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Contents

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A survey on Smarandache notions in number theory: k-th power complements function and k-th power free sequence

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Abstract This paper is a survey on Smarandache k-th power complements function, k-th power free sequence and related problems.

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§1. Introduction

Let $k \geq 2$ be an integer. For any positive integer n, the Smarandache k-th power complements function $S_k(n)$ is defined as the smallest positive integer such that $nS_k(n)$ is a perfect k-th power, i.e.,

$$S_k(n) = \min \left\{ m : nm = u^k,\ u \in \mathbb{N} \right\}.$$

In problems 27-29 of [42], Smarandache proposed some problems about $S_k(n)$. Later, many papers have been written on this subject. For example, Russo [41] presented some properties of $S_2(n)$. Further, Liu and Gou [27] proved

$$\sum_{n \leq x} S_2(n) = \frac{\pi^2}{30} x^2 + O\left(x^{\frac{3}{2}}\right).$$

Similar to the Smarandache k-th power complements function, the additive k-th power complements function $T_k(n)$ is defined as the smallest nonnegative integer such that $T_k(n) + n$ is a perfect k-th power, i.e.,

$$T_k(n) = \min \left\{ m : n + m = u^k,\ u \in \mathbb{N} \right\}.$$

Xu [45] studied the mean value of $T_k(n)$ and proved that

$$\sum_{n \leq x} T_k(n) = \frac{k^2}{4k-2} x^{2-\frac{1}{k}} + O\left(x^{2-\frac{2}{k}}\right).$$

Furthermore, various mean values involving $S_k(n)$ and $T_k(n)$ were studied.

A nature number n is called a k-th power free number if it can not be divided by any p^k, where p is a prime. On the other hand, If $p \mid n$ implies $p^k \mid n$, we call n is a k-th power full number. Let $\mathcal{A}(k)$ denote the set of k-th power free numbers, $\mathcal{B}(k)$ denote the set of k-th power full numbers. Let $x \geq 1$ be a real number, it is well-known that

$$\sum_{\substack{n \leq x \\ n \in \mathcal{A}(r)}} 1 = \frac{x}{\zeta(k)} + O\left(x^{\frac{1}{k}}\right).$$

In problem 31 of [42], Smarandache asked us to study the properties of the k-power free number sequence. Later, many scholars focused on the mean values of arithmetical functions over $\mathcal{A}(k)$ and $\mathcal{B}(k)$, i.e.,

$$\sum_{\substack{n \leq x \\ n \in \mathcal{A}(k)}} f(n), \qquad \sum_{\substack{n \leq x \\ n \in \mathcal{B}(k)}} f(n).$$

In this paper, we make a survey on the Smarandache k-th power complements function, k-th power free sequence and related problems. In Section 2 and Section 3, we introduce some properties of $S_k(n)$ and $T_k(n)$. In Section 4, we introduce some other complements functions. In Section 5, we make a survey on the k-th power free and k-th power full sequences. Finally, in Section 6, some other functions related to k-th power will be shown.

Throughout this paper, we let x be a sufficiently large positive real number. By ε we denote an arbitrary small positive number, not necessarily the same in different occurrence. Let $c_0, c_1, c_2, \cdots$ be constants which can be calculated. We also remark that c_i are not the same in different occurrence.

As usual ϕ is the Euler function, ζ is the Riemann zeta function, μ is the Möbius function, d is the divisor function and Λ is the Mangoldt function. If $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ denotes the factorization of n into prime powers, we define

$$\Omega(n) = \alpha_1 + \alpha_2 + \cdots + \alpha_s,$$

and $\omega(n)$ is the number of distinct prime factors of n. In addition, let m be a positive integer, the arithmetical function $\delta_m(n)$ is defined as

$$\delta_m(n) = \max\{d \in \mathbb{N} : d \mid n, (d, m) = 1\}.$$

§2. Smarandache k-th power complements function

§2.1 Properties of $S_2(n)$

In [41], Russo considered the difference of $S_2(n)$. Later, Le [17] and Wang [43] proved that the difference $|S_2(n+1) - S_2(n)|$ is infinite as $n \to \infty$. Furthermore, Hu and Yang [13] proved that for any positive integer b, when $n \to \infty$, $|S_2(n+b) - S_2(n)|$ is also infinite.

On the other hand, Wang [43] studied some diophantine equations related to $S_2(n)$ and concluded the following results:

(i) $S_2(n) = S_2(n+1)S_2(n+2)$ has no solution.

(ii) $S_2(n)S_2(n+1) = S_2(n+2)$ has no solution.

(iii) For arbitrary positive integer m, the equation $S_2(n_1n_2) = n_1^m S_2(n_2)$ has infinitely many solutions (n_1, n_2).

(iv) For arbitrary integer m with $m \geq 2$, there are only two solutions to this equation

$$S_2(n)^m + S_2(n)^{m-1} + \cdots + S_2(n) = n.$$

§2.2 Series involving $S_k(n)$

In [41], Russo also proposed some problems in terms of the series related to $S_2(n)$. In particular, Russo showed that the series

$$\sum_{n=1}^{\infty} \frac{S_2(n)}{n}$$

diverges. Le [18] [19] proved that

$$\sum_{n=1}^{\infty} \frac{1}{S_2(n)^s} \quad (s \leq 1), \qquad \sum_{n=1}^{\infty} (-1)^n \frac{1}{S_2(n)}$$

are divergence as well. Furthermore, more series related to Smarandache k-th power complements function were studied. The results are as follows.

Liu and Wang [29], **Lu and Wei [35]**.

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n=2}^{x} \frac{\log(S_2(n))}{\log n} = 1.$$

Fan [5].

$$\lim_{x\to\infty} \frac{S_2(x)}{\sum\limits_{n\leq x} \log S_2(n)} = 0.$$

Qi [38].

$$\lim_{x\to\infty} \frac{\sum\limits_{n\leq x} d(S_2(n))}{\sum\limits_{n\leq x} \log(S_2(n))} = \frac{6}{\pi^2} \prod_p \left(1 - \frac{1}{(p+1)^2}\right).$$

Moreover, let s be a complex number with $\mathrm{Re}(s) \geq 1$. Zhang [57] focused on the value of

$$\sum_{n=1}^{\infty} \frac{1}{(nS_k(n))^s}.$$

In [57], Zhang obtained some identities:

$$\sum_{n=1}^{\infty} \frac{1}{(nS_2(n))^s} = \frac{\zeta^2(2s)}{\zeta(4s)}$$

$$\sum_{n=1}^{\infty} \frac{1}{(nS_3(n))^s} = \frac{\zeta^2(3s)}{\zeta(6s)} \prod_p \left(1 + \frac{1}{p^{3s}+1}\right),$$

and

$$\sum_{n=1}^{\infty}\frac{1}{(nS_4(n))^s}=\frac{\zeta^2(4s)}{\zeta(8s)}\prod_p\left(1+\frac{1}{p^{4s}+1}\right)\prod_p\left(1+\frac{1}{p^{4s}+2}\right).$$

Inspired by the work of Russo [41] and Zhang [57], many scholars studied some similar problems.

Liu and Ma [31].

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{nS_k(n)}=\frac{2^k-k-1}{2^k+k-1}\zeta(k)\prod_p\left(1+\frac{k-1}{p^k}\right).$$

Zhang [53]. *For any complex numbers* s_1, s_2 *with* $\mathrm{Re}(s_1)\geq 1$ *and* $\mathrm{Re}(s_2)\geq 1$, *we have*

$$\sum_{n=1}^{\infty}\frac{1}{n^{s_1}S_k(n)^{s_2}}=\zeta(ks_1)\prod_p\left(1+\frac{1-\frac{1}{p^{(k-1)s_1+(k-1)^2s_2}}}{p^{s_1+(k-1)s_2}-1}\right),$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{s_1}S_k(n)^{s_2}}=\left(1-\frac{2\left(2^{ks_1}-1\right)\left(2^{s_1+(k+1)s_2}-1\right)}{2^{(k+1)s_1+(k-1)s_2}-2^{s_1-(k-1)^2s_2}}\right)\sum_{n=1}^{\infty}\frac{1}{n^{s_1}S_k(n)^{s_2}}.$$

Lou [33]. *Let* s *be an real number with* $s>1$. *We have*

$$\sum_{\substack{n=1\\ \delta_m(n)=S_k(n)}}^{\infty}\frac{1}{n^s}=\frac{\zeta\left(\frac{k}{2}s\right)}{\zeta(ks)}\prod_{p|m}\frac{p^{\frac{3}{2}ks}}{(p^{mk}-1)\left(p^{\frac{1}{2}mk}-1\right)}.$$

§2.3 Mean values of $S_k(n)$

Let $f(n)$ be an arithmetical function, many scholars focused on the mean value of $f(S_k(n))$ and $\frac{1}{f(S_k(n))}$. In particular, Liu and Gou [27] proved that

$$\sum_{n\leq x}S_2(n)=\frac{\pi^2}{30}x^2+O\left(x^{\frac{3}{2}}\right),\qquad \sum_{n\leq x}\frac{1}{S_2(n)}=\frac{\zeta(\frac{3}{2})}{\zeta(3)}\sqrt{x}+O\left(\log x\right).$$

When $f(n)$ is the divisor function, Lou [32] obtained the following asymptotic formula

$$\sum_{n\leq x}d(S_2(n))=c_1x\log x+c_2x+O\left(x^{\frac{1}{2}+\varepsilon}\right),$$

where

$$c_1=\frac{6}{\pi^2}\prod_p\left(1-\frac{1}{(p+1)^2}\right),$$

$$c_2=\frac{6}{\pi^2}\prod_p\left(1-\frac{1}{(p+1)^2}\right)\left(\sum_p\frac{2(2p+1)\log p}{(p-1)(p+1)(p+2)}+2\gamma-1\right).$$

For a general k, Chen [2] proved that

$$\sum_{n\leq x}d(S_k(n))=\frac{6\zeta\left(k^2\right)C_1(k)}{k\pi^2}x^k+O\left(x^{k-\frac{1}{2}}\log x\right),$$

where $C_1(k)$ is a constant depends on k. Later, Huang and Ma [15] improved the error term of Chen and obtained

$$\sum_{n\le x} d(S_k(n)) = c_0 x log^k x + \cdots + c_{k-1} x\log x + c_k x + O\left(x^{\frac{1}{2}+\varepsilon}\right).$$

In addition, Yao [51] proved that

$$\sum_{n\le x} d(nS_k(n)) = x\left(c_0\log^k x + c_1\log^{k-1} x + \cdots + c_k\right) + O\left(x^{\frac{1}{2}+\varepsilon}\right).$$

For other arithmetical functions, many scholars also obtained interesting asymptotic formulas. Their results are as follows.

Liu and Lou [28].

$$\sum_{n\le x} \frac{d(S_k(n))}{\phi(S_k(n))} = kx^{\frac{1}{k}} C_2(k) + O\left(x^{\frac{1}{2k}+\varepsilon}\right),$$

where $C_2(k)$ is a constant depends on k.

Yang and Fu [48].

$$\sum_{n\le x} \frac{1}{\delta_m(S_k(n))} = \frac{x^2}{2\zeta(k)} \prod_{p\nmid m} \frac{p^m(p^m - p^{m-1}+1)}{p^{2m}-1} \prod_{p\mid m} \frac{p^{m+1}}{(p+1)(p^m-1)} + O\left(x^{\frac{3}{2}+\varepsilon}\right).$$

Huang [14]. *Let $D(n)$ denote the number of the solutions of the equation $n = n_1 n_2$ with $(n_1, n_2) = 1$. That is*

$$D(n) = \sum_{\substack{d\mid n\\ (d,\frac{n}{d})=1}} 1.$$

We have

$$\sum_{n\le x} D(S_k(n)) = \frac{6\zeta(k) x\log x}{\pi^2} \prod_p \left(1 - \frac{2}{p^k + p^{k-1}}\right) + C_3(k)x + O\left(x^{\frac{1}{2}+\varepsilon}\right).$$

Xu [46]. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$. The arithmetic function $I_1(n)$ and $I_2(n)$ are defined as*

$$I_1(n) = \alpha_1 p_1^{\alpha_1 - 1} \cdots \alpha_s p^{\alpha_s - 1},$$
$$I_2(n) = \frac{1}{(\alpha_1+1)\cdots(\alpha_s+1)} p_1^{\alpha_1+1} \cdots p^{\alpha_s+1}.$$

Then we have

$$\sum_{n\le x} \frac{1}{I_1(S_k(n))} = \frac{6(k-1)\zeta\left(\frac{k}{k-1}\right) x^{\frac{1}{k-1}}}{\pi^2} \prod_p \left(1 + \frac{p}{p+1} \sum_{i=1}^{k-2} \frac{1}{(k-i)p^{k-1-i} p^{\frac{i}{k-1}}}\right) + O\left(x^{\frac{2k-1}{2k(k-1)}} + \varepsilon\right),$$

and

$$\sum_{n\le x} I_2(S_k(n)) d(S_k(n)) = \frac{6\zeta(k(k+1)) x^{k+1}}{(k+1)\pi^2} \prod_p \left(1 + \frac{p}{p+1}\left(\sum_{i=2}^{k} \frac{p^{k+1-i}}{p^{(k+1)i}} - \frac{1}{p^{k(k+1)}}\right)\right)$$

$$+O\left(x^{k+\frac{1}{2}+\varepsilon}\right).$$

Feng [6]. *Let p be an odd prime, and let $e_p(n)$ denote the largest exponent of power p which divide n. We have*

$$\sum_{n\le x} e_p(S_k(n)) = \frac{(k-1)p^k - kp^{k-1}+1}{(p^k-1)(p-1)} + O\left(x^{\frac{1}{2}+\varepsilon}\right).$$

Ding [3]. *Let $H_k(n)=\min\{m : m\in\mathbb{N},\ m^k \mid n\}$. Then we have*

$$\sum_{n\le x} H_2(S_3(n)) = \frac{x^2\pi^4}{315}\prod_p\left(1+\frac{1}{p^4+p^3}\right) + O\left(x^{\frac{3}{2}+\varepsilon}\right).$$

Zhang [58]. *Let $h(n)=\min\{k : k\in\mathbb{N},\ n \mid k!\}$. We have*

$$\sum_{n\le x} S_k(h(n)) = \frac{x^k\zeta(k)}{k\log x} + O\left(\frac{x^k}{\log^2 x}\right).$$

Xue [47]. *Let $\mathcal{A}=\{n\in\mathbb{N} : n \mid S_k(n)\}$. Then we have*

$$\sum_{\substack{n\le x\\ n\in\mathcal{A}}} d(n) = \frac{x\log x}{\zeta(l+1)}\prod_p\left(1-\frac{(l+1)(p-1)}{p^{l+2}-p}\right) + c_1 x + O\left(x^{\frac{1}{2}+\varepsilon}\right),$$

where $l_1=[\frac{k}{2}]$. Let $\mathcal{B}=\{n\in\mathbb{N} : S_k(n) \mid n\}$. We also have

$$\sum_{\substack{n\le x\\ n\in\mathcal{B}}} d(n) = \frac{l_2 x^{\frac{1}{l_2}}}{\zeta^{l_2+1}(2)} C\left(p,l_2\right) f(\log x) + O\left(x^{\frac{1}{2l_2}+\varepsilon}\right),$$

where $C(P,l_2)$ is a constant depend on p and l_2, and $f(y)$ is a polynomial with degree $l_2=[\frac{k+1}{2}]$.

§2.4 values of $\log(S_k(n!))$

In [7], Fu and Yang proved

$$\log(S_2(n!)) = n\log 2 + O\left(n\exp\left(\frac{-c_1\log^{\frac{3}{5}} n}{(\log\log n)^{\frac{1}{5}}}\right)\right),$$

For a general k, Li [23] proved

$$\log(S_k(n!)) = n\left(k-\sum_{i=1}^{\infty}\frac{1}{i(ki+1)}\right) + O\left(n\exp\left(\frac{-c_2\log^{\frac{3}{5}} n}{(\log\log n)^{\frac{1}{5}}}\right)\right).$$

§3. Additive k-th power complements function

Similar to the Smarandache k-th power complements function, many scholars also focused on the mean values of $f(T_k(n))$ and $\frac{1}{f(T_k(n))}$, where $f(n)$ is an arithmetical function. In particular, Xu [45] studied the case for f is the divisor function, he obtained

$$\sum_{n\leq x} d(T_k(n)) = \left(1-\frac{1}{k}\right) x\log x + \left(2\gamma + \log k - 2 + \frac{1}{k}\right) x + O\left(x^{1-\frac{1}{k}}\log x\right).$$

Yi and Liang [52] proved that

$$\sum_{n\leq x} d(n+T_2(n)) = \frac{3}{4\pi^2} x\log^2 x + c_1 x\log x + c_2 x + O\left(x^{\frac{3}{4}+\varepsilon}\right).$$

For other arithmetical functions, many scholars also obtained interesting asymptotic formulas. Their results are as follows.

Liang and Yi [26].

$$\sum_{n\leq x} \Omega(n+T_3(n)) = 3x\log\log x + 3(c_1 - \log 3)x + O\left(\frac{x}{\log x}\right),$$

where $c_1 = \gamma + \sum_p \left(\log\left(1-\frac{1}{p}\right) + \frac{1}{p}\right) + \sum_p \frac{1}{p(p-1)}$, *and* γ *denotes the Euler constant.*

Guo [11].

$$\sum_{n\leq x} \Omega(n+T_k(n)) = kx\log\log x + k(c_2 - \log k)x + O\left(\frac{x}{\log x}\right),$$

where $c_2 = \gamma + \sum_p \left(\log\left(1-\frac{1}{p}\right) + \frac{1}{p-1}\right)$.

Ding [4]. *If* $k \geq 3$ *we have*

$$\sum_{n\leq x} \delta_m(T_k(n)) = \frac{k^2}{2(2k-1)} x^{2-\frac{1}{k}} \prod_{p|m} \frac{p}{p+1} + O\left(x^{2-\frac{2}{k}}\right).$$

Moreover, Lu [34] studied the infinity series

$$\sum_{n=1}^{\infty} \frac{1}{(n+T_k(n))^s},$$

where s is a real number. Lu showed that the series is divergent if $s > 1$. For $s > 1$, we have

$$\sum_{n=1}^{\infty} \frac{1}{(n+T_k(n))^s} = \sum_{i=1}^{k} (-1)^{i+1}\binom{k}{i}\zeta(k\alpha - k + i).$$

§4. Other complements functions

In addition to the k-th power complements function, Smarandache proposed other complements functions. For example, In problem 45 of [42], Smarandache asked us to study the factorial complements. The factorial complements function $L_1(n)$ is defined as

$$L_1(n) = \min\{m : mn = u!,\ m \geq 0,\ u \in \mathbb{N}\}.$$

Similarly, we can defined the additive factorial complements function:

$$L_2(n) = \min\{m : m + n = u!,\ m \geq 0,\ u \in \mathbb{N}\}.$$

In [8], Liu and Gao study the hybird mean value of $L_1(n)$ and Mangoldt function. They proved

$$\sum_{n\leq x} \Lambda(n)\log(L_1(n)) = \frac{1}{2}x^2\log x + O\left(x^2\right).$$

Yang and Yang [50] obtained

$$\sum_{n\leq x} \frac{1}{L_2(n)+1} = \frac{2\log^2 x}{2\log\log x} + O\left(\frac{\log^2 x\log\log\log x}{(\log\log x)^2}\right).$$

Inspired by these complements functions, many scholars constructed various forms of complements functions. Li and Li [20] defined the double factorial number complements function. That is

$$L_3(n) = \min\{m : mn = u!!,\ m \geq 0,\ u \in \mathbb{N}\}.$$

Li and Li [20] proved

$$\sum_{n\leq x} \Lambda(n)\log(L_3(n)) = \frac{1}{2}x^2\log x + O\left(x^2\right).$$

Li and Yang [21] defined the additive hexagon number complements function $L_4(n)$:

$$L_4(n) = \min\{m : m + n = u(2u-1),\ m \geq 0,\ u \in \mathbb{N}\}.$$

They proved

$$\sum_{n\leq x} L_4(n) = \frac{2\sqrt{2}}{3}x^{\frac{3}{2}} + O(x),$$

$$\sum_{n\leq x} d(L_4(n)) = \frac{1}{2}x\log x + \left(\frac{3}{2}\log 2 + 2\gamma - \frac{3}{2}\right)x + O\left(x^{\frac{2}{3}}\right).$$

Moreover, the prime additive complement function $L_5(n)$ is defined as

$$L_5(n) = \min\{m : m + n = p,\ m \geq 0,\ p \text{ is a prime}\}.$$

In [42], Smarandache conjectured that it is possible to have k as large as we want

$$k, k-1, \cdots, 2, 1, 0$$

included in the sequence $\{L_5(n)\}$. Le [16] and Guo [10] proved that this conjecture is correct.

§5. k-th power free and k-th power full number sequences

§5.1 k-th power free number sequence

Many papers have been written on the mean values of arithmetical functions over $\mathcal{A}(k)$. That is

$$\sum_{\substack{n\le x\\ n\in\mathcal{A}(k)}} f(n),$$

where $f(n)$ is an arithmetical function. When $f(n)=n$, Zhu [60] proved the following asymptotic formula,

$$\sum_{\substack{n\le x\\ n\in\mathcal{A}(3)}} n=\frac{x^2}{2\zeta(3)}+O\left(x^{\frac{3}{2}+\varepsilon}\right).$$

In [60], Zhu also studied the cases for Euler function and divisor function, he obtained

$$\sum_{\substack{n\le x\\ n\in\mathcal{A}(3)}} \phi(n)=\frac{x^2}{2\zeta(3)}\prod_p\left(1-\frac{p+1}{p^3+p^2+1}\right)+O\left(x^{\frac{3}{2}+\varepsilon}\right)$$

and

$$\sum_{\substack{n\le x\\ n\in\mathcal{A}(3)}} d(n)=\frac{36x}{\pi^4}\prod_p\frac{p^2+2p+3}{(1+p)^2}\left(\log x+(2\gamma-1)-\frac{24}{\pi^2}\sum_{n=2}^{\infty}\frac{\log n}{n^2}\right.$$
$$\left.-4\sum_p\frac{p\log p}{(p^2+2p+3)(p+1)}\right)+O\left(x^{\frac{1}{2}+\varepsilon}\right).$$

Zhang [54] studied the case for $f(n)=\omega(n)$. Qing [39] improved Zhang's result and obtained

$$\sum_{\substack{n\le x\\ n\in\mathcal{A}(k)}} \omega^2(n)=\frac{1}{\zeta(k)}\left(x(\log\log x)^2+c_1x\log\log x+c_2x\right)+O\left(\frac{x\log\log x}{\log x}\right).$$

The results for other arithmetical functions are as follows.

Hong [12]. *Let $n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}$. $\xi_1(n)$ is defined as*

$$\xi_1(1)=1,\quad \xi_1(n)=p_1p_2\cdots p_s,$$

and $\xi_2(n)$ is defined as

$$\xi_2(n)=(p_1^{\alpha_1}-1)(p_2^{\alpha_2}-1)\cdots(p_s^{\alpha_s}-1).$$

We have

$$\sum_{\substack{n\le x\\ n\in\mathcal{A}(k+1)}} \xi_1(n)=\frac{3x^2}{\pi^2}\prod_p\left(1+\frac{p^{2k-2}-1}{p^{2k+1}+p^{2k}-p^{2k-1}-p^{2k-2}}\right)+O\left(x^{\frac{3}{2}+\varepsilon}\right),$$

$$\sum_{\substack{n\leq x\\ n\in\mathcal{A}(k+1)}}\xi_2(n)=\frac{x^2}{2}\prod_p\left(1-\frac{1}{p^{k+1}}-\frac{p^{2k+1}+p^{2k}-p-1}{p^{2k+3}+p^{2k+1}}\right)+O\left(x^{\frac{3}{2}+\varepsilon}\right).$$

Ma [37]. *Let q be a positive integer and let $g_q(n)=(q,n)$. We have*

$$\sum_{\substack{n\leq x\\ n\in\mathcal{A}(k)}}\delta_m(g_q(n))=\frac{C(q,m,k)}{\zeta(k)}x+O\left(x^{\frac{1}{2}+\varepsilon}\right),$$

where $C(q,m,k)$ is a constant depend on q, m, and k.

Li and Gao [24].

$$\sum_{\substack{n\leq x\\ n\in\mathcal{A}(k)}}S_m(n)=\frac{6x^{m+1}}{(m+1)\pi^2}R(m,k)+O\left(x^{m+\frac{1}{2}+\varepsilon}\right).$$

where $C(k)$ is a constant depends on k.

Weiyi Zhu [61].

$$\sum_{\substack{n\leq x\\ n\in\mathcal{A}(k)}}\Omega(S_k(n))=\frac{(k-1)x\log\log x}{\zeta(k)}+C(k)+O\left(\frac{x}{\log x}\right),$$

where $C(k)$ is a constant depends on k.

§5.2 k-th power full number sequence

Xu [44] studied the mean values of some arithmetic functions over k-full number sequences, i.e.,

$$\sum_{\substack{n\leq x\\ n\in B(k)}}f(n).$$

He studied the cases for the Euler function, the divisor function and and obtained the following results.

$$\sum_{\substack{n\leq x\\ n\in B(k)}}n=\frac{6kx^{1+\frac{1}{k}}}{(k+1)\pi^2}\prod_p\left(1+\frac{1}{(p+1)(p^{\frac{1}{k}}-1)}\right)+O\left(x^{1+\frac{1}{2k}+\varepsilon}\right),$$

$$\sum_{\substack{n\leq x\\ n\in\mathcal{B}(k)}}\phi(n)=\frac{6kx^{1+\frac{1}{k}}}{(k+1)\pi^2}\prod_p\left(1+\frac{p-p^{\frac{1}{k}}}{p^{2+\frac{1}{k}}-p^2+p^{1+\frac{1}{k}}-p}\right)+O\left(x^{1+\frac{1}{2k}+\varepsilon}\right),$$

$$\sum_{\substack{n\leq x\\ n\in\mathcal{B}(k)}}d(n)=\frac{6kx^{\frac{1}{k}}}{\pi^2}\prod_p\left(1+\frac{C(p,k)}{(p+1)^{k+1}(p^{\frac{1}{k}}-1)^2}\right)f(\log x)+O\left(x^{\frac{1}{2k}+\varepsilon}\right),$$

where $C(p,k)$ is a constant depend on p and k, and $f(y)$ is a polynomial with degree k.

§6. Other functions related to k-th power

§6.1 Smarandache k-th power free part function

In problem 65 of [42], Smarandache proposed a k-th power free part function $M_k(n)$. If $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ denotes the factorization of n into prime powers, then

$$M_k(n) = p_1^{\beta_1} p_2^{\beta_2} \cdots p_s^{\beta_s},$$

where $\beta_i = \min(\alpha_i, k-1)$. In other word, $M_k(n)$ denotes the largest k-th power free number which divides n. Gou [9] studied the mean value of $M_k(n)$ and proved

$$\sum_{n \le x} M_k(n) = \frac{1}{2} \prod_p \left(1 - \frac{1}{p(p+1)}\right) x^2 + O\left(x^{\frac{3}{2}+\varepsilon}\right).$$

Other conclusions related to $M_k(n)$ are as follows.

Li and Zhao [25].

$$\log(M_k(n!)) = n \sum_{i=1}^{k-1} \frac{1}{i} + O\left(n \exp\left(\frac{-c_1 \log^{\frac{3}{5}} n}{(\log\log n)^{\frac{1}{5}}}\right)\right).$$

Chen [1]. *Define* $M_k'(n) = p_1^{\beta_1} p_2^{\beta_2} \cdots p_s^{\beta_s}$*, where* $\beta_i = \alpha_i$ *if* $\alpha_i \le k-1$*, and* $\beta_i = 0$ *if* $\alpha_i \ge k$*. For any real number* $s > 1$*, we have the identity*

$$\sum_{\substack{n=1 \\ \delta_m(n) = M_k'(n)}}^{\infty} \frac{1}{n^s} = \frac{\zeta(s)}{\zeta(ks)} \prod_{p|m} \frac{p^{ks} - p^{(k-1)s} + 1}{p^{ks} - 1}.$$

In addition, Zhang [55] studied the mean value of $M_3(n)$ and $S_k(n)$, he obtained the following asymptotic formulas.

$$\sum_{n \le x} M_3(n) S_k(n) = \frac{6x^{k+1}}{(k+1)\pi^2} C_1(k) + O\left(x^{k+\frac{1}{2}+\varepsilon}\right),$$

$$\sum_{n \le x} \phi(M_3(n) S_k(n)) = \frac{6x^{k+1}}{(k+1)\pi^2} C_2(k) + O\left(x^{k+\frac{1}{2}+\varepsilon}\right),$$

$$\sum_{n \le x} d(M_3(n) S_k(n)) = \frac{6x}{\pi^2} c_1 f(\log x) + O\left(x^{\frac{1}{2}+\varepsilon}\right),$$

where $C_1(k)$ and $C_2(k)$ are constants depend on k, and $f(y)$ is a polynomial with degree k.

§6.2 k-th power part residue function

Similar to the k-th power complements function, the k-th power part residue function $G_k(n)$ is defined as follows,

$$G_k(n) = \min\left\{\frac{n}{u^k} : u^k \mid n,\ u \in \mathbb{N}\right\}.$$

Many papers have been written on the relations between $G_k(n)$ and the arithmetical function $\delta_m(n)$. In particular, Liu and Gao [30] proved

$$\sum_{n\le x} \delta_m(G_k(n)) = \frac{x^2\zeta(2k)}{2\zeta(k)} \prod_{p|m} \frac{p^k+1}{p^{k-1}(p+1)} + O\left(x^{\frac{3}{2}+\varepsilon}\right).$$

Zhao and Ren [59] obtained the identity

$$\sum_{\substack{n=1\\ G_k(n)=\delta_m(n)}}^{\infty} \frac{1}{n^s} = \frac{\zeta(s)}{\zeta(ks)} \prod_{p|m} \frac{1-\frac{1}{p^s}}{\left(1-\frac{1}{p^{sk}}\right)^2},$$

where $s > 1$ is an real number.

§6.3 Additive k-th power part residue function

Similar to the additive k-th power complements function, the additive k-th power part residue function $F_k(n)$ is defined as follows,

$$F_k(n) = \min\left\{m : m \ge 0,\ m = n - u^k,\ u \in \mathbb{N}\right\}.$$

The conclusions related to $F_k(n)$ are as follows.

Zhang [58]

$$\sum_{n\le x} F_k(n) = \frac{k^2}{2(2k-1)} x^{2-\frac{1}{k}} + O\left(x^{2-\frac{2}{k}}\right).$$

$$\sum_{n\le x} d(F_k(n)) = \left(1-\frac{1}{k}\right) x\log x + \left(2\gamma + \log k - 2 + \frac{1}{k}\right) x + O\left(x^{1-\frac{1}{k}}\log x\right).$$

Yang and Fu [49].

$$\sum_{n\le x} \delta_m(F_k(n)) = \frac{k^2}{2(2k-1)} \prod_{p|m} \frac{p}{p+1} x^{2-\frac{1}{k}} + O\left(x^{2-\frac{2}{k}}\right).$$

Ma [36]. *Let p be an odd prime, and let $e_p(n)$ denote the largest exponent of power p which divide n. We have*

$$\sum_{n\le x} e_p(F_k(n)) = \frac{1}{p-1} x + O\left(\frac{k}{p-1} x^{1-\frac{1}{k}}\right).$$

$$\sum_{n\leq x}\frac{1}{F_k(n)+1}=\frac{k-1}{k}x^{\frac{1}{k}}\log k+(\log x+\gamma-k+1)x^{\frac{1}{k}}+O(\log x).$$

Li [22]. *Let $r \geq 2$ be an integer, we have*

$$\sum_{\substack{n\leq x\\ F_k(n)\text{ is } r\text{-free}}} 1=\frac{x}{\zeta(r)}+O_{r,k}\left(x^{\frac{1}{k}+\frac{1}{r}-\frac{1}{rk}}\right),$$

where the implied constants defend on r and k. Furthermore, assuming the Riemann Hypothesis, there holds

$$\sum_{\substack{n\leq x\\ F_2(n)\text{ is square-free}}} 1=\frac{6}{\pi^2}x+O\left(x^{\frac{29}{44}+\varepsilon}\right).$$

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A survey on Smarandache notions in number theory: the Smarandache digit sum function

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Abstract In this paper we give a survey on recent results on the Smarandache digit sum function.
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§1. The Smarandache digit sum function

In this part, we will study the distribution properties of the sequences of digital function. First, we will consider a special case, when base 10. Many scholars gave exact calculating formulae for the mean value of digital function.

Definition 1. *For any positive integer n, let $d_s(n)$ denotes the sum of the base 10 digits of n. That is,*

$$n = a_1 10^{k_1} + a_2 10^{k_2} + ... + a_s 10^{k_s},$$
$$d_s(n) = a_1 + a_2 + ... + a_s.$$

X. Pan and X. Guo [7]. *For any positive integer N, let $N = \frac{10^n+2}{3}$. Then for $n = 3k + i\,(i = 0, 1, 2)$, we have the calculating formulas*

$$d_s(N^3) = 9 \cdot (4k + i) + 1.$$

For natural number $x \geq 2$ and arbitrary fixed exponent $m \in \mathbb{N}$, let

$$A_m(x) = \sum_{n<x} d_s^m(n).$$

W. Zhang [14]. *For any positive integer x, let $x = a_1 10^{k_1} + a_2 10^{k_2} + ... + a_s 10^{k_s}$ with $k_1 > k_2 > ... > k_s \geq 0$ and $1 \leq a_i \leq 9$, $i = 2, 3, ..., s$. Then we have the calculating formulas*

$$A_1(x) = \sum_{i=1}^{s} a_i \cdot \left(\frac{9}{2}k_i + \sum_{j=1}^{i} a_j - \frac{a_i+1}{2}\right) \cdot 10^{k_i};$$

$$\begin{aligned} A_2(x) &= \sum_{i=1}^{s} a_i \cdot \left[\frac{k_i(81k_i+33)}{4} + \frac{9k_i}{2}(a_i-1) + \sum_{j=1}^{i} a_j^2 - \frac{(4a_i-1)(a_i+1)}{6} \right] \cdot 10^{k_i} \\ &+ \sum_{i=2}^{s} a_i \cdot \left[(9k_i - a_i - 1)10^{k_i} + 2\sum_{j=i}^{s} a_j 10^{k_j} \right] \cdot \left(\sum_{j=1}^{i-1} a_j \right). \end{aligned}$$

For general integer $m \geq 3$, using the methods we can also give an exact calculating formula for $A_m(x)$. But in these cases, the computations are more complex.

Definition 2. *For any positive integer n, let $b(n)$ denotes the product of base 10 digits of n. That is,*

$$n = a_1 10^{k_1} + a_2 10^{k_2} + ... + a_s 10^{k_s},$$
$$b(n) = a_1 \cdot a_2 \cdot ... \cdot a_s.$$

For natural number $x \geq 2$ and completely multiplicative function $f(n)$, let

$$A(n) = \sum_{n<x} f(b(n)).$$

J. Gao and H. Liu [3]. *For any positive integer x, let $x = a_s 10^s + a_{s-1}10^{s-1} + ... + a_1 10 + a_0$, where $1 \leq a_s \leq 9$, $0 \leq a_i \leq 9$, $i = 0, 1, ..., s-1$. Then we have the indentity*

$$A(x) = \sum_{i=0}^{s} f\left(\prod_{j=i+1}^{s} a_j \right) \cdot F(a_i) \cdot F^i(10) + \frac{F^{s+1}(10) - F(10)}{F(10) - 1},$$

where $F(N) = \sum_{n<N} f(n)$.

Definition 3. *For any positive integer n, let $a(n)$ denotes the product of all non-zero digits in base 10 of n. For natural number $x \geq 2$ and arbitrary fixed exponent $m \in \mathbb{N}$, let*

$$A_m(x) = \sum_{n<x} a^m(n).$$

W. Zhang [13]. *For any positive integer x, let $x = a_1 10^{k_1} + a_2 10^{k_2} + ... + a_s 10^{k_s}$ with $k_1 > k_2 > ... > k_s \geq 0$ and $1 \leq a_i \leq 9$, $i = 2, 3, ..., s$. Then we have the calculating formulas*

$$\begin{aligned} A_1(x) &= \frac{a_1 a_2 ... a_s}{2} \cdot \sum_{i=1}^{s} \frac{a_i^2 - a_i + 2}{\prod_{j=i}^{s} a_j} \left(45 + \left[\frac{1}{k_i+1} \right] \right) \cdot 46^{k_i - 1}; \\ A_2(x) &= \frac{a_1^2 a_2^2 ... a_s^2}{6} \cdot \sum_{i=1}^{s} \frac{2a_i^3 - 3a_i^2 + a_i + 6}{\prod_{j=i}^{s} a_j^2} \left(285 + \left[\frac{1}{k_i+1} \right] \right) \cdot 286^{k_i - 1}, \end{aligned}$$

where $[x]$ denotes the greatest integer not exceeding x.

For general integer $m \geq 3$, using the methods we can also give an exact calculating formula for $A_m(x)$. That is, we have the calculating formula

$$A_m(x) = a_1^m a_2^m ... a_s^m \cdot \sum_{i=1}^{s} \frac{1 + B_m(a_i)}{\prod_{j=i}^{s} a_j^m} \left(\left[\frac{1}{k_i+1} \right] + B_m(10) \right) \cdot (1 + B_m(10))^{k_i - 1},$$

where $B_m(N) = \sum_{1\le n<N} n^m$.

Next, we consider the general case, when base n. H. Li, Q. Yang and J. Zhao gave many calculating formulae for the mean value of the Smarandache digit sum function base n.

Definition 4. *Assume* $n(n \ge 2)$ *be a fixed positive integer, for any positive integer* m *in base* n, *let* $m = a_1 n^{k_1} + a_2 n^{k_2} + ... + a_s n^{k_s}$ *where* $k_1 > k_2 > ... > k_s \ge 0$ *and* $1 \le a_i \le n$, $i = 1, 2, ..., s$. *Let*

$$a(m, n) = a_1 + a_2 + ... + a_s.$$

For any positive integer r, *let*

$$A_r(N, n) = \sum_{m<N} a^r(m, n).$$

H. Li and Q. Yang [5]. *Let* $N = a_1 n^{k_1} + a_2 n^{k_2} + ... + a_s n^{k_s}$ *where* $k_1 > k_2 > ... > k_s \ge 0$ *and* $1 \le a_i \le n$, $i = 1, 2, ..., s$. *Then*

$$A_1(N, n) = \sum_{i=1}^{s} \left(\frac{n-1}{2} k_i + \sum_{j=1}^{i} a_j - \frac{a_{i+1}}{2} \right) a_i n^{k_i}$$

For convenience, let

$$\phi_k(n) = \sum_{i=1}^{n-1} i^k, \quad \phi_1(n) = \frac{n(n-1)}{2}, \quad \phi_2(n) = \frac{n(n-1)(2n-1)}{6}.$$

Q. Yang and H. Li [9]. *Let* $N = a_1 n^{k_1} + a_2 n^{k_2} + ... + a_s n^{k_s}$ *where* $k_1 > k_2 > ... > k_s \ge 0$ *and* $1 \le a_i \le n$, $i = 1, 2, ..., s$. *Then*

$$\begin{aligned} A_2(N, n) &= \sum_{i=1}^{s} \{ a_i k_i \phi_2(n) + n\phi_2(a_i) + (n-1)\phi_1(k_i)\phi_1(n) + 2k_i\phi_1(a_i)\phi_1(n) \\ &\quad + 2a_i k_i \phi_1(n) + n\phi_1(a_i) \cdot \sum_{j=1}^{i-1} a_j + n a_i (\sum_{j=1}^{i-1} a_j)^2 \} n^{k_i}. \end{aligned}$$

H. Li [4]. *Let* $N = a_1 n^{k_1} + a_2 n^{k_2} + ... + a_s n^{k_s}$ *where* $k_1 > k_2 > ... > k_s \ge 0$ *and* $1 \le a_i \le n$, $i = 1, 2, ..., s$. *Then*

$$\begin{aligned} A_3(N, n) &= \sum_{i=1}^{s} (a_i k_i \phi_1^2(n)((2n-1) + \frac{1}{2}(n-1)(k_i - 3)k_i) + 3\phi_2(n)(2a_i\phi_1(n)\phi_1(k_i) + nk_i\phi_1(a_i)) \\ &\quad + 3n\phi_1(n)((n-1)\phi_1(a_i)\phi_1(k_i) + k_i\phi_2(a_i)) + n^2\phi_1^2(a_i) \\ &\quad + 3n(\sum_{j=1}^{i-1} a_j)(k_i a_i \phi_2(n) + n\phi_2(a_i) + (n-1)a_i\phi_1(n)\phi_1(k_i) + 2k_i\phi_1(a_i)\phi_1(n)) \\ &\quad + \frac{3}{2} n^2 a_i (\sum_{j=1}^{i-1} a_j)^2 ((n-1)k_i + (a_i - 1)) + n^2 a_i (\sum_{j=1}^{i-1} a_j)^3) n^{k_i - 2}. \end{aligned}$$

Definition 5. *Assume $n (n \geq 2)$ be a fixed positive integer, for any positive integer m in base n, let $m = a_1 n^{k_1} + a_2 n^{k_2} + ... + a_s n^{k_s}$ where $k_1 > k_2 > ... > k_s \geq 0$ and $1 \leq a_i \leq n$, $i = 1, 2, ..., s$. Let*

$$a(m, n) = \frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_s}.$$

For any positive integer r, let

$$A_r(N, n) = \sum_{m<N} a^r(m, n).$$

For convenience, let

$$\phi_r(n) = \sum_{i=1}^{n-1} \frac{1}{i^r}$$

J. Zhao [10]. *Let $N = a_1 n^{k_1} + a_2 n^{k_2} + ... + a_s n^{k_s}$ where $k_1 > k_2 > ... > k_s \geq 0$ and $1 \leq a_i \leq n$, $i = 1, 2, ..., s$. Then*

$$\begin{aligned} A_2(N, n) \quad = \quad & \sum_{i=1}^{s} \{[2k_i a_i \phi_1(n)(\phi_1(k_i)\phi_1(n) + \sum_{j=1}^{i-1} \frac{1}{a_j}) - \phi_1(k_i)a_i]/n + k_i a_i \phi_2(n) \\ & + [2k_i \phi_2(n) - 1]\phi_1(a_i) + n\phi_2(a_i) + a_i(\sum_{j=1}^{i-1} \frac{1}{a_j})^2\} \cdot n^{k_i - 1}. \end{aligned}$$

Definition 6. *Assume $n (n \geq 2)$ be a fixed positive integer, for any positive integer m in base n, let $m = a_1 n^{k_1} + a_2 n^{k_2} + ... + a_s n^{k_s}$ where $k_1 > k_2 > ... > k_s \geq 0$ and $1 \leq a_i \leq n$, $i = 1, 2, ..., s$. Let*

$$a(m, n) = \frac{1}{a_1^2} + \frac{1}{a_2^2} + ... + \frac{1}{a_s^2}.$$

$$A(N, n) = \sum_{m<N} a(m, n).$$

and

$$\phi_r(n) = \sum_{i=1}^{n-1} \frac{1}{i^r}$$

J. Zhao [11]. *Let $N = a_1 n^1 + a_2 n^2 + ... + a_s n^s$ where $k_1 > k_2 > ... > k_s \geq 0$ and $1 \leq a_i \leq n$, $i = 1, 2, ..., s$. Then*

$$A(N, n) = \sum_{i=1}^{s} \left(k_i a_i \phi_2(n) + n\phi_2 \left(\frac{1}{a_i} \right) + n a_i \sum_{j=1}^{i-1} \frac{1}{a_j^2} \right) \cdot n^{k_i - 1}.$$

§2. The Smarandache digit sum function based on special sequences

Next, we use a particular base, we refer to it as special squences, W. Zhang combined the base with the Lucas sequence $\{L_n\}$ and the Fibonacci sequence $\{F_n\}$, B. Liu combined the base with the F.Smarandache deconstructive sequence $\{a_n\}$. They gave good results.

Definition 7. *The Lucas sequence* $\{L_n\}$ *and the Fibonacci sequence* $\{F_n\}$ $(n = 0, 1, 2, ...)$ *are defined by the second-order linear recurrence sequences*

$$L_{n+2} = L_{n+1} + L_n \qquad \text{and} \qquad F_{n+2} = F_{n+1} + F_n$$

for $n \geq 0, L_0 = 2, L_1 = 1, F_0 = 0$ *and* $F_1 = 1$.

Then we introduce a new counting function $a(m)$ related to the Lucas numbers. By Professor F.Smarandache's research on the Smarandache's generalized base, we take the base as the Lucas sequence, then

Definition 8. *For any positive integer* m *may be uniquely written in the Smarandache Lucas base as:*

$$m = \sum_{i=1}^{n} a_i L_i, \text{ with all } a_i = 0 \text{ or } 1,$$

We define the counting function $a(m) = a_1 + a_2 + ... + a_n$. *For natural number* N, *let*

$$A_r(N) = \sum_{n<N} a^r(n), \ r = 1, 2.$$

W. Zhang [12]. *1. For any positive integer* k, *we have the calculating formulae*

$$A_1(L_k) = \sum_{n<L_k} a(n) = kF_{k-1}$$

and

$$A_2(L_k) = \frac{1}{5}[(k-1)(k-2)L_{k-2} + 5(k-1)F_{k-2} + 7(k-1)F_{k-3} + 3F_{k-1}].$$

2. For any positive integer N, *let* $N = L_{k_1} + L_{k_2} + ... + L_{k_s}$ *with* $k_1 > k_2 > ... > k_s$ *under the Smarandache Lucas base. Then we have*

$$A_1(N) = A_1(L_{k_1}) + N - L_{k_1} + A_1(N - L_{k_1})$$

and

$$A_2(N) = A_2(L_{k_1}) + N - L_{k_1} + A_2(N - L_{k_1}) + 2A_1(N - L_{k_1}).$$

Further,

$$A_1(N) = \sum_{i=1}^{s} [k_i F_{k_i - 1} + (i-1)L_{k_i}].$$

Definition 9. *F. Smarandache deconstructive sequence is defined as*

$$\{a_n\} = \{1, 23, 456, 7891, 23456, 789123, ...\}.$$

B. Liu [6]. *1. Let $\{a_n\}$ be F. Smarandache deconstructive sequence and $S(n)$ denotes the sum of the base 10 digits of n, then for any real number $x > 1$, we have*

$$\sum_{a_n \leq x} S(a_n) = \frac{5}{2} \cdot \left[\frac{\ln x}{\ln 10}\right]^2 + \frac{15}{2} \cdot \left[\frac{\ln x}{\ln 10}\right] + O(1),$$

where $[x]$ denotes the greatest integer not exceeding x.

2. Let $\{a_n\}$ be F. Smarandache deconstructive sequence, then for any real number $x > 1$, we have

$$\sum_{n \leq x} S(a_n) = \frac{5}{2} \cdot x^2 + \frac{5}{2} \cdot x + O(1).$$

Obviously, his results can be generalized. Let $S_k(n)$ denote the k-th power sum of the base 10 digits of n. That is,

$$n = a_1 10^{k_1} + a_2 10^{k_2} + ... + a_s 10^{k_s},$$
$$S_k(n) = a_1^k + a_2^k + ... + a_s^k.$$

Then B. Liu generalized the results to k times.

3. Let $\{a_n\}$ be F. Smarandache deconstructive sequence and k be any fixed positive integer, then for any real number $x > 1$, we have

$$\sum_{a_n \leq x} S_k(a_n) = \frac{c(k)}{18} \cdot \left[\frac{\ln x}{\ln 10}\right]^2 + \frac{c(k)}{6} \cdot \left[\frac{\ln x}{\ln 10}\right] + O(1),$$

where $c(k) = 1^k + 2^k + 3^k + 4^k + 5^k + 6^k + 7^k + 8^k + 9^k$ is a computable constant.

4. Let $\{a_n\}$ be F. Smarandache deconstructive sequence and k be any fixed positive integer, then for any real number $x > 1$, we have

$$\sum_{n \leq x} S_k(a_n) = \frac{c(k)}{18} \cdot x^2 + \frac{c(k)}{6} \cdot x + O(1).$$

§3. The Smarandache digit sum function in finite fields

Finally, we consider the Smarandache digit sum function in finite fields and Swaenepoel, Dartyge, Mauduit and Sárközy gave some interesting results.

Definition 10. *Let p be a prime number, $q = p^r$ with $r \geq 2$, and consider the field F_q. Let $\mathcal{B} = \{a_1, a_2, ..., a_r\}$ be a basis of the linear vector space formed by F_q over F_p, i.e.. Then every $x \in F_q$ has a unique representation*

$$x = \sum_{j=1}^{r} c_j a_j$$

with $c_j \in F_p$. The sum of digits function is defined as

$$S_{\mathcal{B}}(x) = \sum_{j=1}^{r} c_j.$$

Dartyge and Sárközy [2]. *1. Let $c \in F_p$. We define Q_c as the set of the squares of F_q such that their sum of digits is equal to c:*

$$Q_c = \{x \in F_q : S_{\mathcal{B}}(x) = c \text{ and } \exists\, y \in F_q \text{ such that } y^2 = x\}.$$

Then we have:

$$\left| |Q_c| - \frac{p^{r-1}}{2} \right| \leq \sqrt{q}.$$

2. Let $f \in F_q[X]$ be of degree n with $(n,q) = 1$. For all $c \in F_p$, We define the sets:

$$D(f,c) = \{x \in F_q : S_{\mathcal{B}}(f(x)) = c\}.$$

Then we have:

$$\big| |D(f,c)| - p^{r-1} \big| \leq (n-1)\sqrt{q}.$$

3. We denote $\mathcal{G}$ as the set of the generators (or primitive elements) of F_q^*. Let $f \in F_q[X]$ be of degree n with $(n,q) = 1$ and for $c \in F_p$ we consider the sets

$$G(f,c) = \{g \in \mathcal{G} : S_{\mathcal{B}}(f(g)) = c\}.$$

Then we have:

$$\left| |G(f,c)| - \frac{\phi(q-1)}{p} \right| \leq (n-1)\tau(q-1)\sqrt{q}.$$

where $\tau(n)$ denotes the divisor function.

Definition 11. *Let p be a prime number, $q = p^r$ with $r \geq 2$, and let $\mathcal{B} = \{a_1, a_2, ..., a_r\}$ be a basis of F_q over F_p. For $1 \leq j \leq r$, we define the j-th digit function ϵ_j on F_q by*

$$\epsilon_j \left(\sum_{j=1}^{r} c_j a_j \right) = c_j$$

with $c_j \in F_p$.

Swaenepoel [8]. *1. For $P \in F_q[X]$ is a polynomial of degree $n \geq 1$ with $(n,k) = 1$, for $1 \leq k \leq r$, for $J \subset \{1, ..., r\}$ with $|J| = k$ and for $\boldsymbol{\alpha} = (\alpha_j)_{j \in J}$. We define the sets:*

$$\mathcal{F}_q(P,k,J,\boldsymbol{\alpha}) = \{x \in F_q : \epsilon_j P((x)) = \boldsymbol{\alpha}_j \text{ for all } j \in J\}.$$

Then we have:

$$\left| |\mathcal{F}_q(P,k,J,\boldsymbol{\alpha})| - \frac{q}{p^k} \right| \leq \frac{p^k - 1}{p^k}(n-1)\sqrt{q},$$

in particular, if

$$(n-1)(p^k - 1) < \sqrt{q} = p^{\frac{r}{2}}$$

then $\mathcal{F}_q(P,k,J,\boldsymbol{\alpha}) \neq \emptyset$.

2. If $p \geq 3$ *then, for any* $a \in F_q^*$, *for* $1 \leq k \leq r$, *for* $J \subset \{1,...,r\}$ *with* $|J| = k$ *and for* $\boldsymbol{\alpha} = (\alpha_j)_{j\in J}$. *We have*

$$\left| |\mathcal{F}_q(aX^2,k,J,\boldsymbol{\alpha})| - \frac{q}{p^k} \right| \leq \begin{cases} \frac{\sqrt{q}}{\sqrt{p}}, & \text{if } \boldsymbol{\alpha} \neq 0 \text{ and } r \text{ is odd,} \\ \left(\frac{2}{p} - \frac{1}{p^k}\right)\sqrt{q}, & \text{if } \boldsymbol{\alpha} \neq 0 \text{ and } r \text{ is even,} \\ 0, & \text{if } \boldsymbol{\alpha} = 0 \text{ and } r \text{ is odd,} \\ \frac{p^k-1}{p^k}\sqrt{q}, & \text{if } \boldsymbol{\alpha} = 0 \text{ and } r \text{ is even.} \end{cases}$$

3. We denote $\mathcal{G}$ *as the set of the generators (or primitive elements) of* F_q^*. *For* $P \in F_q[X]$ *is a polynomial of degree* $n \geq 1$ *with* $(n,k) = 1$, *for* $1 \leq k \leq r$, *for* $J \subset \{1,...,r\}$ *with* $|J| = k$ *and for* $\boldsymbol{\alpha} = (\alpha_j)_{j\in J}$. *We define the sets:*

$$\mathcal{G} \cap \mathcal{F}_q(P,k,J,\boldsymbol{\alpha}) = \{g \in \mathcal{G} : \epsilon_j P((g)) = \boldsymbol{\alpha}_j \text{ for all } j \in J\}.$$

Then we have:

$$\left| |\mathcal{G} \cap \mathcal{F}_q(P,k,J,\boldsymbol{\alpha})| - \frac{\phi(q-1)}{p^k} \right| \leq \frac{p^k-1}{p^k}\frac{\phi(q-1)}{q-1}\left((n2^{\omega(q-1)}-1)\sqrt{q}+1\right),$$

where $\omega(m)$ *denotes the number of distinct prime factors of* m.

In particular, if

$$n(p^k-1) < \sqrt{q}/2^{\omega(q-1)}$$

then $\mathcal{G} \cap \mathcal{F}_q(P,k,J,\boldsymbol{\alpha}) \neq \emptyset$.

Definition 12. *Let* p *be a prime number,* $q = p^r$ *with* $r \geq 2$, *and let* $\mathcal{B} = \{a_1, a_2, ..., a_r\}$ *be a basis of* F_q *over* F_p. *Let us fix a set* $\mathcal{D} \subset \{0,1,2,...,p-1\}$ *with* $2 \leq |\mathcal{D}| \leq p-1$. *We define the set:*

$$W_{\mathcal{D}} = \{x = \sum_{j=1}^{r} c_j a_j \text{ with } (c_1,...,c_r) \in \mathcal{D}^r\}.$$

For convenience, we will use the notation

$$C(p,t) = \begin{cases} \frac{\log p}{t} + \frac{1}{t}\left(\frac{4}{3} - \frac{\log 3}{2}\right) + \frac{1}{p}, & \text{if } 2 \leq t < p-1, \\ \frac{2}{p} + \frac{2}{\pi(p-1)}\left(1 - \log(2\sin\frac{\pi}{2p})\right), & \text{if } t = p-1. \end{cases}$$

Dartyge, Mauduit and Sárközy [1]. *1. We denote* Q *as the set of the squares of* F_q. *Let* $\mathcal{D} \subset \mathbb{F}_p$ *with* $2 \leq |\mathcal{D}| \leq p-1$. *Then we have:*

$$\left| |W_{\mathcal{D}} \cap Q| - \frac{|W_{\mathcal{D}}|}{2} \right| \leq \frac{1}{2\sqrt{q}}\left(|\mathcal{D}| + p\sqrt{p-|\mathcal{D}|}\right)^r.$$

2. We suppose that $\mathcal{D} = \{0,1,...,t\}$ *with* $2 \leq t \leq p-1$. *Then we have:*

$$\left| |W_{\mathcal{D}} \cap Q| - \frac{|W_{\mathcal{D}}|}{2} \right| \leq \frac{1}{2}\left(C(p,t)t\sqrt{p}\right)^r.$$

3. Let $\mathcal{D} \subset \mathbb{F}_p$ with $2 \leq |\mathcal{D}| \leq p-1$ and $f(x) \in \mathbb{Z}[x]$ with degree $n \geq 2$. Then we have:

$$||W_{\mathcal{D}}(f)| - |W_{\mathcal{D}}|| \leq \frac{n-1}{\sqrt{q}} \left(|\mathcal{D}| + p\sqrt{p - |\mathcal{D}|}\right)^r.$$

4. We suppose that $\mathcal{D} = \{0, 1, ..., t\}$ with $2 \leq t \leq p-1$ and $f(x) \in \mathbb{Z}[x]$ with degree $n \geq 2$. Then we have:

$$||W_{\mathcal{D}}(f)| - |W_{\mathcal{D}}|| \leq (n-1) \left(C(p,t)t\sqrt{p}\right)^r.$$

We denote $\mathcal{G}$ as the set of the generators (or primitive elements) of F_q^*. For $f(x) \in F_q[X]$ we define the sets:

$$W_{\mathcal{D}}(f, \mathcal{G}) = \{g \in \mathcal{G} : f(g) \in W_{\mathcal{D}}\}.$$

5. Let $\mathcal{D} \subset \mathbb{F}_p$ with $2 \leq |\mathcal{D}| \leq p-1$ and $f(x) \in \mathbb{Z}[x]$ with degree $n \geq 2$. Then we have:

$$\left| |W_{\mathcal{D}}(f, \mathcal{G})| - |\mathcal{D}|^r \cdot \frac{\phi(q-1)}{q} \right| \leq \left(\frac{1}{q} + \frac{(n-1)\tau(q-1)}{\sqrt{q}}\right) \cdot \left(|\mathcal{D}| + p\sqrt{p - |\mathcal{D}|}\right)^r.$$

6. We suppose that $\mathcal{D} = \{0, 1, ..., t\}$ with $2 \leq t \leq p-1$ and $f(x) \in \mathbb{Z}[x]$ with degree $n \geq 2$. Then we have:

$$\left| |W_{\mathcal{D}}(f, \mathcal{G})| - |\mathcal{D}|^r \cdot \frac{\phi(q-1)}{q} \right| \leq (1 + (n-1)\tau(q-1)) \cdot \left(C(p,t)t\sqrt{p}\right)^r.$$

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Soft b-locally open sets in soft bitopological spaces

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Abstract In this paper we introduce the notion of soft b-locally open sets, soft bLO^*-sets, soft bLO^{**}-sets in soft bitopological spaces and obtain several characterizations and some properties of these sets.

Keywords Soft bitopological spaces, soft b-open sets, soft b-closed sets, soft b-locally open (closed) sets, soft bLO^*-sets, bLO^{**}-sets.
2010 Mathematics Subject Classification 54A05, 54A10, 54C05, 54C08, 54C10.

§1. Introduction

None mathematical tools can successfully deal with the several kinds of uncertainties in complicated problems in engineering, economics, environment, sociology, medical science, etc, so Molodtsov [18] introduced the concept of a soft set in order to solve these problems in 1999. However, there are some theories such as theory of probability, theory of fuzzy sets [26], theory of intuitionistic fuzzy sets [4], theory of vague sets [10], theory of interval mathematics [11] and the theory of rough sets [20], which can be taken into account as mathematical tools for dealing with uncertainties. But these theories have their own difficulties. Maji et al. [16] introduced a few operators for soft set theory and made a more detailed theoretical study of the soft set theory. Recently, study on the soft set theory and its applications in different fields has been making progress rapidly [9, 22, 25]. Shabir and Naz [24] introduced the concept of soft topological spaces which are defined over an inital universe with fixed set of parameter. Later, Zorlutuna et al. [27], Aygunoglu and Aygun [5] and Hussain et al [13] are continued to study the properties of soft topological space. They got many important results in soft topological spaces. Weak forms of soft open sets were first studied by Chen [8]. He investigated soft semi-open sets in soft topological spaces and studied some properties of it. Arockiarani and Arokialancy [3] are defined soft β-open sets and continued to study weak forms of soft open sets in soft topological space. Later, Akdag and Ozkan [1, 2] defined soft α-open (resp. soft b-open) (soft α-closed (resp. soft b-closed)) sets.

The concept of bitopological spaces was introduced by Kelly [14]. A non-empty set X, equipped with two topologies τ_1 and τ_2 is called a bitopological space, denoted by (X, τ_1, τ_2). Later on several authors were attracted by the notion of bitopological space. Many notions of topological spaces were studied on considering bitopological space. In 2014, Ittanagi [6] introduced the concept of soft bitopological space subsequently Guzide Senel and Naim Cagman [12] introduced the concept of soft bitopological space. The notion of locally closed set in a topological space was introduced by Kuratowski and Sierpienski [15]. It is also found in Bourbaki [7]. In 2001, Nasef [19] have introduced and studies b-locally closed sets in topological space.

In this paper we introduce the notion of soft b-locally open sets, soft bLO^*-sets, soft bLO^{**}-sets in soft bitopological spaces and obtain several characterizations and some properties of these sets.

§2. Preliminaries

In this section, we recall some definition and concepts discussed in [13,17,24,27]. Throughout this study X and Y denote universal sets, E denote the set of parameters, $A, B, C, D, K, T \subseteq E$. Let X be an initial universe and E be a set of parameters. Let $\mathbb{P}(X)$ denote the power set of X and A be a nonempty subset set of E. A pair (F, A) is called a soft set over X, where F is a mapping given by $F : A \to \mathbb{P}(X)$. For two soft sets (F, A) and (G, B) over common universe X, we say that (F, A) is a soft subset (G, B) if $A \subseteq B$ and $F(e) \subseteq G(e)$, for all $e \in A$. In this case, we write $(F, A)\widetilde{\subseteq}(G, B)$ and (G, B) is said to be a soft super set of (F, A). Two soft sets (F, A) and (G, B) over a common universe X are said to be soft equal if $(F, A)\widetilde{\subseteq}(G, B)$ and $(G, B)\widetilde{\subseteq}(F, A)$. The soft set (F, A) over X such that $F(e) = \{x\}\ \forall\ e \in E$ is called singleton soft point and denoted by x_E or (x, E). A soft set (F, A) over X is called null soft set, denoted by $\tilde{\Phi}$, if for each $e \in A$, $F(e) = \Phi$. Similarly, it is called absolute soft set, denoted by $\widetilde{X}$, if for each $e \in A$, $F(e) = X$.

The union of two soft sets (F, A) and (G, B) over the common universe X is the soft set (H, C), where $C = A \cup B$ and for each $e \in C$,

$$H(e) = \begin{cases} F(e) & e \in A - B \\ G(e) & e \in B - A \\ F(e) \cup G(e) & e \in A \cap B \end{cases}$$

We write $(F, A) \cup (G, B) = (H, C)$. Moreover, the intersection (H, C) of two soft sets (F, A) and (G, B) over a common universe X, denoted by $(F, A) \cap (G, B)$, is defined as $C = A \cap B$ and $H(e) = F(e) \cap G(e)$ for each $e \in C$. The difference (H, E) of two soft sets (F, E) and (G, E) over X, denoted by $(F, E) - (G, E)$, is defined as $H(e) = F(e) - G(e)$, for each $e \in E$. Let Y be nonempty subset of X. Then $\widetilde{Y}$ denotes the soft set (Y, E) over X where $Y(e) = Y$ for each $e \in E$. In particular, (X, E) will be denoted by $\widetilde{X}$. Let (F, E) be a soft set over X and $x \in X$. We say that $x \in (F, E)$, whenever $x \in F(e)$, for each $e \in E$ [21].

The relative complement of a soft set (F, A) is denoted by $(F, A)'$ and is defined by $(F, A)' = (F, A)$ where $F' : A \to \mathbb{P}(X)$ is defined by following

$$F^{'}(e) = X - F(e), \quad \forall e \in A$$

In this paper, for convenience, let $SS(X, E)$ be the family of soft sets over X with set of parameters E.

Let τ be the collection of soft sets over X. Then τ is called a soft topology [24] on X if τ satisfies the following axioms:

(i) $\tilde{\Phi}$ and $\widetilde{X}$ belongs to τ.

(ii) The union of any number of soft sets in τ belongs to τ.

(iii) The intersection of any two soft sets in τ belongs to τ.

The triple (X, τ, E) is called soft topological space over X. The members of τ are said to be soft open in X, and the soft set (F, E) is called soft closed in X if its relative complement $(F, E)'$ belongs to τ. Let (X, τ, E) be a soft topological space and (F, A) be a soft set over X. Soft closure of a soft set (F, A) in X is denoted by

$scl(F, A) = \widetilde{\bigcap}\{(F, E) \mid (F, E)\widetilde{\supseteq}(F, A),\ (F, E) \text{ is a soft closed set of } X\}$.

Soft interior of a soft set (F, A) in X is denoted by

$sint(F, A) = \widetilde{\bigcup}\{(F, B) \mid (F, B)\widetilde{\subseteq}(F, A),\ (F, B) \text{ is a soft open set of } X\}$.

Definition 2.1. *[2] Let (X, τ, E) be a soft topological space, the soft set (F, A) is said to be soft b-open if $(F, A)\widetilde{\subseteq}scl(sint(F, A))\widetilde{\vee}sint(scl(F, A))$.*

Definition 2.2. *[23] Let $\tilde{X}$ be a nonempty soft set on the universe X, τ_1 and τ_2 be two different soft topologies on $\tilde{X}$. Then $(\tilde{X}, \tau_1, \tau_2, E)$ is called a soft bitopological spaces (for short, sbts).*

§3. Soft b-locally open sets in soft bitopological space

In this section we introduce the notions of soft b-locally open sets (in short, *SbLO*-sets), $SbLO^*$-sets, $SbLO^{**}$-sets in soft bitopological spaces.

Definition 3.1. *A soft set (F, A) of a sbts (X, τ_1, τ_2, E) is called (τ_1, τ_2)-soft locally open (in short, (τ_1, τ_2)-SLO) if $(F, A) = (F, B)\tilde{\cup}(F, C)$ where (F, B) is τ_1-soft closed and (F, C) is τ_2-soft open in (X, τ_1, τ_2, E).*

Definition 3.2. *A soft set (F, A) of a sbts (X, τ_1, τ_2, E) is called (τ_1, τ_2)-soft b-locally open (in short, (τ_1, τ_2)-SbLO) if $(F, A) = (F, B)\tilde{\cup}(F, C)$ where (F, B) is τ_1-soft b-closed and (F, C) is τ_2-soft b-open is (X, τ_1, τ_2, E).*

Definition 3.3. *A soft set (F, A) of a sbts (X, τ_1, τ_2, E) is called (τ_1, τ_2)-$SbLO^*$) if there exist a τ_1-soft b-closed set (F, B) and a τ_2-soft open set (F, C) of (X, τ_1, τ_2, E) such that $(F, A) = (F, B)\tilde{\cup}(F, C)$.*

Definition 3.4. *A soft set (F, A) of a sbts (X, τ_1, τ_2, E) is called (τ_1, τ_2)-$SbLO^{**}$) if there exist a τ_1-soft closed set (F, B) and a τ_2-soft b-open set (F, C) of (X, τ_1, τ_2, E) such that $(F, A) = (F, B)\tilde{\cup}(F, C)$.*

The collection of all (τ_1, τ_2)-*SLO* (respectively (τ_1, τ_2)-*SbLO*, (τ_1, τ_2)-$SbLO^*$, (τ_1, τ_2)-$SbLO^{**}$-sets of (X, τ_1, τ_2, E) will be denoted by (τ_1, τ_2)-$SLO(X)$ (respectively (τ_1, τ_2)-$SbLO(X)$, (τ_1, τ_2)-$SbLO^*(X)$, (τ_1, τ_2)-$SbLO^{**}(X)$).

Theorem 3.5. *Let (F,A) be a soft set of a sbts (X,τ_1,τ_2,E). Then if $(F,A)\tilde{\in}(\tau_1,\tau_2)$-$SLO(X)$, then*

(i) $(F,A)\tilde{\in}(\tau_1,\tau_2)$-$SbLO^*(X)$.

(ii) $(F,A)\tilde{\in}(\tau_1,\tau_2)$-$SbLO^{**}(X)$.

Proof. (i) Since $(F,A)\tilde{\in}(\tau_1,\tau_2)$-$SLO(X)$, so there exist a τ_1-soft closed set (F,B) and a τ_2-soft open set (F,C) such that $(F,A)=(F,B)\tilde{\cup}(F,C)$. Since (F,B) is τ_1-soft closed, we have $sint(scl(F,B))\tilde{\subseteq}(F,B)$ and $scl(sint(F,B)\tilde{\subseteq}(F,B)$.

Therefore $sint(scl(F,B))\tilde{\cap}scl(sint(F,A))\tilde{\subseteq}(F,B)$. Hence (F,B) is τ_1-soft b-closed. Thus we have $(F,A)=(F,B)\tilde{\cup}(F,C)$, where (F,B) is τ_1-soft b-closed and (F,C) is τ_2-soft open. Hence $(F,A)\tilde{\in}(\tau_1,\tau_2)$-$SbLO^*(X)$.

(ii) Let $(F,A)\tilde{\in}(\tau_1,\tau_2)$-$SLO(X)$. Then we have $(F,A)=(F,B)\tilde{\cup}(F,C)$, where (F,B) is τ_1-soft closed and (F,C) is τ_2-soft open. Since (F,C) is τ_2-soft open , we have $(F,C)\tilde{\subseteq}sint(scl(F,C))$ and $(F,C)\tilde{\subseteq}scl(sint(F,C))$. Therefore $(F,C)\tilde{\subseteq}scl(sint(F,C))\tilde{\cup}sint(scl(F,C))$. Hence (F,C) is τ_2-soft b-open. Now we have $(F,A)=(F,B)\tilde{\cup}(F,C)$, where (F,B) is τ_1-soft closed and (F,C) is τ_2-soft b-open. Hence $(F,A)\tilde{\in}(\tau_1,\tau_2)$-$SbLO^{**}(X)$. This completes the proof. □

Remark 3.6. *The converse of Theorem is not necessarily true. It is clear from the following example.*

Example 3.7. *Let $X=\{h_1,h_2,h_3\}$, $E=\{e_1,e_2\}$, $\tau_1=\{\tilde{\Phi},\tilde{X},(F_1,E),(F_2,E),(F_3,E)\}$ and $\tau_2=\{\tilde{\Phi},\tilde{X},(F_2,E)\}$, where (F_1,E), (F_2,E) and (F_3,E) are soft sets over X defined as follows:*

$F_1(e_1)=\{h_1\},\quad F_1(e_2)=\{h_2\}$
$F_2(e_1)=\{h_2\},\quad F_2(e_2)=\{h_3\}$
$F_3(e_1)=\{h_1,h_2\},\quad F_3(e_2)=\{h_2,h_3\}$

Clearly τ_1 and τ_2 are defines a soft topology on X and thus (X,τ_1,τ_2,E) is sbts. The soft set (F_4,E) which defined as follows

$F_4(e_1)=\{h_1\},\quad F_4(e_2)=\{h_1,h_2\}$

is τ_1-soft b-closed set and (F_2,E) is τ_2-soft open set then $(F_4,E)\tilde{\cup}(F_2,E)=(F,E)(=\{F(e_1)=\{h_1,h_2\},\quad F(e_2)=\tilde{X}\})\tilde{\in}(\tau_1,\tau_2)$-$SbLO^*(X)$ but $(F,E)\tilde{\notin}(\tau_1,\tau_2)$-$SLO(X)$.

Example 3.8. *Let $X=\{h_1,h_2,h_3\}$, $E=\{e_1,e_2\}$, $\tau_1=\{\tilde{\Phi},\tilde{X},(F_2,E)\}$ and $\tau_2=\{\tilde{\Phi},\tilde{X},(F_1,E),(F_2,E),(F_3,E)\}$, where (F_1,E), (F_2,E) and (F_3,E) are soft sets over X defined as follows:*

$F_1(e_1)=\{h_1\},\quad F_1(e_2)=\{h_2\}$
$F_2(e_1)=\{h_2\},\quad F_2(e_2)=\{h_3\}$
$F_3(e_1)=\{h_1,h_2\},\quad F_3(e_2)=\{h_2,h_3\}$

Clearly τ_1 and τ_2 are defines a soft topology on X and thus (X,τ_1,τ_2,E) is sbts. The soft set (F_4,E) which defined as follows

$F_4(e_1)=\{h_1\},\quad F_4(e_2)=\{h_1,h_2\}$

is τ_2-soft b-open set and $(F_2,E)'$ is τ_1-soft closed set then $(F_2,E)'\tilde{\cup}(F_4,E)=(F,E)(=\{F(e_1)=\{h_1,h_3\},\quad F(e_2)=\{h_1,h_2\}\})\tilde{\in}(\tau_1,\tau_2)$-$SbLO^{**}(X)$ but $(F,E)\tilde{\notin}(\tau_1,\tau_2)$-$SLO(X)$.

Theorem 3.9. *Let (F,A) be a soft set of the sbts (X,τ_1,τ_2,E). If $(F,A)\tilde{\in}(\tau_1,\tau_2)$-$SbLO^*(X)$, then $(F,A)\tilde{\in}(\tau_1,\tau_2)$-$SbLO(X)$.*

Proof. Let $(F,A)\tilde{\in}(\tau_1,\tau_2)$-$SbLO^*(X)$. Then there exists a τ_1-soft b-closed set (F,B) and a τ_2-soft open set (F,C) such that $(F,A)=(F,B)\tilde{\cup}(F,C)$. Since (F,C) is τ_2-soft open, we have $(F,C)\tilde{\subseteq}sint(scl(F,C))$.

Further we have $(F,C)\tilde{\subseteq}scl(sint(F,C))$. Thus we have $(F,C)\tilde{\subseteq}scl(sint(F,C))\tilde{\cup}sint(scl(F,C))$. Hence (F,C) is a τ_2-soft b-open set. Thus there exist a τ_1-soft b-closed set (F,B) and τ_2-soft b-open set (F,C) such that $(F,A)=(F,B)\tilde{\cup}(F,C)$. Therefore $(F,A)\tilde{\in}(\tau_1,\tau_2)$-$SbLO(X)$. □

Remark 3.10. *The converse of Theorem is not always true. It follows from the following example.*

Example 3.11. *Let $X=\{h_1,h_2,h_3\}$, $E=\{e_1,e_2\}$, $\tau_1=\{\tilde{\Phi},\tilde{X},(F_1,E)\}$ and $\tau_2=\{\tilde{\Phi},\tilde{X},(F_2,E)\}$, where (F_1,E) and (F_2,E) are soft sets over X defined as follows:*

$F_1(e_1)=\{h_3\},\quad F_1(e_2)=\{h_1\}$

$F_2(e_1)=\{h_1\},\quad F_2(e_2)=\{h_3\}$

Clearly τ_1 and τ_2 are defines a soft topology on X and thus (X,τ_1,τ_2,E) is sbts. The soft set (F,E) which defined as follows

$F(e_1)=\{h_2\},\quad F(e_2)=\{h_3\}$

is τ_1-soft b-closed set and τ_2-soft b-open set then $(F,E)\tilde{\in}(\tau_1,\tau_2)$-$SbLO(X)$ but $(F,E)\tilde{\notin}(\tau_1,\tau_2)$-$SbLO^*(X)$.

Theorem 3.12. *Let (F,A) be a soft set of a sbts (X,τ_1,τ_2,E). If $(F,A)\tilde{\in}(\tau_1,\tau_2)$-$SbLO^{**}(X)$, then $(F,A)\tilde{\in}(\tau_1,\tau_2)$-$SbLO(X)$.*

Proof. The proof is easy, so omitted. □

Remark 3.13. *The converse of Theorem is not always true. It follows from the following example.*

Example 3.14. *In Example , the soft set $(F,E)\tilde{\in}(\tau_1,\tau_2)$-$SbLO(X)$ but $(F,E)\tilde{\notin}(\tau_1,\tau_2)$-$SbLO^{**}(X)$.*

Theorem 3.15. *Let (F,A) and (F,B) be any two soft sets of a sbts (X,τ_1,τ_2,E). If $(F,A)\tilde{\in}(\tau_1,\tau_2)$-$SbLO(X)$ and (F,B) is τ_1-soft b-closed and τ_2-soft b-open, then $(F,A)\tilde{\cap}(F,B)\tilde{\in}(\tau_1,\tau_2)$-$SbLO(X)$.*

Proof. Since $(F,A)\tilde{\in}(\tau_1,\tau_2)$-$SbLO(X)$, then there exist a τ_1-soft b-closed set (G,C) and a τ_2-soft b-open set (G,D) such that $(F,A)=(G,C)\tilde{\cup}(G,D)$.

We have $(F,A)\tilde{\cap}(F,B)=((G,C)\tilde{\cup}(G,D))\tilde{\cap}(F,B)=((G,C)\tilde{\cap}(F,B))\tilde{\cup}((G,D)\tilde{\cap}(F,B))$. Since (F,B) is τ_1-soft b-closed, then $(G,C)\tilde{\cap}(F,B)$ is τ_1-soft b-closed. Since (F,B) is τ_2-soft b-open, then $(G,D)\tilde{\cap}(F,B)$ is τ_2-soft b-open. Then there exist a τ_1-soft b-closed set $(G,C)\tilde{\cap}(F,B)$ and a τ_2-soft b-open set $(G,D)\tilde{\cap}(F,B)$ such that $(F,A)\tilde{\cap}(F,B)=((G,C)\tilde{\cap}(F,B))\tilde{\cup}((G,D)\tilde{\cap}(F,B))$. Hence $(F,A)\tilde{\cap}(F,B)\tilde{\in}(\tau_1,\tau_2)$-$SbLO(X)$. □

Theorem 3.16. *Let $(F,A)\tilde{\in}(\tau_1,\tau_2)$-$SbLO^*(X)$ and (F,B) be a τ_1-soft closed and τ_2-soft open sets of (X,τ_1,τ_2,E), then $(F,A)\tilde{\cap}(F,B)\tilde{\in}(\tau_1,\tau_2)$-$SbLO^*(X)$.*

Proof. Since $(F,A)\tilde{\in}(\tau_1,\tau_2)$-$SbLO^*(X)$. Then there exist a τ_1-soft b-closed set (G,C) and a τ_2-soft open set (G,D) such that $(F,A)=(G,C)\tilde{\cup}(G,D)$. We have $(F,A)\tilde{\cap}(F,B)=((G,C)\tilde{\cup}(G,D))\tilde{\cap}(F,B)=((G,C)\tilde{\cap}(F,B))\tilde{\cup}((G,D)\tilde{\cap}(F,B))$. Since (F,B) is τ_1-soft closed,

$(G,C)\tilde{\cap}(F,B)$ is τ_1-soft b-closed set. Further (F,B) is τ_2-soft open, therefore $(G,D)\tilde{\cap}(F,B)$ is τ_2-soft open. Thus there exist a τ_1-soft b-closed set $(G,C)\tilde{\cap}(F,B)$ and a τ_2-soft open set $(G,D)\tilde{\cap}(F,B)$ such that $(F,A)\tilde{\cap}(F,B) = ((G,C)\tilde{\cap}(F,B))\tilde{\cup}((G,D)\tilde{\cap}(F,B))$.

Hence $(F,A)\tilde{\cap}(F,B)\tilde{\in}(\tau_1,\tau_2)$-$SbLO^*(X)$. □

Theorem 3.17. *Let $(F,A)\tilde{\in}(\tau_1,\tau_2)$-$SbLO^{**}(X)$ and (F,B) be a τ_1-soft closed and τ_2-soft open sets of (X,τ_1,τ_2,E), then $(F,A)\tilde{\cap}(F,B)\tilde{\in}(\tau_1,\tau_2)$-$SbLO^{**}(X)$.*

Proof. Since $(F,A)\tilde{\in}(\tau_1,\tau_2)$-$SbLO^{**}(X)$. Then there exist a τ_1-soft closed set (G,C) and a τ_2-soft b-open set (G,D) such that $(F,A) = (G,C)\tilde{\cup}(G,D)$. Clearly $(F,A)\tilde{\cap}(F,B) = ((G,C)\tilde{\cup}(G,D))\tilde{\cap}(F,B) = ((G,C)\tilde{\cap}(F,B))\tilde{\cup}((G,D)\tilde{\cap}(F,B))$. Since (F,B) is τ_1-soft closed, therefore $(G,C)\tilde{\cap}(F,B)$ is τ_1-soft closed set. Again (F,B) is τ_2-soft open, therefore $(G,D)\tilde{\cap}(F,B)$ is τ_2-soft b-open. Then there exist a τ_1-soft closed set $(G,C)\tilde{\cap}(F,B)$ and a τ_2-soft b-open set $(G,D)\tilde{\cap}(F,B)$ such that $(F,A)\tilde{\cap}(F,B) = ((G,C)\tilde{\cap}(F,B))\tilde{\cup}((G,D)\tilde{\cap}(F,B))$.

Hence $(F,A)\tilde{\cap}(F,B)\tilde{\in}(\tau_1,\tau_2)$-$SbLO^{**}(X)$. □

Theorem 3.18. *Let (F,A) be a soft set of a sbts (X,τ_1,τ_2,E). Then $(F,A)\tilde{\in}(\tau_1,\tau_2)$-$SbLO^*(X)$ if and only if $(F,A) = (F,B)\tilde{\cup}\tau_2$-$sint(F,A)$ for some τ_1-soft b-closed set (F,B).*

Proof. Let $(F,A)\tilde{\in}(\tau_1,\tau_2)$-$SbLO^*(X)$. Then $(F,A) = (F,B)\tilde{\cup}(F,C)$, where (F,B) is τ_1-soft b-closed and (F,C) is τ_2-soft open set in (X,τ_1,τ_2,E). Since $(F,B)\tilde{\subseteq}(F,A)$ and τ_2-$sint(F,A)\tilde{\subseteq}(F,A)$. We have

$$(F,B)\tilde{\cup}\tau_2\text{-}sint(F,A)\tilde{\subseteq}(F,A). \tag{1}$$

Further τ_2-$sint(F,A)\tilde{\supseteq}(F,C)$. Therefore

$$(F,B)\tilde{\cup}\tau_2\text{-}sint(F,A)\tilde{\supseteq}(F,B)\tilde{\cup}(F,C) = (F,A). \tag{2}$$

From (1) and (2) we have $(F,A) = (F,B)\tilde{\cup}\tau_2$-$sint(F,A)$.

Conversely, given that (F,B) is τ_1-soft b-closed. We have τ_2-$sint(F,A)$ is τ_2-soft open. Thus there exist a τ_1-soft b-closed set (F,B) and a τ_2-soft open set τ_2-$sint(F,A)$ in (X,τ_1,τ_2,E) such that $(F,A) = (F,B)\tilde{\cup}\tau_2$-$sint(F,A)$.

Hence $(F,A)\tilde{\in}(\tau_1,\tau_2)$-$SbLO^*(X)$. □

Theorem 3.19. *Let (F,A) and (F,B) be any two soft sets of the sbts (X,τ_1,τ_2,E). If $(F,A)\tilde{\in}(\tau_1,\tau_2)$-$SbLO(X)$ and (F,B) is either τ_1-soft b-closed or τ_2-soft b-open, then $(F,A)\tilde{\cup}(F,B)$ $\tilde{\in}(\tau_1,\tau_2)$-$SbLO(X)$.*

Proof. Since $(F,A)\tilde{\in}(\tau_1,\tau_2)$-$SbLO(X)$, then there exist a τ_1-soft b-closed set (G,C) and a τ_2-soft b-open set (G,D) such that $(F,A) = (G,C)\tilde{\cup}(G,D)$.

We have $(F,A)\tilde{\cup}(F,B) = ((G,C)\tilde{\cup}(G,D))\tilde{\cup}(F,B) = ((G,C)\tilde{\cup}(F,B))\tilde{\cup}(G,D)$. If (F,B) is τ_1-soft b-closed, then $(G,C)\tilde{\cup}(F,B)$ is also τ_1-soft b-closed. Hence $(F,A)\tilde{\cup}(F,B)\tilde{\in}(\tau_1,\tau_2)$-$SbLO(X)$. Let (F,B) be τ_2-soft b-open, then $(F,A)\tilde{\cup}(F,B) = ((G,C)\tilde{\cup}(G,D))\tilde{\cup}(F,B) = (G,C)\tilde{\cup}((G,D)\tilde{\cup}(F,B))$, where $(G,D)\tilde{\cup}(F,B)$ is τ_2-soft b-open. Thus $(F,A)\tilde{\cup}(F,B)\tilde{\in}(\tau_1,\tau_2)$-$SbLO(X)$. □

Theorem 3.20. *If* $(F,A)\tilde{\in}(\tau_1,\tau_2)$*-*$SbLO^*(X)$ *and* (F,B) *is either* τ_1*-soft closed or* τ_2*-soft open set of* (X,τ_1,τ_2,E) *then* $(F,A)\tilde{\cup}(F,B)\tilde{\in}(\tau_1,\tau_2)$*-*$SbLO^*(X)$*.*

Proof. Since $(F,A)\tilde{\in}(\tau_1,\tau_2)$-$SbLO^*(X)$, then $(F,A)=(G,C)\tilde{\cup}(G,D)$, where (G,C) is τ_1-soft b-closed set and (G,D) is τ_2-soft open set of (X,τ_1,τ_2,E).

Now $(F,A)\tilde{\cup}(F,B)=((G,C)\tilde{\cup}(G,D))\tilde{\cup}(F,B)=((G,C)\tilde{\cup}(F,B))\tilde{\cup}(G,D)$. Let (F,B) be τ_1-soft closed, then $(G,C)\tilde{\cup}(F,B)$ is also τ_1-soft b-closed, where (G,C) is τ_1-soft b-closed set. Hence $(F,A)\tilde{\cup}(F,B)\tilde{\in}(\tau_1,\tau_2)$-$SbLO^*(X)$. If (F,B) is τ_2-soft open, then $(G,D)\tilde{\cup}(F,B)$ is τ_2-soft open. Now $(F,A)\tilde{\cup}(F,B)=((G,C)\tilde{\cup}(G,D))\tilde{\cup}(F,B)=(G,C)\tilde{\cup}((G,D)\tilde{\cup}(F,B)$. Thus $(F,A)\tilde{\cup}(F,B)\tilde{\in}(\tau_1,\tau_2)$-$SbLO^*(X)$. □

Theorem 3.21. *If* $(F,A)\tilde{\in}(\tau_1,\tau_2)$*-*$SbLO^{**}(X)$ *and* (F,B) *is either* τ_1*-soft closed or* τ_2*-soft open set of* (X,τ_1,τ_2) *then* $(F,A)\tilde{\cup}(F,B)\tilde{\in}(\tau_1,\tau_2)$*-*$SbLO^{**}(X)$*.*

Proof. The proof is easy, so omitted. □

Theorem 3.22. *If* $(F,A),(F,B)\tilde{\in}(\tau_1,\tau_2)$*-*$SbLO(X)$*, then* $(F,A)\tilde{\cup}(F,B)\tilde{\in}$ (τ_1,τ_2)*-*$SbLO(X)$*.*

Proof. Let $(F,A),(F,B)\tilde{\in}(\tau_1,\tau_2)$-$SbLO(X)$. Then there exist τ_1-soft b-closed sets $(G,C),(G,K)$ and τ_2-soft b-open sets $(G,D),(G,T)$ such that $(F,A)=(G,C)\tilde{\cup}(G,D)$ and $(F,B)=(G,K)\tilde{\cup}(G,T)$. We have $(F,A)\tilde{\cup}(F,B)=((G,C)\tilde{\cup}(G,D))\tilde{\cup}((G,K)\tilde{\cup}(G,T))=((G,C)\tilde{\cup}(G,K))\tilde{\cup}((G,D)\tilde{\cup}(G,T))$, where $(G,C)\tilde{\cup}(G,K)$ is τ_1-soft b-closed set and $(G,D)\tilde{\cup}(G,T)$ is τ_2-soft b-open.

Hence $(F,A)\tilde{\cup}(F,B)\tilde{\in}(\tau_1,\tau_2)$-$SbLO(X)$. □

Theorem 3.23. *If* $(F,A),(F,B)\tilde{\in}(\tau_1,\tau_2)$*-*$SbLO^*(X)$*, then* $(F,A)\tilde{\cup}(F,B)\tilde{\in}$ (τ_1,τ_2)*-*$SbLO^*(X)$*.*

Proof. Since $(F,A),(F,B)\tilde{\in}(\tau_1,\tau_2)$-$SbLO^*(X)$, then by Theorem , there exist τ_1-soft b-closed sets (G,C) and (G,D) such that $(F,A)=(G,C)\tilde{\cup}\tau_2$-$sint(F,A)$ and $(F,B)=(G,D)\tilde{\cup}\tau_2$-$sint(F,B)$. We have

$$\begin{aligned}(F,A)\tilde{\cup}(F,B)&=[(G,C)\tilde{\cup}\tau_2\text{-}sint(F,A)]\tilde{\cup}[(G,D)\tilde{\cup}\tau_2\text{-}sint(F,B)]\\&=((G,C)\tilde{\cup}(G,D))\tilde{\cup}(\tau_2\text{-}sint(F,A)\tilde{\cup}\tau_2\text{-}sint(F,B)),\end{aligned}$$

where $(G,C)\tilde{\cup}(G,D)$ is τ_1-soft b-closed and τ_2-$sint(F,A)\tilde{\cup}\tau_2$-$sint(F,B)$ is τ_2-soft open set. Hence $(F,A)\tilde{\cup}(F,B)\tilde{\in}(\tau_1,\tau_2)$-$SbLO^*(X)$. □

Theorem 3.24. *If* $(F,A),(F,B)\tilde{\in}(\tau_1,\tau_2)$*-*$SbLO^{**}(X)$*, then* $(F,A)\tilde{\cup}(F,B)\tilde{\in}$ (τ_1,τ_2)*-*$SbLO^{**}(X)$*.*

Proof. Easy, so omitted. □

Conclusion

Generalized open sets play a very important role in general and soft topology and they are now the research topics of many topologist worldwide. Indeed a significant theme in general, soft topology and real analysis concerns the variously modified forms of continuity, separation axioms, etc. by utilizing generalized open sets. The concept of a soft bitopological spaces was introduced by Ittanagi [6]. In this paper we introduced and studied the notions of soft b-locally open sets, soft bLO^*-sets, soft bLO^{**}-sets in soft bitopological spaces and obtain several characterizations and some properties of these sets. In the end, we hope that this paper is just a beginning of a new structure, it will be necessary to carry out more theoretical research to promote a general framework for the practical application.

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Results on differential subordination involving Ruscheweyh operator

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Abstract In the present paper, we find certain results on Ruscheweyh operator using differential subordination. In particular, we find sufficient conditions for close-to-convex, starlike and convex functions.

Keywords Analytic function, convex function, close-to-convex function, Ruscheweyh operator, starlike function.
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§1. Introduction and preliminaries

Let $\mathcal{H}$ denote the class of functions f, analytic in the open unit disk $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$ in the complex plane $\mathbb{C}$. Let $\mathcal{A}$ be the subclass of $\mathcal{H}$, consisting of functions f, analytic in the open unit disk $\mathbb{E}$ and normalized by the conditions $f(0) = 0 = f'(0) - 1$. A function $f \in \mathcal{A}$ is said to be starlike of order α if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha,\ 0 \leq \alpha < 1,\ z \in \mathbb{E}.$$

The class of such functions is denoted by $\mathcal{S}^*(\alpha)$. A function $f \in \mathcal{A}$ is said to be convex of order α in $\mathbb{E}$, if and only if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha,\ 0 \leq \alpha < 1,\ z \in \mathbb{E}.$$

Let $\mathcal{K}(\alpha)$ denote the class of all those functions $f \in \mathcal{A}$ that are convex of order α in $\mathbb{E}$.
A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{C}(\alpha)$ of close-to-convex of order α in $\mathbb{E}$ if and only if it satisfies

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > \alpha,\ 0 \leq \alpha < 1,\ \text{where } g \in \mathcal{S}^*.$$

Let f and g be two analytic functions in open unit disk $\mathbb{E}$. Then we say f is subordinate to g in $\mathbb{E}$ written as $f \prec g$ if there exists a Schwarz function w, analytic in $\mathbb{E}$ with $w(0) = 0$ and $|w(z)| < 1,\ z \in \mathbb{E}$ such that $f(z) = g(w(z)),\ z \in \mathbb{E}$. In case the function g is univalent, the

above subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{E}) \subset g(\mathbb{E})$.
The Taylor's series expansions of $f,\ g \in \mathcal{A}$ are given as

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \text{ and } g(z) = z + \sum_{k=2}^{\infty} b_k z^k.$$

Then the convolution/Hadamard product of f and g is denoted by $f * g$, and defined as

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k.$$

Ruscheweyh [4] introduced a differential operator R^λ, (known as Ruscheweyh differential operator) for $f \in \mathcal{A}$ is defined as follows

$$R^\lambda f(z) = \frac{z}{(1-z)^{\lambda+1}} * f(z),\ \lambda \geq -1,\ z \in \mathbb{E}. \tag{1}$$

For $\lambda = n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$

$$R^n f(z) = \frac{z(z^{n-1} f(z))^{(n)}}{n!},\ z \in \mathbb{E}.$$

Lecko et al. [2] observed that for $\lambda \geq -1$, the expression given in (1) becomes

$$R^\lambda f(z) = z + \sum_{k=2}^{\infty} \frac{(\lambda+1)(\lambda+2)\dots(\lambda+k-1)}{(k-1)!} a_k z^k,\ z \in \mathbb{E},$$

and for every $\lambda > -1$

$$\begin{aligned} R^1 R^\lambda f(z) &= z(R^\lambda f)'(z) = z\left(\frac{z}{(1-z)^{\lambda+1}} * f(z)\right)' \\ &= \frac{z}{(1-z)^{\lambda+1}} * (zf'(z)) = R^\lambda(zf'(z)) = R^\lambda R^1 f(z),\ z \in \mathbb{E}. \end{aligned}$$

We notice that

$$R^{-1} f(z) = z,\ R^0 f(z) = f(z),\ R^1 f(z) = zf'(z),\ R^2 f(z) = zf'(z) + \frac{z^2}{2} f''(z)$$

and so on. For $\lambda \geq -1$, we have

$$z(R^\lambda f)'(z) = (\lambda+1)R^{\lambda+1} f(z) - \lambda R^\lambda f(z),\ z \in \mathbb{E}. \tag{2}$$

Recently, Shams et al. [5] studied the Ruscheweyh derivative operator for $f \in \mathcal{A}_n$, which satisfies the condition given below:

$$\left| \left(1 - \alpha + \alpha(\lambda+2)\frac{R^{\lambda+2} f(z)}{R^{\lambda+1} f(z)}\right) \left(\frac{R^{\lambda+1} f(z)}{R^\lambda f(z)}\right)^\mu - \alpha(\lambda+1) \left(\frac{R^{\lambda+1} f(z)}{R^\lambda f(z)}\right)^{\mu+1} - 1 \right| < M,$$

where $\mathcal{A}_n$ is the subclass of $\mathcal{H}$ and an analytic function $f \in \mathcal{A}_n$ having Taylor's series expansion of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$$

in the unit disk $\mathbb{E}$. Note that $\mathcal{A}_1 = \mathcal{A}$. They obtained the values of $M,\ \alpha,\ \gamma$ and μ for which the function had become starlike of order γ.

In the present paper, using differential subordination we are studying the following operator

$$\left(\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}\right)^{\beta}\left[1-\alpha+\alpha\left((\lambda+2)\frac{R^{\lambda+2}f(z)}{R^{\lambda+1}f(z)}-(\lambda+1)\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}\right)\right],$$

where $\lambda \geq -1$ and $\alpha,\ \beta$ are complex numbers. We obtain some previously known results and certain conditions for starlike, convex and close-to-convex functions. As a particular case of our main result, we obtain the best dominant for $(zf'(z)/f(z))^{\beta},\ (f'(z))^{\beta}$ and $\left(1+\frac{zf''(z)}{f(z)}\right)^{\beta}$.

To prove our main result, we shall make use of the following lemma of Miller and Macanu [3].

Lemma 1.1 *[3, Theorem 3.4h, p. 132]. Let q be univalent in $\mathbb{E}$ and let θ and ϕ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{E})$, with $\phi(w) \neq 0$, when $w \in q(\mathbb{E})$. Set $Q(z) = zq'(z)\phi[q(z)],\ h(z) = \theta[q(z)] + Q(z)$ and suppose that either*

(i) h is convex, or

(ii) Q is starlike.

In addition, assume that

(iii) $\Re\left(\frac{zh'(z)}{Q(z)}\right) > 0$.

If p is analytic in $\mathbb{E}$, with $p(0) = q(0),\ p(\mathbb{E}) \subset \mathbb{D}$ and

$$\theta[p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)],$$

then $p \prec q$ and q is the best dominant.

§2. Main Results

In what follows, all the powers taken are the principal ones.

Theorem 2.1 *Let $\alpha,\ \beta$ be non-zero complex numbers such that $\Re(\beta/\alpha) > 0$ and let $f \in \mathcal{A}$, $\left(\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}\right)^{\beta} \neq 0,\ z \in \mathbb{E}$, satisfy*

$$\left(\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}\right)^{\beta}\left[1-\alpha+\alpha\left((\lambda+2)\frac{R^{\lambda+2}f(z)}{R^{\lambda+1}f(z)}-(\lambda+1)\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}\right)\right] \prec \frac{1+Az}{1+Bz}+\frac{\alpha}{\beta}\frac{(A-B)z}{(1+Bz)^2}, \quad (3)$$

then

$$\left(\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}\right)^{\beta} \prec \frac{1+Az}{1+Bz},\quad -1 \leq B < A \leq 1,\ z \in \mathbb{E}.$$

The dominant $\frac{1+Az}{1+Bz}$ is the best dominant.

Proof. Define $\left(\dfrac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}\right)^{\beta} = u(z), \quad z \in \mathbb{E}.$
On differentiating logarithmically, we get

$$\left[\frac{z(R^{\lambda+1}f(z))'}{R^{\lambda+1}f(z)} - \frac{z(R^{\lambda}f(z))'}{R^{\lambda}f(z)}\right] = \frac{zu'(z)}{\beta u(z)}. \tag{4}$$

Using the equality (2), the above equation reduces to

$$(\lambda+2)\frac{R^{\lambda+2}f(z)}{R^{\lambda+1}f(z)} - (\lambda+1)\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)} = 1 + \frac{zu'(z)}{\beta u(z)}.$$

Now, from (3), we obtain

$$\left(\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}\right)^{\beta}\left[1-\alpha+\alpha\left((\lambda+2)\frac{R^{\lambda+2}f(z)}{R^{\lambda+1}f(z)} - (\lambda+1)\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}\right)\right] = u(z) + \frac{\alpha}{\beta}zu'(z).$$

Define the functions θ and ϕ as:

$$\theta(w) = w \text{ and } \phi(w) = \frac{\alpha}{\beta}.$$

Obviously, the functions θ and ϕ are analytic in the domain $\mathbb{D} = \mathbb{C}$ and $\phi(w) \neq 0, \ w \in \mathbb{D}$. Selecting $q(z) = \dfrac{1+Az}{1+Bz}, \ -1 \leq B < A \leq 1, \ z \in \mathbb{E}$ and defining functions Q and h as under:

$$Q(z) = zq'(z)\phi(q(z)) = \frac{\alpha}{\beta}zq'(z) = \frac{\alpha}{\beta}\frac{(A-B)z}{(1+Bz)^2}$$

and

$$h(z) = \theta(q(z)) + Q(z) = q(z) + \frac{\alpha}{\beta}zq'(z) = \frac{1+Az}{1+Bz} + \frac{\alpha}{\beta}\frac{(A-B)z}{(1+Bz)^2}.$$

We can easily check that

$$\Re\left(\frac{zQ'(z)}{Q(z)}\right) = \Re\left(\frac{1-Bz}{1+Bz}\right) > 0, \ z \in \mathbb{E},$$

and

$$\left(\frac{zh'(z)}{Q(z)}\right) = \Re\left(\frac{1-Bz}{1+Bz}\right) + \Re\left(\frac{\beta}{\alpha}\right) > 0, \ z \in \mathbb{E}.$$

Hence conditions of Lemma 1.1 are satisfied and proof now follows from this lemma. □

Setting $\lambda = -1$ in Theorem 2.1, we obtain the following result of S. S. Billing [1] for $p = 1$:

Corollary 2.2 *Let $\alpha, \ \beta$ be non-zero complex numbers such that $\Re(\beta/\alpha) > 0$. If $f \in \mathcal{A}$, $\left(\dfrac{f(z)}{z}\right)^{\beta} \neq 0, \ z \in \mathbb{E}$, satisfies*

$$(1-\alpha)\left(\frac{f(z)}{z}\right)^{\beta} + \alpha\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\beta} \prec \frac{1+Az}{1+Bz} + \frac{\alpha}{\beta}\frac{(A-B)z}{(1+Bz)^2},$$

then

$$\left(\frac{f(z)}{z}\right)^{\beta} \prec \frac{1+Az}{1+Bz}, \ -1 \leq B < A \leq 1, \ z \in \mathbb{E}.$$

Taking $\lambda = -1$ and replacing $f(z)$ with $zf'(z)$ in Theorem 2.1, we obtain the following result:

Corollary 2.3 *Let $\alpha,\ \beta$ be non-zero complex numbers such that $\Re(\beta/\alpha) > 0$. If $f \in \mathcal{A},\ (f'(z))^\beta \neq 0,\ z \in \mathbb{E}$, satisfies*

$$(f'(z))^\beta \left(1 + \alpha \frac{zf''(z)}{f'(z)}\right) \prec \frac{1+Az}{1+Bz} + \frac{\alpha}{\beta} \frac{(A-B)z}{(1+Bz)^2},$$

then

$$(f'(z))^\beta \prec \frac{1+Az}{1+Bz},\ -1 \leq B < A \leq 1,\ z \in \mathbb{E}.$$

Selecting $\beta = 1,\ B = -1,\ A = 1 - 2\gamma$, where $0 \leq \gamma < 1$, in the above corollary, we have:

Example 2.4 *Let α be non-zero complex number such that $\Re(1/\alpha) > 0$. If $f \in \mathcal{A},\ f'(z) \neq 0,\ z \in \mathbb{E}$, satisfies*

$$f'(z) + \alpha z f''(z) \prec \frac{1+(1-2\gamma)z}{1-z} + \frac{2\alpha(1-\gamma)z}{(1-z)^2},$$

then

$$f'(z) \prec \frac{1+(1-2\gamma)z}{1-z},\ 0 \leq \gamma < 1,\ z \in \mathbb{E}.$$

Hence $f \in \mathcal{C}(\gamma)$.

Choosing $\lambda = 0$ in Theorem 2.1, we get the following result:

Corollary 2.5 *Let $\alpha,\ \beta$ be non-zero complex numbers such that $\Re(\beta/\alpha) > 0$. If $f \in \mathcal{A},\ \left(\frac{zf'(z)}{f(z)}\right)^\beta \neq 0,\ z \in \mathbb{E}$, satisfies*

$$(1+\alpha)\left(\frac{zf'(z)}{f(z)}\right)^\beta + \alpha\left(\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)\left(\frac{zf'(z)}{f(z)}\right)^\beta \prec \frac{1+Az}{1+Bz} + \frac{\alpha}{\beta}\frac{(A-B)z}{(1+Bz)^2},$$

then

$$\left(\frac{zf'(z)}{f(z)}\right)^\beta \prec \frac{1+Az}{1+Bz},\ -1 \leq B < A \leq 1,\ z \in \mathbb{E}.$$

In the above corollary, setting $\beta = 1,\ B = -1,\ A = 1 - 2\gamma$, where $0 \leq \gamma < 1$, we obtain:

Example 2.6 *Let α be non-zero complex number such that $\Re(1/\alpha) > 0$. If $f \in \mathcal{A},\ \frac{zf'(z)}{f(z)} \neq 0,\ z \in \mathbb{E}$, satisfies*

$$(1+\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\gamma)z}{1-z} + \frac{2\alpha(1-\gamma)z}{(1-z)^2},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\gamma)z}{1-z},\ 0 \leq \gamma < 1,\ z \in \mathbb{E}.$$

Hence f is a starlike function of order γ.

For $\lambda = 0$ and on replacing $f(z)$ with $zf'(z)$ in Theorem 2.1, we have:

Corollary 2.7 *Let* α, β *be non-zero complex numbers such that* $\Re(\beta/\alpha) > 0$. *If* $f \in \mathcal{A}$, $\left(1 + \frac{zf''(z)}{f'(z)}\right)^{\beta} \neq 0$, $z \in \mathbb{E}$, *satisfies*

$$\left(1 + \frac{zf''(z)}{f'(z)}\right)^{\beta}\left[1 + \alpha\left(\frac{2zf''(z) + z^2 f'''(z)}{f'(z) + zf''(z)} - \frac{zf''(z)}{f'(z)}\right)\right] \prec \frac{1 + Az}{1 + Bz} + \frac{\alpha}{\beta}\frac{(A - B)z}{(1 + Bz)^2},$$

then

$$\left(1 + \frac{zf''(z)}{f'(z)}\right)^{\beta} \prec \frac{1 + Az}{1 + Bz}, \ -1 \leq B < A \leq 1, \ z \in \mathbb{E}.$$

Choosing $\beta = 1$, $B = -1$, $A = 1 - 2\gamma$, where $0 \leq \gamma < 1$, in the above corollary, we have the following result:

Example 2.8 *Let* α *be non-zero complex number such that* $\Re(1/\alpha) > 0$. *If* $f \in \mathcal{A}$, *satisfies*

$$\left(1 + \frac{zf''(z)}{f'(z)}\right)\left[1 + \alpha\left(\frac{2zf''(z) + z^2 f'''(z)}{f'(z) + zf''(z)} - \frac{zf''(z)}{f'(z)}\right)\right] \prec \frac{1 + (1 - 2\gamma)z}{1 - z} + \frac{2\alpha(1 - \gamma)z}{(1 - z)^2},$$

then

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{1 + (1 - 2\gamma)z}{1 - z}, \ 0 \leq \gamma < 1, \ z \in \mathbb{E}.$$

Hence $f \in \mathcal{K}(\gamma)$.

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On class of entire Dirichlet series with variable sequence of complex exponents

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Abstract Various classes of Dirichlet Series with constant sequence of real exponents have been considered in the past. Researchers have provided several space structures to class of such series. In the present paper we deal with class $\mathbb{M}$ of multiple Dirichlet series having variable sequence of complex exponents. With a well defined norm on $\mathbb{M}$, various results are established.

Keywords Multiple Dirichlet Series, Banach Algebra, Quasi singular, Skew field
2010 Mathematics Subject Classification 30B50, 46J15, 12E15

§1. Introduction and preliminaries

Let L be a set of sequences $\{\lambda^k\}$, $\lambda^k = ({\lambda_1}^k, {\lambda_2}^k, .., {\lambda_n}^k)$, k=1,2,.. of complex numbers in $\mathbb{C}^n$ with $|\lambda^k| \to \infty$ as $k \to \infty$ and

$$\limsup_{k\to\infty} \frac{\log k}{|\lambda^k|} < \infty$$

where $|\lambda^k| = \sqrt{{\lambda_1}^k\overline{{\lambda_1}^k} + {\lambda_2}^k\overline{{\lambda_2}^k} + .. + {\lambda_n}^k\overline{{\lambda_n}^k}}$.

Consider a Dirichlet series

$$\alpha(s) = \sum_{k=1}^{\infty} c_k e^{<\lambda^k, s>} \tag{1}$$

where $s = (s_1, s_2, .., s_n) \in \mathbb{C}^n$,$\{c_k\}$ is the sequence of complex numbers and $\{\lambda^k\} \in$ L.
Also $< \lambda^k, s >= {\lambda_1}^k s_1 + {\lambda_2}^k s_2 + .. + {\lambda_n}^k s_n$.
If (1) satisfies

$$\limsup_{k\to\infty} \frac{\log|c_k|}{|\lambda^k|} = -\infty \tag{2}$$

then from [1] (1) converges in the whole complex plane.

Several investigations on class of entire Dirichlet series with constant sequence of real exponents have been made by many researchers in the past. Kamthan [3] proved the class of entire functions represented by Dirichlet series to be an FK Space. Srivastava [4] provided a Banach algebraic structure to class of Dirichlet series in one complex variable and having constant real frequencies $\{\lambda^k\}$ for which the sequence $e^{k\lambda_k}|c_k|$ is bounded.
Shaker, Hussein and Srivastava [2] investigated the bornological aspects of class of entire functions represented by multiple Dirichlet series with constant frequencies. They introduced a bornology on the class and proved it to be a separated convex bornological vector space. Singh and Rastogi [7] characterised the Goldberg q^{th} order and Goldberg q^{th} type of entire function represented by multiple Dirichlet series in terms of its real exponents and coefficients.Khoi[1] studied coefficient multiplers on class of series of form (1).
In this paper we prove some results on class of Dirichlet series having variable sequence of complex exponents.

1 Class $\mathbb{M}$

Let $\mathbb{M}$ be a class of series of form (1) with variable complex frequencies $\{\lambda^k\}$ for which the sequence

$$|\lambda^k|^{e_1|\lambda^k|}(k!)^{e_2}|c_k{}^{\frac{1}{|\lambda^k|}}|$$

is bounded where $e_1, e_2 \geq 0$ and are not simultaneously zero,then every element of $\mathbb{M}$ becomes entire.
We consider two elements $\alpha(s) = \sum_{k=1}^{\infty} c_k e^{<\lambda^k,s>}$ and $\beta(s) = \sum_{k=1}^{\infty} d_k e^{<\mu^k,s>}$ of class $\mathbb{M}$ to be equivalent i.e. $\alpha \equiv \beta$ if and only if

$$|\lambda^k|^{e_1|\lambda^k|}(k!)^{e_2}c_k{}^{\frac{1}{|\lambda^k|}} = |\mu^k|^{e_1|\mu^k|}(k!)^{e_2}d_k{}^{\frac{1}{|\mu^k|}}\ ,\ k \geq 1$$

Clearly relation "≡" is an equivalence relation on class $\mathbb{M}$. Hence the class $\mathbb{M}$ can be treated as the set of so formed equivalence classes. For the sake of brevity, we consider "the member $\alpha(s)$ of class $\mathbb{M}$" same as "equivalence class generated by $\alpha(s)$ of $\mathbb{M}$".
Next we define binary operations on set $\mathbb{M}$ for $\alpha(s) = \sum_{k=1}^{\infty} c_k e^{<\lambda^k,s>}$ and $\beta(s) = \sum_{k=1}^{\infty} d_k e^{<\mu^k,s>}$ as

$$\alpha(s) + \beta(s) = \sum_{k=1}^{\infty}\left[\left(\frac{|\lambda^k|^{|\lambda^k|}}{|x^k|^{|x^k|}}\right)^{e_1} c_k{}^{\frac{1}{|\lambda^k|}} + \left(\frac{|\mu^k|^{|\mu^k|}}{|x^k|^{|x^k|}}\right)^{e_1} d_k{}^{\frac{1}{|\mu^k|}}\right]^{|x^k|} e^{<x^k,s>}$$

$$r\alpha(s) = \sum_{k=1}^{\infty}\left[\left(\frac{|\lambda^k|^{|\lambda^k|}}{|x^k|^{|x^k|}}\right)^{e_1} rc_k{}^{\frac{1}{|\lambda^k|}}\right]^{|x^k|} e^{<x^k,s>}\ ; r \in \mathbb{C}$$

$$\alpha(s).\beta(s) = \sum_{k=1}^{\infty}\left[(k!)^{e_2}\left(\frac{|\lambda^k|^{|\lambda^k|}|\mu^k|^{|\mu^k|}}{|x^k|^{|x^k|}}\right)^{e_1} c_k^{\frac{1}{|\lambda^k|}} d_k^{\frac{1}{|\mu^k|}}\right]^{|x^k|} e^{<x^k,s>}$$

where $\{x^k\}$ denotes the arbitrary element of set L.
Throughout the paper, we assume $\{x^k\}$ to be an arbitrary element of L.
We define the norm in $\mathbb{M}$ as

$$\|\alpha\| = \sup_{k\geq 1}|\lambda^k|^{e_1|\lambda^k|}(k!)^{e_2}|c_k^{\frac{1}{|\lambda^k|}}| \tag{3}$$

where $\alpha(s) = \sum_{k=1}^{\infty} c_k e^{<\lambda^k,s>}$

For the definitions used refer [5] and [6].

2 Main Results

Theorem 3.1. *$\mathbb{M}$ is a commutative and unital Banach Algebra.*

Proof. Clearly, $\mathbb{M}$ is a vector space over the field of complex numbers.
Let $\alpha_p(s) = \sum_{k=1}^{\infty} c_{kp} e^{<\lambda^{kp},s>}$ be a Cauchy sequence in $\mathbb{M}$.
Then for $\epsilon > 0$, $\exists$ some t such that

$$\|\alpha_p - \alpha_q\| < \epsilon \textit{ whenever } p,q \geq t$$

$$\Rightarrow \sup_{k\geq 1}\left||\lambda^{kp}|^{e_1|\lambda^{kp}|}(k!)^{e_2} c_{kp}^{\frac{1}{|\lambda^{kp}|}} - |\lambda^{kq}|^{e_1|\lambda^{kq}|}(k!)^{e_2} c_{kq}^{\frac{1}{|\lambda^{kq}|}}\right| < \epsilon \textit{ whenever } p,q \geq t$$

$$\Rightarrow \left||\lambda^{kp}|^{e_1|\lambda^{kp}|}(k!)^{e_2} c_{kp}^{\frac{1}{|\lambda^{kp}|}} - |\lambda^{kq}|^{e_1|\lambda^{kq}|}(k!)^{e_2} c_{kq}^{\frac{1}{|\lambda^{kq}|}}\right| < \epsilon \textit{ whenever } p,q \geq t$$

As $\{|\lambda^{kq}|^{e_1|\lambda^{kq}|}(k!)^{e_2} c_{kq}^{\frac{1}{|\lambda^{kq}|}}\}$ is a Cauchy sequence in $\mathbb{C}$ and owing to the completeness of $\mathbb{C}$, let $\{|\lambda^{kq}|^{e_1|\lambda^{kq}|}(k!)^{e_2} c_{kq}^{\frac{1}{|\lambda^{kq}|}}\}$ converges to d_k.

Taking $q \to \infty$ in above inequality, we get

$$\sup_{k\geq 1}\left||\lambda^{kp}|^{e_1|\lambda^{kp}|}(k!)^{e_2} c_{kp}^{\frac{1}{|\lambda^{kp}|}} - d_k\right| < \epsilon \textit{ whenever } p \geq t$$

Let $h(s) = \sum_{k=1}^{\infty}\left(|x^k|^{-e_1|x^k|}(k!)^{-e_2} d_k\right)^{|x^k|} e^{<x^k,s>}$. Then clearly, $\alpha_p \to h$.

Also, h $\in \mathbb{M}$ as

$|x^k|^{e_1|x^k|}(k!)^{e_2}\left||x^k|^{-e_1|x^k|}(k!)^{-e_2} d_k\right| = \left|d_k\right|$

$$= \left| d_k + |\lambda^{kp}|^{e_1|\lambda^{kp}|}(k!)^{e_2} c_k^{\frac{1}{|\lambda^{kp}|}} - |\lambda^{kp}|^{e_1|\lambda^{kp}|}(k!)^{e_2} c_k^{\frac{1}{|\lambda^{kp}|}} \right|$$
$$\leq \left| |\lambda^{kp}|^{e_1|\lambda^{kp}|}(k!)^{e_2} c_{kp}^{\frac{1}{|\lambda^{kp}|}} - d_k \right| + \left| |\lambda^{kp}|^{e_1|\lambda^{kp}|}(k!)^{e_2} c_{kp}^{\frac{1}{|\lambda^{kp}|}} \right|$$

The identity element in $\mathbb{M}$ is

$$e(s) = \sum_{k=1}^{\infty} \left(|x^k|^{-e_1|x^k|}(k!)^{-e_2} \right)^{|x^k|} e^{<x^k,s>}$$

□

Theorem 3.2. *The class $\mathbb{M}$ is not a Division Algebra. Infact, an element $\alpha(s) = \sum_{k=1}^{\infty} c_k e^{<\lambda^k,s>}$ in $\mathbb{M}$ becomes invertible if and only if*

$$\frac{1}{|\lambda^k|^{e_1\lambda^k}(k!)^{e_2}|e_k^{\frac{1}{|\lambda^k|}}|} < \infty$$

for all k.

Proof. The inverse of element

$$f(s) = \sum_{k=1}^{\infty} \left(k^{-1}|x^k|^{-e_1|x^k|}(k!)^{-e_2} \right)^{|x^k|} e^{<x^k,s>} \text{ in } \mathbb{M}$$

is

$$g(s) = \sum_{k=1}^{\infty} \left(k|y^k|^{-e_1|y^k|}(k!)^{-e_2} \right)^{|y^k|} e^{<y^k,s>} \;; \{y^k\} \in L$$

which does not belong to $\mathbb{M}$.

For $\alpha(s)$ to be invertible in $\mathbb{M}$ there must exist some $\beta(s) = \sum_{k=1}^{\infty} d_k e^{<\mu^k,s>}$ such that

$$\alpha(s).\beta(s) = e(s)$$
$$|\mu^k|^{e_1|\mu^k|}(k!)^{e_2} d_k^{\frac{1}{|\mu^k|}} = \frac{1}{|\lambda^k|^{e_1|\lambda^k|}(k!)^{e_2} c_k^{\frac{1}{|\lambda^k|}}}$$

As $\beta(s) \in \mathbb{M}$ therefore $\exists$ some N such that

$$\frac{1}{|\lambda^k|^{e_1|\lambda^k|}(k!)^{e_2}|c_k^{\frac{1}{|\lambda^k|}}|} \leq N \;\forall\; k \geq 1$$

Conversely, Define

$$\beta(s) = \sum_{k=1}^{\infty} \left(\frac{1}{(|x^k|^{|x^k|}|\lambda^k|^{|\lambda^k|})^{e_1}(k!)^{2e_2}|c_k^{\frac{1}{|\lambda^k|}}|} \right)^{|x^k|} e^{<x^k,s>}$$

Clearly, $\beta(s) \in \mathbb{M}$.
Also, $\alpha(s).\beta(s) = \beta(s).\alpha(s) = e(s)$.
Hence the theorem. $\square$

Theorem 3.3. *An element* $\alpha(s) = \sum_{k=1}^{\infty} c_k e^{<\lambda^k, s>}$ *in* $\mathbb{M}$ *is quasi singular if and only if*

$$\inf_{k\geq 1}\{|1 + |\lambda^k|^{c_1|\lambda^k|}(k!)^{c_2} c_k^{\frac{1}{|\lambda^k|}}|\} = 0.$$

Proof. We suppose that $\alpha(s)$ is not quasi singular. Then, $\alpha(s)$ is quasi invertible i.e. $\exists$ some $\beta(s) = \sum_{k=1}^{\infty} d_k e^{<\mu^k, s>}$ in $\mathbb{M}$ such that

$$\alpha(s) * \beta(s) = 0$$

$$\Rightarrow \alpha(s) + \beta(s) + \alpha(s).\beta(s) = 0$$

$$|\lambda^k|^{e_1|\lambda^k|} c_k^{\frac{1}{|\lambda^k|}} + |\mu^k|^{e_1|\mu^k|} d_k^{\frac{1}{|\mu^k|}} + (k!)^{e_2} |\lambda^k|^{e_1|\lambda^k|} |\mu^k|^{e_1|\mu^k|} c_k^{\frac{1}{|\lambda^k|}} d_k^{\frac{1}{|\mu^k|}} = 0 \ , k \geq 1$$

$$\Rightarrow |\mu^k|^{e_1|\mu^k|}(k!)^{e_2} d_k^{\frac{1}{|\mu^k|}} = \frac{-|\lambda^k|^{e_1|\lambda^k|}(k!)^{e_2} c_k^{\frac{1}{|\lambda^k|}}}{1 + |\lambda^k|^{e_1|\lambda^k|}(k!)^{e_2} c_k^{\frac{1}{|\lambda^k|}}}$$

If $\inf_{k\geq 1}\{|1 + |\lambda^k|^{e_1|\lambda^k|}(k!)^{e_2} c_k^{\frac{1}{|\lambda^k|}}|\} = 0$ then $\exists$ a subsequence $\{k_n\}$ of $\{k\}$ such that $\|\alpha_n\| = 1$ where $\alpha_n(s) = \sum_{n=1}^{\infty} c_{k_n} e^{<\lambda^{k_n}, s>}$ and $|\lambda^{k_n}|^{e_1|\lambda^{k_n}|}(k_n!)^{e_2}|c_{k_n}^{\frac{1}{|\lambda^{k_n}|}}| \to 1$ as $n \to \infty$.
Here

$$\|\beta_n\| = \sup_{k\geq 1} |\mu^{k_n}|^{e_1|\mu^{k_n}|}(k_n!)^{e_2}|d_{k_n}^{\frac{1}{|\mu^{k_n}|}}| = \sup_{k\geq 1} \frac{|\lambda^{k_n}|^{e_1|\lambda^{k_n}|}(k_n!)^{e_2}|c_{k_n}^{\frac{1}{|\lambda^{k_n}|}}|}{|1 + (k_n!)^{e_2}|\lambda^{k_n}|^{e_1|\lambda^{k_n}|} c_{k_n}^{\frac{1}{|\lambda^{k_n}|}}|}$$

does not belong to $\mathbb{M}$ which is a contradiction where $\beta_n(s) = \sum_{n=1}^{\infty} d_{k_n} e^{<\mu^{k_n}, s>}$.

Hence, $\inf_{k\geq 1}\{|1 + |\lambda^k|^{e_1|\lambda^k|}(k)!^{e_2} c_k^{\frac{1}{|\lambda^k|}}|\} \neq 0$.

Conversely, Let $\inf_{k\geq 1}\{|1 + |\lambda^k|^{e_1|\lambda^k|}(k!)^{e_2} c_k^{\frac{1}{|\lambda^k|}}|\} > 0$.
Define $\beta(s) = \sum_{k=1}^{\infty} d_k e^{<\mu^k, s>}$ where

$$|\mu^k|^{e_1|\mu^k|}(k!)^{e_2} d_k^{\frac{1}{|\mu^k|}} = \frac{-|\lambda^k|^{e_1|\lambda^k|}(k!)^{e_2} c_k^{\frac{1}{|\lambda^k|}}}{1 + (k!)^{e_2}|\lambda^k|^{e_1|\lambda^k|} c_k^{\frac{1}{|\lambda^k|}}}$$

Clearly, $\beta(s) \in \mathbb{M}$ and also $\alpha(s) * \beta(s) = \alpha(s) * \beta(s) = 0$ which implies that α is not quasi singular.
Hence the theorem. □

Theorem 3.4. *$\mathbb{M}$ is a Banach algebra with continous quasi inverse.*

Proof. Let $V_\epsilon(0)$ denotes the ϵ-neighbourhood of 0.
If $\beta(s) = \sum_{k=1}^{\infty} d_k e^{<\mu^k,s>} \in V_\epsilon(0)$ then $|\mu^k|^{e_1|\mu_k|}(k!)^{e_2}|d_k{}^{\frac{1}{|\mu^k|}}| < \epsilon$
So, $\inf_{k\geq 1}\{|1+|\mu^k|^{e_1|\mu_k|}(k!)^{e_2}d_k{}^{\frac{1}{|\mu^k|}}|\} \geq 1-\epsilon > 0$. Hence, by previous theorem, $\beta(s)$ is not quasi singular and thus has a quasi inverse, say $\omega(s) = \sum_{k=1}^{\infty} a_k e^{<x^k,s>}$.
Then $\beta(s) * \omega(s) = 0$ i.e.

$$|x^k|^{e_1|x_k|}a_k{}^{\frac{1}{|x^k|}} = \frac{-|\mu^k|^{e_1|\mu_k|}d_k{}^{\frac{1}{|\mu^k|}}}{1+|\mu^k|^{e_1|\mu_k|}(k!)^{e_2}d_k{}^{\frac{1}{|\mu^k|}}}.$$

Now,

$$\|\omega\| = \sup_{k\geq 1}\frac{|\mu_k|^{e_1|\mu_k|}(k!)^{e_2}|d_k{}^{\frac{1}{|\mu^k|}}|}{|1+|\mu_k|^{e_1|\mu_k|}(k!)^{e_2}d_k{}^{\frac{1}{|\mu^k|}}|}$$

$$< \frac{\epsilon}{1-\epsilon}$$

Hence the theorem. □

Theorem 3.5. *Spectrum $\sigma(\alpha)$ of an element $\alpha(s) = \sum_{k=1}^{\infty} c_k e^{<\lambda^k,s>}$ is of the form*

$$\sigma(\alpha) = cl\{|\lambda^k|^{e_1|\lambda^k|}(k!)^{e_2}c_k{}^{\frac{1}{|\lambda^k|}} : k \geq 1\}$$

Proof. $\sigma(\alpha)$ is the set of all complex numbers z such that $\alpha - ze$ is not invertible. Here,

$$\alpha(s) - ze(s) = \sum_{k=1}^{\infty}\left[\left(\frac{|\lambda^k|^{|\lambda^k|}}{|x^k|^{|x^k|}}\right)^{e_1} c_k{}^{\frac{1}{|\lambda^k|}} - \frac{z}{|x^k|^{e_1|x^k|}(k!)^{e_2}}\right]^{|x^k|} e^{<x^k,s>}$$

From previous Theorem , for $\alpha(s) - ze(s)$ to be not invertible

$$\frac{1}{|\lambda^k|^{e_1|\lambda^k|}(k!)^{e_2}c_k{}^{\frac{1}{|\lambda^k|}} - z}$$

is not bounded.
Then $\exists$ subsequence (k_n) of (k) such that

$$\lim_{n\to\infty}|\lambda^{k_n}|^{e_1|\lambda^{k_n}|}(k_n!)^{e_2}c_{k_n}{}^{\frac{1}{|\lambda^{k_n}|}} - z = 0$$

Hence the theorem. □

Theorem 3.6 *Every continous linear functional*

$$\theta : \mathbb{M} \to \mathbb{C}$$

is of the form

$$\theta(\alpha) = \sum_{k=1}^{\infty} \left(|\lambda^k|^{e_1|\lambda^k|} (k!)^{e_2} \right)^{|\lambda^k|} c_k d_k$$

where

$$\alpha(s) = \sum_{k=1}^{\infty} c_k e^{<\lambda^k, s>}$$

and d_k is a bounded sequence in $\mathbb{C}$.

Proof. Let $\theta : \mathbb{M} \to \mathbb{C}$ be a continous linear functional. So,

$$\theta(\alpha) = \theta(\sum_{k=1}^{\infty} c_k e^{<\lambda^k, s>}) = \sum_{k=1}^{\infty} c_k \theta(e^{<\lambda^k, s>}) \tag{4}$$

Now, we define a sequence $\{\alpha_k\}$ in $\mathbb{M}$ as

$$\alpha_k(z) = (|x^k|^{e_1|x^k|} (k!)^{e_2})^{-|x^k|} e^{<x^k, z>}$$

As $(|x^k|^{e_1|x^k|} (k!)^{e_2})^{-|x^k|} e^{<x^k, z>} \equiv (|\lambda^k|^{e_1|\lambda^{k^k}|} (k!)^{e_2})^{-|\lambda^k|} e^{<\lambda^k, z>}$ for all k therefore

$$\theta(\alpha) = \sum_{k=1}^{\infty} c_k \left(|\lambda^k|^{e_1|\lambda^k|} (k!)^{e_2} \right)^{|\lambda^k|} \theta(\alpha_k(z))$$

Since θ is a continous linear functional, therefore $|\theta(\alpha_k)| \leq K\|\alpha_k\|$ for some K.
As $\|\alpha_k\| = 1$ therefore $|\theta(\alpha_k)| \leq$ K.
Let $d_k = \theta(\alpha_k)$.
Then $\theta(\alpha) = \sum_{k=1}^{\infty} \left(|\lambda^k|^{e_1|\lambda^k|} (k!)^{e_2} \right)^{|\lambda^k|} c_k d_k$ where d_k is a bounded sequence. □

Theorem 3.7. *An element α of $\mathbb{M}$ is a topological divisor of zero if and only if*

$$\lim_{k \to \infty} |\lambda^k|^{e_1|\lambda^k|} (k!)^{e_2} |c_k^{\frac{1}{|\lambda^k|}}| = 0.$$

Proof. Let $\alpha(s) = \sum_{k=1}^{\infty} c_k e^{<\lambda^k, s>}$ be a topological zero divisor of zero. We suppose that

$$\lim_{k \to \infty} |\lambda^k|^{e_1|\lambda^k|} (k!)^{e_2} |c_k^{\frac{1}{|\lambda^k|}}| = \alpha > 0$$

Then for a given ϵ, $0 < \epsilon < \alpha$, $\exists$ a natural number N such that

$$|\lambda^k|^{e_1|\lambda^k|} (k!)^{e_2} |c_k^{\frac{1}{|\lambda^k|}}| > \alpha - \epsilon \textit{ whenever } k \geq N \tag{5}$$

As $\alpha \in \mathbb{M}$ is a topological divisor of zero therefore $\exists$ a sequence $\{\beta_t\}$ of elements in $\mathbb{M}$ having unit norm such that for all $t \geq 1$ we have

$$\sup_{k\geq 1} |\mu^{kt}|^{e_1|\mu^{kt}|}(k!)^{e_2}|d_{kt}{}^{\frac{1}{|\mu^{kt}|}}| = 1 \text{ for } \beta_t(s) = \sum_{k=1}^{\infty} d_{kt}e^{<\mu^{kt},s>}.$$

For some δ ,$0 < \delta < 1$ we can find an integer N_t and a subsequence $\{k_i\}$ of $\{k\}$ such that

$$|\mu^{kt}|^{e_1|\mu^{kt}|}(k!)^{e_2}|d_{kt}{}^{\frac{1}{|\mu^{kt}|}}| > 1-\delta \;\forall\, k = k_i \geq N_t. \tag{6}$$

From (5) and (6), we have
Thus

$$|\mu^{kt}|^{e_1|\mu^{kt}|}(k!)^{e_2}|d_{kt}{}^{\frac{1}{|\mu^{kt}|}}||\lambda^k|^{e_1|\lambda^k|}(k!)^{e_2}|c_k{}^{\frac{1}{|\lambda^k|}}| > 0 \;\forall\; k = k_i \geq N_t.$$

Therefore $\|\alpha(s).\beta_t(s)\| \nrightarrow 0$ which is a contradiction to the fact that $\alpha(s)$ is a topological divisor of zero. Hence, $\lim_{k\to\infty} |\lambda^k|^{e_1|\lambda^k|}(k!)^{e_2}|c_k{}^{\frac{1}{|\lambda^k|}}| = 0$.

Conversely
Let

$$\lim_{k\to\infty} |\lambda^k|^{e_1|\lambda^k|}(k!)^{e_2}|c_k{}^{\frac{1}{|\lambda^k|}}| = 0.$$

Construct a sequence $\{\beta_k\}$ such that

$$\beta_k(s) = \left(|\mu^k|^{-e_1|\mu^k|}(k!)^{-e_2}\right)^{|\mu^k|} e^{<\mu^k,s>}.$$

Clearly, $\beta_k(s) \in \mathbb{M}$ for all $k \geq 1$ and $\|\beta_k\| = 1$.
Now,

$$\beta_k(s).\alpha(s) = \alpha(s).\beta_k(s) = \left(\frac{|\lambda^k|^{e_1|\lambda^k|}}{|x^k|^{e_1|x^k|}} c_k{}^{\frac{1}{|\lambda^k|}}\right)^{|x^k|} e^{<x^k,s>}$$

therefore $\|\beta_k.\alpha\| = \|\alpha.\beta_k\| = |\lambda^k|^{e_1|\lambda^k|}(k!)^{e_2}|c_k{}^{\frac{1}{|\lambda^k|}}|$.
Here $\|\beta_k.\alpha\| = \|\alpha.\beta_k\| \to 0$ as $k \to \infty$ therefore $\alpha(s)$ is a topological divisor of zero. Hence the theorem. $\square$

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r-(τ_i, τ_j)-generalized regular fuzzy closed sets in smooth bitopological spaces

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Abstract In this paper, a new class of fuzzy sets, namely r-(τ_i, τ_j)-generalized regular fuzzy closed (briefly, r-(τ_i, τ_j)-grfc) sets is introduced for smooth bitopological spaces and some notions of these sets are investigated. By using r-(τ_i, τ_j)-grfc sets, we define a new fuzzy closure operator referred to as (i, j)-GRC which generates a new smooth topology, $\tau_{(i,j)\text{-}GRC}$. An application of these sets the definition of (i, j)-$FRT_{1/2}$ spaces. Finally, (i, j)-generalized regular fuzzy continuous and (i, j)-generalized regular fuzzy irresolute mappings are introduce, we show that (i, j)-generalized regular fuzzy continuous properly fits in between j-fuzzy regular continuous and (i, j)-generalized fuzzy continuous.

Keywords r-(τ_i, τ_j)-generalized regular fuzzy closed sets, r-(τ_i, τ_j)-generalized regular fuzzy closure operator, (i, j)-$FRT_{1/2}$ spaces, (i, j)-generalized regular fuzzy continuous (irresolute) maps.
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§1. Introduction

Kubiak [14] and Šostak [22], introduced the fundamental concept of a fuzzy topological structure, as an extension of both crisp topology and Chang's fuzzy topology [4], indicating that not only the object were fuzzified, but also the axiomatics. Subsequently, Badard [3], introduced the concept of smooth topological space. Chattopadhyay et al. [5] have redefined the same concept under the name gradation of openness. Ramadan [19] introduced a similar definition, namely, smooth topological space for lattice $L = [0, 1]$. Following Ramadan, several authors have reintroduced and further studied smooth topological space [5–7, 9, 23]. Thus, the terms 'fuzzy topology', in Šostak sense, 'gradation of openness' and 'smooth topology' are essentially referring to the same concept. In our paper, we adopt the term smooth topology. Lee et al. [15] introduced the concept of smooth bitopological space as a generalization of smooth topological space and Kandil's fuzzy bitopological spaces [10]. The concept of generalized closed sets in topological spaces introduced by Levine [16]. Subsequently, Fukutake [8], introduced the concept of generalized closed sets in bitopological spaces. Balasubramanian and Sundaram [1] gave the concept of generalized fuzzy closed sets in Chang's fuzzy topology as an extension of generalized closed sets of Levine. Jin Han Park and Jin Keun Park [18] introduced weaker form of generalized fuzzy closed set and generalized fuzzy continuous mappings i.e, regular generalized fuzzy closed set and generalizations of fuzzy continuous functions. Bhattacharya and Chakraborty [2] introduced another generalizatioin of fuzzy closed set i.e., generalized regular fuzzy closed set which is the stronger form of the previous two generalizations. Kim and Ko [12] defined r-generalized fuzzy closed sets in smooth topological spaces. Osama et al. [24] in 2015 introduced the concept of r-(τ_i, τ_j)-generalized fuzzy closed sets in smooth bitopological spaces. Recently, we [25] introduced the concept of r-generalized regular fuzzy closed set in smooth topological spaces.

The aim of this paper is to continue the study of generalized regular fuzzy closed sets in smooth bitopological spaces and study its basic properties. Moreoevr, we define a new fuzzy closure operator by using this class of r-generalized regular fuzzy closed sets, which is induced a smooth topology. Finally, we introduce and study the concept of a new class of fuzzy mappings, namely (i, j)-generalized regular fuzzy continuous and (i, j)-generalized regular fuzzy irresolute mappings and give the relations between them.

§2. Preliminaries

Throughout this paper, let X be a non-empty set, $I = [0, 1]$, $I_0 = (0, 1]$. A fuzzy set μ of X is a mapping $\mu : X \to I$, and I^X be the family of all fuzzy sets on X. For any $\mu_1,\ \mu_2 \in I^X$, $\mu_1 \wedge \mu_2 = min\{\mu_1(x),\ \mu_2(x) : x \in X\}$, $\mu_1 \vee \mu_2 = max\{\mu_1(x),\ \mu_2(x) : x \in X\}$. The complement of a fuzzy set λ is denoted by $\overline{1} - \lambda$. For $\alpha \in I$, $\overline{\alpha}(x) = \alpha\ \forall x \in X$. By $\overline{0}$ and $\overline{1}$, we denote constant maps on X with value 0 and 1, respectively. For each $x \in X$ and $t \in I_0$, the fuzzy set x_t of X whose value t at x and 0 otherwise is called the fuzzy point in X. Let $Pt(X)$ be a family of all fuzzy points in X. $x_t \in \lambda$ iff $\lambda(x) \geq t$. For $\lambda \in I^X$, $\overline{1} - \lambda$ denotes the complement of λ. All other notations and definitions are standard in the fuzzy set theory.

Definition 2.1. *[3, 5, 19, 22] A smooth topology on X is a mapping $\tau : I^X \to I$ which satisfies the following properties:*

(1) $\tau(\overline{0}) = \tau(\overline{1}) = 1$,

(2) $\tau(\bigvee_{i\in J} \mu_i) \geq \bigwedge_{i\in J} \tau(\mu_i)$, *for any* $\{\mu_i : i \in J\} \subseteq I^X$.

(3) $\tau(\mu_1 \wedge \mu_2) \geq \tau(\mu_1) \wedge \tau(\mu_2)$, *for all* $\mu_1,\ \mu_2 \in I^X$,

The pair (X, τ) is called a smooth topological space. For $r \in I_0$, μ is an r-fuzzy open set of X if $\tau(\mu) \geq r$, and μ is an r-fuzzy closed set of X if $\tau(\overline{1} - \mu) \geq r$. Note, Šostak [22] used the term 'fuzzy topology' and Chattopadhyay [5], the term 'gradation of openness' for a smooth topology τ.

Subsequently, the fuzzy closure for any fuzzy set in smooth topological space is given as follows:

Definition 2.2. *[6] Let (X, τ) be a smooth topological space. For $\lambda \in I^X$ and $r \in I_0$, a fuzzy closure of λ is a mapping $C_\tau : I^X \times I_0 \to I^X$ such that*

$$C_\tau(\lambda, r) = \bigwedge\{\mu \in I^X \mid \mu \geq \lambda,\ \tau(\overline{1} - \mu) \geq r\}.$$

Definition 2.3. *[6] A mapping $C : I^X \times I_0 \to I^X$ is called a fuzzy closure operator if, for $\lambda,\ \mu \in I^X$ and $r, s \in I_0$ the mapping C satisfies the following conditions:*

(C1) $C(\overline{0}, r) = \overline{0}$,

(C2) $\lambda \leq C(\lambda,\ r)$,

(C3) $C(\lambda,\ r) \vee C(\mu,\ r) = C(\lambda \vee \mu,\ r)$,

(C4) $C(\lambda,\ r) \leq C(\lambda,\ s)$ *if* $r \leq s$,

(C5) $C(C(\lambda,\ r),\ r) = C(\lambda,\ r)$.

The fuzzy closure operator C generates a smooth topology $\tau_C : I^X \to I$ given by

$$\tau_C(\lambda) = \bigvee\{r \in I | C(\overline{1} - \lambda, r) = \overline{1} - \lambda\}.$$

In a similar pattern, a fuzzy interior operator was defined.

Definition 2.4. *[11, 20] A mapping $I : I^X \times I_0 \to I^X$ is called a fuzzy interior operator if, for $\lambda,\ \mu \in I^X$ and $r, s \in I_0$ the mapping I satisfies the following conditions:*

(I1) $I_\tau(\overline{1}, r) = \overline{1}$,

(I2) $\lambda \geq I_\tau(\lambda,\ r)$,

(I3) $I_\tau(\lambda,\ r) \wedge I_\tau(\mu,\ r) = I_\tau(\lambda \wedge \mu,\ r)$,

(I4) $I_\tau(\lambda,\ r) \geq I_\tau(\lambda,\ s)$ *if* $r \leq s$,

(I5) $I_\tau(I_\tau(\lambda,\ r),\ r) = I_\tau(\lambda,\ r)$.

The fuzzy interior operator I generates a smooth topology $\tau_I : I^X \to I$ as follows

$$\tau_I(\lambda) = \bigvee\{r \in I|\ I(\lambda, r) = \lambda\}.$$

Lemma 2.5. *[17] Let $f : X \to Y$ be a mapping and let λ and μ be fuzzy sets in X and Y , respectively, then the following properties hold:*

(1) $\lambda \leq f^{-1}(f(\lambda))$ *and equality holds if f is injective.*

(2) $f(f^{-1}(\mu)) \leq \mu$ *and equality holds if f is surjective.*

(3) For any fuzzy point $x_t \in X$, $f(x_t)$ is a fuzzy point in Y and $f(x_t) = (f(x))_t$.

(4) When $f(\lambda) \leq \mu,\ \lambda \leq f^{-1}(\mu)$.

Definition 2.6. *[21] Let $(X,\ \tau)$ be a smooth topological space, $\lambda \in I^X$ and $r \in I_0$. Then*

(1) A fuzzy set λ is called r-fuzzy regular open (for short, r-fro) if $\lambda = I_\tau(C_\tau(\lambda, r), r)$.

(2) A fuzzy set λ is called r-fuzzy regular closed (for short, r-frc) if $\lambda = C_\tau(I_\tau(\lambda, r), r)$.

Definition 2.7. *[12] Let $(X,\ \tau)$ be a smooth topological space, let $\lambda,\ \mu \in I^X$ and $r \in I_0$. A fuzzy set λ is called r-generalized fuzzy closed (r-gfc, for short) if $C_\tau(\lambda, r) \leq \mu$, whenever $\lambda \leq \mu$ and $\tau(\mu) \geq s$ for all $0 < s \leq r$. The complement of r-gfc is called an r-generalized fuzzy open (r-gfo, for short) if $\overline{1} - \lambda$ is r-gfc.*

Definition 2.8. *[25] Let $(X,\ \tau)$ and $(Y,\ \eta)$ be a smooth topological space's. Let $f:(X,\ \tau)\to(Y,\ \eta)$ be a function. Then f is called fuzzy regular continuous (FR-continuous) iff $f^{-1}(\mu)$ is r-frc set in X for each $\mu\in I^Y$ with $\eta(\overline{1}-\mu)\geq r$.*

Definition 2.9. *[25] Let $(X,\ \tau)$ be a smooth topological space. For $\lambda,\mu\in I^X$ and $r\in I_0$.*

(1) The r-fuzzy regular closure of λ, denoted by $RC_\tau(\lambda,r)$, and is defined by $RC_\tau(\lambda,r)=\bigwedge\{\mu\in I^X\mid\mu\geq\lambda,\mu$ is r-frc $\}$.

(2) The r-fuzzy regular interiror of λ, denoted by $RI_\tau(\lambda,r)$, and is defined by $RI_\tau(\lambda,r)=\bigvee\{\mu\in I^X\mid\mu\leq\lambda,\mu$ is r-fro $\}$.

Proposition 2.10. *[25] A function $RC: I^X\times I_0\to I^X$ is called a fuzzy regular closure operator if it satisfies the following conditions: for $\lambda,\ \mu\in I^X$ and $r,s\in I_0$,*

(C1) $RC(\overline{0},r)=\overline{0}$,

(C2) $\lambda\leq RC(\lambda,\ r)$,

(C3) $RC(\lambda,\ r)\vee RC(\mu,\ r)=RC(\lambda\vee\mu,\ r)$,

(C4) $RC(\lambda,\ r)\leq RC(\lambda,\ s)$ *if* $r\leq s$,

(C5) $RC(RC(\lambda,\ r),\ r)=RC(\lambda,\ r)$.

The fuzzy regular closure operator RC generates a fuzzy topology $\tau_{RC}(\lambda): I^X\to I$ given by

(C6) $\tau_{RC}(\lambda)=\bigvee\{r\in I|\ RC(\overline{1}-\lambda,r)=\overline{1}-\lambda\}$.

Proposition 2.11. *[25] A mapping $RI: I^X\times I_0\to I^X$ is called a fuzzy regular interior operator if, for $\lambda,\ \mu\in I^X$ and $r,s\in I_0$, it satisfies the following conditions:*

(I1) $RI(\overline{1},r)=\overline{1}$,

(I2) $\lambda\geq RI_\tau(\lambda,\ r)$,

(I3) $RI(\lambda,\ r)\wedge RI(\mu,\ r)=RI(\lambda\wedge\mu,\ r)$,

(I4) $RI(\lambda,\ r)\geq RI(\lambda,\ s)$ *if* $r\leq s$,

(I5) $RI(RI(\lambda,\ r),\ r)=RI(\lambda,\ r)$,

(I6) $RI(\overline{1}-\lambda,\ r)=\overline{1}-RC(\lambda,\ r)$.

The fuzzy regular interior operator RI generates a fuzzy topology $\tau_{RI}(\lambda): I^X\to I$ given by

(I7) $\tau_{RI}(\lambda)=\bigvee\{r\in I|\ RI(\lambda,r)=\lambda\}$.

Definition 2.12. *[15] A triple (X,τ_1,τ_2) consisting of the set X endowed with smooth topologies τ_1 and τ_2 on X is called a smooth bitopological space (smooth bts, for short). For $\lambda\in I^X$ and $r\in I_0$, r-τ_i-fuzzy open (resp. fuzzy closed) set denotes the r-fuzzy open (resp. fuzzy closed) set in (X,τ_i), for $i=1,2$.*

Theorem 2.13. *[6,13] Let (X, τ_1, τ_2) be a smooth bts. For $\lambda \in I^X$ and $r \in I_0$, a τ_i-fuzzy closure of λ is a mapping $C_{\tau_i} : I^X \times I_0 \to I^X$ defined as*

$$C_{\tau_i}(\lambda, r) = \bigwedge\{\mu \in I^X \mid \mu \geq \lambda,\ \tau_i(\overline{1} - \mu) \geq r\}.$$

And, a τ_i-fuzzy interior of λ is a mapping $I_{\tau_i} : I^X \times I_0 \to I^X$ defined as

$$I_{\tau_i}(\lambda, r) = \bigvee\{\mu \in I^X \mid \mu \leq \lambda,\ \tau_i(\mu) \geq r\}.$$

Then:
(1) C_{τ_i} (resp. I_{τ_i}) is a fuzzy closure (resp. fuzzy interior) operator.
(2) $\tau_{C_{\tau_i}} = \tau_{I_{\tau_i}} = \tau_i$.
(3) $I_{\tau_i}(\overline{1} - \lambda, r) = \overline{1} - C_{\tau_i}(\lambda, r)$, $\forall r \in I_0,\ \lambda \in I^X$.

Definition 2.14. *[24] Let (X, τ_1, τ_2) be a smooth bts, $\lambda \in I^X$ and $r \in I_0$. Then λ is called:*

(1) an r-(τ_i, τ_j)-generalized fuzzy closed (briefly, r-(τ_i, τ_j)-gfc), if $C_{\tau_j}(\lambda, s) \leq \mu$, whenever $\lambda \leq \mu$ such that $\tau_i(\mu) \geq s$ for all $0 < s \leq r$.

(2) an r-(τ_i, τ_j)-generalized fuzzy open (briefly, r-(τ_i, τ_j)-gfo), if $\overline{1} - \lambda$ is an r-(τ_i, τ_j)-gfc.

Definition 2.15. *[24] A mapping $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called:*

(1) (i, j)-generalized fuzzy continuous ((i, j)-GF-continuous, for short) if $f^{-1}(\mu)$ is an r-(τ_i, τ_j)-gfc in X for each $\mu \in I^Y$ with $\sigma_j(\overline{1} - \mu) \geq r$.

(2) (i, j)-generalized fuzzy irresolute ((i, j)-GF-irresolute, for short) if $f^{-1}(\mu)$ is an r-(τ_i, τ_j)-gfc in X for each r-(σ_i, σ_j)-gfc in $\mu \in I^Y$.

§3. r-(τ_i, τ_j)-generalized regular fuzzy closed sets

In this section we introduce and investigate the concept of r-(τ_i, τ_j)-generalized regular fuzzy closed sets in smooth bts (X, τ_1, τ_2).

Definition 3.1. *Let (X, τ_1, τ_2) be a smooth bts, $\lambda \in I^X$ and $r \in I_0$. Then λ is called:*

(1) r-(τ_i, τ_j)-generalized regular fuzzy closed (briefly, r-(τ_i, τ_j)-grfc), if $RC_{\tau_j}(\lambda, s) \leq \mu$, whenever $\lambda \leq \mu$ such that $\tau_i(\mu) \geq s$ for all $0 < s \leq r$.

(2) r-(τ_i, τ_j)-generalized regular fuzzy open (briefly, r-(τ_i, τ_j)-grfo), if $\overline{1} - \lambda$ is an r-(τ_i, τ_j)-grfc.

The set of all r-(τ_i, τ_j)-grfc and r-(τ_i, τ_j)-grfo sets of a smooth bts (X, τ_1, τ_2) will be denoted by r-(τ_i, τ_j)-GRFC(X) and r-(τ_i, τ_j)-GRFO(X) respectively.

Remark 3.2. *If $\tau_1 = \tau_2$ in Definition , then r-(τ_i, τ_j)-grfc is an r-grfc in Definition 3.1 in the sense of [12].*

Proposition 3.3. *Let (X, τ_1, τ_2) be a smooth bts, $\lambda \in I^X$ and $r \in I_0$. Then*

(1) If λ is an r-τ_j-frc set, then λ is an r-(τ_i, τ_j)-grfc .

(2) If λ is an r-τ_j-fro set, then λ is an r-(τ_i, τ_j)-grfo .

Proof. To show (1), let $\lambda \leq \mu$ such that $\tau_i(\mu) \geq s$ for $0 < s \leq r$. Since λ is a r-τ_j-frc set, then $RC_{\tau_j}(\lambda, r) = \lambda$. In view of Proposition (C4), we get $RC_{\tau_j}(\lambda, s) \leq RC_{\tau_j}(\lambda, r) = \lambda$ for all $s \leq r$. Thus, $RC_{\tau_j}(\lambda, s) \leq \mu$. Hence, λ is an r-(τ_i, τ_j)-grfc.

To prove (2), clearly $\overline{1} - \lambda$ is an r-τ_j-frc set. By using (1), we get that λ is an r-(τ_i, τ_j)-grfo. □

The converse of the above Proposition is not true as seen from the following example.

Example 3.4. *Let $X = \{a, b, c\}$, $\lambda, \mu, \delta \in I^X$ are defined as $\lambda(a) = 0.6, \lambda(b) = 0.4, \lambda(c) = 0.7; \mu(a) = 0.8, \mu(b) = 0.4, \mu(c) = 0.7; \delta(a) = 0.7, \delta(b) = 0.5, \delta(c) = 0.7$. We define smooth topologies $\tau_1, \tau_2 : I^X \to I$ as follows:* $\tau_1(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{2} & \text{if } \lambda = \lambda, \\ 0 & \text{otherwise}, \end{cases}$ $\tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{2} & \text{if } \lambda = \mu, \\ 0 & \text{otherwise}. \end{cases}$
Then for $r = \frac{1}{2}$ the fuzzy set δ is r-(τ_1, τ_2)-grfc but not r-τ_2-frc set.

Theorem 3.5. *Let (X, τ_1, τ_2) be a smooth bts, $\lambda \in I^X$ and $r \in I_0$. If λ is both r-τ_i-fro set and r-(τ_i, τ_j)-grfc then λ is an r-τ_j-frc set.*

Proof. Since λ is an r-τ_i-fro set. Since $\lambda \leq \lambda$ and λ is an r-(τ_i, τ_j)-grfc, then from Definition (1), $RC_{\tau_j}(\lambda, s) \leq \lambda$ for $0 < s \leq r$. However, $\lambda \leq RC_{\tau_j}(\lambda, s)$. Thus, $RC_{\tau_j}(\lambda, s) = \lambda$ for $0 < s \leq r$. Consequently, $RC_{\tau_j}(\lambda, r) = \lambda$. Hence, λ is an r-τ_j-frc set. □

Proposition 3.6. *Let (X, τ_1, τ_2) be a smooth bts, $\lambda_1, \lambda_2 \in I^X$ and $r \in I_0$. Then:*

(1) If λ_1, λ_2 are r-(τ_i, τ_j)-grfc sets, then $\lambda_1 \vee \lambda_2$ is an r-(τ_i, τ_j)-grfc.

(2) If λ_1, λ_2 are r-(τ_i, τ_j)-grfo sets, then $\lambda_1 \wedge \lambda_2$ is an r-(τ_i, τ_j)-grfo.

Proof. To prove part (1), let $\lambda_1 \vee \lambda_2 \leq \mu$ such that $\tau_i(\mu) \geq s$ for $0 < s \leq r$. This yields, $\lambda_1 \leq \mu$ and $\lambda_2 \leq \mu$. Since λ_1, λ_2 are r-(τ_i, τ_j)-grfc sets, then $RC_{\tau_j}(\lambda_1, s) \leq \mu$ and $RC_{\tau_j}(\lambda_2, s) \leq \mu$, imply $RC_{\tau_j}(\lambda_1, s) \vee RC_{\tau_j}(\lambda_2, s) \leq \mu$. It implies $RC_{\tau_j}(\lambda_1 \vee \lambda_2, s) = RC_{\tau_j}(\lambda_1, s) \vee RC_{\tau_j}(\lambda_2, s) \leq \mu$. Hence, $\lambda_1 \vee \lambda_2$ is r-(τ_i, τ_j)-grfc. The proof of (2), follows from the duality of (1). □

Remark 3.7. *The intersection (resp., union) of two r-(τ_i, τ_j)-grfc (resp. grfo) sets cannot to be an r-(τ_i, τ_j)-grfc (resp. grfo) set as seen from the following example.*

Example 3.8. *Let $X = \{a, b, c\}$, $\lambda, \mu, \delta \in I^X$ are defined as $\lambda(a) = 0.8, \lambda(b) = 0.4, \lambda(c) = 0.7; \mu(a) = 0.6, \mu(b) = 0.5, \mu(c) = 0.8; \delta(a) = 0.6, \delta(b) = 0.4, \delta(c) = 0.7$. We define smooth topologies $\tau_1, \tau_2 : I^X \to I$ as follows:* $\tau_1(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{2} & \text{if } \lambda = \lambda, \\ 0 & \text{otherwise}, \end{cases}$ $\tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{2} & \text{if } \lambda = \mu, \\ 0 & \text{otherwise}. \end{cases}$
Then for $r = \frac{1}{2}$ the fuzzy sets λ and μ are r-(τ_1, τ_2)-grfc but $\lambda \wedge \mu = \delta$ is not r-(τ_1, τ_2)-grfc set.

Next we introduce some properities of r-(τ_i, τ_j)-grfc (resp. r-(τ_i, τ_j)-grfo) sets.

Proposition 3.9. *Let (X, τ_1, τ_2) be a smooth bts. If $\tau_1 \leq \tau_2$, then r-(τ_2, τ_1)-$GRFC(X) \leq r$-$(\tau_1, \tau_2) - GRFC(X)$.*

Proof. Let $\lambda \in r$-(τ_2, τ_1)-$GRFC(X)$, i.e., λ is an r-(τ_2, τ_1)-grfc. Let $\lambda \leq \mu$ such that $\tau_1(\mu) \geq s$ for $0 < s \leq r$. Since $\tau_1 \leq \tau_2$, then $\tau_2(\mu) \geq s$ for $0 < s \leq r$. Since λ is an r-(τ_2, τ_1)- grfc, we have

$RC_{\tau_1}(\lambda, s) \leq \mu$. Again since $\tau_1 \leq \tau_2$, then $RC_{\tau_2}(\lambda, s) \leq RC_{\tau_1}(\lambda, s) \leq \mu$. So, $RC_{\tau_2}(\lambda, s) \leq \mu$. Hence, $\lambda \in r$-$(\tau_1, \tau_2) - GRFC(X)$. □

Remark 3.10. *Let (X, τ_1, τ_2) be a smooth bts, $\lambda \in I^X$ and $r \in I_0$. Then*

(1) r-(τ_1, τ_2)-$GRFC(X)$ is generally not equal to r-(τ_2, τ_1)-$GRFC(X)$. To show this consider Example .

(2) If $\lambda \in r$-(τ_1, τ_2)-$GRFC(X) \cap r$-(τ_2, τ_1)-$GRFC(X)$ then λ is called pairwise grfc.

Theorem 3.11. *Let (X, τ_1, τ_2) be a smooth bts, $\lambda, \mu \in I^X$ and $r \in I_0$. Then:*

(1) If λ is an r-(τ_i, τ_j)-grfc such that $\lambda \leq \mu \leq RC_{\tau_j}(\lambda, r)$, then μ is an r-(τ_i, τ_j)-grfc.

(2) λ is an r-(τ_i, τ_j)-grfo if and only if $\mu \leq RI_{\tau_j}(\lambda, r)$, whenever $\mu \leq \lambda$ and μ is an r-τ_i-frc set.

(3) If λ is an r-(τ_i, τ_j)-grfo such that $RC_{\tau_j}(\lambda, r) \leq \mu \leq \lambda$, then μ is an r-(τ_i, τ_j)-grfo.

Proof. To prove (1), let $\mu \leq \nu$ such that $\tau_i(\nu) \geq s$ for $0 < s \leq r$. Since $\lambda \leq \mu$, we obtain $\lambda \leq \nu$. Since λ is an r-(τ_2, τ_1)-grfc, this yields $RC_{\tau_j}(\lambda, s) \leq \nu$ for $0 < s \leq r$. From Definition (1) and Proposition (C5), we have $RC_{\tau_j}(\mu, s) \leq RC_{\tau_j}(RC_{\tau_j}(\lambda, s), s) = RC_{\tau_j}(\lambda, s) \leq \nu$. Thus, $RC_{\tau_j}(\mu, s) \leq \nu$ and consequently, μ is an r-(τ_i, τ_j)-grfc.

Next to prove (2), for the necessity, let $\overline{1} - \lambda \leq \overline{1} - \mu$ and $\tau_i(\overline{1} - \mu) \geq s$ for $0 < s \leq r$ and apply Definition (1) and Proposition (6), giving the required result.

Conversely, let $\overline{1} - \lambda \leq \mu$ such that $\tau_i(\mu) \geq s$ for $0 < s \leq r$. i.e., $\overline{1} - \mu \leq \lambda$ such that $\overline{1} - \mu$ is an s-fuzzy closed set for $0 < s \leq r$. Assuming we have $\overline{1} - \mu \leq RI_{\tau_j}(\lambda, s)$, this implies $\overline{1} - RI_{\tau_j}(\lambda, s) \leq \mu$. In view of Proposition (6), we then have $RC_{\tau_j}(\overline{1} - \lambda, s) \leq \mu$. Thus, $\overline{1} - \lambda$ is an r-(τ_i, τ_j)-grfc. Hence, λ is an r-(τ_i, τ_j)-grfo.

Finally, to prove (3), taking $\overline{1} - \lambda$ as an r-(τ_i, τ_j)-grfc and then applying (1), we have the required result. □

Theorem 3.12. *Let (X, τ_1, τ_2) be a smooth bts. Then for each $x \in X$ and $t = 1$, x_t is an r-τ_i-frc set or $\overline{1} - x_t$ is an r-(τ_i, τ_j)-grfc.*

Proof. If x_t is not an r-τ_i-frc set, then $\overline{1} - x_t$ is not an r-τ_i-fro set, implying that the only r-τ_i-fro set in X which containing $\overline{1} - x_t$ is $\overline{1}$. Thus, $RC_{\tau_j}(\overline{1} - x_t, s) \leq \overline{1}$ for all $0 < s \leq r$. Therefore, $\overline{1} - x_t$ is an r-(τ_i, τ_j)-grfc. □

§4. Characterization of (i, j) - generalized regular fuzzy closure operator

In this section, we introduce a new fuzzy closure operator by using r-(τ_i, τ_j)-grfc sets and study some of their properties. Also, we introduce a new smooth topology by using the fuzzy closure operator.

Definition 4.1. *Let (X, τ_1, τ_2) be a smooth bts, $\lambda \in I^X$ and $r \in I_0$. The (i, j)- generalized regular fuzzy closure of λ is a map, (i, j)-$GRC : I^X \times I_0 \to I^X$ defined by*

(i,j)-$GRC(\lambda,r) = \wedge\{\rho \in I^X | \rho \geq \lambda, \rho$ *is* r-(τ_i,τ_j)-*grfc* $\}$,
and the (i,j)*-generalized regular fuzzy interior of* λ *is a map,* (i,j)-$GRI : I^X \times I_0 \to I^X$ *defined by*

(i,j)-$GRI(\lambda,r) = \vee\{\rho \in I^X | \rho \leq \lambda, \rho$ *is* r-(τ_i,τ_j)-*grfo*$\}$.

Proposition 4.2. *Let* (X,τ_1,τ_2) *be a smooth bts,* $\lambda \in I^X$ *and* $r \in I_0$*. Then,* $RI_{\tau_j}(\lambda,r) \leq (i,j)$-$GRI(\lambda,r) \leq \lambda \leq (i,j)$-$GRC(\lambda,r) \leq RC_{\tau_j}(\lambda,r)$.

Proof. Since every r-τ_j-frc (resp. r-τ_j-fro) set is an r-(τ_i,τ_j)-grfc (resp. r-(τ_i,τ_j)-grfo) set, the proof is established. □

Next, we state some basic properties of (i,j)-GRC and (i,j)-GRI in the following proposition.

Proposition 4.3. *Let* (X,τ_1,τ_2) *be a smooth bts,* $\lambda,\lambda_1,\lambda_2 \in I^X$ *and* $r \in I_0$*. Then:*

(1) (i,j)-$GRI(\overline{1}-\lambda,r) = \overline{1} - (i,j) - GRC(\lambda,r)$.

(2) If $\lambda_1 \leq \lambda_2$*, then* (i,j)-$GRC(\lambda_1,r) \leq (i,j)$-$GRC(\lambda_2,r)$.

(3) If λ *is an* r-(τ_i,τ_j)*-grfc, then* (i,j)-$GRC(\lambda,r) = \lambda$.

(4) If $\lambda_1 \leq \lambda_2$*, then* (i,j)-$GRI(\lambda_1,r) \leq (i,j)$-$GRI(\lambda_2,r)$.

(5) If λ *is an* r-(τ_i,τ_j)*-grfo, then* (i,j)-$GRI(\lambda,r) = \lambda$.

Proof. We prove (1) using Definition

$$\begin{aligned}\overline{1} - (i,j)\text{-}GRC(\lambda,r) &= \overline{1} - \wedge\{\rho \in I^X | \rho \geq \lambda,\ \rho \text{ is } r\text{-}(\tau_i,\tau_j)\text{-grfc }\},\\ &= \vee\{\overline{1}-\rho \in I^X | \overline{1}-\rho \leq \overline{1}-\lambda,\ \overline{1}-\rho \text{ is } r\text{-}(\tau_i,\tau_j)\text{-grfo }\},\\ &= (i,j)\text{-}GRI(\overline{1}-\lambda,r).\end{aligned}$$

The proof of (2), follows from Definition while the proof of (3), follows from Definition and Proposition . The proof of (4), comes by taking the complement of (2) and from (1). Finally, the proof of (5) is from the same elements as are in (3). □

In Proposition the converse of (3) and (5) is not true as the following example shows.

Example 4.4. *Let* $X = \{a,b\}$*. Define smooth topologies* $\tau_1,\tau_2 : I^X \to I$ *as follows:*

$$\tau_1(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0},\overline{1}\},\\ \frac{1}{2} & \text{if } \lambda = a_{0.7},\\ 0 & \text{otherwise,}\end{cases} \quad \tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0},\overline{1}\},\\ \frac{1}{2} & \text{if } \lambda = a_{0.8},\\ 0 & \text{otherwise.}\end{cases}$$

Then (X,τ_1,τ_2) *is a smooth bts. The fuzzy set* $a_{0.7}$ *is not a* $1/2$-(τ_1,τ_2)*-grfc set on* X *because* $a_{0.7} \leq a_{0.7}$, $\tau_1(a_{0.7}) \geq s$, $0 < s \leq 1/2$, $RC_{\tau_2}(a_{0.7},s) = \overline{1} \nleq a_{0.7}$*. Since* $a_{0.7} \vee b_s$ *is a* $1/2$-(τ_1,τ_2)*-grfc set for* $s \in I_0$*, then* $(1,2)$-$GRC(a_{0.7},1/2) = \bigwedge(a_{0.7} \vee b_s) = a_{0.7} \vee \bigwedge_{s \in I_0} b_s = a_{0.7}$.

Theorem 4.5. *Let* (X,τ_1,τ_2) *be a smooth bts,* $\lambda \in I^X$ *and* $r \in I_0$*. Then:*

(1) (i,j)-GRC *(resp.* (i,j)-GRI*) is a generalized regular fuzzy closure (resp. generalized regular fuzzy interior) operator.*

(2) define $\tau_{(i,j)\text{-}GRC} : I^X \to I$ *as*

$\tau_{(i,j)\text{-}GRC}(\lambda) = \bigvee\{r \in I | (i,j)\text{-}GRC(\overline{1}-\lambda,r) = \overline{1}-\lambda\}$.

Then, $\tau_{(i,j)\text{-}GRC}$ *is a smooth topology on* X *such that* $\tau_j \leq \tau_{(i,j)\text{-}GRC}$

Proof. We have proven that (i, j)-GRC is a generalized regular fuzzy closure operator and in a similar way can prove that (i, j)-GRI is a generalized regular fuzzy interior operator. To prove (1), we need to satisfy conditions (C1)-(C5) in Proposition

(C1) Since $\overline{0}$ is an r-τ_j-frc set in X, then from Proposition (1), $\overline{0}$ is an r-(τ_i, τ_j)-grfc in X and, from Proposition (3), we have (i, j)-$GRC(\overline{0}, r) = \overline{0}$.

(C2) Follows immediately from Definition

(C3) Since $\lambda \le \lambda \vee \mu$ and $\mu \le \lambda \vee \mu$, then from Proposition (2),

(i, j)-$GRC(\lambda, r) \le (i, j)$-$GRC(\lambda \vee \mu, r)$ and (i, j)-$GRC(\mu, r) \le (i, j)$-$GRC(\lambda \vee \mu, r)$. This implies that (i, j)-$GRC(\lambda, r) \vee (i, j)$-$GRC(\mu, r) \le (i, j)$-$GRC(\lambda \vee \mu, r)$.

Suppose (i, j)-$GRC(\lambda \vee \mu, r) \not\le (i, j)$-$GRC(\lambda, r) \vee (i, j)$-$GRC(\mu, r)$. Consequently, $x \in X$ and $t \in (0, 1)$ exist such that

$$(i, j)\text{-}GRC(\lambda, r)(x) \vee (i, j)\text{-}GRC(\mu, r)(x) < t < (i, j)\text{-}GRC(\lambda \vee \mu, r)(x). \tag{1}$$

Since (i, j)-$GRC(\lambda, r)(x) < t$ and (i, j)-$GRC(\mu, r)(x) < t$, there exist r-(τ_i, τ_j)-grfc sets ρ_1, ρ_2 with $\lambda \le \rho_1$ and $\mu \le \rho_2$ such that $\rho_1(x) < t, \rho_2(x) < t$. Since $\lambda \vee \mu \le \rho_1 \vee \rho_2$ and $\rho_1 \vee \rho_2$ is an r-(τ_i, τ_j)-grfc from Proposition (1), we have (i, j)-$GRC(\lambda \vee \mu, r)(x) \le (\rho_1 \vee \rho_2)(x) < t$. This, however, contradicts (1). Hence, (i, j)-$GRC(\lambda, r) \vee (i, j)$-$GRC(\mu, r) = (i, j)$-$GRC(\lambda \vee \mu, r)$.

(C4) Let $r \le s$, $r, s \in I_0$. Suppose (i, j)-$GRC(\lambda, r) \not\le (i, j)$-$GRC(\lambda, s)$. Consequently, $x \in X$ and $t \in (0, 1)$ exist such that

$$(i, j)\text{-}GRC(\lambda, r)(x) < t < (i, j)\text{-}GRC(\lambda, r)(x). \tag{2}$$

Since (i, j)-$GRC(\lambda, s)(x) < t$, there is an s-(τ_i, τ_j)-grfc set ρ with $\lambda \le \rho$ such that $\rho(x) < t$. This yields $RC_{\tau_j}(\rho, s_1) \le \mu$, whenever $\rho \le \mu$ and $\tau_i(\mu) \ge s_1$, for $0 < s_1 \le s$. Since $r \le s$, then $RC_{\tau_j}(\rho, r_1) \le \mu$ whenever $\rho \le \mu$ and $\tau_i(\mu) \ge r_1$, for $0 < r_1 \le r \le s_1 \le s$. This implies ρ is an r-(τ_i, τ_j)-grfc. From Definition , we have (i, j)-$GRC(\lambda, r)(x) \le \rho(x) < t$. This contradicts (2). Hence, (i, j)-$GRC(\lambda, r) \le (i, j)$-$GRC(\lambda, s)$.

(C5) Let ρ be any r-(τ_i, τ_j)-grfc containing λ. Then, from Definition , we have (i, j)-$GRC(\lambda, r) \le \rho$. From Proposition (2), we obtain (i, j)-$GRC((i, j)$-$GRC(\lambda, r), r) \le (i, j)$-$GRC(\rho, r) = \rho$. This mean that (i, j)-$GRC((i, j)$-$GRC(\lambda, r), r)$ is contained in every r-(τ_i, τ_j)-grfc set containing λ. Hence, (i, j)-$GRC((i, j)$-$GRC(\lambda, r), r) \le (i, j)$-$GRC(\lambda, r)$. However, (i, j)-$GRC(\lambda, r) \le (i, j)$-$GRC((i, j)$-$GRC(\lambda, r), r)$. Therefore, (i, j)-$GRC((i, j)$-$GRC(\lambda, r), r) = (i, j)$-$GRC(\lambda, r)$. Thus (i, j)-GRC is a generalized regular fuzzy closure operator.

To prove (2), using (1) and Proposition , we get $\tau_{(i,j)\text{-}GRC}$, which is a smooth topology. By Proposition , we have (i, j)-$GRC(\lambda, r) \le RC_{\tau_j}(\lambda, r)$. This means that $RC_{\tau_j}(\overline{1} - \lambda, r) = \overline{1} - \lambda$ and implies (i, j)-$GRC(\overline{1} - \lambda, r) = \overline{1} - \lambda$. Thus, $\tau_j(\lambda) \le \tau_{(i,j)\text{-}GRC}(\lambda)\ \forall \lambda \in I^X$. □

Proposition 4.6. *Let (X, τ_1, τ_2) be a smooth bts, $\lambda \in I^X$ and $r \in I_0$. Then:*

(1) If $\tau_1 \le \tau_2$, then $(1, 2)$-$GRC(\lambda, r) \le (2, 1)$-$GRC(\lambda, r)$.

(2) If λ is an r-(τ_i, τ_j)-grfc, then λ is an r-$\tau_{(i,j)\text{-}GRC}$-fuzzy set.

(3) If $\tau_1 \le \tau_2$, then $\tau_{(2,1)\text{-}GRC} \le \tau_{(2,1)\text{-}GRC}$.

Proof. To show (1), suppose $(1,2)$-$GRC(\lambda,r) \nleq (2,1)$-$GRC(\lambda,r)$. There exists $x \in X$ and $t \in (0,1)$ such that

$$(2,1)\text{-}GRC(\lambda,r)(x) < t < (1,2)\text{-}GRC(\lambda,r)(x). \tag{3}$$

Since $(2,1)$-$GRC(\lambda,r)(x) < t$, there exists an r-(τ_2,τ_1)-grfc set ρ such that $\lambda \leq \rho$ and $\rho(x) < t$. From Proposition , ρ is an r-(τ_1,τ_2)-grfc, which implies $(1,2)$-$GRC(\lambda,r)(x) < \rho(x) < t$. This contradicts (3).

The proof of (2) follows from Proposition (3). Finally (3), follows directly from (1). □

The converse of Proposition (2) is not true as shown in Example .

§5. (i,j) - GRF - continuous and (i,j) - GRF - irresolute mappings

In this section we introduce the concepts of (i,j)-generalized regular fuzzy continuous (resp. irresolute) mappings in smooth bts and study the relationship between them. We also investigate some of their properties and also, we introduce the definition of (i,j)-$FRT_{1/2}$ space in smooth bts (X,τ_1,τ_2). Throughout this section consider (X,τ_1,τ_2), (Y,σ_1,σ_2) and (Z,η_1,η_2) and as smooth bts's. For a mapping f from (X,τ_1,τ_2) into (Y,σ_1,σ_2), we shall denote the fuzzy regular continuous (resp., closed, open) mapping from (X,τ_j) into (Y,σ_j), $j \in \{1,2\}$ by j-fuzzy regular continuous (resp., closed, open) mapping. Firstly, we state the definition of (i,j)-generalized regular fuzzy continuous (resp. irresolute) mappings.

Definition 5.1. *A mapping $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is called:*

(1) (i,j)-generalized regular fuzzy continuous ((i,j)-GRF-continuous, for short) if $f^{-1}(\mu)$ is an r-(τ_i,τ_j)-grfc in X for each $\mu \in I^Y$ with $\sigma_j(\overline{1}-\mu) \geq r$.

(2) (i,j)-generalized regular fuzzy irresolute ((i,j)-GRF-irresolute, for short) if $f^{-1}(\mu)$ is an r-(τ_i,τ_j)-grfc in X for each r-(σ_i,σ_j)-grfc in $\mu \in I^Y$.

Remark 5.2. *[24] Every j-fuzzy continuous function is (i,j)-generalized fuzzy continuous, but converse is not true.*

Remark 5.3.

(1) Every j-fuzzy regular continuous function is (i,j)-generalized regular fuzzy continuous, but converse is not true.

(2) Every (i,j)-generalized regular fuzzy continuous function is (i,j)-generalized fuzzy continuous, but converse is not true.

Example 5.4. *Let $X = \{a,b\}$ and $Y = \{p,q\}$. $\lambda_1,\lambda_2 \in I^X$, $\lambda_3,\lambda_4 \in I^Y$ are defined as* $\lambda_1(a) = 0.5, \lambda_1(b) = 0.7; \lambda_2(a) = 0.5, \lambda_2(b) = 0.8; \lambda_3(p) = 0.7, \lambda_3(q) = 0.4; \lambda_4(p) = 0.9, \lambda_4(q) = 0.2$. *We define smooth topologies $\tau_1,\tau_2,\sigma_1,\sigma_2 : I^X \to I$ as follows:*

$$\tau_1(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_1, \\ 0 & \text{otherwise}, \end{cases} \qquad \tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_2, \\ 0 & \text{otherwise}, \end{cases}$$

$$\sigma_1(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_3, \\ 0 & \text{otherwise}, \end{cases} \qquad \sigma_2(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_4, \\ 0 & \text{otherwise}. \end{cases}$$

Consider the mapping $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ defined by $f(a) = p$ and $f(b) = q$. Then f is $(1,2)$-GRF-continuous but not 2-FR-continuous, for $r = \frac{1}{2}$, $\sigma(\lambda_4) \geq r$, $\overline{1} - \lambda_4$ is r-(τ_1, τ_2)-grfc set but not r-τ_2-frc set.

Example 5.5. *Let $X = \{a, b, c\}$ and $Y = \{p, q, r\}$. $\lambda_1, \lambda_2 \in I^X$, $\lambda_3, \lambda_4 \in I^Y$ are defined as $\lambda_1(a) = 0.5, \lambda_1(b) = 0.7, \lambda_1(c) = 0.9; \lambda_2(a) = 0.5, \lambda_2(b) = 0.7, \lambda_2(c) = 0.9; \lambda_3(p) = 0.7, \lambda_3(q) = 0.4, \lambda_1(r) = 0.7; \lambda_4(p) = 0.5, \lambda_4(q) = 0.7, \lambda_4(r) = 0.9$. We define smooth topologies $\tau_1, \tau_2, \sigma_1, \sigma_2 : I^X \to I$ as follows:*

$$\tau_1(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_1, \\ 0 & \text{otherwise}, \end{cases} \qquad \tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_2, \\ 0 & \text{otherwise}, \end{cases}$$

$$\sigma_1(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_3, \\ 0 & \text{otherwise}, \end{cases} \qquad \sigma_2(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_4, \\ 0 & \text{otherwise}. \end{cases}$$

Consider the mapping $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ defined by $f(a) = p, f(b) = q$ and $f(c) = r$. Then f is $(1,2)$-GF-continuous but not $(1,2)$-GRF-continuous, for $r = \frac{1}{2}$, $\sigma(\overline{1} - (\overline{1} - \lambda_4)) = \sigma(\lambda_4) \geq r$, $\overline{1} - \lambda_4$ is r-(τ_1, τ_2)-gfc set but not r-(τ_1, τ_2)-grfc.

The following Theorem gives an equivalent definition of (i, j)-GRF-continuous mapping.

Theorem 5.6. *A mapping $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is (i, j)-GRF-continuous if and only if $f^{-1}(\mu)$ is an r-(τ_i, τ_j)-grfo in X for each $\mu \in I^Y$ with $\sigma_j(\mu) \geq r$.*

Proof. This follows directly from Definition (2) and Definition (1). □

Theorem 5.7. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a j-FR-continuous, then f is (i, j)-GRF-continuous.*

Proof. Let $\mu \in I^Y$, such that $\sigma_j(\overline{1} - \mu) \geq r$. Since f is a j-FR-continuous, then $f^{-1}(\mu)$ is an r-τ_j-frc set in X. From Proposition (1), we have that $f^{-1}(\mu)$ is an r-(τ_i, τ_j)-grfc. Hence, f is (i, j)-*GRF*- continuous. □

The converse of above Theorem is not true as seen from the above following Example .

Thus we have the following implication and none of them is reversible.

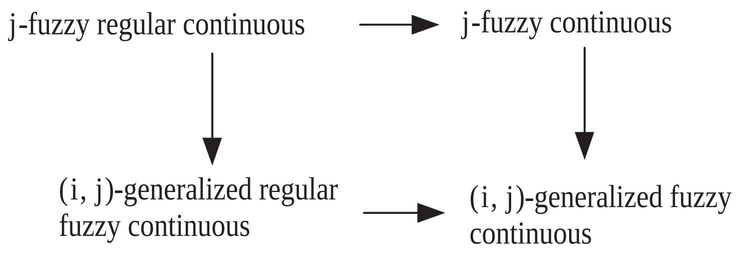

Diagram - I

Theorem 5.8. *Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ be a mapping. If f is (i,j)-GRF-irresolute, then f is (i,j)-GRF-continuous.*

Proof. This follows directly from Proposition (1) and Definition (2). □

Then converse of above Theorem is not true as seen from the following example.

Example 5.9. *Let $X=\{a,b\}$ and $Y=\{p,q\}$. $\lambda_1,\lambda_2\in I^X$, $\lambda_3,\lambda_4\in I^Y$ are defined as* $\lambda_1(a)=0.5,\lambda_1(b)=0.2;\lambda_2(a)=0.5,\lambda_2(b)=0.4;\lambda_3(p)=0.9,\lambda_3(q)=0.6;\lambda_4(p)=0.1,\lambda_4(q)=0.8$. *We define smooth topologies $\tau_1,\tau_2,\sigma_1,\sigma_2:I^X\to I$ as follows:*

$$\tau_1(\lambda)=\begin{cases}1 & \text{if } \lambda\in\{\overline{0},\overline{1}\},\\ \frac{1}{2} & \text{if } \lambda=\lambda_1,\\ 0 & \text{otherwise},\end{cases}\qquad \tau_2(\lambda)=\begin{cases}1 & \text{if } \lambda\in\{\overline{0},\overline{1}\},\\ \frac{1}{2} & \text{if } \lambda=\lambda_2,\\ 0 & \text{otherwise},\end{cases}$$

$$\sigma_1(\lambda)=\begin{cases}1 & \text{if } \lambda\in\{\overline{0},\overline{1}\},\\ \frac{1}{2} & \text{if } \lambda=\lambda_3,\\ 0 & \text{otherwise},\end{cases}\qquad \sigma_2(\lambda)=\begin{cases}1 & \text{if } \lambda\in\{\overline{0},\overline{1}\},\\ \frac{1}{2} & \text{if } \lambda=\lambda_4,\\ 0 & \text{otherwise}.\end{cases}$$

Consider the mapping $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ defined by $f(a)=p$ and $f(b)=q$. Then f is $(1,2)$-GRF-continuous but not $(1,2)$-GRF-irresolute, for $r=\frac{1}{2}$, $\sigma(\lambda_4)\geq r$, $\overline{1}-\lambda_4$ is r-(τ_1,τ_2)-grfc set in X but not r-(τ_1,τ_2)-grfc set in Y.

Theorem 5.10. *Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ be a mapping. Consider the following statements:*

(1) f is (i,j)-GRF-continuous.

(2) $f((i,j)\text{-}GRC(\lambda,r))\leq RC_{\sigma_j}(f(\lambda),r)$, for each $\lambda\in I^X$, $r\in I_0$.

(3) $(i,j)\text{-}GRC(f^{-1}(\mu),r)\leq f^{-1}(RC_{\sigma_j}(\mu,r))$, for each $\mu\in I^Y$.
Then (1)⇒(2)⇒(3).

Proof. (1)⇒(2) Let $\lambda\in I^X$. Since $f(\lambda)\in I^Y$, then $f(\lambda)\leq RC_{\sigma_j}(f(\lambda),r)$. Then, $\lambda\leq f^{-1}(RC_{\sigma_j}(f(\lambda),r))$. Since f is (i,j)-GRF-continuous, then $f^{-1}(RC_{\sigma_j}(f(\lambda),r))$ is an r-(τ_i,τ_j)-grfc in X. Hence, (i,j)-$GRC(\lambda,r)\leq f^{-1}(RC_{\sigma_j}(f(\lambda),r))$ implies $f((i,j)\text{-}GRC(\lambda,r))\leq f(f^{-1}(RC_{\sigma_j}(f(\lambda),r)))$. Thus, $f((i,j)\text{-}GRC(\lambda,r))\leq RC_{\sigma_j}(f(\lambda),r)$.

(2)⇒(3) Letting $\lambda = f^{-1}(\mu)$ and applying (2), we arrive at $f((i,j)\text{-}GRC(f^{-1}(\mu), r)) \leq RC_{\sigma_j}(f(f^{-1}(\mu)), r) \leq RC_{\sigma_j}(\mu, r)$. Consequently, $f((i,j)\text{-}GRC(f^{-1}(\mu), r)) \leq RC_{\sigma_j}(\mu, r)$ implies $f^{-1}(f((i,j)\text{-}GRC(f^{-1}(\mu), r))) \leq f^{-1}(RC_{\sigma_j}(\mu, r))$, which yields $(i,j)\text{-}GRC(f^{-1}(\mu), r) \leq f^{-1}(RC_{\sigma_j}(\mu, r))$. □

Next, we give an example to show that (3) does not lead to (1) in above theorem.

Example 5.11. *Let $X = \{a, b\}$ and $Y = \{p, q\}$. $\lambda_1, \lambda_2 \in I^X$, $\lambda_3, \lambda_4 \in I^Y$ are defined as $\lambda_1(a) = 0.6, \lambda_1(b) = 0.3; \lambda_2(a) = 0.7, \lambda_2(b) = 0.6; \lambda_3(p) = 0.4, \lambda_3(q) = 0.6; \lambda_4(p) = 0.4, \lambda_4(q) = 0.7$. We define smooth topologies $\tau_1, \tau_2, \sigma_1, \sigma_2 : I^X \to I$ as follows:*

$$\tau_1(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_1, \\ 0 & \text{otherwise}, \end{cases} \qquad \tau_2(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_2, \\ 0 & \text{otherwise}, \end{cases}$$

$$\sigma_1(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_3, \\ 0 & \text{otherwise}, \end{cases} \qquad \sigma_2(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_4, \\ 0 & \text{otherwise}. \end{cases}$$

Consider the mapping $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ defined by $f(a) = p$ and $f(b) = q$. Then $(1,2)\text{-}GRC(f^{-1}(\lambda, \frac{1}{2}) \leq f^{-1}(RC_{\sigma_2}(\lambda, \frac{1}{2}))$ for each $\lambda \in I^Y$, but f is not $(1,2)$-GRF-continuous since $\overline{1} - \lambda_4$ is a $\frac{1}{2}$-σ_2-fuzzy closed set in Y, but $f^{-1}(\overline{1} - \lambda_4)$ is not a $\frac{1}{2}$-(τ_1, τ_2)-grfc set in X.

Theorem 5.12. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a mapping. If f is (i,j)-GRF-continuous, then for each $x_t \in Pt(X)$ and for each r-σ_j-fro set $\nu \in Y$ such that $f(x_t) \in \nu$, there exists an r-(τ_i, τ_j)-grfo η in X such that $x_t \in \eta$ and $f(\eta) \leq \nu$.*

Proof. Let $x_t \in Pt(X)$, let ν be an r-σ_j-fro set in Y such that $f(x_t) \in \nu$. Since f is (i,j)-GRF-continuous then, by Theorem , $f^{-1}(\nu)$ is an r-(τ_i, τ_j)-grfo in X such that $x_t \in f^{-1}(\nu)$, let $\eta = f^{-1}(\nu)$, then $f(\eta) \leq \nu$. □

Theorem 5.13. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ and $g : (Y, \sigma_1, \sigma_2) \to (Z, \eta_1, \eta_2)$ be mappings. Then:*

(1) If g is j-FR-continuous and f is (i,j)-GRF-continuous, then $g \circ f$ is (i,j)- GRF-continuous.

(2) If g is (i,j)-GRF-irresolute and f is (i,j)-GRF-irresolute, then $g \circ f$ is (i,j)- GRF-irresolute.

(3) If g is (i,j)-GRF-continuous and f is (i,j)-GRF-irresolute, then $g \circ f$ is (i,j)- GRF-continuous.

Proof. We prove (1), and the proof of (2) and (3) are similar to (1). Let μ be an r-η_j- fuzzy closed set of Z. Since g is a j-fuzzy regular continuous, then $g^{-1}(\mu)$ is an r-σ_j-frc set of Y. When f is (i,j)-GRF-continuous, then $(g \circ f)^{-1}(\mu) = f^{-1}(g^{-1}(\mu))$ is an r-(τ_i, τ_j)-grfc of X. Hence, $g \circ f$ is (i,j)-GRF-continuous. □

We now introduce the definition of (i,j)-$FRT_{1/2}$ space in a smooth bts (X, τ_1, τ_2).

Definition 5.14. *A smooth bts (X, τ_1, τ_2) is said to be (i,j)-$FRT_{1/2}$ space if every r-(τ_i, τ_j)-grfc is an r-τ_j-frc set of X.*

Theorem 5.15. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is (i, j)-GRF-irresolute and X is (i, j)-$FRT_{1/2}$ space, then f is a j-FR-continuous.*

Proof. Let μ be an r-σ_j-frc set of Y. Then, from Proposition (1), we have that μ is an r-(τ_i, τ_j)-grfc of Y. Since f is (i, j)-GRF-irresolute, then $f^{-1}(\mu)$ is an r-(τ_i, τ_j)-grfc of X, but X is (i, j)-$FRT_{1/2}$ space, which implies $f^{-1}(\mu)$ is an r-τ_j-frc set of X. Hence, f is a j-FR-continuous, since every r-σ_j-frc set is r-σ_j-fuzzy closed. □

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Pairwise fuzzy D-Baire spaces

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Abstract In this paper we introduce the concept of D-Baire bitopological spaces and several properties are investigated.

Keywords Pairwise fuzzy dense, Pairwise fuzzy open, Pairwise fuzzy closed, Pairwise fuzzy nowhere dense, Pairwise fuzzy first category, Pairwise fuzzy residual, Pairwise fuzzy Baire, Pairwise fuzzy D-Baire spaces
2010 Mathematics Subject Classification 54A40, 03E72.

§1. Introduction

The theory of fuzzy sets was initiated by by L.A.ZADEH in his classical paper [12] in the year 1965 as an attempt to develop a mathematically precise framework in which to treat systems or phenomena which cannot themselves be characterized precisely. The potential of fuzzy notion was realized by the researchers and has successfully been applied for investigations in all the branches of Science and Technology. The paper of C.L.CHANG [3] in 1968 paved the way for the subsequent tremendous growth of the numerous fuzzy topological concepts. In 1989, KANDIL[4] introduced the concept of fuzzy bitopological space as an extension and generalization of fuzzy topological space.Rene Baire introduced the concept of first and second category in topology. To define first category Baire, relied on Cantor's definition of dense sets and P.du Bois-Reymond's definition of nowhere dense sets.Denjoy introduced the concept residual as the sets which are complements of first category sets around 1912.

The concept of Baire spaces in fuzzy setting was introduced and studied by G. Thangaraj and S. Anjalmose in [6]. The concept of Baire spaces in fuzzy bitopological setting was introduced and studied by the authors in [8]. In this paper we introduce the concept of D-Baire bitopological spaces in fuzzy setting and investigate several characterizations of pairwise fuzzy D-Baire spaces.

§2. Preliminaries

Now we introduce some basic notions and results used in the sequel. In this work by (X, T_1, T_2) or simply by X,we will denote a fuzzy bitopological space due to KANDIL[4] . By a fuzzy Bitopological space we mean an ordered triple (X, T_1, T_2) where T_1 and T_2 are fuzzy topologies on the non-empty set X.

Definition 2.1. Let λ and μ be any two fuzzy sets in a fuzzy topological space (X, T). Then we define:

(i) $\lambda \vee \mu$: X $\to$ [0,1] as follows: $(\lambda \vee \mu)(x)$ = max $\{\lambda(\text{x}), \mu(\text{x})\}$;

(ii) $\lambda \wedge \mu$: X $\to [0, 1]$ as follows: $(\lambda \wedge \mu)(x)$ = min $\{\lambda(\text{x}), \mu(\text{x})\}$;

(iii) $\mu = \lambda^c \Leftrightarrow \mu(x) = 1 - \lambda(x)$.

For a family $\{\lambda_i/i \in I\}$ of fuzzy sets in (X, T), the union $\psi = \vee_i \lambda_i$ and intersection $\delta = \wedge_i \lambda_i$ are defined respectively as $\psi(x) = sup_i\{\lambda_i(x), x \in X\}$ and $\delta(x) = inf_i\{\lambda_i(x), x \in X\}$.

Definition 2.2.[1] Let (X, T) be a fuzzy topological space. For a fuzzy set λ of X, the interior and the closure of λ are defined respectively as $int(\lambda) = \vee\{\mu/\mu \leq \lambda, \mu \in T\}$ and $cl(\lambda) = \wedge\{\mu/\lambda \leq \mu, 1 - \mu \in T\}$.

Definition 2.3.[8] A fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2) is called pairwise fuzzy nowhere dense if $int_{T_1}(cl_{T_2}(\lambda)) = int_{T_2}(cl_{T_1}(\lambda)) = 0$.

Definition 2.4. [11] Let (X, T_1, T_2) be a fuzzy bitopological space.A fuzzy set λ in (X, T_1, T_2) is called a *pairwise fuzzy open set* if $\lambda \in T_1$ and $\lambda \in T_2$.

Definition 2.5. [11] Let (X, T_1, T_2) be a fuzzy bitopological space.A fuzzy set λ in (X, T_1, T_2) is called a *pairwise fuzzy closed set* if $1 - \lambda \in T_1$ and $1 - \lambda \in T_2$.

Definition 2.6. [5] Let (X, T_1, T_2) be a fuzzy bitopological space.A fuzzy set λ in (X, T_1, T_2) is called a *pairwise fuzzy dense set* if $cl_{T_1}(cl_{T_2}(\lambda)) = cl_{T_2}(cl_{T_1}(\lambda)) = 1$.

Definition 2.7 [8] Let (X, T_1, T_2) be a fuzzy bitopological space.A fuzzy set λ in (X, T_1, T_2) is called *pairwise fuzzy first category set* if $\lambda = \bigvee_{i=1}^{\infty} \lambda_i$ where λ_i's are pairwise fuzzy nowhere dense sets in (X, T_1, T_2). A fuzzy set which is not pairwise fuzzy first category set is called a pairwise fuzzy second category set in (X, T_1, T_2).

Definition 2.8 [8] Let (X, T_1, T_2) be a fuzzy bitopological space.A fuzzy set λ in (X, T_1, T_2) is called a *pairwise fuzzy residual set* if its complement is a pairwise fuzzy first category set.

Definition 2.9 [8] A fuzzy bitopological space (X, T_1, T_2) is called a *pairwise fuzzy Baire* if $int(\bigvee_{i=1}^{\infty} \lambda_i) = 0$ where λ_i's are pairwise fuzzy nowhere dense sets in (X, T_1, T_2).

§3. Pairwise fuzzy D-Baire spaces

Definition 3.1 : A fuzzy bitopological space (X, T_1, T_2) is called a *pairwise fuzzy D-Baire space* if $int_{T_1}(cl_{T_2}(\bigvee_{i=1}^{\infty}(\lambda_i))) = int_{T_2}(cl_{T_1}(\bigvee_{i=1}^{\infty}(\lambda_i))) = 0$, where λ_i's are pairwise fuzzy nowhere dense sets in (X, T_1, T_2).

Example 3.1 : Let X = $\{a, b, c\}$. The fuzzy sets λ , μ and ν are defined on X as follows :
λ : X $\longrightarrow$ [0, 1] is defined as λ (a) = 0.5; λ (b) = 0.7; λ (c) = 0.6.
μ : X $\longrightarrow$[0, 1] is defined as μ(a) = 0.4; μ (b) = 0.6; μ(c) = 0.5.
ν: X $\longrightarrow$ [0,1] is defined as ν(a) = 0.6; ν(b) = 0.5; ν(c) = 0.4.
Clearly $T_1 = \{0, \lambda, \mu, \nu, \lambda \vee \nu, \mu \vee \nu, \lambda \wedge \nu, \mu \wedge \nu, \lambda \wedge (\mu \vee \nu), 1\}$ and
$T_2 = \{0, \lambda, \mu, 1\}$ are fuzzy topologies on X and (X, T_1, T_2) is a fuzzy Bitopological space . Clearly

$1-\lambda$, $1-\mu$, $1-(\lambda\vee\nu)$,$1-(\mu\vee\nu)$, and $1-(\lambda\wedge(\mu\vee\nu))$ are pairwise fuzzy nowhere dense sets in (X,T_1,T_2) . Also $1-\mu=(1-\lambda)\vee(1-\mu)\vee(1-(\lambda\vee\nu))\vee(1-(\mu\vee\nu))\vee(1-(\lambda\wedge(\mu\vee\nu)))$ is a pairwise fuzzy first category set in (X,T_1,T_2) . Also $int_{T_1}(cl_{T_2}(1-\mu))=int_{T_2}(cl_{T_1}(1-\mu))=0$. Hence the bitopological space (X,T_1,T_2) is a pairwise fuzzy D-Baire space

Example 3.2 : Let X = $\{a,b,c\}$. The fuzzy sets λ_i(i=1,2,3) ,μ_j(j=1,2,3), are defined on X as follows :
λ_1 : X $\longrightarrow$ [0, 1] is defined as λ_1 (a) = 0.5; λ_1 (b) = 0.7; λ_1 (c) = 0.6.
λ_2 : X $\longrightarrow$ [0, 1] is defined as λ_2 (a) = 0.4; λ_2 (b) = 0.6; λ_2 (c) = 0.5.
λ_3 : X $\longrightarrow$ [0, 1] is defined as λ_3 (a) = 0.6; λ_3 (b) = 0.5; λ_3 (c) = 0.4.
μ_1 : X $\longrightarrow$[0, 1] is defined as μ_1(a) = 0.8; μ_1 (b) = 0.5; μ_1(c) = 0.7.
μ_2 : X $\longrightarrow$[0, 1] is defined as μ_2(a) = 0.6; μ_1 (b) = 0.9; μ_1(c) = 0.4.
μ_3 : X $\longrightarrow$[0, 1] is defined as μ_3(a) = 0.4; μ_3 (b) = 0.7; μ_3(c) = 0.8.
Clearly $T_1=\{0,\lambda_1,\lambda_2,\lambda_3,\lambda_1\vee\lambda_3,\lambda_2\vee\lambda_3,\lambda_1\wedge\lambda_3,\lambda_2\wedge\lambda_3,\lambda_2\wedge(\lambda_1\wedge\lambda_3),1\}$ and $T_2=\{0,\mu_1,\mu_2,\mu_3,\mu_1\vee\mu_2,\mu_1\vee\mu_3,\mu_2\vee\mu_3,\mu_1\wedge\mu_2,\mu_1\wedge\mu_3,\mu_2\wedge\mu_3,\mu_1\vee(\mu_2\wedge\mu_3),\mu_2\vee(\mu_1\wedge\mu_3),\mu_3\vee(\mu_1\wedge\mu_2),\mu_1\wedge(\mu_2\vee\mu_3),\mu_2\wedge(\mu_1\vee\mu_3),\mu_3\wedge(\mu_1\vee\mu_2),(\mu_1\vee\mu_2\vee\mu_3),1\}$ are fuzzy topologies on X and (X,T_1,T_2) is a fuzzy Bitopological space .α,β and ν are defined on X as follows :
α : X $\longrightarrow$[0, 1] is defined as α(a) = 0.6; α (b) = 0.3; α(c) = 0.4.
β : X $\longrightarrow$[0, 1] is defined as β(a) = 0.4; β (b) = 0.3; β(c) = 0.6.
ν: X $\longrightarrow$ [0,1] is defined as ν(a) = 0.6; ν(b) = 0.5; ν(c) = 0.6.
Clearly α, β, $1-\lambda_1$, $1-\mu_1$, $1-\mu_3$ and $1-(\mu_1\vee\mu_2)$ are pairwise fuzzy nowhere dense sets in (X,T_1,T_2) . Hence $\nu=\alpha\vee\beta\vee(1-\lambda_1)\vee(1-\mu_1)\vee(1-\mu_3)\vee(1-(\mu_1\vee\mu_2))$ is a pairwise fuzzy first category set in (X,T_1,T_2). But $int_{T_2}(cl_{T_1}(\nu))=\mu_1\wedge\mu_2\neq 0$. Hence the bitopological space (X,T_1,T_2) is not a pairwise fuzzy D-Baire space.

Proposition 3.1. Let (X,T_1,T_2) be a fuzzy bitopological space .Then the following are equivalent:
(i)(X,T_1,T_2) is a pairwise fuzzy D-Baire space.
(ii)$int_{T_1}(cl_{T_2}(\lambda))=int_{T_2}(cl_{T_1}(\lambda))=0$, for every pairwise fuzzy first category set λ in (X,T_1,T_2)
(iii)$cl_{T_1}(int_{T_2}(\mu))=cl_{T_2}(int_{T_1}(\mu))=1$, for every pairwise fuzzy residual set μ in (X,T_1,T_2)

Proof. (i)$\Longrightarrow$ (ii). Let λ be a pairwise fuzzy first category set in the pairwise fuzzy D-Baire space (X,T_1,T_2). Then $\lambda=\bigvee_{i=1}^{\infty}(lambda_i)$ where λ_i's are pairwise fuzzy nowhere dense sets in (X,T_1,T_2) . Sine (X,T_1,T_2) is a pairwise fuzzy D-Baire space, $int_{T_1}(cl_{T_2}(\bigvee_{i=1}^{\infty}(\lambda_i)))=int_{T_2}(cl_{T_1}(\bigvee_{i=1}^{\infty}(\lambda_i)))=0$. Hence $int_{T_1}(cl_{T_2}(\lambda))=int_{T_2}(cl_{T_1}(\lambda))=0$, for every pairwise fuzzy first category set λ in (X,T_1,T_2) .
(ii)$\Longrightarrow$ (iii). Let μ be a pairwise fuzzy residual set in (X,T_1,T_2). Then $1-\mu$ is a pairwise fuzzy first category set and hence, by hypothesis, $int_{T_1}(cl_{T_2}(1-\mu))=int_{T_2}(cl_{T_1}(1-\mu))=0$. This implies that, $cl_{T_1}(int_{T_2}(\mu))=cl_{T_2}(int_{T_1}(\mu))=1$. Hence we have $cl_{T_1}(int_{T_2}(\mu))=cl_{T_2}(int_{T_1}(\mu))=1$, for every pairwise fuzzy residual set μ in (X,T_1,T_2)
(iii)$\Longrightarrow$ (i). Let λ_i's be pairwise fuzzy nowhere dense sets in (X,T_1,T_2). Then $\lambda=\bigvee_{i=1}^{\infty}(\lambda_i)$ is a pairwise fuzzy first category set and hence, $1-\lambda$ is a pairwise fuzzy residual set in (X,T_1,T_2). By hypothesis $cl_{T_1}(int_{T_2}(1-\lambda))=cl_{T_2}(int_{T_1}(1-\lambda))=1$. This implies $int_{T_1}(cl_{T_2}(\lambda))=int_{T_2}(cl_{T_1}(\lambda))=0$. That is, $int_{T_1}(cl_{T_2}(\bigvee_{i=1}^{\infty}(\lambda_i)))=int_{T_2}(cl_{T_1}(\bigvee_{i=1}^{\infty}(\lambda_i)))=0$. Hence (X,T_1,T_2) is a pairwise fuzzy D-Baire space.

Theorem 3.1.[8] Let (X, T_1, T_2) be a fuzzy bitopological space .Then the following are equivalent:
(i)(X, T_1, T_2) is a pairwise fuzzy Baire space.
(ii)$int_{T_j}(\lambda) = 0$,(j=1,2) for every pairwise fuzzy first category set λ in (X, T_1, T_2) .
(iii)$cl_{T_j}(\mu) = 1$, (j=1,2) for every pairwise fuzzy residual set μ in (X, T_1, T_2)

Proposition 3.2. If (X, T_1, T_2) is a pairwise fuzzy D-Baire space then (X, T_1, T_2) is a pairwise fuzzy Baire space .

Proof. Let λ be a pairwise fuzzy first category set in a pairwise fuzzy D-Baire space (X, T_1, T_2) . By Proposition 1.1, $int_{T_1}(cl_{T_2}(\lambda)) = int_{T_2}(cl_{T_1}(\lambda)) = 0$. Now $int_{T_1}(\lambda) \leq int_{T_1}(cl_{T_2}(\lambda))$ and $int_{T_2}(\lambda) \leq int_{T_2}(cl_{T_1}(\lambda))$ implies that $int_{T_1}(\lambda) = int_{T_2}(\lambda) = 0$, and by Theorem 3.1, (X, T_1, T_2) is a fuzzy Baire space.

Proposition 3.3. If the fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy D-Baire space , then no nonzero pairwise fuzzy open set is a pairwise fuzzy first category set in (X, T_1, T_2).

Proof. Suppose that the nonzero pairwise fuzzy open set λ is a pairwise fuzzy first category set in (X, T_1, T_2) . Since (X, T_1, T_2) is a pairwise fuzzy D-Baire space and λ is a pairwise fuzzy first category set implies $int_{T_1}(cl_{T_2}(\lambda)) = int_{T_2}(cl_{T_1}(\lambda)) = 0$. But λ is a pairwise fuzzy open set in (X, T_1, T_2), $int_{T_i}(\lambda) = \lambda$ (i=1,2) . This gives $int_{T_1}(\lambda) \leq int_{T_1}(cl_{T_2}(\lambda))$ and $int_{T_2}(\lambda) \leq int_{T_2}(cl_{T_1}(\lambda))$. This implies that $int_{T_1}(\lambda) = int_{T_2}(\lambda) = 0$ and so $\lambda = 0$, a contradiction to λ, being a nonzero pairwise fuzzy open set . Hence no nonzero pairwise fuzzy open set is a pairwise fuzzy first category set in a pairwise fuzzy D-Baire space (X, T_1, T_2).

Proposition 3.4. If (X, T_1, T_2) is a pairwise fuzzy D-Baire space and if $\vee_{i=1}^{\infty}(\lambda_i)$ =1 then there is exists atleast one fuzzy set λ_i such that either $int_{T_1}(cl_{T_2}(\lambda_i)) \neq 0$ or $int_{T_2}(cl_{T_1}(\lambda_i)) \neq 0$.

Proof. Suppose $int_{T_1}(cl_{T_2}(\lambda_i)) = 0$ and $int_{T_2}(cl_{T_1}(\lambda_i)) = 0$ for all i, then λ_i's are pairwise fuzzy nowhere dense sets in (X, T_1, T_2) . Then $\vee_{i=1}^{\infty}(\lambda_i) = 1$ implies that $int_{T_1}cl_{T_2}(\vee_{i=1}^{\infty}(\lambda_i)) = int_{T_1}cl_{T_2}(1) = 1 \neq 0$, a contradiction to (X, T_1, T_2) being a pairwise fuzzy D-Baire space in which $int_{T_1}cl_{T_2}(\vee_{i=1}^{\infty}(\lambda_i)) = int_{T_2}cl_{T_1}(\vee_{i=1}^{\infty}(\lambda_i)) = 0$. Hence either $int_{T_1}(cl_{T_2}(\lambda_i)) \neq 0$ or $int_{T_2}(cl_{T_1}(\lambda_i)) \neq 0$ for atleast one i in (X, T_1, T_2).

Proposition 3.5. If $int_{T_1}(cl_{T_2}(\bigvee_{i=1}^{\infty}(\lambda_i))) = int_{T_2}(cl_{T_1}(\bigvee_{i=1}^{\infty}(\lambda_i))) = 0$ where $int_{T_j}(\lambda_i) = 0$,(j=1,2) and λ_i's are pairwise fuzzy closed sets in (X, T_1, T_2), then the fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy D-Baire space.

Proof. Let λ_i's be pairwise fuzzy closed sets. Then $cl_{T_j}(\lambda_i) = \lambda_i$ (j=1,2) . Now $int_{T_j}(\lambda_i) = 0$ (j=1,2) implies that $int_{T_1}(cl_{T_2}(\lambda_i)) = int_{T_2}(cl_{T_1}(\lambda_i)) = 0$ .Therefore λ_i's are pairwise fuzzy nowhere dense sets in (X, T_1, T_2). Hence $int_{T_1}(cl_{T_2}(\bigvee_{i=1}^{\infty}(\lambda_i))) = int_{T_2}(cl_{T_1}(\bigvee_{i=1}^{\infty}(\lambda_i))) = 0$ where λ_i's are pairwise fuzzy nowhere dense sets in (X, T_1, T_2) implies the pairwise fuzzy D-Baire space.

Theorem 3.2.[10] If $\lambda \leq \mu$ and μ is a pairwise fuzzy first category set in a fuzzzy bitopological space (X, T_1, T_2) then λ is also a pairwise fuzzy first category set .

Proposition 3.6. If μ is any fuzzy set such that $\mu \leq \lambda$, where λ is any pairwise fuzzy first category set in a pairwise fuzzzy D-Baire space (X, T_1, T_2) then μ is a pairwise fuzzy nowhere dense set.

Proof. Let λ be a pairwise fuzzy first category set in (X, T_1, T_2) and μ be any fuzzy set in (X, T_1, T_2) such that $\mu \leq \lambda$. By Theorem 3.2, μ is also a pairwise fuzzy first category set. Since μ is a pairwise fuzzy first category set in the pairwise fuzzy D-Baire space (X, T_1, T_2), by Proposition 3.1, we have $int_{T_1}(cl_{T_2}(\lambda)) = int_{T_2}(cl_{T_1}(\lambda)) = 0$. Hence μ is a pairwise fuzzy nowhere dense set.

Theorem 3.3.[8] If λ is a pairwise fuzzy nowhere dense set in a fuzzy bitopological space (X, T_1, T_2), then $1 - \lambda$ is a pairwise fuzzy dense set in (X, T_1, T_2).

Proposition 3.7. If (X, T_1, T_2) is a pairwise fuzzy D-Baire space then every pairwise fuzzy residual set in (X, T_1, T_2) is a pairwise fuzzy dense set.

Proof. Let λ be a pairwise fuzzy residual set in (X, T_1, T_2). Then $1 - \lambda$ is a pairwise fuzzy first category set. Since (X, T_1, T_2) is a pairwise fuzzy D-Baire space, $1 - \lambda$ is a pairwise fuzzy nowhere dense set. By Theorem 3.3, $\lambda = 1 - (1 - \lambda)$ is a pairwise fuzzy dense set.

Proposition 3.8. If μ is any fuzzy set such that $\lambda \leq \mu$, where λ is any pairwise fuzzy residual set in a pairwise fuzzzy D-Baire space (X, T_1, T_2), then μ is a pairwise fuzzy dense set.

Proof. Let λ be a pairwise fuzzy residual set in (X, T_1, T_2) and μ be any fuzzy set in (X, T_1, T_2) such that $\lambda \leq \mu$. Now $1 - \mu \leq 1 - \lambda$ and $1 - \lambda$ is a pairwise fuzzy first category set. Hence by Theorem 3.2, $1 - \mu$ is a pairwise fuzzy first category set in pairwise fuzzy D-Baire space (X, T_1, T_2) . Then μ is a pairwise fuzzy residual set and hence by Proposition 3.7, μ is a pairwise fuzzy dense set.

Proposition 3.9. If the pairwise fuzzy first category set λ , is a pairwise fuzzy closed set, in a pairwise fuzzy Baire space (X, T_1, T_2), then (X, T_1, T_2) is a pairwise fuzzy D-Baire space .

Proof. Let λ be a pairwise fuzzy first category set in a pairwise fuzzy Baire space (X, T_1, T_2) and $cl_{T_i}(\lambda) = \lambda$ ...(1) (i = 1,2) By theorem 3.1, $int_{T_i}(\lambda) = 0$ ...(2) (i = 1, 2) , for the pairwise fuzzy first category set λ in (X, T_1, T_2). Then, from (1) and (2), we have $int_{T_1}(cl_{T_2}(\lambda)) = int_{T_2}(cl_{T_1}(\lambda)) = 0$. Hence, by proposition 3.1, (X, T_1, T_2) is a pairwise fuzzy D-Baire space .

Proposition 3.10. If the pairwise fuzzy residual set μ , is a pairwise fuzzy open set, in a pairwise fuzzy Baire space (X, T_1, T_2), then (X, T_1, T_2) is a pairwise fuzzy D-Baire space .

Proof. Let λ be a pairwise fuzzy first category set in a pairwise fuzzy Baire space (X, T_1, T_2). Then $1 - \lambda$ is a pairwise fuzzy residual set. By hypothesis $1 - \lambda$ is a pairwise fuzzy open set in (X, T_1, T_2). Hence λ is a pairwise fuzzy closed set. This implies that the pairwise fuzzy first category set λ , is a pairwise fuzzy closed set, in the pairwise fuzzy Baire space (X, T_1, T_2). By proposition 3.9, (X, T_1, T_2) is a pairwise fuzzy D-Baire space .

Proposition 3.11. If the fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy first category space then (X, T_1, T_2) is not a pairwise fuzzy D-Baire space .

Proof. Let (X, T_1, T_2) be a pairwise fuzzy first category space . Then $\vee_{i=1}^{\infty}\lambda_i = 1_X$ where λ_i's are pairwise fuzzy nowhere dense sets in (X, T_1, T_2) . Then $int_{T_1}(cl_{T_2}(\vee_{i=1}^{\infty}(\lambda_i)) = int_{T_1}(cl_{T_2}(1)) = int_{T_1}(1) = 1 \neq 0$ and $int_{T_2}(cl_{T_1}(\vee_{i=1}^{\infty}(\lambda_i)) = int_{T_2}(cl_{T_1}(1)) = int_{T_2}(1) = 1 \neq 0$. Hence (X, T_1, T_2) is not a pairwise fuzzy D-Baire space .

§4. Inter-relations between pairwise fuzzy strongly irresolvable spaces, pairwise fuzzy submaximal spaces and pairwise Fuzzy Baire spaces

Proposition 4.1. If (X, T_1, T_2) is a pairwise fuzzy submaximal space ,then (X, T_1, T_2) is not a pairwise fuzzy D-Baire space .

Proof. Let (X, T_1, T_2) be a pairwise fuzzy submaximal space.Suppose that (X, T_1, T_2) is a pairwise fuzzy D-Baire space. Let $\lambda = \vee_{i=1}^{\infty} \lambda_i$, be a pairwise fuzzy first category set in (X, T_1, T_2) . Then λ_i's are pairwise fuzzy nowhere dense sets. This implies $int_{T_1}(cl_{T_2}(\lambda_i) = 0$ and $int_{T_2}(cl_{T_1}(\lambda_i) = 0$. Now $int_{T_1}(\lambda_i) \leq int_{T_1}(cl_{T_2}(\lambda_i))$ and $int_{T_2}(\lambda_i) \leq int_{T_2}(cl_{T_1}(\lambda_i))$, implies that $int_{T_1}(\lambda_i) = 0$ and $int_{T_2}(\lambda_i) = 0$. Then $1 - int_{T_1}(\lambda_i) = 1$ and $1 - int_{T_2}(\lambda_i) = 1$ implies that $cl_{T_1}(1 - \lambda_i) = 1$ and $cl_{T_2}(1 - \lambda_i) = 1$. This implies that $cl_{T_1}(cl_{T_2}(1 - \lambda_i)) = 1$ and $cl_{T_2}(cl_{T_1}(1 - \lambda_i)) = 1$. Hence $1 - \lambda_i$'s are pairwise fuzzy dense sets in (X, T_1, T_2) .Now $int_{T_1}(1 - \lambda_i) = 1 - (cl_{T_1}(\lambda_i)) < (1 - \lambda_i)$ and $int_{T_2}(1 - \lambda_i) = 1 - (cl_{T_2}(\lambda_i)) < (1 - \lambda_i)$. Hence $int_{T_1}(1 - \lambda_i) \neq (1 - \lambda_i)$ and $int_{T_2}(1 - \lambda_i) \neq (1 - \lambda_i)$ and therefore $(1 - \lambda_i)$'s are not pairwise fuzzy open sets in (X, T_1, T_2) . But this is a contradiction to (X, T_1, T_2), being a pairwise fuzzy submaximal space, in which each pairwise fuzzy dense set is pairwise fuzzy open set in (X, T_1, T_2). Hence our assumption that (X, T_1, T_2) is a pairwise fuzzy D-Baire space does not hold. Thus every pairwise fuzzy submaximal space is not a pairwise fuzzy D-Baire space .

Under what conditions, a pairwise fuzzy submaximal space is a pairwise fuzzy D-Baire space? The answer, for this question,is given in the following proposition.

Proposition 4.2. If the fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy submaximal space and pairwise fuzzy Baire space, in which every pairwise fuzzy residual set is a pairwise fuzzy dense set in (X, T_1, T_2), then (X, T_1, T_2) is a pairwise fuzzy D-Baire space .

Proof. Let (X, T_1, T_2) be a pairwise fuzzy submaximal Baire space and λ be a pairwise fuzzy residual set in (X, T_1, T_2). By hypothesis, λ is a pairwise fuzzy dense set . Also since (X, T_1, T_2) is a pairwise fuzzy submaximal space, for the pairwise fuzzy dense set λ , we have $\lambda \in T_i$ (i = 1,2). Hence the pairwise fuzzy residual set λ is a pairwise fuzzy open set in (X, T_1, T_2). Then, by proposition 3.11, (X, T_1, T_2) is a pairwise fuzzy D-Baire space .

Definition 4.1.[2] A fuzzy bitopological space (X, T_1, T_2) is said to be a *pairwise fuzzy strongly irresolvable space* if for each pairwise fuzzy dense set λ in (X, T_1, T_2), $cl_{T_1}(int_{T_2}(\lambda)) = cl_{T_2}(int_{T_1}(\lambda)) = 1$.

Theorem 4.1.[9] If the fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy Baire space, then each pairwise fuzzy residual set is a pairwise fuzzy dense set in (X, T_1, T_2).

Proposition 4.3. If the fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy strongly irresolvable Baire space, then (X, T_1, T_2) is a pairwise fuzzy D-Baire space .

Proof. Let (X, T_1, T_2) be a pairwise fuzzy strongly irresolvable Baire space and λ be a pairwise fuzzy residual set in (X, T_1, T_2). Since (X, T_1, T_2) is a pairwise fuzzy Baire space, by Theorem 4.1, λ is a pairwise fuzzy dense set . Also since (X, T_1, T_2) is a pairwise fuzzy strongly irresolvable space, for the pairwise fuzzy dense set λ , we have $cl_{T_1}(int_{T_2}(\lambda)) = cl_{T_2}(int_{T_1}(\lambda)) = 1$. Then by Proposition 3.1, (X, T_1, T_2) is a pairwise fuzzy D-Baire space .

Definition 4.2.[9] A fuzzy bitopological space (X, T_1, T_2) is said to be a *pairwise fuzzy almost resolvable space* , if $\vee_{k=1}^{\infty}(\lambda_k) = 1$, where the fuzzy sets λ_k's in (X, T_1, T_2) are such that $int_{T_i}(\lambda_k) = 0$,(i=1,2) .

Theorem 4.2.[8] If the fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy Baire space, then(X, T_1, T_2) is a pairwise fuzzy second category space.

Proposition 4.4. If the fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy D-Baire space then (X, T_1, T_2) is not a pairwise fuzzy almost resolvable space.

Proof. Let (X, T_1, T_2) be a pairwise fuzzy D-Baire space. Then, by proposition 3.2,(X, T_1, T_2) is a pairwise fuzzy Baire space. By Theorem 4.2, is a pairwise fuzzy second category space, and hence (X, T_1, T_2) is not a pairwise fuzzy first category space. This implies that $\bigvee_{i=1}^{\infty}(\lambda_k) \neq 1$, where λ_k's (k = 1 to ∞) are pairwise fuzzy nowhere dense sets in (X, T_1, T_2). Since λ_k's (k = 1 to ∞) are pairwise fuzzy nowhere dense sets in (X, T_1, T_2), $int_{T_1}(cl_{T_2}(\lambda_k)) = int_{T_2}(cl_{T_1}(\lambda_k)) = 0$. Also, since $int_{T_1}(\lambda_k) \leq int_{T_1}(cl_{T_2}(\lambda_k))$ and $int_{T_2}(\lambda_k) \leq int_{T_2}(cl_{T_1}(\lambda_k))$,$int_{T_i}(\lambda_k) = 0$ (i=,2). Hence $\bigvee_{i=1}^{\infty}(\lambda_k) \neq 1$,where $int_{T_i}(\lambda_k) = 0$, (i = 1,2). Therefore (X, T_1, T_2) is not a pairwise fuzzy almost irresolvable space.

Definition 4.3.[9] A fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy nodec space if every non - zero pairwise fuzzy nowhere dense set in (X, T_1, T_2), is a pairwise fuzzy closed set in (X, T_1, T_2) .That is, if λ is a pairwise fuzzy nowhere dense set in a fuzzy bitopological space (X, T_1, T_2), then $1 - \lambda \in T_i$ (i = 1,2).

Proposition 4.5. If (X, T_1, T_2) is a pairwise fuzzy nodec space , then (X, T_1, T_2) is not a pairwise fuzzy D-Baire space .

Proof. Let $\lambda = \vee_{i=1}^{\infty}\lambda_i$, be a pairwise fuzzy first category set in (X, T_1, T_2) .Then λ_i's are pairwise fuzzy nowhere dense sets in (X, T_1, T_2) . But (X, T_1, T_2) is a pairwise fuzzy nodec space, hence λ_i's are pairwise fuzzy closed sets and $cl_{T_j}(\lambda_i) = \lambda_i$, j=1,2 . Now $int_{T_1}(\lambda) = int_{T_1}(\vee_{i=1}^{\infty}(\lambda_i)) = int_{T_1}(\vee_{i=1}^{\infty}cl_{T_2}(\lambda_i) > \vee_{i=1}^{\infty}int_{T_1}(cl_{T_2}(\lambda_i)$. Since λ_i's are pairwise fuzzy nowhere dense sets in $(X, T_1, T_2))$, $int_{T_1}(cl_{T_2}(\lambda_i) = 0$. Hence we have $int_{T_1}(\lambda) \neq 0$ and $0 \neq int_{T_1}(\lambda) \leq int_{T_1}(cl_{T_2}(\lambda))$ implies that $int_{T_1}(cl_{T_2}(\lambda)) \neq 0$. Therefore by proposition 3.1, (X, T_1, T_2) is not a pairwise fuzzy D-Baire space .

Theorem 4.4.[10] Let (X, T_1, T_2) be a pairwise fuzzy strongly irresolvable space. Then λ is a pairwise fuzzy dense set in (X, T_1, T_2) if and only if $1 - \lambda$ is a pairwise fuzzy nowhere dense set.

Proposition 4.6. Let (X, T_1, T_2) be a pairwise fuzzy strongly irresolvable space. Then (X, T_1, T_2) is a pairwise fuzzy D-Baire space if and only if (X, T_1, T_2) is a pairwise fuzzy Baire space.

Proof. Let (X, T_1, T_2) be a pairwise fuzzy D-Baire space. By proposition 3.2, (X, T_1, T_2) is a pairwise fuzzy Baire space.
Conversely (X, T_1, T_2) is a pairwise fuzzy Baire space and pairwise fuzzy strongly irresolvable space. Let λ be a pairwise fuzzy first category set in (X, T_1, T_2) . Then $1 - \lambda$ is a pairwise fuzzy residual set in (X, T_1, T_2) . Since (X, T_1, T_2) is a pairwise fuzzy Baire space, by Theorem 4.1, $1 - \lambda$ is a pairwise fuzzy dense set in (X, T_1, T_2). Since (X, T_1, T_2), is a pairwise fuzzy strongly irresolvable space, $cl_{T_1}(int_{T_2}(1 - \lambda)) = 1$ and $cl_{T_2}(int_{T_1}(1 - \lambda)) = 1$. Then $int_{T_1}(cl_{T_2}(\lambda)) = 0$ and $int_{T_2}(cl_{T_1}(\lambda)) = 0$, hence by proposition 3.1, (X, T_1, T_2) is a pairwise fuzzy D-Baire space.

Proposition 4.7. Let (X, T_1, T_2) be a pairwise fuzzy strongly irresolvable space. Then (X, T_1, T_2) is a pairwise fuzzy D-Baire space if and only if $\lambda = \wedge_{i=1}^{\infty}\lambda_i$, where λ_i's are pairwise fuzzy dense sets, is a pairwise fuzzy dense set in (X, T_1, T_2).

Proof. Let (X, T_1, T_2) be a pairwise fuzzy strongly irresolvable and pairwise fuzzy D-Baire space . Let $\lambda = \wedge_{i=1}^{\infty}\lambda_i$, where λ_i's are pairwise fuzzy dense sets in (X, T_1, T_2). We have to prove that λ is pairwise fuzzy dense set. Now $1 - \lambda = \vee_{i=1}^{\infty}(1 - \lambda_i)$ and since (X, T_1, T_2) is a pairwise fuzzy strongly irresolvable space by Theorem 4.4, $(1 - \lambda_i)'s$ are pairwise fuzzy nowhere dense sets in (X, T_1, T_2). Hence $1 - \lambda$ is a pairwise fuzzy first category set. Since (X, T_1, T_2) is a pairwise fuzzy D-Baire space, by proposition 3.1, $int_{T_1}(cl_{T_2}(1 - \lambda)) = 0$ and $int_{T_2}(cl_{T_1}(1-\lambda)) = 0$. This implies that $1-int_{T_1}(cl_{T_2}(1-\lambda)) = 1$ and $1-int_{T_2}(cl_{T_1}(1-\lambda)) = 1$. Hence $cl_{T_1}(int_{T_2}(\lambda)) = 1$ and $cl_{T_2}(int_{T_1}(\lambda)) = 1$. Since $cl_{T_1}(int_{T_2}(\lambda)) \leq cl_{T_1}(cl_{T_2}(\lambda))$ and $cl_{T_2}(int_{T_1}(\lambda)) \leq cl_{T_2}(cl_{T_1}(\lambda))$, we have, $cl_{T_1}(cl_{T_2}(\lambda)) = cl_{T_2}(cl_{T_1}(\lambda)) = 1$ and λ is pairwise fuzzy dense set.Conversely suppose $\lambda = \wedge_{i=1}^{\infty}\lambda_i$, where λ_i's are pairwise fuzzy nowhere dense sets, is a pairwise fuzzy dense set in (X, T_1, T_2). We have to prove that (X, T_1, T_2) is a pairwise fuzzy D-Baire space. Let μ be a pairwise fuzzy first category set. Then $\mu = \bigvee_{i=1}^{\infty}\mu_i$, where μ_i's are pairwise fuzzy nowhere dense sets in (X, T_1, T_2). Now $1 - \mu = \wedge_{i=1}^{\infty}(1 - \mu_i)$. Since (X, T_1, T_2) is a pairwise fuzzy strongly irresolvable space and μ_i's are pairwise fuzzy nowhere dense sets, by theorem 4.4, $(1 - \mu_i)'s$ are pairwise fuzzy dense sets . Therefore, by hypothesis, $1 - \mu$ is a pairwise fuzzy dense set in a pairwise fuzzy strongly irresolvable space (X, T_1, T_2). Hence $cl_{T_1}(int_{T_2}(1-\mu)) = cl_{T_2}(int_{T_1}((1-\mu)) = 1$. This implies that $int_{T_1}(cl_{T_2}(\mu)) = int_{T_2}(cl_{T_1}(\mu)) = 0$. Hence, by proposition 3.1, (X, T_1, T_2) is a pairwise fuzzy D-Baire space.

Remark 4.1. In view of proposition 4.6 and proposition 4.7, we have, the following result. Let (X, T_1, T_2) be a pairwise fuzzy strongly irresolvable space. Then the following are equivalent.
(i)(X, T_1, T_2) is a pairwise fuzzy Baire space.
(ii)(X, T_1, T_2) is a pairwise fuzzy D-Baire space.
(iii) $\lambda = \wedge_{i=1}^{\infty}\lambda_i$, where λ_i's are pairwise fuzzy dense sets in (X, T_1, T_2), is a pairwise fuzzy dense set in (X, T_1, T_2).

Theorem 4.5.[10] If every pairwise fuzzy G_δ set is fuzzy pairwise dense in a pairwise fuzzy submaximal and pairwise fuzzy strongly irresolvable space (X, T_1, T_2), then (X, T_1, T_2) is a pairwise fuzzy Baire space.

Proposition 4.8. If every pairwise fuzzy G_δ set is fuzzy pairwise dense in a pairwise fuzzy submaximal and pairwise fuzzy strongly irresolvable space (X, T_1, T_2), then (X, T_1, T_2) is a pairwise fuzzy D-Baire space.

Proof. Proof follows from Remark 4.2 and Theorem 4.5.

Theorem 4.6.[10] If every pairwise fuzzy G_δ set is a pairwise fuzzy dense set in a pairwise fuzzy strongly irresolvable and pairwise fuzzy nodec space (X, T_1, T_2), then (X, T_1, T_2) is a pairwise fuzzy Baire space.

Proposition 4.9. If every pairwise fuzzy G_δ set is a pairwise fuzzy dense set in a pairwise fuzzy strongly irresolvable and pairwise fuzzy nodec space (X, T_1, T_2), then (X, T_1, T_2) is a pairwise fuzzy D-Baire space.

Proof. Proof follows from Remark 4.2 and Theorem 4.6.

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Mean value theorems on bounded variation of Henstock-Kurzweil-Stieltjes-◊-Integral for normed linear space-valued functions on time scales

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Abstract In this paper, we employ the concept of bounded variation in the study of integration on time scales in the sense of Henstock-Kurzweil-Stietljes-◊-integral to prove mean value theorems for normed linear space-valued functions.

Keywords Bounded variation, Henstock-Kurzweil integral, Stieltjes integral, Normed linear space, Time scales.
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§1. Introduction and preliminaries

The dual of the space of functions of bounded variation was studied by K. K. Aye and P. Y. Lee [2] which was dedicated to Prof. J. Kurzweil on the occasion of his 80th birthday. Hildebrandt [8] has characterized continuous linear functionals on the space of bounded variation (BV) regarding BV as a two-norm space. The Henstock delta integral on time scales was introduced by Allan Peterson and Bevan Thompson [11] and Henstock-Kurzweil integrals on time scales was studied by Brian S. Thomson [13]. We relate the time scales version of integration to the usual form. This relation shows that most of the properties of a time scale integral can be realized by using the techniques tailored to the time scale setting. See ([1], [3], [4], [9], [10] and [13]).

A time scale $\mathbb{T}$ is any closed non-empty subset of $\mathbb{R}$, with the topology inherited from the standard topology on the real numbers $\mathbb{R}$.
Let $\mathbb{T}$ be a time scale, $a, b \in \mathbb{T}, a < b$, and $I = [a, b]_{\mathbb{T}}$. A partition of I is any finite ordered subset $P = t_0, t_1, ..., t_n \subset [a, b]_{\mathbb{T}}$, where $a = t_0 < t_1 < ... < t_n = b$. Each partition $P = t_0, t_1, ..., t_n$ of I decomposes I into subintervals $I_{\Diamond_i} = [t_{i-1}, t_i]_{\Diamond}$,
$i = 1, 2, ..., n$, such that $I_{\Diamond_i} \cap I_{\Diamond_k} = \varnothing$ for any $k \neq i$. By $\triangle t_i = t_i - t_{i-1}$, we denote the length

of the i^{th} subinterval in the partition P; by $P(I)$ the set of all partitions of I. See ([1]- [6]).

Let us employ diamond symbol to represent delta and nabla operators in order to avoid repetition with respect to the approach promoted by Bartosiewicz and Piotrowska [3]. We denote one of them by $I_\Diamond$ where $\Diamond$ means either $\triangle$ or $\triangledown$. Similarly, we use "$\Diamond$" as a common notation for the two kinds of derivatives on time scales. We can read $f^\Diamond$ as either f^Δ or f^∇.

Definition 1.1. *Let X be a normed linear space and $f : [a,b]_{\mathbb{T}} \to X$. Let g be a non-decreasing function defined on $[a,b]_{\mathbb{T}}$ and let $P = \{t_0, t_1, ..., t_n\}$ be a tagged partition of $[a,b]_{\mathbb{T}}$. The Henstock-Kurzweil-Stieltjes sum $S(P_\delta, f, g)$ of f with respect to g on partition P, is defined by*

$$S(P_\delta, f, g) = \sum_{i=1}^{n} f(\xi_i)[g(t_i) - g(t_{i-1})].$$

Since $\Diamond_{g_i} = g(t_i) - g(t_{i-1})$, therefore, the Henstock-Kurzweil-Stieltjes sum can be written as

$$S(P_\delta, f, g) = \sum_{i=1}^{n} f(\xi_i)\Diamond_{g_i}.$$

Definition 1.2. *Let X be a normed linear space and $f : [a,b]_{\mathbb{T}} \to X$ is Henstock-Kurzweil-Stieltjes-$\Diamond$-integrable with respect to a monotone increasing function g on $[a,b]_{\mathbb{T}}$ if there is a number L of member of X such that for every $\varepsilon > 0$, there exists a $\delta(t) > 0$ for $[a,b]_{\mathbb{T}}$ such that*

$$\|S(P_\delta, f, g) - L\| < \varepsilon,$$

for each define partition of $[a,b]_{\mathbb{T}}$ such that $\|P\| < \delta$ and $t_{i-1} \leq \xi_i \leq t_i, \quad i = 1, 2, ..., n$ and ξ_i is arbitrarily chosen in $[t_{i-1}, t_i]$.

We say that L is the Henstock-Kurzweil-Stieltjes-$\Diamond$-integral of f with respect to a monotone increasing function g over $[a,b]_{\mathbb{T}}$, and write

$$\| \int_a^b f(t)\Diamond g(t)\| = L.$$

Definition 1.3. *Let $f : [a,b]_{\mathbb{T}} \to X$ and $P = \{t_0, t_1, ..., t_n\}$ be any partition of $[a,b]_{\mathbb{T}}$. Define*

$$\|B(P)\| = \sum_{i=1}^{n} |f(t)[g(t_i) - g(t_{i-1})]|.$$

The function f is said to be of bounded variation on $[a,b]_{\mathbb{T}}$ iff there is a real number M such that $\|B(P)\| \leq M$ for all partitions P of $[a,b]_{\mathbb{T}}$.

Definition 1.4. *([7]). If f is of bounded variation on $[a,b]_{\mathbb{T}}$, the total variation of f over $[a,b]_{\mathbb{T}}$ is defined as $\|V(f; a,b)\| = \sup\{|B(P)| : P$ is a partition of $[a,b]_{\mathbb{T}}\}$.*
If f is monotone, then for each partition P,

$$\|B(P)\| = \sum_{i=1}^{n} |f(t)[g(t_i) - g(t_{i-1})]|$$

collapses to $\|f(b) - f(a)\|$ *or* $\|f(a) - f(b)\|$*, depending on whether* f *is increasing or decreasing respectively.*
Thus, if f *is monotone on* $[a, b]_{\mathbb{T}}$*, then* f *is of bounded variation on* $[a, b]_{\mathbb{T}}$ *and* $\|V(f; a, b)\| = \|f(b) - f(a)\|$.

§2. The Main Results

In this section, we prove some mean value theorems on bounded variation of the Henstock-Kurzweil-Stieltjes-◊-integral for normed linear space-valued functions on time scales. The following theorems will be used in the prove of mean value theorems.

Theorem 2.1. *Suppose that* $f : [a, b]_{\mathbb{T}} \to X$ *and* $g : [a, b]_{\mathbb{T}} \to X$ *are bounded, that* $\varphi : [a, b]_{\mathbb{T}} \to X$ *and* $\psi : [a, b]_{\mathbb{T}} \to X$ *are of bounded variation, and that* $f, g \in \mathbb{R}(\varphi) \cap \mathbb{R}(\psi)$*. Then*
(i) for all real numbers m *and* n, $mf(t) + ng(t) \in \mathbb{R}(\varphi)$ *and*

$$\int_a^b (mf(t) + ng(t))\Diamond\varphi(t) = m\int_a^b f(t)\Diamond\varphi(t) + n\int_a^b g(t)\Diamond\varphi(t);$$

(ii) for all real numbers m *and* n, $f(t) \in \mathbb{R}(m\varphi(t) + n\psi(t))$ *and*

$$\int_a^b f(t)\Diamond(m\varphi(t) + n\psi(t)) = m\int_a^b f(t)\Diamond\varphi(t) + n\int_a^b f(t)\Diamond\psi(t).$$

Proof. (i). Suppose $f, g \in \mathbb{R}(\varphi)$ and that m and n are real numbers.
Let $f, g \in \mathbb{R}(v_\varphi) \cap \mathbb{R}(v_\varphi - \varphi)$. Thus,
$mf(t) + ng(t) \in \mathbb{R}(v_\varphi) \cap \mathbb{R}(v_\varphi - \varphi)$; that is, $mf + ng \in \mathbb{R}(\varphi)$ and

$$\begin{aligned}
\int_a^b (mf(t) + ng(t))\Diamond\varphi(t) &= \int_a^b (mf(t) + ng(t))\Diamond v_{\varphi(t)} - \int_a^b (mf(t) + ng(t))\Diamond(v_{\varphi(t)} - \varphi(t)) \\
&= m\int_a^b f(t)\Diamond v_{\varphi(t)} + n\int_a^b g(t)\Diamond v_{\varphi(t)} \\
&\quad -m\int_a^b f(t)\Diamond(v_{\varphi(t)} - \varphi(t)) - n\int_a^b g(t)\Diamond(v_{\varphi(t)} - \varphi(t)) \\
&= m\int_a^b f(t)\Diamond\varphi(t) + n\int_a^b g(t)\Diamond\varphi(t).
\end{aligned}$$

(ii). If $f \in \mathbb{R}(\varphi) \cap \mathbb{R}(\psi)$, then there are increasing functions $\varphi_1, \varphi_2, \psi_1, \psi_2$ such that

$$f \in \mathbb{R}(\varphi_1) \cap \mathbb{R}(\varphi_2) \cap \mathbb{R}(\psi_1) \cap \mathbb{R}(\psi_2)$$

and $\varphi = \varphi_1 - \varphi_2, \psi = \psi_1 - \psi_2$.
Then

$$f \in \mathbb{R}(\varphi_1 + \psi_1), f \in \mathbb{R}(\varphi_2 + \psi_2),$$

$$\int_a^b f(t)\Diamond(\varphi_1(t)+\psi_1(t)) = \int_a^b f(t)\Diamond\varphi_1(t) + \int_a^b f(t)\Diamond\psi_1(t),$$

and

$$\int_a^b f(t)\Diamond(\varphi_2(t)+\psi_2(t)) = \int_a^b f(t)\Diamond\varphi_2(t) + \int_a^b f(t)\Diamond\psi_2(t).$$

Now, $\varphi+\psi = (\varphi_1+\psi_1)-(\varphi_2+\psi_2)$, hence $f \in \mathbb{R}(\varphi+\psi)$ and

$$\int_a^b f(t)\Diamond(\varphi(t)+\psi(t)) = \int_a^b f(t)\Diamond(\varphi_1(t)+\psi_1(t)) - \int_a^b f(t)\Diamond(\varphi_2(t)+\psi_2(t))$$

$$= \int_a^b f(t)\Diamond(\varphi_1(t) + \int_a^b f(t)\Diamond(\psi_1(t) - \int_a^b f(t)\Diamond(\varphi_2(t) - \int_a^b f(t)\Diamond(\psi_2(t)$$

$$= \int_a^b f(t)\Diamond(\varphi(t) + \int_a^b f(t)\Diamond(\psi(t).$$

It is now remains to show that $f \in \mathbb{R}(\varphi)$ implies $f \in \mathbb{R}(m\varphi)$ and

$$\int_a^b f(t)\Diamond(m\varphi(t)) = m\int_a^b f(t)\Diamond\varphi(t).$$

As above, let's assume that φ_1, φ_2 are increasing with $f \in \mathbb{R}(\varphi_1)\cap\mathbb{R}(\varphi_2)$ and $\varphi = \varphi_1 - \varphi_2$. If $m \geq 0$, then $m\varphi_1$ and $m\varphi_2$ are increasing, $f \in \mathbb{R}(m\varphi_1)\cap\mathbb{R}(m\varphi_2)$ and

$$\int_a^b f(t)\Diamond(m\varphi_i(t)) = m\int_a^b f(t)\Diamond(\varphi_i(t))$$

for $i = 1, 2$. If $m < 0$, then $-m\varphi_1$ and $-m\varphi_2$ are increasing,
$f \in \mathbb{R}(-m\varphi_1)\cap\mathbb{R}(-m\varphi_2)$ and

$$\int_a^b f(t)\Diamond(-m\varphi_i(t)) = -m\int_a^b f(t)\Diamond(\varphi_i(t)).$$

Now $m\varphi = m\varphi_1 - m\varphi_2 = -m\varphi_2 - (-m\varphi_1)$. Hence, in either case, $m \geq 0$ or $m < 0$, $f \in \mathbb{R}(m\varphi)$, and

$$\int_a^b f(t)\Diamond(m\varphi(t)) = m\int_a^b f(t)\Diamond\varphi(t).$$

□

Theorem 2.2. *(Partial Integration Formula).*
Suppose that $f : [a,b]_{\mathbb{T}} \to X$ and $g : [a,b]_{\mathbb{T}} \to X$ are of bounded variation and that $f \in \mathbb{R}(g)$. Then $g \in \mathbb{R}(f)$ and

$$\|\int_a^b f(t)\Diamond g(t)\| = \|f(b)g(b) - f(a)g(a)\| - \|\int_a^b g(t)\Diamond f(t)\|.$$

Proof. Choose $\varepsilon > 0$. There is a partition P of $[a,b]_{\mathbb{T}}$ such that if Q is a refinement of P, then

$$\|S(Q,f,g) - \int_a^b f(t)\Diamond g(t)\| < \varepsilon.$$

Suppose $Q = \{t_0, t_1, ..., t_n\}$ is a refinement of P and $s_k \in [t_{k-1} - t_k]_{\mathbb{T}}$ is chosen for $k = 1, 2, ..., n$. Then

$$Q^{'} = Q \cup \{t_0, t_1, ..., t_n\}$$

is a partition of $[a, b]_{\mathbb{T}}$, which is a refinement of P.
Now

$$\|f(b)g(b) - f(a)g(a)\| = \sum_{k=1}^{n} |f(t_k)g(t_k)| - \sum_{k=1}^{n} |f(t_{k-1})g(t_{k-1})|,$$

and

$$S(Q, g, f) = \sum_{k=1}^{n} g(t_k)[f(t_k) - f(t_{k-1})];$$

hence

$$\begin{aligned} &\|f(b)g(b) - f(a)g(b)\| - S(Q, g, f) \\ &= \sum_{k=1}^{n} [f(t_k)g(t_k) - f(t_{k-1})g(t_{k-1})] - \sum_{k=1}^{n} g(t_k)[f(t_k) - f(t_{k-1})] \\ &= \sum_{k=1}^{n} f(t_k)[g(t_k) - g(s_k)] + \sum_{k=1}^{n} f(t_{k-1})[g(s_k) - g(t_{k-1})] \\ &= S(Q^{'}, f, g). \end{aligned}$$

Thus,

$$\begin{aligned} &\|S(Q, g, f) - [f(b)g(b) - f(a)g(a) - \int_a^b f(t)\Diamond g(t)]\| \\ &= \|\int_a^b f(t)\Diamond g(t) - S(Q^{'}, f, g)\| < \varepsilon. \end{aligned}$$

So by Theorem 2.1, $g \in \mathbb{R}(f)$ and

$$\|\int_a^b f(t)\Diamond g(t)\| = \|f(b)g(b) - f(a)g(a) - \int_a^b g(t)\Diamond f(t)\|.$$

We shall now prove the mean value theorems on bounded variation of the Henstock-Kurzweil-Stieltjes-$\Diamond$-integral for normed linear space-valued functions on time scales.

□

Theorem 2.3. *(First Mean-Value Theorem).*
Let $f : [a, b]_{\mathbb{T}} \to X$ be continuous and $g : [a, b]_{\mathbb{T}} \to X$ be increasing, then there is $c \in [a, b]_{\mathbb{T}}$ such that

$$\int_a^b f(t)\Diamond g(t) = f(c)[g(b) - g(a)].$$

Proof. Let $m = \inf\{f(t) : t \in [a, b]_{\mathbb{T}}\}$ and $M = \sup\{f(t) : t \in [a, b]_{\mathbb{T}}\}$.
Then

$$m\|[g(b) - g(a)]\| \le \int_a^b |f(t)|\Diamond g(t) \le M\|[g(b) - g(a)]\|;$$

hence there is a real number λ such that $m \leq \lambda \leq M$ and

$$\lambda\|[g(b) - g(a)]\| = \|\int_a^b f(t)\Diamond g(t)\|.$$

Now, if f is continuous on $[a, b]_{\mathbb{T}}$, there is $c \in [a, b]_{\mathbb{T}}$ such that $f(c) = \lambda$.
Therefore,

$$\|f(c)[g(b) - g(a)]\| = \|\int_a^b f(t)\Diamond g(t)\|.$$

□

Theorem 2.4. *(Second Mean-Value Theorem).*
Suppose $f : [a, b]_{\mathbb{T}} \to X$ is increasing and $g : [a, b]_{\mathbb{T}} \to X$ is continuous and of bounded variation on$[a, b]_{\mathbb{T}}$. Then there is $c \in [a, b]_{\mathbb{T}}$ such that

$$\|\int_a^b f(t)\Diamond g(t)\| = \|f(a)[g(c) - g(a)] + f(b)[g(b) - g(c)]\|.$$

Proof. Assume that $g : [a, b]_{\mathbb{T}} \to X$ is continuous and of bounded variation on$[a, b]_{\mathbb{T}}$. Then the continuity of g guarantees that v_g and $v_g - g$ are continuous; hence, $f \in \mathbb{R}(v_g)$ and $f \in \mathbb{R}(v_g - g)$. Therefore, $f \in \mathbb{R}(g)$. By Theorem 2.2, $g \in \mathbb{R}(f)$ and

$$\|\int_a^b f(t)\Diamond g(t)\| = \|f(b)g(b) - f(a)g(a)\| - \|\int_a^b g(t)\Diamond f(t)\|.$$

We may now apply the first mean-value theorem to $\int_a^b g(t)\Diamond f(t)$ to conclude that there is $c \in [a, b]_{\mathbb{T}}$ such that

$$\|g(c)[f(b) - f(a)]\| = \|\int_a^b g(t)\Diamond f(t)\|.$$

Thus, we have

$$\|\int_a^b f(t)\Diamond g(t)\| = \|f(b)g(b) - f(a)g(a) - g(c)[f(b) - f(a)]\|$$

$$= \|f(b)[g(b) - g(c)] + f(a)[g(c) - g(a)]\|.$$

□

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A short interval result for the function $(\tau_3^{(e)}(n))^r$

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Abstract Let $n > 1$ be an integer. The integer $d = \prod_{i=1}^{s} p_i^{b_i}$ is called an exponential divisor of $n = \prod_{i=1}^{s} p_i^{a_i}$, if $b_i|a_i$ for every $i \in 1, 2, \cdots, s$. Let $\tau^{(e)}(n)$ denote the exponential divisor function. Similar to the generalization from $d(n)$ to $d_k(n)$, $\tau^{(e)}(n)$ can be extended to $\tau_k^{(e)}(n)$. In this paper, we investigate the case $k = 3$ and establish a short interval result for the $r - th$ power of the function $\tau_3^{(e)}(n)$.

Keywords Exponential divisor function, Generalized divisor function, Short interval.
2010 Mathematics Subject Classification 11L07, 11N80, 11L26.
[1]

§1. Introduction and preliminaries

Many scholars are interested in researching the divisor problem and they have got a large number of good results. The study of the exponential divisor function is one of the most important problems in analytic number theory. In 1972, Subbarao [1] established the definition of exponential divisor: Let $n > 1$ be an integer of canonical from $n = \prod_{i=1}^{s} p_i^{a_i}$. If $d = \prod_{i=1}^{s} p_i^{b_i}$ satisfies $b_i|a_i$, $i \in 1, 2, \cdots, s$, then d is called an exponential divisor of n, notation $d|_e n$. By convention $1|_e 1$. Besides, he also studied the mean value problem of exponential divisor function $\tau^{(e)}(n) = \sum_{d|_e n} 1$ and got

$$\sum_{n\leq x} \tau^{(e)}(n) = Ax + E(x),$$

where

$$E(x) = O(x^{\frac{1}{2}}).$$

J. Wu [2] improved the above result got the following result:

$$\sum_{n\leq x} \tau^{(e)}(n) = Ax + Bx^{\frac{1}{2}} + O(x^{\frac{2}{9}} logx),$$

where

$$A = \prod_{p} \left(1 + \sum_{2}^{\infty} \frac{d(a) - d(a-1)}{p^a}\right),$$

[1]This work is Supported by National Natural Science Foundation of China (Grant No. 11771256).

$$B = \prod_p \left(1 + \sum_{a=5}^{\infty} \frac{d(a) - d(a-1) - d(a-2) + d(a-3)}{p^{\frac{a}{2}}}\right).$$

Subbarao [1] also proved that any positive integer r,

$$\sum_{n \le x} \left(\tau^{(e)}(n)\right)^r \sim A_r x,$$

where

$$A_r = \prod_p \left(1 + \sum_{a=2}^{\infty} \frac{(d(a))^r - (d(a-1))^r}{p^a}\right).$$

L. Toth [4] proved

$$\sum_{n \le x} \left(\tau^{(e)}(n)\right)^r = A_r(x) + x^{\frac{1}{2}} P_{2^r-2}(\log x) + O(x^{u_r+\varepsilon}),$$

where $P_{2^r-2}(t)$ is a polynomial of degree $2^r - 2$ of t, $u_r = \frac{2^{r+1}-1}{2^{r+1}+1}$.

Similarly to the generalization of $d_k(n)$ from $d(n)$, we extended $\tau^{(e)}(n)$ and established a definition as follows:

$$\tau_k^{(e)}(n) = \prod_{p_i^{a_i} \| n} d_k(a_i), \ \ k \ge 2.$$

Obviously $\tau_2^{(e)}(n) = \tau^{(e)}(n)$. $\tau_3^{(e)}(n)$ is obviously a multiplicative function. The aim of this short text is to study the short interval case and prove the following.

Theorem *If $x^{\frac{1}{4}+2\varepsilon} < y \le x$, then*

$$\sum_{x < n \le x+y} \left(\tau_3^{(e)}(n)\right)^r = C_1 y + O(yx^{-\frac{\varepsilon}{4}} + O(x^{\frac{1}{4}+\frac{5}{4}\varepsilon})),$$

where $C_1 = Res_{s=1} V(s)$ and $V(s) = \sum_{n=1}^{\infty} \frac{(\tau_3^{(e)}(n))^r}{n^s}$.

Notations *Throughout this paper, ε always denotes a fixed but sufficiently small positive constant. We assume that $1 \le a \le b$ are fixed integers, and we denote by*

$$d(a, b; k) = \sum_{k = n_1^a n_2^b} 1$$

and $d(a, b; k) \ll k^{\varepsilon^2}$ will be used freely.

§2. Some lemmas

In order to prove theorem, we need the following lemmas.

Lemma 1. *For $r > 1$, $s = \sigma + it$ is a complex number, then we have*

$$\sum_{n=1}^{\infty} \frac{(\tau_3^{(e)}(n))^r}{n^s} = \zeta(s)\zeta^{3^r-1}(2s)V(s),$$

where the infinite series $V(s) := \sum_{n=1}^{\infty} \frac{v(n)}{n^s}$ is absolutely convergent for $\Re s > \frac{1}{4}$.

Proof. By Euler's product formula, we can get

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(\tau_3^{(e)}(n))^r}{n^s} &= \prod_p\left(1+\frac{(\tau_3^{(e)}(p))^r}{p^s}+\frac{(\tau_3^{(e)}(p^2))^r}{p^{2s}}+\frac{(\tau_3^{(e)}(p^3))^r}{p^{3s}}+\frac{\tau_3^{(e)}(p^4))^r}{p^{4s}}+\cdots\right)\\ &= \prod_p\left(1+\frac{d_3^r(1)}{p^s}+\frac{d_3^r(2)}{p^{2s}}+\frac{d_3^r(3)}{p^{3s}}+\frac{d_3^r(4)}{p^{4s}}+\frac{d_3^r(5)}{p^{5s}}+\cdots\right)\\ &= \prod_p\left(1+\frac{1}{p^s}+\frac{3^r}{p^{2s}}+\frac{3^r}{p^{3s}}+\frac{6^r}{p^{4s}}+\frac{3^r}{p^{5s}}+\cdots\right)\\ &= \zeta(s)\prod_p\left(1+\frac{3^r-1}{p^{2s}}+\frac{6^r-3^r}{p^{4s}}+\cdots\right)\\ &= \zeta(s)\zeta^{3^r-1}(2s)V(s),\end{aligned} \tag{1}$$

where the infinite series $V(s) := \sum_{n=1}^{\infty}\frac{v(n)}{n^s}$ is absolutely convergent for $\Re s > \frac{1}{4}$. □

Lemma 2. *Let $k \geq 2$ be a fixed integer, $1 < y \leq x$ be large real numbers and*

$$B(x,y;k,\varepsilon) := \sum_{\substack{x<nm^k\leq x+y\\ m>x^{\varepsilon}}} 1.$$

Then we have $B(x,y;k,\varepsilon) \ll yx^{-\varepsilon} + x^{\frac{1}{2k+1}}logx$.

Proof. This Lemma is very important when studying the short interval distribution, see [5]. □

Let $f(n)$, $h(n)$ be arithmetic functions defined by the following Dirichlet series for $\Re s > 1$.

$$\sum_{n=1}^{\infty}\frac{f(n)}{n^s} = \zeta(s)V(s), \tag{2}$$

$$\sum_{n=1}^{\infty}\frac{h(n)}{n^{2s}} = \zeta^{3^r-1}(2s). \tag{3}$$

Lemma 3. *Let f(n) be an arithmetic function defined by (2), then we have*

$$\sum_{n\leq x} f(n) = Cx + O(x^{\frac{1}{4}+\varepsilon}),$$

where $C = Res_{s=1}\zeta(s)V(s)$.

Proof. Since the infinite series $\sum_{n=1}^{\infty}\frac{v(n)}{n^s}$ is absolutely convergent for $\sigma > \frac{1}{4}$, we have

$$\sum_{n\leq x}|v(n)| \ll x^{\frac{1}{4}+\varepsilon}.$$

Therefore, from the definition of $v(n)$ and (2), it follows that

$$\sum_{n\leq x} f(n) = \sum_{km\leq x} f(km) = \sum_{k\leq x} v(k)\sum_{m\leq \frac{x}{k}} 1 = \sum_{k\leq x} v(k)(\frac{x}{k}+O(1)) = Cx + O(x^{\frac{1}{4}+\varepsilon})$$

where $C = Res_{s=1}\zeta(s)V(s)$. □

§2. Proof of the Theorem

From the definition of $f(n)$ and $h(n)$, we get

$$\sum_{n=1}^{\infty}\frac{[\tau_3^{(e)}(km^2)]^r}{(km^2)^s}=\sum_{k=1}^{\infty}\frac{f(k)}{k^s}\sum_{m=1}^{\infty}\frac{h(m)}{m^{2s}}=\sum_{\substack{n=1\\n=km^2}}^{\infty}\frac{f(k)h(m)}{(km^2)^s},$$

then

$$(\tau_3^{(e)}(n))^r=\sum_{n=km^2}f(k)h(m)$$

and

$$f(n)\ll n^{\varepsilon^2},h(n)\ll n^{\varepsilon^2},$$

so we have

$$\sum_{n\le x+y}\left(\tau_3^{(e)}(n)\right)^r-\sum_{n\le x}(\tau_3^{(e)}(n))^r=\sum_{x<km^2\le x+y}f(k)h(m)=\sum_1+O(\sum_2), \tag{4}$$

where

$$\sum_1=\sum_{m\le x^{\varepsilon}}h(m)\sum_{\frac{x}{m^2}<k\le\frac{x+y}{m^2}}f(k),$$

$$\sum_2=\sum_{\substack{x<nm^2\le x+y\\m>x^{\varepsilon}}}|f(k)h(m)|.$$

In view of Lemma 3,

$$\begin{aligned}\sum_1&=\sum_{m\le x^{\epsilon}}h(m)\left[C\frac{y}{m^2}+O((\frac{x}{m^2})^{\frac14+\varepsilon})\right]\\&=C_1y+O(y\sum_{m>x^{\varepsilon}}\frac{h(m)}{m^2})+O(x^{\frac14+\varepsilon}\sum_{m\le x^{\varepsilon}}\frac{h(m)}{m^{\frac12+2\varepsilon}})\\&=C_1y+O(yx^{-\frac{\varepsilon}{4}})+O(x^{\frac14+\varepsilon}x^{\frac{\varepsilon}{4}})\\&=C_1y+O(yx^{-\frac{\varepsilon}{4}})+O(x^{\frac14+\frac{5\varepsilon}{4}}),\end{aligned} \tag{5}$$

where $C_1=Res_{s=1}\zeta(s)\zeta^{3^r-1}(2s)V(s)$.

$$\begin{aligned}\sum_2&\ll\sum_{\substack{x<km^2\le x+y\\m>x^{\varepsilon}}}(km)^{\varepsilon^2}\ll x^{\varepsilon^2}\sum_{\substack{x<km^2\le x+y\\m>x^{\varepsilon}}}1\\&=x^{\varepsilon^2}B(x,y;2,\varepsilon)\ll x^{\varepsilon^2}(yx^{-\varepsilon}+x^{\frac15}logx)\\&\ll yx^{-\frac{\varepsilon}{2}}+x^{\frac15+\frac{\varepsilon}{2}}.\end{aligned} \tag{6}$$

From (4)−(6), we get

$$\sum_{x<n\le x+y}\left(\tau_3^{(e)}(n)\right)^r=C_1y+O(yx^{-\frac{\varepsilon}{4}}+O(x^{\frac14+\frac54\varepsilon})),$$

so the theorem is proved.

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On the mean value of $\phi^{(e)}(n)$ with a negative r-th power

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Abstract Many scholars are interested in researching the divisor problem, and they have obtained a large number of good results. However, there are many problems have not been solved. In this paper we shall study the mean value of $\phi^{(e)}(n)$ with a negative r-th power by the convolution method.

Keywords Dirichlet convolution, Euler formula, exponential divisor function.
2010 Mathematics Subject Classification 11M32, 11B68, 11L07.
[1]

§1. Introduction and preliminaries

Let $n > 1$ be an integer. The integer $d = \prod_{i=1}^{s} p_i^{b_i}$ is called an exponential divisor of $n = \prod_{i=1}^{s} p_i^{a_i}$, if $b_i|a_i$ for every $i \in \{1, 2, \cdots, s\}$, notation: $d|_e n$. By convention $1|_e 1$.

Let $\tau^{(e)}(n)$ denote the number of exponential divisors of n. The function $\tau^{(e)}$ is called the exponential divisor function. Similarly to the generalization of $d_k(n)$ from $d(n)$, we define the function $\tau_k^{(e)}(n)$:

$$\tau_k^{(e)}(n) = \prod_{{p_i}^{a_i}||n} d_k(a_i), k \geq 2, \tag{1}$$

Obviously when $k = 2$, that is $\tau^{(e)}(n)$. $\tau_3^{(e)}(n)$ is obviously a multiplicative function.

Throughout this paper, ε always denotes a fixed but sufficiently small positive constant.

J.Wu [1] got the following result:

$$\sum_{n \leq x} \tau^{(e)}(n) = A(x) + Bx^{\frac{1}{2}} + O(x^{\frac{2}{9}} \log x), \tag{2}$$

where

$$A = \prod_p (1 + \sum_{a=2}^{\infty} \frac{d(a) - d(a-1)}{p^a}),$$

$$B = \prod_p (1 + \sum_{a=5}^{\infty} \frac{d(a) - d(a-1) - d(a-2) + d(a-3)}{p^{\frac{a}{2}}}).$$

[1]This work is Supported by National Natural Science Foundation of China (Grant No. 11771256).

M.V.Subbarao [3] also proved for some positive integer r:

$$\sum_{n\leq x}(\tau^{(e)}(n))^r \sim A_r x, \tag{3}$$

where

$$A_r = \prod_p (1 + \sum_{a=2}^{\infty} \frac{(d(a))^r - (d(a-1))^r}{p^a}).$$

László Tóth [4] improved the result above and established a more precise asymptotic formula for the r-th power of the function $\tau^{(e)}(n)$:

$$\sum_{n\leq x}(\tau^{(e)}(n))^r = A_r x + x^{\frac{1}{2}} P_{2^r-2}(\log x) + O(x^{u_r+\varepsilon}). \tag{4}$$

Let $\phi^{(e)}(n)$ denote the number of divisors d of n such that d and n have no common exponential divisors. $\phi^{(e)}$ is multiplicative and for every prime power p^a $(a \geqslant 1)$, $\phi^{(e)}(p^a) = \phi(a)$, where ϕ is the Euler function.

In this paper, we will study the asymptotic formula for the mean value of the r-th power of the function $\phi^{(e)}(n)$, where $r > 1$ is an integer.

Theorem 1.1. For every integer $r > 1$ and $N \geq 1$, then we have

$$\sum_{n\leq x}(\phi^{(e)}(n))^{-r} = B_r x + x^{\frac{1}{3}} \log^{2^{-r}-2} \sum_{j=0}^{N} d_j(r) \log^{-j} x + O(x^{t_r+\varepsilon}), \tag{5}$$

for every $\varepsilon > 0$, where $d_0(r), d_1(r), \cdots, d_N(r)$ are computable constants, $t_r := \frac{1}{4-\alpha_{2^{-r}-1}}$, $\alpha_{2^{-r}-1}$ is as defined in Lemma 2.2, and

$$B_r := \prod_p (1 + \sum_{a=3}^{\infty} \frac{(\phi(a))^{-r} - (\phi(a-1))^{-r}}{p^a}).$$

§2. Some lemmas

In order to prove our theorem, we define for an arbitrary complex number k the general divisor function $d_k(n)$ by

$$\sum_{n=1}^{\infty} d_k(n) n^{-s} = \zeta^k(s) = \prod_p (1-p^{-s})^{-k}, \Re s > 1, \tag{6}$$

where a branch of $\zeta^k(s)$ is defined by

$$\zeta^k(s) = exp\{k \log \zeta(s)\} = exp(-k \sum_p \sum_{j=1}^{\infty} j^{-1} p^{-js}), \Re s > 1. \tag{7}$$

The definition shows that $d_k(n)$ is multiplicative function of n which generalizes $d_k(n)$. The divisor function $d_k(n)$ ($k \geq 2$ a fixed integer) may be defined by

$$\sum_{n=1}^{\infty} d_k(n) n^{-s} = \zeta^k(s) = \prod_p (1-p^{-s})^{-k}, \Re s > 1. \tag{8}$$

In this section, we give some lemmas which will be used in the proof of our theorem. Lemma 2.2 and Lemma 2.3 can be found in [5] and [6].

Lemma 2.1. For $r > 1$, then we have

$$\sum_{n=1}^{\infty} \frac{(\phi^{(e)}(n))^{(-r)}}{n^s} = \zeta(s)\zeta^{2^{-r}-1}(3s)H(s),$$

where the infinite series $H(s) := \sum_{n=1}^{\infty} \frac{h(n)}{n^s}$ is absolutely convergent for $\Re s > \frac{1}{5}$.

Proof. By Euler's product formula, we can get

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(\phi^{(e)}(n))^{-r}}{n^s} &= \prod_p \left(1 + \frac{(\phi^{(e)}(p))^{-r}}{p^s} + \frac{(\phi^{(e)}(p^2))^{-r}}{p^{2s}} + \frac{(\phi^{(e)}(p^3))^{-r}}{p^{3s}} + \cdots\right) \\
&= \prod_p \left(1 + \frac{(\phi(1))^{-r}}{p^s} + \frac{(\phi(2))^{-r}}{p^{2s}} + \frac{(\phi(3))^{-r}}{p^{3s}} + \frac{(\phi(4))^{-r}}{p^{4s}} + \cdots\right) \\
&= \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{2^{-r}}{p^{3s}} + \frac{2^{-r}}{p^{4s}} + \frac{4^{-r}}{p^{5s}} + \frac{2^{-r}}{p^{6s}} + \frac{6^{-r}}{p^{7s}} + \cdots\right) \qquad (9) \\
&= \zeta(s) \prod_p \left(1 + \frac{2^{-r}-1}{p^{3s}} + \frac{4^{-r}-2^{-r}}{p^{5s}} + \frac{2^{-r}-4^{-r}}{p^{6s}} + \frac{6^{-r}-2^{-r}}{p^{7s}} + \cdots\right) \\
&= \zeta(s)\zeta^{2^{-r}-1}(3s)H(s),
\end{aligned}$$

where the infinite series $H(s) := \sum_{n=1}^{\infty} \frac{h(n)}{n^s}$ is absolutely convergent for $\Re s > \frac{1}{5}$. □

Lemma 2.2. Suppose $k \geq 2$ is an integer. Then

$$D_k(x) = \sum_{n \leq x} d_k(n) = x \sum_{j=0}^{k-1} c_j (\log x)^j + O(x^{\alpha_k + \varepsilon}),$$

where c_j is a calculable constant, ε is a sufficiently small positive constant, α_k is the infimum of numbers α_k, such that

$$\Delta_k(x) = \sum_{n \leq x} d_k(n) - x P_{k-1}(\log x) \ll x^{\alpha_k + \varepsilon}, \qquad (10)$$

and

$$\begin{aligned}
&\alpha_2 \leq \frac{131}{416}, \quad \alpha_3 \leq \frac{43}{94}, \\
&\alpha_k \leq \frac{3k-4}{4k}, \quad 4 \leq k \leq 8, \\
&\alpha_9 \leq \frac{35}{54}, \quad \alpha_{10} \leq \frac{41}{61}, \quad \alpha_{11} \leq \frac{7}{10}, \\
&\alpha_k \leq \frac{k-2}{k+2}, \quad 12 \leq k \leq 25, \\
&\alpha_k \leq \frac{k-1}{k+4}, \quad 26 \leq k \leq 50, \\
&\alpha_k \leq \frac{31k-98}{32k}, \quad 51 \leq k \leq 57, \\
&\alpha_k \leq \frac{7k-34}{7k}, \quad k \geq 58.
\end{aligned}$$

Lemma 2.3. Suppose $f(m), g(n)$ are arithmetical functions such that

$$\sum_{m\le x} f(m)=\sum_{j=1}^{J} x^{\alpha_j}P_j(\log x)+O(x^{\alpha}),\quad \sum_{n\le x}\mid g(n)\mid=O(x^{\beta}),$$

where $\alpha_1\ge\alpha_2\ge\cdots\ge\alpha_J>\alpha>\beta>0, P_j(t)$ is a polynomial in t. If $h(n)=\sum_{n=md} f(m)g(d)$, then

$$\sum_{n\le x} h(n)=\sum_{j=1}^{J} x^{\alpha_j}Q_j(\log x)+O(x^{\alpha}),$$

where $Q_j(t)$ $\{j=1,\cdots,J\}$ is a polynomial in t.

§3. The mean value of $d_k(m)$

Theorem 3.1. Let $A>0$ be arbitrary but fixed real number. If $|k|\le A$, then uniformly in k

$$\sum_{m^3l\le x} d_k(m)=\zeta^k(3)x+x^{\frac{1}{3}}Q_{k-1}(\log x)+O(x^{\frac{1}{4-\alpha_k}+\varepsilon}), \tag{11}$$

where α_k is as defined in Lemma 2.2, $Q_{k-1}(\log x)$ is a polynomial of degree $k-1$.

Proof. By hyperbolic summation formula, we have

$$\begin{aligned}\sum_{m^3l\le x} d_k(m)&=\sum_{m\le y} d_k(m)\sum_{m^3l\le x}1+\sum_{l\le z}\sum_{m^3l\le x} d_k(m)-\sum_{m\le y} d_k(m)\sum_{l\le z}1\\&:=S_1+S_2-S_3,\end{aligned} \tag{12}$$

where y,z are parameters that will be determined later, and satisfy that $y^3z=x, 1\le y\le x$. Now, we deal with S_1, S_2 and S_3 separately,

$$\begin{aligned}S_1&=\sum_{m\le y} d_k(m)\sum_{m^3l\le x}1=\sum_{m\le y} d_k(m)[\frac{x}{m^3}]\\&=x\sum_{m\le y}\frac{d_k(m)}{m^3}+O\left(\sum_{m\le y} d_k(m)\right)\\&=x\sum_{m=1}^{\infty}\frac{d_k(m)}{m^3}-x\sum_{m>y}\frac{d_k(m)}{m^3}+O\left(\sum_{m\le y} d_k(m)\right)\\&=\zeta^k(3)x-x\sum_{m>y}\frac{d_k(m)}{m^3}+O(y^{1+\varepsilon}).\end{aligned} \tag{13}$$

Using Lemma 2.2 and partial summation formula, we have

$$
\begin{aligned}
\sum_{m>y}\frac{d_k(m)}{m^3} &= \int_{y^+}^{\infty}\frac{1}{t^3}d\left(\sum_{m\le t}d_k(m)\right)\\
&= \int_{y^+}^{\infty}\frac{1}{t^3}d\left(t\sum_{j=0}^{k-1}c_j(\log t)^j+O(t^{\alpha_k+\varepsilon})\right)\\
&= \sum_{j=0}^{k-1}c_j\int_{y^+}^{\infty}\frac{1}{t^3}d(t(\log t)^j)+O(y^{-3+\alpha_k+\varepsilon})\\
&= \sum_{j=0}^{k-1}\frac{1}{2}c_jy^{-2}[(\log y)^j+\frac{3}{2}j(\log y)^{j-1}+\frac{3}{2}j(j-1)(\log y)^{j-2}+\cdots+\frac{3}{2}j(j-1)\cdots 1]\\
&\quad +O(y^{-3+\alpha_k+\varepsilon}).
\end{aligned}
$$

Since $y=\sqrt[3]{\frac{x}{z}}$, we have $\log y=\frac{1}{3}(\log x-\log z)$, inserting this into (13), we can get

$$
S_1=\zeta^k(3)x-S_{11}-S_{12}+O(y^{1+\varepsilon}+xy^{-3+\alpha_k+\varepsilon}), \tag{14}
$$

where

$$
\begin{aligned}
S_{11} &= \frac{1}{2}x^{\frac{1}{3}}z^{\frac{2}{3}}\sum_{j=1}^{k-1}\frac{c_j}{3^j}\sum_{i=0}^{j}C_j^i(\log x)^{j-i}(-1)^i(\log z)^i,\\
S_{12} &= \frac{3}{4}x^{\frac{1}{3}}z^{\frac{2}{3}}\sum_{j=1}^{k-1}c_j\sum_{i=1}^{j}\frac{j!}{(j-i)!3^{j-i}}\sum_{s=0}^{j-i}C_{j-i}^s(\log x)^{j-i-s}(-1)^s(\log z)^s.
\end{aligned}
$$

By Lemma 2.2, we get

$$
\begin{aligned}
S_2 &= \sum_{l\le z}\sum_{m\le\sqrt[3]{\frac{x}{l}}}d_k(m)=\sum_{l\le z}\left(\sqrt[3]{\frac{x}{l}}\sum_{j=0}^{k-1}c_j\left(\log\sqrt[3]{\frac{x}{l}}\right)^j+O\left(\left(\sqrt[3]{\frac{x}{l}}\right)^{\alpha_k+\varepsilon}\right)\right)\\
&= x^{\frac{1}{3}}\sum_{j=0}^{k-1}\frac{c_j}{3^j}\sum_{i=0}^{j}C_j^i(\log x)^{j-i}(-1)^i\sum_{l\le z}l^{-\frac{1}{3}}(\log l)^i+O(xy^{-3+\alpha_k+\varepsilon}),
\end{aligned} \tag{15}
$$

where

$$
\sum_{l\le z}l^{-\frac{1}{3}}(\log l)^i=\int_{1^-}^{z}t^{-\frac{1}{3}}(\log t)^id[t]=\int_{1^-}^{z}t^{-\frac{1}{3}}(\log t)^idt+\int_{1^-}^{z}t^{-\frac{1}{3}}(\log t)^id\Delta(t). \tag{16}
$$

We can easily get that $\Delta(t)=O(1)$. Using partial integral formula, we have

$$
\int_{1^-}^{z}t^{-\frac{1}{3}}(\log t)^id\Delta(t)=\omega_i+O(z^{-\frac{1}{3}+\varepsilon}), \tag{17}
$$

where ω_i is a constant. We can also obtain that

$$
\int_{1^-}^{z}t^{-\frac{1}{3}}(\log t)^idt=\frac{3}{2}z^{\frac{2}{3}}(\log z)^i-(\frac{3}{2})^2iz^{\frac{2}{3}}(\log z)^{i-1}+\cdots+(-1)^i(\frac{3}{2})^{i+1}i!. \tag{18}
$$

Combing (15)−(18), we have

$$S_2 = x^{\frac{1}{3}}\widetilde{Q}_{k-1}(\log x) + S_{21} + S_{22} + O(xy^{-3+\alpha_k+\varepsilon}), \tag{19}$$

where

$$\widetilde{Q}_{k-1}(\log x) = \sum_{j=0}^{k-1}\frac{c_j}{3^j}\sum_{i=0}^{j}C_j^i(\log x)^{j-i}(-1)^i(\omega_i - (-1)^i(\frac{3}{2})^{i+1}i!),$$

$$S_{21} = \frac{3}{2}x^{\frac{1}{3}}z^{\frac{2}{3}}\sum_{j=0}^{k-1}\frac{c_j}{3^j}\sum_{i=0}^{j}C_j^i(\log x)^{j-i}(-1)^i(\log z)^i,$$

$$S_{22} = x^{\frac{1}{3}}z^{\frac{2}{3}}\sum_{j=0}^{k-1}\frac{c_j}{3^j}\sum_{i=0}^{j}C_j^i(\log x)^{j-i}(-1)^i\sum_{s=0}^{i-1}(-1)^{s-i}(\frac{3}{2})^{i-s+1}\frac{i!}{s!}(\log z)^s.$$

For S_3, we have

$$\begin{aligned} S_3 &= \sum_{m\le y}d_k(m)\sum_{l\le z}1 \\ &= yz\sum_{j=0}^{k-1}c_j(\log y)^j + O(y^{\alpha_k+\varepsilon}z + y^{1+\varepsilon}). \end{aligned} \tag{20}$$

Inserting $y = \sqrt[3]{\frac{x}{z}}$, and $\log y = \frac{1}{3}(\log x - \log z)$ into (20), then

$$S_3 = S_{31} + O(y^{\alpha_k+\varepsilon}z + y^{1+\varepsilon}), \tag{21}$$

where

$$S_{31} = x^{\frac{1}{3}}z^{\frac{2}{3}}\sum_{j=0}^{k-1}\frac{c_j}{3^j}\sum_{i=0}^{j}C_j^i(\log x)^{j-i}(-1)^i(\log z)^i.$$

Note that $C_j^i = \frac{j!}{i!(j-i)!}$. After some simplification we can easily get that $S_{11} + S_{31} = S_{21}, S_{12} = S_{22}$. Taking $y = x^{\frac{1}{4-\alpha_k}}, z = x^{\frac{1-\alpha_k}{4-\alpha_k}}$, then Theorem 3.1 is proved. □

§4. Proof of Theorem 1.1

For $r \geq 1$, from Lemma 2.1, we have $H(s) := \sum_{n=1}^{\infty}\frac{h(n)}{n^s}$ is absolutely convergent for $\Re s > \frac{1}{5}$, and then

$$\sum_{n\le x} | \ h(n) \ | \ll x^{\frac{1}{5}+\varepsilon}.$$

Let $F(s) = \zeta(s)\zeta^{2^{-r}-1}(3s) := \sum_{n=1}^{\infty}\frac{f(n)}{n^s}$, where $f(n) = d_k(m)$. From Theorem 3.1, we have

$$\sum_{n\le x}f(n) = \sum_{m^3l\le x}d_k(m) = \zeta^k(3)x + x^{\frac{1}{3}}Q_{k-1}(\log x) + O(x^{\frac{1}{4-\alpha_k}+\varepsilon}), \tag{22}$$

and we choose $k = 2^{-r} - 1$. From Lemma 2.1, we have

$$(\phi^{(e)}(n))^{-r} = \sum_{n=kl}h(k)f(l), \tag{23}$$

then, by Lemma 2.3 we can get the Theorem 1.1.

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Stress-Sum index for graphs

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Abstract The stress of a vertex is a node centrality index, which has been introduced by Shimbel (1953). The stress of a vertex in a graph is the number of geodesics (shortest paths) passing through it. A topological index of a chemical structure (graph) is a number that correlates the chemical structure with chemical reactivity or physical properties. In this paper, we introduce a new topological index for graphs called stress-sum index using stresses of vertices. Further, we establish some inequalities, prove some results and compute stress-sum index for some standard graphs.

Keywords Graph, stress of a vertex, path, geodesic, stress, topological index.
2010 Mathematics Subject Classification 05Cxx.

§1. Introduction

For standard terminology and notion in graph theory, we follow the text-book of Harary [5]. The non-standard will be given in this paper as and when required.

Let $G = (V, E)$ be a graph (finite and undirected). The distance between two vertices u and v in G, denoted by $d(u, v)$ is the number of edges in a shortest path (also called a graph geodesic) connecting them. We say that a graph geodesic P is passing through a vertex v in G if v is an internal vertex of P (i.e., v is a vertex in P, but not an end vertex of P). The degree of a vertex v in G is denoted by $d(v)$.

The concept of stress of a node (vertex) in a network (graph) has been introduced by Shimbel as centrality measure in 1953 [9]. This centrality measure has applications in biology,

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sociology, psychology, etc., (See [6,8]). The stress of a vertex v in a graph G, denoted by $\text{str}_G(v)$ $\text{str}(v)$, is the number of geodesics passing through it. We denote the maximum stress among all the vertices of G by Θ_G and minimum stress among all the vertices of G by θ_G. Further, the concepts of stress number of a graph and stress regular graphs have been studied by K. Bhargava, N.N. Dattatreya, and R. Rajendra in their paper [1]. A graph G is k-stress regular if $\text{str}(v) = k$ for all $v \in V(G)$.

The Zagreb indices have been defined using degrees of vertices in a graph to explain some properties of chemical compounds at molecular level [2,3]. The first Zagreb index $M_1(G)$ and the second Zagreb index $M_2(G)$ of a simple graph G are defined as:

$$M_1(G) = \sum_{v \in V(G)} d(v)^2 \tag{1}$$

$$M_2(G) = \sum_{uv \in E(G)} d(u)d(v). \tag{2}$$

By the motivation of these indices, Rajendra et al. [7] have introduced two topological indices of for graphs called first stress index and second stress index, using stresses of vertices. The first stress index $\mathcal{S}_1(G)$ and the second stress index $\mathcal{S}_2(G)$ of a simple graph G are defined as

$$\mathcal{S}_1(G) = \sum_{v \in V(G)} \text{str}(v)^2 \tag{3}$$

$$\mathcal{S}_2(G) = \sum_{uv \in E(G)} \text{str}(u)\text{str}(v). \tag{4}$$

We note that the first Zagreb index $M_1(G)$ satisfies the identity

$$M_1(G) = \sum_{uv \in E(G)} d(u) + d(v) \tag{5}$$

but $\mathcal{S}_1(G)$ does not satisfy such identity. For instance, consider the path P_3 on 3 vertices.

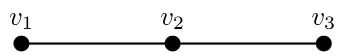

Figure 1: The path P_3.

The stresses of the vertices of P_3 are as follows: $\text{str}(v_1) = \text{str}(v_3) = 0$ and $\text{str}(v_2) = 1$. The first stress index of P_3 is,

$$\mathcal{S}_1(P_3) = \text{str}(v_1)^2 + \text{str}(v_2)^2 + \text{str}(v_3)^2 = 0^2 + 1^2 + 0^2 = 1.$$

But

$$\sum_{uv \in E(P_3)} \text{str}(u) + \text{str}(v) = \text{str}(v_1) + \text{str}(v_2) + \text{str}(v_2) + \text{str}(v_3) = 0 + 1 + 1 + 0 = 2.$$

Therefore there is a scope for introducing a new topological index using stress on vertices which is motivated by the identity (5). In this paper we introduce such topological index for graphs using stress on vertices called stress-sum index. Further, we establish some inequalities and compute stress-sum index for some standard graphs.

§2. Stress-Sum Index for Graphs

Definition 2.1. *The stress-sum index $\mathcal{SS}(G)$ of a simple graph G is defined as*

$$\mathcal{SS}(G) = \sum_{uv \in E(G)} str(u) + str(v) \tag{6}$$

Observation: From the Definition 2.1, it follows that, for any graph G,

$$2m\theta_G \leq \mathcal{SS}(G) \leq 2m\Theta_G$$

where m is the number of edges in G.
Example 2.2. Consider the graph G given in Figure 2.

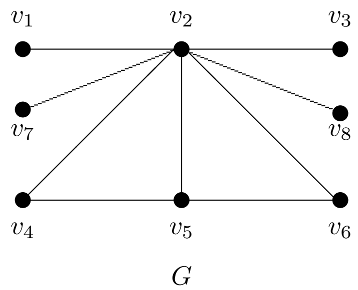

Figure 2: A graph G

The stresses of the vertices of G are as follows:
$\text{str}(v_1) = \text{str}(v_3) = \text{str}(v_7) = \text{str}(v_8) = 0$,
$\text{str}(v_2) = 19$,
$\text{str}(v_5) = 1$,
$\text{str}(v_4) = \text{str}(v_6) = 0$.
The stress-sum index of G is:

$$\begin{aligned}
\mathcal{SS}(G) =& (\text{str}(v_2) + \text{str}(v_1)) + (\text{str}(v_2) + \text{str}(v_3)) + (\text{str}(v_2) + \text{str}(v_7)) \\
& + (\text{str}(v_2) + \text{str}(v_8) + (\text{str}(v_2) + \text{str}(v_4)) + (\text{str}(v_2) + \text{str}(v_5)) \\
& + (\text{str}(v_2) + \text{str}(v_6)) + (\text{str}(v_4) + \text{str}(v_5)) + (\text{str}(v_5) + \text{str}(v_6)) \\
=& (19 + 0) + (19 + 0) + (19 + 0) + (19 + 0) + (19 + 0) + (19 + 1) \\
& + (19 + 0) + (0 + 1) + (1 + 0) \\
=& 136.
\end{aligned}$$

Proposition 2.3. *Let N be the number of geodesics of length ≥ 2 in a graph G. Then*

$$0 \leq \mathcal{SS}(G) \leq 2N(|E| - t), \tag{7}$$

where t is the number of edges with end vertices having zero stress in G.

Proof. If N is the number of all geodesics of length ≥ 2 in a graph G, then by the definition of stress of a vertex, for any vertex v in G, $0 \leq \text{str}(v) \leq N$. Hence by the Definition 2.1, we have

$$0 \leq \mathcal{SS}(G) \leq 2N(|E| - t), \tag{8}$$

where t is the number of edges with end vertices having zero stress in G. □

Corollary 2.4. *If there is no geodesic of length ≥ 2 in a graph G, then $\mathcal{SS}(G) = 0$. Moreover, for a complete graph K_n, $\mathcal{SS}(K_n) = 0$.*

Proof. If there is no geodesic of length ≥ 2 in a graph G, then $N = 0$. Hence, by the Proposition 2.3., we have $\mathcal{SS}(G) = 0$.

In K_n, there is no geodesic of length ≥ 2 and so $\mathcal{SS}(K_n) = 0$. □

Theorem 2.5. *For a graph G, $\mathcal{SS}(G) = 0$ if and only if neighbours of every vertex induce a complete subgraph of G.*

Proof. Suppose that $\mathcal{SS}(G) = 0$. Then by the Definition 2.1(Eq.(3)), $\text{str}(u) + \text{str}(v) = 0$, $\forall uv \in E(G)$. Hence $\text{str}(v) = 0$, $\forall v \in V(G)$. Let $v \in V(G)$. We need to show that neighbors of v induce a complete subgraph of G. If v is a pendant vertex, then there is nothing to prove. Suppose that v is not a pendant vertex. We claim that any two neighbouring vertices are adjacent in G. If there are two neighbours u and w of v that are not adjacent in G, then uvw is a graph geodesic passing through v, which implies $\text{str}(v) \geq 1$, a contradiction. Hence our claim holds. Thus neighbours of v induce a complete subgraph of G. Since v is arbitrary in $V(G)$, the neighbours of every vertex induce a complete subgraph of G.

Conversely, suppose that neighbours of every vertex in G induce a complete subgraph of G. Let $v \in V(G)$. Since neighbors of v induce a complete subgraph of G, any two neighbouring vertices are adjacent and so there is no geodesic of length ≥ 2 passing through v. Since v is an arbitrary vertex in G, by the Corollary 2.4, it follows that $\mathcal{SS}(G) = 0$. □

Proposition 2.6. *For the complete bipartite $K_{m,n}$,*

$$\mathcal{SS}(K_{m,n}) = \frac{mn}{2}\left[n(n-1) + m(m-1)\right].$$

Proof. Let $V_1 = \{v_1, \ldots, v_m\}$ and $V_2 = \{u_1, \ldots, u_n\}$ be the partite sets of $K_{m,n}$. We have,

$$\text{str}(v_i) = \frac{n(n-1)}{2} \text{ for } 1 \leq i \leq m \tag{9}$$

and

$$\mathrm{str}(u_j) = \frac{m(m-1)}{2} \text{ for } 1 \leq j \leq n. \tag{10}$$

Using (9) and (10) in the Definition 2.1, we have

$$\begin{aligned}
\mathcal{SS}(K_{m,n}) &= \sum_{uv \in E(G)} \mathrm{str}(u) + \mathrm{str}(v) \\
&= \sum_{1 \leq i \leq m,\ 1 \leq j \leq m} \mathrm{str}(v_i) + \mathrm{str}(u_j) \\
&= \sum_{1 \leq i \leq m,\ 1 \leq j \leq n} \left[\frac{n(n-1)}{2} + \frac{m(m-1)}{2} \right] \\
&= mn \left[\frac{n(n-1)}{2} + \frac{m(m-1)}{2} \right] \\
&= \frac{mn}{2} \left[n(n-1) + m(m-1) \right].
\end{aligned}$$

□

Proposition 2.7. *If $G = (V, E)$ is a k-stress regular graph, then*

$$\mathcal{SS}(G) = 2k|E|.$$

Proof. Suppose that G is a k-stress regular graph. Then

$$\mathrm{str}(v) = k \text{ for all } v \in V(G).$$

By the Definition 2.1, we have

$$\begin{aligned}
\mathcal{SS}(G) &= \sum_{uv \in E(G)} \mathrm{str}(u) + \mathrm{str}(v) \\
&= \sum_{uv \in E(G)} k + k \\
&= 2k|E|.
\end{aligned}$$

□

Corollary 2.8. *For a cycle C_n,*

$$\mathcal{SS}(C_n) = \begin{cases} \dfrac{n(n-1)(n-3)}{4}, & \textit{if } n \textit{ is odd} \\ \dfrac{n^2(n-2)}{4}, & \textit{if } n \textit{ is even.} \end{cases}$$

Proof. For any vertex v in C_n, we have,

$$\mathrm{str}(v) = \begin{cases} \dfrac{(n-1)(n-3)}{8}, & \text{if } n \text{ is odd} \\ \dfrac{n(n-2)}{8}, & \text{if } n \text{ is even.} \end{cases}$$

Hence C_n is

$$\begin{cases} \dfrac{(n-1)(n-3)}{8}\text{-stress regular}, & \text{if } n \text{ is odd} \\ \dfrac{n(n-2)}{8}\text{-stress regular}, & \text{if } n \text{ is even.} \end{cases}$$

Since C_n has n vertices and n edges, by the Proposition 2.7, we have

$$\begin{aligned} \mathcal{SS}(C_n) &= 2n \times \begin{cases} \dfrac{(n-1)(n-3)}{8}, & \text{if } n \text{ is odd} \\ \dfrac{n(n-2)}{8}, & \text{if } n \text{ is even.} \end{cases} \\ &= \begin{cases} \dfrac{n(n-1)(n-3)}{4}, & \text{if } n \text{ is odd} \\ \dfrac{n^2(n-2)}{4}, & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

□

Proposition 2.9. *Let T be a tree on n vertices. Then*

$$\mathcal{SS}(T) = \sum_{uv\in J}\left[\sum_{1\le i<j\le m(u)} |C_i^u||C_j^u| + \sum_{1\le i<j\le m(v)} |C_i^v||C_j^v|\right] + \sum_{w\in Q}\sum_{1\le i<j\le m(w)} |C_i^w||C_j^w|.$$

where J is the set of internal(non-pendant) edges in T, Q denotes the set of all vertices adjacent to pendent vertices in T, and the sets $C_1^v, \ldots, C_m^v$ denotes the vertex sets of the components of $T - v$ for an internal vertex v of degree $m = m(v)$.

Proof. We know that a pendant vertex in T has zero stress. Let v be an internal vertex of T of degree $m = m(v)$. Let $C_1^v, \ldots, C_m^v$ be the components of $T - v$. Since there is only one path between any two vertices in a tree, it follows that,

$$\text{str}(v) = \sum_{1\le i<j\le m} |C_i^v||C_j^v| \tag{11}$$

Let J denotes the set of internal(non-pendant) edges, and P denotes pendant edges and Q denotes the set of all vertices adjacent to pendent vertices in T. Then using (11) in the Definition 2.1 (6), we have

$$\begin{aligned} \mathcal{SS}(T) &= \sum_{uv\in J} \text{str}(u) + \text{str}(v) + \sum_{uv\in P} \text{str}(u) + \text{str}(v) \\ &= \sum_{uv\in J} \text{str}(u) + \text{str}(v) + \sum_{w\in Q} \text{str}(w) \\ &= \sum_{uv\in J}\left[\sum_{1\le i<j\le m(u)} |C_i^u||C_j^u| + \sum_{1\le i<j\le m(v)} |C_i^v||C_j^v|\right] \\ &\qquad + \sum_{w\in Q}\sum_{1\le i<j\le m(w)} |C_i^w||C_j^w|. \end{aligned}$$

□

Corollary 2.10. *For the path P_n on n vertices*

$$\mathcal{SS}(P_n) = \frac{1}{3}n(n-1)(n-2).$$

Proof. The proof of this corollary follows by above Proposition 2.9. We follow the proof of the Proposition 2.9 to compute the index. Let P_n be the path with vertex sequence $v_1, v_2, \ldots, v_n$ (shown in Figure 3).

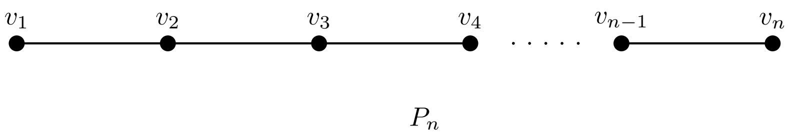

Figure 3: The path P_n on n vertices.

We have,

$$\text{str}(v_i) = (i-1)(n-i), \qquad 1 \leq i \leq n.$$

Then

$$\begin{aligned}\mathcal{SS}(P_n) &= \sum_{uv \in E(P_n)} \text{str}(u) + \text{str}(v) \\ &= \sum_{i=1}^{n-1} \text{str}(v_i) + \text{str}(v_{i+1}) \\ &= \sum_{i=1}^{n-1} [(i-1)(n-i) + (i)(n-i-1)] \\ &= \frac{1}{3}n(n-1)(n-2).\end{aligned}$$

□

Proposition 2.11. *Let $Wd(n,m)$ denotes the windmill graph constructed for $n \geq 2$ and $m \geq 2$ by joining m copies of the complete graph K_n at a shared universal vertex v. Then*

$$\mathcal{SS}(Wd(n,m)) = \frac{m^2(m-1)(n-1)^3}{2}.$$

Hence, for the friendship graph F_k on $2k+1$ vertices,

$$\mathcal{SS}(F_k) = 4k^2(k-1).$$

Proof. Clearly the stress of any vertex other than universal vertex is zero in $Wd(n,m)$, because neighbors of that vertex induces a complete subgraph of $Wd(n,m)$. Also, since there are m copies of K_n in $Wd(n,m)$ and their vertices are adjacent to v, it follows that, the only geodesics passing through v are of length 2 only. So, $\text{str}(v) = \dfrac{m(m-1)(n-1)^2}{2}$. Note that there are

$m(n-1)$ edges incident on v and the edges that are not incident on v have end vertices of stress zero. Hence by the Definition 2.1, we have

$$\begin{aligned}\mathcal{SS}(Wd(n,m)) &= m(n-1)\text{str}(v)\\ &= m(n-1)\frac{m(m-1)(n-1)^2}{2}\\ &= \frac{m^2(m-1)(n-1)^3}{2}.\end{aligned}$$

Since the friendship graph F_k on $2k+1$ vertices is nothing but $Wd(3,k)$, it follows that

$$\mathcal{SS}(F_k) = \frac{2^3k^2(k-1)}{2} = 4k^2(k-1). \qquad \square$$

Proposition 2.12. *Let W_n denotes the wheel graph constructed on $n \geq 4$ vertices. Then*

$$\mathcal{SS}(W_n) = \begin{cases} \dfrac{(n-1)(7n-10)(n-4)}{8}, & \textit{if } n \textit{ is even;}\\ \dfrac{(n-1)^2(7n-25)}{8}, & \textit{if } n \textit{ is odd.}\end{cases}.$$

Proof. In W_n with $n \geq 4$, there are $(n-1)$ peripheral vertices and one central vertex, say v. It is easy to see that

$$\text{str}(v) = \frac{(n-1)(n-4)}{2} \tag{12}$$

Let p be a peripheral vertex. Since v is adjacent to all the peripheral vertices in W_n, there is no geodesic passing through p and containing v. Hence contributing vertices for $\text{str}(p)$ are the rest peripheral vertices. So, by denoting the cycle $W_n - p$ (on $n-1$ vertices) by C_{n-1}, we have

$$\begin{aligned}\text{str}_{W_n}(p) &= \text{str}_{W_n - v}(p)\\ &= \text{str}_{C_{n-1}}(p)\\ &= \begin{cases}\dfrac{(n-2)(n-4)}{8}, & \text{if } n-1 \text{ is odd;}\\ \dfrac{(n-1)(n-3)}{8}, & \text{if } n-1 \text{ is even,}\end{cases}\\ &= \begin{cases}\dfrac{(n-2)(n-4)}{8}, & \text{if } n \text{ is even;}\\ \dfrac{(n-1)(n-3)}{8}, & \text{if } n \text{ is odd.}\end{cases}\end{aligned} \tag{13}$$

Let us denote the set of all radial edges in W_n by R, and the set of all peripheral edges by Q. Note that there are $(n-1)$ radial edges and $(n-1)$ peripheral edges in W_n. Using (12) and (13) in the Definition 2.1, we have

$$\begin{aligned}\mathcal{SS}(W_n) &= \sum_{xy\in R}[\text{str}(x)+\text{str}(y)] + \sum_{xy\in Q}[\text{str}(x)+\text{str}(y)]\\ &=(n-1)[\text{str}(v)+\text{str}(p)] + (n-1)\cdot 2\cdot \text{str}(p)\end{aligned}$$

$$
\begin{aligned}
&= (n-1)\left[\frac{(n-1)(n-4)}{2} + \begin{cases} \frac{(n-2)(n-4)}{8}, & \text{if } n \text{ is even;} \\ \frac{(n-1)(n-3)}{8}, & \text{if } n \text{ is odd.} \end{cases}\right] \\
&\quad + 2(n-1) \times \begin{cases} \frac{(n-2)(n-4)}{8}, & \text{if } n \text{ is even;} \\ \frac{(n-1)(n-3)}{8}, & \text{if } n \text{ is odd.} \end{cases} \\
&= \begin{cases} \frac{(n-1)^2(n-4)}{2} + \frac{3(n-1)(n-2)(n-4)}{8}, & \text{if } n \text{ is even;} \\ \frac{(n-1)^2(n-4)}{2} + \frac{3(n-1)^2(n-3)}{8}, & \text{if } n \text{ is odd.} \end{cases} \\
&= \begin{cases} \frac{(n-1)(7n-10)(n-4)}{8}, & \text{if } n \text{ is even;} \\ \frac{(n-1)^2(7n-25)}{8}, & \text{if } n \text{ is odd.} \end{cases} \qquad \square
\end{aligned}
$$

Conclusion

We have introduced a new topological index for graphs called stress-sum index using stresses of vertices. Further, we established some inequalities, proved some results and computed the stress-difference index for some standard graphs. A large number of molecular-graph-based structure descriptors (topological indices) have been defined, depending on vertex degrees. But in this paper, we have defined the new topological index for graphs without using the degrees of vertices. This index can be used to determine $\mathcal{SS}(G)$ for other classes of graphs and results in this direction will be reported in a subsequent paper.

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