n\}$ be an orthonormal basis of a Hilbert space $\mathcal{H}$. The Hilbert-Schmidt norm is given by

$$\|T\|_2 = \left(\sum_{n=1}^{\infty} \|Te_n\|^2\right)^{\frac{1}{2}}.$$

An operator T is said to be a Hilbert-Schmidt operator if $\|T\|_2 < \infty$ (see [3] for details). $C_2(\mathcal{H})$ denotes a set of all Hilbert-Schmidt operators. For $S, T \in B(\mathcal{H})$, the operator $\Gamma_{S,T}$ defined as $\Gamma_{S,T} : C_2(\mathcal{H}) \ni X \to SXT \in C_2(\mathcal{H})$ has been studied in [2]. It is known that $|\Gamma| \leq \|S\|\|T\|$ and $(\Gamma_{S,T})^*X = S^*XT^* = \Gamma_{S^*,T^*}X$. If $T \geq 0$ and $S \geq 0$, then $\Gamma_{T,S} \geq 0$. For more information see [2].

The organization of the paper is as follows; in Section 2, we give some properties for $p-(\alpha,\beta)$-normal operators and totally $p-(\alpha,\beta)$-normal operators needed in the sequel. In Section 3, we prove the following assertions:

1. if T is $p-(\alpha,\beta)$-normal operator, S is an invertible operator and X is a Hilbert-Schmidt operator such that $TX = XS$, then $T^*X = XS^*$, and

2. if T is totally $p-(\alpha,\beta)$-normal operator, then the range of generalized derivation δ_T : $B(\mathcal{H}) \ni X \to TX - XT \in B(\mathcal{H})$ is orthogonal to its kernel.

§2. Some properties of $p-(\alpha,\beta)$-normal operators

In this section we give several preliminary results which will be used in the sequel.

Definition 2.1. *An operator $T \in B(\mathcal{H})$ is called $p-(\alpha,\beta)$-normal operator $0 < p \leq 1$, $(0 \leq \alpha \leq 1 \leq \beta)$ if*

$$\alpha^2 (T^*T)^p \leq (TT^*)^p \leq \beta^2 (T^*T)^p.$$

Lemma 2.2. [6] *Every invertible operator A is $p-(\alpha,\beta)$-normal operator.*

Theorem 2.3. [19] *If $T \in B(\mathcal{H})$ is an $p-(\alpha,\beta)$-normal, then the following hold:*
(i)*If $Tx = \lambda x$, $\lambda \neq 0$, then $T^*x = \bar{\lambda}x$.*
(ii) *$\sigma_{jp}(T) - \{0\} = \sigma_p(T) - \{0\}$.*
(iii) *If $Tx = \lambda x$ and $Ty = \mu y$ with $\lambda \neq \mu$, then $x \perp y$.*

Definition 2.4. *An operator T is called totally $p-(\alpha,\beta)$-normal, if the translation $T-\lambda$ is $p-(\alpha,\beta)$-normal for all $\lambda \in \mathbb{C}$.*

§3. Main Results

Given nonzero operators $T, S \in B(\mathcal{H})$, let $T \otimes S$ denote the tensor products on the tensor product space $\mathcal{H} \otimes \mathcal{H}$. The operation of taking tensor products $T \otimes S$ preserves many properties of $T, S \in B(\mathcal{H})$. $T \otimes S$ is hyponormal if and only if T and S are hyponormal [11]. Now we obtain an analogous result for $p-(\alpha,\beta)$-normal operators.

Lemma 3.1. [21] *Let $A_1, A_2 \in B(\mathcal{H}), B_1, B_2 \in B(\mathcal{H})$ be non-negative operators. If A_1 and B_1 are non-zero, then the following assertions are equivalent*

(i) $A_1 \otimes B_1 \leq A_2 \otimes B_2$
(ii) *There exists* $c > 0$ *for which* $A_1 \leq A_2$ *and* $B_1 \leq c^{-1}B_2$.

Theorem 3.2. *If* $T \in B(\mathcal{H})$ *is* $p-(\alpha_1, \beta_1)$*-normal operator and* $S \in B(\mathcal{H})$ *is* $p-(\alpha_2, \beta_2)$*-normal operator, then* $T \otimes S$ *is* $p-(\alpha_1\alpha_2, \beta_1\beta_2)$*-normal operator.*

Proof. Note that $0 \leq \alpha_1\alpha_2 \leq 1 \leq \beta_1\beta_2$ and by Lemma 3.1, we have

$$\begin{aligned}
\alpha_1^2\alpha_2^2[(T \otimes S)^*(T \otimes S)]^p &= \alpha_1^2\alpha_2^2(T^*T \otimes S^*S)^p \\
&= \alpha_1^2\alpha_2^2(T^*T)^p \otimes (S^*S)^p \\
&= \alpha_1^2(TT^*)^p \otimes \alpha_2^2(SS^*)^p \\
&\leq (TT^*)^p \otimes (SS^*)^p \\
&= [(T \oplus S)(T \oplus S)^*]^p.
\end{aligned}$$

Similarly,

$$[(T \otimes S)(T \otimes S)^*]^p \leq \quad \beta_1^2\beta_2^2[(T \otimes S)^*(T \otimes S)]^p.$$

□

In the following, we show that if X is a Hilbert-Schmidt operator, T is $p-(\alpha, \beta)$-normal operator and S is an invertible operator such that $TX = XS$, then $T^*X = XS^*$.

Lemma 3.3. *Let* $T, S \in \mathcal{B}(\mathcal{H})$. *If* T *is* $p-(\alpha, \beta)$*-normal and* S^* *is* $p-(\alpha', \beta')$*-normal, then the operator* $\Gamma_{T,S}$ *defined on* $C_2(\mathcal{H})$ *by* $\Gamma_{T,S}(X) = TXS$ *is* $p-(\alpha\alpha', \beta\beta')$*-normal operator.*

Proof. The unitary operator $\mathcal{U} : C_2(\mathcal{H}) \to \mathcal{H} \otimes \mathcal{H}$ by a map $x \otimes y^* \to x \otimes y$ induces the $*$-isomorphism $\Psi : \mathcal{B}(C_2(\mathcal{H})) \to \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ by a map $X \to \mathcal{U}X\mathcal{U}^*$. Then we can obtain $\Psi(\Gamma_{T,S}) = T \otimes S^*$ [8]. The complete proof comes from Theorem 3.2. □

Theorem 3.4. *Let* T *be an* $p-(\alpha, \beta)$*-normal operator and* S^* *be an invertible operator. If* $TX = XS$ *for some* $X \in C_2(\mathcal{H})$, *then* $T^*X = XS^*$.

Proof. Let Γ be defined on $C_2(\mathcal{H})$ by $\Gamma(Y) = TYS^{-1}$. The operator S is invertible, then S is $p-(\alpha, \beta)$-normal operator by Lemma 2.1.

Since T and S are $p-(\alpha, \beta)$-normal operators, we observe that Γ is also $p-(\alpha^2, \beta^2)$-normal operator by Lemma 3.3. Moreover, $\Gamma(X) = TXS^{-1} = X$ because of $TX = XS$. Hence X is an eigenvector of Γ. By Theorem 2.3, we have $\Gamma^*(X) = T^*X(S^{-1})^* = X$, that is, $T^*X = XS^*$ as desired. □

Theorem 3.5. *Let* $T \in B(\mathcal{H})$ *be a* $p-(\alpha, \beta)$*-normal operator and* $\mathcal{M} \subseteq \mathcal{H}$ *be an invariant subspace of* T, *then the restriction* $T|_{\mathcal{M}}$ *is also* $p-(\alpha, \beta)$*-normal.*

Proof. Let P be the orthogonal projection on $\mathcal{M}$ and Let $A = T|_{\mathcal{M}}$, then $TP = PTP = A \oplus 0$ on $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$. Since T is $p-(\alpha, \beta)$-normal

$$\alpha^2 P(T^*T)^p P \leq P(TT^*)^p P$$

and so

$$\alpha^2(|A|^{2p}\oplus 0) \leq P(TT^*)^pP$$

From Hansen's inequality

$$\begin{aligned}
\alpha^2|A|^{2p} &\leq (PTT^*P)^p \\
&\leq (PTPT^*P)^p \\
&\leq [(A\oplus 0)(A^*\oplus 0)]^p \\
&= (AA^*)^p
\end{aligned}$$

Similarly, we prove that

$$(AA^*)^p \leq \beta^2(A^*A)^p$$

This means that the restriction A is $p-(\alpha,\beta)$-normal. □

Lemma 3.6. *If T is $p-(\alpha,\beta)$-normal for some $0<p\leq 1$, then T is $q-(\alpha,\beta)$-normal for every $0<q\leq p$.*

Proof. The result follows from the Lowner-Heinz's inequality. □

A complex number λ is said to be in the joint spectrum of T if there exist a joint eigenvector x of T and T^* such that $Tx=\lambda x$ and $T^*x=\bar{\lambda}x$. The following result of totally $p-(\alpha,\beta)$-normal operators are finite has been obtained in [19].

Theorem 3.7. [19] *If T is totally $p-(\alpha,\beta)$-normal operator, then T is finite.*

Lemma 3.8. *If $T\in B(\mathcal{H})$ is totally $p-(\alpha,\beta)$-normal and if S is a normal operator such that $TS=ST$, then*

$$\|S-(TX-XT)\| \geq |\mu|$$

for all $\mu\in\sigma_p(S)$ and for all $X\in B(\mathcal{H})$.

Proof. Let $\mathcal{M}_\mu$ be the eigenspace associated to $\mu\in\sigma_p(S)$. Since S is normal, the Fuglede-Putnam theorem ensures $TS=ST$ implies $S^*T=TS^*$. Hence $\mathcal{M}_\mu$ reduces both S and T. Now we write matrix representations of T,S and X as

$$T=\begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix},\quad S=\begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix},\quad \text{and } X=\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix},$$

on $\mathcal{H}=\mathcal{M}_\mu\oplus\mathcal{M}_\mu^\perp$. Hence we have

$$S-(TX-XT)=\begin{pmatrix} \mu-(T_1X_1-X_1T_1) & P \\ Q & R \end{pmatrix},$$

for some operators P,Q and R and hence

$$\|S-(TX-XT)\| \geq \|\mu-(T_1X_1-X_1T_1)\| \tag{1}$$

Since T is totally $p-(\alpha,\beta)$-normal operator and $\mathcal{M}_\mu$ is a reducing subspace of T, the restriction $T_1 = T\mid_{\mathcal{M}_\mu}$ is totally $p-(\alpha,\beta)$-normal by Theorem 3.5. Since T_1 is finite by Theorem 3.7, we have

$$\|(T_1X_1 - X_1T_1) - \mu\| \geq \|\left\|\left(T_1\frac{X_1}{\mu} - \frac{X_1}{\mu}T_1\right) - 1\right\| \geq |\mu|. \tag{2}$$

From (1) and (2), we have

$$\|S - (TX - XT)\| \geq |\mu|$$

for all $X \in \mathcal{B}(\mathcal{H})$. □

The following result due to S.K. Berberian [7] is well known.

Proposition 3.9. [7] [Berberian technique] *Let $\mathcal{H}$ be a complex Hilbert space. Then there exist a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and $\psi : B(\mathcal{H}) \to B(\mathcal{K})$ such that ψ is an $*$-isometric isomorphism preserving the order satisfying*

(i) $\psi(T^*) = \psi(T)^*$, $\psi(I_{\mathcal{H}}) = I_{\mathcal{K}}$, $\psi(\alpha T + \beta S) = \alpha\psi(T) + \beta\psi(S)$, $\psi(TS) = \psi(T)\psi(S)$, $\|\psi(T)\| = \|T\|$, $\psi(T) \leq \psi(S)$ *if* $T \leq S$ *for all* $T, S \in B(\mathcal{H})$ *and for all* $\alpha, \beta \in \mathbb{C}$.

(ii) $\sigma(T) = \sigma(\psi(T)), \sigma_a(T) = \sigma_a(\psi(T)) = \sigma_p(\psi(T))$, *where* $\sigma_p(T)$ *is the point spectrum of* T.

Now, we prove that the range of $\delta_{T,S}$ is orthogonal to the null space of $\delta_{T,S}$ when T is totally $p-(\alpha,\beta)$-normal and S is a normal operator.

Theorem 3.10. *Let $T \in B(\mathcal{H})$ be totally $p-(\alpha,\beta)$-normal and let S be a normal operator such that $TS = ST$. Then*

$$\|S\| \leq \|S - (TX - XT)\|$$

for all $X \in \mathcal{B}(\mathcal{H})$.

Proof. By Proposition 3.9, it follows that $\psi(T)$ is $p-(\alpha,\beta)$-normal, $\psi(S)$ is normal and $\psi(T)\psi(S) = \psi(S)\psi(T)$. Since $\psi(T)$ is finite by Theorem 3.7, we have for all $\mu \in \sigma_p(S) = \sigma_a(\psi(S)) = \sigma_a(S) = \sigma(S)$,

$$|\mu| \leq \|\psi(S) - \psi(T)\psi(X) - \psi(X)\psi(T)\| = \|S - (TX - XT)\|$$

for all $X \in \mathcal{B}(\mathcal{H})$ by Lemma 3.8. Hence

$$\|S\| = r(S) = \sup_{\mu\in\sigma(S)} |\mu| \leq \|S - (TX - XT)\|.$$

□

Acknowledgements

The author would like to express his cordial gratitude to the referee for valuable comments which improved the paper.

References

[1] A. Aluthge, On p-hyponormal operatorf for $0 < p < 1$, Integral Equations Operators Theory, 13(1990), 307–315.

[2] S. K. Berberian, Extensions of a theorem of Fuglede and Putnam, Proc. Amer. Math. Soc., 71(1978), 113-114.

[3] J. B. Conway, A Course in Functional Analysis, 2nd ed, Springer, New York, 1990.

[4] A. Bachir and F. Lombarkia, Fuglede-Putnam's theorem for w-hyponormal operators, Math. Inequal. Appl., 15(2012), 777–786.

[5] A. Bachir, Fuglede-Putnam Theorem for w-hyponormal or class $\mathcal{Y}$ operators. Ann. Funct. Anal. 4 (2013), no. 1, 53–60.

[6] A. Bachir and T. Prasad, Putnam-Fuglede Theorem for (α,β)-Normal Operator, Rendiconti del Circolo Mathematico di Palermo Serie 2, 68(2019).

[7] S.K. Berberian, Approximate proper vectors, Proc. Amer. Math. Soc., 13(1962), 111–114.

[8] A. Brown, C. Percy, Spectra of Tensor Products of Operators, Proc. Amer. Math. Soc., 17(1966), 162–166.

[9] S.S. Dragomir and M.S. Moslehian, Some inequalities for (α,β)-normal operators in Hilbert spaces, Ser. Math. Inform., (2008), 39–47.

[10] T. Furuta, An extenstion of the Fuglede theorem to subnormal operators using a Hilbert-Schmidt norm inequality, Proc. Amer. Math. Soc., 81(1981), 240–242.

[11] J.C. Hou, On tensor Products of Operators, Acta Math. Sin., 91 (1993), 195–202.

[12] R.L. Moore, D.D. Rogers and T.T. Trent, A note on intertwining M -hyponormal operators, Proc. Amer. Math. Soc., 83(1981), 514–516.

[13] S. Mecheri, Finite operators and orthogonality, Nihonkai Math., 19(2008) 53-60.

[14] S. Mecheri, A generalization of Fuglede-Putnam theorem, J. Pure Math. 21(2004), 31–38.

[15] M.S. Moslehian, On (α,β)-normal operators in Hilbert spaces, IMAGE, Problem (2007), 39–47.

[16] P. Pagacz, The Putnam-Fuglede property for paranormal and *-paranormal operators, Opuscula Math. 33, no. 3 (2013), 565–574.

[17] C.R. Putnam, On the normal operators on Hilbert space, Amer. J. Math., 73(1951), 357–362.

[18] M. H. M. Rashid and M. S. M. Noorani, On relaxation normality in the Fulgede-Putnam's theorem for a quasi-class $\mathcal{A}$ operators, Tamkang. J. Math., 40(2009), 307–312.

[19] D. Senthilkumar and R. Santhi, On p-(α,β)-Normal Operators, Applied Mathematical Sciences, Vol. 8, 41(2014), 2014–2052.

[20] D. Senthilkumar and S. M. Sherin Joy, On totally (α,β)-normal operators, Far East Journal of Mathematical Sciences (FJMS), Volume 71, Number 1, 2012, Pages 151-167,

[21] J. Stochel, Seminormality of Operators from their Tensor Product, Proc. Amer. Math. Soc., 124(1996), 435–440.

[22] A. Uchiyama and K. Tanahashi, Fuglede-Putnam's theorem for p-hyponormal or log-hyponormal operators, Glasgow Math. J., 44(2002), 397–410.

[23] K. Takahashi, M. Uchiyama, Every C_{00} contractions with Hilbert-Schmidt defect operator is of class C_0, J. Operator Theory 12 (1983) 331–335.

[24] T. Yoshino, Remark on the generalized Putnam–Fuglede theorem, Proc. Amer. Math. Soc., 95(1985), 571–572.

[25] B.I. Wadhwa, M-hyponormal Operators, Duke Math. J., 41(3)(1974), 655–660.

[26] J.P. Williams, Finite operators, Proc. Amer. Math. Soc., 26 (1970), 129-135.

[27] H. Wielandt, Helmut ber die Unbeschrnktheit der Operatoren der Quantenmechanik, (German) Math. Ann., 121(1949), 21.

Scientia Magna
Vol. 16 (2021), No. 1, 59-66

On Γ-ring for a class of Entire Dirichlet series in two variables

Lakshika Chutani[1] , Garima Manocha[2] and Niraj Kumar[3]

[1]Vivekananda School of Economics, Vivekananda Institute of Professional Studies
Pitam Pura,New Delhi - 110034, India
E-mail: lakshika91.chutani@gmail.com
[2]Department of Mathematics, Netaji Subhas Institute of Technology
Sector 3 Dwarka, New Delhi-110078, India
E-mail:garima89.manocha@gmail.com
[3] Department of Mathematics, Netaji Subhas Institute of Technology
Sector 3 Dwarka, New Delhi-110078, India
E-mail:nirajkumar2001@hotmail.com

Abstract In the present paper, Y denotes the set of all Entire Dirichlet series in two variables which forms a Γ-ring, where Γ denotes the set of those Entire Dirichlet series in two variables for which $((m+n)e)^{(\lambda_m+\mu_n)e}\left|\dfrac{a_{m,n}}{\alpha_{m,n}}\right|$ is bounded. We thereby prove the commutativity conditions and the relation among primitivity, primeness and simplicity for the set Y.

Keywords Dirichlet series, Gamma ring, Simple Gamma ring, Primitive Gamma ring, Prime Gamma ring.
2010 Mathematics Subject Classification 30B50,17D20,17C20.

§1. Introduction and preliminaries

Let

$$f(s_1,s_2)=\sum_{m,n=1}^{\infty} a_{m,n}\, e^{(\lambda_m s_1+\mu_n s_2)}, \qquad (s_j=\sigma_j+it_j,\ j=1,2) \tag{0.1}$$

be a Dirichlet series of two complex variables s_1 and s_2 and $a_{m,n}{}'s\in\mathbb{C}$. Also $\lambda_m{}'s,\ \mu_n{}'s\in\mathbb{R}$ satisfying

$$\begin{aligned} &0<\lambda_1<\lambda_2<\ldots<\lambda_m\to\infty \text{ as } m\to\infty\\ \text{and}\quad &0<\mu_1<\mu_2<\ldots<\mu_n\to\infty \text{ as } n\to\infty.\end{aligned}$$

If

$$\limsup_{m+n\to\infty}\frac{\log(m+n)}{\lambda_m+\mu_n}=Q<\infty \tag{0.2}$$

and

$$\limsup_{m+n\to\infty}\frac{\log|a_{m,n}|}{\lambda_m+\mu_n}=-\infty \tag{0.3}$$

Then from [1], the series (0.1) represents an entire function. Let Y be the set of all entire functions and Γ denotes the set of those Entire Dirichlet series in two variables for which $((m+n)e)^{(\lambda_m+\mu_n)e}\left|\frac{a_{m,n}}{\alpha_{m,n}}\right|$ is bounded.

The norm in Γ is defined as follows

$$\|f\| = \sup_{m,n\geq 1}((m+n)e)^{(\lambda_m+\mu_n)e}\left|\frac{a_{m,n}}{\alpha_{m,n}}\right|. \tag{0.4}$$

Uptil now a lot of research has been done in the field of dirichlet series which can be seen in [1]- [2]. Luh in [4] and [5] studied various results on primitive Γ-rings. Kumar and Manocha in [7] worked on the generalized weighted norm for a Dirichlet series of one variable and proved results on continuous linear functional. Then in [6] the series is considered and proved that it fails to form divison algebra and conditions for topological zero divisor has been obtained. In the present paper we consider the set of all Entire Dirichlet series which forms a Γ-ring and obtain various results on commutativity conditions and the equivalence among primitivity, primeness and simplicity for this set Y. The purpose of this paper is to establish a link between analysis and algebra.

§2. Basic Results

In the sequel following definitions are required to prove the main results. For all notions relevant to ring theory, refer [3] and [10].

Definition 0.1. *Let N and Γ be two additive abelian groups. If there exists a mapping $N\ X\ \Gamma\ X\ N \to N$ such that for all $x, y, z \in N$ and $\alpha, \beta \in \Gamma$ the conditions*
$(1.a)$ $(x+y)\alpha z = x\alpha z + y\alpha z,$
$(1.b)$ $x(\alpha+\beta)z = x\alpha z + x\beta z,$
$(1.c)$ $x\alpha(y+z) = x\alpha y + x\alpha z,$
$(1.d)$ $(x\alpha y)\beta z = x\alpha(y\beta z)$
are satisfied then N is a Γ-ring.
An additive subgroup J of N is a left (right) ideal of N if $N\Gamma J \subset J$ ($J\Gamma N \subset J$). If J is both a left and a right ideal of N then J is a two- sided ideal or simply an ideal of N. For all other concepts refer [8] and [9].

Definition 0.2. *Let N be a Γ-ring. If for any non-zero element n of N, there exists an element γ (depending on n) in Γ such that $n\gamma n \neq 0$, we say N is semi-simple.*

Definition 0.3. *If for any non-zero elements m and n of N, there exists γ (depending on m and n) in Γ such that $n\gamma m \neq 0$, we say N is simple.*

Definition 0.4. *Let N be a Γ-ring. An ideal Q of N is prime if for all pairs of ideals G and H of N, $G\Gamma H \subseteq Q$ implies $G \subseteq Q$ or $H \subseteq Q$. A Γ ring N is prime if the zero ideal is prime.*

Definition 0.5. *An ideal P of M is semi-prime if for any ideal V, $V\Gamma V \subseteq P$ implies $V \subseteq P$. A Γ ring N is semi-prime if the zero ideal is semi-prime.*

Definition 0.6. *A* Γ*-ring* Y *is left primitive if*
(1) *The left operator ring of* Y *is a left primitive ring.*
(2) $y(s_1, s_2).\Gamma.Y = 0$ *implies* $y(s_1, s_2) = 0$.
Y *is a two-sided primitive* Γ*-ring if it is both left and right primitive.*

Now let $x(s_1, s_2), y(s_1, s_2) \in Y$ and $\beta(s_1, s_2) \in \Gamma$ such that

$$\begin{aligned} x(s_1, s_2) &= \sum_{m,n=1}^{\infty} a_{m,n}\, e^{(\lambda_m s_1 + \mu_n s_2)}, y(s_1, s_2) = \sum_{m,n=1}^{\infty} b_{m,n}\, e^{(\lambda_m s_1 + \mu_n s_2)}, \beta(s_1, s_2) \\ &= \sum_{m,n=1}^{\infty} \beta_{m,n} e^{(\lambda_m s_1 + \mu_n s_2)} \end{aligned} \tag{0.5}$$

The binary operations that is addition and multiplication in Y X Γ X Y is defined as

$$x(s_1, s_2) + \beta(s_1, s_2) + y(s_1, s_2) = \sum_{m,n=1}^{\infty} (a_{m,n} + \beta_{m,n} + b_{m,n}) e^{(\lambda_m s_1 + \mu_n s_2)},$$

$$x(s_1, s_2).\beta(s_1, s_2).y(s_1, s_2) = \sum_{m,n=1}^{\infty} (a_{m,n}.\beta_{m,n}.b_{m,n})\ e^{(\lambda_m s_1 + \mu_n s_2)}.$$

Clearly the set Y forms a Γ-ring. Now for any non-zero elements $x(s_1, s_2)$ and $y(s_1, s_2)$ of Y there exists an element $\beta(s_1, s_2)$ (depending on $x(s_1, s_2)$ and $y(s_1, s_2)$) in Γ such that $a_{m,n}.\beta_{m,n}.a_{m,n} \neq 0$ and $a_{m,n}.\beta_{m,n}.b_{m,n} \neq 0$. Thus Y is semi-simple and simple respectively.
Let K be a free abelian group generated by the set of all ordered pairs $\{x(s_1, s_2), \beta(s_1, s_2)\}$ where $x(s_1, s_2) \in Y$ and $\beta(s_1, s_2) \in \Gamma$.
Let F be a subgroup of elements $\sum_i m_i\{x_i(s_1, s_2), \beta_i(s_1, s_2)\} \in K$ where m_i are integers such that

$$\sum_i m_i[x_i(s_1, s_2).\beta_i(s_1, s_2).a(s_1, s_2)] = 0 \text{ for all } a(s_1, s_2) \in Y.$$

Denote by L the factor group K/F and by $[x(s_1, s_2), \beta(s_1, s_2)]$ the coset $F+\{x(s_1, s_2), \beta(s_1, s_2)\}$.
Clearly every element in L can be expressed as a finite sum $\sum_i [x_i(s_1, s_2), \beta_i(s_1, s_2)]$.
Also for all $x(s_1, s_2), y(s_1, s_2) \in Y$ and $\alpha(s_1, s_2), \beta(s_1, s_2) \in \Gamma$

$$[x(s_1, s_2), \alpha(s_1, s_2)] + [x(s_1, s_2), \beta(s_1, s_2)] = [x(s_1, s_2), \alpha(s_1, s_2) + \beta(s_1, s_2)]$$

$$[x(s_1, s_2), \beta(s_1, s_2)] + [y(s_1, s_2), \beta(s_1, s_2)] = [x(s_1, s_2) + y(s_1, s_2), \beta(s_1, s_2)]$$

Define multiplication in L as

$$\sum_i [x_i(s_1, s_2), \beta_i(s_1, s_2)]. \sum_j [y_j(s_1, s_2), \alpha_j(s_1, s_2)] = \sum_{i,j} [x_i(s_1, s_2).\beta_i(s_1, s_2).y_j(s_1, s_2), \alpha_j(s_1, s_2)]$$

Then L forms a ring. Furthermore Y is a left L-module with the definition

$$\sum_i [x_i(s_1, s_2), \beta_i(s_1, s_2)].a(s_1, s_2) = \sum_i [x_i(s_1, s_2).\beta_i(s_1, s_2).a(s_1, s_2)] \text{ for all } a(s_1, s_2) \in Y.$$

We call the ring L the left operator ring of Y. Similarly we can define the right operator ring R of Y.
It is known that every one-sided primitive Γ-ring having minimal one sided ideals is a two-sided primitive Γ-ring. Since no left primitive Γ-ring has non-zero strongly nilpotent one-sided ideals, every minimal left ideal of a primitive Γ-ring Y is of the form $Y.\epsilon(s_1, s_2).e(s_1, s_2)$ where

$$e(s_1, s_2) = \sum_{m,n=1}^{\infty} e_{m,n} e^{(\lambda_m s_1 + \mu_n s_2)} \in J, \ \epsilon(s_1, s_2) = \sum_{m,n=1}^{\infty} \epsilon_{m,n} e^{(\lambda_m s_1 + \mu_n s_2)} \in \Gamma$$

and

$$e_{m,n}.\epsilon_{m,n}.e_{m,n} = e_{m,n}.$$

§2. Main Results

Theorem 2.1. ***Commutativity conditions on a Γ-ring Y-***
A Γ-ring Y is commutative if for every element $x(s_1, s_2) \in Y$ we have $x(s_1, s_2).\gamma(s_1, s_2).x(s_1, s_2) = x(s_1, s_2)$ for $\gamma(s_1, s_2) \in \Gamma$ (fixed).

Proof. Suppose

$$\{x(s_1, s_2) + y(s_1, s_2)\}.\gamma(s_1, s_2).\{x(s_1, s_2) + y(s_1, s_2)\} = x(s_1, s_2) + y(s_1, s_2).$$

This implies

$$x(s_1, s_2).\gamma(s_1, s_2).y(s_1, s_2) = -y(s_1, s_2).\gamma(s_1, s_2).x(s_1, s_2).$$

Now

$$x(s_1, s_2).\gamma(s_1, s_2).\{x(s_1, s_2).\gamma(s_1, s_2).y(s_1, s_2)\} = x(s_1, s_2).\gamma(s_1, s_2)\{-y(s_1, s_2).\gamma(s_1, s_2).x(s_1, s_2)\}$$

further implies

$$x(s_1, s_2).\gamma(s_1, s_2).y(s_1, s_2) = -x(s_1, s_2).\gamma(s_1, s_2)\{y(s_1, s_2).\gamma(s_1, s_2).x(s_1, s_2)\}$$

Similarly

$$x(s_1, s_2).\gamma(s_1, s_2)\{y(s_1, s_2).\gamma(s_1, s_2).x(s_1, s_2)\} = -y(s_1, s_2).\gamma(s_1, s_2).x(s_1, s_2)$$

Thus

$$x(s_1, s_2).\gamma(s_1, s_2).y(s_1, s_2) = y(s_1, s_2).\gamma(s_1, s_2).x(s_1, s_2)$$

Clearly Y is a commutative Γ-ring. Hence the theorem. □

Equivalence of Primeness, Primitivity and Simplicity of a Γ-ring Y-
Theorem 2.2. *If Y is a simple Γ-ring then Y is prime.*

Proof. Suppose that Y is not prime and we have $U\Gamma V = 0$ where U and V are non-zero ideals of Y. Since Y is simple implies $U = V = Y$ which further implies

$$Y\Gamma Y = 0.$$

This contradicts the simplicity of Y. Thus Y is prime which completes the proof of the theorem. □

Theorem 2.3. *If Y is a left primitive Γ-ring then Y is prime.*

Proof. Given Y is a left primitive Γ-ring. Thus L be the left operator ring of Y. Let N be the faithful irreducible left L-module. Let us suppose that Y is not prime. Let U and V be two non-zero ideals of Y and $U\Gamma V = 0$. We claim that $[V,\Gamma]N = N$. If $[V,\Gamma]N = 0$ implies $[V,\Gamma] = 0$ which implies $V\Gamma Y = 0$. Since Y is primitive implies $V = 0$ which is a contradiction. Thus $[V,\Gamma]N = N$. Similarly $[U,\Gamma]N = N$.
Consider

$$\begin{aligned} 0 &= [U\Gamma V,\Gamma]N \\ &= [U,\Gamma][V,\Gamma]N \\ &= [U,\Gamma]N \\ &= N \end{aligned}$$

which contradicts the assumption which implies that Y is a prime Γ-ring. □

Theorem 2.4. *If Y is a Γ-ring having minimal left ideals then Y is primitive if and only if Y is prime.*

Proof. By above Theorem, primitivity implies primeness. Let us assume that Y is prime and J be the minimal left ideal of Y. Clearly J is an irreducible left L-module. $J\Gamma J \neq 0$, $J = Y.\gamma(s_1,s_2).e(s_1,s_2)$ and $e_{m,n}\gamma_{m,n}e_{m,n} = e_{m,n}$ where $e(s_1,s_2) = \sum_{m,n=1}^{\infty} e_{m,n}e^{(\lambda_m s_1+\mu_n s_2)}$.
Suppose $\sum_i [x_i(s_1,s_2),\gamma_i(s_1,s_2)]J = 0$. This implies

$$\sum_i x_i(s_1,s_2).\gamma_i(s_1,s_2).Y.\Gamma.Y.\gamma(s_1,s_2).e(s_1,s_2) \subseteq \sum_i [x_i(s_1,s_2),\gamma_i(s_1,s_2)]J$$

which implies

$$\sum_i x_i(s_1,s_2).\gamma_i(s_1,s_2).Y.\Gamma.Y.\gamma(s_1,s_2).e(s_1,s_2) = 0.$$

Therefore $\sum_i x_i(s_1,s_2).\gamma_i(s_1,s_2).Y = 0$ or simply $\sum_i [x_i(s_1,s_2),\gamma_i(s_1,s_2)] = 0$. Thus, J is a faithful irreducible left L-module and L is a left primitive ring. Moreover, if $x(s_1,s_2).\Gamma.Y = 0$ implies $x(s_1,s_2).\Gamma.Y.\Gamma.x(s_1,s_2) = 0$. Since Y is prime one gets $x(s_1,s_2) = 0$. Therefore Y is a primitive Γ-ring. Hence the theorem. □

Theorem 2.5. *If Y is a primitive Γ-ring with minimum condition on left ideals then*

$$Y = Y.\gamma_1(s_1, s_2).e_1(s_1, s_2) + Y.\gamma_2(s_1, s_2).e_2(s_1, s_2) + \ldots + Y.\gamma_k(s_1, s_2).e_k(s_1, s_2) \ (direct\ sum)$$

where $e_i.\gamma_{k_i}.e_{k_i} = e_{k_i}$ and $e_{k_i}.\gamma_{k_j}.e_{k_j} = 0$ if $i > j$ where $Y.\gamma_i(s_1, s_2).e_i(s_1, s_2)$ are minimal left ideals of Y.

Proof. Let $J_1 = Y.\gamma_1(s_1, s_2).e_1(s_1, s_2)$ be a minimal left ideal of Y where $e_{k_1}.\gamma_{k_1}.e_{k_1} = e_{k_1}$ and let $Y_1 = \{y(s_1, s_2) \in Y : y(s_1, s_2).\gamma_1(s_1, s_2).e_1(s_1, s_2) = 0\}$. Each $a(s_1, s_2) \in Y$ has the form

$$a(s_1, s_2) = a(s_1, s_2).\gamma_1(s_1, s_2).e_1(s_1, s_2) + \{a(s_1, s_2) - a(s_1, s_2).\gamma_1(s_1, s_2).e_1(s_1, s_2)\}.$$

This implies

$$Y = Y.\gamma_1(s_1, s_2).e_1(s_1, s_2) + Y_1 \ (direct\ sum)$$

If $Y_1 \neq 0$, then by the minimum condition, Y_1 contains a minimal left ideal $Y.\gamma_2(s_1, s_2).e_2(s_1, s_2)$ of Y where $e_{k_2}.\gamma_{k_2}.e_{k_2} = e_{k_2}$ and $e_{k_2}.\gamma_{k_1}.e_{k_1} = 0$. This implies

$$Y = Y.\gamma_1(s_1, s_2).e_1(s_1, s_2) + Y.\gamma_2(s).e_2(s_1, s_2) + Y_2 \ (direct\ sum)$$

where $Y_2 = \{y(s_1, s_2) \in Y_1 : x(s_1, s_2).\gamma_2(s_1, s_2).e_2(s_1, s_2) = 0\}$. Continuing this process we find that $Y_k = 0$ for some positive integer k. Thus,

$$Y = Y.\gamma_1(s_1, s_2).e_1(s_1, s_2) + Y.\gamma_2(s_1, s_2).e_2(s_1, s_2) + \ldots + Y.\gamma_k(s_1, s_2).e_k(s_1, s_2) \ (direct\ sum).$$

□

Theorem 2.6. *Let Y be a left primitive Γ-ring and let J be a non-zero left ideal of Y. If $e(s_1, s_2).\gamma(s_1, s_2).e(s_1, s_2) = e(s_1, s_2) \neq 0$ where $e(s_1, s_2) \in Y$ and $\gamma(s_1, s_2) \in \Gamma$ then $e(s_1, s_2).\gamma(s_1, s_2).J \neq 0$.*

Proof. Let N be a faithful irreducible left L-module where L is the left operator ring of Y. By the primitivity of Y, we have $[J, \Gamma]N = N$. Now, suppose on the contrary $e(s_1, s_2).\gamma(s_1, s_2).J = 0$. Then

$$\begin{aligned} [e(s_1, s_2), \gamma(s_1, s_2)]N &= [e(s_1, s_2), \gamma(s_1, s_2)][J, \Gamma]N \\ &= [e(s_1, s_2).\gamma(s_1, s_2).J, \Gamma]N \\ &= 0. \end{aligned}$$

Since N is faithful implies $[e(s_1, s_2), \gamma(s_1, s_2)] = 0$. Now,

$$e(s_1, s_2) = e(s_1, s_2).\gamma(s_1, s_2).e(s_1, s_2) = [e(s_1, s_2), \gamma(s_1, s_2)]e(s_1, s_2) = 0$$

which is a contradiction therefore $e(s_1, s_2).\gamma(s_1, s_2).J \neq 0$.

□

Theorem 2.7. *If Y is a primitive Γ-ring with minimum condition on left ideals then Y is simple.*

Proof. We know that,

$$Y = Y.\gamma_1(s_1,s_2).e_1(s_1,s_2) + Y.\gamma_2(s_1,s_2).e_2(s_1,s_2) + \ldots + Y.\gamma_k(s_1,s_2).e_k(s_1,s_2)\ (direct\ sum)$$

where $e_{k_i}.\gamma_{k_i}.e_{k_i} = e_{k_i}$ and $e_{k_i}.\gamma_{k_j}.e_{k_j} = 0\ if\ i > j$ where $Y.\gamma_i(s_1,s_2).e_i(s_1,s_2)$ are minimal left ideals of Y. Let J be a non-zero ideal of Y. Each $x(s_1,s_2) \in J$ has the form

$$x(s_1,s_2) = x_1(s_1,s_2).\gamma_1(s_1,s_2).e_1(s_1,s_2) + x_2(s_1,s_2).\gamma_2(s_1,s_2).e_2(s_1,s_2) + \ldots$$
$$+ x_k(s_1,s_2).\gamma_k(s_1,s_2).e_k(s_1,s_2)$$

where $x_i(s_1,s_2) \in Y$ $(i = 1,2,3,....,k)$. Assume that

$$x_p(s_1,s_2).\gamma_p(s_1,s_2).e_p(s_1,s_2) + \ldots + x_k(s_1,s_2).\gamma_k(s_1,s_2).e_k(s_1,s_2) \in J\ where\ 1 \le p < k.$$

Then $\{x_p(s_1,s_2).\gamma_p(s_1,s_2).e_p(s_1,s_2) + \ldots + x_k(s_1,s_2).\gamma_k(s_1,s_2).e_k(s_1,s_2)\}\gamma_p(s_1,s_2).e_p(s_1,s_2) \in J$ which implies

$$x_p(s_1,s_2).\gamma_p(s_1,s_2).e_p(s_1,s_2) \in J.$$

Again

$$x_{p+1}(s_1,s_2).\gamma_{p+1}(s_1,s_2).e_{p+1}(s_1,s_2) + \ldots + x_k(s_1,s_2).\gamma_k(s_1,s_2).e_k(s_1,s_2) \in J.$$

Hence, by induction $x_p(s_1,s_2).\gamma_p(s_1,s_2).e_p(s_1,s_2) \in J$ $(p = 1,2,....,k)$. But

$$\begin{aligned} x_p(s_1,s_2).\gamma_p(s_1,s_2).e_p(s_1,s_2) &= \{x_p(s_1,s_2).\gamma_p(s_1,s_2).e_p(s_1,s_2)\}.\gamma_p(s_1,s_2).e_p(s_1,s_2) \\ &\in J.\gamma_p(s_1,s_2).e_p(s_1,s_2). \end{aligned}$$

This implies

$$J \subseteq J.\gamma_1(s_1,s_2).e_1(s_1,s_2) + J.\gamma_2(s_1,s_2).e_2(s_1,s_2) + J.\gamma_3(s_1,s_2).e_3(s_1,s_2) + \ldots + J.\gamma_p(s_1,s_2).e_p(s_1,s_2).$$

Since, J is a two-sided ideal of Y implies $J.\gamma_p(s_1,s_2).e_p(s_1,s_2) \subseteq J$. Hence

$$J = J.\gamma_1(s_1,s_2).e_1(s_1,s_2) + J.\gamma_2(s_1,s_2).e_2(s_1,s_2) + J.\gamma_3(s_1,s_2).e_3(s_1,s_2) + \ldots + J.\gamma_p(s_1,s_2).e_p(s_1,s_2).$$

We now assert that $J.\gamma_p(s_1,s_2).e_p(s_1,s_2) \ne 0\ for\ all\ p$. For otherwise

$$\begin{aligned} & \{e_p(s_1,s_2).\gamma_p(s_1,s_2).J\}\Gamma\{e_p(s_1,s_2).\gamma_p(s_1,s_2).J\} \\ = \ & e_p(s_1,s_2).\gamma_p(s_1,s_2).\{J.\Gamma.e_p(s_1,s_2)\}.\gamma_p(s_1,s_2).J \\ = \ & 0 \end{aligned}$$

while from above, $e_p(s_1,s_2).\gamma_p(s_1,s_2).J \ne 0$. Thus, $e_p(s_1,s_2).\gamma_p(s_1,s_2).J$ is a non-zero strongly nilpotent right ideal of Y which is a contradiction to the fact that a primitive Γ-ring has no non-zero strongly nilpotent one-sided ideal. This implies

$$J.\gamma_p(s_1,s_2).e_p(s_1,s_2) = Y.\gamma_p(s_1,s_2).e_p(s_1,s_2)$$

which further implies

$$J = Y.\gamma_1(s_1,s_2).e_1(s_1,s_2) + Y.\gamma_2(s_1,s_2).e_2(s_1,s_2) + \ldots + Y.\gamma_k(s_1,s_2).e_k(s_1,s_2) = Y.$$

Thus Y is simple. Hence the theorem.

□

Theorem 2.8. *For a Γ-ring Y with minimum condition on left ideals the conditions*
(1) Y *is prime.*
(2) Y *is primitive.*
(3) Y *is simple.*
are equivalent.

References

[1] S. Daoud. On a class of entire Dirichlet functions of several complex variables having finite order point. Portugaliae Mathematica 43 (1986), 417 – 427.

[2] S. Daoud. On the class of entire functions defined by Dirichlet series of several complex variables. J. Math. Anal. Appl. 162 (1991), 294 – 299.

[3] N. Jacobson. Structure of rings. Amer. Math. Soc. Colloquium Publ. 37. Providence, 1964.

[4] J. Luh. On primitive Γ-rings with minimal one-sided ideals. Osaka J. Math. 5 (1968), 165 – 173.

[5] J. Luh. The structure of primitive gamma rings. Osaka J. Math. 7 (1970), 267 – 274.

[6] N. Kumar, L. Chutani and G. Manocha. Certain results on a class of Entire Dirichlet series in two variables. Scientia Magna 11 (2016), No. 2, 33 – 40.

[7] N. Kumar and G. Manocha. On a class of entire functions represented by Dirichlet series. J. Egypt. Math. Soc. 21 (2013), 21 – 24.

[8] S. Kyuno. On prime gamma rings. Pacific J. Math. 75 (1978), 185 – 190.

[9] S. Kyuno. Prime ideals in gamma rings. Pacific J. Math. 98 (1982), 375 – 379.

[10] F. A. Szasz. Radicals of Rings. Wiley, Chichester, 1981.

Scientia Magna
Vol. 16 (2021), No. 1, 67-71

A short interval result for the function $\tau_3^{(e)}(n)$ with a negative r-th power

Yang Li

School of Mathematics and Statistics, Shandong Normal University
Shandong Jinan, China
E-mail: liyangsdnu@163.com

Abstract If $b_i|a_i$ for every $i \in \{1, 2, \cdots, s\}$, then the integer $d = \prod_{i=1}^{s} p_i^{b_i}$ is called an exponential divisor of $n = \prod_{i=1}^{s} p_i^{a_i} > 1$. The number of exponential divisors of n is denoted by $\tau^{(e)}(n)$. Similarly to the generalization of $d_k(n)$ from $d(n)$, $\tau^{(e)}(n)$ can be extended to $\tau_k^{(e)}(n)$. In this paper, we shall establish a short interval result for the function $(\tau_3^{(e)}(n))^{-r}$.

Keywords Exponential divisor function, generalized divisor function, short interval.
2010 Mathematics Subject Classification 11L07, 11N80, 11L26.

§1. Introduction and preliminaries

Let $n > 1$ be an integer. If $b_i|a_i$ for every $i \in \{1, 2, \cdots, s\}$, then the integer $d = \prod_{i=1}^{s} p_i^{b_i}$ is called an exponential divisor of $n = \prod_{i=1}^{s} p_i^{a_i} > 1$, notation: $d|_e n$. By convention $1|_e 1$.

The number of exponential divisors of n is denoted by $\tau^{(e)}(n)$. The function $\tau^{(e)}$ is called the exponential divisor function.

Wu [4] got the following result:

$$\sum_{n\le x} \tau^{(e)}(n) = Ax + Bx^{\frac{1}{2}} + O(x^{\frac{2}{9}} \log x), \tag{1}$$

where

$$A = \prod_p (1 + \sum_{a=2}^{\infty} \frac{d(a) - d(a-1)}{p^a}),$$

$$B = \prod_p (1 + \sum_{a=5}^{\infty} \frac{d(a) - d(a-1) - d(a-2) + d(a-3)}{p^{\frac{a}{2}}}).$$

For any positive integers r, Subbarao [2] proved the following result:

$$\sum_{n\le x} (\tau^{(e)}(n))^r \sim B_r x, \tag{2}$$

where

$$B_r = \prod_p (1 + \sum_{a=2}^{\infty} \frac{(d(a))^r - (d(a-1))^r}{p^a}).$$

For this result, Tóth [3] improved it and proved a more precise asymptotic formula for the function $(\tau^{(e)}(n))^r$:

$$\sum_{n\leq x}(\tau^{(e)}(n))^r = B_r x + x^{\frac{1}{2}} P_{2^r-2}(\log x) + O(x^{u_r+\varepsilon}), \tag{3}$$

where $P_{2^r-2}(t)$ is a polynomial of degree 2^r-2 of t, $u_r = \frac{2^{r+1}-1}{2^{r+1}+1}$.

Let $r \geq 1$ be a fixed integer. Zheng [5] proved that the asymptotic formula for the function $(\tau^{(e)}(n))^{-r}$:

$$\sum_{n\leq x}(\tau^{(e)}(n))^{-r} = D_r x + x^{\frac{1}{2}} \log^{2^{-r}-2} \sum_{j=0}^{N} d_j(r) \log^{-j} x + O(\log^{-N-1} x) \tag{4}$$

holds for any fixed integer $N \geq 1$, where $d_0(r), d_1(r), \cdots, d_N(r)$ are computable constants, and

$$D_r := \prod_p (1 + \sum_{a=2}^{\infty} \frac{(\tau(a))^{-r} - (\tau(a-1))^{-r}}{p^a}).$$

Similarly to the generalization of $d_k(n)$ from $d(n)$, $\tau^{(e)}(n)$ can be extended to $\tau_k^{(e)}(n)$. We established the following definition:

$$\tau_k^{(e)}(n) = \prod_{{p_i}^{a_i}||n} d_k(a_i), k \geq 2. \tag{5}$$

Obviously if $k = 2$, then $\tau_2^{(e)}(n) = \tau^{(e)}(n)$. $\tau_3^{(e)}(n)$ is obviously a multiplicative function.

The aim of this paper is to study the short interval case of $(\tau_3^{(e)}(n))^{-r}$ and prove the following.

Theorem. If $x^{\frac{1}{5}+2\epsilon} < y \leq x$, then

$$\sum_{x<n\leq x+y} (\tau_3^{(e)}(n))^{-r} = A_r y + O(yx^{-\frac{2\varepsilon}{5}} + x^{\frac{1}{5}+\frac{3}{2}\varepsilon}), \tag{6}$$

where $A_r = Res_{s=1} G(s)$ and $G(s) = \sum_{n=1}^{\infty} \frac{(\tau_3^{(e)}(n))^{-r}}{n^s}$.

Throughout this paper, ε always denotes a fixed but sufficiently small positive constant. Suppose that $1 \leq a \leq b$ are fixed integers, the divisor function $d(a, b; k)$ is defined by

$$d(a, b; k) = \sum_{k=n_1^a n_2^b} 1. \tag{7}$$

The estimate $d(a, b; k) \ll k^{\varepsilon^2}$ will be used freely.

§2. Some lemmas

In order to prove theorem, we need the following lemmas.

Lemma 1. Support $s = \sigma + it$ is a complex number, then we have

$$\sum_{n=1}^{\infty} \frac{(\tau_3^{(e)}(n))^{-r}}{n^s} = \zeta(s)\zeta^{3^{-r}-1}(2s)\zeta^{-c_r}(4s)G(s), \tag{8}$$

where $c_r = \frac{3^{-2r}}{2} + \frac{3^{-r}}{2} - 6^{-r} > 0$ and the Dirichlet series $G(s) := \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$ is absolutely convergent for $\Re s > \frac{1}{5}$.

Proof. By Euler's product formula, we can get

$$
\begin{aligned}
\sum_{n=1}^{\infty}\frac{(\tau_3^{(e)}(n))^{-r}}{n^s} &= \prod_p\left(1+\frac{(\tau_3^{(e)}(p))^{-r}}{p^s}+\frac{(\tau_3^{(e)}(p^2))^{-r}}{p^{2s}}+\frac{(\tau_3^{(e)}(p^3))^{-r}}{p^{3s}}+\frac{\tau_3^{(e)}(p^4))^{-r}}{p^{4s}}+\cdots\right)\\
&= \prod_p\left(1+\frac{(d_3(1))^{-r}}{p^s}+\frac{(d_3(2))^{-r}}{p^{2s}}+\frac{(d_3(3))^{-r}}{p^{3s}}+\frac{(d_3(4))^{-r}}{p^{4s}}+\frac{(d_3(5))^{-r}}{p^{5s}}+\cdots\right)\\
&= \prod_p\left(1+\frac{1}{p^s}+\frac{3^{-r}}{p^{2s}}+\frac{3^{-r}}{p^{3s}}+\frac{6^{-r}}{p^{4s}}+\frac{3^{-r}}{p^{5s}}+\cdots\right)\\
&= \zeta(s)\prod_p\left(1+\frac{3^{-r}-1}{p^{2s}}+\frac{6^{-r}-3^{-r}}{p^{4s}}+\cdots\right)\\
&= \zeta(s)\zeta^{3^{-r}-1}(2s)\prod_p(1-\frac{1}{p^{2s}})^{3^{-r}-1}\left(1+\frac{3^{-r}-1}{p^{2s}}+\frac{6^{-r}-3^{-r}}{p^{4s}}+\cdots\right)\\
&= \zeta(s)\zeta^{3^{-r}-1}(2s)\zeta^{-c_r}(4s)G(s),
\end{aligned}
\tag{9}
$$

where $c_r = \frac{3^{-2r}}{2}+\frac{3^{-r}}{2}-6^{-r}$ and the Dirichlet series $G(s) := \sum_{n=1}^{\infty}\frac{g(n)}{n^s}$ is absolutely convergent for $\Re s > \frac{1}{5}$. □

Lemma 2. *Let $k \geq 2$ be a fixed integer, $1 < y \leq x$ be large real numbers and*

$$B(x,y;k,\varepsilon) := \sum_{\substack{x<nm^k\leq x+y\\ m>x^{\varepsilon}}} 1.$$

Then we have $B(x,y;k,\varepsilon) \ll yx^{-\varepsilon} + x^{\frac{1}{2k+1}}logx$.

Proof. This Lemma is a result of k-free number [1]. □

Let $a(n)$,$b(n)$,$c(n)$ be arithmetic functions defined by the following Dirichlet series for $\Re s > 1$.

$$\sum_{n=1}^{\infty}\frac{a(n)}{n^s} = \zeta(s)G(s), \tag{10}$$

$$\sum_{n=1}^{\infty}\frac{b(n)}{n^{2s}} = \zeta^{3^{-r}-1}(2s), \tag{11}$$

$$\sum_{n=1}^{\infty}\frac{c(n)}{n^{4s}} = \zeta^{-c_r}(4s). \tag{12}$$

Lemma 3. *Let $a(n)$ be an arithmetic function defined by* (10)*, then we have*

$$\sum_{n\leq x} a(n) = A_1x + O(x^{\frac{1}{5}+\varepsilon}),$$

where $A_1 = Res_{s=1}\zeta(s)G(s)$.

Proof. Using Lemma 1, we have
$$\sum_{n\le x}|g(n)|\ll x^{\frac{1}{5}+\varepsilon}.$$

Therefore from the definition of $g(n)$ and (10), it follows that
$$\sum_{n\le x}a(n)=\sum_{mn\le x}g(x)=\sum_{n\le x}g(x)\sum_{m\le \frac{x}{n}}1=\sum_{n\le x}g(n)(\frac{x}{n}+O(1))=A_1x+O(x^{\frac{1}{5}+\varepsilon}),$$
and $A_1=Res_{s=1}\zeta(s)G(s)$. □

§3. Proof of the Theorem

From Lemma 3 and the definition of $a(n)$, $b(n)$, $c(n)$, we get
$$(\tau_3^{(e)}(n))^{-r}=\sum_{n=n_1n_2^2n_3^4}a(n_1)b(n_2)c(n_3),$$
and
$$a(n)\ll n^{\varepsilon^2},b(n)\ll n^{\varepsilon^2},c(n)\ll n^{\varepsilon^2},$$
so we have
$$\sum_{n\le x+y}\left(\tau_3^{(e)}(n)\right)^{-r}-\sum_{n\le x}\left(\tau_3^{(e)}(n)\right)^{-r}=\sum_{x<n_1n_2^2n_3^4\le x+y}a(n_1)b(n_2)c(n_3)=\sum_1+O(\sum_2+\sum_3),\tag{13}$$
where
$$\sum_1=\sum_{\substack{n_2\le x^{\varepsilon}\\ n_3\le x^{\varepsilon}}}b(n_2)c(n_3)\sum_{\frac{x}{n_2^2n_3^4}<n_1\le\frac{x+y}{n_2^2n_3^4}}a(n_1),$$
$$\sum_2=\sum_{\substack{x<n_1n_2^2n_3^4\le x+y\\ n_2>x^{\varepsilon}}}|a(n_1)b(n_2)c(n_3)|,$$
$$\sum_3=\sum_{\substack{x<n_1n_2^2n_3^4\le x+y\\ n_3>x^{\varepsilon}}}|a(n_1)b(n_2)c(n_3)|.$$

In view of Lemma 3
$$\begin{aligned}\sum_1&=\sum_{\substack{n_2\le x^{\varepsilon}\\ n_3\le x^{\varepsilon}}}b(n_2)c(n_3)\left[A_1\frac{y}{n_2^2n_3^4}+O((\frac{x}{n_2^2n_3^4})^{\frac{1}{5}+\varepsilon})\right]\\&=A_ry+O(y\sum_{n_2>x^{\varepsilon}}\frac{b(n_2)}{n_2^2}\sum_{n_3>x^{\varepsilon}}\frac{c(n_3)}{n_3^4})+O(x^{\frac{1}{5}+\varepsilon}\sum_{n_2\le x^{\varepsilon}}\frac{b(n_2)}{n_2^{\frac{2}{5}+2\varepsilon}}\sum_{n_3\le x^{\varepsilon}}\frac{c(n_3)}{n_3^{\frac{4}{5}+4\varepsilon}})\\&=A_ry+O(yx^{-\frac{2\varepsilon}{5}})+O(x^{\frac{1}{5}+\frac{7\varepsilon}{5}})\\&=A_ry+O(yx^{-\frac{2\varepsilon}{5}}+x^{\frac{1}{5}+\frac{3\varepsilon}{2}}),\end{aligned}\tag{14}$$
where $A_r=Res_{s=1}\zeta(s)\zeta^{3^{-r}-1}(2s)\zeta^{-c_r}(4s)G(s)$. For $\sum_2$ we have by Lemma 2 that

$$
\begin{aligned}
\sum_2 &\ll \sum_{\substack{x<n_1n_2^2n_3^4\le x+y\\ n_2>x^{\varepsilon}}} (n_1n_2n_3)^{\varepsilon^2}\\
&\ll x^{\varepsilon^2}\sum_{\substack{x<n_1n_2^2n_3^4\le x+y\\ n_2>x^{\varepsilon}}} 1\\
&\ll x^{2\varepsilon^2}B(x,y;2,\varepsilon)\\
&\ll x^{2\varepsilon^2}(yx^{-\varepsilon}+x^{\frac{1}{5}+\varepsilon}logx)\\
&\ll yx^{-\frac{\varepsilon}{2}}+x^{\frac{1}{5}+\frac{3\varepsilon}{2}}, \qquad (15)
\end{aligned}
$$

if $\varepsilon < \frac{1}{4}$.

Similarly we have

$$\sum_3 \ll yx^{-\frac{\varepsilon}{2}}+x^{\frac{1}{5}+\frac{3\varepsilon}{2}}. \qquad (16)$$

From (13)-(16), the theorem is proved.

Acknowledgements

The author wishes to thank the referee for reading the manuscript carefully and many valuable suggestions.

References

[1] M. Filaseta and O. Trifonov. The distribution of square fulll numbers in short intervals. Acta Arith. 67 (1944), 323 – 333.

[2] M. V. Subbarao. On some arithmetic convolutions. In: The Theory of Arithmetic Functions. Lecture Notes in Mathematics. Vol. 251, Springer, 1972, 247 – 271.

[3] L. Tóth. An order result for the exponential divisor function. Publ. Math. Debrecen 71 (2007), 165 – 171.

[4] J. Wu. Problème de diviseurs exponentiels at entiers exponentiellement sans facteur carré. Théor Nombres Bordeaux 7 (1995), 133 – 141.

[5] C. Zheng and L. Li. A negative order result for the exponential divisor function. Scientia Magna 5 (2009), 85 – 90.

Scientia Magna
Vol. 16 (2021), No. 1, 72-78

Minimal and maximal open sets in ditopological texture space

Senol Dost

Department of Secondary Science and Mathematics Education, Hacettepe University
06800 Beytepe, Ankara, Turkey
E-mail: dost@hacettepe.edu.tr

Abstract The aim of this paper is to introduce and study minimal and maximal open sets, and minimal and maximal closed sets in the context of ditopological texture spaces. Some properties of these notions are presented, and some results on topological spaces are given.

Keywords Texture, Fuzzy set, Ditopology, Minimal open set, Maximal open set.
2010 Mathematics Subject Classification 06D72, 54A05, 54D99.

§1. Introduction

Generalized open sets are used in solving various problems in many areas of mathematics as well as in topology. Further,the concepts of minimal open sets and maximal closed sets in topological spaces are introduced by Nakaoka and Oda in [8, 9].

The theory of texture spaces is an alternative setting for fuzzy sets and therefore, many properties of Hutton algebras (known as fuzzy lattices) can be discussed in terms of textures [1–3]. Ditopologies (dichotomous topologies) on textures unify the fuzzy topologies, topologies and bitopologies in a non-complemented setting by means of duality in the textural concepts [4, 5]. In the last 10 years, some generalizations of open sets as well as some other mathematical concepts in ditopological spaces have been studied [6, 7]. In this paper we present a discussion on minimal/maximal open and closed sets on ditopological texture spaces.

In the next section, we shall briefly the basic motivation and its study for texture spaces and ditopologies. For more details, we refer to [1–5].

§2. Texture spaces and Ditopology

Definition 2.1. Let U be a set. A *texturing* $\mathcal{U}$ of U is a subset of $\mathcal{P}(U)$ which is a point-separating, complete, completely distributive lattice containing U and $\emptyset$, and for which meet coincides with intersection and finite joins with union. The pair $(U, \mathcal{U})$ is then called a *texture space*, or shortly *texture*.

For $u \in U$, the *p-sets* and the *q-sets* are defined by

$$P_u = \bigcap\{A \in \mathcal{U} \mid u \in A\}, \quad Q_u = \bigvee\{A \in \mathcal{U} \mid u \notin A\}, \quad \text{respectively.}$$

In a texture, arbitrary joins need not coincide with unions, and clearly this will be so if and only if $\mathcal{U}$ is closed under arbitrary unions. In this case $(U, \mathcal{U})$ is said to be *plain*. Equivalently, a texture $(U, \mathcal{U})$ is plain if and only if $P_u \not\subseteq Q_u$, $\forall u \in U$.

We note in particular that $A^\flat = \{s \in S \mid A \not\subseteq Q_s\}$ is called the *core* of $A \in \mathcal{S}$.

In general a texturing of U need not be closed under set complementation, but it may be that there exists a mapping $\sigma : \mathcal{U} \to \mathcal{U}$ satisfying $\sigma(\sigma(A)) = A$, $\forall A \in \mathcal{U}$ and $A \subseteq B \implies \sigma(B) \subseteq \sigma(A), \forall A, B \in \mathcal{U}$. In this case σ is called a *complementation* on $(U, \mathcal{U})$, and $(U, \mathcal{U}, \sigma)$ is said to be a *complemented texture*.

Examples 2.2. (1) For any set U, $(U, \mathcal{P}(U), \pi)$, $\pi(A) = U \setminus A$ for $A \subseteq U$, is the complemented *discrete* texture representing the usual set structure of U. Clearly, $P_u = \{u\}$, $Q_u = U \setminus \{u\}$ for all $u \in U$.

(2) Let $L = (0, 1]$, $\mathcal{L} = \{(0, r] \mid r \in [0, 1]\}$ and $\lambda((0, r]) = (0, 1 - r]$, $r \in [0, 1]$. Then $(L, \mathcal{L}, \lambda)$ is complemented texture space. Here $P_r = Q_r = (0, r]$ for all $r \in L$.

(3) For $\mathbb{I} = [0, 1]$ define $\mathcal{J} = \{[0, t] \mid t \in [0, 1]\} \cup \{[0, t) \mid t \in [0, 1]\}$, $\iota([0, t]) = [0, 1 - t)$ and $\iota([0, t)) = [0, 1 - t]$, $t \in [0, 1]$. $(\mathbb{I}, \mathcal{J}, \iota)$ is a complemented texture, which we will refer to as the *unit interval texture*. Here $P_t = [0, t]$ and $Q_t = [0, t)$ for all $t \in \mathbb{I}$.

(4) Let $U = \{a, b, c\}$. Then $\mathcal{U} = \{U, \{b\}, \{b, c\}, \emptyset\}$ is a texturing on U. Here, $P_a = U$, $P_b = \{b\}$, $P_c = \{b, c\}$ and $Q_a = \{b, c\}$, $P_b = \emptyset$, $Q_c = \{b\}$. The mapping $\sigma : \mathcal{U} \to \mathcal{U}$ defined by $\sigma(U) = \emptyset$, $\sigma(\emptyset) = U$, $\sigma(\{b\}) = \{b, c\}$, $\sigma(\{b, c\}) = \{b\}$ is a complementation on $(U, \mathcal{U})$.

Definition 2.3. A *ditopology* on a texture $(U, \mathcal{U})$ is a pair (τ, κ) of subsets of $\mathcal{U}$ where the set of *open sets* τ and the set of *closed sets* κ satisfy

$$\begin{array}{rr} U, \emptyset \in \tau, & U, \emptyset \in \kappa \\ G_1, G_2 \in \tau \implies G_1 \cap G_2 \in \tau, & K_1, K_2 \in \kappa \implies K_1 \cup K_2 \in \kappa \\ G_i \in \tau, i \in I \implies \bigvee_{i \in I} G_i \in \tau, & K_i \in \kappa, i \in I \implies \bigcap_{i \in I} K_i \in \kappa. \end{array}$$

Hence a ditopology is essentially a "topology" for which there is no *a priori* relation between the open and closed sets. If (τ, κ) is a ditopology on $(U, \mathcal{U})$ then $(S, \mathcal{S}, \tau, \kappa)$ is called ditopological texture space or shortly, ditopological space.

Examples 2.4. (1) For any texture $(U, \mathcal{U})$ a ditopology (τ, κ) with $\tau = \mathcal{U}$ is called *discrete*, and one with $\kappa = \mathcal{U}$ is called *codiscrete*.

(2) For any texture $(U, \mathcal{U})$ a ditopology (τ, κ) with $\tau = \{\emptyset, U\}$ is called *indiscrete*, and one with $\kappa = \{\emptyset, U\}$ is called *co-indiscrete*.

(3) For any topology $\mathcal{T}$ on X, $(\mathcal{T}, \mathcal{T}^c)$, $\mathcal{T}^c = \{X \setminus G \mid G \in \mathcal{T}\}$, is a complemented ditopology on $(X, \mathcal{X}, \pi_X)$.

(4) $\tau_{\mathbb{I}} = \{[0, r) \mid 0 \le r \le 1\} \cup \{\mathbb{I}\}$, $\kappa_{\mathbb{I}} = \{[0, r] \mid 0 \le r \le 1\} \cup \{\emptyset\}$ defines a complemented ditopology, called the *natural ditopology* on $(\mathbb{I}, \mathcal{J}, \iota)$.

Note that there is no relation between open and closed sets of a ditopology. However, if σ is a complementation on $(U, \mathcal{U})$ and $\kappa = \sigma(\tau)$, then we say (τ, κ) a complemented ditopology on $(U, \mathcal{U}, \sigma)$.

For $A \in \mathcal{U}$ we define the *closure* and the *interior* of A under (τ, κ) by the equalities

$$int(A) = \bigvee\{G \in \tau \mid G \subseteq A\}, \quad cl(A) = \bigcap\{K \in \kappa \mid A \subseteq K\}$$

Definition 2.5. Let (τ, κ) be a ditopology on $(U, \mathcal{U})$.

(i) If $u \in U^\flat$, a neighbourhood(nhds) of u is a set $N \in \mathcal{U}$ for which there exists $G \in \tau$ satisfying $P_u \subseteq G \subseteq N \not\subseteq Q_u$.

(ii) If $u \in U$, a coneighbourhood(conhds) of u is a set $M \in \mathcal{U}$ for which there exists $K \in \kappa$ satisfying $P_u \not\subseteq M \subseteq K \subseteq Q_u$.

We denote the set of nhds (conhds) of u by $\eta(u)\,(\,\mu(u)\,)$, respectively. We will also refer to $(\,\eta(u), \mu(u)\,)$, $u \in U^\flat$ as the dinhd system of (τ, κ).

Finally, we also recall from [6, 7] the some classes of ditopological texture spaces: For a ditopological texture space $(U, \mathcal{U}, \tau, \kappa)$:

1. $A \in \mathcal{U}$ is called pre-open (semi-open) if $A \subseteq intclA$ ($A \subseteq clintA$).

2. $B \in \mathcal{S}$ is called pre-closed (semi-closed) if $clintB \subseteq B$ ($intclB \subseteq B$).

§3. Minimal and Maximal Open Sets in Ditopological Spaces

Definition 3.1. Let $(U, \mathcal{U}, \tau, \kappa)$ be a ditopological texture space.

(1) A set $A \in \tau \setminus \{\emptyset\}$ is called minimal open if $G \subseteq A$, then $G = \emptyset$, or $G = A$ for all $G \in \tau$.

(2) A set $B \in \tau \setminus \{U\}$ is called maximal open if $B \subseteq H$, then $H = U$, or $H = B$ for all $H \in \tau$.

We denote by $\mathcal{MNO}(U, \mathcal{U}, \tau, \kappa)$, or when there can be no confusion by $\mathcal{MNO}(U)$ or even just $\mathcal{MNO}$, the set of minimal open sets in $\mathcal{U}$. Likewise, $\mathcal{MXO}(U, \mathcal{U}, \tau, \kappa)$, $\mathcal{MXO}(S)$ or $\mathcal{MXO}$ will denote the set of maximal open sets.

Lemma 3.2. *Let $(U, \mathcal{U}, \tau, \kappa)$ be a ditopological texture space. Then:*

(i) If $A \in \mathcal{MNO}$ and $B \in \tau$, then $A \cap B = \emptyset$ or $A \subseteq B$.

(ii) If $A, B \in \mathcal{MNO}$, then $A \cap B = \emptyset$ or $A = B$.

(iii) If $A \in \mathcal{MXO}$ and $B \in \tau$, then $A \cup B = U$ or $B \subseteq A$.

(iv) If $A, B \in \mathcal{MXO}$, then $A \cup B = U$ or $A = B$.

Proof. (i) Suppose that $A \cap B \neq \emptyset$. Since A is minimal open set and $A \cap B \in \tau$ and $A \cap B \subseteq A$, we have $A \cap B = A$, and so $A \subseteq B$.

(ii) Let $A \cap B \neq \emptyset$. Then $A \subseteq B$ and $B \subseteq A$ by (i), and so $A = B$.

(iii) Suppose that $A \cup B \neq U$. Since A is maximal open set and $A \subseteq A \cup B$, we have $A \cup B = A$. Therefore $B \subseteq A$.

(iv) Let $A \cup B \neq U$. Then $A \subseteq B$ and $B \subseteq A$ by (iii), and so $A = B$. □

Note that the set of open (closed) nbhd of u will denote by $\eta_\tau(u)\,(\eta_\kappa(u)\,)$.

Proposition 3.3. *Let $A \in \mathcal{MNO}$. If $A \not\subseteq Q_u$ and $N \in \eta_\tau(u)$, then A is proper subset of N.*

Proof. Suppose that $N \in \eta_\tau(u)$, but $A \not\subset N$. Then $A \cap N \in \tau$ and $A \cap N$ is proper subset of A and $A \cap N \neq \emptyset$. This is a contradiction, since A is minimial open set. □

Now we consider any topological space $(U, \mathcal{T})$. Then the pair $(\mathcal{T}, \mathcal{T}^c)$ is a complemented ditopology on the discrete texture $(U, \mathcal{P}(U), \pi)$ (see Examples 2.2 (3)). For $A \subseteq U$ and $u \in U$, we have:

$$cl(A) \not\subseteq Q_u \Longleftrightarrow A \cap N \neq \emptyset, \quad \forall\, N \in \eta(u).$$

by the closure definition.

Note that we will consider $(U, \mathcal{P}(U), \mathcal{T}, \mathcal{T}^c)$ ditopological space throughout this subsection.

Proposition 3.4. *Let $A \in \mathcal{T}$. The following are equivalent.*

(i) $A \in \mathcal{MNO}$.

(ii) $A \subseteq cl(B)$ for every $B \in \mathcal{P}(U) \setminus \{\emptyset\}$ and $B \subseteq A$.

(iii) $cl(A) = cl(B)$ for every $B \in \mathcal{P}(U) \setminus \{\emptyset\}$ and $B \subseteq A$.

Proof. (i) $\Longrightarrow$ (ii) Let $B \in \mathcal{P}(U) \setminus \{\emptyset\}$ and $B \subseteq A$. Suppose that $A \not\subseteq Cl(B)$. Then there exist $u \in U$ such that $A \not\subseteq Q_u$ and $P_u \not\subseteq cl(B)$. Then if $A \not\subseteq Q_u$ then $P_u \subseteq A$ and so $u \in A$. Now let $N \in \eta_{\mathcal{T}}(u)$. Then

$$B = A \cap B \subset N \cap B$$

and so $N \cap B \neq \emptyset$. Hence we have $cl(B) \not\subseteq Q_u$, that is $u \in cl(B)$. This contradicts $P_u \subseteq cl(B)$.

(ii)$\Longrightarrow$ (iii) Let $B \in \mathcal{P}(U)$ and $B \subseteq A$. Then $cl(B) \subseteq cl(A)$. Further, by (ii), $cl(A) \subseteq cl(cl(B)) = cl(B)$. Then we have $cl(A) = cl(B)$.

(iii)$\Longrightarrow$ (i) Let $A \in \mathcal{T}$ be a not minimal set. From the Definition 3.1, there exists $B \in \mathcal{T} \setminus \{\emptyset\}$ such that $B \subseteq A$ and $B \neq A$. Hence $A \not\subseteq Q_u$ and $P_u \not\subseteq B$ for some $u \in U$. Then we have $B^c \in \mathcal{T}^c$ and $B^c \not\subseteq Q_u$, and so $cl(\{u\}) \subseteq B^c$. It follows that $cl(\{u\}) \neq cl(A)$. □

Proposition 3.5. *If $A \in \mathcal{MNO}$ and $\emptyset \neq B \subseteq A$, then A is preopen set.*

Proof. Suppose that A is minimal open set and $\emptyset \neq B \subseteq A$. Then $int(A) \subseteq int(cl(B))$, by Proposition 3.4 (2). It follows that we have $B \subset A = int(A) \subseteq int(cl(B))$, since A is open set. □

Proposition 3.6. *Let $A \in \mathcal{MXO}$, and $U \setminus A \nsubseteq Q_u$ for some $u \in U$. Then $U \setminus A \subset N$ for any $N \in \eta_{\mathcal{T}}(u)$.*

Proof. Suppose that $U \setminus A \nsubseteq Q_u$. Then $u \in U \setminus A$, and so $N \subset A$ for any $N \in \eta_{\mathcal{T}}(u)$. By Lemma 3.2 (iii), $U = N \cup A$ and so $U \setminus A \subseteq N$. □

Corollary 3.7. *Let $A \in \mathcal{MXO}$. Then either of the following (i) and (ii) is provided:*

(i) If $U \setminus A \nsubseteq Q_u$ for some $u \in U$, then $N = U$ for all $N \in \eta_{\mathcal{T}}(u)$.

(ii) There exists a set $G \in \mathcal{T} \setminus \{U\}$ such that $U \setminus A \subseteq G$.

Proof. We suppose that (i) does not provided. Then there exists $u \in U$ and $N \in \eta_{\mathcal{T}}(u)$ such that $P_u \nsubseteq A$ and $N \subset U$. By Proposition 3.6, we have $U \setminus A \subseteq N$. □

Corollary 3.8. *Let $A \in \mathcal{MXO}$. Then either of the following (i) and (ii) is provided:*

(i) If $U \setminus A \nsubseteq Q_u$ for some $u \in U$, then $U \setminus A \subsetneq N$ for all $N \in \eta_{\mathcal{T}}(u)$.

(ii) There exists a set $G \in \mathcal{T}$ such that $U \setminus A = G \neq U$.

Proof. We suppose that (ii) does not provided. By Proposition 3.6, we have $U \setminus A \subseteq N$, for all $U \setminus A \nsubseteq Q_u$ and $N \in \eta_{\mathcal{T}}(u)$. It follows that we have $U \setminus A \subsetneq N$. □

Proposition 3.9. *Let $A \in \mathcal{MXO}$. The following are satisfied.*

(i) $int(U \setminus A) = U \setminus A$ or $int(U \setminus A) = \emptyset$.

(ii) If $\emptyset \neq B \subseteq U \setminus A$, then $cl(B) = U \setminus A$.

(iiii) If $A \subsetneq B$, then $cl(B) = U$.

Proof. (i) It is obtained immediately by Corollary 3.8.

(ii) Let $\emptyset \neq B \subseteq U \setminus A$. By Proposition 3.6, we have $N \cap B \neq \emptyset$ for $U \setminus A \nsubseteq Q_u$ and $N \in \eta_{\mathcal{T}}(u)$. Hence, $U \setminus A \subseteq cl(B)$. Since $U \setminus A \in \mathcal{T}^c$ and $B \subseteq U \setminus A$, we obtain that $cl(B) \subseteq cl(U \setminus A) = U \setminus A$, and so $cl(B) = U \setminus A$.

(iii) Let $A \subsetneq B \subseteq U$. Then there exists $\emptyset \neq M \subseteq U \setminus A$ such that $B = A \cup M$. By (ii), we have $cl(B) = cl(A) \cup cl(M) \supseteq (U \setminus A) \cup A = U$. Hence, $cl(B) = U$. □

Proposition 3.10. *Let $A \in \mathcal{MXO}$. The following are satisfied.*

(i) If B is a proper subset of U and $A \subset B$, then $int(B) = A$.

(ii) If $\emptyset \neq B \subseteq U \setminus A$, then $U \setminus cl(B) = int(U \setminus B) = A$.

Proof. (i) If $B = A$, then $int(B) = int(A) = A$. Now, suppose that $B \neq A$, and so $A \subsetneq B$. Then $A \subset int(B)$. It follows that we have $int(B) \subsetneq A$, since A is maximal open set. Hence, $int(B) = A$.

(ii) By Proposition 3.9 (ii) and the above (i), the proof is obvious, since $A \subset U \setminus B \subsetneq U$. □

Corollary 3.11. *If $A \in \mathcal{MXO}$ and $A \subset B$, then B is a preopen set.*

Proof. If $B = A$, then B is an open set, hence it is preopen set. Suppose that $A \subsetneq B$. By Proposition 3.9 (iii), we have $int(cl(B)) = intU = U \supseteq B$. Hence, B is a preopen set. □

§4. Minimal and Maximal Closed Sets

Definition 4.1. Let $(U, \mathcal{U}, \tau, \kappa)$ be a ditopological texture space.

(1) A set $A \in \kappa \setminus \{\emptyset\}$ is called minimal closed if $F \subseteq A$, then $F = \emptyset$, or $F = A$ for all $F \in \kappa$.

(2) A set $B \in \kappa \setminus \{U\}$ is called maximal closed if $B \subseteq H$, then $H = U$, or $H = B$ for all $H \in \kappa$

We denote by $\mathcal{MNC}(U, \mathcal{U}, \tau, \kappa)$, or when there can be no confusion by $\mathcal{MNC}(U)$ or even just $\mathcal{MNC}$, the set of minimal closed sets in $\mathcal{U}$. Likewise, $\mathcal{MXC}(U, \mathcal{U}, \tau, \kappa)$, $\mathcal{MXC}(S)$ or $\mathcal{MXC}$ will denote the set of maximal closed sets.

The proof of the next results is omitted, since it is obtained by an argument similar to the proof of Lemma 3.2.

Lemma 4.2. *Let $(U, \mathcal{U}, \tau, \kappa)$ be a ditopological texture space. The following are satisfied.*

(i) If $A \in \mathcal{MNC}$ and $B \in \kappa$, then $A \cap B = \emptyset$ or $A \subseteq B$.

(ii) If $A, B \in A \in \mathcal{MNC}$, then $A \cap B = \emptyset$ or $A = B$.

(iii) If $A \in \mathcal{MXC}$ and $B \in \kappa$ then $A \cup B = U$ or $B \subseteq A$.

(iv) $A, B \in A \in \mathcal{MXC}$, then $A \cup B = U$ or $A = B$.

Proposition 4.3. *Let $(U, \mathcal{U}, \tau, \kappa)$ be a ditopological space.*

(a) Let $A \in \mathcal{MNC}$ and $\{A_j\}_{j \in J} \subseteq \mathcal{MNC}$. Then:

(i) If $A \subseteq \bigvee_{j \in J} A_j$, then there exists $k \in J$ such that $A = A_k$.

(ii) If $A \neq A_k$ for some $k \in J$, then $(\bigvee_{j \in J} A_j) \cap A = \emptyset$.

(b) Let $A \in \mathcal{MXC}$ and $\{A_j\}_{j \in J} \subseteq \mathcal{MXC}$. Then

(iii) If $\bigcap_{j \in J} A_j \subset A$, then there exists $k \in J$ such that $A = A_k$.

Proof. (i) Let $A \subseteq \bigvee_{j\in J} A_j$. Then $A = A \cap (\bigvee_{j\in J} A_j) = \bigvee_{j\in J}(A \cap A_j)$. By Lemma 4.2 (ii), if $A \neq A_j$ for some $j \in J$, then $A \cap A_j = \emptyset$, and so $\emptyset = \bigvee_{j\in J}(A \cap A_j) = A$. This is a contradiction, since A is minimal closed set. It follows that there exists $k \in J$ such that $A = A_k$.

(ii) Suppose that $(\bigvee_{j\in J} A_j) \cap A \neq \emptyset$. Then there exists $k \in J$ such that $A_k \cap A \neq \emptyset$. From Lemma 4.2 (ii), we have $A_k = A$, but it is a contradiction.

(iii) Let $\bigcap_{j\in J} A_j \subset A$.Then $A = A \cup (\bigcap_{j\in J} A_j) = \bigcap_{j\in J}(A \cup A_j)$. If $A \cup A_k = U$ for some $k \in J$, then we have $U = \bigcap_{j\in J}(A \cup A_j) = A$.It foolows that we have a contradiction, since A is maximal closed set. Thus there exists $j \in J$ such that $A \cup A_j \neq U$.By Lemma 4.2 (iii), the result is obtained. □

Proposition 4.4. *Let $(U, \mathcal{U}, \sigma, \tau, \kappa)$ be a complemented ditopological space and $A \in \mathcal{U}$.*

(i) A is a minimal closed set if and only if $\sigma(A)$ is a maximal open set.

(ii) A is maximal closed set if and only if $\sigma(A)$ is a minimal open set.

Proof. (i) Suppose that A is a minimal closed set. Then $A \neq \emptyset$ and $\sigma(A) \in \kappa \setminus \{U\}$. Now let $H \in \tau$ and $\sigma(A) \subseteq H$. From the complemented ditopology, we have $\sigma(H) \in \kappa$ and $\sigma(H) \subseteq \sigma(\sigma(A)) = A$. Since A is minimal closed set, $\sigma(H) = \emptyset$ or $\sigma(H) = A$. Then it is obtained $H = \sigma\sigma(H)) = \sigma(es) = U$ or $H = \sigma(\sigma(H)) = \sigma(A)$.

(ii) The proof is obtained by an argument similar to the proof (i). □

References

[1] L. M. Brown and M. Diker. *Ditopological texture spaces and intuitionistic sets*, Fuzzy sets and systems **98**, (1998), 217–224.

[2] L. M. Brown and R. Ertürk. Fuzzy Sets as Texture Spaces, I. Representation Theorems, Fuzzy Sets and Systems **110** (2) (2000), 227–236.

[3] L. M. Brown and R. Ertürk. Fuzzy sets as texture spaces, II. Subtextures and quotient textures, Fuzzy Sets and Systems **110** (2) (2000), 237–245.

[4] L. M. Brown, R. Ertürk and Ş Dost. Ditopological texture spaces and fuzzy topology, I. Basic Concepts, Fuzzy Sets and Systems **147** (2) (2004), 171–199.

[5] L. M. Brown, R. Ertürk and Ş Dost. *Ditopological texture spaces and fuzzy topology, II. Topological Considerations*, Fuzzy Sets and Systems **147** (2) (2004), 201–231.

[6] Ş. Dost. Semi-open and semi-closed sets in ditopological texture spaces, J. Adv. Math. Stud. Vol. **5** (2012), No. 1, 97–110.

[7] Ş. Dost, L. M. Brown and R. Ertürk. β-open and β-closed sets in ditopological setting, Filomat, 24:2 (2010), 11-26, DOI: 10.2298/FIL1002011D.

[8] F. Nakaoka and N. Oda. Minimal closed sets and maximal closed sets, Int. J. Math. Math. Sci., (2006), 1–8.

[9] F. Nakaoka and N. Oda. Some applications of minimal open sets, Int. J. Math. Math. Sci., **27** (8)(2001), 471–476.

Scientia Magna
Vol. 16 (2021), No. 1, 79-83

The mean value for the function $\left(t^{(e)}(n)\right)^2$ over square-full numbers

Fuxia Xu

School of Mathematics and Statistics, Shandong Normal University
Jinan, Shandong, China
E-mail: xufuxia20@stu.sdnu.edu.cn

Abstract Let $t^{(e)}(n)$ denote the number of e-squarefree e-divisor of n. In this paper we shall use the complex integration method to study the mean value of $\left(t^{(e)}(n)\right)^2$ and give its asymptotic formula $\sum\limits_{n\le x}\left(t^{(e)}(n)\right)^2 = x^{\frac{1}{2}}(P_1(\log x)) + x^{\frac{1}{3}}(P_2(\log x)) + O(x^{\sigma_0+\varepsilon})$.

Keywords Square-full integers, square-free divisors, mean value.
2010 Mathematics Subject Classification 11H60.

§1. Introduction and preliminaries

An integer n is a full square number if p is the prime factor of n, then $p|n$, that is, its prime factorization formula must have the following form

$$n = p_1^{a_1}p_2^{a_2}\cdots p_s^{a_s}, (a_1 \ge 2, a_2 \ge 2, \cdots, a_s \ge 2).$$

Let $n > 1$ be an integer, $n = \prod_{i=1}^{s} p_i^{a_i}$, $d = \prod_{i=1}^{s} p_i^{b_i}$, where $b_i|a_i$. The integer d is called an exponential divisor of n if $b_i|a_i$, $i \in \{1, 2, \cdots, s\}$, notation: $d|_e n$. By convention $1|_e 1$. If all exponents $a_1, a_2, \cdots, a_s$ are squarefree, the integer $n > 1$ called e-squarefree. Now we consider the exponential squarefree exponential divisor of n, if $b_i|a_i$, $i \in \{1, 2, \cdots, s\}$ and $b_1, b_2, \cdots, b_s$ are squarefree, then $d = \prod_{i=1}^{s} p_i^{b_i}$ is an e-squarefree e-divisor of $n = \prod_{i=1}^{s} p_i^{a_i}$. Note that the integer 1 is e-squarefree but is not an e-divisor of $n > 1$.

Let $t^{(e)}(n)$ denote the number of e-squarefree e-divisoe of n. We know that $t^{(e)}(n)$ is multiplicative and if $n = \prod_{i=1}^{s} p_i^{a_i} > 1$, $i \in \{1, 2, \cdots, s\}$, then (see [4])

$$t^{(e)}(n) = 2^{\omega(a_1)}\cdots 2^{\omega(a_s)},$$

where $\omega(n)$ denote the number of distinct prime factors of n, with $\omega(1) = 0$ and $\omega(n) = s$. In particular, for each prime p,

$$t^{(e)}(p) = 1,\ t^{(e)}(p^2) = t^{(e)}(p^3) = t^{(e)}(p^4) = t^{(e)}(p^5) = 2,\ t^{(e)}(p^6) = 4, \cdots.$$

The Dirichlet series of $t^{(e)}(n)$ is of form

$$\sum_{n=1}^{\infty} \frac{t^{(e)}(n)}{n^s} = \zeta(s)\zeta(2s)V(s),\ \Re s > 1,$$

where $V(s) = \sum_{n=1}^{\infty} \frac{v(n)}{n^s}$ is absolutely convergent for $\Re s > \frac{1}{4}$. For a sufficiently small positive constant ε, Tóth [4] showed that

$$\sum_{n\le x} t^{(e)}(n) = C_1 x + C_2 x^{\frac{1}{2}} + R(x), \tag{1}$$

where

$$C_1 := \prod_p \left(1 + \sum_{a=6}^{\infty} \frac{2^{\omega(a)} - 2^{\omega(a-1)}}{p^a}\right),$$

$$C_2 := \zeta\left(\frac{1}{2}\right) \prod_p \left(1 + \sum_{a=4}^{\infty} \frac{2^{\omega(a)} - 2^{\omega(a-1)} - 2^{\omega(a-2)} + 2^{\omega(a-4)}}{p^{\frac{a}{2}}}\right),$$

and $R(x) = O(x^{\frac{1}{4}+\varepsilon})$. Suppose RH is ture, then in Liu and Dong [2] it is proved that $R(x) = O(x^{\frac{7}{29}+\varepsilon})$.

Many scholars are interested in researching the divisor problem, and they have obtained a large number of good results. However, there are many problems have not been solved.

In this paper, we will use the complex integration method to estimate the mean value of $\left(t^{(e)}(n)\right)^2$ over square-full numbers, that is

$$\sum_{\substack{n\le x\\ n\, is\, square-full}} \left(t^{(e)}(n)\right)^2 = \sum_{n\le x} \left(t^{(e)}(n)\right)^2 f(n),$$

where $f(n)$ is the characteristic function of square-full integers, i.e.,

$$f(n) = \begin{cases} 1, & n \text{ is square-full}, \\ 0, & \text{otherwise}. \end{cases}$$

We establish the following theorem.

Theorem 1.1. *For any $\varepsilon > 0$, then we have*

$$\sum_{n\le x} \left(t^{(e)}(n)\right)^2 = x^{\frac{1}{2}}(P_1(\log x)) + x^{\frac{1}{3}}(P_2(\log x)) + O(x^{\sigma_0+\varepsilon}), \tag{2}$$

where $\sigma_0 = \frac{10741}{39424} = 0.272\cdots$, and $P_k(\log x), k = 1, 2$ are polynomials of degree 1 in $\log x$.

§2. Some lemmas

In this part, in order to prove our theorem, we give some lemmas.

Lemma 2.1. *Suppose $\frac{1}{2} \le \sigma \le 1$, $t \ge t_0 \ge 2$, we have*

$$\zeta(\sigma + it) \ll t^{\frac{(1-\sigma)}{3}} \log t. \tag{3}$$

Proof. This lemma can be founed in Titchmarsh [3]. □

Lemma 2.2. *Suppose* $\frac{1}{2} < \sigma < 1$, *define*

$$\begin{aligned} m(\sigma) &= \frac{4}{(3-4\sigma)},\ \frac{1}{2} < \sigma \le \frac{5}{8}, \\ m(\sigma) &= \frac{12408}{(4537-4890\sigma)},\ \frac{3}{4} \le \sigma \le \frac{5}{6}, \end{aligned} \tag{4}$$

then for any $\varepsilon > 0$, *we have*

$$\int_1^T \mid \zeta(\sigma+it) \mid^{m(\sigma)} dt \ll T^{1+\varepsilon}. \tag{5}$$

Proof. This lemma can be found in Ivić [1]. □

§3. Proof of Theorem 1.1

Let

$$F(s) = \sum_{n=1}^{\infty} \frac{\left(t^{(e)}(n)\right)^2}{n^s},\ \Re s > \frac{1}{2}. \tag{6}$$

For $\Re s > \frac{1}{2}$, since $t^{(e)}(n)$ is a multiplicative function, than we use Euler product formula and get

$$\begin{aligned} F(s) &= \sum_{\substack{n=1 \\ n\, is\, square-full}}^{\infty} \left(t^{(e)}(n)\right)^2 = \sum_{n=1}^{\infty} \left(t^{(e)}(n)\right)^2 f(n) \\ &= \prod_p \left(1 + \frac{(t^{(e)}(p))^2 f(p)}{p^s} + \frac{(t^{(e)}(p^2))^2 f(p^2)}{p^{2s}} + \frac{(t^{(e)}(p^3))^2 f(p^3)}{p^{3s}} + \cdots\right) \\ &= \prod_p \left(1 + \frac{4}{p^{2s}} + \frac{4}{p^{3s}} + \frac{4}{p^{4s}} + \frac{4}{p^{5s}} + \frac{16}{p^{6s}} + \cdots\right) \\ &= \zeta(2s) \prod_p \left(1 + \frac{3}{p^{2s}} + \frac{4}{p^{3s}} + \frac{12}{p^{6s}} + \cdots\right) \\ &= \zeta^4(2s) \prod_p \left(1 + \frac{4}{p^{3s}} - \frac{6}{p^{4s}} - \frac{12}{p^{5s}} + \frac{20}{p^{6s}} + \cdots\right) \\ &= \zeta^4(2s)\zeta(3s) \prod_p \left(1 + \frac{3}{p^{3s}} - \frac{6}{p^{4s}} - \frac{12}{p^{5s}} + \frac{16}{p^{6s}} + \cdots\right) \\ &= \zeta^4(2s)\zeta^4(3s) \prod_p \left(1 - \frac{6}{p^{4s}} - \frac{12}{p^{5s}} + \cdots\right) \\ &= \zeta^4(2s)\zeta^4(3s)G(s), \end{aligned} \tag{7}$$

where $G(s) = \sum\limits_{n=1}^{\infty} \frac{g(s)}{n^s}$, $G(s)$ is absolutely convergent for $\Re s > \frac{1}{4}$. Then

$$\sum_{n \le x} \mid g(n) \mid \ll x^{\frac{1}{4}+\varepsilon}.$$

By Perron's formula, we have

$$\sum_{n\le x}\left(t^{(e)}(n)\right)^2=\frac{1}{2\pi i}\int_{b-iT}^{b+iT}\zeta^4(2s)\zeta^4(3s)G(s)\cdot\frac{x^s}{s}ds+O\left(\frac{x^{\frac{1}{2}+\varepsilon}}{T}\right),\tag{8}$$

where $b=\frac{1}{2}+\varepsilon$, $T=x^c$, $c=5$. We move the line of integration to $\Re s=\sigma_0$, $\frac{1}{4}<\sigma_0<\frac{1}{3}$, whose exact value will be determined later. According to the residue theorem, we get

$$S(x)=x^{\frac{1}{2}}(P_1(\log x))+x^{\frac{1}{3}}(P_2(\log x))+I_1+I_2+I_3+O(1),$$

where

$$I_1=\frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT}\zeta^4(2s)\zeta^4(3s)G(s)\frac{x^s}{s}ds,$$
$$I_2=\frac{1}{2\pi i}\int_{b-iT}^{\sigma_0-iT}\zeta^4(2s)\zeta^4(3s)G(s)\frac{x^s}{s}ds,$$
$$I_3=\frac{1}{2\pi i}\int_{\sigma_0+iT}^{b+iT}\zeta^4(2s)\zeta^4(3s)G(s)\frac{x^s}{s}ds.$$

Applying Lemma 2.1, we can obtain

$$\begin{aligned}I_2+I_3&\ll\int_{\sigma_0}^{\frac{1}{2}}|\zeta(2\sigma+2iT)|^4\,|\zeta(3\sigma+3iT)|^4\,x^{\sigma}T^{-1}d\sigma\\&\ll x^{\frac{1}{2}+\varepsilon}T^{\frac{5-20\sigma_0}{3}+\varepsilon}\\&\ll 1,\end{aligned}\tag{9}$$

if $\sigma_0>\frac{53}{200}$, where $s=\sigma+it$. On the other hand, we have

$$I_1\ll x^{\sigma_0}\int_1^T|\zeta(2\sigma_0+2it)|^4\,|\zeta(3\sigma_0+3it)|^4\,t^{-1}dt.\tag{10}$$

According to the partial integral formula, it suffices to prove that

$$\begin{aligned}I&=\int_1^T|\zeta(2\sigma_0+2it)|^4\,|\zeta(3\sigma_0+3it)|^4\,dt\\&\ll T^{1+\varepsilon}.\end{aligned}\tag{11}$$

Let $p>0,\ q>0$ are real number satisfying $\frac{1}{p}+\frac{1}{q}=1$. According to the Hölder's inequality, we have

$$I\ll\left(\int_1^T\left|\zeta^4(2\sigma_0+2it)\right|^p dt\right)^{\frac{1}{p}}\left(\int_1^T\left|\zeta^4(3\sigma_0+3it)\right|^q dt\right)^{\frac{1}{q}}.\tag{12}$$

Let $4p=m(2\sigma_0)$, $4q=m(3\sigma_0)$ satisfying

$$\frac{4}{m(2\sigma_0)}+\frac{4}{m(3\sigma_0)}=1,\tag{13}$$

applying Lemma 2.2 we can get (11), if $\sigma_0=\frac{10741}{39486}=0.272\cdots$.

Thus we complete the proof of Theorem 1.1.

Acknowledgements

This paper quotes the research literature of several scholars, and the author expresses his gratitude to the scholars in the references. Without the help and inspiration of the research results of each scholar, I would have been difficult to complete the writing of this paper.

References

[1] A. Ivić. The Riemann Zeta-function. John Wiley and Sons, 1985.

[2] H. Liu and Y. Dong, On the mean value of the e-squarefree e-divisor function. Scientia Magna 5 (2009), No. 4, 46 – 50.

[3] E. C. Titchmarsh. The theory of the Riemannzeta-function. Oxford: Clarendon Press, 1951.

[4] L. Tóth. On certain arithmetic functions involving exponential divisors. AnnalesUniv. Sci. Budapest. Sect. Comp. 27 (2007), 155 – 166.

[5] D. Zhang and W. Zhai. On certain divisor sum over squarefull integers. Journal of shandong university 41 (2006), No. 2, 66 – 68.

Scientia Magna
Vol. 16 (2021), No. 1, 84-96

Norms over intuitionistic fuzzy SU-subalgebras

Rasul Rasuli

Department of Mathematics
Payame Noor University (PNU), Tehran, Iran.
E-mail: rasulirasul@yahoo.com

Abstract Using norms (T and C) this paper introduces a new notion of intuitionistic fuzzy SU-algebra and some of its properties are discussed in detail. Next, we define the intersection and direct product of them and investigate some properties of them. Finally, we obtain some results of them under homomorphisms.

Keywords Theory of fuzzy sets, norms, intuitionistic fuzzy set, intuitionistic fuzzy SU-algebra, homomorphisms.
2010 Mathematics Subject Classification 03E72, 47A30, 20K30.

§1. Introduction

Algebra (from Arabic: al-jabr, meaning "reunion of broken parts and "bonesetting) is one of the broad parts of mathematics, together with number theory, geometry and analysis. In its most general form, algebra is the study of mathematical symbols and the rules for manipulating these symbols; it is a unifying thread of almost all of mathematics. It includes everything from elementary equation solving to the study of abstractions such as groups, rings, and fields. The more basic parts of algebra are called elementary algebra; the more abstract parts are called abstract algebra or modern algebra. Elementary algebra is generally considered to be essential for any study of mathematics, science, or engineering, as well as such applications as medicine and economics. Abstract algebra is a major area in advanced mathematics, studied primarily by professional mathematicians. Several algebraic structure have been introduced in the recent past. Iseki and Tanaka [6], introduced a class of abstract algebra: BCK-algebra. Also Imai and Iseki [5], dealt about BCI-algebras. Then, Hu and Li [4], have given the notion of BCH-algebra which is the generalization of BCI and BCK-algebras. Neggers et.al [9, 10, 11] introduced the notions of B-algebras, Q-algebras and d-algebras. In 2010, Megalai and Tamilarasi [47], introduced TM-algebra. During 2011, Keawrahun and Leerawat [7] introduced new structured algebra called SU-Algebra. The concept of a fuzzy set was introduced by Zadeh [48]. In 1986, Atanassov [2] introduced the notion of intuitionistic fuzzy sets as a generalization of fuzzy sets. Muralikrishna and Chandramouleeswarna [8] introduced the notion of intuitionistic fuzzy SU-subalgebra. Triangular norms and conorms are operations which generalize the logical conjunction and logical disjunction to fuzzy logic. They are a natural interpretation of the conjunction and disjunction in the semantics of mathematical fuzzy logics [Hjek (1998)] and they are used to combine criteria in multi-criteria decision making. The author by using norms, investigated some properties of fuzzy algebraic structures [12-46]. In this paper, by using norms (T and C) a new notion of intuitionistic fuzzy SU-algebra is defined and some substructures are also established in this algebra.

§2. Preliminaries

This section contains some basic definitions and preliminary results which will be needed in the sequal. For more details we refer to [1, 2, 3, 7, 8, 30, 42].

Definition 2.1. *For sets X, Y and Z, $f = (f_1, f_2) : X \to Y \times Z$ is called a complex mapping if $f_1 : X \to Y$ and $f_2 : X \to Z$ are mappings.*

Definition 2.2. *Let X be a nonempty set. A complex mapping $A = (\mu_A, \nu_A) : X \to [0,1] \times [0,1]$ is called an intuitionistic fuzzy set (in short, IFS) in X if $\mu_A + \nu_A \leq 1$ where the mappings $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) for each $x \in X$ to A, respectively. In particular $\emptyset_X$ and U_X denote the intuitionistic fuzzy empty set and intuitionistic fuzzy whole set in X defined by $\emptyset_X(x) = (0,1)$ and $U_X(x) = (1,0)$, respectively. We will denote the set of all $IFSs$ in X as $IFS(X)$.*

Definition 2.3. *Let X be a nonempty set and let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be $IFSs$ in X. Then*
(1) Inclusion: $A \subseteq B$ iff $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$.
(2) Equality:$A = B$ iff $A \subseteq B$ and $B \subseteq A$.
(3) $\sqcup A = (\mu_A, \mu_{\bar{A}})$.
(4) $\sqcap A = (\nu_{\bar{A}}, \nu_A)$.

Definition 2.4. *A t-norm T is a function $T : [0,1] \times [0,1] \to [0,1]$ having the following four properties:*
(T1) $T(x,1) = x$ (neutral element)
(T2) $T(x,y) \leq T(x,z)$ if $y \leq z$ (monotonicity)
(T3) $T(x,y) = T(y,x)$ (commutativity)
(T4) $T(x, T(y,z)) = T(T(x,y), z)$ (associativity),
for all $x, y, z \in [0,1]$.

Example 2.1. *(1) Standard intersection t-norm $T_m(x,y) = \min\{x,y\}$.*
(2) Bounded sum t-norm $T_b(x,y) = \max\{0, x+y-1\}$.
(3) algebraic product t-norm $T_p(x,y) = xy$.
(4) Drastic t-norm

$$T_D(x,y) = \begin{cases} y & \text{if } x = 1 \\ x & \text{if } y = 1 \\ 0 & \text{otherwise.} \end{cases}$$

(5) Nilpotent minimum t-norm

$$T_{nM}(x,y) = \begin{cases} \min\{x,y\} & \text{if } x + y > 1 \\ 0 & \text{otherwise.} \end{cases}$$

(6) Hamacher product t-norm

$$T_{H_0}(x,y) = \begin{cases} 0 & \text{if } x = y = 0 \\ \frac{xy}{x+y-xy} & \text{otherwise.} \end{cases}$$

The drastic t-norm is the pointwise smallest t-norm and the minimum is the pointwise largest t-norm: $T_D(x,y) \leq T(x,y) \leq T_{\min}(x,y)$ for all $x, y \in [0,1]$.

Lemma 2.1. *Let T be a t-norm. Then*

$$T(T(x,y), T(w,z)) = T(T(x,w), T(y,z)),$$

for all $x, y, w, z \in [0,1]$.

Definition 2.4. *A t-conorm C is a function $C : [0,1] \times [0,1] \to [0,1]$ having the following four properties:*
(C1) $C(x,0) = x$
(C2) $C(x,y) \leq C(x,z)$ if $y \leq z$
(C3) $C(x,y) = C(y,x)$
(C4) $C(x, C(y,z)) = C(C(x,y), z)$,
for all $x, y, z \in [0,1]$.

Example 2.2. *(1) Standard union t-conorm $C_m(x,y) = \max\{x,y\}$.*
(2) Bounded sum t-conorm $C_b(x,y) = \min\{1, x+y\}$.
(3) Algebraic sum t-conorm $C_p(x,y) = x + y - xy$.
(4) Drastic T-conorm

$$C_D(x,y) = \begin{cases} y & \text{if } x = 0 \\ x & \text{if } y = 0 \\ 1 & \text{otherwise,} \end{cases}$$

dual to the drastic T-norm.
(5) Nilpotent maximum T-conorm , dual to the nilpotent minimum T-norm:

$$C_{nM}(x,y) = \begin{cases} \max\{x,y\} & \text{if } x + y < 1 \\ 1 & \text{otherwise.} \end{cases}$$

(6) Einstein sum (compare the velocity-addition formula under special relativity) $C_{H_2}(x,y) = \dfrac{x+y}{1+xy}$ is a dual to one of the Hamacher t-norms. Note that all t-conorms are bounded by the maximum and the drastic t-conorm: $C_{\max}(x,y) \leq C(x,y) \leq C_D(x,y)$ for any t-conorm C and all $x, y \in [0,1]$.

Recall that t-norm T(t-conorm C) is idempotent if for all $x \in [0,1]$, $T(x,x) = x(C(x,x) = x)$.

Lemma 2.2. *Let C be a t-conorm. Then*

$$C(C(x,y), C(w,z)) = C(C(x,w), C(y,z)),$$

for all $x, y, w, z \in [0, 1]$.

Definition 2.5. *Let* $A = (\mu_A, \nu_A) \in IFS(X)$ *and* $B = (\mu_B, \nu_B) \in IFS(X)$. *Define intesection* A *and* B *as*

$$A \cap B = (\mu_A, \nu_A) \cap (\mu_B, \nu_B) = (\mu_{A\cap B}, \nu_{A\cap B})$$

such that $\mu_{A\cap B}(x) = T(\mu_A(x), \mu_B(x))$ *and* $\nu_{A\cap B}(x) = C(\nu_A(x), \nu_B(y))$ *for all* $x \in X$.

Definition 2.6. *Let* φ *be a function from set* X *into set* Y *such that* $A = (\mu_A, \nu_A)$ *and* $B = (\mu_B, \nu_B)$ *be two intuitionistic fuzzy sets in* X *and* Y *respectively.*
For all $x \in X, y \in Y$, *we define*

$$\varphi(A)(y) = (\varphi(\mu_A)(y), \varphi(\nu_A)(y))$$
$$= \begin{cases} (\sup\{\mu_A(x) \mid x \in X, \varphi(x) = y\}, \inf\{\nu_A(x) \mid x \in X, \varphi(x) = y\}) & \text{if } \varphi^{-1}(y) \neq \emptyset \\ (0, 1) & \text{if } \varphi^{-1}(y) = \emptyset \end{cases}$$

Also $\varphi^{-1}(B)(x) = (\varphi^{-1}(\mu_B)(x), \varphi^{-1}(\nu_B)(x)) = (\mu_B(\varphi(x)), \nu_B(\varphi(x)))$.

Definition 2.7. *A SU-algebra is a non-empty set* X *with a consonant* 0 *and a single binary operation* $*$ *(denoted by* $(X, *, 0)$ *) satisfying the following axioms for any* $x, y, z \in X$*:*
(1) $((x * y) * (x * z)) * (y * z) = 0$,
(2) $x * 0 = x$,
(3) if $x * y = 0$, *then* $x = y$.

Definition 2.8. *A non-empty subset* S *of a SU-algebra* X *is said to be a subalgebra if* $x * y \in S$ *for all* $x, y \in S$.

Definition 2.9. *A function* $f : X \to Y$ *of SU-algebras* X *and* Y *is called homomorphism if* $f(x * y) = f(x) * f(y)$ *for all* $x, y \in X$.

§3. Norms over intuitionistic fuzzy SU-subalgebras

Definition 3.1. *A fuzzy subset* $\mu : (X, *, 0) \to [0, 1]$ *in a SU-algebra* $(X, *, 0)$ *is said to be a fuzzy SU-subalgebra of* X *under t-norm* T *if* $\mu(x * y) \geq T(\mu(x), \mu(y))$ *for all* $x, y \in X$. *We denote the set of all fuzzy SU-subalgebras of* X *under t-norm* T *by* $FSUT(X)$.

Definition 3.2. *Let* $A = (\mu_A, \nu_A) \in IFS(X)$ *in a SU-algebra* $(X, *, 0)$. *We say that* $A = (\mu_A, \nu_A)$ *is fuzzy SU-subalgebra of* X *under norms* (T, C) *if*
(1) $\mu_A(x * y) \geq T(\mu_A(x), \mu_A(y))$ *and*
(2) $\nu_A(x * y) \leq C(\nu_A(x), \nu_A(y))$
for all $x, y \in X$. *We denote the set of all intuitionistic fuzzy SU-subalgebras of* X *under norms* (T, C) *by* $IFSUN(X)$.

Example 3.1. *Let* $X = \{0, 1, 2, 3\}$ *be a set with the following table:*

*	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then $(X,*,0)$ is a SU-algebra. Define fuzzy subset $\mu : (X,*,0) \to [0,1]$ as

$$\mu(x) = \begin{cases} 0.1 & \textit{if } x = 0 \\ 0.2 & \textit{if } x = 1 \\ 0.3 & \textit{if } x = 2 \\ 0.4 & \textit{if } x = 3 \end{cases}$$

and let T be algebraic product t-norm as $T_p(a,b) = ab$ for all $a,b \in [0,1]$. Then $\mu \in FSUT(X)$. Also let $A = (\mu_A, \nu_A) \in IFS(X)$ in a SU-algebra $(X,*,0)$ such that

$$\mu_A(x) = \begin{cases} 0.2 & \textit{if } x = 0 \\ 0.3 & \textit{if } x = 1 \\ 0.4 & \textit{if } x = 2 \\ 0.45 & \textit{if } x = 3 \end{cases}$$

and

$$\nu_A(x) = \begin{cases} 0.3 & \textit{if } x = 0 \\ 0.4 & \textit{if } x = 1 \\ 0.5 & \textit{if } x = 2 \\ 0.55 & \textit{if } x = 3 \end{cases}$$

and let $T_p(a,b) = ab$ and $C_p(a,b) = a + b - ab$ for all $a,b \in [0,1]$. Then $A = (\mu_A, \nu_A) \in IFSUN(X)$.

Proposition 3.1. *Let $A = (\mu_A, \nu_A) \in IFSUN(X)$. If T and C be idempotent, then $A(0) \supseteq A(x)$ for all $x \in X$.*

Proof. Let $x \in X$. Then

$$\mu_A(0) = \mu_A(x * x) \geq T(\mu_A(x), \mu_A(x)) = \mu_A(x)$$

and

$$\nu_A(0) = \nu_A(x * x) \leq C(\nu_A(x), \nu_A(x)) = \nu_A(x)$$

and thus

$$A(0) = (\mu_A(0), \nu_A(0)) \supseteq (\mu_A(x), \nu_A(x)) = A(x).$$

□

Proposition 3.2. *Let $A = (\mu_A, \nu_A) \in IFSUN(X)$ and $B = (\mu_B, \nu_B) \in IFSUN(X)$. Then $A \cap B \in IFSUN(X)$.*

Proof. Let $x, y \in X$. Then
(1)

$$\begin{aligned}\mu_{A\cap B}(x*y) &= T(\mu_A(x*y), \mu_B(x*y))\\ &\geq T(T(\mu_A(x), \mu_A(y)), T(\mu_B(x), \mu_B(y)))\\ &= T(T(\mu_A(x), \mu_B(x)), T(\mu_A(y), \mu_B(y)))\\ &= T(\mu_{A\cap B}(x), \mu_{A\cap B}(y))\end{aligned}$$

so $\mu_{A\cap B}(x*y) \geq T(\mu_{A\cap B}(x), \mu_{A\cap B}(y))$.
(2)

$$\begin{aligned}\nu_{A\cap B}(x*y) &= C(\nu_A(x*y), \nu_B(x*y))\\ &\leq C(C(\nu_A(x), \nu_A(y)), C(\nu_B(x), \nu_B(y)))\\ &= C(C(\nu_A(x), \nu_B(x)), C(\nu_A(y), \nu_B(y)))\\ &= C(\nu_{A\cap B}(x), \nu_{A\cap B}(y))\end{aligned}$$

and then $\nu_{A\cap B}(x*y) \leq C(\nu_{A\cap B}(x), \nu_{A\cap B}(y))$.
Therefore (1)-(2) give us that $A \cap B \in IFSUN(X)$. □

Recall that if $A = (\mu_A, \nu_A) \in IFS(X)$, then $\bar{A} = (\mu_{\bar{A}}, \nu_{\bar{A}}) = (1-\mu_A, 1-\nu_A) \in IFS(X)$.

Proposition 3.3. *$A = (\mu_A, \nu_A) \in IFSUN(X)$ if and only if $\mu_A \in FSUT(X)$ and $\nu_{\bar{A}} \in FSUT(X)$.*

Proof. Let $x \in X$. If $A = (\mu_A, \nu_A) \in IFSUN(X)$, then $\mu_A(x*y) \geq T(\mu_A(x), \mu_A(y))$ and $\nu_A(x*y) \leq C(\nu_A(x), \nu_A(y))$. Thus $\mu_A \in FSUT(X)$. Also

$$\nu_{\bar{A}}(x*y) = 1-\nu_A(x*y) \geq 1-C(\nu_A(x), \nu_A(y)) = T(1-\nu_A(x), 1-\nu_A(y)) = T(\nu_{\bar{A}}(x), \nu_{\bar{A}}(y))$$

and so $\nu_{\bar{A}} \in FSUT(X)$.
Conversely, let $\mu_A \in FSUT(X)$ and $\nu_{\bar{A}} \in FSUT(X)$. As $\mu_A \in FSUT(X)$ so

$$\mu_A(x*y) \geq T(\mu_A(x), \mu_A(y)).$$

As $\nu_{\bar{A}} \in FSUT(X)$ so $\nu_{\bar{A}}(x*y) \geq T(\nu_{\bar{A}}(x), \nu_{\bar{A}}(y))$ then $-\nu_{\bar{A}}(x*y) \leq -T(\nu_{\bar{A}}(x), \nu_{\bar{A}}(y))$ and so $1-\nu_{\bar{A}}(x*y) \leq 1-T(\nu_{\bar{A}}(x), \nu_{\bar{A}}(y))$ which means that $\nu_A(x*y) \leq 1-T(1-\nu_A(x), 1-\nu_A(y))$ which implies that

$$\nu_A(x*y) \leq C(\nu_A(x), \nu_A(y)).$$

Therefore $A = (\mu_A, \nu_A) \in IFSUN(X)$. □

Proposition 3.4. *$A = (\mu_A, \nu_A) \in IFSUN(X)$ if and only if $\sqcup A = (\mu_A, \mu_{\bar{A}}) \in IFSUN(X)$ and $\sqcap A = (\nu_{\bar{A}}, \nu_A) \in IFSUN(X)$.*

Proof. Let $x \in X$. Let $A = (\mu_A, \nu_A) \in IFSUN(X)$ then

$$\mu_A(x * y) \geq T(\mu_A(x), \mu_A(y)). \quad \text{(a)}$$

and

$$\nu_A(x * y) \leq C(\nu_A(x), \nu_A(y)). \quad \text{(d)}$$

Also

$$\begin{aligned}
\mu_A(x * y) &\geq T(\mu_A(x), \mu_A(y)) \\
&\Longleftrightarrow -\mu_A(x * y) \leq -T(\mu_A(x), \mu_A(y)) \\
&\Longleftrightarrow 1 - \mu_A(x * y) \leq 1 - T(\mu_A(x), \mu_A(y)) \\
&\Longleftrightarrow 1 - \mu_A(x * y) \leq 1 - T(1 - \mu_{\bar{A}}(x), 1 - \mu_{\bar{A}}(y)) \\
&\Longleftrightarrow \mu_{\bar{A}}(x * y) \leq C(\mu_{\bar{A}}(x), \mu_{\bar{A}}(y)). \quad \text{(b)}
\end{aligned}$$

Then from (a) and (b) we will have that $\sqcup A = (\mu_A, \mu_{\bar{A}}) \in IFSUN(X)$.
Also

$$\begin{aligned}
\nu_A(x * y) &\leq C(\nu_A(x), \nu_A(y)) \\
&\Longleftrightarrow -\nu_A(x * y) \geq -C(\nu_A(x), \nu_A(y)) \\
&\Longleftrightarrow 1 - \nu_A(x * y) \geq 1 - C(\nu_A(x), \nu_A(y)) \\
&\Longleftrightarrow 1 - \nu_A(x * y) \geq 1 - C(1 - \nu_{\bar{A}}(x), 1 - \nu_{\bar{A}}(y)) \\
&\Longleftrightarrow \nu_{\bar{A}}(x * y) \geq T(\nu_{\bar{A}}(x), \nu_{\bar{A}}(y)). \quad \text{(c)}
\end{aligned}$$

Now from (c) and (d) we get that $\sqcap A = (\nu_{\bar{A}}, \nu_A) \in IFSUN(X)$.
Conversely, let $\sqcup A = (\mu_A, \mu_{\bar{A}}) \in IFSUN(X)$ and $\sqcap A = (\nu_{\bar{A}}, \nu_A) \in IFSUN(X)$. As $\sqcup A = (\mu_A, \mu_{\bar{A}}) \in IFSUN(X)$ so

$$\mu_A(x * y) \geq T(\mu_A(x), \mu_A(y)) \quad \text{(a)}$$

and since $\sqcap A = (\nu_{\bar{A}}, \nu_A) \in IFSUN(X)$ so

$$\nu_A(x * y) \leq C(\nu_A(x), \nu_A(y)). \quad \text{(b)}$$

Then from (a) and (b) we get that $A = (\mu_A, \nu_A) \in IFSUN(X)$. □

Proposition 3.5. *Let φ be a function from SU-algebra of X into SU-algebra of Y and $A = (\mu_A, \nu_A) \in IFSUN(X)$. Then $\varphi(A) = (\varphi(\mu_A), \varphi(\nu_A)) \in IFSUN(Y)$.*

Proof. Let $y_1, y_2 \in Y$. Then

$$\begin{aligned}
\varphi(\mu_A)(y_1 * y_2) &= \sup\{\mu_A(x_1 * x_2) \mid x_1, x_2 \in X, \varphi(x_1) = y_1, \varphi(x_2) = y_2\} \\
&\geq \sup\{T(\mu_A(x_1), \mu_A(x_2)) \mid x_1, x_2 \in X, \varphi(x_1) = y_1, \varphi(x_2) = y_2\}
\end{aligned}$$

$$\geq T(\sup\{\mu_A(x_1) \mid x_1 \in X, \varphi(x_1) = y_1\}, \sup\{\mu_A(x_2) \mid x_2 \in X, \varphi(x_2) = y_2\})$$
$$= T(\varphi(\mu_A)(y_1), \varphi(\mu_A)(y_2))$$

and so

$$\varphi(\mu_A)(y_1 * y_2) \geq T(\varphi(\mu_A)(y_1), \varphi(\mu_A)(y_2)). \quad \text{(a)}$$

Also

$$\begin{aligned}\varphi(\nu_A)(y_1 * y_2) &= \inf\{\nu_A(x_1 * x_2) \mid x_1, x_2 \in X, \varphi(x_1) = y_1, \varphi(x_2) = y_2\}\\ &\leq \inf\{C(\nu_A(x_1), \nu_A(x_2)) \mid x_1, x_2 \in X, \varphi(x_1) = y_1, \varphi(x_2) = y_2\}\\ &\leq C(\inf\{\nu_A(x_1) \mid x_1 \in X, \varphi(x_1) = y_1\}, \inf\{\nu_A(x_2) \mid x_2 \in X, \varphi(x_2) = y_2\})\\ &= C(\varphi(\nu_A)(y_1), \varphi(\nu_A)(y_2))\end{aligned}$$

thus

$$\varphi(\nu_A)(y_1 * y_2) \leq C(\varphi(\nu_A)(y_1), \varphi(\nu_A)(y_2)). \quad \text{(b)}$$

Therefore (a) and (b) give us that $\varphi(A) = (\varphi(\mu_A), \varphi(\nu_A)) \in IFSUN(Y)$. □

Proposition 3.6. *Let φ be a function from SU-algebra of X into SU-algebra of Y and $B = (\mu_B, \nu_B) \in IFSUN(Y)$. Then $\varphi^{-1}(B) = (\varphi^{-1}(\mu_B), \varphi^{-1}(\nu_B)) \in IFSUN(X)$.*

Proof. Let $x_1, x_2 \in X$. Then

$$\varphi^{-1}(\mu_B)(x_1 * x_2) = \mu_B(\varphi)(x_1 * x_2) = \mu_B(\varphi(x_1) * \varphi(x_2))$$

$$\geq T(\mu_B(\varphi(x_1)), \mu_B(\varphi(x_2))) = T(\varphi^{-1}(\mu_B)(x_1), \varphi^{-1}(\mu_B)(x_2))$$

and then

$$\varphi^{-1}(\mu_B)(x_1 * x_2) \geq T(\varphi^{-1}(\mu_B)(x_1), \varphi^{-1}(\mu_B)(x_2)). \quad \text{(a)}$$

Also

$$\varphi^{-1}(\nu_B)(x_1 * x_2) = \nu_B(\varphi)(x_1 * x_2) = \nu_B(\varphi(x_1) * \varphi(x_2))$$

$$\leq C(\nu_B(\varphi(x_1)), \nu_B(\varphi(x_2))) = C(\varphi^{-1}(\nu_B)(x_1), \varphi^{-1}(\nu_B)(x_2))$$

and thus

$$\varphi^{-1}(\nu_B)(x_1 * x_2) \leq C(\varphi^{-1}(\nu_B)(x_1), \varphi^{-1}(\nu_B)(x_2)). \quad \text{(b)}$$

Then (a) and (b) imply that $\varphi^{-1}(B) = (\varphi^{-1}(\mu_B), \varphi^{-1}(\nu_B)) \in IFSUN(X)$. □

Proposition 3.7. *Let $A = (\mu_A, \nu_A) \in IFSUN(X)$ and $B = (\mu_B, \nu_B) \in IFSUN(Y)$. Then $A \times B = (\mu_{A\times B}, \nu_{A\times B}) \in IFSUN(X \times Y)$ for every SU-algebra of X and SU-algebra of Y.*

Proof. Let $(x_1, y_1), (x_2, y_2) \in X \times Y$. Then

$$\begin{aligned}
\mu_{A\times B}((x_1, y_1) * (x_2, y_2)) &= \mu_{A\times B}(x_1 * x_2, y_1 * y_2) \\
&= T(\mu_A(x_1 * x_2), \mu_B(y_1 * y_2)) \\
&\geq T(T(\mu_A(x_1), \mu_A(x_2)), T(\mu_B(y_1), \mu_B(y_2))) \\
&= T(T(\mu_A(x_1), \mu_B(y_1)), T(\mu_A(x_2), \mu_B(y_2))) \\
&= T(\mu_{A\times B}(x_1, y_1), \mu_{A\times B}(x_2, y_2))
\end{aligned}$$

then

$$\mu_{A\times B}((x_1, y_1) * (x_2, y_2)) \geq T(\mu_{A\times B}(x_1, y_1), \mu_{A\times B}(x_2, y_2)). \qquad \text{(a)}$$

Now

$$\begin{aligned}
\nu_{A\times B}((x_1, y_1) * (x_2, y_2)) &= \nu_{A\times B}(x_1 * x_2, y_1 * y_2) \\
&= C(\nu_A(x_1 * x_2), \nu_B(y_1 * y_2)) \\
&\leq C(C(\nu_A(x_1), \nu_A(x_2)), C(\nu_B(y_1), \nu_B(y_2))) \\
&= C(C(\nu_A(x_1), \nu_B(y_1)), C(\nu_A(x_2), \nu_B(y_2))) \\
&= C(\nu_{A\times B}(x_1, y_1), \nu_{A\times B}(x_2, y_2))
\end{aligned}$$

thus

$$\nu_{A\times B}((x_1, y_1) * (x_2, y_2)) \leq C(\nu_{A\times B}(x_1, y_1), \nu_{A\times B}(x_2, y_2)). \qquad \text{(b)}$$

Thus from (a) and (b) we will have that $A \times B = (\mu_{A\times B}, \nu_{A\times B}) \in IFSUN(X \times Y)$. $\square$

Proposition 3.8. *Let $A = (\mu_A, \nu_A) \in IFS(X)$ and $B = (\mu_B, \nu_B) \in IFS(Y)$. If $A \times B = (\mu_{A\times B}, \nu_{A\times B}) \in IFSUN(X \times Y)$, then at least one of the following statements hold:*
(1) $A(0) \supseteq B(y)$ for all $y \in Y$,
(2) $B(0) \supseteq A(x)$ for all $x \in X$.

Proof. Let none of the statements holds, then we can find $x \in X$ and $y \in Y$ such that $A(0) \subset B(y)$ and $B(0) \subset A(x)$. Then $\mu_A(0) < \mu_B(y)$ and $\nu_A(0) > \nu_B(y)$ and $\mu_A(x) > \mu_B(0)$ and $\nu_A(x) < \nu_B(0)$. Now

$$\mu_{A\times B}(x, y) = T(\mu_A(x), \mu_B(y)) > T(\mu_B(0), \mu_A(0)) = T(\mu_A(0), \mu_B(0)) = \mu_{A\times B}(0, 0)$$

and

$$\nu_{A\times B}(x, y) = C(\nu_A(x), \nu_B(y)) < C(\nu_B(0), \nu_A(0)) = C(\nu_A(0), \nu_B(0)) = \nu_{A\times B}(0, 0)$$

thus

$$(A \times B)(0, 0) = (\mu_{A\times B}(0, 0), \nu_{A\times B}(0, 0)) \subset (\mu_{A\times B}(x, y), \nu_{A\times B}(x, y)) = (A \times B)(x, y)$$

so $(A \times B)(0, 0) \subset (A \times B)(x, y)$ and this is contradiction with Proposition 3.1 and then at least one of the statements hold. $\square$

Proposition 3.9. *Let $A = (\mu_A, \nu_A) \in IFS(X)$ and $B = (\mu_B, \nu_B) \in IFS(Y)$. If $A \times B = (\mu_{A\times B}, \nu_{A\times B}) \in IFSUN(X \times Y)$ and $A(x) \subseteq B(0)$ for all $x \in X$. Then $A = (\mu_A, \nu_A) \in IFSUN(X)$.*

Proof. As $A(x) \subseteq B(0)$ so $\mu_A(x) \leq \mu_B(0)$ and $\mu_A(y) \leq \mu_B(0)$ and $\mu_A(x*y) \leq \mu_B(0) = \mu_B(0*0)$ for all $x, y \in X$. Then

$$\begin{aligned}\mu_A(x * y) &= T(\mu_A(x * y), \mu_B(0 * 0)) \\ &= \mu_{A\times B}((x, 0) * (y, 0)) \\ &\geq T(\mu_{A\times B}(x, 0), \mu_{A\times B}(y, 0)) \\ &= T(T(\mu_A(x), \mu_B(0)), T(\mu_A(y), \mu_B(0))) \\ &= T(\mu_A(x), \mu_A(y))\end{aligned}$$

and thus

$$\mu_A(x * y) \geq T(\mu_A(x), \mu_A(y)). \qquad \text{(a)}$$

Also since $A(x) \subseteq B(0)$ so $\nu_A(x) \geq \nu_B(0)$ and $\nu_A(y) \geq \nu_B(0)$ and $\nu_A(x*y) \geq \nu_B(0) = \nu_B(0*0)$ for all $x, y \in Y$. Thus

$$\begin{aligned}\nu_A(x * y) &= C(\nu_A(x * y), \nu_B(0 * 0)) \\ &= \nu_{A\times B}((x, 0) * (y, 0)) \\ &\leq C(\nu_{A\times B}(x, 0), \nu_{A\times B}(y, 0)) \\ &= C(C(\nu_A(x), \nu_B(0)), C(\nu_A(y), \nu_B(0))) \\ &= C(\nu_A(x), \nu_A(y))\end{aligned}$$

therefore

$$\nu_A(x * y) \leq C(\nu_A(x), \nu_A(y)). \qquad \text{(b)}$$

Now (a) and (b) give us that $A = (\mu_A, \nu_A) \in IFSUN(X)$. □

Proposition 3.10. *Let $A = (\mu_A, \nu_A) \in IFS(X)$ and $B = (\mu_B, \nu_B) \in IFS(Y)$. If $A \times B = (\mu_{A\times B}, \nu_{A\times B}) \in IFSUN(X \times Y)$ and $B(x) \subseteq A(0)$ for all $x \in Y$. Then $B = (\mu_B, \nu_B) \in IFSUN(Y)$.*

Proof. The proof is similar to Proposition 3.9. □

Proposition 3.11. *Let $A = (\mu_A, \nu_A) \in IFS(X)$ and $B = (\mu_B, \nu_B) \in IFS(Y)$. If $A \times B = (\mu_{A\times B}, \nu_{A\times B}) \in IFSUN(X \times Y)$, then either $A = (\mu_A, \nu_A) \in IFSUN(X)$ or $B = (\mu_B, \nu_B) \in IFSUN(Y)$.*

Proof. From Proposition 3.8 we have $A(0) \supseteq B(y)$ for all $y \in Y$ or $B(0) \supseteq A(x)$ for all $x \in X$. Now using Propositions 3.9 and 3.10 we get that either $A = (\mu_A, \nu_A) \in IFSUN(X)$ or $B = (\mu_B, \nu_B) \in IFSUN(Y)$. □

Proposition 3.12. *Let* $A = (\mu_A, \nu_A) \in IFSUN(X)$ *and* $B = (\mu_B, \nu_B) \in IFSUN(Y)$. *Then* $A \times B = (\mu_{A\times B}, \nu_{A\times B}) \in IFSUN(X \times Y)$ *if and only if* $\mu_{A\times B} \in FSUT(X \times Y)$ *and* $\nu_{A\bar{\times}B} \in FSUT(X \times Y)$.

Proof. The proof is similar to Proposition 3.3. $\square$

Proposition 3.13. *Let* $A = (\mu_A, \nu_A) \in IFS(X)$ *and* $B = (\mu_B, \nu_B) \in IFS(Y)$. *Then* $A = (\mu_A, \nu_A) \in IFSUN(X)$ *and* $B = (\mu_B, \nu_B) \in IFSUN(Y)$ *if and only if* $\sqcup(A \times B) = (\mu_{A\times B}, \mu_{A\bar{\times}B}) \in IFSUN(X \times Y)$ *and* $\sqcap(A \times B) = (\nu_{A\bar{\times}B}, \nu_{A\times B}) \in IFSUN(X \times Y)$.

Proof. The proof is similar to Proposition 3.4. $\square$

Acknowledgements

We would like to thank the reviewers for carefully reading the manuscript and making several helpful comments to increase the quality of the paper.

References

[1] M. T. Abu Osman, *on some products of fuzzy subgroups*, Fuzzy Sets and Systems, **24** (1987), 79-86.

[2] K. T. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems, **20**(1986), 87-96.

[3] J. J. Buckley and E. Eslami, *An introduction to fuzzy logic and fuzzy sets*, Springer-Verlag Berlin Heidelberg GmbH, (2002).

[4] Q. P. Hu and X. Li, *On BCH-algebras*, Math. Seminor Notes, **11**(1980), 313-320.

[5] K. Iseki and Y. Imai, *On BCI-algebras*, Math. Seminor Notes., **80**(1980), 125-130.

[6] K. Iseki and S. Tanaka, *An Introduction to theory of BCK-algebras*, Math. Japo., **23**(1987), 1-26.

[7] S. Keawrahun and U. Leerawat, *On a classification of a structure algebra: SU-Algebra*, Scientia Magna, **7** (2011), 69-76.

[8] P. Muralikrishna and M. Chandramouleeswaran, *On Bifuzzy SU-Subalgebras*, Walailak J Sci Tech, **10(4)** (2013), 415-422.

[9] J. Neggers, S. S. Ahn and H.S. Kim, *On Q-algebras*, Int. J. Math. and Math. Sci., **27**(2001), 749-757.

[10] J. Neggers and H. S. Kim, *On d-algebras*, Math.Solvaca, **49**(1999), 19-26.

[11] J. Neggers and H. S. Kim, *On B-algebras*, Math. Vesnik., **54**(2002), 21-29.

[12] R. Rasuli, *Fuzzy Ideals of Subtraction Semigroups with Respect to A t-norm and A t-conorm*, The Journal of Fuzzy Mathematics Los Angeles, **24** (4) (2016), 881-892.

[13] R. Rasuli, *Fuzzy modules over a t-norm*, Int. J. Open Problems Compt. Math., **9** (3) (2016), 12-18.

[14] R. Rasuli, *Fuzzy Subrings over a t-norm* , The Journal of Fuzzy Mathematics Los Angeles, **24** (4) (2016), 995-1000.

[15] R. Rasuli, *Norms over intuitionistic fuzzy subrings and ideals of a ring*, Notes on Intuitionistic Fuzzy Sets, **22** (5) (2016), 72-83.

[16] R. Rasuli, *Norms over fuzzy Lie algebra*, Journal of New Theory, **15**(2017), 32-38.

[17] R. Rasuli, *Fuzzy subgroups on direct product of groups over a t-norm*, Journal of Fuzzy Set Valued Analysis, **3**(2017), 96-101.

[18] R. Rasuli, *Characterizations of intuitionistic fuzzy subsemirings of semirings and their homomorphisms by norms*, Journal of New Theory, **18**(2017), 39-52.

[19] R. Rasuli, *Intuitionistic fuzzy subrings and ideals of a ring under norms*, LAP LAMBERT Academic publishing, 2017, ISBN: 978-620-2-06926-7.

[20] R. Rasuli, *Characterization of Q-Fuzzy subrings (Anti Q-Fuzzy Subrings) with respect to a T-norm (T-Conorms)*, Journal of Information and Optimization Science, **31**(2018), 1-11.

[21] R. Rasuli, *T-Fuzzy Submodules of $R \times M$*, Journal of New Theory, **22**(2018), 92-102.

[22] R. Rasuli, *Fuzzy subgroups over a T-norm*, Journal of Information and Optimization Science, **39**(2018), 1757-1765.

[23] R. Rasuli, *Fuzzy Sub-vector Spaces and Sub-bivector Spaces under t-Norms*, General Letters in Mathematics, **5**(2018), 47-57.

[24] R. Rasuli, *Anti Fuzzy Submodules over A t -conorm and Some of Their Properties*, The Journal of Fuzzy Mathematics Los Angles, **27**(2019), 229-236.

[25] R. Rasuli, *Artinian and Noetherian Fuzzy Rings*, Int. J. Open Problems Compt. Math., **12**(2019), 1-7.

[26] R. Rasuli and H. Narghi, *T-Norms Over Q-Fuzzy Subgroups of Group*, Jordan Journal of Mathematics and Statistics (JJMS), **12**(2019), 1-13.

[27] R. Rasuli, *Fuzzy equivalence relation, fuzzy congrunce relation and fuzzy normal subgroups on group G over t-norms*, Asian Journal of Fuzzy and Applied Mathematics, **7**(2019), 14-28.

[28] R. Rasuli, *Norms over anti fuzzy G-submodules*, MathLAB Journal, **2**(2019), 56-64.

[29] R. Rasuli, *Norms over bifuzzy bi-ideals with operators in semigroups*, Notes on Intuitionistic Fuzzy Sets , **25**(2019), 1-11.

[30] R. Rasuli, *Norms Over Basic Operations on Intuitionistic Fuzzy Sets*, The Journal of Fuzzy Mathematics Los Angles, **27**(3)(2019), 561-582.

[31] R. Rasuli, *T-fuzzy Bi-ideals in Semirings*, Earthline Journal of Mathematical Sciences, **27**(1)(2019), 241-263.

[32] R. Rasuli, *Norms Over Intuitionistic Fuzzy Vector Spaces*, Algebra Letters, **1**(1)(2019), 1-19.

[33] R. Rasuli, *Some Results of Anti Fuzzy Subrings Over tConorms*, MathLAB Journal, **1**(4)(2019), 25-32.

[34] R. Rasuli, *Anti Fuzzy Equivalence Relation on Rings with respect to t-conorm C*, Earthline Journal of Mathematical Sciences, **3**(1)(2020), 1-19.

[35] R. Rasuli, *Anti Fuzzy Subbigroups of Bigroups under t -conorms*, The Journal of Fuzzy Mathematics Los Angles, **28**(1)(2020), 181-200.

[36] R. Rasuli, *t-norms over Fuzzy Multigroups*, Earthline Journal of Mathematical Sciences, **3**(2)(2020), 207-228.

[37] R. Rasuli, *Anti Q-fuzzy subgroups under t-conorms*, Earthline Journal of Mathematical Sciences, **4**(1)(2020), 13-28.

[38] R. Rasuli, *Anti Fuzzy Congruence on Product Lattices with respect to S-norms*, The Second National Congress on Mathematics and Statistics Conbad Kavous University, 2020.

[39] R. Rasuli, *Direct product of fuzzy multigroups under t-norms*, Open Journal of Discrete Applied Mathematics (ODAM), **3**(1)(2020), 75-85.

[40] R. Rasuli, *Level subsets and translations of QF ST(G)*, MathLAB Journal, **5**(1)(2020), 1-11.

[41] R. Rasuli, *Conorms over anti fuzzy vector spaces*,Open Journal of Mathematical Sciences, **4**(2020), 158-167.

[42] R. Rasuli, *Intuitionistic fuzzy subgroups with respect to norms (T, S)*, Eng. Appl. Sci. Lett. (EASL), **3(2)**(2020), 40-53.

[43] Rasul Rasuli, Mohammad Mehde Moatamedi nezhad and Hossien Naraghi, *Characterization of TF(G) and direct product of it*,The First National Conference on Soft Computing and Cognitive Science Conbad Kavous University, July 2020.

[44] R. Rasuli, *Anti complex fuzzy subgroups under s-norms*, Eng. Appl. Sci. Lett. (EASL), **3(4)**(2020), 1-10.

[45] Rasul Rasuli and Mohammad Mehde Moatamedi nezhad, *Characterization of Fuzzy modules and anti fuzzy modules under norms*, The First International Conference on Basic Sciences, Tehran, October 21, 2020.

[46] Rasul Rasuli and Mohammad Mehde Moatamedi nezhad, *Fuzzy subrings and anti fuzzy subrings under norms*, The First International Conference on Basic Sciences, Tehran, October 21, 2020.

[47] A. Tamilarasi and K. Megalai, *TM-algebra an introduction*, IJCA special issue on "CASCT", (2010), 17-23.

[48] L. A. Zadeh, *Fuzzy sets*, Inform. and Control, **8**(1965), 338-353.

Scientia Magna
Vol. 16 (2021), No. 1, 97-102

The mean value of function $\left(q_k^{(e)}(n)\right)^r$ over cube-full number

Ruiyang Yue

School of Mathematics and Statistics, Shandong Normal University
Shandong Jinan, China
E-mail: yueruiyangsdnu@163.com

Abstract Let $n > 1$ be an integer, $\left(q_k^{(e)}(n)\right)^r$ is a multiplicative function. In this paper, we shall study the mean value of exponential divisor function $\left(q_k^{(e)}(n)\right)^r$ over cube-full number, that is

$$\sum_{\substack{n\le x\\ n \text{ is cube-full}}} \left(q_k^{(\mathrm{e})}(n)\right)^r = \sum_{n\le x} \left(q_k^{(\mathrm{e})}(n)\ f_3(\mathrm{n})\right)^r,$$

where $f_3(n)$ is the characteristic function of cube-full integers, i.e.

$$f_3(n) = \begin{cases} 1, & n \text{ is cube-full;} \\ 0, & \text{otherwise.} \end{cases}$$

Keywords Dirichlet convolution, mean value, cube-full number, divisor function, residue theorem.
2010 Mathematics Subject Classification 44A35, 11H60, 11N37.

§1. Introduction

Let $n > 1$ be an integer of canonical form $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$. The integer n is called a k-full number if $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$, where $a_1 \ge k, a_2 \ge k, \cdots, a_r \ge k$. Let $f_k(n)$ be the characteristic function of k-full integers, i.e.

$$f_k(n) = \begin{cases} 1, & n \text{ is } k \text{ -full;} \\ 0, & \text{otherwise.} \end{cases}$$

An integer $n = \prod_{i=1}^r p_i^{a_i}$ is called exponentially k -free if all the exponents $a_i (1 \le i \le r)$ are k-free, i.e. n is not divisible by the k-th power of any prime $(k \ge 2)$. So let $q_k^{(e)}(n)$ denote the characteristic function of exponentially k-free integers.

Obviously the function $q_k^{(e)}(n)$ is multiplicative, and for every prime power p^a there are $q_k^{(e)}(p) = q_k^{(e)}\left(p^2\right) = q_k^{(e)}\left(p^3\right) = \ldots = q_k^{(e)}\left(p^{2^k-1}\right) = 1, q_k^{(e)}\left(p^{2^k}\right) = 0.$

The properties of the exponential divisor function $q_k^{(e)}(n)$ have been studied by many authors. L. Tóth [5] proved the following result:

$$\sum_{n\le x} q_k^{(e)}(n) = D_k x + O\left(x^{1/2^k}\delta(x)\right),$$

where

$$D_k = \prod_p \left(1 + \sum_{a=2^k}^{\infty} \frac{q_k(a) - q_k(a-1)}{p^a}\right),$$

$q_k(n)$ denoting the characteristic function of k-free integers.

In this paper, we will study the mean value of function $\left(q_k^{(e)}(n)\right)^r$ over cube-full number.

Theorem 1.1. For some $D > 0$,

$$\sum_{\substack{n\le x\\ n \text{ is cube-full}}} \left(q_k^{(e)}(n)\right)^r = \frac{\zeta\left(\frac{4}{3}\right)\zeta\left(\frac{5}{3}\right)G\left(\frac{1}{3}\right)}{\zeta\left(\frac{8}{3}\right)} x^{\frac{1}{3}} + \frac{\zeta\left(\frac{3}{4}\right)\zeta\left(\frac{5}{4}\right)G\left(\frac{1}{4}\right)}{\zeta(2)} x^{\frac{1}{4}} + \frac{\zeta\left(\frac{3}{5}\right)\zeta\left(\frac{4}{5}\right)G\left(\frac{1}{5}\right)}{\zeta\left(\frac{8}{5}\right)} x^{\frac{1}{5}}$$
$$+ O\left(x^{\frac{1}{8}} \exp\left(-D(\log x)^{\frac{3}{5}}(\log\log x)^{-\frac{1}{5}}\right)\right),$$

where

$$G(s) = \prod_p \left(1 - \frac{1}{p^{9s}} + \cdots\right),$$

which is absolutely convergent for $\Re s > \frac{1}{9}$.

Notation Throughout this paper, ϵ always denotes a fixed but sufficiently small positive constant.

§2. Some lemmas

In order to prove our theorem, we need the following lemmas.

Lemma 2.1.

$$\Delta(3,4,5;x) \ll x^{\frac{22}{177}} \log^3 x.$$

Proof. The proof of this bound depends on the theory of two-dimensional exponent pairs. □

Lemma 2.2. Suppose $f(n)$ is arithmetical function, and satisfy

$$\sum_{n\le x} f(n) = \sum_{j=1}^{l} x^{a_j} P_j(\log x) + O\left(x^a\right),$$

$$\sum_{n\le x} |f(n)| = O\left(x^{a_1}\log^r x\right), \tag{1}$$

here $a_1 \ge a_2 \ge \cdots \ge a_l > 1/c > a \ge 0, r \ge 0, P_1(t), \cdots, P_l(t)$ are polynomials on t whose degree are not exceed r, and $c \ge 1, b \ge 1$ are fixed integers.

Suppose for $\Re s > 1$,we have

$$\sum_{n=1}^{\infty} \frac{\mu_b(n)}{n^s} = \frac{1}{\zeta^b(s)}, \tag{2}$$

if

$$h(n) = \sum_{d^c|n} \mu_b(d) f\left(n/d^c\right), \tag{3}$$

then

$$\sum_{n\le x} h(n) = \sum_{j=1}^{l} x^{a_j} R_j(\log x) + E_c(x), \tag{4}$$

here $R_1(t), \cdots, R_l(t)$ are polynomials on t whose degree are not exceed r, and for some $D > 0$,

$$E_c(x) \ll x^{1/c} \exp\left(\left(-D(\log x)^{3/5}(\log\log x)^{-1/5}\right)\right). \tag{5}$$

Proof. If $b = 1$, see theorem 14.2 in A. Ivić [1]. If $b \ge 2$, it can be proved in the same method. □

Lemma 2.3. Let $s = \sigma + it$ is a complex number, then we have

$$\sum_{\substack{n=1\\ n \text{ is cube-full}}} \frac{\left(q_k^{(e)}(n)\right)^r}{n^s} = \frac{\zeta(3s)\zeta(4s)\zeta(5s)}{\zeta(8s)} G(s), \tag{6}$$

where the Dirichlet series $G(s) = \prod_p \left(1 - \frac{1}{p^{9s}} + \cdots\right)$ is absolutely convergent for $\Re s > \frac{1}{9}$.

Proof. Let

$$F(s) := \sum_{\substack{n=1\\ n \text{ is } cube- \text{ full}}} \frac{\left(q_k^{(e)}(n)\right)^r}{n^s} = \sum_{n=1}^{\infty} \frac{\left(q_k^{(e)}(n) f_3(n)\right)^r}{n^s}, (\Re s > 1)$$

here

$$f_3(n) = \begin{cases} 1, & n \text{ is cube-full };\\ 0, & \text{otherwise }. \end{cases}$$

From $\left(q_k^{(e)}(n)\right)^r$ is multiplicative we have for $\Re s > 1$, we have

$$\begin{aligned}
F(s) = & \sum_{\substack{n=1 \\ n \text{ is } cube- \text{ full}}}^{\infty} \frac{\left(q_k^{(e)}(n)\right)^r}{n^s} = \sum_{n=1}^{\infty} \frac{\left(q_k^{(e)}(n) f_3(n)\right)^r}{n^s} \\
= & \prod_p \left(1 + \frac{\left(q_k^{(e)}(p) f_3(p)\right)^r}{p^s} + \frac{\left(q_k^{(e)}\left(p^2\right) f_3\left(p^2\right)\right)^r}{p^{2s}}\right. \\
& \left. + \frac{\left(q_k^{(e)}\left(p^3\right) f_3\left(p^3\right)\right)^r}{p^{3s}} + \cdots + \frac{\left(q_k^{(e)}\left(p^{2^k-1}\right) f_3\left(p^{2^k-1}\right)\right)^r}{p^{(2^k-1)s}}\right) \\
= & \prod_p \left(1 + \frac{1}{p^{3s}} + \frac{1}{p^{4s}} + \frac{1}{p^{5s}} + \cdots + \frac{1}{p^{(2^k-1)s}}\right) \\
= & \zeta(3s) \prod_p \left(1 + \frac{1}{p^{4s}} + \frac{1}{p^{5s}} - \frac{1}{p^{2^k} s} - \frac{1}{p^{(2^k+1)s}} - \frac{1}{p^{(2^k+2)s}}\right) \\
= & \zeta(3s)\zeta(4s) \prod_p \left(1 + \frac{1}{p^{5s}} - \frac{1}{p^{8s}} - \frac{1}{p^{9s}} - \frac{1}{p^{k^k}} - \frac{1}{p^{(2^k+1)s}} + \cdots\right) \\
= & \zeta(3s)\zeta(4s)\zeta(5s) \prod_p \left(1 - \frac{1}{p^{8s}} - \frac{1}{p^{9s}} + \cdots\right) \\
= & \frac{\zeta(3s)\zeta(4s)\zeta(5s)}{\zeta(8s)} \prod_p \left(1 - \frac{1}{p^{9s}} + \cdots\right) \\
= & \frac{\zeta(3s)\zeta(4s)\zeta(5s)}{\zeta(8s)} G(s),
\end{aligned}$$

where $G(s) = \prod_p \left(1 - \frac{1}{p^{9s}} + \cdots\right)$, and it is absolutely convergent for $\sigma > \frac{1}{9} + \epsilon$. □

§3. Proof of Theorem 1.1

Now we prove Theorem 1.1.

Proof. From Lemma 2.3, we have known that

$$F(s) := \sum_{\substack{n=1 \\ n \text{ is } cube- \text{ full}}}^{\infty} \frac{\left(q_k^{(e)}(n)\right)^r}{n^s} = \frac{\zeta(3s)\zeta(4s)\zeta(5s)}{\zeta(8s)} G(s),$$

where $G(s)$ is absolutely convergent for $\sigma > \frac{1}{9} + \epsilon$.

Define

$$G(s) := \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$$

and

$$H(s) := \zeta(3s)\zeta(4s)\zeta(5s)G(s) := \sum_{n=1}^{\infty} \frac{h(n)}{n^s} = \sum_{n=1}^{\infty} \frac{\sum_{n=ml} d(3,4,5;m) g(l)}{n^s}, (\Re s > 1)$$

where
$$h(n) = \sum_{n=ml} d(3,4,5;m)g(l).$$
Then we can get
$$\sum_{n\le x} h(n) = \sum_{ml\le x} d(3,4,5;m)g(l) = \sum_{l\le x} g(l) \sum_{m\le \frac{x}{l}} d(3,4,5;m).$$
From Perron's formula, Residue theorem and Lemma 2.1, we can get
$$\begin{aligned}\sum_{n\le x} d(3,4,5;n) &= \zeta\left(\frac{4}{3}\right)\zeta\left(\frac{5}{3}\right)x^{\frac{1}{3}} + \zeta\left(\frac{3}{4}\right)\zeta\left(\frac{5}{4}\right)x^{\frac{1}{4}} + \zeta\left(\frac{3}{5}\right)\zeta\left(\frac{4}{5}\right)x^{\frac{1}{5}} + \Delta(3,4,5;x)\\ &= \zeta\left(\frac{4}{3}\right)\zeta\left(\frac{5}{3}\right)x^{\frac{1}{3}} + \zeta\left(\frac{3}{4}\right)\zeta\left(\frac{5}{4}\right)x^{\frac{1}{4}} + \zeta\left(\frac{3}{5}\right)\zeta\left(\frac{4}{5}\right)x^{\frac{1}{5}} + O\left(x^{\frac{22}{177}}\right).\end{aligned} \tag{7}$$
Then from (7) and Abel integral formula, we have the relation:
$$\begin{aligned}\sum_{n\le x} h(n) =& \sum_{l\le x} g(l)\left[\zeta\left(\frac{4}{3}\right)\zeta\left(\frac{5}{3}\right)\left(\frac{x}{l}\right)^{\frac{1}{3}} + \zeta\left(\frac{3}{4}\right)\zeta\left(\frac{5}{4}\right)\left(\frac{x}{l}\right)^{\frac{1}{4}}\right.\\ &\left. +\zeta\left(\frac{3}{5}\right)\zeta\left(\frac{4}{5}\right)\left(\frac{x}{l}\right)^{\frac{1}{5}} + O\left(\left(\frac{x}{l}\right)^{\frac{22}{17}}\right)\right]\\ =&\zeta\left(\frac{4}{3}\right)\zeta\left(\frac{5}{3}\right)x^{\frac{1}{3}}\sum_{l\le x}\frac{g(l)}{l^{1/3}} + \zeta\left(\frac{3}{4}\right)\zeta\left(\frac{5}{4}\right)x^{\frac{1}{4}}\sum_{l\le x}\frac{g(l)}{l^{1/4}}\\ &+ \zeta\left(\frac{3}{5}\right)\zeta\left(\frac{4}{5}\right)x^{\frac{1}{5}}\sum_{l\le x}\frac{g(l)}{l^{1/5}} + O\left(x^{\frac{22}{17}}\sum_{l\le x}\frac{|g(l)|}{l^{2/177}}\right)\\ =&\zeta\left(\frac{4}{3}\right)\zeta\left(\frac{5}{3}\right)x^{\frac{1}{3}}\sum_{l=1}^{\infty}\frac{g(l)}{l^{1/3}} + \zeta\left(\frac{3}{4}\right)\zeta\left(\frac{5}{4}\right)x^{\frac{1}{4}}\sum_{l=1}^{\infty}\frac{g(l)}{l^{1/4}} + \zeta\left(\frac{3}{5}\right)\zeta\left(\frac{4}{5}\right)x^{\frac{1}{5}}\sum_{l=1}^{\infty}\frac{g(l)}{l^{1/5}}\\ &+ O\left(x^{\frac{1}{3}}\sum_{l>x}\frac{|g(l)|}{l^{1/3}}\right) + O\left(x^{\frac{1}{4}}\sum_{l>x}\frac{|g(l)|}{l^{1/4}}\right)\\ &+ O\left(x^{\frac{1}{5}}\sum_{l>x}\frac{|g(l)|}{l^{1/5}}\right) + O\left(x^{\frac{22}{177}}\sum_{k<x}\frac{|g(l)|}{l^{22/177}}\right)\\ =&\zeta\left(\frac{4}{3}\right)\zeta\left(\frac{5}{3}\right)G\left(\frac{1}{3}\right)x^{\frac{1}{3}} + \zeta\left(\frac{3}{4}\right)\zeta\left(\frac{5}{4}\right)G\left(\frac{1}{4}\right)x^{\frac{1}{4}}\\ &+ \zeta\left(\frac{3}{5}\right)\zeta\left(\frac{4}{5}\right)G\left(\frac{1}{5}\right)x^{\frac{1}{5}} + O\left(x^{\frac{1}{9}}\right).\end{aligned}$$
From Perron's formula and Lemma 2.1, we can get:
$$\begin{aligned}\sum_{\substack{n\le x\\ n \text{ is cube-full}}} \left(q_k^{(e)}\right)^r(n) =& \frac{\zeta\left(\frac{4}{3}\right)\zeta\left(\frac{5}{3}\right)G\left(\frac{1}{3}\right)}{\zeta\left(\frac{8}{3}\right)}x^{\frac{1}{3}} + \frac{\zeta\left(\frac{3}{4}\right)\zeta\left(\frac{5}{4}\right)G\left(\frac{1}{4}\right)}{\zeta(2)}x^{\frac{1}{4}} + \frac{\zeta\left(\frac{3}{5}\right)\zeta\left(\frac{4}{5}\right)G\left(\frac{1}{5}\right)}{\zeta\left(\frac{8}{5}\right)}x^{\frac{1}{5}}\\ &+ O\left(x^{\frac{1}{8}}\exp\left(-D(\log x)^{\frac{3}{5}}(\log\log x)^{-\frac{1}{5}}\right)\right),\end{aligned}$$
where $D > 0$ and $G(s) = \prod_p\left(1 - \frac{1}{p^{9s}} + \cdots\right)$ is absolutely convergent for $\Re s > \frac{1}{9}$. □

Now we have completed the proof of Theorem 1.1.

Acknowledgements

The authors express their gratitude to the referee for a careful reading of the manuscript and many valuable suggestions which highly improve the quality of this paper.

References

[1] A. Ivić. The Riemann Zeta-function. John Wiley and Sons, 1985.

[2] A. Ivić. The Riemann zeta-function: theory and applications. Oversea Publishing House, 2003.

[3] M. V. Subbarao. On some airthmetic convolutions. in "The Theory of Arithmetic Functions", Lecture Notes in Mathematic, Springer, 251 (1972), 247 – 271.

[4] E. C. Titchmarsh. The theory of the Riemann Zeta-function (revised by D. R. Heath-Brown), Oxford university Press, 1986.

[5] L. Tóth. On certain arithmetical functions involving exponential divisors. Ann. Univ. Sci. Budapest. Sect. Comput 24 (2004), 285 – 294.

[6] L. Tóth. On certain arithmetical functions involving exponential divisors. Ann. Univ. Sci. Budapest. Sect. Comput 27 (2007), 155 – 166.

[7] J. Wu. Problème de diviseurs exponentiels at entiers exponentiellement sans facteur carré. Théor Nombres Bordeaux 7 (1995), 133 – 141.

Scientia Magna
Vol. 16 (2021), No. 1, 103-109

On some Somos's η-function identities of level 27 and their partition interpretations

G. Sharath

Department of Mathematics, Vidya Vikas First Grade College
Mysuru, INDIA
E-mail: sharathgns@gmail.com

Abstract Micheal Somos in his recordings has discovered round 6200+ η-function identities of different levels using computer and has offered no proofs for them. These identities are akin to the ones established by Srinivasa Ramanujan. Motivated by these, in this paper, we give proofs of nine Somos's η-function identities of level 27 and establish certain partition theoretic interpretations of these.

Keywords Dedekind η-functions, Colored partitions.
2010 Mathematics Subject Classification 33C05, 11F20.

§1. Introduction and preliminaries

Let τ be a complex number satisfying $\mathrm{Im}(\tau) > 0$ and let $q = e^{2\pi i\tau}$. The Dedekind η-function is defined by

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n=0}^{\infty}(1-q^n), \qquad q \in \mathbb{C}, \qquad |q| < 1.$$

We denote

$$f(-q) := q^{\frac{-1}{24}}\eta(\tau) \qquad \text{and} \qquad f_n := f(-q^n).$$

In his Notebooks [12], Ramanujan recorded without proofs 25 beautiful η-function identities, proofs of all these identities can be seen in [4]. These η-function identities play a very important role in evaluations of class invariants, continued fractions and ratios of theta functions. See for example [1, 5, 6, 10, 11, 13, 20–22, 24].

A η-function identity which relates f_1, f_{n_1}, f_{n_2} and f_n (where $n = n_1 n_2$) is called a η-function identity of level n. M. Somos [14] in his recordings has obtained around 6200+ η-function identities of different levels using computer and has offered no proofs for them. Recently many mathematicians, for example B. Yuttanam [25], K. R. Vasuki and R. G. Veeresha [23], B. R. Srivatsa Kumar et. al. [15–19], E. N. Bhuvan [7] have obtained proofs for levels 6, 8, 10, 12, 14, 16 and 21. Motivated by these, in Section 2 of this paper, we derive proofs of nine η-function identities of level 27 recorded by Somos. Further in Section 3 of this paper, we establish combinatorial interpretations for these results.

§2. Somos's identities of Level 27

In this Section, we prove nine η-function identities of Somos's of level 27. First, we state a Lemma employing which proof of the said identities shall be established.

Lemma 2.1. *[2] If x and y are defined by*

$$x := \frac{f_1}{q^{\frac{1}{3}} f_9} \qquad \textit{and} \qquad y := \frac{f_3}{q f_{27}},$$

then

$$(xy)^3 + \frac{9^3}{(xy)^3} + 27\left[\left(\frac{x}{y}\right)^3 + \left(\frac{y}{x}\right)^3\right] + 9(x^3+y^3) + 243\left(\frac{1}{x^3}+\frac{1}{y^3}\right) + 81 = \left(\frac{x}{y}\right)^6. \tag{1}$$

Theorem 2.1. *[14] If x and y are as defined in Lemma 2.1, then we have*

$$\frac{1}{q}\left(\frac{f_3}{f_9}\right)^4 = \frac{y^3(3+x^3)}{y^3+9}. \tag{2}$$

Proof. From [12, p. 366] [3, Entry 1(iv), p. 345] we have

$$3 + \frac{f_1^3}{q f_9^3} = \left(27 + \frac{f_3^{12}}{q^3 f_9^{12}}\right)^{\frac{1}{3}}.$$

Multiplying the above identity throughout by $\frac{f_9}{f_3}$, we obtain

$$\begin{aligned} \frac{f_9}{f_3}\left(3 + \frac{f_1^3}{q f_9^3}\right) &= \left(27\frac{f_9^3}{f_3^3} + \frac{f_3^9}{q^3 f_9^9}\right)^{\frac{1}{3}} \\ &= \frac{f_3^3}{q f_9^3}\left(27 q^3 \frac{f_9^{12}}{f_3^{12}} + 1\right)^{\frac{1}{3}} \\ &= \frac{f_3^3}{q f_9^3}\left(1 + 9q^3\frac{f_{27}^3}{f_3^3}\right), \end{aligned} \tag{3}$$

where in the righthand side we have employed the identity [3, Entry 1(iv), p. 346]

$$1 + 9q\frac{f_9^3}{f_1^3} = \left(1 + 27q\frac{f_3^{12}}{f_1^{12}}\right)^{\frac{1}{3}},$$

after changing q to q^3. Now, employing the definition of x and y as in Lemma 2.1 in (3), we immediately deduce the required result. □

Remark. We note that Theorem 2.1 is equivalent to :

$$f_1^3 f_9 + 3q f_9^4 - f_3^4 - 9q^3 f_3 f_{27}^3 = 0.$$

Theorem 2.2. *[14] We have*

$$f_1^9 f_9^3 + 81q^4 f_1^3 f_3 f_9^5 f_{27}^3 + 9q f_1^3 f_3^4 f_9^5 - f_3^{12} = 0.$$

Proof. Dividing (1) throughout by $y^9(3+x^3)^3$ and after rearranging the terms, we obtain

$$x^9\left(\frac{y^3+9}{y^3(3+x^3)}\right)^3 + 81\frac{x^3}{y^3}\left(\frac{y^3+9}{y^3(3+x^3)}\right)^2 + 9x^3\left(\frac{y^3+9}{y^3(3+x^3)}\right)^2 - 1 = 0,$$

where x and y are as defined in Lemma 2.1. Now using (2) in the above identity, we arrive at

$$x^9 q^3\left(\frac{f_9}{f_3}\right)^{12} + 81q^2\frac{x^3}{y^3}\left(\frac{f_9}{f_3}\right)^8 + 9q^2x^3\left(\frac{f_9}{f_3}\right)^8 - 1 = 0.$$

Next, employing the definitions of x and y in the above and then simplifying we find that

$$\frac{f_1^9 f_9^3}{f_3^{12}} + 81q^4\frac{f_1^3 f_3 f_9^5 f_{27}^3}{f_3^{12}} + 9q\frac{f_1^3 f_3^4 f_9^5}{f_3^{12}} - 1 = 0.$$

Multiplying the above identity throughout by f_3^{12}, we arrive at the required result. □

Theorem 2.3. *[14] We have*

$$f_1^3 f_3^5 f_9 f_{27}^3 + 27q^6 f_3^3 f_{27}^9 + 3q f_3^5 f_9^4 f_{27}^3 - f_9^{12} = 0.$$

Proof. Dividing (1) throughout by $(y^3+9)^3$ and after rearranging the terms, we deduce that

$$\frac{x^3}{y^3}\left(\frac{y^3(3+x^3)}{y^3+9}\right)^2 + \frac{27}{y^9}\left(\frac{y^3(3+x^3)}{y^3+9}\right)^3 + \frac{3}{y^3}\left(\frac{y^3(3+x^3)}{y^3+9}\right)^2 - 1 = 0.$$

Using (2) in the above identity, we obtain

$$\frac{x^3}{y^3q^2}\left(\frac{f_3}{f_9}\right)^8 + \frac{27}{y^9q^3}\left(\frac{f_3}{f_9}\right)^{12} + \frac{3}{y^3q^2}\left(\frac{f_3}{f_9}\right)^8 - 1 = 0.$$

Next, employing the definitions of x and y in the above and then simplifying we see that

$$\frac{f_1^3 f_3^5 f_9 f_{27}^3}{f_9^{12}} + 27q^6\frac{f_3^3 f_{27}^9}{f_9^{12}} + 3q\frac{f_3^5 f_9^4 f_{27}^3}{f_9^{12}} - 1 = 0.$$

Multiplying the above identity throughout by f_9^{12}, we deduce the required result. □

Theorem 2.4. *[14] We have*

$$f_1^3 f_3^8 f_9 + 243q^9 f_3^3 f_{27}^9 + 3q f_3^8 f_9^4 + 81q^6 f_3^6 f_{27}^6 - f_3^{12} - 9q^3 f_9^{12} = 0.$$

Proof. Dividing (1) throughout by $(y^3+9)^3$, sorting and clubbing the terms, we obtain

$$x^3\left(\frac{y^3(3+x^3)}{y^3+9}\right)^2+\frac{243}{y^9}\left(\frac{y^3(3+x^3)}{y^3+9}\right)^3+3\left(\frac{y^3(3+x^3)}{y^3+9}\right)^2+\frac{81}{y^6}\left(\frac{y^3(3+x^3)}{y^3+9}\right)^3$$

$$-\left(\frac{y^3(3+x^3)}{y^3+9}\right)^3-9=0.$$

Next, employing (2) in the above and using the definitions of x and y, we arrive at

$$\frac{f_1^3f_3^8}{f_9^{11}}+243q^9\frac{f_3^3f_{27}^9}{f_9^{12}}+3q\frac{f_3^8}{f_9^8}+81q^6\frac{f_3^6f_{27}^6}{f_9^{12}}-\frac{f_3^{12}}{f_9^{12}}-9=0.$$

Now, multiplying throughout by $q^3f_9^{12}$ in the above identity, we obtain the required result. □

Theorem 2.4. *[14] We have*

$$f_1^9f_9^3+243q^5f_3f_9^8f_{27}^3+27q^2f_3^4f_9^8+9qf_1^6f_9^6-f_3^{12}-81q^3f_9^{12}=0.$$

Proof. Dividing (1) throughout by $(y^3+9)^3$ and then by proper rearranging of terms, we deduce that

$$x^9+\frac{243}{y^3}\frac{y^3(3+x^3)}{y^3+9}+27\frac{y^3(3+x^3)}{y^3+9}+9x^6-\left(\frac{y^3(3+x^3)}{y^3+9}\right)^3-81=0.$$

Now employing (2) and the definitions of x and y in the above, after some simplification we see that

$$\frac{f_1^9}{f_9^9}+243q^5\frac{f_3f_{27}^3}{f_9^4}+27q^2\frac{f_3^4}{f_9^4}+9q\frac{f_1^6}{f_9^6}-\frac{f_3^{12}}{f_9^{12}}-81=0.$$

Next, multiplying the above identity throughout by $q^3f_9^{12}$, we obtain the required result. □

Remark. We omit the proof of the following four identities as the proofs are similar to the previously established results

$$f_3^{12}+2f_1^9f_9^3+3f_1^3f_3^8f_9+9qf_1^6f_9^6-18qf_1^3f_3^4f_9^5-243q^6f_1^3f_3^2f_9f_{27}^6-6f_1^6f_3^4f_9^2=0,$$

$$f_1^6f_3f_9^2f_{27}^3+54q^4f_3^2f_9^4f_{27}^6+6qf_3^5f_9^4f_{27}^3-f_9^{12}-54q^6f_3^3f_{27}^9-9q^2f_3f_9^8f_{27}^3-9q^3f_3^6f_{27}^6=0,$$

$$f_1^9f_3^4f_9^3+3qf_3^{12}f_9^4+9qf_1^6f_3^4f_9^6-f_3^{16}-243q^5f_1^3f_3f_9^9f_{27}^3-3qf_1^9f_9^7=0$$

and

$$f_1^3f_3^9f_9f_{27}^3+27q^6f_3^7f_{27}^9+3qf_9^{16}-f_3^4f_9^{12}-27q^4f_3^6f_9^4f_{27}^6-81q^7f_3^3f_9^4f_{27}^9=0.$$

§3. Application to Colored Partitions

In this Section, we extract partition theoretic interpretation of Theorem 2.1 proved in Section 2. Partition interpretation of other identities follow in the same way.

Following [8, 9], we define the concept of colored partitions by : 'A positive integer n has k colors if there are k copies of n available colors and all of them are viewed as distinct objects'. Partitions of positive integer into parts with colors are called colored partitions.

Further, we adopt the standard notation

$$(a_1, a_2, \cdots, a_n; q)_\infty := \prod_{k=1}^{n} (a_k; q)_\infty$$

and define

$$(q^{r\pm}; q^s)_\infty := (q^r, q^{s-r}; q^s)_\infty, \tag{4}$$

where r and s are positive integers and $r < s$.

Theorem 3.1.. *Let $A(n)$ be the number of partitions of n being divided into parts congruent to $\pm 3, \pm 6, \pm 12$ modulo 27 with 1 color each. Let $B(n)$ indicate the number of partitions of n being split into parts congruent to ± 1, ± 2, ± 3, ± 4, ± 5, ± 6, ± 7, ± 8, ± 10, ± 11, ± 12, ± 13 modulo 27 with 3, 3, 4, 3, 3, 4, 3, 3, 3, 3, 4 and 3 colors respectively. Let $C(n)$ be taken to represent the number of partitions of n into several parts congruent to $\pm 1, \pm 2, \pm 4, \pm 5, \pm 7, \pm 8, \pm 10, \pm 11, \pm 13$ modulo 27 with 3 colors each. If $D(n)$ stands for the number of partitions of n into many parts that are congruent to ± 1, ± 2, ± 3, ± 4, ± 5, ± 6, ± 7, ± 8, ± 9, ± 10, ± 11, ± 12, ± 13 modulo 27 with 3 colors each. Then the following relation holds true:*

$$A(n) + 3B(n-1) - C(n) - 9D(n-3) = 0, \qquad n \geq 3.$$

Proof. On dividing the identity in Theorem 2.1 by $f_1^3 f_3^4 f_9^4 f_{27}^3$, simplifying and then employing (4), we obtain

$$\frac{1}{(q_1^{3\pm}, q_1^{6\pm}, q_1^{12\pm}; q^{27})_\infty} + \frac{3q}{(q_3^{1\pm}, q_3^{2\pm}, q_4^{3\pm}, q_3^{4\pm}, q_3^{5\pm}, q_4^{6\pm}, q_3^{7\pm}, q_3^{8\pm}, q_3^{10\pm}, q_3^{11\pm}, q_4^{12\pm}, q_3^{13\pm}; q^{27})_\infty}$$
$$- \frac{1}{(q_3^{1\pm}, q_3^{2\pm}, q_3^{4\pm}, q_3^{5\pm}, q_3^{7\pm}, q_3^{8\pm}, q_3^{10\pm}, q_3^{11\pm}, q_3^{13\pm}; q^{27})_\infty}$$
$$- \frac{9q^3}{(q_3^{1\pm}, q_3^{2\pm}, q_3^{3\pm}, q_3^{4\pm}, q_3^{5\pm}, q_3^{6\pm}, q_3^{7\pm}, q_3^{8\pm}, q_3^{9\pm} q_3^{10\pm}, q_3^{11\pm}, q_3^{12\pm}, q_3^{13\pm}; q^{27})_\infty} = 0.$$

We see that the above identity generates $A(n), B(n), C(n)$ and $D(n)$ as generating functions and hence we have

$$\sum_{n=0}^{\infty} A(n)q^n + 3q \sum_{n=0}^{\infty} B(n)q^n - \sum_{n=0}^{\infty} C(n)q^n - 9q^3 \sum_{n=0}^{\infty} D(n)q^n = 0,$$

where we set the values $A(0) = B(0) = C(0) = D(0) = 1$. Now on equating the coefficients of q^n in the above, we obtain the required result.

We verify the case for $n = 3$ for Theorem 3.1 by the following table:

Table

$A(3) = 1$	1_o
$B(2) = 9$	$2_o, 2_y, 2_w, 1_o + 1_o, 1_y + 1_y, 1_w + 1_w, 1_o + 1_y, 1_o + 1_w, 1_y + 1_w$
$C(3) = 19$	$2_o + 1_o, 2_y + 1_y, 2_w + 1_w, 2_o + 1_y, 2_o + 1_w, 2_y + 1_w, 2_y + 1_o$ $2_w + 1_o, 2_w + 1_y, 1_o + 1_o + 1_o, 1_y + 1_y + 1_y, 1_w + 1_w + 1_w,$ $1_o + 1_o + 1_y, 1_o + 1_y + 1_w, 1_o + 1_o + 1_w, 1_y + 1_y + 1_o,$ $1_y + 1_y + 1_w, 1_w + 1_w + 1_y, 1_w + 1_w + 1_o$
$D(0) = 1$	1_o

□

Acknowledgements

This work is supported by Extra Mural Research Funding by SERB (DST), Government of India (No. EMR/2016/001601).

References

[1] C. Adiga, K. R. Vasuki and M. S. M. Naika. Some new explicit evaluations of Ramanujan's cubic continued fraction. New Zealand J. Math. 31 (2002), 109 – 114.

[2] N. D. Baruah and N. Saikia. Some general theorems on the explicit evaluations of Ramanujan's cubic continued fraction. J. Comp. Appl. Math. 160 (2003), 37 – 51.

[3] B. C. Berndt. Ramanujan's Notebooks, Part III. Springer-Verlag, New York, 1991.

[4] B. C. Berndt. Ramanujan's Notebooks, Part IV. Springer-Verlag, New York, 1994.

[5] B. C. Berndt and L.-C. Zhang. Ramanujan's identities for eta-functions. Math. Ann. 292 (1992), 561 – 573.

[6] S. Bhargava, K. R. Vasuki and T. G. Sreeramamurthy. Some evaluations of Ramanujan's cubic continued fraction. Indian J. of Pure Appl. Math. 35 (2004), 1003 – 1025.

[7] E. N. Bhuvan. On Somos's identities of level twenty one and their partition interpretations. Matematički Vesnik. 72 (2020), no.1, 1 – 5.

[8] S-L. Chen and S-S. Huang. New modular relations for the Göllnitz-Gordon functions. J. Number Theory. 93 (2002), 58 – 75.

[9] S. -S. Huang. On modular relations for the Göllnitz-Gordon functions with applications to partitions. J. Number Theory. 68 (1998), 178 – 216.

[10] S.-Y. Kang. Ramanujan's formulas for explicit evaluation of the Rogers-Ramanujan continued fraction and theta-functions. Acta Arith. 90 (1999), 49 – 68.

[11] M. S. Mahadeva Naika and B. N. Dharmendra. On some new general theorems for the explicit evaluations of Ramanujan's remarkable product of theta-function. Ramanujan J. 15 (2008), 349 – 366.

[12] S. Ramanujan. *Notebooks* (2 volumes). Tata Institute of Fundamental Research, Bombay (1957).

[13] N. Saikia. Some new explicit values of quotients of Ramanujan's theta functions and continued fractions. International Journal of Mathematics and Mathematical Sciences. 2 (2014), 1 – 7.

[14] Somos M. Personal communication.

[15] B. R. Srivatsa Kumar and R. G. Veeresha. Partition identities arising from Somos's theta-function identities. Annali Dell Universita Di Ferrara. 63 (2017), 303 – 313 .

[16] B. R. Srivatsa Kumar and Anu Radha. Somos's theta-function identities of level 10. Turk. J. Math. 42 (2018), 768 – 773.

[17] B. R. Srivatsa Kumar, K. R. Rajanna and R. Narendra. Theta function identities of level 6 and their application to partitions. Filomat. 33 (2019), no. 11, 3323 – 3335.

[18] B. R. Srivatsa Kumar, K. R. Rajanna and R. Narendra. Theta-function identities of level 8 and its application to partitions. Afrika Mathematika. 30 (2019), 257 – 267.

[19] B. R. Srivatsa Kumar, K. R. Rajanna and R. Narendra. New theta-function identities of level 6 in the spirit of Ramanujan. Mathematical Notes. 106 (2019), 922 – 929.

[20] K. R. Vasuki and T. G. Sreeramamurthy. A note on explicit evaluations of Ramanujan's continued fraction. Adv. Stud. Contemp. Math. 9 (2004), 63 – 80.

[21] K. R. Vasuki and K. Shivashankara. A Note on explicit evaluations of products and ratios of class invariants. Math. Forum. 13 (2000), 45 – 46.

[22] K. R. Vasuki and B. R. Srivatsa Kumar. Evaluations of the Ramanujan-Göllnitz-Gordon continued fraction $H(q)$ by modular equations. Indian J. Math. 48 (2006), no. 3, 275 – 300.

[23] K. R. Vasuki and R. G. Veeresha. On Somos's theta-function identities of level 14. Ramanujan J. 42 (2017), 131 – 144.

[24] J. Yi. Theta-function identities and the explicit formulas for theta-function and their applications. J. Math. Ana. Appl. 292 (2004), 381 – 400.

[25] B. Yuttanam. New modular equations in the spirit of Ramanujan. Ramanujan J. 29 (2012), 257 – 272.

Scientia Magna
Vol. 16 (2021), No. 1, 110-113

A short interval for the function $(\phi^{(e)}(n))^r$

Lu Zhang

School of Mathematics and Statistics, Shandong Normal University
Shandong Jinan, China
E-mail: zhanglu@stu.sdnu.edu.cn

Abstract Let $n > 1$ be an integer. $\phi^{(e)}(n)$ is a multiplicative function. The integer $d = \prod_{i=1}^{s} p_i^{b_i}$ is called an exponential divisor of $n = \prod_{i=1}^{s} p_i^{a_i}$, if $b_i | a_i$ for every $i \in \{1, 2, \cdots, s\}$. Let $\phi^{(e)}(n)$ denote the number of divisors d of n such that d and n have no common exponential divisors. The aim of the paper is to establish a short interval for r–th power of the function $\phi^{(e)}(n)$ for any fixed integer $r \geq 1$.

Keywords The exponential divisor function, arithmetic function, short interval.
2010 Mathematics Subject Classification 11E45, 11A25.

§1. Introduction

Let $n > 1$ be an integer. For $n = \prod_{i=1}^{s} p_i^{a_i}$ $a_i \geq 1 (1 \leq i \leq r)$, denote by $\phi^{(e)}(n)$ the number of integers $\prod_{i=1}^{s} p_i^{c_i}$ such that $1 \leq c_i \leq a_i$ and $(c_i, a_i) = 1$ for $1 \leq i \leq r$, and let $\phi^{(e)}(1) = 1$. Thus $\phi^{(e)}(n)$ counts the number of divisors d of n such that d and n are exponentially coprime.

It is easy to see that $\phi^{(e)}(n)$ is a prime independent multiplicative function and for $n > 1$,

$$\phi^{(e)}(n) = \prod_{i} \phi(a_i), \tag{1}$$

where ϕ is the Euler-function.

Throughout this paper, ε always denotes a fixed but sufficiently small positive constant.

Liu [3] got the following result:

$$\sum_{x < n \leq x+y} \phi^{(e)}(n) = C(y) + O(yx^{\frac{-\varepsilon}{3}} + x^{2\varepsilon + \frac{1}{7}}) \tag{2}$$

where $C = G(1)\zeta(3)\zeta^2(5)$ is a constant.

In this paper, we shall prove the following short interval result.

Theorem 1.1 If $x^{\frac{1}{5} + \frac{4\varepsilon}{3}} < y \leq x$, then we have

$$\sum_{x < n \leq x+y} (\phi^{(e)}(n))^r = A_r y + O(yx^{\frac{-\varepsilon}{6}} + x^{\frac{1}{5} + \frac{7\varepsilon}{6}}), \tag{3}$$

where $A_r = Res_{s=1} F(s)$ and $F(s) := \sum_{n=1}^{\infty} \frac{(\phi^{(e)}(n))^r}{n^s}$.

§2. Some lemmas

In order to prove our theorem, we need the following lemmas.

Lemma 2.1. For $r \geq 1$, then we have

$$F(s) = \sum_{n=1}^{\infty} \frac{(\phi^{(e)}(n))^r}{n^s} = \zeta(s)\zeta^{2^{-r}-1}(3s)H(s),$$

where the infinite series $H(s) := \sum_{n=1}^{\infty} \frac{h(n)}{n^s}$ is absolutely convergent for $\Re s > \frac{1}{5}$.

Proof. By Euler's product formula, we can get

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(\phi^{(e)}(n))^r}{n^s} &= \prod_p \left(1 + \frac{(\phi^{(e)}(p))^r}{p^s} + \frac{(\phi^{(e)}(p^2))^r}{p^{2s}} + \frac{(\phi^{(e)}(p^3))^r}{p^{3s}} + \cdots\right) \\
&= \prod_p \left(1 + \frac{(\phi(1))^r}{p^s} + \frac{(\phi(2))^r}{p^{2s}} + \frac{(\phi(3))^r}{p^{3s}} + \frac{(\phi(4))^r}{p^{4s}} + \cdots\right) \\
&= \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{2^r}{p^{3s}} + \frac{2^r}{p^{4s}} + \frac{4^r}{p^{5s}} + \cdots\right) \qquad (4) \\
&= \zeta(s) \prod_p \left(1 + \frac{2^r - 1}{p^{3s}} + \frac{4^r - 2^r}{p^{5s}} + \cdots\right) \\
&= \zeta(s)\zeta^{2^r-1}(3s)H(s),
\end{aligned}$$

where the infinite series $H(s) := \sum_{n=1}^{\infty} \frac{h(n)}{n^s}$ is absolutely convergent for $\Re s > \frac{1}{5}$. □

Lemma 2.2. Let $k \geq 2$ be a fixed integer. $1 < y \leq x$ be large real numbers and

$$B(x, y; k, \varepsilon) := \sum_{\substack{x < nm^k \leq x+y \\ m > x^{\varepsilon}}} 1,$$

then we have

$$B(x, y; k, \varepsilon) \ll yx^{-\varepsilon} + x^{\frac{1}{2k+1}} \log x. \qquad (5)$$

Proof. This lemma is very important when studying the short interval distribution of k-free number, see in [4]. □

Let $a_1(n), a_2(n)$ be arithmetic functions defined by the following Dirichlet series (for $\Re s > 1$):

$$\sum_{n=1}^{\infty} \frac{a_1(n)}{n^s} = \zeta(s)H(s), \qquad (6)$$

$$\sum_{n=1}^{\infty} \frac{a_2(n)}{n^{3s}} = \zeta^{2^r-1}(3s). \qquad (7)$$

Lemma 2.3. Let $a_1(n)$ be arithmetical function defined by (6), then we have

$$\sum_{n \leq x} a_1(n) = A_1 x + O(x^{\frac{1}{5}+\varepsilon}),$$

where $A_1 = Res_{s=1}\zeta(s)H(s)$.

Proof. Using Lemma 2.1, it is easy to see that

$$\sum_{n\leq x} \mid h(n) \mid \ll x^{\frac{1}{5}+\varepsilon}. \tag{8}$$

Therefore from the definition of $h(n)$ and (6),it follows that

$$\begin{aligned}\sum_{n\leq x} a_1(n) &= \sum_{mk\leq x} h(k) = \sum_{k\leq x} h(k) \sum_{m\leq \frac{x}{k}} 1 \\ &= \sum_{k\leq x} h(k)(\frac{x}{k} + O(1)) = A_1 x + O(x^{\frac{1}{5}+\varepsilon}),\end{aligned} \tag{9}$$

where $A_1 = Res_{s=1}\zeta(s)H(s)$

□

§3. Proof of the theorem

From Lemma 2.3 and the definition of $a_1(n)$ and $a_2(n)$, we get

$$\phi^{(e)}(n))^r = \sum_{n=n_1 n_2^3} a_1(n_1)a_2(n_2) \tag{10}$$

and

$$a_1(n) \ \ll n^{\varepsilon^2}, a_2(n) \ \ll n^{\varepsilon^2}. \tag{11}$$

So we have

$$\begin{aligned}\sum_{x<n\leq x+y} (\phi^{(e)}(n))^r &= \sum_{x<n_1 n_2^3\leq x+y} a_1(n_1)a_2(n_2) \\ &= \sum_1 + O(\sum_2),\end{aligned} \tag{12}$$

where

$$\sum_1 = \sum_{n_2\leq n^{\varepsilon}} a_2(n_2) \sum_{\frac{x}{n_2^3}<n_1\leq \frac{x+y}{n_2^3}} a_1(n_1),$$

and

$$\sum_2 = \sum_{\substack{x<n_1 n_2^3\leq x+y \\ n_2>x^{\varepsilon}}} \mid a_1(n_1)a_2(n_2) \mid . \tag{13}$$

By Lemma 2.3 we get

$$\begin{aligned}\sum_1 &= \sum_{n_2 \ \ll n^{\varepsilon^2}} a_2(n_2)(\frac{A_1 y}{n_2^3} + O(\frac{x}{n_2^3})^{\frac{1}{5}+\varepsilon}) \\ &= A_r y + O(y \sum_{n_2>x^{\varepsilon}} \frac{a_2(n_2)}{n_2^3}) + O(x^{\frac{1}{5}+\varepsilon} \sum_{n_2\leq x^{\varepsilon}} \frac{a_2(n_2)}{n_2^{\frac{3}{5}+3\varepsilon}}) \\ &= A_r y + O(yx^{\frac{-\varepsilon}{6}}) + O(x^{\frac{1}{5}+\varepsilon}x^{\frac{\varepsilon}{6}}) \\ &= A_r y + O(yx^{\frac{-\varepsilon}{6}} + x^{\frac{1}{5}+\frac{7\varepsilon}{6}}),\end{aligned} \tag{14}$$

where $A_r = Res_{s=1}F(s)$. For $\sum_2$ we have by Lemma 2.2 and (11) that

$$\begin{aligned}
\sum_2 &\ll \sum_{\substack{x<n_1n_2^3\le x+y\\ n_2>x^{\varepsilon}}} (n_1n_2)^{\varepsilon^2}\\
&\ll x^{\varepsilon^2} \sum_{\substack{x<n_1n_2^3\le x+y\\ n_2>x^{\varepsilon}}} 1\\
&\ll x^{\varepsilon^2}(yx^{-\varepsilon} + x^{\frac{1}{7}}\log x)\\
&\ll yx^{\varepsilon^2-\varepsilon} + x^{\frac{1}{7}+\varepsilon^2}\\
&\ll yx^{\frac{-\varepsilon}{2}} + x^{\frac{1}{7}+\frac{\varepsilon}{2}}.
\end{aligned} \tag{15}$$

Now our theorem follows from (12)–(15).

Acknowledgements

The authors wish to express their gratitude to the referee for a careful reading of the manuscript and many valuable suggestions which highly improve the quality of this paper.

References

[1] A. Ivić. The Riemann Zeta-function, John Wiley and Sons, 1985.

[2] A. Ivić. The Riemann zeta-function: theory and applications. Oversea Publishing House, 2003.

[3] X. F. Liu and P. M. Zhao. A short interval result for the function $\phi^{(e)}(n)$. Scientia Magna 8 (2012), 80 – 84.

[4] M. V. Subbarao. On some arithmetic convolutions. In "The Theory of Arithmetic Functions", Lecture Notes in Mathematic, Springer, 251 (1972), 247 – 271.

[5] L. Tóth. An order result for the exponential divisor function. Publ. Math. Debrean 71 (2007), 165 – 171.

[6] J. Wu. Problème de diviseurs exponentiels at entiers exponentiellement sans facteur carré. J. Théor. Nombres Bordeaux 7 (1995), 133 – 141.

SCIENTIA MAGNA

An international journal

ISSN 1556- 6706