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A survey on Smarandache notions in number theory VI: Smarandache Ceil function

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Abstract In this paper we give a survey on recent results on Smarandache Ceil function.

Keywords Smarandache notion, Smarandache Ceil function, sequence, mean value.
2010 Mathematics Subject Classification 11A07, 11B50, 11L20, 11N25.

§1. Definition and simple properties

For any fixed positive integer k and any positive integer n, the famous Smarandache ceil function $S_k(n)$ is defined as follows:

$$S_k(n) = \min\left\{m \in \mathbb{N} : \ n \mid m^k\right\}. \tag{1.1}$$

Many people had studied elementary properties of $S_k(n)$, and obtained some interesting results.

Z. Xu [18]. *Define $\Omega(n) = \Omega(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}) = \alpha_1 + \alpha_2 + \cdots + \alpha_r$. Let k be a given positive integer. Then for any real number $x \geq 3$, we have the asymptotic formula*

$$\sum_{n \leq x} \Omega\left(S_k(n)\right) = x \ln\ln x + Ax + O\left(\frac{x}{\ln x}\right),$$

where $A = \gamma + \sum\limits_{p}\left(\ln\left(1 - \frac{1}{p}\right) + \frac{1}{p}\right)$, γ is the Euler constant and $\sum\limits_{p}$ denotes the sum over all the primes.

J. Li [8]. *Define $\Omega(n) = \Omega(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}) = \alpha_1 + \alpha_2 + \cdots + \alpha_r$. Let k be a given positive integer. Then for any integer $n \geq 3$, we have the asymptotic formula*

$$\Omega\left(S_k(n!)\right) = \frac{n}{k}(\ln\ln n + C) + O\left(\frac{n}{\ln n}\right),$$

where C is a computable constant.

Y. Wang [15]. *Let k be a fixed positive integer, then for any integer $n \geq 3$, we have the asymptotic formula*

$$\ln(S_k(n!)) = \frac{n \ln n}{k} + O\left(n\right).$$

§2. Mean values of the Smarandache Ceil function

L. Ding [1]. *Let $x \geq 2$, for any fixed positive integer k, we have the asymptotic formula*

$$\sum_{n\leq x} S_k(n) = \frac{x^2\zeta(2k-1)}{2}\prod_p\left[1-\frac{1}{p(p+1)}\left(1+\frac{1}{p^{2k-3}}\right)\right]+O\left(x^{\frac{3}{2}+\varepsilon}\right),$$

where $\zeta(s)$ is the Riemann zeta function, $\prod\limits_p$ denotes the product over all prime p, and ε is any fixed positive number.

C. Wu [16]. 1) *For any fixed positive integer $k \geq 2$ and any positive integer n, let $a_k(n)$ denote the k-th power complements of n. Then we have*

$$(S_k(n))^k = a_k(n)\cdot n.$$

2) *Let k be a fixed positive integer. For any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{n\leq x} S_k(n) = \frac{\zeta(2k-1)}{2}x^2\prod_p\left(1-\frac{1}{p^2+p}-\frac{1}{p^{2k-1}+p^{2k-2}}\right)+O\left(x^{\frac{3}{2}+\varepsilon}\right),$$

where $\zeta(s)$ is the Riemann zeta function, $\varepsilon > 0$ is any fixed positive number.

X. Wang [13]. *For any real number $x \geq 2$, we have the asymptotic formula*

$$\sum_{n\leq x}\frac{1}{S_2(n)} = \frac{3\ln^2 x}{2\pi^2}+A_1\ln x+A_2+O\left(x^{-\frac{1}{4}+\varepsilon}\right),$$

where A_1 and A_2 are two computable constants, ε is any fixed positive integer.

Y. Wang [14]. 1) *For any real number $\alpha > 1$ and integer $k \geq 2$, we have the identity*

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{S_k^{\alpha}(n)} = \frac{2^{\alpha}-k-1}{2^{\alpha}+k-1}\prod_p\left(1+\frac{k}{p^{\alpha}-1}\right),$$

where $\prod\limits_p$ denotes the product over all prime p.

2) *For any positive integer n, the dual function of $S_k(n)$ is defined as $\overline{S_k}(n) = \max\left\{m\in\mathbb{N}:\ m^k \mid n\right\}$. For any real number $\alpha > 1$ and integer $k \geq 2$, we have the identities*

$$\sum_{n=1}^{\infty}\frac{\overline{S_k}(n)}{n^{\alpha}} = \frac{\zeta(\alpha)\zeta(k\alpha-1)}{\zeta(k\alpha)},$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\overline{S_k}(n)}{n^{\alpha}} = \frac{\zeta(\alpha)\zeta(k\alpha-1)}{\zeta(k\alpha)}\left[\frac{(2^{\alpha}-1)(2^{k\alpha-1}-1)}{2^{\alpha-2}(2^{k\alpha}-1)}-1\right],$$

where $\zeta(s)$ is the Riemann zeta function.

D. Ren [12]. *Let $d(n)$ denote the Dirichlet divisor function, and let k be a given positive integer with $k \geq 2$. Then for any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{n\leq x} d\left(S_k(n)\right) = \frac{6\zeta(k)x\ln x}{\pi^2}\prod_p\left(1-\frac{1}{p^k+p^{k-1}}\right)+Cx+O\left(x^{\frac{1}{2}+\varepsilon}\right),$$

where $\zeta(s)$ is the Riemann zeta function, C is a computable constant, and ε is any fixed positive number.

X. He and J. Guo [7]. 1) *Let $\alpha > 0$, $\sigma_\alpha(n) = \sum_{d|n} d^\alpha$. Then for any real number $x \geq 2$, and any fixed positive integer $k \geq 2$, we have the asymptotic formula*

$$\sum_{n\leq x} \sigma_\alpha\left(S_k(n)\right) = \frac{6x^{\alpha+1}\zeta(\alpha+1)\zeta(k(\alpha+1)-\alpha)}{(\alpha+1)\pi^2} R(\alpha+1) + O\left(x^{\alpha+\frac{1}{2}}+\varepsilon\right),$$

where $\zeta(s)$ is the Riemann zeta function, ε is any fixed positive number, and

$$R(\alpha+1) = \prod_p \left(1 - \frac{1}{p^{k(\alpha+1)-\alpha} - p^{(k-1)(\alpha+1)}}\right).$$

2) *Let $d(n)$ denote the Dirichlet divisor function. Then for any real number $x \geq 1$, and any fixed positive integer $k \geq 2$, we have the asymptotic formula*

$$\sum_{n\leq x} d\left(S_k(n)\right) = \frac{6\zeta(k)x\ln x}{\pi^2} \prod_p \left(1 - \frac{1}{p^k + p^{k-1}}\right) + Cx + O\left(x^{\frac{1}{2}+\varepsilon}\right),$$

where $\zeta(s)$ is the Riemann zeta function, C is a computable constant, and ε is any fixed positive number.

L. Zhang, M. Lv and W. Zhai [20]. *Let $d_3(n)$ denote the Piltz divisor function of dimensional 3, then for any real number $x \geq 2$, we have*

$$\sum_{n\leq x} d_3\left(S_k(n)\right) = xP_{2,k}(\log x) + O\left(x^{\frac{1}{2}} e^{-c\delta(x)}\right),$$

where $P_{2,k}(\log x)$ is a polynlmial of degree 2 in $\log x$, $\delta(x) = \log^{\frac{3}{5}} x(\log\log x)^{-\frac{1}{5}}$, $c > 0$ is an absulute constant.

Y. Zhang, H. Liu and P. Zhao [21]. *Let $d(n)$ denote the Dirichlet divisor function, $S_k(n)$ denote the Smarandache ceil function, then for any real number $\frac{1}{4} < \theta < \frac{1}{3}$, $x^{\theta+2\varepsilon} \leq y \leq x$, we have*

$$\sum_{x<n\leq x+y} d(S_k(n)) = H(x+y) - H(x) + O\left(yx^{-\frac{\varepsilon}{2}} + x^{\theta+\varepsilon}\right),$$

where $H(x) = t_1 x\log x + t_2 x$, ε denotes a fixed but sufficiently small positive constant.

Q. Feng and R. Wang [4]. *For any positive integer n, we define*

$$a_k(n) = \left[n^{\frac{1}{k}}\right], \qquad n = 0, 1, 2, 3, \cdots.$$

Let $\zeta(s)$ be the Riemann zeta function, X be any positive number, and

$$g(s) = \prod_p (1 + p^{1-s} - p^{1-ks} - p^{-s}).$$

1) *For any real number $x \geq 1$, $k \geq 3$, we have*

$$\sum_{n\leq x} S_k\left(a_k(n)\right) = \frac{1}{k}\zeta(k-1)g(1)x + O\left(x^{1-\frac{1}{2k}+X}\right).$$

2) *For any real number $x \geq 1$, $k \leq 2$, we have*

$$\sum_{n\leq x} S_k\left(a_k(n)\right) = \frac{k}{k^2-k+2}\zeta\left(\frac{2}{k}\right) g\left(\frac{2}{k}\right) x^{\frac{k^2-k+2}{k^2}} + O\left(x^{\frac{k^2-k+2}{k^2}+X}\right).$$

Q. Feng, J. Guo and R. Wang [5]. *For any positive integer n and any natural number m, we define*

$$a_m(n) = \max\left\{i^m:\ i^m \leq n,\ i \in \mathbb{N}\right\}.$$

1) *For any real number $x \geq 1$, $n, m, k, t \in \mathbb{N}$, $m, t \geq 2$, $k = tm+1$, we have*

$$\begin{aligned}\sum_{n\leq x} S_k\left(a_m(n)\right) &= \frac{m}{m+1}x^{1+\frac{1}{m}}\zeta(2t-1)\zeta((2t-1)m+2)\\ &\times\prod_p\left[1-\frac{1}{p(p+1)}\left(1+\frac{1}{p^{2t-3}}+\frac{1}{p^{(2t-1)m-1}}\left(1-\frac{1}{p^{2t}}\right)\right)\right] + O\left(x^{1+\frac{1}{2m}+\varepsilon}\right),\end{aligned}$$

where $\zeta(s)$ is the Riemann zeta function, ε is any positive real number.

2) *For any real number $x \geq 1$, $n, m, k, t \in \mathbb{N}$, $m = 2$, $t \geq 2$, $k = 2t+1$, we have*

$$\sum_{n\leq x} S_k\left(a_m(n)\right) = \frac{2}{3}x^{\frac{3}{2}}\zeta(4t)\prod_p\left[1-\frac{1}{p(p+1)}\left(1+\frac{1}{p^{2t-1}}+\frac{1}{p^{2(t-1)}}\left(1-\frac{1}{p^{2t}}\right)\right)\right] + O\left(x^{\frac{5}{4}+\varepsilon}\right),$$

where $\zeta(s)$ is the Riemann zeta function, ε is any positive real number.

3) *For any real number $x \geq 1$, $n, m, k, t \in \mathbb{N}$, $m, t \geq 2$, $k = tm$, we have*

$$\sum_{n\leq x} S_k\left(a_m(n)\right) = \frac{m}{m+1}x^{1+\frac{1}{m}}\zeta(2t-1)\prod_p\left(1-\frac{p^{2t}+p^3}{p^{2t+2}+p^{2t+1}}\right) + O\left(x^{1+\frac{1}{2m}+\varepsilon}\right),$$

where $\zeta(s)$ is the Riemann zeta function, ε is any positive real number.

4) *For any real number $x \geq 1$, $n, m, k, t \in \mathbb{N}$, $m, t \geq 2$, $m = kt$, we have*

$$\sum_{n\leq x} S_k\left(a_m(n)\right) = \frac{m}{m+1}x^{1+\frac{t}{m}} + O\left(x^{1+\frac{t}{2m}+\varepsilon}\right),$$

where ε is any positive real number.

J. Xu [17]. *For any fixed positive integer k and any integer n, we define*

$$\begin{aligned} c_k(n) &= \min\left\{m^k:\ m^k \geq n,\ m \in \mathbb{N}^+\right\},\\ d_k(n) &= \max\left\{m^k:\ m^k \leq n,\ m \in \mathbb{N}^+\right\}.\end{aligned}$$

For any real number $x > 2$, we have the asymptotic formula

$$\sum_{n\leq x} S_k\left(c_k(n)\right) = \frac{x^2}{2} + O\left(x^{\frac{2k-1}{k}}\right), \qquad \sum_{n\leq x} S_k\left(d_k(n)\right) = \frac{x^2}{2} + O\left(x^{\frac{2k-1}{k}}\right).$$

L. Qi and Y. Zhao [11]. *Let $k \geq 2$, $m \geq 1$ be two positive integers. For any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{n\leq x} \varphi^m\left(S_k(n)\right) = \frac{6\zeta(m+1)\zeta(k(m+1)-m)R(m+1)x^{m+1}}{\pi^2(m+1)} + O\left(x^{m+\frac{1}{2}+\varepsilon}\right),$$

where $\zeta(s)$ is the Riemann zeta function, $\varphi(n)$ is the Euler function, ε is any positive real number, and

$$R(m+1) = \prod_p \left[1 - \frac{1}{1+p}\left(\frac{1}{p} + \frac{1}{p^{k(m+1)-m}} + \frac{1}{p^{m-1}} - \frac{1}{p^{(k-1)(m+1)-m}} - \left(1 - \frac{1}{p^{k(m+1)}}\right)\cdot\frac{1}{p}\left(1-\frac{1}{p}\right)^m\right)\right].$$

E. Lv [10]. *Define*

$$U(1) = 1, \qquad U(n) = \prod_{p|n} p.$$

Let $k \geq 2$ be a fixed positive integer. For any real number $x \geq 1$, we have the asymptotic formula

$$\begin{aligned}\sum_{n\leq x}\left(S_k(n) - U(n)\right)^2 \quad = \quad & \frac{2\zeta(3)\zeta(3k-2)x^3}{\pi^2}\prod_p\left(1 - \frac{1+p^{5-3k}}{p^2+p^3}\right) + \frac{2\zeta(3)x^3}{\pi^2}\prod_p\left(1-\frac{1}{p^2+p^3}\right)\\ & -\frac{4\zeta(3)\zeta(3k-1)x^3}{\pi^2}\prod_p\left(1+\frac{p-p^2-p^4-p^{3k}}{p^{3k+3}+p^{3k+2}}\right) + O\left(x^{\frac{5}{2}+\varepsilon}\right),\end{aligned}$$

where $\zeta(s)$ is the Riemann zeta function, $\varepsilon > 0$ is any positive real number.

Y. Xue and L. Gao [19]. *Define*

$$U(1) = 1, \qquad U(n) = \prod_{p|n} p.$$

Let $k \geq 2$ be a fixed positive integer. For any real number $x \geq 1$, we have the asymptotic formula

$$\begin{aligned}\sum_{n\leq x}\left(S_k(n) + U(n)\right)^3 \quad = \quad & \frac{3\zeta(4)\zeta(4k-3)x^4}{2\pi^2}\prod_p\left(1-\frac{1+p^{7-4k}}{p^3+p^4}\right)\\ & +\frac{9\zeta(4)\zeta(4k-2)x^4}{2\pi^2}\prod_p\left(1-\frac{1+p^{3-4k}+p^{6-4k}-p^{2-4k}}{p^3+p^4}\right)\\ & +\frac{9\zeta(4)\zeta(4k-1)x^4}{2\pi^2}\prod_p\left(1-\frac{1+p^{5-4k}-p^{1-4k}+p^{3-4k}}{p^3+p^4}\right)\\ & +\frac{3\zeta(4)x^4}{2\pi^2}\prod_p\left(1-\frac{1}{p^3+p^4}\right) + O\left(x^{\frac{7}{2}+\varepsilon}\right),\end{aligned}$$

where $\zeta(s)$ is the Riemann zeta function, ε is any positive real number.

§3. The dual function of the Smarandache Ceil function

For any positive integer n and any fixed positive integer k, the dual function of $S_k(n)$ is defined as follows:

$$\overline{S_k}(n) = \max\left\{m \in \mathbb{N} : \ m^k \mid n\right\}.$$

J. Guo and Y. He [6]. 1) *Let $\alpha > 0$, $\sigma_\alpha(n) = \sum_{d|n} d^\alpha$. Then for any real number $x \geq 1$ and any fixed positive integer $k \geq 2$, we have the asymptotic formula*

$$\sum_{n\leq x}\sigma_\alpha\left(\overline{S_k}(n)\right) = \begin{cases} \frac{k\zeta\left(\frac{\alpha+1}{k}\right)}{\alpha+1}x^{\frac{\alpha+1}{k}} + O\left(x^{\frac{\alpha+1}{2k}+\varepsilon}\right), & \textit{if} \ \ \alpha > k-1,\\ \zeta(k-\alpha)x + O\left(x^{\frac{1}{2}+\varepsilon}\right), & \textit{if} \ \ \alpha \leq k-1,\end{cases}$$

where $\zeta(s)$ is the Riemann zeta function, and ε is any fixed positive number.

2) *Let $d(n)$ denote the Dirichlet divisor function. Then for any real number $x \geq 1$ and any fixed positive integer $k \geq 2$, we have*

$$\sum_{n\leq x} d\left(\overline{S_k}(n)\right) = \zeta(k)x + O\left(x^{\frac{1}{2}+\varepsilon}\right),$$

where $\zeta(s)$ is the Riemann zeta function, and ε is any fixed positive number.

Y. Lu [9]. *Let $d(n)$ denote the Dirichlet divisor function, and let $k \geq 2$ be a fixed integer. Then for any real number $x > 1$, we have the asymptotic formula*

$$\sum_{n\leq x} d\left(\overline{S_1}(n)\right) = x\ln x + (2\gamma - 1)x + O\left(x^{\frac{1}{3}}\right),$$

$$\sum_{n\leq x} d\left(\overline{S_k}(n)\right) = \zeta(k)x + \zeta\left(\frac{1}{k}\right)x^{\frac{1}{k}} + O\left(x^{\frac{1}{k+1}}\right),$$

where γ is the Euler constant, and $\zeta(s)$ is the Riemann zeta function.

L. Ding [2]. 1) *Let $x \geq 2$, for any fixed positive integer $k > 2$, we have the asymptotic formula*

$$\sum_{n\leq x} \overline{S_k}(n) = \frac{\zeta(k-1)}{\zeta(k)}x + O\left(x^{\frac{1}{2}+\varepsilon}\right),$$

where $\zeta(s)$ is the Riemann zeta function, and ε is any fixed positive number.

2) *For $k = 2$, we have the asymptotic formula*

$$\sum_{n\leq x} \overline{S_2}(n) = x\left(\frac{3}{\pi^2}\ln x + C\right) + O\left(x^{\frac{3}{4}+\varepsilon}\right),$$

where C is a computable constant, and ε is any fixed positive number.

Q. Feng and J. Guo [3]. *For any positive integer n and any fixed positive integer $k \geq 2$, we define*

$$c_k(n) = \min\left\{m \in \mathbb{N} : \ nm = t^k, \ t \in \mathbb{N}\right\}.$$

1) *For any real number $x \geq 1$, k, $n \in \mathbb{N}$, $k \geq 2$, we have*

$$\begin{aligned}\sum_{n\leq x} S_k(n)c_k(n) \quad &= \quad \frac{6}{(k+1)\pi^2}x^{k+1}\zeta(k+2)\zeta(k^2+k-1) \\ &\quad \times \prod_p \left(1 - \frac{1}{p^{k-1}(p+1)}\left(\frac{1}{p^2} + \frac{1}{p^{k^2-1}}\right)\right) + O\left(x^{k+\frac{1}{2}+\varepsilon}\right),\end{aligned}$$

where $\zeta(s)$ is the Riemann zeta function, and ε is any fixed positive number.

2) *For any real number $x \geq 1$, k, $n \in \mathbb{N}$, $k \geq 2$, we have*

$$\sum_{n\leq x} S_k(c_k(n)) = \frac{3}{\pi^2}x^2\prod_p\left(1 + \frac{R(2)}{(p+1)(p^2-2)}\right) + O\left(x^{\frac{3}{2}+\varepsilon}\right),$$

where ε is any fixed positive number, and

$$R(2) = 1 - \frac{1}{p^{2(k-2)}} + \left(p^2\left(1 - \frac{1}{p^{2(k-1)}}\right) + p^3 - p\right)\frac{1}{p^{2k-1}}.$$

3) *For any real number $x \geq 1$, k, $n \in \mathbb{N}$, $k \geq 2$, we have*

$$\begin{aligned}\sum_{n\leq x} \overline{S_k}(n)c_k(n) &= \frac{6}{k\pi^2}x^k\zeta(k+1)\zeta(k^2-1)\\ &\times\prod_p\left(1-\frac{1}{p^k(p+1)}\left(1+\frac{1}{p^{k^2-k-1}}-\frac{1}{p^{k^2-1}}\right)\right)+O\left(x^{k+\frac{1}{2}+\varepsilon}\right),\end{aligned}$$

where $\zeta(s)$ is the Riemann zeta function, and ε is any fixed positive number.

4) *For any real number $x \geq 1$, k, $n \in \mathbb{N}$, $k \geq 2$, we have*

$$\sum_{n\leq x} \overline{S_k}(c_k(n)) = x + O\left(x^{\frac{1}{2}+\varepsilon}\right),$$

where ε is any fixed positive number

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A survey on Smarandache notions in number theory VII: Smarandache multiplicative function

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Abstract In this paper we give a survey on recent results on Smarandache multiplicative function.

Keywords Smarandache multiplicative function, sequence, mean value.
2010 Mathematics Subject Classification 11A07, 11B50, 11L20, 11N25.

§1. Definition and the mean value properties of the Smarandache multiplicative function

For any positive integer n, $f(n)$ is called a Smarandache multiplicative function if $f(ab) = \max(f(a), f(b)), (a, b) = 1$, and if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ is the prime powers factorization of n, then

$$f(n) = \max_{1 \leq i \leq k} \{f(p_i^{\alpha_i})\}, \tag{1.1}$$

for any prime p and any positive integer α, $f(n)$ is a new Smarandache multiplicative function if $f(p^\alpha) = \alpha p$. That is

$$f(n) = \max_{1 \leq i \leq k} \{f(p_i^{\alpha_i})\} = \max_{1 \leq i \leq k} \{\alpha_i p_i\}.$$

J. Ma [11]. *For any real number $x \geq 2$, we have the asymptotic formula*

$$\sum_{n \leq x} f(n) = \frac{\pi^2}{12} \cdot \frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

Y. Liu, P. Gao [10]. *A new arithmetical function $P_d(n)$ is defined as*

$$P_d(n) = \prod_{d|n} d = n^{\frac{d(n)}{2}},$$

where $d(n) = \sum_{d|n} 1$ is the Dirichlet divisor function. For any real number $x \geq 2$, we have the asymptotic formula

$$\sum_{n \leq x} f(P_d(n)) = \frac{\pi^4}{72} \cdot \frac{x^2}{\ln x} + C \cdot \frac{x^2}{\ln^2 x} + O\left(\frac{x^2}{\ln^3 x}\right),$$

where $C = \frac{5\pi^4}{288} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{d(n) \ln n}{n^2}$ is a constant.

X. Zhang [24]. *For any integer $n \in \mathbb{N}^+$, n is named as a simple number if the product of all proper divisors of n is no more than n. Now let A be a simple number set, that is $A = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 17, 19, 21, \ldots\}$. For any real number $x \geq 2$ we have the asymptotic formula*

$$\sum_{\substack{n \leq x \\ n \in A}} f(n) = D_1 \frac{x^2}{\ln x} + D_2 \frac{x^2}{\ln^2 x} + \frac{2x}{\ln x} + \frac{9x^{2/3}}{2 \ln x} + O\left(\frac{x^2}{\ln^3 x}\right),$$

where D_1, D_2 are computable constants.

W. Xiong [19]. *Let $OF(N)$ denotes the number of all integers $1 \leq k \leq n$ such that $f(n)$ is odd, $EF(n)$ denotes the number of all integer $1 \leq k \leq n$ such that $f(n)$ is even. For any positive integer $n > 1$, we have the asymptotic formula*

$$\frac{EF(n)}{OF(n)} = O\left(\frac{1}{\ln n}\right).$$

From the formula above, it can be immediately deduced the following

$$\lim_{n \to \infty} \frac{EF(n)}{OF(n)} = 0.$$

J. Li [6]. *For any real number $x > 1$, we have the asymptotic formula*

$$\sum_{\substack{n \in \mathbb{N} \\ f(n) \leq x}} = \mathrm{e}^{c \frac{x}{\ln x} + O\left(\frac{x (\ln \ln x)^2}{\ln^2 x}\right)},$$

where $c = \sum_{n=1}^{\infty} \frac{\ln(n+1)}{n(n+1)}$ is a constant.

Z. Feng [1]. *A natural number n is of the k-full number if for any prime p, $p \mid n$ implies $p^k \mid n$. Let A_k be a simple number set, for any real number $x \geq 2$ we have the asymptotic formula*

$$\sum_{\substack{n \leq x \\ n \in A_k}} f(n) = C_1 \frac{x^2}{\ln x} + C_2 \frac{x^2}{\ln^2 x} + \frac{2x}{\ln x} + \frac{9x^{2/3}}{2 \ln x} + O\left(\frac{x^2}{\ln^3 x}\right),$$

where C_1, C_2 are computable constants.

Y. Men [12]. *Let $Smd(n) = \sum_{d \mid n} \frac{1}{f(d)}$, for any real number $x \geq 1$, when $n \neq 1, 24$, we have*

(1). *If $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s} p$, $p_1^{\alpha_1} < p_2^{\alpha_2} < \cdots < p_s^{\alpha_s} < p$, and p, $p_i (i = 1, 2, \ldots, s)$ are primes, then $Smd(n)$ is not a positive integer;*

(2). *If $n = p_1 p_2 \cdots p_s, p_1 < p_2 < \cdots < p_s$, $p_i (i = 1, 2, \ldots, s)$ are primes, then $Smd(n)$ is not a positive integer.*

R. Guo and X. Zhao [2]. 1. *For any real number $x \geq 1$ and any fixed positive integer $k \geq 2$, we have the asymptotic formula*

$$\sum_{n \leq x} \Lambda(n) f(n) = x^2 \sum_{i=1}^{k} \frac{c_i}{\ln^{i-1} x} + O\left(\frac{x^2}{\ln^k x}\right),$$

where $\Lambda(n)$ *is the Mangoldt function,* $c_i(i = 1, 2, \ldots, k)$ *are computable constants and* $c_1 = \frac{1}{2}$.

2. *For any real number* $x \geq 1$ *and any fixed positive integer* $k \geq 2$, *we have the asymptotic formula*

$$\sum_{n \leq x} \Lambda(n)S(n) = x^2 \sum_{i=1}^{k} \frac{c_i}{\ln^{i-1} x} + O\left(\frac{x^2}{\ln^k x}\right),$$

where $S(n)$ *is the famous Smarandache function,* $S(n) = \min\{m : m \in \mathbb{N}, n \mid m!\}$, $c_i(i = 1, 2, \ldots, k)$ *are computable constants and* $c_1 = \frac{1}{2}$.

For any positive integers m and n, an arithmetical function $h(n)$ is defined as follows

$$(m, n) = 1 \Rightarrow h(mn) = \max\{h(m), h(n)\}.$$

If $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ is the prime powers factorization of n, defining

$$h(1) = 1, h(n) = \max_{1 \leq i \leq k} \{\frac{1}{\alpha_i + 1}\}, \tag{1.2}$$

then $h(n)$ is also a Smarandache multiplicative function.

J. Zhang and P. Zhang [22]. 1. *For any real number* $x > 1$, *we have the asymptotic formula*

$$\sum_{n \leq x} h(n) = \frac{1}{2} \cdot x + O(x^{\frac{1}{2}}).$$

2. *For any real number* $x > 1$, *we have the asymptotic formula*

$$\sum_{n \leq x} \left(h(n) - \frac{1}{2}\right)^2 = \frac{1}{36} \cdot \frac{\zeta(\frac{3}{2})}{\zeta(3)} \cdot \sqrt{x} + O(x^{\frac{1}{3}}),$$

where $\zeta(n)$ *is the Riemann Zeta-function.*

The Smarandache multiplicative function $g(n)$ can also be defined as follows

$$g(1) = 0, (m, n) = 1 \Rightarrow g(mn) = \min\{g(m), g(n)\}. \tag{1.3}$$

If $n = p_1^{t_1} p_2^{t_2} \cdots p_r^{t_r}$ is the prime powers factorization of n, then

$$g(n) = \min_{1 \leq i \leq r} \{f(p_i^{t_i})\}, \tag{1.4}$$

specifically let $g(p^t) = \min\{t, p\}$, then $g(n)$ is a new Smarandache multiplicative function.

Z. Ren [13]. *For any real number* $x > 1$, *we have the asymptotic formula*

$$\sum_{n \leq x} g(n) = x + \frac{12x^{1/2}}{\pi^2} \prod_p \left(1 + \frac{1}{(p+1)(p^{\frac{1}{2}} - 1)}\right) + \frac{18x^{1/3}}{\pi^2} \prod_p \left(1 + \frac{1}{(p+1)(p^{\frac{1}{3}} - 1)}\right) + O(x^{\frac{1}{4}+X}),$$

where X *is any fixed positive number.*

L. Li [8]. 1. *For any positive integer* n, *if* $n = p_1^{t_1} p_2^{t_2} \cdots p_r^{t_r}$ *is the prime powers factorization of* n, *let* $\lambda = \max_{1 \leq i \leq r} \{t_i\}, i = 1, \ldots, r$ *and*

$$F(1) = 1, F(n) = \min_{1 \leq i \leq r} \{\frac{1}{t_i + 1}\} = \frac{1}{\lambda + 1}, \tag{1.5}$$

then $F(n)$ is a Smarandache multiplicative function. For any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\leq x} F(n) = \frac{1}{\lambda+1}x + O(x^{\frac{1}{2}}).$$

2. *For any real number $x > 1$, we have the asymptotic formula*

$$\sum_{n\leq x}\left(F(n) - \frac{1}{2}\right)^2 = \frac{12}{\pi^2}\sqrt{x} + O(x^{\frac{1}{3}}).$$

T. Zhang [23]. *Let p be a prime and for any positive real number m, $U_m(n)$ is defined as follows*

$$U(1) = 1, U_m(p^\alpha) = p^\alpha + m, \tag{1.6}$$

if $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ is the prime powers factorization of n, $U_m(n)$ is defined as $U_m(n) = U_m(p_1^{\alpha_1})\cdots U_m(p_k^{\alpha_k})$. For any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\leq x} U_m(n) = \frac{1}{2}x^2\prod_p\left(1 + \frac{m}{p(p+1)}\right) + O(x^{\frac{3}{2}+\varepsilon}).$$

X. Wang [18]. *Let $I(n)$ be the multiplicative function such that for any prime p and any integer $\alpha \geq 1$, one has*

$$I(p^\alpha) = \frac{p^{\alpha+1}}{\alpha+1},$$

then we have

$$\sum_{mn\leq x} I(m)I(n) = Cx^3 + O(x^{\frac{5}{2}+\varepsilon}),$$

where C is an explicit constant.

L. Wang [16]. *Let $N_0 \geq 1$ be a fixed integer and for the multiplicative function $I(n)$, we have*

$$\sum_{n\leq x} I(n) = x^3\log^{\frac{1}{2}} x\left(\sum_{i=1}^{N_0} c_i\log^{-i} x + O(\log^{-N_0-1} x)\right),$$

where $c_i (i \geq 1)$ are computable constants.

§2. Some hybrid mean values involving the Smarandache multiplicative function

Y. Yi [21]. *For any prime p and positive integer α, the Smarandache multiplicative function $f(n)$ is defined as $f(p^\alpha) = p^{\frac{1}{\alpha}}$. Let $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$ is the prime powers factorization of n, then from the definition of $f(p^\alpha)$ we have*

$$f(n) = \max_{1\leq i\leq r}\{f(p_i^{\alpha_i})\} = \max_{1\leq i\leq r}\left\{p_i^{\frac{1}{\alpha_i}}\right\}.$$

For any real number $x \geq 3$, we have the asymptotic formula

$$\sum_{n\leq x}(f(n)-P(n))^2 = \frac{2\zeta(\frac{3}{2})x^{\frac{3}{2}}}{3\ln x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x}\right),$$

where $\zeta(n)$ denotes the Riemann zeta-function and $P(n)$ is the greatest prime divisor of n.

W. Lu and L. Gao [9]. *For any real number $x \geq 3$ and any real number or complex number α, we have the asymptotic formula*

$$\sum_{n\leq x}\delta_\alpha(n)(f(n)-P(n))^2 = \frac{\zeta(\alpha+3)\zeta(2\alpha+3)x^{2\alpha+3}}{(2\alpha+3)\ln x} + \sum_{i=2}^{r}\frac{c_i\cdot x^{2\alpha+3}}{\ln^i x} + O\left(\frac{x^{2\alpha+3}}{\ln^{r+1} x}\right),$$

where $\zeta(n)$ denotes the Riemann zeta-function and all c_i are computable constants.

H. Shen [14]. *For any positive integer n, if $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$ is the prime powers factorization of n, the Smarandache multiplicative function $V(n)$ is defined as follows*

$$V(1) = 1, V(n) = \max_{1\leq i\leq r}\{\alpha_1 p_1, \ldots, \alpha_r p_r\}. \tag{2.1}$$

For any real number $x \geq 1$ and any fixed positive integer r, we have the asymptotic formula

$$\sum_{n\leq x}(V(n)-p(n))^2 = x^{\frac{3}{2}}\sum_{i=1}^{r}\frac{c_i}{\ln^i x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^{r+1} x}\right),$$

where $p(n)$ is the least prime divisor of n and all c_i are computable constants.

H. Liu and W. Cui [3]. *Let $n \geq 1$ is a positive integer, we have the asymptotic formula*

$$\sum_{n\leq x}V(n)p(n) = \sum_{i=1}^{r}\frac{x^3 a_i}{\ln^i x} + O\left(\frac{x^3}{\ln^{r+1} x}\right),$$

where all $a_i (i = 1, \ldots, r)$ are computable constants.

§3. Mean values involving the Smarandache-type multiplicative function

The Smarandache-type multiplicative function $C_m(n)$ is defined as the m-th root of the largest m-th power dividing n, $J_m(n)$ is denoted as m-th root of the smallest m-th power divisible by n.

H. Liu and J. Gao [5]. 1. *For any integer $m \geq 3$ and real number $x \geq 1$, we have*

$$\sum_{n\leq x}C_m(n) = \frac{\zeta(m-1)}{\zeta(m)}x + O\left(x^{\frac{1}{2}+\epsilon}\right).$$

2. *For any integer $m \geq 1$ and real number $x \geq 1$, we have*

$$\sum_{n\leq x}J_m(n) = \frac{x^2}{2\zeta(2)}\prod_p\left[1+\frac{\frac{1}{p^{2m}}+\frac{1}{p^3}-\frac{1}{p^{2m+1}}-\frac{1}{p^{2m+2}}}{(1+\frac{1}{p})(1-\frac{1}{p^2})(1-\frac{1}{p^{2m-1}})}\right] + O(x^{\frac{3}{2}+\epsilon}).$$

H. Liu and J. Gao [4]. 1. *For any integer $m \geq 3$ and real number $x \geq 1$, we have*

$$\sum_{n\leq x}\Lambda(n)C_m(n) = x + O\left(\frac{x}{\log x}\right),$$

where $\Lambda(n)$ is the Mangoldent function.

2. *For any integer $m \geq 2$ and real number $x \geq 1$, we have*

$$\sum_{n\leq x} \Lambda(n) J_m(n) = x^2 + O\left(\frac{x^2}{\log x}\right),$$

The Smarandache-type multiplicative function $K_m(n)$ is the largest m-th power-free number dividing n, $L_m(n)$ is denoted as: n divided by the largest m-th power-free number dividing n. That is, if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ is the prime powers factorization of n, it follows that

$$K_m(n) = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k},$$
$$L_m(n) = p_1^{\gamma_1} p_2^{\gamma_2} \cdots p_k^{\gamma_k},$$

where $\beta_i = \min(\alpha_i, m-1)$, $\gamma_i = \max(0, \alpha_i - m + 1)$

C. Yang and C. Li [20]. 1. *Let $m \geq 2$ is a given integer, then for any real number $x \geq 1$, we have*

$$\sum_{n\leq x} K_m(n) = \frac{x^2}{2\zeta(m)} \prod_p \left(1 + \frac{1}{(p^m-1)(p+1)}\right) + O\left(x^{\frac{3}{2}+\epsilon}\right).$$

2. *Let $m \geq 2$ is a given integer, then for any real number $x \geq 1$, we have*

$$\sum_{n\leq x} \frac{1}{L_m(n)} = \frac{x}{\zeta(m)} \prod_p \left(1 + \frac{1}{(p^m-1)(p+1)}\right) + O\left(x^{\frac{1}{2}+\epsilon}\right),$$

where $\zeta(s)$ is the Riemann Zeta-function.

J. Wang [15]. *The asymptotic formula*

$$\sum_{n\leq x} K_m(n) = \frac{x^2}{2\zeta(m)} \prod_p \left(1 + \frac{1}{(p^m-1)(p+1)}\right) + O\left(x^{1+\frac{1}{m}} e^{-c_0\delta(x)}\right).$$

holds, where c_0 is an absolute positive constant and $\delta(x) = (\log x)^{3/5}(\log\log x)^{-1/5}$.

For any fixed positive integer n with the normal factorization $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, $(1 \leq i \leq k)$, the Smarandache-type multiplicative function $F_m(n)$, $G_m(n)$ are denoted as

$$F_m(p_i^{\alpha_i}) = \begin{cases} 1, & \text{if } \alpha_i = mk, \\ p_i^m, & \text{otherwise} . \end{cases}$$

and

$$G_m(p_i^{\alpha_i}) = \begin{cases} 1, & \text{if } \alpha_i = mk, \\ p_i, & \text{otherwise} . \end{cases}$$

J. Li and D. Liu [7]. 1. *For any integer $m \geq 2$ and real number $x \geq 1$, we have*

$$\sum_{n\leq x} F_m(n) = \frac{6\zeta(m^2+m)\zeta(m+1)R(m+1)x^{m+1}}{\pi^2} + O\left(x^{m+\frac{1}{2}+\epsilon}\right),$$

where ϵ be any fixed positive integer,and

$$R(m+1) = \prod_p \left(1 - \frac{1}{p^{m+1} + p^m} - \frac{1}{p^{m^2} + p^{m^2-1}}\right).$$

2. *For any integer $m \geq 2$ and real number $x \geq 1$, we have*

$$\sum_{n \leq x} G_m(n) = \zeta(2m)R(2)x^2 + O(x^{\frac{3}{2}+\epsilon}),$$

where

$$R(2) = \prod_p \left(1 - \frac{1}{p^2 + p} - \frac{1}{p^{2m-1} + p^{2m-2}}\right).$$

M. Wang [17]. 1. *For any integer $m \geq 2$, A be a set without m-th power factor number, we have*

$$\sum_{\substack{n \leq x \\ n \in A}} F_m(n) = \frac{6\zeta(m+1)x^{m+1}}{\pi^2} \prod_p \left(1 - \frac{1}{p^{m-1} + p^m} - \frac{1}{p^{m^2} + p^{m^2-1}}\right) + O\left(x^{m+\frac{1}{2}-\epsilon}\right),$$

where ϵ be any fixed positive number.

2. *For any positive integer $m \geq 2$, A be a set without m-th power factor number, we have*

$$\sum_{\substack{n \leq x \\ n \in A}} G_m(n) = x^2 \prod_p \left(1 - \frac{1}{p^2 + p^m} - \frac{1}{p^{2m-1} + p^{2m-2}}\right) + O\left(x^{\frac{3}{2}-\epsilon}\right).$$

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Fuzzy e^*-open sets in $\hat{S}$ostak's topological spaces

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Abstract We introduce r-fuzzy e^*-open and r-fuzzy e^*-closed sets in fuzzy topological spaces in the sense of $\hat{S}$ostak's. Also we introduce r-fuzzy e^*-interior, r-fuzzy e^*-closure and investigate some of their properties.

Keywords r-fuzzy e^*-open, r-fuzzy e^*-closed, r-fuzzy e^*-interior and r-fuzzy e^*-closure.
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§1. Introduction and preliminaries

$\hat{S}$ostak [23] introduced the fuzzy topology as an extension of Chang's fuzzy topology [4]. It has been developed in many directions [11, 12, 21]. Weaker forms of fuzzy continuity between fuzzy topological spaces have been considered by many authors [2, 3, 5, 8, 10, 18, 19] using the concepts of fuzzy semi-open sets [2], fuzzy regular open sets [2], fuzzy preopen sets, fuzzy strongly semiopen sets [3], fuzzy γ-open sets [10], fuzzy δ-semiopen sets [1], fuzzy δ-preopen sets [1], fuzzy semi δ-preopen sets [25] and fuzzy e-open sets [22]. Recently, Bin Shahna [3] introduced and investigated fuzzy strong semi-continuity and fuzzy precontinuity between fuzzy topological spaces, one of which was independent and the other strictly stronger than fuzzy semi-continuity [2]. Ganguly and Saha [9] introduced the notions of fuzzy δ-cluster points in fuzzy topological spaces in the sense of Chang [4]. Kim and Park [14] introduced r-δ-cluster points and δ-closure operators in fuzzy topological spaces in view of the definition of $\hat{S}$ostak. It is a good extension of the notions of Ganguly and Saha [9]. Park et al. [17] introduced the concept of fuzzy semi-preopen sets which is weaker than any of the concepts of fuzzy semi-open or fuzzy preopen sets. Using these concepts he defined and studied fuzzy semi-precontinuous mappings between fuzzy topological spaces in Chang's sense. Sobana et al. [24], defined r-fuzzy e-open and r-fuzzy e-closed sets in a fuzzy topological space in the sense of $\hat{S}$ostak. In 2008, the initiations of e^*-open sets in topological spaces was introduced by Erdal Ekici [6].

In this paper, we define r-fuzzy e^*-open and r-fuzzy e^*-closed sets in a fuzzy topological space in the sense of $\hat{S}$ostak [23]. Using these concepts, we define and study fuzzy e^*-interior, fuzzy e^*-closure and some of their properties.

Throughout this paper, nonempty sets will be denoted by X, Y etc., $I = [0,\ 1]$ and $I_0 = (0,\ 1]$. For $\alpha \in I$, $\overline{\alpha}(x) = \alpha$ for all $x \in X$. A fuzzy point x_t for $t \in I_0$ is an element of I^X such that $x_t(y) = \begin{cases} t & \text{if } y = x \\ 0 & \text{if } y \notin x. \end{cases}$

The set of all fuzzy points in X is denoted by $Pt(X)$. A fuzzy point $x_t \in \lambda$ iff $t < \lambda(x)$. A fuzzy set λ is quasi-coincident with μ, denoted by $\lambda q \mu$, if there exists $x \in X$ such that $\lambda(x) + \mu(x) > 1$. If λ is not quasi-coincident with μ, we denoted $\lambda \bar{q} \mu$. If $A \subset X$, we define the characteristic function χ_A on X by $\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$

All other notations and definitions are standard, for all in the fuzzy set theory.

Lemma 1.1. [23] *Let X be a nonempty set and $\lambda,\ \mu \in I^X$. Then*

(i) $\lambda q \mu$ *iff there exists* $x_t \in \lambda$ *such that* $x_t q \mu$.

(ii) $\lambda q \mu$, *then* $\lambda \wedge \mu \neq \underline{0}$.

(iii) $\lambda \bar{q} \mu$ *iff* $\lambda \leq \underline{1} - \mu$.

(iv) $\lambda \leq \mu$ *iff* $x_t \in \lambda$ *implies* $x_t \in \mu$ *iff* $x_t q \lambda$ *implies* $x_t q \mu$ *implies* $x_t \bar{q} \lambda$.

(v) $x_t \bar{q} \bigvee\limits_{i \in \Lambda} \mu_i$ *iff there exists* $i_0 \in \Lambda$ *such that* $x_t \bar{q} \mu_{i_0}$.

Definition 1.1. [23] *A function $\tau : I^X \to I$ is called a fuzzy topology on X if it satisfies the following conditions:*

(1) $\tau(\underline{0}) = \tau(\underline{1}) = 1$,

(2) $\tau(\bigvee_{i \in \Gamma} \mu_i) \geq \bigwedge_{i \in \Gamma} \tau(\mu_i)$, *for any* $\{\mu_i\}_{i \in \Gamma} \subset I^X$,

(3) $\tau(\mu_1 \wedge \mu_2) \geq \tau(\mu_1) \wedge \tau(\mu_2)$, *for any* $\mu_1,\ \mu_2 \in I^X$.

The pair $(X,\ \tau)$ is called a fuzzy topological space (for short, fts).

Remark 1.1. [20] *Let (X, τ) be a fuzzy topological space. Then, for each $r \in I_0$, $\tau_r = \{\mu \in I^X : \tau(\mu) \geq r\}$ is a Change's fuzzy topology on X.*

Theorem 1.1. [21] *Let $(X,\ \tau)$ be a fts. Then for each $\lambda \in I^X$, $r \in I_0$ we define an operator $C_\tau : I^X \times I_0 \to I^X$ as follows: $C_\tau(\lambda,\ r) = \bigwedge\{\mu \in I^X : \lambda \leq \mu,\ \tau(\underline{1} - \mu) \geq r\}$. For $\lambda,\ \mu \in I^X$ and $r, s \in I_0$, the operator C_τ satisfies the following conditions: (1) $C_\tau(\underline{0}, r) = \underline{0}$, (2) $\lambda \leq C_\tau(\lambda,\ r)$, (3) $C_\tau(\lambda,\ r) \vee C_\tau(\mu,\ r) = C_\tau(\lambda \vee \mu,\ r)$, (4) $C_\tau(\lambda,\ r) \leq C_\tau(\lambda,\ s)$ if $r \leq s$, (5) $C_\tau(C_\tau(\lambda,\ r),\ r) = C_\tau(\lambda,\ r)$.*

Theorem 1.2. [21] *Let $(X,\ \tau)$ be a fts. Then for each $r \in I_0$, $\lambda \in I^X$ we define an operator $I_\tau : I^X \times I_0 \to I^X$ as follows: $I_\tau(\lambda,\ r) = \bigvee\{\mu \in I^X : \lambda \geq \mu,\ \tau(\mu) \geq r\}$. For $\lambda,\ \mu \in I^X$ and $r, s \in I_0$, the operator I_τ satisfies the following conditions: (1) $I_\tau(\underline{1}, r) = \underline{1}$, (2)*

$\lambda \geq I_\tau(\lambda,\ r)$, *(3)* $I_\tau(\lambda,\ r) \wedge I_\tau(\mu,\ r) = I_\tau(\lambda \wedge \mu,\ r)$, *(4)* $I_\tau(\lambda,\ r) \leq I_\tau(\lambda,\ s)$ *if* $s \leq r$, *(5)* $I_\tau(I_\tau(\lambda,\ r),\ r) = I_\tau(\lambda,\ r)$, *(6)* $I_\tau(\underline{1} - \lambda, r) = \underline{1} - C_\tau(\lambda,\ r)$ *and* $C_\tau(\underline{1} - \lambda, r) = \underline{1} - I_\tau(\lambda,\ r)$

Definition 1.2. [15] *Let* $(X,\ \tau)$ *be a fts. Then for each* $\mu \in I^X,\ x_t \in P_t(X)$ *and* $r \in I_0$,

(i) μ *is called r-open* Q_τ*-neighbourhood of* x_t *if* $x_t q \mu$ *with* $\tau(\mu) \geq r$.

(ii) μ *is called r-open* R_τ*-neighbourhood of* x_t *if* $x_t q \mu$ *with* $\mu = I_\tau(C_\tau(\lambda,\ r),\ r)$.
We denote $Q_\tau(x_t,\ r) = \{\mu \in I^X : x_t q\mu, \tau(\mu) \geq r\}$, $R_\tau(x_t,\ r) = \{\mu \in I^X : x_t q \mu = I_\tau(C_\tau(\lambda,\ r),\ r)\}$.

Definition 1.3. [15] *Let* $(X,\ \tau)$ *be a fts. Then for each* $\lambda \in I^X,\ x_t \in P_t(X)$ *and* $r \in I_0$,

(i) x_t *is called r-*τ *cluster point of* λ *if for every* $\mu \in Q_\tau(x_t,\ r)$*, we have* $\mu q \lambda$.

(ii) x_t *is called r-*δ *cluster point of* λ *if for every* $\mu \in R_\tau(x_t,\ r)$*, we have* $\mu q \lambda$.

(iii) An δ*-closure operator is a mapping* $D_\tau : I^X \times I \to I^X$ *defined as follows:* δ-$C_\tau(\lambda,\ r)$ *or* $D_\tau(\lambda,\ r) = \bigvee\{x_t \in P_t(X) : x_t$ *is r-*δ*-cluster point of* $\lambda\}$

Definition 1.4. *Let* $(X,\ \tau)$ *be a fuzzy topological space. For* $\lambda \in I^X$ *and* $r \in I_0$,

(i) λ *is called an r-fuzzy semiopen (resp. r-fuzzy semi-closed) [16] set if* $\lambda \leq C_\tau(I_\tau(\lambda,\ r),\ r)$ *(resp.* $I_\tau(C_\tau(\lambda,\ r),\ r) \leq \lambda$*).*

(ii) λ *is called an r-fuzzy preopen (resp. r-fuzzy preclosed) [13] set if* $\lambda \leq I_\tau(C_\tau(\lambda,\ r),\ r)$ *(resp.* $C_\tau(I_\tau(\lambda,\ r),\ r) \leq \lambda$*).*

(iii) λ *is called r-fuzzy* δ*-closed [13] iff* $\lambda = D_\tau(\lambda,\ r)$.

(iv) The complement of r-fuzzy semiopen (resp. r-fuzzy preopen, r-fuzzy semi-preopen and r-fuzzy δ*-closed) is r-fuzzy semi-closed (resp. r-fuzzy preclosed, r-fuzzy semi-preclosed and r-fuzzy* δ*-open).*

Definition 1.5. *Let* $(X,\ \tau)$ *be a fuzzy topological space.* $\lambda,\ \mu \in I^X$ *and* $r \in I_0$,

(i) λ *is called an r-fuzzy* δ*-semiopen (resp. r-fuzzy* δ*-semiclosed) [24] set if* $\lambda \leq C_\tau(\delta\text{-}I_\tau(\lambda,\ r),\ r)$ *(resp.* $I_\tau(\delta\text{-}C_\tau(\lambda,\ r),\ r) \leq \lambda$*).*

(ii) λ *is called an r-fuzzy* δ*-preopen (resp. r-fuzzy* δ*-preclosed) [24] set if* $\lambda \leq I_\tau(\delta\text{-}C_\tau(\lambda,\ r),\ r)$ *(resp.* $C_\tau(\delta\text{-}I_\tau(\lambda,\ r),\ r) \leq \lambda$*).*

(iii) λ *is called an r-fuzzy e-open (resp. r-fuzzy e-closed) [24] set if* $\lambda \leq C_\tau(\delta\text{-}I_\tau(\lambda,\ r),\ r) \vee I_\tau(\delta\text{-}C_\tau(\lambda,\ r),\ r)$ *(resp.* $C_\tau(\delta I_\tau(\lambda,\ r),\ r) \wedge I_\tau(\delta C_\tau(\lambda,\ r),\ r) \leq \lambda$*).*

(iv) λ *is called an r-fuzzy* β*-open (resp.r-fuzzy* β*-closed) set if* $\lambda \leq C_\tau(I_\tau(C_\tau(\lambda,\ r),\ r),\ r)$ *(resp.* $I_\tau(C_\tau(I_\tau(\lambda,\ r),\ r),\ r) \leq \lambda$*).*

Definition 1.6. *[24] Let* (X, τ) *be a fuzzy topological space.* $\lambda, \mu \in I^X$ *and* $r \in I_0$,

(i) $eI_\tau(\lambda,\ r) = \bigvee\{\mu \in I^X : \mu \leq \lambda,\ \mu$ *is a r-feo set* $\}$ *is called the r-fuzzy e-interior of* λ.

(ii) $eC_\tau(\lambda,\ r) = \bigwedge\{\mu \in I^X : \mu \geq \lambda,\ \mu$ *is a r-fe***o set* $\}$ *is called the r-fuzzy e-closure of* λ.

§2. r-fuzzy e^*-open sets

We introduce the following definitions.

Definition 2.1. *Let $(X,\ \tau)$ be a fuzzy topological space. For $\lambda,\ \mu \in I^X$ and $r \in I_0$, λ is called an r-fuzzy e*-open (resp. r-fuzzy e*-closed) set if* $\lambda \leq C_\tau(I_\tau(\delta\text{-}C_\tau(\lambda,\ r),\ r),\ r)$ *(resp.* $I_\tau(C_\tau(\delta\text{-}I_\tau(\lambda,\ r),\ r),\ r) \leq \lambda$*).*

Definition 2.2. *Let (X,τ) be a fuzzy topological space. $\lambda,\ \mu \in I^X$ and $r \in I_0$,*

(i) $e^*I_\tau(\lambda,\ r) = \bigvee\{\mu \in I^X : \mu \leq \lambda,\ \mu \text{ is a } r\text{-}fe^*o \text{ set } \}$ *is called the r-fuzzy e*-interior of λ.*

(ii) $e^*C_\tau(\lambda,\ r) = \bigwedge\{\mu \in I^X : \mu \geq \lambda,\ \mu \text{ is a } r\text{-}fe^*c \text{ set } \}$ *is called the r-fuzzy e*-closure of λ.*

Obviously, $e^*C_\tau(\lambda,\ r)$ is the smallest r-fe*c set which contains λ, and $e^*I_\tau(\lambda,\ r)$ is the largest r-fe*o set which is contained in λ. Also $e^*C_\tau(\lambda,\ r) = (\lambda,\ r)$ for any r-fe*c set λ and $e^*I_\tau(\lambda,\ r) = (\lambda,\ r)$ for any r-fe*o set λ.

Hence we have

$$I_\tau(\lambda,\ r) \leq \delta s I_\tau(\lambda,\ r) \leq eI_\tau(\lambda,\ r) \leq \beta I_\tau(\lambda,\ r) \leq e^*I_\tau(\lambda,\ r) \leq (\lambda,\ r).$$

$$(\lambda,\ r) \leq e^*C_\tau(\lambda,\ r) \leq \beta C_\tau(\lambda,\ r) \leq eC_\tau(\lambda,\ r) \leq \delta s C_\tau(\lambda,\ r) \leq C_\tau(\lambda,\ r).$$

and

$$I_\tau(\lambda,\ r) \leq \delta p I_\tau(\lambda,\ r) \leq eI_\tau(\lambda,\ r) \leq \beta I_\tau(\lambda,\ r) \leq e^*I_\tau(\lambda,\ r) \leq (\lambda,\ r).$$

$$(\lambda,\ r) \leq e^*C_\tau(\lambda,\ r) \leq \beta C_\tau(\lambda,\ r) \leq eC_\tau(\lambda,\ r) \leq \delta p C_\tau(\lambda,\ r) \leq C_\tau(\lambda,\ r).$$

Lemma 2.1. *The following hold for a subset λ of a fts X.*

(i) $e^*C_\tau(\lambda,\ r)$ *is r-fe*c.*

(ii) $1 - e^*C_\tau(\lambda,\ r) = e^*I_\tau(1 - (\lambda,\ r))$.

Theorem 2.1. *The following holds for a subset λ of a fts X.*

(i) $(\lambda,\ r)$ *is r-fe*o* $\Leftrightarrow (\lambda,\ r) = (\lambda,\ r) \wedge C_\tau(I_\tau(\delta C_\tau(\lambda,\ r),\ r),\ r)$.

(ii) $(\lambda,\ r)$ *is r-fe*c* $\Leftrightarrow (\lambda,\ r) = (\lambda,\ r) \vee I_\tau(C_\tau(\delta I_\tau(\lambda,\ r),\ r),\ r)$.

(iii) $e^*C_\tau(\lambda,\ r) = (\lambda,\ r) \vee I_\tau(C_\tau(\delta I_\tau(\lambda,\ r),\ r),\ r)$.

(iv) $e^*I_\tau(\lambda,\ r) = (\lambda,\ r) \wedge C_\tau(I_\tau(\delta C_\tau(\lambda,\ r),\ r),\ r)$.

Proof. (i) Let λ be r-fe*o. Then $\lambda \leq C_\tau(I_\tau(\delta C_\tau(\lambda,\ r),\ r),\ r)$. We obtain

$$(\lambda,\ r) = (\lambda,\ r) \wedge C_\tau(I_\tau(\delta C_\tau(\lambda,\ r),\ r),\ r).$$

Conversely, let $(\lambda,\ r) = (\lambda,\ r) \wedge C_\tau(I_\tau(\delta C_\tau(\lambda,\ r),\ r),\ r)$. We have

$$\begin{aligned}(\lambda,\ r) &= (\lambda,\ r) \wedge C_\tau(I_\tau(\delta C_\tau(\lambda,\ r),\ r),\ r).\\ &\leq C_\tau(I_\tau(\delta C_\tau(\lambda,\ r),\ r),\ r).\end{aligned}$$

Hence $(\lambda,\ r)$ is r-fe*o.

(ii) Taking complements, proof is similar to (i).
(iii) Since $e^*C_\tau(\lambda,\ r)$ is r-fe*c, we have,

$$I_\tau(C_\tau(\delta I_\tau(\lambda,\ r), r), r) \le I_\tau(C_\tau(\delta I_\tau(e^*C_\tau(\lambda,\ r), r), r),\ r) \le e^*C_\tau(\lambda,\ r).$$

Hence, $(\lambda,\ r) \bigvee I_\tau(C_\tau(\delta I_\tau(\lambda,\ r),\ r),\ r) \le e^*C_\tau(\lambda,\ r)$. On the other way, since

$$\begin{aligned}
&I_\tau(C_\tau(\delta I_\tau(\lambda \bigvee I_\tau(C_\tau(\delta I_\tau(\lambda,\ r),\ r),\ r),\ r),\ r),\ r)\\
&\quad = I_\tau(C_\tau(\delta I_\tau(\lambda \bigvee \delta I_\tau(\delta C_\tau(\delta I_\tau(\lambda,\ r),\ r),\ r),\ r),\ r),\ r)\\
&\quad = I_\tau(C_\tau(\delta I_\tau(\lambda,\ r) \bigvee \delta I_\tau(\delta C_\tau(\delta I_\tau(\lambda,\ r),\ r),\ r),\ r),\ r)\\
&\quad = I_\tau(C_\tau(\delta I_\tau(\delta C_\tau(\delta I_\tau(\lambda,\ r),\ r),\ r),\ r),\ r)\\
&\quad = I_\tau(C_\tau(\delta I_\tau(\lambda,\ r),\ r),\ r)\\
&\quad \le (\lambda,\ r) \bigvee I_\tau(C_\tau(\delta I_\tau(\lambda,\ r),\ r),\ r),
\end{aligned}$$

then $(\lambda,\ r) \bigvee I_\tau(C_\tau(\delta I_\tau(\lambda,\ r),\ r),\ r)$ is r-fe*c containing λ and hence

$$e^*C_\tau(\lambda,\ r) \le (\lambda,\ r) \bigvee I_\tau(C_\tau(\delta I_\tau(\lambda,\ r),\ r),\ r).$$

Thus, we obtain $e^*C_\tau(\lambda,\ r) = (\lambda,\ r) \bigvee I_\tau(C_\tau(\delta I_\tau(\lambda,\ r),\ r),\ r)$.
(iv) Similar to the proof of (iii). □

Theorem 2.2. *Let λ be a subset of a fts X. Then the following hold*

(i) $e^*C_\tau(\delta I_\tau(\lambda,\ r),\ r) = I_\tau(C_\tau(\delta I_\tau(\lambda,\ r),\ r),\ r)$.

(ii) $\delta I_\tau(e^*C_\tau(\lambda,\ r),\ r) = I_\tau(C_\tau(\delta I_\tau(\lambda,\ r),\ r),\ r)$.

(iii) $e^*I_\tau(\delta C_\tau(\lambda,\ r),\ r) = \delta C_\tau(e^*I_\tau(\lambda,\ r),\ r) = C_\tau(I_\tau(\delta C_\tau(\lambda,\ r),\ r),\ r)$.

(iv) $e^*I_\tau(eC_\tau(\lambda,\ r),\ r) = \delta sI_\tau(\delta sC_\tau(\lambda,\ r),\ r) \bigwedge \delta pC_\tau(\lambda,\ r)$.

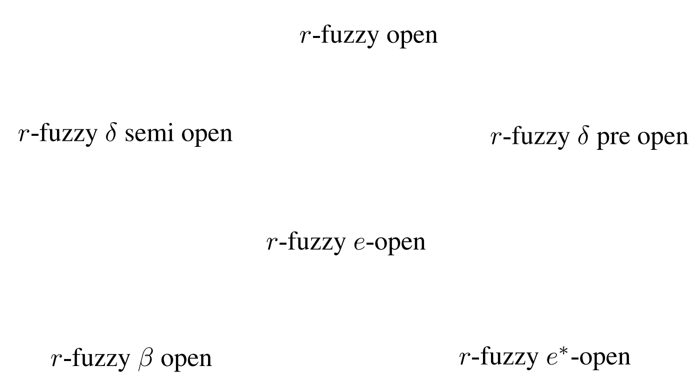
(v) $e^*C_\tau(eI_\tau(\lambda,\ r),\ r) = \delta sC_\tau(\delta sI_\tau(\lambda,\ r),\ r) \bigvee \delta pI_\tau(\lambda,\ r)$.

(vi) $eC_\tau(e^*I_\tau(\lambda,\ r),\ r) = \delta sI_\tau(\delta sC_\tau(\lambda,\ r),\ r) \bigwedge \delta pC_\tau(\lambda,\ r)$.

(vii) $eI_\tau(e^*C_\tau(\lambda,\ r),\ r) = \delta sC_\tau(\delta sI_\tau(\lambda,\ r),\ r) \bigvee \delta pI_\tau(\lambda,\ r)$.

Proof. The Proof is similar to the proof of Theorem 2.15 in [7]. □

Remark 2.1. *From the above definitions it is clear that the following implications are true for $r \in I_0$.*



where r-fo, r-fδso, r-fδsc, r-fδpo, r-fδpc, r-feo, r-fec, r-fβo, r-fβc, r-fe*o, r-fe*care abbreviated by r-fuzzy open, r-fuzzy δ- semiopen, r-fuzzy δ-semiclosed, r-fuzzy δ-preopen, r-fuzzy δ-preclosed, r-fuzzy e-open, r-fuzzy e-closed, r-fuzzy β-open, r-fuzzy β-closed, r-fuzzy e^*-open, r-fuzzy e^*-closed respectively.

From the above definitions, it is clear that every r-fδpo is r-feo and every r-fδso is r-feo. Also, it is clear that every r-feo set is r-fβo set and r-fe*o set. Also, every r-fβo set is r-fe*o set. The converses need not be true in general.

The converses of the above implications are not true as the following examples show:

Example 2.1. *Let λ_1, λ_2, λ_3 and λ_4 be fuzzy subsets of $X = \{a,\ b\}$ defined as follows*

$\lambda_1(a) = 0.2,\ \lambda_1(b) = 0.1;$

$\lambda_2(a) = 0.3,\ \lambda_2(b) = 0.5;$

$\lambda_3(a) = 0.7,\ \lambda_3(b) = 0.7;$

$\lambda_4(a) = 0.2,\ \lambda_4(b) = 0.8.$

Then $\tau : I^X \to I$ defined as

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ \frac{1}{2}, & \text{if } \lambda = \lambda_1,\ \lambda_2,\ \lambda_3, \\ 0, & \text{otherwise}, \end{cases}$$

Then λ_4 is $\frac{1}{2}$-fβo but λ_4 is not $\frac{1}{2}$-feo set.

Example 2.2. *Let λ and μ be fuzzy subsets of $X = \{a,\ b,\ c\}$ defined as follows*

$\lambda(a) = 0.4,\ \lambda(b) = 0.5,\ \lambda(c) = 0.5;$

$\mu(a) = 0.4,\ \mu(b) = 0.5,\ \mu(c) = 0.4.$

Then $\tau : I^X \to I$ defined as

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ \frac{1}{2}, & \text{if } \lambda = \lambda, \\ 0, & \text{otherwise}, \end{cases}$$

Then μ is $\frac{1}{2}$-feo set but μ is not $\frac{1}{2}$-fδso set.

Example 2.3. *Let λ and μ be fuzzy subsets of $X = \{a,\ b,\ c\}$ defined as follows*

$\lambda(a) = 0.5,\ \lambda(b) = 0.3,\ \lambda(c) = 0.2;$

$\mu(a) = 0.5,\ \mu(b) = 0.4,\ \mu(c) = 0.4.$

Then $\tau : I^X \to I$ defined as

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ \frac{1}{2}, & \text{if } \lambda = \lambda, \\ 0, & \text{otherwise}, \end{cases}$$

Then μ is $\frac{1}{2}$-feo set but μ is not $\frac{1}{2}$-fδpo set.

Example 2.4. *Let λ, μ and ω be fuzzy subsets of $X = \{a,\ b,\ c\}$ defined as follows*

$\lambda(a) = 0.3,\ \lambda(b) = 0.5,\ \lambda(c) = 0.2;$

$\mu(a) = 0.4,\ \mu(b) = 0.5,\ \mu(c) = 0.5;$

$\omega(a) = 0.7$, $\omega(b) = 0.4$, $\omega(c) = 0.8$.

Then $\tau : I^X \to I$ *defined as*

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ \frac{1}{2}, & \text{if } \lambda = \lambda,\ \mu, \\ 0, & \text{otherwise}, \end{cases}$$

Then ω *is* $\frac{1}{2}$*-fe***o set but* ω *is not* $\frac{1}{2}$*-f*β*o set.*

Example 2.5. *Let* λ *and* μ *be fuzzy subsets of* $X = \{a,\ b,\ c\}$ *defined as follows*

$\lambda(a) = 0.4$, $\lambda(b) = 0.5$, $\lambda(c) = 0.2$;

$\mu(a) = 0.5$, $\mu(b) = 0.4$, $\mu(c) = 0.7$.

Then $\tau : I^X \to I$ *defined as*

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ \frac{1}{2}, & \text{if } \lambda = \lambda, \\ 0, & \text{otherwise}, \end{cases}$$

Then μ *is* $\frac{1}{2}$*-fe***o set but* μ *is not* $\frac{1}{2}$*-feo set. Also* μ *is not* $\frac{1}{2}$*-fuzzy open set.*

Theorem 2.3. *Let* $(X,\ \tau)$ *be a fts and* $r \in I_o$.

(i) Any union of r-fe**o sets is an r-fe***o set.*

(ii) Any intersection of r-fe**c sets is an r-fe***c set.*

Proof. (i) Let $\{\lambda_\alpha : \alpha \in \Gamma\}$ be a family of r-fe*o sets.

For each $\alpha \in \Gamma$, $\lambda_\alpha \leq C_\tau(I_\tau(\delta\text{-}C_\tau(\lambda_\alpha,\ r),\ r),\ r)$.

$$\begin{aligned} \bigvee_{\alpha\in\Gamma} \lambda_\alpha &\leq \bigvee_{\alpha\in\Gamma} C_\tau(I_\tau(\delta\text{-}C_\tau(\lambda_\alpha,\ r),\ r),\ r). \\ &\leq C_\tau(I_\tau(\delta\text{-}C_\tau(\vee\lambda_\alpha,\ r),\ r),\ r). \end{aligned}$$

(ii) Similar to the proof of (i). □

Theorem 2.4. *Let* $(X,\ \tau)$ *be a fts. For* $\lambda,\ \mu \in I^X$ *and* $r \in I_0$*. then,*

(i) If $\tau(\mu) \geq r$*, where* μ *is a crisp subset and* λ *is an r-fe***o set, then* $\lambda \wedge \mu$ *is an r-fe***o set.*

(ii) If $\tau(1-\mu) \geq r$*, where* μ *is a crisp subset and* λ *is an r-fe***c set, then* $\lambda \vee \mu$ *is an r-fe***c set.*

Proof. (i) Let λ be r-fe*o and $\mu \in I^X$ with $\tau(\mu) \geq r$ which is a crisp subset. Then

$$\begin{aligned} \lambda \wedge \mu &\leq C_\tau(I_\tau(\delta\text{-}C_\tau(\lambda,\ r),\ r),\ r) \wedge \mu. \\ &\leq C_\tau(I_\tau(\delta\text{-}C_\tau(\lambda \wedge \mu,\ r),\ r),\ r). \end{aligned}$$

Hence $\lambda \wedge \mu$ is r-fe*o.

(ii) Similar to the proof of (i). □

Theorem 2.5. *Let* $(X,\ \tau)$ *be a fts,* $\lambda,\ \mu \in I^X$ *and* $r \in I_0$.

(i) If λ *is r-fe***o with* $\tau(1-\lambda) \geq r$*, then* λ *is r-f*δ*po.*

(ii) If λ *is r-fe***c with* $\tau(\lambda) \geq r$*, then* λ *is r-f*δ*pc.*

Proof. (i) Let λ be an r-fe*o set and $\tau(1-\lambda) \geq r$. Then

$$\lambda \leq C_\tau(I_\tau(\delta\text{-}C_\tau(\lambda,\ r),\ r),\ r).$$
$$\leq I_\tau(\delta\text{-}C_\tau(\lambda,\ r),\ r).$$

Hence λ is an r-fδpo set of X.

(ii) is similar to (i). □

Theorem 2.6. *Let $(X,\ \tau)$ be a fts, For $\lambda,\ \mu \in I^X$ and $r \in I_0$.*

(i) λ is r-fe*o iff $1-\lambda$ is r-fe*c.

(ii) If $\tau(\lambda) \geq r$ then λ is r-fe*o set.

(iii) $I_\tau(\lambda,\ r)$ is an r-fe*o set.

(iv) $C_\tau(\lambda,\ r)$ is an r-fe*c set.

Proof. (i) and (ii) are trivial.
(iii) From the Definition of I_τ of Theorem 1.2 and Definition 1.1(3), since $\tau(I_\tau(\lambda,\ r)) \geq r$, by (ii) $I_\tau(\lambda,\ r)$ is an r-fe*o set.
(iv) Since $1-C_\tau(\lambda,\ r) = I_\tau(1-\lambda,\ r)$, from Theorem 1.2 (6), by (iii) we have $\tau(1-C_\tau(\lambda,\ r)) \geq r$. Hence $1-C_\tau(\lambda,\ r)$ is r-fe*o. By (i) $C_\tau(\lambda,\ r)$ is an r-fe*c set. □

Theorem 2.7. *Let $(X,\ \tau)$ be a fts. Let $\lambda \in I^X$ and $r \in I_o$.*

(i) λ is r-fe*o iff $\lambda = e^*I_\tau(\lambda,\ r)$.

(ii) λ is r-fe*c iff $\lambda = e^*C_\tau(\lambda,\ r)$.

Theorem 2.8. *Let $(X,\ \tau)$ be a fts. Let $\lambda \in I^X$ and $r \in I_o$, the following statements hold:*

(i) $e^*C_\tau(0,\ r) = 0$ and $e^*I_\tau(1,\ r) = 1$.

(ii) $I_\tau(\lambda,\ r) \leq e^*I_\tau(\lambda,\ r) \leq \lambda \leq e^*C_\tau(\lambda,\ r) \leq C_\tau(\lambda,\ r)$.

(iii) $\lambda \leq \mu \Rightarrow e^*I_\tau(\lambda,\ r) \leq e^*I_\tau(\mu,\ r)$ and $e^*C_\tau(\lambda,\ r) \leq e^*C_\tau(\mu,\ r)$.

(iv) $e^*C_\tau(\lambda,\ r) \vee e^*C_\tau(\mu,\ r) \leq e^*C_\tau(\lambda \vee \mu,\ r)$.

(v) $e^*C_\tau(e^*C_\tau(\lambda,\ r),\ r) = e^*C_\tau(\lambda,\ r)$ and $e^*I_\tau(e^*I_\tau(\lambda,\ r),\ r) = e^*I_\tau(\lambda,\ r)$.

(vi) $C_\tau(e^*C_\tau(\lambda,\ r),\ r) = e^*C_\tau(C_\tau(\lambda,\ r),\ r) = C_\tau(\lambda,\ r)$.

Proof. (i) It is trivial from the Definitions of e^*C_τ and e^*I_τ.
(ii) and (iii) can be easily proved from Theorem 2.6.
(iv) Since $\lambda \leq \lambda \vee \mu$, by the definition of e^*C_τ, we have

$$e^*C_\tau(\lambda,\ r) \leq e^*C_\tau(\lambda \vee \mu,\ r).$$

Similarly, $e^*C_\tau(\lambda,\ r) \leq e^*C_\tau(\lambda \vee \mu,\ r)$. Hence,

$$e^*C_\tau(\lambda,\ r) \vee eC_\tau(\mu,\ r) \leq e^*C_\tau(\lambda \vee \mu,\ r).$$

(v) It is trivial from Theorem 2.7.

(vi) From Theorem 2.6 (iv), and Theorem 2.7 (ii), $e^*C_\tau(C_\tau(\lambda,\ r),\ r) = C_\tau(\lambda,\ r)$. We only show that $C_\tau(e^*C_\tau(\lambda,\ r),\ r) = C_\tau(\lambda,\ r)$. Since $\lambda \leq e^*C_\tau(\lambda,\ r)$, $C_\tau(\lambda,\ r) \leq C_\tau(e^*C_\tau(\lambda,\ r),\ r)$. Suppose that $C_\tau(\lambda,\ r) < C_\tau(e^*C_\tau(\lambda,\ r),\ r)$. There exist $x \in X$ and $\mu \in I^X$ with $\lambda \leq \mu$ and $\tau(1-\mu) \geq r$ such that $C_\tau(e^*C_\tau(\lambda,\ r),\ r)(x) > \mu(x) \geq C_\tau(\lambda,\ r)(x)$. On the other hand, since $\mu = C_\tau(\lambda,\ r)$, $\lambda \leq \mu$ implies

$$e^*C_\tau(\lambda,\ r) \leq e^*C_\tau(\mu,\ r) = e^*C_\tau(\lambda,\ r) = C_\tau(\lambda,\ r) = \mu.$$

Thus $C_\tau(e^*C_\tau(\lambda,\ r),\ r) \leq \mu$. This is a contradiction. Hence $C_\tau(e^*C_\tau(\lambda,\ r),\ r) = C_\tau(\lambda,\ r)$. □

Theorem 2.9. *Let $(X,\ \tau)$ be a fts. For $\lambda \in I^X$ and $r \in I_0$ we have*

(i) $e^*I_\tau(1-\lambda,\ r) = 1 - (e^*C_\tau(\lambda,\ r))$.

(ii) $e^*C_\tau(1-\lambda,\ r) = 1 - (e^*I_\tau(\lambda,\ r))$.

Proof. (i) For all $\lambda \in I^X$, $r \in I_0$ we have the following:

$$\begin{aligned} 1 - (e^*C_\tau(\lambda,\ r)) &= 1 - \bigwedge\{\mu : \mu \geq \lambda,\ \mu \text{ is } r\text{-fe}^*\text{o}\} \\ &= \bigvee\{1-\mu : 1-\mu \leq 1-\lambda,\ 1-\mu \text{ is } r\text{-fe}^*\text{o}\} \\ &= e^*I_\tau(1-\lambda,\ r). \end{aligned}$$

(ii) Similar to the proof of (i). □

Theorem 2.10. *Let $(X,\ \tau)$ be a fts, $\lambda,\ \mu \in I^X$ and $r \in I_0$.*

(i) If λ is r-fβo set, $\tau(1-\lambda) \geq r$ and λ is r-fδc then λ is r-feo.

(ii) If λ is r-fβc set, $\tau(\lambda) \geq r$ and λ is r-fδo then λ is r-fec.

Proof. (i) Let λ be an r-fβo set and $\tau(1-\lambda) \geq r$. Then

$$\begin{aligned} \lambda &\leq C_\tau(I_\tau(C_\tau(\lambda,\ r),\ r),\ r) \\ &\leq C_\tau(I_\tau(\lambda,\ r),\ r) \\ &= C_\tau(I_\tau(\lambda,\ r) \vee I_\tau(\lambda,\ r),\ r) \\ &\leq C_\tau(\delta\text{-}I_\tau(\lambda,\ r),\ r) \vee I_\tau(\lambda,\ r) \\ &= C_\tau(\delta\text{-}I_\tau(\lambda,\ r),\ r) \vee I_\tau(\delta\text{-}C_\tau(\lambda,\ r),\ r). \end{aligned}$$

Hence λ is an r-feo set of X.

(ii) is similar to (i).

□

Theorem 2.11. *Let $(X,\ \tau)$ be a fts, $\lambda,\ \mu \in I^X$ and $r \in I_0$.*

(i) If λ is r-fe*o with $\tau(1-\lambda) \geq r$, then λ is r-feo set.

(ii) If λ is r-fe*c with $\tau(\lambda) \geq r$, then λ is r-fec set.

Proof. (i) Let λ be an r-fe*o set and $\tau(1-\lambda) \geq r$. Then

$$\begin{aligned} \lambda &\leq C_\tau(I_\tau(\delta\text{-}C_\tau(\lambda,\ r),\ r),\ r). \\ &= I_\tau(\delta\text{-}C_\tau(\lambda,\ r),\ r). \\ &\leq C_\tau(\delta\text{-}I_\tau(\lambda,\ r),\ r) \vee I_\tau(\delta\text{-}C_\tau(\lambda,\ r),\ r). \end{aligned}$$

Hence λ is an r-feo set of X.

(ii) is similar to (i). □

Theorem 2.12. *Let $(X,\ \tau)$ be a fts, $\lambda,\ \mu \in I^X$ and $r \in I_0$.*

(i) If λ is r-fe*o, $\tau(1-\lambda) \geq r$ and λ is r-fδc, then λ is r-fβo set.

(ii) If λ is r-fe*c $\tau(\lambda) \geq r$ and λ is r-fδo, then λ is r-fβc set.

Proof. (i) Let λ be an r-fe*o set and $\tau(1-\lambda) \geq r$. Then

$$\lambda \leq C_\tau(I_\tau(\delta\text{-}C_\tau(\lambda,\ r),\ r),\ r).$$
$$= C_\tau(I_\tau(C_\tau(\lambda,\ r),\ r),\ r)$$

Hence λ is an r-fβo set of X.

(ii) is similar to (i). □

Conclusion

In this paper, r-fuzzy e^*-open and r-fuzzy e^*-closed sets are introduced in fuzzy topological spaces in the sense of $\hat{S}$ostak's. We also introduce r-fuzzy e^*-interior and r-fuzzy e^*-closure. Moreover, we investigated the relationships between r-fuzzy e^*-open sets, r-fuzzy beta open sets, r-fuzzy e-open sets, r-fuzzy δ-semiopen sets and r-fuzzy δ-preopen sets.

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Somewhat fuzzy I_{rw}-continuous functions

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Abstract In this paper, we introduce and study the concept of somewhat fuzzy I_{rw}-continuous functions, somewhat fuzzy I_{rw}-open functions and Somewhat fuzzy I_{rw}-irresolute open functions in fuzzy ideal topological spaces and obtain some of its basic properties and characterizations. Also we have introduce the concept of somewhat fuzzy I_{rw}-homeomorphism, fuzzy I_{rw}-resolvable and fuzzy I_{rw}-irresolvabe spaces and we have given characterizations of fuzzy I_{rw}-resolvable and fuzzy I_{rw}-irresolvable spaces.

Keywords Fuzzy I_{rw}-open sets, Somewhat fuzzy I_{rw}-continuous functions, Somewhat fuzzy I_{rw}-open functions, Somewhat fuzzy I_{rw}-irresolute open functions, somewhat fuzzy I_{rw}-homeomorphism, fuzzy I_{rw}-resolvable and fuzzy I_{rw}-irresolvable spaces.
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§1. Introduction

In 1945 R. Vaidyanathaswamy [25] introduced the concept of ideal topological spaces. Hayashi [13] defined the local function and studied some topological properties using local function in ideal topological spaces in 1964. Since then many mathematicians like Erdal Ekici et. al. [9], Hatir and Jafari [12], Naseef and Hatir [15] studied various topological concepts in ideal topological spaces. After the introduction of fuzzy sets by Zadeh [29] in 1965 and fuzzy topology by Chang [4] in 1968, several researches were conducted on the generalization of the notions of fuzzy sets and fuzzy topology. The hybridization of fuzzy topology and fuzzy ideal theory was initiated by Mahmoud [14] and Sarkar [17] independently in 1997. They ([14], [17]) introduced the concept of fuzzy ideal topological spaces as an extension of fuzzy topological spaces and ideal topological spaces. The concept of fuzzy topology may be relevent to quantum particle physics particularly in connection with string theory and E-infinite theory [5–8]. Hatir and Jafari [12], Naseef and Hatir [15] introduced the concept of fuzzy semi-I-open sets and fuzzy pre-I-open sets in fuzzy ideal topological spaces. Yuksel et. al. [28] introduced and studied fuzzy α-I-open sets and consequently Gupta and Rajneesh [11] introduced the concept of fuzzy γ-I-open sets in fuzzy ideal topological spaces. In 2003, G. Thangaraj and

G. Balasubramanian [20] introduced the concept of somewhat fuzzy continuous functions and many others [1, 3, 10, 18, 19, 21, 22, 27] have turned their attention to the various concepts of fuzzy topology by considering somewhat fuzzy ideal topological spaces instead of somewhat fuzzy topological spaces. Recently, A. Vadivel and E. Elavarasan [23] introduced and studied the concept of fuzzy I_{rw}-closed sets in fuzzy ideal topological spaces which simultaneously generalizes the concepts of I_{rw}-closed sets due to A. Vadivel and Mohanrao Navuluri [24] and fuzzy rw-closed sets due to R. S. Wali [26]. In the present paper, to introduce and study the concept of somewhat fuzzy I_{rw}-continuous functions, somewhat fuzzy I_{rw}-open functions and somewhat fuzzy I_{rw}-irresolute open functions in fuzzy ideal topological spaces. Also we have introduced the concept of somewhat fuzzy I_{rw}-homeomorphism, fuzzy I_{rw}-resolvable and fuzzy I_{rw}-irresolvabe spaces and we have given characterizations of fuzzy I_{rw}-resolvable and fuzzy I_{rw}-irresolvable spaces in fuzzy ideal topological spaces.

§2. Preliminaries

Throughout this paper, (X, τ) always means fuzzy topological space in the sense of Chang [4]. For a fuzzy subset λ of X, the fuzzy interior of λ is denoted by $Int(\lambda)$ and is defined as $Int(\lambda) = \bigvee\{\mu | \mu \leq \lambda,\ \mu$ is a fuzzy open subset of $X\}$ and the fuzzy closure of λ is denoted by $Cl(\lambda)$ and is defined as $Cl(\lambda) = \bigwedge\{\mu|\mu \geq \lambda,\ \mu$ is a fuzzy closed subset of $X\}$. A fuzzy set λ in $(X,\ \tau)$ is said to be quasi-coincident with a fuzzy set μ, denoted by $\lambda q \mu$, if there exists a point $x \in X$ such that $\lambda(x) + \mu(x) > 1$ [12]. A fuzzy set μ in $(X,\ \tau)$ is called a Q-neighborhood of a fuzzy point x_β if there exists a fuzzy open set λ of X such that $x_\beta q \lambda \leq \mu$ [12].

A nonempty collection of fuzzy sets I of a set X is called a fuzzy ideal [11, 12] if and only if (i) $\lambda \in I$ and $\mu \leq \lambda$, then $\mu \in I$, (ii) if $\lambda \in I$ and $\mu \in I$, then $\lambda \vee \mu \in I$. The triple (X, τ, I) means a fuzzy ideal topological space with a fuzzy ideal I and fuzzy topology τ. The local function for a fuzzy set λ of X with respect to τ and I denoted by $\lambda^*(\tau,\ I)$ (briefly λ^*) in a fuzzy ideal topological space $(X,\ \tau,\ I)$ is the union of all fuzzy points x_β such that if μ is a Q-neighborhood of x_β and $\delta \in I$ then for at least one point $y \in X$ for which $\mu(y) + \lambda(y) - 1 > \delta(y)$ [16]. The $*$-closure operator of a fuzzy set λ denoted by $Cl^*(\lambda)$ in $(X,\ \tau,\ I)$ defined as $Cl^*(\lambda) = \lambda \bigvee \lambda^*$ [16].

Definition 2.1. *A fuzzy set λ of fuzzy topological space $(X,\ \tau)$ is called fuzzy regular open [2] if $\lambda = int(cl(\lambda))$. The complement of a fuzzy regular open set is called fuzzy regular closed.*

Definition 2.2. *A fuzzy set λ of fuzzy topological space $(X,\ \tau)$ is said to be fuzzy regular semi-open [26] if there is a fuzzy regular open set μ such that $\mu \leq \lambda \leq cl(\mu)$. The complement of a fuzzy regular semi-open set is called fuzzy regular semi-closed.*

Definition 2.3. *A fuzzy set λ of a fuzzy ideal topological space (X, τ, I) is called fuzzy I_{rw}-closed [23] if $\lambda^* \leq \mu$, whenever $\lambda \leq \mu$ and μ is fuzzy regular semi-open. The complement of a fuzzy I_{rw}-closed set is called fuzzy I_{rw}-open.*

The family of all fuzzy I_{rw}-closed (resp. fuzzy I_{rw}-open) subsets of (X, τ, I) is denoted by FI_{rw}-$C(X)$ (resp. FI_{rw}-$O(X)$).

The fuzzy I_{rw}-closure and fuzzy I_{rw}-interior of a fuzzy set λ are respectively, denoted by

I_{rw}-$Cl(\lambda)$ and I_{rw}-$Int(\lambda)$ and is defined as

I_{rw}-$Cl(\lambda) = \wedge\{\mu \mid \lambda \leq \mu, \mu \in FI_{rw}\text{-}C(X)\}$ and

I_{rw}-$Int(\lambda) = \vee\{\mu \mid \lambda \geq \mu, \mu \in FI_{rw}\text{-}O(X)\}$.

A fuzzy set λ is said to be fuzzy I_{rw}-closed (resp. fuzzy I_{rw}-open) if and only if I_{rw}-$Cl(\lambda) = \lambda$(resp. I_{rw}-$Int(\lambda) = \lambda$). Clearly, I_{rw}-$Cl(1-\lambda) = 1 - I_{rw}$-$Int(\lambda)$ and I_{rw}-$Int(1-\lambda) = I_{rw}$-$Cl(\lambda)$.

Definition 2.4. *[23] A fuzzy ideal topological space $(X,\ \tau,\ I)$ is fuzzy I_{rw}-$T_{1/2}$ if every fuzzy I_{rw}-closed set in X is fuzzy closed in X.*

Definition 2.5. *A function $f : (X, \tau, I) \to (Y, \sigma)$ is called fuzzy continuous [4] if $f^{-1}(\mu)$ is fuzzy open in X for every fuzzy open set $\mu \in Y$.*

Definition 2.6. *A function $f : (X, \tau) \to (Y, \sigma)$ is called fuzzy open [4] if and only if for any fuzzy open subset λ of X, $f(\lambda) \in \sigma$.*

Definition 2.7. *A function $f : (X, \tau, I) \to (Y, \sigma)$ is called fuzzy I_{rw}-continuous [23] if $f^{-1}(\mu)$ is fuzzy I_{rw}-open in X for every fuzzy open set $\mu \in Y$.*

Definition 2.8. *A function $f : (X, \tau, I) \to (Y, \sigma)$ is called fuzzy I_{rw}-irresolute [23] if $f^{-1}(\mu)$ is fuzzy I_{rw}-open in X for every fuzzy I_{rw}-open set $\mu \in Y$.*

Definition 2.9. *A function $f : (X, \tau) \to (Y, \sigma)$ is called somewhat fuzzy continuous [20] if for every fuzzy open set λ in Y such that $f^{-1}(\lambda) \neq 0$, there exists a fuzzy open set $\mu \neq 0$ in (X, τ) such that $\mu \leq f^{-1}(\lambda)$. That is, $int[f^{-1}(\lambda)] \neq 0$.*

Definition 2.10. *A function $f : (X, \tau) \to (Y, \sigma)$ is called somewhat fuzzy open [20] if for every fuzzy open set λ in (X, τ) such that $\lambda \neq 0$, there exists a fuzzy open set $\mu \neq 0$ in (Y, σ) such that $\mu \leq f(\lambda)$. That is, $int[f(\lambda)] \neq 0$.*

Lemma 2.1. *[2] Let $g : X \to X \times Y$ be the graph of a function $f : X \to Y$. Then, if λ is a fuzzy set of X and μ is a fuzzy set of Y, $g^{-1}(\lambda \times \mu) = \lambda \wedge f^{-1}(\mu)$.*

§3. Somewhat fuzzy I_{rw}-continuous functions

Definition 3.1. *A function $f : (X, \tau, I) \to (Y, \sigma)$ is called somewhat fuzzy I_{rw}-continuous if for every fuzzy open set λ in Y such that $f^{-1}(\lambda) \neq 0$, there exists a fuzzy I_{rw}-open set $\mu \neq 0$ in (X, τ) such that $\mu \leq f^{-1}(\lambda)$.*

It is clear that every fuzzy continuous function is somewhat fuzzy I_{rw}-continuous and also every somewhat fuzzy continuous function is somewhat fuzzy I_{rw}-continuous but the converses is not true as the following example shows.

Example 3.1. *Let $X = \{a,\ b,\ c\}$, $Y = \{p,\ q,\ r\}$ and the fuzzy sets λ and μ are defined as follows: $\lambda(a) = 0.6$, $\lambda(b) = 0.4$, $\lambda(c) = 0.5$; $\mu(p) = 0.7$, $\mu(q) = 0.6$, $\mu(r) = 0.5$. Let $\tau = \{0,\ 1,\ \lambda\}$, $\sigma = \{0,\ 1,\ \mu\}$ be the fuzzy topology on X and Y respectively. Let $I = \{0\}$ be the fuzzy ideal on X and λ^c is fuzzy I_{rw}-open set in X. Then the mapping $f : (X,\ \tau,\ I) \to (Y,\ \sigma)$ defined by $f(a) = p$, $f(b) = q$ and $f(c) = r$ is somewhat fuzzy I_{rw}-continuous but it not fuzzy continuous.*

Example 3.2. *Let $X = \{a,\ b,\ c\}$, $Y = \{p,\ q,\ r\}$ and the fuzzy sets λ and μ are defined as follows: $\lambda(a) = 0.6$, $\lambda(b) = 0.4$, $\lambda(c) = 0.5$; $\mu(p) = 0.4$, $\mu(q) = 0.6$, $\mu(r) = 0.5$. Let $\tau = \{0,\ 1,\ \lambda\}$, $\sigma = \{0,\ 1,\ \mu\}$ be the fuzzy topology on X and Y respectively. Let $I = \{0\}$ be the* *fuzzy ideal on X and λ^c is fuzzy I_{rw}-open set in X. Then the mapping $f : (X,\ \tau,\ I) \to (Y,\ \sigma)$ defined by $f(a) = p$, $f(b) = q$ and $f(c) = r$ is somewhat fuzzy I_{rw}-continuous but it not somewhat fuzzy continuous.*

Remark 3.1. *The implications contained in the following diagram are true and the reverse implications need not be true.*

Definition 3.2. *A fuzzy set λ in a fuzzy ideal topological space (X, τ, I) is called fuzzy I_{rw}-dense if there exists no fuzzy I_{rw}-closed set μ such that $\lambda < \mu < 1$ or equivalently $I_{rw}\text{-}Cl(\lambda) = 1$.*

Theorem 3.1. *If $f : (X, \tau, I) \to (Y, \sigma)$ is a somewhat fuzzy I_{rw}-continuous surjection and $g : (Y, \sigma) \to (Z, \eta)$ is somewhat fuzzy continuous, then $g \circ f : (X, \tau, I) \to (Z, \eta)$ somewhat fuzzy I_{rw}-continuous.*

Proof. Let λ be any non zero fuzzy open set of (Z, η) and $(g \circ f)^{-1}(\lambda) \neq 0$. Then $g^{-1}(\lambda) \neq 0$. Since g is somewhat fuzzy continuous, there exists $\mu \in \sigma$ such that $0 \neq \mu \leq g^{-1}(\lambda)$. Since f is surjective, $0 \neq f^{-1}(\mu) \leq f^{-1}(g^{-1}(\lambda))$. Since f is somewhat fuzzy I_{rw}-continuous, There exists an fuzzy I_{rw}-open set δ in (X, τ, I) such that $0 \neq \delta \leq f^{-1}(\mu)$. Therefore, we have $0 \neq \delta \leq (g \circ f)^{-1}(\lambda))$. This shows that $g \circ f$ is somewhat fuzzy I_{rw}-continuous. □

Proposition 3.1. *If $f : (X, \tau, I) \to (Y, \sigma)$ is a somewhat fuzzy I_{rw}-continuous function and $g : (Y, \sigma) \to (Z, \eta)$ is fuzzy continuous function, then $g \circ f : (X, \tau, I) \to (Z, \eta)$ is somewhat fuzzy I_{rw}-continuous.*

Proof. Let λ be any non zero fuzzy open set of (Z, η), then $g^{-1}(\lambda) \neq 0$. Since g is fuzzy continuous function, $g^{-1}(\lambda)$ in (Y, σ). Suppose that $f^{-1}(g^{-1}(\lambda)) \neq 0$. Since by hypothesis, f is somewhat fuzzy I_{rw}-continuous function, there exists a fuzzy I_{rw}-open set μ in X such that $\mu \neq 0$ and $\mu \leq f^{-1}(g^{-1}(\lambda))$. But $f^{-1}(g^{-1}(\lambda)) = (g \circ f)^{-1}(\lambda)$, which implies that $\mu \leq (g \circ f)^{-1}(\lambda)$. Therefore $(g \circ f)$ is somewhat fuzzy I_{rw}-continuous. □

Theorem 3.2. *For a function $f : (X, \tau, I) \to (Y, \sigma)$, the following statements are equivalent:*

(i) *f is somewhat fuzzy I_{rw}-continuous.*

(ii) *If λ is a fuzzy closed set of Y such that $f^{-1}(\lambda) \neq 1$, then there exists a proper fuzzy I_{rw}-closed set μ of X such that $\mu \geq f^{-1}(\lambda)$.*

(iii) *If λ is a fuzzy I_{rw}-dense set, then $f(\lambda)$ is a fuzzy dense set in Y.*

Proof. (i)⇒(ii): Suppose f is somewhat fuzzy I_{rw}-continuous and λ is any fuzzy closed set in Y such that $f^{-1}(\lambda) \neq 1$. Therefore, clearly $1-\lambda$ is a fuzzy open set and $f^{-1}(1-\lambda) = 1 - f^{-1}(\lambda) \neq 0$. But by (i), there exists a fuzzy I_{rw}-open set μ in (X, τ, I) such that $\mu \neq 0$ and $\mu \leq f^{-1}(1-\lambda)$. Therefore, $1-\mu \geq 1 - f^{-1}(1-\lambda) = 1-(1-f^{-1}(\lambda)) = f^{-1}(\lambda)$. Put $1-\mu = \delta$. Clearly, δ is a proper fuzzy I_{rw}-closed set such that $\delta \geq f^{-1}(\lambda)$.

(ii)⇒(iii): Let λ be a fuzzy I_{rw}-dense set in X and suppose $f(\lambda)$ is not fuzzy dense in Y. Then there exists a fuzzy closed set, say, μ such that $f(\lambda) < \mu < 1$. Now, $\mu < 1 \Rightarrow f^{-1}(\mu) \neq 1$. Then by $f(\lambda) < \mu < 1$, there exists a proper fuzzy I_{rw}-closed set δ in (X, τ, I) such that $\delta \geq f^{-1}(\mu)$. But by (i), $f^{-1}(\mu) > f^{-1}(f(\lambda)) \geq \lambda$, that is, $\delta > \lambda$. This implies that there exists a proper fuzzy I_{rw}-closed set δ such that $\delta > \lambda$, which is a contradiction, since λ is fuzzy I_{rw}-dense.

(iii)⇒(i): Let λ be any fuzzy open set in (Y, σ) and suppose $f^{-1}(\lambda) \neq 0$ and hence $\lambda \neq 0$. Suppose I_{rw}-$Int(f^{-1}(\lambda)) = 0$. Then I_{rw}-$Cl(1-f^{-1}(\lambda)) = 1 - I_{rw}$-$Int(f^{-1}(\lambda)) = 1 - 0 = 1$. This means that $1-f^{-1}(\lambda)$ is a fuzzy I_{rw}-dense set in X. By (iii), $f(1-f^{-1}(\lambda))$ is a fuzzy dense in Y. That is, $Cl(f(1-f^{-1}(\lambda))) = 1$, but $f(1-f^{-1}(\lambda)) = f(f^{-1}(1-\lambda)) \leq 1-\lambda = 1$, since $\lambda \neq 0$. Since $1-\lambda$ is fuzzy closed and $f(1-f^{-1}(\lambda)) \leq 1-\lambda$, $Cl(f(f^{-1}(\lambda))) \leq 1-\lambda$. That is, $1 \leq 1-\lambda \Rightarrow \lambda \leq 0$ and hence $\lambda = 0$, which is a contradiction to the fact that $\lambda \neq 0$. Therefore, we must have I_{rw}-$Int(f^{-1}(\lambda)) \neq 0$. This means that, there exists a fuzzy I_{rw}-open set μ in (X, τ, I) such that $0 \neq \mu \leq f^{-1}(\lambda)$ and consequently f is somewhat fuzzy I_{rw}-continuous. □

Theorem 3.3. *Let $f : (X, \tau, I) \to (Y, \sigma)$ be a function, where X is product related to Y, and $g : X \to X \times Y$, the graph function of f. If g is somewhat fuzzy I_{rw}-continuous, then f is so.*

Proof. Let λ be a non-zero fuzzy open set in Y. Then by Lemma 2.4 of [2], we have $f^{-1}(\lambda) = 1 \wedge f^{-1}(\lambda) = g^{-1}(1 \times \lambda)$. Since g is somewhat fuzzy I_{rw}-continuous and $1 \times \lambda$ is a non-zero fuzzy open set in $X \times Y$, there exists a non-zero fuzzy I_{rw}-open set μ of (X, τ, I) such that $\mu \leq g^{-1}(1 \times \lambda) = f^{-1}(\lambda)$. This proves that f is a somewhat fuzzy I_{rw}-continuous function. □

Proposition 3.2. *Let (X, τ, I) and (Y, σ, I) be any two fuzzy ideal topological spaces. If the function $f : (X, \tau, I) \to (Y, \sigma, I)$ is somewhat fuzzy I_{rw}-continuous, onto and if I_{rw}-$Int(\lambda) = 0$ for any non-zero fuzzy set λ in (X, τ, I), then I_{rw}-$Int\big(f(\lambda)\big) = 0$ in (Y, σ, I).*

Proof. Let $\lambda \neq 0$ be a non-zero fuzzy set in (X, τ, I) such that I_{rw}-$Int(\lambda) = 0$. Then $1 - I_{rw}$-$Int(\lambda) = 1 - 0 = 1$ implies that I_{rw}-$Cl(1-\lambda) = 1$. Since f is somewhat fuzzy I_{rw}-continuous and $1-\lambda$ is fuzzy I_{rw}-dense in (X, τ, I), $f(1-\lambda)$ is fuzzy I_{rw}-dense in (Y, σ, I) [by Theorem]. That is, I_{rw}-$Cl[f(1-\lambda)] = 1$. Then I_{rw}-$Cl[1-f(\lambda)] = 1$. [since f is onto]. Therefore we have $[1 - I_{rw}$-$Int(f(\lambda)] = 1$ which implies that I_{rw}-$Int(f(\lambda)) = 0$. Hence the proposition. □

Definition 3.3. *A fuzzy ideal topological space (X, τ, I) is called a fuzzy $D_{I_{rw}}$-space (D-space) if for every nonzero fuzzy I_{rw}-open (fuzzy open) set in X is fuzzy I_{rw}-dense (fuzzy dense) in X.*

Proposition 3.3. *If $f : (X, \tau, I) \to (Y, \sigma, I)$ is a somewhat fuzzy I_{rw}-continuous surjection and (X, τ, I) is a fuzzy $D_{I_{rw}}$-space, then Y is a fuzzy D-space.*

Proof. Let λ be a nonzero fuzzy open set in Y. We want to show that λ is fuzzy dense in Y. Suppose not, then there exists a fuzzy closed set $\mu \in Y$ such that $\lambda < \mu < 1$. Therefore, $f^{-1}(\lambda) < f^{-1}(\mu) < f^{-1}(1) = 1$. Since $\lambda \neq 0$, $f^{-1}(\lambda) \neq 0$ and since f is somewhat fuzzy I_{rw}-continuous there exists a fuzzy I_{rw}-open set $\delta \neq 0$ in X such that $\delta < f^{-1}(\lambda)$. Hence $\delta < f^{-1}(\lambda) < f^{-1}(\mu) < I_{rw}\text{-}Cl(f^{-1}(\mu)) < 1$. That is, $\delta < I_{rw}\text{-}Cl(f^{-1}(\mu)) < 1$. This contradicts the fact that (X, τ, I) is a fuzzy $D_{I_{rw}}$-space, hence Y is a fuzzy D-space. □

Theorem 3.4. *Let (X, τ, I) be any fuzzy ideal topological space and (Y, σ) any fuzzy ideal topological space. If λ is an fuzzy open set in X and $f : (\lambda, \tau/\lambda, I/\lambda) \to (Y, \sigma, I)$ is a somewhat fuzzy I_{rw}-continuous function such that $f(\lambda)$ is fuzzy I_{rw}-dense in Y, then any extension $F : (X, \tau, I) \to (Y, \sigma)$ of f is somewhat fuzzy I_{rw}-continuous.*

Proof. Let μ be any fuzzy open set in (Y, σ) such that $F^{-1}(\mu) \neq 0$. Since $f(\lambda) < Y$ is dense in Y and $\mu \wedge f(\lambda) \neq 0$, it follows that $F^{-1}(\mu) \wedge \lambda \neq 0$. That is $f^{-1}(\mu) \wedge \lambda \neq 0$. Hence by hypothesis on f, there exists an fuzzy I_{rw}-open set δ in λ such that $\delta \neq 0$ and $\delta < f^{-1}(\mu) < F^{-1}(\mu)$ which implies F is somewhat fuzzy I_{rw}-continuous. □

Theorem 3.5. *Let (X, τ, I) and (Y, σ, J) be any two fuzzy ideal topological spaces, $X = \lambda \vee \mu$ where λ and μ are fuzzy I_{rw}-open subsets of X and $f : (X, \tau, I) \to (Y, \sigma, J)$ be a function such that f/λ and f/μ are somewhat fuzzy I_{rw}-continuous. Then f is somewhat fuzzy I_{rw}-continuous.*

Proof. Let δ be any fuzzy open set in (Y, σ, J) such that $f^{-1}(\delta) \neq 0$. Then $(f/\lambda)^{-1}(\delta) \neq 0$ or $(f/\mu)^{-1}(\delta) \neq 0$ or both $(f/\lambda)^{-1}(\delta) \neq 0$ and $(f/\mu)^{-1}(\delta) \neq 0$.

Case (1) Suppose $(f/\lambda)^{-1}(\delta) \neq 0$. Since f/λ is somewhat fuzzy I_{rw}-continuous, there exists an fuzzy I_{rw}-open set $\gamma \leq \lambda$ such that $\gamma \neq 0$ and $\gamma \leq (f/\lambda)^{-1}(\delta) \leq f^{-1}(\delta)$. Since γ is fuzzy I_{rw}-open in λ and λ is fuzzy I_{rw}-open in X, γ is fuzzy I_{rw}-open in X. Thus f is somewhat fuzzy I_{rw}-continuous.

Case (2) the proof is similar with Case (1).

Case (3) Suppose $(f/\lambda)^{-1}(\delta) \neq 0$ and $(f/\mu)^{-1}(\delta) \neq 0$. This follows from both the Cases (1) and (2). Thus f is somewhat fuzzy I_{rw}-continuous. □

§4. Fuzzy I_{rw}-Weakly Equivalent Topologies

Definition 4.1. *Let X be a set and τ and σ be topologies for X. Then τ is said to be fuzzy I_{rw}-weakly equivalent to σ provided that if a fuzzy I_{rw}-open set λ in (X, τ) and $\lambda \neq 0$, then there is an fuzzy I_{rw}-open set μ in (X, σ) such that $\mu \neq 0$ and $\mu \leq \lambda$ and a fuzzy I_{rw}-open set λ in (X, σ) and $\lambda \neq 0$, then there is an fuzzy I_{rw}-open set set μ in (X, τ) such that $\mu \neq 0$ and $\mu \leq \lambda$.*

Theorem 4.1. *Let $f : (X, \tau, I) \to (Y, \sigma_1, I)$ be a somewhat fuzzy I_{rw}-continuous surjective function and let σ_2 be a fuzzy topology for Y. If σ_2 is weakly equivalent to σ_1, then the function $f : (X, \tau, I) \to (Y, \sigma_2)$ is somewhat fuzzy I_{rw}-continuous.*

Proof. Since σ_2 is weakly equivalent to σ_1, the identity function $i : (Y, \sigma_1) \to (Y, \sigma_2)$ is somewhat continuous. Therefore, by Theorem , $f = f \circ i : (X, \tau, I) \to (Y, \sigma_2)$ is somewhat fuzzy I_{rw}-continuous. □

Theorem 4.2. *Let $f : (X, \tau_1, I) \to (Y, \sigma)$ be a somewhat fuzzy continuous function and let τ_2 be a fuzzy topology for X. If τ_2 is fuzzy I_{rw}-weakly equivalent to τ_1, then the function $f : (X, \tau_2, I) \to (Y, \sigma)$ is somewhat fuzzy I_{rw}-continuous.*

Proof. Since τ_2 is fuzzy I_{rw}-weakly equivalent to τ_1, the identity function $i : (X, \tau_2, I) \to (X, \tau_1, I)$ is somewhat fuzzy I_{rw}-continuous. Therefore, by Theorem , $f = f \circ i : (X, \tau_2, I) \to (Y, \sigma)$ is somewhat fuzzy I_{rw}-continuous. □

§5. Somewhat fuzzy I_{rw}-open function

Definition 5.1. *A function $f : (X, \tau, I) \to (Y, \sigma, I)$ is called somewhat fuzzy I_{rw}-open if and only if for any fuzzy open set λ, $\lambda \neq 0$ in (X, τ, I) implies that there exists a fuzzy I_{rw}-open set μ in (Y, σ, I) such that $\mu \neq 0$ and $\mu \leq f(\lambda)$.*

It is clear that every fuzzy open function is somewhat fuzzy I_{rw}-open and also every somewhat fuzzy open function is somewhat fuzzy I_{rw}-open but the converses is not true as it can be seen from the following example.

Example 5.1. *Let $X = \{a, b, c\}$, $Y = \{p, q, r\}$ and the fuzzy sets λ and μ are defined as follows: $\lambda(a) = 0.4$, $\lambda(b) = 0.6$, $\lambda(c) = 0.5$; $\mu(p) = 0.7$, $\mu(q) = 0.8$, $\mu(r) = 0.9$. Let $\tau = \{0, 1, \lambda\}$, $\sigma = \{0, 1, \mu\}$ be the fuzzy topology on X and Y respectively. Let $I = \{0\}$ be the fuzzy ideal on X, λ^c and μ^c is fuzzy I_{rw}-open sets in X and Y respectively. Then the mapping $f : (X, \tau, I) \to (Y, \sigma, I)$ defined by $f(a) = p$, $f(b) = q$ and $f(c) = r$ is somewhat fuzzy I_{rw}-open but not fuzzy open.*

Example 5.2. *In Example . Then the mapping f is somewhat fuzzy I_{rw}-open but not somewhat fuzzy open.*

Remark 5.1. *The implications contained in the above diagram are true and the reverse implications need not be true.*

Proposition 5.1. *If $f : (X, \tau, I) \to (Y, \sigma, I)$ is fuzzy open function and $g : (Y, \sigma, I) \to (Z, \eta, I)$ is somewhat fuzzy I_{rw}-open functions, then $g \circ f : (X, \tau) \to (Z, \eta, I)$ is somewhat fuzzy I_{rw}-open.*

Proof. Clear. □

Proposition 5.2. *Let (X,τ,I) and (Y,σ,I) be any two fuzzy ideal topological spaces. If the function $f:(X,\tau,I)\to(Y,\sigma,I)$ is somewhat fuzzy I_{rw}-open and if I_{rw}-$Int(\lambda)=0$ for any non-zero fuzzy set λ in (Y,σ,I), then I_{rw}-$Int(f^{-1}(\lambda))=0$ in (X,τ,I).*

Proof. Let $\lambda\neq 0$ be a nonzero fuzzy set in (Y,σ,I) such that I_{rw}-$Int(\lambda)=0$. Then $1-I_{rw}$-$Int(\lambda)=1-0=1$ implies that I_{rw}-$Cl(1-\lambda)=1$. Since the function f is somewhat fuzzy I_{rw}-open and $1-\lambda$ is fuzzy I_{rw}-dense in (Y,σ,I), $f^{-1}(1-\lambda)$ is fuzzy I_{rw}-dense in (X,τ,I). That is, I_{rw}-$Cl(f^{-1}(1-\lambda))=1$. Then I_{rw}-$Cl[1-f^{-1}(\lambda)]=1$. Therefore $[1-I_{rw}$-$Int(f^{-1}(\lambda))]=1$ implies that I_{rw}-$Int(f^{-1}(\lambda))=0$. Hence the proposition. □

Theorem 5.2. *For a surjective function $f:(X,\tau,I)\to(Y,\sigma,I)$, the following statements are equivalent:*

(i) f is somewhat fuzzy I_{rw}-open.

(ii) If λ is a fuzzy closed set in X such that $f(\lambda)\neq 1$, then there exists a fuzzy I_{rw}-closed set μ in Y such that $\mu\neq 1$ and $\mu>f(\lambda)$.

Proof. (i)⇒(ii): Let λ be a fuzzy closed set in X such that $f(\lambda)\neq 1$. Then $1-\lambda$ is a fuzzy open set such that $f(1-\lambda)=1-f(\lambda)\neq 0$. Since f is somewhat fuzzy I_{rw}-open, there exists a fuzzy I_{rw}-open set γ in (Y,σ,I) such that $\gamma\neq 0$ and $\gamma\leq f(1-\lambda)$. Now $1-\gamma$ is fuzzy I_{rw}-closed set in Y such that $1-\gamma\neq 1$ and $\gamma<f(1-\lambda)$. Put $1-\gamma=\mu$. Then $\gamma>1-f(1-\lambda)=f(\lambda)$.

(ii)⇒(i): Let λ be a fuzzy open of X such that $\lambda\neq 0$. Then $1-\lambda$ is fuzzy closed and $1-\lambda\neq 1$, $f(1-\lambda)=1-f(\lambda)\neq 1$. Hence by hypothesis, there exists a fuzzy I_{rw}-closed set μ in Y such that $\mu\neq 1$ and $\mu>f(1-\lambda)=1-f(\lambda)$, that is, $f(\lambda)>1-\mu$ and let $1-\mu=\delta$. Clearly, δ is a fuzzy I_{rw}-open set of Y such that $\delta<f(\lambda)$ and $\delta\neq 0$. Hence f is somewhat fuzzy I_{rw}-open. □

Theorem 5.3. *For a surjective function $f:(X,\tau,I)\to(Y,\sigma,I)$, the following statements are equivalent:*

(i) f is somewhat fuzzy I_{rw}-open.

(ii) If λ is a fuzzy I_{rw}-dense set of Y, then $f^{-1}(\lambda)$ is fuzzy I_{rw}-dense set in X.

Proof. (i)⇒(ii): Suppose λ is fuzzy I_{rw}-dense and fuzzy I_{rw}-closed set of (Y,τ,I). We must to show that $f^{-1}(\lambda)$ is fuzzy I_{rw}-dense in (X,τ,I). Suppose not, then there exists a fuzzy I_{rw}-closed set μ in X such that $f^{-1}(\mu)<\mu<1$. Since f is somewhat fuzzy I_{rw}-open and $1-\mu$ is fuzzy I_{rw}-open, there exists a fuzzy I_{rw}-open set γ in Y such that $\gamma<f(1-\mu)$ and $\gamma<1-f(\mu)$. From $f^{-1}(\lambda)<\mu<1$, we have $\lambda<f(\mu)<1$. Then $\gamma<1-f(\mu)<1-\lambda$. That is, $\lambda<1-\gamma<1$. Since $1-\gamma$ is fuzzy I_{rw}-closed set in Y, this implies that λ is not a fuzzy I_{rw}-dense, which is a contradicition. Therefore, $f^{-1}(\lambda)$ must be a fuzzy I_{rw}-dense set in X.

(ii)⇒(i): Suppose $f^{-1}(\lambda)$ is fuzzy I_{rw}-dense in (X,τ,I), where λ is fuzzy I_{rw}-dense set in Y. We want to show that f is somewhat fuzzy I_{rw}-open. Assume that $\lambda\neq 0$ is fuzzy open and fuzzy I_{rw}-open set in (X,τ,I). We have to show that I_{rw}-$Int(f(\lambda))\neq 0$. Suppose not, then I_{rw}-$Int(f(\lambda))=0$ whenever λ is fuzzy I_{rw}-open. Then I_{rw}-$Cl(1-f(\lambda))=1-I_{rw}$-$Int(f(\lambda))=1-0=1$. That is, $1-f(\lambda)$ is fuzzy I_{rw}-dense in Y. Therefore by assumption $f^{-1}(1-f(\lambda))$

is fuzzy I_{rw}-dense in X. Therefore, $1 = I_{rw}\text{-}Cl(f^{-1}(1 - f(\lambda))) = I_{rw}\text{-}Cl(1 - \lambda) = 1 - \lambda$. This shows that $\lambda = 0$, which is a contradiction and so $I_{rw}\text{-}Int(f(\lambda)) \neq 0$. □

§6. Somewhat fuzzy I_{rw}-irresolute open function

Definition 6.1. *A function $f : (X, \tau, I) \to (Y, \sigma, I)$ is called somewhat fuzzy I_{rw}-irresolute open if and only if for any fuzzy I_{rw}-open set λ, $\lambda \neq 0$ in (X, τ, I) implies that there exists a fuzzy I_{rw}-open set μ in (Y, σ, I) such that $\mu \neq 0$ and $\mu \leq f(\lambda)$.*

Proposition 6.1. *If $f : (X, \tau, I) \to (Y, \sigma, I)$ and $g : (Y, \sigma, I) \to (Z, \eta, I)$ are somewhat fuzzy I_{rw}-irresolute open functions, then $g \circ f : (X, \tau) \to (Z, \eta, I)$ is somewhat fuzzy I_{rw}-irresolute open.*

Proof. Clear. □

Theorem 6.1. *For a surjective function $f : (X, \tau, I) \to (Y, \sigma, I)$, the following statements are equivalent:*

(i) f is somewhat fuzzy I_{rw}-irresolute open.

(ii) If λ is a fuzzy I_{rw}-closed set in X such that $f(\lambda) \neq 1$, then there exists a fuzzy I_{rw}-closed set μ in Y such that $\mu \neq 1$ and $\mu > f(\lambda)$.

Proof. (i)⇒(ii): Let λ be a fuzzy I_{rw}-closed set in X such that $f(\lambda) \neq 1$. Then $1 - \lambda$ is a fuzzy I_{rw}-open set such that $f(1 - \lambda) = 1 - f(\lambda) \neq 0$. Since f is somewhat fuzzy I_{rw}-open, there exists a fuzzy I_{rw}-open set γ in (Y, σ, I) such that $\gamma \neq 0$ and $\gamma \leq f(1 - \lambda)$. Now $1 - \gamma$ is fuzzy I_{rw}-closed set in Y such that $1 - \gamma \neq 1$ and $\gamma < f(1 - \lambda)$. Put $1 - \gamma = \mu$. Then $\gamma > 1 - f(1 - \lambda) = f(\lambda)$.

(ii)⇒(i): Let λ be a fuzzy I_{rw}-open of X such that $\lambda \neq 0$. Then $1 - \lambda$ is fuzzy I_{rw}-closed and $1 - \lambda \neq 1$, $f(1 - \lambda) = 1 - f(\lambda) \neq 1$. Hence by hypothesis, there exists a fuzzy I_{rw}-closed set μ in Y such that $\mu \neq 1$ and $\mu > f(1 - \lambda) = 1 - f(\lambda)$, that is, $f(\lambda) > 1 - \mu$ and let $1 - \mu = \delta$. Clearly, δ is a fuzzy I_{rw}-open set of Y such that $\delta < f(\lambda)$ and $\delta \neq 0$. Hence f is somewhat fuzzy I_{rw}-open. □

Theorem 6.2. *For a surjective function $f : (X, \tau, I) \to (Y, \sigma, I)$, the following statements are equivalent:*

(i) f is somewhat fuzzy I_{rw}-irresolute open.

(ii) If λ is a fuzzy I_{rw}-dense set of Y, then $f^{-1}(\lambda)$ is fuzzy I_{rw}-dense set in X.

Proof. (i)⇒(ii): Suppose λ is fuzzy I_{rw}-dense and fuzzy I_{rw}-closed set of (Y, τ, I). We must to show that $f^{-1}(\lambda)$ is fuzzy I_{rw}-dense in (X, τ, I). Suppose not, then there exists a fuzzy I_{rw}-closed set μ in X such that $f^{-1}(\mu) < \mu < 1$. Since f is somewhat fuzzy I_{rw}-open and $1 - \mu$ is fuzzy I_{rw}-open, there exists a fuzzy I_{rw}-open set γ in Y such that $\gamma < f(1 - \mu)$ and $\gamma < 1 - f(\mu)$. From $f^{-1}(\lambda) < \mu < 1$, we have $\lambda < f(\mu) < 1$. Then $\gamma < 1 - f(\mu) < 1 - \lambda$. That is, $\lambda < 1 - \gamma < 1$. Since $1 - \gamma$ is fuzzy I_{rw}-closed set in Y, this implies that λ is not a fuzzy I_{rw}-dense, which is a contradicition. Therefore, $f^{-1}(\lambda)$ must be a fuzzy I_{rw}-dense set in X.

(ii)⇒(i): Suppose $f^{-1}(\lambda)$ is fuzzy I_{rw}-dense in (X, τ, I), where λ is fuzzy I_{rw}-dense set in Y. We want to show that f is somewhat fuzzy I_{rw}-open. Assume that $\lambda \neq 0$ and a fuzzy I_{rw}-open set in (X, τ, I). We have to show that I_{rw}-$Int(f(\lambda)) \neq 0$. Suppose not, then I_{rw}-$Int(f(\lambda)) = 0$ whenever λ is fuzzy I_{rw}-open. Then I_{rw}-$Cl(1 - f(\lambda)) = 1 - I_{rw}$-$Int(f(\lambda)) = 1 - 0 = 1$. That is, $1 - f(\lambda)$ is fuzzy I_{rw}-dense in Y. Therefore by assumption $f^{-1}(1 - f(\lambda))$ is fuzzy I_{rw}-dense in X. Therefore, $1 = I_{rw}$-$Cl(f^{-1}(1 - f(\lambda))) = I_{rw}$-$Cl(1 - \lambda) = 1 - \lambda$. This shows that $\lambda = 0$, which is a contradiction and so I_{rw}-$Int(f(\lambda)) \neq 0$. □

§7. Somewhat fuzzy I_{rw}-homeomorphism

Definition 7.1. *A mapping $f : (X, \tau, I) \to (Y, \sigma, I)$ is called somewhat fuzzy I_{rw}-homeomorphism if f and f^{-1} are somewhat fuzzy I_{rw}-continuous.*

Definition 7.2. *A mapping $f : (X, \tau, I) \to (Y, \sigma, I)$ is called somewhat fuzzy I_{rw}*-homeomorphism if f and f^{-1} are somewhat fuzzy I_{rw}-irresolute.*

Theorem 7.1. *Let $f : (X, \tau, I) \to (Y, \sigma, I)$ be a bijective mapping. Then the following are equivalent*

(i) f is somewhat fuzzy I_{rw}-homeomorphism.

(ii) f is somewhat fuzzy I_{rw}-continuous and somewhat fuzzy I_{rw}-open map.

(iii) f is somewhat fuzzy I_{rw}-continuous and somewhat fuzzy I_{rw}-closed map.

Proof. (i)⇒ (ii) Let f be somewhat fuzzy I_{rw}-homeomorphism. Then f and f^{-1} are somewhat fuzzy I_{rw}-continuous. To prove that f is somewhat fuzzy I_{rw}-open map, let λ be a fuzzy open set in X. Since $f^{-1} : Y \to X$ is somewhat fuzzy I_{rw}-continuous, $(f^{-1})^{-1}(\lambda) = f(\lambda)$ is somewhat fuzzy I_{rw}-open in Y. Therefore $f(\lambda)$ is somewhat fuzzy I_{rw}-open in Y. Hence f is somewhat fuzzy I_{rw}-open.

(ii)⇒ (i) Let f be somewhat fuzzy I_{rw}-open and somewhat fuzzy I_{rw}-continuous map. To prove that $f^{-1} : Y \to X$ is somewhat fuzzy I_{rw}-continuous. Let λ be a fuzzy open set in X. Then $f(\lambda)$ is somewhat fuzzy I_{rw}-open set in Y since f is somewhat fuzzy I_{rw}-open map. Now $(f^{-1})^{-1}(\lambda) = f(\lambda)$ is somewhat fuzzy I_{rw}-open set in Y. Therefore $f^{-1} : Y \to X$ is somewhat fuzzy I_{rw}-continuous. Hence f is somewhat fuzzy I_{rw}-homeomorphism.

(ii)⇒ (iii) Let f be somewhat fuzzy I_{rw}-continuous and somewhat fuzzy I_{rw}-open map. To prove that f is somewhat fuzzy I_{rw}-closed map. Let μ be a fuzzy closed set in X. Then $1 - \mu$ is fuzzy open set in X. Since f is somewhat fuzzy I_{rw}-open map, $f(1 - \mu)$ is somewhat fuzzy I_{rw}-open set in Y. Now $f(1 - \mu) = 1 - f(\mu)$. Therefore $f(\mu)$ is somewhat fuzzy I_{rw}-closed in Y. Hence f is a somewhat fuzzy I_{rw}-closed.

(iii)⇒ (ii) Let f be somewhat fuzzy I_{rw}-continuous and somewhat fuzzy I_{rw}-closed map. To prove that f is somewhat fuzzy I_{rw}-open map. Let λ be a fuzzy open set in X. Then $1 - \lambda$ is a fuzzy closed set in X. Since f is somewhat fuzzy I_{rw}-closed map, $f(1 - \lambda)$ is somewhat fuzzy I_{rw}-closed in Y. Now $f(1 - \lambda) = 1 - f(\lambda)$. Therefore $f(\lambda)$ is somewhat fuzzy I_{rw}-open in Y. Hence f is somewhat fuzzy I_{rw}-open. □

Theorem 7.2. *Let $f : (X, \tau, I) \to (Y, \sigma, I)$ be a bijective function. Then the following are equivalent:*

(i) f is somewhat fuzzy I_{rw}-homeomorphism.*

(ii) f is somewhat fuzzy I_{rw}-irresolute and somewhat fuzzy I_{rw}-open.*

(iii) f is somewhat fuzzy I_{rw}-irresolute and somewhat fuzzy I_{rw}-closed.*

Proof. Similar by above Theorem . □

Theorem 7.3. *If $f : (X, \tau, I) \to (Y, \sigma, I)$ is somewhat fuzzy I_{rw}-homeomorphism and $g : (Y, \sigma, I) \to (Z, \eta)$ is somewhat fuzzy I_{rw}-homeomorphism and Y is fuzzy I_{rw}-$T_{1/2}$ space, then $g \circ f : X \to Z$ is somewhat fuzzy I_{rw}-homeomorphism.*

Proof. Clear. □

Theorem 7.4. *If $f : (X, \tau, I) \to (Y, \sigma, I)$, $g : (Y, \sigma, I) \to (Z, \eta, I)$ are somewhat fuzzy I_{rw}*-homeomorphism then $g \circ f : X \to Z$ is somewhat fuzzy I_{rw}*-homeomorphism.*

Proof. Clear. □

§8. Fuzzy I_{rw}-resolvable and fuzzy I_{rw}-irresolvable spaces

Definition 8.1. *A fuzzy ideal topological space (X, τ, I) is said to be fuzzy I_{rw}-resolvable if there exists a non-zero fuzzy I_{rw}-dense set λ in (X, τ, I) such that I_{rw}-$Cl(1-\lambda) = 1$. Otherwise (X, τ, I) is called a fuzzy I_{rw}-irresolvable space.*

Theorem 8.1. *A fuzzy ideal topological space (X, τ, I) is a fuzzy I_{rw}-resolvable space if and only if (X, τ, I) has a pair of fuzzy I_{rw}-dense sets λ_1 and λ_2 such that $\lambda_1 \leq 1 - \lambda_2$.*

Proof. Let (X, τ, I) be a fuzzy I_{rw}-resolvable space. Suppose that for all fuzzy I_{rw}-dense sets λ_i and λ_j, we have $\lambda_i \not\leq 1 - \lambda_j$. Then we have $\lambda_i > 1 - \lambda_j$ for some i and j. Then, we have I_{rw}-$Cl(\lambda_i) > I_{rw}$-$Cl(1 - \lambda_j)$ which implies that $1 > I_{rw}$-$Cl(1 - \lambda_j)$. Then I_{rw}-$Cl(1 - \lambda_j) \neq 1$. Also $\lambda_j > 1-\lambda_i$. Then I_{rw}-$Cl(\lambda_j) > I_{rw}$-$Cl(1-\lambda_i)$ which implies that $1 > I_{rw}$-$Cl(1-\lambda_i)$. Then I_{rw}-$Cl(1-\lambda_i) \neq 1$. Hence I_{rw}-$Cl(\lambda_i) = 1$, but I_{rw}-$Cl(1-\lambda_i) \neq 1$ for all fuzzy I_{rw}-dense sets λ_i in (X, τ, I), which is a contradiction to (X, τ, I) being a fuzzy I_{rw}-resolvable space. Therefore (X, τ, I) has a pair of fuzzy I_{rw}-dense sets λ_1 and λ_2 such that $\lambda_1 \leq 1 - \lambda_2$.

Conversely, suppose that the fuzzy ideal topological space (X, τ, I) has a pair of fuzzy I_{rw}-dense sets λ_1 and λ_2 such that $\lambda_1 \leq 1 - \lambda_2$. We want to show that (X, τ, I) is fuzzy I_{rw}-resolvable. Suppose that (X, τ, I) is a fuzzy I_{rw}-irresolvable space. Then for all fuzzy I_{rw}-dense sets λ_i in (X, τ, I), we have I_{rw}-$Cl(1 - \lambda_i) \neq 1$. In particular I_{rw}-$Cl(1 - \lambda_2) \neq 1$ implies that there exist a fuzzy I_{rw}-closed set μ in (X, τ, I) such that $(1 - \lambda_2) < \mu < 1$. Then $\lambda_1 \leq 1 - \lambda_2 < \mu < 1 \Rightarrow \lambda_1 < \mu < 1$, which is a contradiction to I_{rw}-$Cl(\lambda_1) = 1$. Hence our assumption that (X, τ, I) is a fuzzy I_{rw}-irresolvable space, is wrong. Hence (X, τ, I) is a fuzzy I_{rw}-resolvable space. □

Proposition 8.1. *A fuzzy ideal topological space (X, τ, I) is a fuzzy I_{rw}-resolvable space if $\bigvee_{i=1}^{i=n} \lambda_i = 1$ where I_{rw}-$Int(\lambda_i) = 0$.*

Proof. $\bigvee_{i=1}^{i=n} \lambda_i = 1$ where I_{rw}-$Int(\lambda_i) = 0$, implies that $1 - \bigvee_{i=1}^{i=n} \lambda_i = 0$. Then we have $\bigwedge_{i=1}^{i=n} (1 - \lambda_i) = 0$. Then there must be at least two non-zero disjoint fuzzy sets $1 - \lambda_i$, $1 - \lambda_j$ in (X, τ, I). Hence $(1 - \lambda_i) + (1 - \lambda_j) \leq 1$. Therefore $(1 - \lambda_i) \leq \lambda_j$ which implies that I_{rw}-$Cl(1 - \lambda_i) \leq I_{rw}$-$Cl(\lambda_j)$. But I_{rw}-$Int(\lambda_i) = 0$ implies that I_{rw}-$Cl(1 - \lambda_i) = 1$. Hence $1 \leq I_{rw}$-$Cl(\lambda_j)$ which implies that I_{rw}-$Cl(\lambda_j) = 1$. Also I_{rw}-$Int(\lambda_j) = 0$ implies that I_{rw}-$Cl(1 - \lambda_j) = 1$. Therefore (X, τ, I) has a fuzzy I_{rw}-dense set λ_j such that I_{rw}-$Cl(1 - \lambda_j) = 1$. Hence (X, τ, I) is a fuzzy I_{rw}-resolvable space. □

Proposition 8.2. *If (X, τ, I) is fuzzy I_{rw}-irresolvable if and only if I_{rw}-$Int(\lambda) \neq 0$ for all fuzzy I_{rw}-dense sets λ in (X, τ, I).*

Proof. Since (X, τ, I) is fuzzy I_{rw}-irresolvable, for all fuzzy I_{rw}-dense sets λ in (X, τ, I), we have I_{rw}-$Cl(1 - \lambda) \neq 1$. Then $1 - I_{rw}$-$Int(\lambda) \neq 1$ implies that I_{rw}-$int(\lambda) \neq 0$.

Conversely let I_{rw}-$Int(\lambda) \neq 0$ for each fuzzy I_{rw}-dense set λ in (X, τ, I). Suppose that (X, τ, I) is fuzzy I_{rw}-resolvable. Then there exists a non-zero fuzzy I_{rw}-dense set λ in (X, τ, I) such that I_{rw}-$Cl(1 - \lambda) = 1$. Then we have $1 - I_{rw}$-$Int(\lambda) = 1$ and therefore I_{rw}-$Int(\lambda) = 0$ which is a contradiction. Hence (X, τ, I) is a fuzzy I_{rw}-irresolvable space. □

§9. Functions and fuzzy I_{rw}-irresolvable spaces

Definition 9.1. *A function $f : (X, \tau, I) \to (Y, \sigma, I)$ is said to be weakly somewhat fuzzy I_{rw}-open if for each I_{rw}-dense fuzzy set λ in (Y, σ, I) with I_{rw}-$Int(\lambda) \neq 0$, we have that $f^{-1}(\lambda)$ is also a fuzzy I_{rw}-dense set in (X, τ, I).*

The above definition leads to a characterization of fuzzy I_{rw}-irresolvable space as follows:

Theorem 9.1. *The following statements are equivalent for a fuzzy ideal topological space (Y, σ, I).*

(1) (Y, σ, I) is fuzzy I_{rw}-irresolvable

(2) For every fuzzy ideal topological space (X, τ, I), every weakly somewhat fuzzy I_{rw}-open function $f : (X, \tau, I) \to (Y, \sigma, I)$ is somewhat fuzzy I_{rw}-open.

Proof. (1) $\Rightarrow$ (2) Let $f : (X, \tau, I) \to (Y, \sigma, I)$ be a weakly somewhat fuzzy I_{rw}-open function from a fuzzy ideal topological spaces (X, τ, I) to a fuzzy I_{rw}-irresolvable space (Y, σ, I). Since (Y, σ, I) is fuzzy I_{rw}-irresolvable space, (Y, σ, I) has a pair of fuzzy I_{rw}-dense sets λ_1 and λ_2 such that $\lambda_1 \nleq 1 - \lambda_2$. Now I_{rw}-$Int(\lambda_1) \neq 0$ and I_{rw}-$Int(\lambda_2) \neq 0$. For, if I_{rw}-$Int(\lambda_1) = 0$ then, $1 - I_{rw}$-$Cl(1 - \lambda_1) = 0$. Now $\lambda_1 > 1 - \lambda_2 \Rightarrow \lambda_2 > 1 - \lambda_1$. Therefore I_{rw}-$Cl(\lambda_2) > I_{rw}$-$Cl(1 - \lambda_1)$. In other words $1 - I_{rw}$-$Cl(\lambda_2) < 1 - I_{rw}$-$Cl(1 - \lambda_1) = 0$. Then $1 < I_{rw}$-$Cl(\lambda_2)$ implies $1 < 1$, which is a contradiction. Therefore I_{rw}-$Int(\lambda_1) \neq 0$. Similarly we can have I_{rw}-$Int(\lambda_2) \neq 0$. Since f is weakly somewhat fuzzy I_{rw}-open, $f^{-1}(\lambda_1)$ and $f^{-1}(\lambda_2)$ are fuzzy I_{rw}-dense sets in (X, τ, I). Therefore by Theorem , f is somewhat fuzzy I_{rw}-open.

(2) $\Rightarrow$ (1) Suppose that fuzzy ideal topological space (Y, σ, I) is fuzzy I_{rw}-resolvable. This means that there exists a pair of fuzzy I_{rw}-dense sets λ_1 and λ_2 such that $\lambda_1 \leq 1 - \lambda_2$. Let $X = Y$ and $\tau = \{0, 1, \lambda_1\}$. Define $f : (X, \tau, I) \to (Y, \sigma, I)$ to be the identity function. Then f is not somewhat fuzzy I_{rw}-open, since $f^{-1}(\lambda_2)$ is not a fuzzy I_{rw}-dense set in (Y, τ, I). For, $f^{-1}(\lambda_2) = \lambda_2$ and $\lambda_2 \leq 1 - \lambda_1 \neq 1$. Then $\lambda_2 \leq 1 - \lambda_1 \Rightarrow I_{rw}\text{-}Cl(\lambda_2) \leq I_{rw}\text{-}Cl(1 - \lambda_1)$. Since $1 - \lambda_1$ is fuzzy closed and hence I_{rw}-closed in (Y, τ, I), $I_{rw}\text{-}Cl(\lambda_2) \neq 1$. That is, λ_2 is not a fuzzy I_{rw}-dense set. We shall now show that f is weakly somewhat fuzzy I_{rw}-open. Let λ be any fuzzy I_{rw}-dense set in (Y, σ, I) such that $I_{rw}\text{-}Int(\lambda) \neq 0$. Then $f^{-1}(\lambda) = \lambda$. We have to show that $I_{rw}\text{-}Cl[f^{-1}(\lambda)] = I_{rw}\text{-}Cl(\lambda) = 1$ in (Y, τ, I). Now $I_{rw}\text{-}Int(\lambda) \neq 0$ and λ_1 is fuzzy I_{rw}-dense implies that $\lambda \not\leq 1 - \lambda_1$. Therefore $I_{rw}\text{-}Cl(\lambda) = 1$. That is, λ is fuzzy I_{rw}-dense in (Y, τ, I). This proves that f is weakly somewhat fuzzy I_{rw}-open. Hence (2) $\Rightarrow$ (1) is proved. □

Theorem 9.2. *Let (X, τ, I) and (Y, σ, I) be any two fuzzy ideal topological spaces. Let $f : (X, \tau, I) \to (Y, \sigma, I)$ be a somewhat fuzzy I_{rw}-open function. If (X, τ, I) is a fuzzy I_{rw}-irresolvable space, then (Y, σ, I) is a fuzzy I_{rw}-irresolvable space.*

Proof. Let $\lambda \neq 0$ be an arbitrary fuzzy set in (Y, σ) such that $I_{rw}\text{-}Cl(\lambda) = 1$. We claim that $I_{rw}\text{-}Int(\lambda) \neq 0$. Assume the contrary. That is, $I_{rw}\text{-}Int(\lambda) = 0$. Then by Proposition , we have $I_{rw}\text{-}Int(f^{-1}(\lambda)) = 0$ in (X, τ, I). Now λ is fuzzy I_{rw}-dense in (Y, σ, I), then by Theorem , we have $f^{-1}(\lambda)$ is fuzzy I_{rw}-dense in (X, τ, I). Therefore for the fuzzy I_{rw}-dense that $f^{-1}(\lambda)$, we have $I_{rw}\text{-}Int(f^{-1}(\lambda)) = 0$ in (X, τ, I), which is a contradiction. [since (X, τ, I) is fuzzy I_{rw}-irresolvable, by Proposition , $I_{rw}\text{-}Int(\mu) \neq 0$ for all fuzzy I_{rw}-dense sets μ in (X, τ, I)]. Hence we must have $I_{rw}\text{-}Int(\mu) \neq 0$ for all fuzzy I_{rw}-dense sets λ in (Y, σ, I). Hence by Proposition , (Y, σ, I) is a fuzzy I_{rw}-irresolvable space. □

Theorem 9.3. *Let (X, τ, I) and (Y, σ, I) be any two fuzzy ideal topological spaces and $f : (X, \tau, I) \to (Y, \sigma, I)$ be a somewhat fuzzy I_{rw}-continuous and onto function. If (Y, σ, I) is a fuzzy I_{rw}-irresolvable space, then (X, τ, I) is a fuzzy I_{rw}-irresolvable space.*

Proof. Let $\lambda \neq 0$ be an arbitrary fuzzy set in (X, τ, I) such that $I_{rw}\text{-}Cl(\lambda) = 1$. We claim that $I_{rw}\text{-}Int(\lambda) \neq 0$. Assume the contrary. That is, $I_{rw}\text{-}Int(\lambda) = 0$. Then by Proposition , we have $I_{rw}\text{-}Int(f(\lambda)) = 0$. Now λ is fuzzy I_{rw}-dense in (X, τ, I), then by Theorem , we have $f(\lambda)$ is fuzzy I_{rw}-dense in (X, τ, I). Therefore for the fuzzy I_{rw}-dense set $f(\lambda)$ in (Y, σ, I), we have $I_{rw}\text{-}Int(f(\lambda)) = 0$, which is a contradiction. [since (Y, σ, I) is fuzzy I_{rw}-irresolvable, $I_{rw}\text{-}Int(\mu) \neq 0$ for all fuzzy I_{rw}-dense sets μ in (X, τ, I)]. Therefore we must have $I_{rw}\text{-}Int(\lambda) \neq 0$ for all fuzzy I_{rw}-dense sets λ in (Y, τ, I). Hence by Proposition , the fuzzy ideal topological space (X, τ, I) is a fuzzy I_{rw}-irresolvable space. □

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The mean value of $\tau_3^{(e)}(n)$ with a negative r-th power

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Abstract Many scholars are interested in researching the divisor problem, and they have obtained a large number of good results. However, there are many problems have not been solved. In this paper we shall study the mean value of the exponential divisor function involving a negative r-th power by the convolution method.

Keywords The exponential divisor function, Euler product, Dirichlet convolution.
2010 Mathematics Subject Classification 11N37.

§1. Introduction

Let $n > 1$ be an integer. The integer $d = \prod_{i=1}^{s} p_i^{b_i}$ is called an exponential divisor of $n = \prod_{i=1}^{s} p_i^{a_i}$, if $b_i | a_i$ for every $i \in \{1, 2, \cdots, s\}$, notation: $d|_e n$. By convention $1|_e 1$.

Let $\tau^{(e)}(n)$ denote the number of exponential divisors of n. The function $\tau^{(e)}$ is called the exponential divisor function. Similarly to the generalization of $d_k(n)$ from $d(n)$, we define the function $\tau_k^{(e)}(n)$:

$$\tau_k^{(e)}(n) = \prod_{{p_i}^{a_i} || n} d_k(a_i), k \geq 2, \tag{1}$$

Obviously when $k = 2$, that is $\tau^{(e)}(n)$. $\tau_3^{(e)}(n)$ is obviously a multiplicative function.

Throughout this paper, ϵ always denotes a fixed but sufficiently small positive constant.

J.Wu [1] got the following result:

$$\sum_{n \leq x} \tau^{(e)}(n) = A(x) + Bx^{\frac{1}{2}} + O(x^{\frac{2}{9}} \log x), \tag{2}$$

where

$$A = \prod_p (1 + \sum_{a=2}^{\infty} \frac{d(a) - d(a-1)}{p^a}),$$

$$B = \prod_p (1 + \sum_{a=5}^{\infty} \frac{d(a) - d(a-1) - d(a-2) + d(a-3)}{p^{\frac{a}{2}}}).$$

M.V.Subbarao [3] also proved for some positive integer r:

$$\sum_{n \leq x} (\tau^{(e)}(n))^r \sim A_r x, \tag{3}$$

where

$$A_r = \prod_p (1 + \sum_{a=2}^{\infty} \frac{(d(a))^r - (d(a-1))^r}{p^a}).$$

László Tóth [4] improved the result (3) and established a more precise asymptotic formula for the r-th power of the function $\tau^{(e)}(n)$:

$$\sum_{n\leq x} (\tau^{(e)}(n))^r = A_r x + x^{\frac{1}{2}} P_{2^r-2}(\log x) + O(x^{u_r+\epsilon}). \tag{4}$$

Jing Huang and Ping Song [7] also proved that

$$\sum_{n\leq x} (\tau_3^{(e)}(n))^r = A_r x + x^{\frac{1}{2}} R_{3^r-2}(\log x) + O(x_r^b + \epsilon), \tag{5}$$

where $b_r = \frac{1}{3-\alpha_{3^r-1}}$ (see [7], Lemma 2.2), $R_{3^r-2}(x)$ is a polynomial of degree $3^r - 2$ and

$$A_r = \prod_p (1 + \sum_{a=2}^{\infty} \frac{(d_3(a))^r - (d_3(a-1))^r}{p^a}).$$

In this paper, we shall study the mean value of the exponential divisor function involving a negative r-th power of the function $\tau_3^{(e)}(n)$ by the convolution method, where $r > 1$ is an integer.

Theorem 1.1. *For every integer $r > 1$ and $N \geq 1$, then we have*

$$\sum_{n\leq x} (\tau_3^{(e)}(n))^{-r} = C_r x + x^{\frac{1}{2}} \log^{3^{-r}-2} (\sum_{j=0}^{N} d_j(r) \log^{-j} x + O(\log^{-N-1} x)), \tag{6}$$

where $d_0(r), d_1(r), \cdots, d_N(r)$ are computable constants, and

$$C_r := \prod_p (1 + \sum_{a=2}^{\infty} \frac{(d_3(a))^{-r} - (d_3(a-1))^{(-r)}}{p^a}).$$

§2. Preliminaries

In order to prove our theorem, we define for an arbitrary complex number z the general divisor function $d_z(n)$ by

$$\sum_{n=1}^{\infty} d_z(n) n^{-s} = \zeta^z(s) = \prod_p (1 - p^{-s})^{-z}, Res > 1, \tag{7}$$

where a branch of $\zeta^z(s)$ is defined by

$$\zeta^z(s) = exp\{z \log \zeta(s)\} = exp(-z \sum_p \sum_{j=1}^{\infty} j^{-1} p^{-js}), Res > 1. \tag{8}$$

The definition shows that $d_z(n)$ is multiplicative function of n which generalizes $d_k(n)$. The divisor function $d_k(n)$ ($k \geq 2$ a fixed integer) may be defined by

$$\sum_{n=1}^{\infty} d_k(n)n^{-s} = \zeta^k(s) = \prod_p (1-p^{-s})^{-k}, Res > 1. \tag{9}$$

The proof of the Theorem 1.1 is based on the following lemmas.

Lemma 2.1. *Suppose s is a complex number for with $Res > 1$, $r \geq 1$ is a fixed integer, then*

$$F(s) := \sum_{n=1}^{\infty} \frac{(\tau_3^{(e)}(n))^{-r}}{n^s} = \zeta(s)\zeta^{3^{-r}-1}(2s)G(s,r), \tag{10}$$

where the Dirichlet series $G(s,r) := \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$ is absolutely convergent for $Res > \frac{1}{4}$.

Proof. By the Euler product formula, we can get

$$\begin{aligned} F(s) &= \prod_p (1 + \frac{(\tau_3^{(e)}(p))^{-r}}{p^s} + \frac{(\tau_3^{(e)}(p^2))^{-r}}{p^{2s}} + \frac{(\tau_3^{(e)}(p^3))^{-r}}{p^{3s}} + \cdots) \\ &= \prod_p (1 + \frac{(d_3^{(e)}(1))^{-r}}{p^s} + \frac{(d_3^{(e)}(2))^{-r}}{p^{2s}} + \frac{(d_3^{(e)}(3))^{-r}}{p^{3s}} + \frac{(d_3^{(e)}(4))^{-r}}{p^{4s}} + \frac{(\tau_3^{(e)}(p^5))^{-r}}{p^{5s}} + \cdots) \\ &= \prod_p (1 + \frac{1}{p^s} + \frac{3^{-r}}{p^{2s}} + \frac{3^{-r}}{p^{3s}} + \frac{6^{-r}}{p^{4s}} + \frac{3^{-r}}{p^{5s}} + \cdots) \\ &= \prod_p (1 - \frac{1}{p^s})^{-1} \prod_p (1 - \frac{1}{p^s})(1 + \frac{1}{p^s} + \frac{3^{-r}}{p^{2s}} + \frac{3^{-r}}{p^{3s}} + \frac{6^{-r}}{p^{4s}} + \frac{3^{-r}}{p^{5s}} + \cdots) \\ &= \zeta(s) \prod_p (1 + \frac{3^{-r}-1}{p^{2s}} + \frac{6^{-r}-3^{-r}}{p^{4s}} + \frac{3^{-r}-6^{-r}}{p^{5s}} + \cdots) \\ &= \zeta(s)\zeta^{3^{-r}-1}(2s)G(s,r), \end{aligned} \tag{11}$$

where the infinite series

$$G(s,r) = \prod_p (1 - \frac{1}{p^{2s}})^{3^{-r}-1} (1 + \frac{3^{-r}-1}{p^{2s}} + \frac{6^{-r}-3^{-r}}{p^{4s}} + \frac{3^{-r}-6^{-r}}{p^{5s}} + \cdots).$$

Write $G(s,r) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$. It is absolutely convergent for $Res > \frac{1}{4}$. □

Lemma 2.2. *Let $A > 0$ be arbitrary but fixed real number, and let $N_1 \geq 1$ be an arbitrary but fixed integer. If $|z| \leq A$, then uniformly in z*

$$\begin{aligned} \sum_{n \leq x} d_z(n) =& C_1(z)x\log^{z-1}x + C_2(z)x\log^{z-2}x + \cdots \\ &+ C_{N_1}(z)x\log^{z-N_1}x + O(x\log^{Rez-N_1-1}x), \end{aligned} \tag{12}$$

where $C_j(z) = \frac{B_j(z)}{\Gamma(z-j-1)}$, $(j = 1, 2, \cdots, N_1)$ and each $B_j(z)$ is regular for $|z| \leq A$.

Proof. See Ivić [2], Theorem 14.9. □

Lemma 2.3. *Let $A > 0$ be arbitrary but fixed real number, and let $M \geq 1$ be an arbitrary but fixed integer. If $|z| \leq A$, then uniformly in z*

$$\begin{aligned}\sum_{mn^2\leq x} d_z(n) =&\zeta^z(2)x + x^{\frac{1}{2}}(K_1(z)\log^{z-1} x + K_2(z)\log^{z-2} x + \cdots \\ &+ K_M(z)\log^{z-M} x) + O(x^{\frac{1}{2}}\log^{Rez-M-1} x),\end{aligned} \tag{13}$$

where the functions $K_j(z)(j = 1, 2, \cdots, M)$ are regular in $|z| \leq A$.

Proof. Suppose $1 \leq y \leq x$ is a parameter to be determined later. We have

$$\begin{aligned}\sum_{mn^2\leq x} d_z(n) &= \sum_{n\leq y} d_z(n) \sum_{m\leq \frac{x}{n^2}} 1 + \sum_{m\leq \frac{x}{y^2}} \sum_{n^2\leq \frac{x}{m}} d_z(n) - \sum_{m\leq \frac{x}{y^2}} \sum_{n\leq y} d_z(n) \\ &= \sum_1 + \sum_2 - \sum_3,\end{aligned} \tag{14}$$

where

$$\sum_1 = \sum_{n\leq y} d_z(n) \sum_{m\leq \frac{x}{n^2}} 1$$

$$\sum_2 = \sum_{m\leq \frac{x}{y^2}} \sum_{n^2\leq \frac{x}{m}} d_z(n)$$

$$\sum_3 = \sum_{m\leq \frac{x}{y^2}} \sum_{n\leq y} d_z(n).$$

For $\sum_1$, we have

$$\begin{aligned}\sum_1 &= \sum_{n\leq y} d_x(n)[\frac{x}{n^2}] \\ &= x\sum_{n\leq y} \frac{d_z(n)}{n^2} + O(\sum_{n\leq y} |d_z(n)|).\end{aligned} \tag{15}$$

We see that $|d_z(n)| \leq d_k(n)$, if $k = [A] + 1$ and $|z| \leq A$. If we use the weak asymptotic formula (see, Ivić[2])

$$\sum_{n\leq x} d_k(n) = xP_{k-1}(\log x) + O(x^{\frac{k}{k+1}}), \tag{16}$$

the error term in $\sum_1$ is bounded by $O(y\log^{k-1} y)$.

So by lemma2.2 and the partial summation, we have

$$\begin{aligned}\sum_1 =&x\sum_{n=1}^{\infty} \frac{d_z(n)}{n^2} - x\sum_{n>y} \frac{d_z(n)}{n^2} + O(y\log^{k-1} y) \\ =&\zeta^z(2)x + \frac{x}{y}\sum_{j=1}^{N_1} C_j(z)\log^{z-j} y + \frac{2x}{y}\sum_{j=1}^{N_1}(z-j)C_j(z)\log^{z-j-1} y \\ &+ \frac{2x}{y}\sum_{j=1}^{N_1}(z-j)(z-j-1)C_j(z)\log^{z-j-2} y + \cdots \\ &+ O(\frac{x}{y}\log^{Rez-N_1-1} y) + O(y\log^{k-1} y).\end{aligned} \tag{17}$$

Using Lemma 2.2, it is seen that

$$\begin{aligned}\sum_3 &= \sum_{m\le \frac{x}{y^2}} \sum_{n\le y} d_z(n) \\ &= \sum_{n\le y} d_z(n)(\frac{x}{y^2}+O(1)) \\ &= \frac{x}{y}\sum_{j=1}^{N_1} C_j(z)\log^{z-j} y + O(\frac{x}{y}\log^{Rez-N_1-1} y) + O(y\log^{k-1} y).\end{aligned} \tag{18}$$

By similar computation, we can obtain

$$\begin{aligned}\sum_2 &= \sum_{m\le \frac{x}{y^2}} [\sum_{j=1}^{N_1} C_j(z)\sqrt{\frac{x}{m}}\log^{z-j}(\frac{x}{m})^{\frac12} + O(\sqrt{\frac{x}{m}}\log^{Rez-L-1}(\frac{x}{m}))] \\ &=\sqrt{x}\sum_{j=1}^{N_1} C_j(z)\sum_{m\le \frac{x}{y^2}} m^{-\frac12}\log^{z-j}(\frac{x}{m})^{\frac12} + O(\sum_{m\le \frac{x}{y^2}}\sqrt{\frac{x}{m}}\log^{Rez-N_1-1}(\frac{x}{m})) \\ &=\sqrt{x}\sum_{j=1}^{N_1} C_j(z)(\frac12)^{z-j}\log^{z-j} x\sum_{m\le \frac{x}{y^2}} m^{-\frac12}(1-\frac{\log m}{\log x})^{z-j} \\ &\quad + O(\sqrt{x}\log^{Rez-N_1-1} x\sum_{m\le \frac{x}{y^2}} m^{-\frac12}) \\ &=\sum_{2,1} + O(\frac{x}{y}\log^{Rez-N_1-1} x),\end{aligned} \tag{19}$$

where we define

$$\sum_{2,1} = \sqrt{x}\sum_{j=1}^{N_1} C_j(z)(\frac12)^{z-j}\log^{z-j} x\sum_{m\le \frac{x}{y^2}} m^{-\frac12}(1-\frac{\log m}{\log x})^{z-j}.$$

Using Taylor formula and foregoing method, we can obtain

$$\begin{aligned}\sum_{2,1} &=\sqrt{x}\sum_{j=1}^{N_1} C_j(z)(\frac12)^{z-j}\log^{z-j} x\sum_{m\le \frac{x}{y^2}} m^{-\frac12}(1-(z-j)\frac{\log m}{\log x} \\ &\quad + \frac{(z-j)(z-j-1)}{2!}(\frac{\log m}{\log x})^2+\cdots) \\ &=x^{\frac12}\sum_{j=1}^{N_1} K_j(z)\log^{z-j} x - \frac{2x}{y}\sum_{j=1}^{N_1}(z-j)C_j(z)\log^{z-j-1} y \\ &\quad - \frac{2x}{y}\sum_{j=1}^{N_1}(z-j)(z-j-1)C_j(z)\log^{z-j-2} y+\cdots \\ &\quad + O(y\log^{Rez-1} y) + O(\frac{x}{y}\log^{Rez-N_1-1} y),\end{aligned} \tag{20}$$

where $K_1(z), K_2(z), \cdots, K_M(z)$ are regular functions.

So by choosing $y=\sqrt{x}\log^C(x), C = Rez-M-k$ and $N_1 = 2M+k-Rez$ completes the proof of the Lemma 2.3. □

§3. Prove of Theorem 1.1.

Now we go on with the proof of our main Theorem.

Proof. Combining Lemma 2.1 and Lemma 2.3, we get

$$
\begin{aligned}
\sum_{n\le x}(\tau_3^{(e)}(n))^{-r} &= \sum_{n_1n_2^2n_3\le x} d_z(n_2)g(n_3)\\
&= \sum_{n_3\le x} g(n_3)\sum_{n_1n_2^2\le x/n_3} d_z(n_2)\\
&= \sum_{n_3\le x} g(n_3)[\zeta^z(2)(\frac{x}{n_3})+(\frac{x}{n_3})^{\frac{1}{2}}\sum_{j=1}^{M}K_j(z)\log^{z-j}(\frac{x}{n_3})\\
&\quad +O((\frac{x}{n_3})^{\frac{1}{2}}\log^{Rez-M-1}(\frac{x}{n_3}))] \qquad (21)\\
&=x\zeta^z(2)\sum_{n_3\le x} g(n_3){n_3}^{-1}+x^{\frac{1}{2}}\sum_{n_3\le x} g(n_3){n_3}^{-\frac{1}{2}}\sum_{j=1}^{M}K_j\log^{z-j}(\frac{x}{n_3})\\
&\quad +O(\sum_{n_3\le x} g(n_3)(\frac{x}{n_3})^{\frac{1}{2}}\log^{Rez-M-1}(\frac{x}{n_3}))\\
&=S_1(x)+S_2(x)+O(S_3(x)),
\end{aligned}
$$

where

$$
\begin{aligned}
S_1(x) &= x\zeta^z(2)\sum_{n_3\le x} g(n_3){n_3}^{-1},\\
S_2(x) &= x^{\frac{1}{2}}\sum_{n_3\le x} g(n_3){n_3}^{-\frac{1}{2}}\sum_{j=1}^{M}K_j\log^{z-j}(\frac{x}{n_3}),\\
S_3(x) &= \sum_{n_3\le x} g(n_3)(\frac{x}{n_3})^{\frac{1}{2}}\log^{Rez-M-1}(\frac{x}{n_3}),
\end{aligned}
$$

and we choose $z=3^{-r}-1$.

Then we just need to calculate the three sums separately.

$$
\begin{aligned}
S_1(x) &=x\zeta^{3^{-r}-1}(2)\sum_{n_3\le x} g(n_3){n_3}^{-1}\\
&=x\zeta^z(2)\sum_{n_3=1}^{\infty} g(n_3){n_3}^{-1}-x\zeta^z(2)\sum_{n_3>x} g(n_3){n_3}^{-1} \qquad (22)\\
&=C_r(x)+O(x^{\frac{1}{4}+\epsilon})
\end{aligned}
$$

Analogously to $\sum_{2,1}$, we have

$$\begin{aligned}
S_2(x) =& x^{\frac{1}{2}} \sum_{n_3\le x} g(n_3) n_3^{-\frac{1}{2}} \sum_{j=1}^{M} K_j \log^{3^{-r}-1-j}(\frac{x}{n_3}) \\
=& x^{\frac{1}{2}} \sum_{j=1}^{M} K_j \log^{3^{-r}-1-j} x \sum_{n_3\le x} g(n_3) n_3^{-\frac{1}{2}} (1-\frac{\log n_3}{\log x})^{3^{-r}-1-j} \\
=& x^{\frac{1}{2}} \log^{3^{-r}-2} x \sum_{j=0}^{M-1} E_j(r) \log^{-j} x + O(x^{\frac{1}{2}} \log^{3^{-r}-M-2} x) \\
=& x^{\frac{1}{2}} \log^{3^{-r}-2} x \sum_{j=0}^{N} d_j(r) \log^{-j} x + O(x^{\frac{1}{2}} \log^{3^{-r}-N-3} x),
\end{aligned} \tag{23}$$

where $E_1(r), E_2(r), \cdots, E_N(r)$ are computable constants depending on r, and we set $N = M-1$.

Similarly, we also have

$$\begin{aligned}
S_3(x) =& \sum_{n_3\le x} g(n_3) (\frac{x}{n_3})^{\frac{1}{2}} \log^{Rez-M-1}(\frac{x}{n_3}) \\
=& x^{\frac{1}{2}} \log^{Rez-M-1} x \sum_{n_3\le x} g(n_3) n_3^{-\frac{1}{2}} (1-\frac{\log n_3}{\log x})^{Rez-M-1} \\
\ll& x^{\frac{1}{2}} \log^{3^{-r}-M-2} x \\
=& x^{\frac{1}{2}} \log^{3^{-r}-N-3} x.
\end{aligned} \tag{24}$$

Hence, the Theorem 1.1 is proved by (21)-(23). □

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The mean value of $\tau^{(e)}(n)$ over cube-full numbers

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Abstract Let $n > 1$ be an integer, the function $\tau^{(e)}(n)$ denote the exponential divisor function. In this paper, we will study the mean value of $\tau^{(e)}(n)$ over cube-full numbers, that is

$$\sum_{\substack{n \leq x \\ n \text{ is cube-full}}} (\tau_3^{(e)}(n))^2 = \sum_{n \leq x} (\tau_3^{(e)}(n))^2 f_3(n).$$

Keywords asymptotic formula, exponential divisor, Dirichlet convolution.
2010 Mathematics Subject Classification 11N37.

§1. Introduction and preliminaries

An integer $n = \prod_{i=1}^{s} p_i^{a_i}$ is called k-full number if all the exponents $a_1 \geq k, a_2 \geq k, \cdots, a_s \geq k$. when $k = 3$, n is called cube-full integer, i.e.

$$f_3(n) = \begin{cases} 1, & n \text{ is cube-full }, \\ 0, & \text{otherwise }. \end{cases}$$

Many scholars are interested in researching the divisor problem and have obtained a large number of good results. But there are many problems hasn't been solved. For example, F.Smarandache gave some unsolved problems in his book *Only problems, Not solutions*! [6], and one problem is that, a number n is called simple number if the product of its proper divisors is less than or equal to n. Generally speaking, $n = p$, or $n = p^2$, or $n = p^3$, or pq, where p and q are distinct primes. The properties of this simple number sequence has't been studied yet. And other problems are introduced in this book, such as proper divisor products sequence and the largest exponent (of power p) which divides n, where $p \geq 2$ is an integer.

In the definition of exponential divisor: suppose $n > 1$ is an integer, and $n = \prod_{i=1}^{s} p_i^{a_i}$. If $d = \prod_{i=1}^{s} p_i^{b_i}$ satisfies $b_i | a_i, i = 1, 2, \cdots, s$, then d is called an exponential divisor of n, notation $d|_e n$. By convention $1|_e 1$.

J.Wu [4] improved the above result got the following result:

$$\sum_{n \leqslant x} \tau^{(e)}(n) = A(x) + Bx^{\frac{1}{2}} + O(x^{\frac{2}{9}} \log x),$$

where

$$A = \prod_p (1 + \sum_{a=2}^{\infty} \frac{d(a) - d(a-1)}{p^a}),$$

$$B = \prod_p (1 + \sum_{a=5}^{\infty} \frac{d(a) - d(a-1) - d(a-2) + d(a-3)}{p^{\frac{a}{2}}}).$$

M.V.Subbarao [2] also proved for some positive integer r,

$$\sum_{n \le x} (\tau^{(e)}(n))^r \sim A_r x,$$

where

$$A_r = \prod_p (1 + \sum_{a=2}^{\infty} \frac{(d(a))^r - (d(a-1))^r}{p^a})$$

L.Toth [3] proved

$$\sum_{n \le x} (\tau^{(e)}(n))^r = A_r(x) + x^{\frac{1}{2}} P_{2^r-2}(\log x) + O(x^{u_r+\varepsilon})$$

where $P_{2^r-2}(t)$ is a polynomial of degree $2^r - 2$ in t, $u_r = \frac{2^{r+1}-1}{2^{r+1}+1}$.

Similarly to the generalization of $d_k(n)$ from $d(n)$, we define the function $\tau_k^{(e)}(n)$:

$$\tau_k^{(e)}(n) = \prod_{p_i^{a_i} || n} d_k(a_i), k \ge 2,$$

Obviously when $k = 2$, that is $\tau^{(e)}(n)$. $\tau_3^{(e)}(n)$ is obviously a multiplicative function. In this paper we investigate the case $k = 3$, i.e. the properties of the functions $\tau_3^{(e)}(n)$.

In this paper, we will study the asymptotic formula for the mean value of the function $(\tau_3^{(e)}(n))^2$ over cube-full numbers.

Theorem 1.1. *We have the asymptotic formula*

$$\sum_{\substack{n \le x \\ n \text{ is cube-full}}} (\tau_3^{(e)}(n))^2 = x^{\frac{1}{3}} Q_{8,1}(\log x) + x^{\frac{1}{4}} Q_{35,2}(\log x) + O(x^{\sigma_0+\varepsilon})$$

where $Q_{8,1}(t)$ is a polynomial of degree 8 in t, $Q_{35,2}(t)$ is a polynomial of degree 35 in t, $\sigma_0 = \frac{3530376}{14646528} = 0.241038422\cdots$.

Natation Through out this paper, ε always denotes a fixed but sufficiently small positive constant.

§2. Some lemmas

In the section, we give some lemmas which will be used in the proof of our theorem. Lemma 2.2, Lemma 2.3, and Lemma2.4 can be found in [5], [7], and [1].

Lemma 2.1. *Let*

$$\tau_3^{(e)}(n) = \prod_{p_i^{a_i}||n} d_3(a_i),$$

then we have

$$\sum_{\substack{n=1 \\ n \text{ is cube-full}}}^{\infty} \frac{(\tau_3^{(e)}(n))^2}{n^s} = \zeta^9(3s)\zeta^{36}(4s)G(s),$$

where the infinite series $G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$ *is absolutely convergent for* $\Re s > \frac{1}{5}$.

Proof. By Euler's product formula, we can get

$$\begin{aligned}
&\sum_{\substack{n=1 \\ n \text{ is cube-full}}}^{\infty} \frac{(\tau_3^{(e)}(n))^2}{n^s} = \sum_{n=1}^{\infty} \frac{(\tau_3^{(e)}(n))^2 f_3(n)}{n^s} \\
&= \prod_p \left(1 + \frac{d_3^2(1)f_3(p)}{p^s} + \frac{d_3^2(2)f_3(p^2)}{p^{2s}} + \frac{d_3^2(3)f_3(p^3)}{p^{3s}} + \frac{d_3^2(4)f_3(p^4)}{p^{4s}} + \frac{d_3^2(5)f_3(p^5)}{p^{5s}} + \cdots\right) \\
&= \prod_p \left(1 + \frac{d_3^2(3)}{p^{3s}} + \frac{d_3^2(4)}{p^{4s}} + \frac{d_3^2(5)}{p^{5s}} + \cdots\right) \\
&= \prod_p \left(1 + \frac{3^2}{p^{3s}} + \frac{6^2}{p^{4s}} + \frac{3^2}{p^{5s}} + \cdots\right) \\
&= \zeta^9(3s)\prod_p (1 + \frac{36}{p^{4s}} + \frac{9}{p^{5s}} + \cdots) \\
&= \zeta^9(3s)\zeta^{36}(4s)\prod_p (1 + \frac{9}{p^{5s}} + \cdots) \\
&= \zeta^9(3s)\zeta^{36}(4s)G(s)
\end{aligned}$$

where the infinite series $G(s) := \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$ is absolutely convergent for $\Re s > \frac{1}{5}$. □

Lemma 2.2. *Suppose* $f(m)$, $g(n)$ *are arithmetical functions such that*

$$\sum_{m\le x} f(m) = \sum_{j=1}^{J} x_j^{\alpha_j} P_j(\log x) + O(x^{\alpha}), \quad \sum_{n\le x} |g(n)| = O(x^{\beta}),$$

where $\alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_J > \alpha > \beta > 0$, $P_j(t)$ *is a polynomial in* t, *if* $h(n) = \sum_{n=md} f(m)g(d)$, *then*

$$\sum_{n\le x} h(n) = \sum_{j=1}^{J} x^{\alpha_j} Q_j(\log x) + O(x^{\alpha}),$$

where $Q_j(t)$ $j = 1, \cdots, J$ *is a polynomial in* t.

Lemma 2.3. *Let* $\frac{1}{2} \le \sigma \le 1$, $t \ge t_0 \ge 2$, *we have*

$$\zeta(\sigma + it) \ll t^{\frac{1-\sigma}{3}} \log t.$$

Lemma 2.4. *Let* $\frac{1}{2} < \sigma < 1$, *define*

$$\begin{aligned}
&m(\sigma) \ge \frac{4}{3-4\sigma}, && \frac{1}{2} < \sigma \le \frac{5}{8},\\
&m(\sigma) \ge \frac{10}{5-6\sigma}, && \frac{5}{8} < \sigma \le \frac{35}{54},\\
&m(\sigma) \ge \frac{19}{6-6\sigma}, && \frac{35}{54} < \sigma \le \frac{41}{60},\\
&m(\sigma) \ge \frac{2112}{859-948\sigma}, && \frac{41}{60} < \sigma \le \frac{3}{4},\\
&m(\sigma) \ge \frac{12408}{4537-4890\sigma}, && \frac{3}{4} < \sigma \le \frac{5}{6},\\
&m(\sigma) \ge \frac{4324}{1031-1044\sigma}, && \frac{5}{6} < \sigma \le \frac{7}{8},\\
&m(\sigma) \ge \frac{98}{31-32\sigma}, && \frac{7}{8} < \sigma \le 0.91591,\\
&m(\sigma) \ge \frac{24\sigma-9}{(4\sigma-1)(1-\sigma)}, && 0.91591 < \sigma \le 1-\varepsilon.
\end{aligned}$$

Lemma 2.5.

$$\sum_{n\le x} d(\underbrace{3,\cdots,3}_{9},\underbrace{4,\cdots,4}_{36};n) = x^{\frac{1}{3}}P_{8,1}(\log x) + x^{\frac{1}{4}}P_{35,2}(\log x) + O(x^{\sigma_0+\varepsilon})$$

where $P_{8,1}(t)$ *is a polynomial of degree 8 in* t, $P_{35,2}(t)$ *is a polynomial of degree 35 in* t, $\sigma_0 = \frac{3530376}{14646528} = 0.241038422\cdots$.

Proof. By the Perron's formula, we have

$$S(x) = \sum_{n\le x} \delta(n)d(n) = \frac{1}{2\pi i}\int_{b-iT}^{b+iT} \zeta^9(3s)\zeta^{36}(4s)\frac{x^s}{s}ds + O(\frac{x^{\frac{1}{3}+\varepsilon}}{T})$$

where $b = \frac{1}{3}+\varepsilon$, $T = x^c$, c is a very large number of fixed numbers, $\frac{1}{5} < \sigma_0 < \frac{1}{4}$. According to the Residue theorem, we have

$$\begin{aligned}
S(x) &= x^{\frac{1}{3}}P_{8,1}(\log x) + x^{\frac{1}{4}}P_{35,2}(\log x) + I_1 + I_2 + I_3 + O(1),\\
I_1 &= \frac{1}{2\pi i}\int_{b-iT}^{\sigma_0-iT} \zeta^9(3s)\zeta^{36}(4s)\frac{x^s}{s}ds,\\
I_2 &= \frac{1}{2\pi i}\int_{\sigma_0-it}^{\sigma_0+it} \zeta^9(3s)\zeta^{36}(4s)\frac{x^s}{s}ds,\\
I_3 &= \frac{1}{2\pi i}\int_{\sigma_0+iT}^{b+iT} \zeta^9(3s)\zeta^{36}(4s)\frac{x^s}{s}ds.
\end{aligned}$$

For I_1, I_3, since $\sigma_0 > \frac{14}{57} + \delta$, $(s = \sigma + iT)$, and from Lemma 2.3, we have

$$\begin{aligned}
I_1 + I_3 \le & \int_{\sigma_0}^{\frac{1}{3}+\varepsilon} |\zeta(3\sigma + i3T)|^9 |\zeta(4\sigma + i4T)|^{36} x^{\sigma} T^{-1} d\sigma \\
& \ll T^{-1}\Big(\int_{\sigma_0}^{\frac{1}{4}} + \int_{\frac{1}{4}}^{\frac{1}{3}} + \int_{\frac{1}{3}}^{\frac{1}{3}+\varepsilon}\Big)|\zeta(3\sigma + i3T)|^9 |\zeta(4\sigma + i4T)|^{36} x^{\sigma} d\sigma \\
& \ll T^{-1+\varepsilon} \int_{\sigma_0}^{\frac{1}{4}} T^{\frac{9(1-3\sigma)}{3} + \frac{36(1-4\sigma)}{3}} x^{\sigma} d\sigma + T^{-1+\varepsilon} \int_{\frac{1}{4}}^{\frac{1}{3}} T^{\frac{9(1-3\sigma)}{3}} x^{\sigma} d\sigma \\
& + T^{-1+\varepsilon} \int_{\frac{1}{3}}^{\frac{1}{3}+\varepsilon} x^{\sigma} d\sigma \\
& \ll x^{\frac{1}{5}} T^{-\delta+\varepsilon} + x^{\frac{1}{4}} T^{-\frac{1}{4}+\varepsilon} + x^{\frac{1}{3}} T^{-1+\varepsilon} + x^{\frac{1}{3}+\varepsilon} T^{-1+\varepsilon} \\
& \ll x^{\frac{1}{3}+\varepsilon} T^{-\delta+\varepsilon}
\end{aligned}$$

where δ is very small normal number, $\delta > \varepsilon$.

$$I_2 \ll x^{\sigma_0}\Big(1 + \int_1^T |\zeta(3\sigma + i3T)|^9 |\zeta(4\sigma + i4T)|^{36} t^{-1} dt\Big).$$

According to the partial integral formula, we have

$$I_4 = \int_1^T |\zeta(3\sigma + i3T)|^9 |\zeta(4\sigma + i4T)|^{36} dt \ll T^{1+\varepsilon}.$$

If $p_i \ge 0$, $(i = 1, 2)$ are real numbers, and $\frac{1}{p_1} + \frac{1}{p_2} = 1$, by Hölder inequality, we have

$$I_4 \le \Big(\int_1^T |\zeta(3\sigma + i3T)|^{9p_1}\Big)^{\frac{1}{p_1}} \Big(\int_1^T |\zeta(4\sigma + i4T)|^{36p_2}\Big)^{\frac{1}{p_2}}.$$

So we have to prove

$$\begin{aligned}
& \int_1^T |\zeta(3\sigma + i3T)|^{9p_1} dt \ll T^{1+\varepsilon}, \\
& \int_1^T |\zeta(4\sigma + i4T)|^{36p_2} dt \ll T^{1+\varepsilon}.
\end{aligned}$$

Let $m(3\sigma_0) = 9p_1$, $m(4\sigma_0) = 36p_2$, since $\frac{9}{m(3\sigma_0)} + \frac{36}{m(4\sigma_0)} = 1$, and from Lemma 2.4, we have $\sigma_0 = \frac{3530376}{14646528} = 0.241038422\cdots$. □

§3. Proof of Theorem 1.1

Proof. Let

$$\zeta^9(3s)\zeta^{36}(4s)G(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}, \qquad \Re s > 1,$$

$$\zeta^9(3s)\zeta^{36}(4s) = \sum_{n=1}^{\infty} \frac{d(3, \cdots, 3, 4 \cdots, 4; n)}{n^s},$$

such that

$$f(n)=\sum_{n=md} d(\underbrace{3,\cdots,3}_{9},\underbrace{4,\cdots,4}_{36};m)g(d) \tag{1}$$

From Lemma 2.1, we have $G(s)=\sum_{n=1}^{\infty}\frac{g(n)}{n^s}$ is absolutely convergent for $\Re s>\frac{1}{5}$, and then

$$\sum_{n\le x}|g(n)|\ll x^{\frac{1}{5}+\varepsilon}. \tag{2}$$

From Lemma 2.5, we have

$$\sum_{m\le x} d(\underbrace{3,\cdots,3}_{9},\underbrace{4,\cdots,4}_{36};m)=x^{\frac{1}{3}}P_{8,1}(\log x)+x^{\frac{1}{4}}P_{35,2}(\log x)+O(x^{\sigma_0+\varepsilon}), \tag{3}$$

where $P_{8,1}(t)$ is a polynomial of degree 8 in t, $P_{35,2}(t)$ is a polynomial of degree 35 in t, Combining (1), (2) and (3), and applying lemma 2.2, we have

$$\sum_{n\le x} f(n)=x^{\frac{1}{3}}Q_{8,1}(\log x)+x^{\frac{1}{4}}Q_{35,2}(\log x)+O(x^{\sigma_0+\varepsilon})$$

where $Q_{8,1}(t)$ is a polynomial of degree 8 in t, $Q_{35,2}(t)$ is a polynomial of degree 35 in t, From lemma 2.1,we have

$$(\tau_3^{(e)}(n))^2 f_3(n)=\sum_{n=md} d(\underbrace{3,\cdots,3}_{9},\underbrace{4,\cdots,4}_{36};m)g(d)=f(n)$$

Then we complete the proof of Theorem 1.1. □

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On the second order involute of a spacelike curve with timelike binormal in IL^3.

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Abstract We have already defined and worked on the second order involute curve of a unit speed curve in IL^3. In this paper, we consider the second order involute of a spacelike curve with timelike binormal in IL^3 . There are three kinds of casual caharacteristics of the second order involute curve. All Frenet apparatus of their are examined in terms of Frenet apparatus of the curve α.

Keywords Lorentz metric, involute curve, second order involute curve
2010 Mathematics Subject Classification 53A04.

§1. Introduction and preliminaries

Basic properties of involute-evolute curves are very famous studies in differantial geometry. In [5], [6] and [7] the second order involute curves , the second order Mannheim partner curve and the nˆth order Bertrand mate curves in Euclidean 3-space are examined, respectively. In Lorenzt space there are two kind of non-null curve, which are timelike and spacelike. The involutes of the spacelike Curve with a timelike binormal spacelike binormal are examined in [1] and [2], respectively. In this study we will work on the second order involute curves in of a spacelike curve with timelike binormal in Lorenzt 3-space.

$$\langle X, Y\rangle = -x_1y_2 + x_2y_2 + x_3y_3 \tag{1}$$

is known Lorentz metric with index one, and $\left\{IR^3, \langle,\rangle\right\}$ is 3-dimensional Lorentz space with notation IL^3. For $X \in IL^3$ the casual characteristics of any vector X, are if $\langle X, X\rangle > 0$, X is spacelike vector, if $\langle X, X\rangle < 0$, X is timelike vector, if $\langle X, X\rangle = 0$, X is lightlike or null vector. $\|X\| = \sqrt{|\langle X, X\rangle|}$ is norm of X , [9]. Vectorel product of X and Y is

$$X\Lambda Y = (x_3y_2 - x_2y_3, x_1y_3 - x_3y_1, x_1y_2 - x_2y_1). \tag{2}$$

Let $\alpha : I \to \mathbb{E}^3$ be the C^2- class differentiable unit speed curve denote by $\{T, N, B\}$ the moving Frenet frame. For an arbitrary curve $\alpha \in \mathbb{E}^3$, with first and second curvature, κ and

τ respectively. Differential curve with Frenet frame, if tangent vector T is timelike (spacelike) vector is called timelike (spacelike) curve.

• **Frenet formulaes of a timelike curve are**

$$T' = \kappa N , \quad N' = \kappa T - \tau B , \quad B' = \tau N \tag{3}$$

and

$$T \wedge N = -B , \quad N \wedge B = T , \quad B \wedge T = -N.$$

Darboux vector is

$$W = \tau T - \kappa B , \quad \|W\| = \kappa^2 - \tau^2$$

see in , [9]. For any unit speed curve $\alpha : I \to \mathbb{E}^3$, the vector W is called Darboux vector defined by [3] $W = \tau T + \kappa B$. If we consider the normalization of the unit Darboux vector $C = \frac{W}{\|W\|}$, we can write. Let the angle between Darboux vector and binormal vector of first timelike curve be φ and since B is spacelike,

If $|\kappa| > |\tau|$ then, W is spacelike vector and

$$\kappa = \|W\| \cosh\varphi , \quad \tau = \|W\| \sinh\varphi$$

If $|\kappa| < |\tau|$then, is W is timelike vector and

$$\kappa = \|W\| \sinh\varphi , \quad \tau = \|W\| \cosh\varphi$$

•• **Frenet formulaes of spacelike curve with timelike binormal are**

$$T' = \kappa N , \quad N' = -\kappa T + \tau B , \quad B' = \tau N \tag{4}$$

and

$$T \wedge N = B , \quad N \wedge B = -T , \quad B \wedge T = N.$$

Darboux vector is $W = \tau T - \kappa B$, see in [9]. Since B is timelike;

if $|\kappa| < |\tau|$ then W is spacelike vector

$$\kappa = \|W\| \sinh\varphi , \quad \tau = \|W\| \cosh\varphi , \quad \|W\|^2 = \tau^2 - \kappa^2.$$

If $|\kappa| > |\tau|$ then W is timelike vector

$$\kappa = \|W\| \cosh\varphi , \quad \tau = \|W\| \sinh\varphi , \quad \|W\|^2 = \kappa^2 - \tau^2$$

see in [9].

••• **Frenet formulaes of a spacelike curve with timelike normal vector**

$$T' = \kappa N , \quad N' = \kappa T + \tau B , \quad B' = \tau N \tag{5}$$

and

$$T \wedge N = -B\ ,\quad N \wedge B = -T\ ,\quad B \wedge T = N.$$

Darboux vector is $W = -\tau T + \kappa B$. Since B is spacelike,

If $|\kappa| < |\tau|$ then is W timelike vector and

$$\kappa = \|W\| \sinh\varphi\ ,\quad \tau = \|W\| \cosh\varphi$$

If $|\kappa| > |\tau|$ then W is spacelike vector then is timelike vector

$$\kappa = \|W\| \cosh\varphi\ ,\quad \tau = \|W\| \sinh\varphi.$$

The involute of a given curve is a well-known concept in Euclidean $3-space$. We can say that evolute and involute is a method of deriving a new curve based on a given curve. The involute of the curve is called sometimes *the evolvent*. Involvents play a part in the construction of gears. The evolute is the locus of the centers of tangent circles of the given planar curve [8]. Here, we will work on the second order involute of *spacelike* evolute curve with timelike binormal. Let $\alpha : I \to IL^3$be a *spacelike* evolute curve with timelike binormal. If tangent vector of the curve $\alpha_1 : I \to IL^3$ is perpendicular to tangent vector of the curve $\alpha : I \to IL^3$, then $\alpha_1 : I \to IL^3$ is the involute curve of spacelike curve α, and we have the equation,

$$\alpha_1(s) = \alpha(s) + \lambda(s)T(s),\quad \lambda(s) = c - s,\quad [4] \tag{6}$$

where $c = constant$. Also $< T, T_1 >= 0$ *and* $T_1 = N$.

Theorem 1.1. Frenet-Serret apparatus $\{T_1, N_1, B_1, \kappa_1, \tau_1\}$ of involute curve α_1 of a *spacelike* evolute curve α, with timelike binormal , are given based on the Frenet-Serret apparatus $\{T, N, B, \kappa, \tau\}$ of evolute curve are

$$T_1 = N\ ,\ N_1 = \frac{-\kappa}{\sqrt{|\tau^2-\kappa^2|}}T + \frac{\tau}{\sqrt{|\tau^2-\kappa^2|}}B\ ,\ B_1 = \frac{-\tau}{\sqrt{|\tau^2-\kappa^2|}}T + \frac{\kappa}{\sqrt{|\tau^2-\kappa^2|}}B. \tag{7}$$

The curvatures of curve α and the involute α_1, respectively are

$$\kappa_1 = \frac{\epsilon_0\sqrt{\kappa^2-\tau^2}}{|(c-s)\kappa|}\ ,\quad \tau_1 = \frac{\kappa\tau' - \kappa'\tau}{|(c-s)\kappa|\,|\tau^2-\kappa^2|},\quad \epsilon_0 = \begin{cases} +1 & N_1 \text{ is space like} \\ -1 & N_1 \text{ is time like} \end{cases} \tag{8}$$

see in [1].

Theorem 1.2. Frenet-Serret apparatus $\{T_1, N_1, B_1, \kappa_1, \tau_1\}$ of involute curve α_1 of a spacelike evolute curve α, with spacelike binormal, are given based on the Frenet-Serret apparatus $\{T, N, B, \kappa, \tau\}$ of evolute curve are

$$T_1 = N\ ,\ N_1 = \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}T - \frac{\tau}{\sqrt{\kappa^2+\tau^2}}B\ ,\ B_1 = \frac{-\tau}{\sqrt{\kappa^2+\tau^2}}T - \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}B. \tag{9}$$

The curvatures of curve α and the involute, α_1, respectively, are

$$\kappa_1 = \frac{\sqrt{\kappa^2+\tau^2}}{|(k-s)\kappa|}\ ,\quad \tau_1 = \frac{\kappa\tau' - \kappa'\tau}{|(k-s)\kappa|\,\sqrt{\kappa^2+\tau^2}}. \tag{10}$$

§2. Second order involute of a spacelike curve with timelike binormal

Let $\alpha_2(s)$ be the involute of the curve $\alpha_1(s)$. $\{T_1, N_1, B_1, \kappa_1, \tau_1\}$ and $\{T_2, N_2, B_2, \kappa_2, \tau_2\}$ are collectively Frenet-Serret apparatus of the curve α_1 and the involute α_2, respectively. α_1 has the parametrization with arclength s_1 as the involute curve of $\alpha(s)$. $\alpha_2(s) = \alpha_1(s) + \lambda_1 T_1(s)$ is the parametrization of second order involute curve. Hence, we can write

$$\alpha_2(s) = \alpha(s) + \lambda(s)T(s) + \lambda_1(s)N(s) \tag{11}$$

where it is given in terms of Frenet apparatus of evolute α, also λ_2 is constant.

$$< T_1, T_2 >= 0 \quad and \quad T_2 = N_1$$

see in [5].

Theorem 2.1. Involute and second involute curve of a *spacelike* evolute curve with *timelike* normal N or timelike binormal B , has the casual characteristics as in the following forms. Let $\{T, N, B, \kappa, \tau\}$, $\{T_1, N_1, B_1, \kappa_1, \tau_1\}$ and $\{T_2, N_2, B_2, \kappa_2, \tau_2\}$ are collectively Frenet apparatus of the evolute curve α, the involute α_1 and the second order involute α_2, respectively.

$$\begin{array}{ccc} evolute & involute & 2^{nd}involute \\ & sst & < \begin{matrix} sst \\ sts \end{matrix} \\ sst & sts & \to tss \end{array} \tag{12}$$

Proof. For a *spacelike* evolute curve with *timelike* binormal and *spacelike* principal normal, hence

$$\begin{array}{ccc} T \;\; spacelike & N \;\; spacelike & B \;\; timelike \\ s & s & t \end{array} \tag{13}$$

Since $< T, T_1 >= 0$ and $T_1 = N$ (*spacelike*), T_2 must be *spacelike*. Hence the involute of a spacelike curve with *timelike* binormal is always spacelike curve. So normal N_1 or binormal B_1 must be *timelike*, as in the following way,

$$\begin{array}{ccc} Tangent \; T_1 & Normal \; N_1 & Binormal \; B_1 \\ s & s & t \\ s & t & s \end{array} \tag{14}$$

and *spacelike* involute with timelike binormal B_1, there are the following forms;

$$\begin{array}{ccc} evolute & involute & 2^{nd}involute \\ sst & sst & < \begin{matrix} sst \\ sts \end{matrix} \end{array} \tag{15}$$

For a *spacelike* evolute curve with *timelike* normal N, binormal B is *spacelike*.

$$\begin{array}{ccc} T\ spacelike & N\ timelike & B\ spacelike \\ s & t & s \end{array} \tag{16}$$

Since $<T, T_1>= 0$ *and* $T_1 = N$ (*timelike*). It is trivial that T_1 must be *timelike*.The involute of a spacelike curve with *timelike* normal is always timelike curve.

$$\begin{array}{ccc} T_1\ timelike & N_1\ spacelike & B_1\ spacelike \\ t & s & s \end{array} \tag{17}$$

Hence a *spacelike* evolute curve with *timelike* normal N, has the casual characteristics as in the following form

$$\begin{array}{cc} evolute & involute \\ sts \rightarrow & tss \end{array}$$

as a result, we have the proof. □

Theorem 2.2. Frenet apparatus of *second order involute* α_2 of a curve α can be given in terms of Frenet apparatus of α,

$$\begin{aligned}
T_2 &= \frac{-\kappa}{\sqrt{|\tau^2-\kappa^2|}}T + \frac{\tau}{\sqrt{|\tau^2-\kappa^2|}}B \\
N_2 &= \frac{-\dfrac{\sqrt{\kappa^2-\tau^2}}{|(c-s)\kappa|}}{\sqrt{|\tau_1^2-\kappa_1^2|}}N + \frac{\dfrac{\kappa\tau'-\kappa'\tau}{|(c-s)\kappa|\,|\tau^2-\kappa^2|}}{\sqrt{|\tau_1^2-\kappa_1^2|}}\frac{(-\tau T+\kappa B)}{\sqrt{|\tau^2-\kappa^2|}}, \begin{cases} \text{if } |\kappa| > |\tau|, \text{ then } N_1 \text{ spacelike} \\ \text{if } |\kappa| < |\tau|, \text{ then } N_1 \text{ timelike} \end{cases} \\
B_2 &= \frac{-\dfrac{\kappa\tau'-\kappa'\tau}{|(c-s)\kappa|\,|\tau^2-\kappa^2|}}{\sqrt{|\tau_1^2-\kappa_1^2|}}N + \frac{\dfrac{\sqrt{\kappa^2-\tau^2}}{|(c-s)\kappa|}}{\sqrt{|\tau_1^2-\kappa_1^2|}}\frac{(-\tau T+\kappa B)}{\sqrt{|\tau^2-\kappa^2|}}, \begin{cases} \text{if } |\kappa| > |\tau|, \text{ then } B_1 \text{ timelike} \\ \text{if } |\kappa| < |\tau|, \text{ then } B_1 \text{ spacelike} \end{cases}
\end{aligned}$$

for

$$\begin{array}{ccc} evolute & involute & 2^{nd}involute \\ sst & sst & < \begin{array}{c} sst \\ sts \end{array} \end{array}$$

Proof. Since for T_1 spacelike, N_1spacelike, B_1 timelike and T_2 spacelike, N_2 timelike, B_2 spacelike we have already

$$T_2 = N_1$$

$$N_2 = \frac{-\kappa_1}{\sqrt{|\tau_1^2-\kappa_1^2|}}T_1 + \frac{\tau_1}{\sqrt{|\tau_1^2-\kappa_1^2|}}B_1 \begin{cases} \text{if } |\kappa_1| > |\tau_1|, \text{ then } N_2 \text{ spacelike} \\ \text{if } |\kappa_1| < |\tau_1|, \text{ then } N_2 \text{ timelike} \end{cases} \quad (18)$$

$$B_2 = \frac{-\tau_1}{\sqrt{|\tau_1^2-\kappa_1^2|}}T_1 + \frac{\kappa_1}{\sqrt{|\tau_1^2-\kappa_1^2|}}B_1, \begin{cases} \text{if } |\kappa_1| > |\tau_1|, \text{ then } B_2 \text{ timelike} \\ \text{if } |\kappa_1| < |\tau_1|, \text{ then } B_2 \text{ spacelike} \end{cases}$$

hence

$$T_2 = \frac{-\kappa T + \tau B}{\sqrt{|\tau^2-\kappa^2|}}$$

$$N_2 = \frac{-\dfrac{\sqrt{\kappa^2-\tau^2}}{|(c-s)\kappa|}}{\sqrt{|\tau_1^2-\kappa_1^2|}}N + \frac{\dfrac{\kappa\tau'-\kappa'\tau}{|(c-s)\kappa|\,|\tau^2-\kappa^2|}}{\sqrt{|\tau_1^2-\kappa_1^2|}}\frac{(-\tau T+\kappa B)}{\sqrt{|\tau^2-\kappa^2|}} \begin{cases} \text{if } |\kappa| > |\tau|, \text{ then } N_1 \text{ spacelike} \\ \text{if } |\kappa| < |\tau|, \text{ then } N_1 \text{ timelike} \end{cases}$$

$$B_2 = \frac{-\dfrac{\kappa\tau'-\kappa'\tau}{|(c-s)\kappa|\,|\tau^2-\kappa^2|}}{\sqrt{|\tau_1^2-\kappa_1^2|}}N + \frac{\dfrac{\sqrt{\kappa^2-\tau^2}}{|(c-s)\kappa|}}{\sqrt{|\tau_1^2-\kappa_1^2|}}\frac{(-\tau T+\kappa B)}{\sqrt{|\tau^2-\kappa^2|}}, \begin{cases} \text{if } |\kappa| > |\tau|, \text{ then } B_1 \text{ timelike} \\ \text{if } |\kappa| < |\tau|, \text{ then } B_1 \text{ spacelike} \end{cases}$$

$$\begin{aligned} \tau_1^2-\kappa_1^2 &= \left(\frac{\kappa\tau'-\kappa'\tau}{|(c-s)\kappa|\,|\tau^2-\kappa^2|}\right)^2 - \left(\frac{\sqrt{\kappa^2-\tau^2}}{|(c-s)\kappa|}\right)^2 \\ &= \frac{(\kappa\tau'-\kappa'\tau)^2}{|(c-s)\kappa|^2\,|\tau^2-\kappa^2|^2} - \frac{|\kappa^2-\tau^2|}{|(c-s)\kappa|^2} \\ &= \frac{(\kappa\tau'-\kappa'\tau)^2 - |\kappa^2-\tau^2|^3}{|(c-s)\kappa|^2\,|\tau^2-\kappa^2|^2}. \end{aligned}$$

The curvatures of curves , respectively, are

$$\kappa_2 = \frac{\epsilon_1\sqrt{\kappa_1^2-\tau_1^2}}{|(c_1-s)\kappa_1|}, \quad \tau_2 = \frac{\kappa_1\tau_1'-\kappa_1'\tau_1}{|(c_1-s)\kappa_1|\,|\tau_1^2-\kappa_1^2|}, \quad \epsilon_1 = \begin{cases} +1, & N_2 \text{ is space like} \\ -1, & N_2 \text{ is time like} \end{cases} \quad (19)$$

□

Theorem 2.3. Frenet-Serret apparatus $\{T_1, N_1, B_1, \kappa_1, \tau_1\}$ of involute curve α_1, are given based on the Frenet apparatus $\{T, N, B, \kappa, \tau\}$ of evolute curve α;

$$\begin{cases} T_2 = \dfrac{-\kappa}{\sqrt{|\tau^2-\kappa^2|}}T + \dfrac{\tau}{\sqrt{|\tau^2-\kappa^2|}}B \\ \\ N_2 = \dfrac{\kappa 1}{\sqrt{\kappa_1^2+\tau_1^2}}N - \dfrac{\tau_1}{\sqrt{\kappa_1^2+\tau_1^2}}\dfrac{-\tau}{\sqrt{|\tau^2-\kappa^2|}}T + \dfrac{\kappa}{\sqrt{|\tau^2-\kappa^2|}}B \\ \\ B_2 = \dfrac{-\tau_1}{\sqrt{\kappa_1^2+\tau_1^2}}N - \dfrac{\kappa_1}{\sqrt{\kappa_1^2+\tau_1^2}}\dfrac{-\tau}{\sqrt{|\tau^2-\kappa^2|}}T + \dfrac{\kappa}{\sqrt{|\tau^2-\kappa^2|}}B, \end{cases} \tag{20}$$

for

$$\begin{array}{ccc} evolute & involute & 2^{nd} involute \\ sst & sts & \rightarrow tss. \end{array}$$

Proof. Frenet apparatus $\{T_1, N_1, B_1, \kappa_1, \tau_1\}$ of involute curve α_1, are given then [8] based on the Frenet apparatus $\{T, N, B, \kappa, \tau\}$ of evolute curve α; for

$$\begin{array}{cc} evolute & involute \\ sst & sts \end{array}$$

$$\begin{cases} T_1 = N, \\ \\ N_1 = \dfrac{-\kappa}{\sqrt{|\tau^2-\kappa,^2|}}T + \dfrac{\tau}{\sqrt{|\tau^2-\kappa^2|}}B, & \begin{cases} \text{if } |\kappa| > |\tau|, \text{ then } N_1 \text{ spacelike} \\ \text{if } |\kappa| < |\tau|, \text{ then } N_1 \text{ timelike} \end{cases} \\ \\ B_1 = \dfrac{-\tau}{\sqrt{|\tau^2-\kappa^2|}}T + \dfrac{\kappa}{\sqrt{|\tau^2-\kappa^2|}}B, & \begin{cases} \text{if } |\kappa| > |\tau|, \text{ then } B_1 \text{ timelike} \\ \text{if } |\kappa| < |\tau|, \text{ then } B_1 \text{ spacelike.} \end{cases} \end{cases} \tag{21}$$

The curvatures of curve α and the involute α_1, respectively are

$$\kappa_1 = \frac{\epsilon_0\sqrt{\kappa^2-\tau^2}}{|(c-s)\kappa|}, \quad \tau_1 = \frac{\kappa\tau' - \kappa'\tau}{|(c-s)\kappa|\,|\tau^2-\kappa^2|}, \quad \epsilon_0 = \begin{cases} +1, & N_1 \text{ is space like} \\ -1, & N_1 \text{ is time like} \end{cases} \tag{22}$$

also

$$\kappa_1^2 + \tau_1^2 = \left(\frac{\epsilon_0\sqrt{\kappa^2-\tau^2}}{|(c-s)\kappa|}\right)^2 + \left(\frac{\kappa\tau' - \kappa'\tau}{|(c-s)\kappa|\,|\tau^2-\kappa^2|}\right)^2,$$

$$\sqrt{\kappa_1^2 + \tau_1^2} = \frac{\sqrt{|\kappa^2-\tau^2|^3 + (\kappa\tau' - \kappa'\tau)^2}}{|(c-s)\kappa|\,|\tau^2-\kappa^2|}.$$

For

$$\begin{array}{cc} involute & 2^{nd}\ involute \\ sts & tss. \end{array}$$

Since

$$T_2 = N_1, N_2 = \frac{\kappa 1}{\sqrt{\kappa_1^2+\tau_1^2}}T_1 - \frac{\tau_1}{\sqrt{\kappa_1^2+\tau_1^2}}B_1, B_2 = \frac{-\tau_1}{\sqrt{\kappa_1^2+\tau_1^2}}T_1 - \frac{\kappa_1}{\sqrt{\kappa_1^2+\tau_1^2}}B_1,$$

we have

$$\begin{aligned} T_2 &= \frac{-\kappa}{\sqrt{|\tau^2-\kappa^2|}}T + \frac{\tau}{\sqrt{|\tau^2-\kappa^2|}}B, \\ N_2 &= \frac{\kappa 1}{\sqrt{\kappa_1^2+\tau_1^2}}N - \frac{\tau_1}{\sqrt{\kappa_1^2+\tau_1^2}}\frac{-\tau}{\sqrt{|\tau^2-\kappa^2|}}T + \frac{\kappa}{\sqrt{|\tau^2-\kappa^2|}}B, \\ B_2 &= \frac{-\tau_1}{\sqrt{\kappa_1^2+\tau_1^2}}N - \frac{\kappa_1}{\sqrt{\kappa_1^2+\tau_1^2}}\frac{-\tau}{\sqrt{|\tau^2-\kappa^2|}}T + \frac{\kappa}{\sqrt{|\tau^2-\kappa^2|}}B. \end{aligned}$$

The curvatures of the second order curve α_2 based on the involute α_1, respectively are

$$\kappa_2 = \frac{\sqrt{|\kappa^2-\tau^2|^3 + (\kappa\tau' - \kappa'\tau)^2}}{|(c-s)\kappa|\,|\tau^2-\kappa^2|\,|(m-s)\kappa_1|}, \quad \tau_2 = \frac{(\kappa_1\tau_1' - \kappa_1'\tau_1)\,|(c-s)\kappa|\,\left|\tau^2-\kappa^2\right|}{|(m-s)\kappa_1|\sqrt{|\kappa^2-\tau^2|^3 + (\kappa\tau'-\kappa'\tau)^2}}.$$

Since $\kappa_2 = \dfrac{\sqrt{\kappa_1^2+\tau_1^2}}{|(m-s)\kappa_1|}$ and $\tau_2 = \dfrac{\kappa_1\tau_1' - \kappa_1'\tau_1}{|(m-s)\kappa_1|\sqrt{\kappa_1^2+\tau_1^2}}$ it is trivial. □

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On several types of generalized regular fuzzy continuous functions

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Abstract In this paper, we introduce the concept of slightly regular fuzzy continuous, slightly generalized regular fuzzy continuous and somewhat slightly generalized regular fuzzy continuous functions in fuzzy topological spaces in the sense of Šostak's. Several interesting properties and characterizations are introduced and discussed. Furthermore, the relationship among the new concepts are introduced and established with some interesting counter examples.

Keywords Generalized regular fuzzy closed sets, generalized regular fuzzy continuous, slightly regular fuzzy continuous, slightly generalized regular fuzzy continuous and somewhat slightly generalized regular fuzzy continuous functions.
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§1. Introduction

Kubiak [11] and Šostak [18] introduced the fundamental concept of a fuzzy topological structure, as an extension of both crisp topology and fuzzy topology [3], in the sense that not only the objects are fuzzified, but also the axiomatics. In [19, 20], Šostak gave some rules and showed how such an extension can be realized. Chattopadhyay et al., [5] have redefined the same concept under the name gradation of openness. A general approach to the study of topological type structures on fuzzy power sets was developed in [7–9, 11, 12]. Balasubramanian and Sundaram [1] gave the concept of generalized fuzzy closed sets in Chang's fuzzy topology as an extension of generalized closed sets of Levine [13] in topological spaces.

Jin Han Park and Jin Keun Park [16] introduced weaker form of generalized fuzzy closed set and generalized fuzzy continuous mappings i.e, regular generalized fuzzy closed set and generalizations of fuzzy continuous functions. Bhattacharya and Chakraborty [2] introduced another generalization of fuzzy closed set i.e, generalized regular fuzzy closed set which is the stronger form of the previous two generalizations. Recently, Vadivel and Elavarasan [23] introduced the concepts of r-generalized regular fuzzy closed sets in fuzzy topological spaces in the sense of Šostak.

In 1980, jain [10] introduced the notion of slightly continuous functions. Recently, Nour [14] defined slightly semicontinuous functions as a weak form of slightly continuity and investigated its properties. On the other hand, Takashi Noiri [15] introduced the concept of slightly β-continuous functions. In 2004, Ekici and Caldas [6] introduced the notion of slightly γ-continuity (slightly b-continuity). After that slightly fuzzy ω-continuous functions and slightly fuzzy continuous functions are introduced by sudha et al. [21,22]. Recently, [23] introduced the concepts of r-generalized regular fuzzy closed sets, generalized regular fuzzy continuous functions and generalized regular fuzzy irresolute functions and investigate interrelation between them.

In this paper, we introduce the concept of slightly regular fuzzy continuous, slightly generalized regular fuzzy continuous and somewhat slightly generalized regular fuzzy continuous functions in fuzzy topological spaces in the sense of Šostak's. Several interesting properties and characterizations are introduced and discussed. Furthermore, the relationship among the new concepts are introduced and established with some interesting counter examples.

§2. Preliminaries

Throughout this paper, let X be a nonempty set, $I = [0,1]$ and $I_0 = (0,1]$. For $\lambda \in I^X$, $\overline{\lambda}(x) = \lambda$ for all $x \in X$. For $x \in X$ and $t \in I_0$, a fuzzy point x_t is defined by

$$x_t(y) = \begin{cases} t & \text{if } y = x \\ 0 & \text{if } y \neq x. \end{cases}$$

Let $Pt(X)$ be the family of all fuzzy points in X. A fuzzy point $x_t \in \lambda$ iff $t < \lambda(x)$. All other notations and definitions are standard, for all in the fuzzy set theory.

Definition 2.1. *[18] A function $\tau : I^X \to I$ is called a fuzzy topology on X if it satisfies the following conditions:*

(O1) $\tau(\overline{0}) = \tau(\overline{1}) = 1$,

(O2) $\tau(\bigvee_{i\in\Gamma} \mu_i) \geq \bigwedge_{i\in\Gamma} \tau(\mu_i)$, *for any* $\{\mu_i\}_{i\in\Gamma} \subset I^X$,

(O3) $\tau(\mu_1 \wedge \mu_2) \geq \tau(\mu_1) \wedge \tau(\mu_2)$, *for any* $\mu_1,\ \mu_2 \in I^X$.

The pair $(X,\ \tau)$ is called a fuzzy topological space (for short, fts). A fuzzy set λ is called an r-fuzzy open (r-fo, for short) if $\tau(\lambda) \geq r$. A fuzzy set λ is called an r-fuzzy closed (r-fc, for short) set iff $\overline{1} - \lambda$ is an r-fo set.

Theorem 2.1. *[4] Let (X, τ) be a fts. Then for each $\lambda \in I^X$ and $r \in I_0$, we define an operator $C_\tau : I^X \times I_0 \to I^X$ as follows: $C_\tau(\lambda,\ r) = \bigwedge\{\mu \in I^X : \lambda \leq \mu,\ \tau(\overline{1} - \mu) \geq r\}$. For $\lambda,\ \mu \in I^X$ and $r, s \in I_0$, the operator C_τ satisfies the following statements:*

(C1) $C_\tau(\overline{0}, r) = \overline{0}$,

(C2) $\lambda \leq C_\tau(\lambda,\ r)$,

(C3) $C_\tau(\lambda,\ r) \vee C_\tau(\mu,\ r) = C_\tau(\lambda \vee \mu,\ r)$,

(C4) $C_\tau(\lambda,\ r) \le C_\tau(\lambda,\ s)$ *if* $r \le s$,

(C5) $C_\tau(C_\tau(\lambda,\ r),\ r) = C_\tau(\lambda,\ r)$.

Theorem 2.2. *[4] Let* (X, τ) *be a fts. Then for each* $\lambda \in I^X$ *and* $r \in I_0$, *we define an operator* $I_\tau : I^X \times I_0 \to I^X$ *as follows:* $I_\tau(\lambda,\ r) = \bigvee\{\mu \in I^X : \mu \le \lambda,\ \tau(\mu) \ge r\}$. *For* $\lambda,\ \mu \in I^X$ *and* $r, s \in I_0$, *the operator* I_τ *satisfies the following statements:*

(I1) $I_\tau(\overline{1}, r) = \overline{1}$,

(I2) $I_\tau(\lambda,\ r) \le \lambda$,

(I3) $I_\tau(\lambda,\ r) \wedge I_\tau(\mu,\ r) = I_\tau(\lambda \wedge \mu,\ r)$,

(I4) $I_\tau(\lambda,\ r) \le I_\tau(\lambda,\ s)$ *if* $s \le r$,

(I5) $I_\tau(I_\tau(\lambda,\ r),\ r) = I_\tau(\lambda,\ r)$.

(I6) $I_\tau(\overline{1} - \lambda, r) = \overline{1} - C_\tau(\lambda, r)$ *and* $C_\tau(\overline{1} - \lambda, r) = \overline{1} - I_\tau(\lambda,\ r)$

Definition 2.2. *[17] Let* $(X,\ \tau)$ *be a fts,* $\lambda \in I^X$ *and* $r \in I_0$. *Then*

(1) a fuzzy set λ *is called r-fuzzy regular open (for short, r-fro) if* $\lambda = I_\tau(C_\tau(\lambda, r), r)$.

(2) a fuzzy set λ *is called r-fuzzy regular closed (for short, r-frc) if* $\lambda = C_\tau(I_\tau(\lambda, r), r)$.

(3) a fuzzy set λ *is called r-fuzzy regular clopen (for short, r-frco) set iff* λ *is r-frc set and r-fro set.*

Definition 2.3. *[23] Let* $f : (X,\ \tau) \to (Y,\ \sigma)$ *be a function and* $r \in I_0$. *Then* f *is called:*

(1) fuzzy regular continuous (for short, fr-continuous) if $f^{-1}(\lambda)$ *is r-fro set in* I^X *for each* $\lambda \in I^Y$ *with* $\sigma(\lambda) \ge r$.

(2) fuzzy regular open (for short, fr-open) if $f(\lambda)$ *is r-fro set in* I^Y *for each* $\lambda \in I^X$ *with* $\tau(\lambda) \ge r$.

(3) fuzzy regular closed (for short, fr-closed) if $f(\lambda)$ *is r-frc set in* I^Y *for each* $\lambda \in I^X$ *with* $\tau(\overline{1} - \lambda) \ge r$.

Definition 2.4. *[23] Let* $(X,\ \tau)$ *be a fts. For* $\lambda, \mu \in I^X$ *and* $r \in I_0$.

(1) The r-fuzzy regular closure of λ, *denoted by* $RC_\tau(\lambda, r)$, *and is defined by* $RC_\tau(\lambda, r) = \bigwedge\{\mu \in I^X | \mu \ge \lambda, \mu \text{ is r-frc }\}$.

(2) The r-fuzzy regular interiror of λ, *denoted by* $RI_\tau(\lambda, r)$, *and is defined by* $RI_\tau(\lambda, r) = \bigvee\{\mu \in I^X | \mu \le \lambda, \mu \text{ is r-fro }\}$.

Definition 2.5. *[23] Let* $(X,\ \tau)$ *be a fts. For any* $\lambda,\ \mu \in I^X$ *and* $r \in I_0$.

(1) A fuzzy set λ *is called r-generalized regular fuzzy closed (for short, r-grfc) set if* $RC_\tau(\lambda, r) \le \mu$, *whenever* $\lambda \le \mu$ *and* $\tau(\mu) \ge r$.

(2) A fuzzy set λ is called r-generalized regular fuzzy open (for short, r-grfo) set if $\overline{1} - \lambda$ is r-grfc.

(3) A fuzzy set λ is called r-generalized regular fuzzy clopen (for short, r-grfco) set iff λ is r-grfc set and r-grfo set.

Definition 2.6. *[23] Let $(X,\ \tau)$ be a fts. For $\lambda, \mu \in I^X$ and $r \in I_0$.*

(1) The r-generalized regular fuzzy closure of λ, denoted by $GRC_\tau(\lambda, r)$ and is defined by $GRC_\tau(\lambda, r) = \bigwedge\{\mu \in I^X |\ \lambda \leq \mu,\ \mu$ is r-grfc $\}$.

(2) The r-generalized regular fuzzy interiror of λ, denoted by $GRI_\tau(\lambda, r)$ and is defined by $GRI_\tau(\lambda, r) = \bigvee\{\mu \in I^X |\ \lambda \geq \mu,\ \mu$ is r-grfo $\}$.

Definition 2.7. *[23] Let $(X,\ \tau)$ and $(Y,\ \eta)$ be a fts's. Let $f : (X,\ \tau) \to (Y,\ \eta)$ be a function.*

(1) f is called generalized regular fuzzy continuous (for short, grf-continuous) iff $f^{-1}(\mu)$ is r-grfc for each $\mu \in I^Y,\ r \in I_0$ with $\eta(\overline{1} - \mu) \geq r$.

(2) f is called generalized regular fuzzy open (for short, grf-open) iff $f(\lambda)$ is r-grfo for each $\lambda \in I^X,\ r \in I_0$ with $\tau(\lambda) \geq r$.

(3) f is called generalized regular fuzzy closed (for short, grf-closed) iff $f(\lambda)$ is r-grfc for each $\lambda \in I^X,\ r \in I_0$ with $\tau(\overline{1} - \lambda) \geq r$.

(4) generalized regular fuzzy irresolute (grfi, for short) if $f^{-1}(\mu)$ is an r-grfc set, for each r-grfc set $\mu \in I^Y,\ r \in I_0$.

Definition 2.8. *[22] Let $(D,\ \geq)$ be a directed set. Let X be an ordinary set and f be the collection of all fuzzy points in X. The function $S : D \to f$ is called a fuzzy net in X. In other words, a fuzzy net is a pair $(S,\ \geq)$ such that S is a function :$D \to f$ and $\geq$ direct the domain of S. For $n \in D$, $S(n)$ is often denoted by S_n and hence a net S is often denoted by $\{S_n : n \in D\}$.*

§3. Slightly regular fuzzy continuous functions

Definition 3.1. *Let $(X,\ \tau)$ and $(Y,\ \eta)$ be fts's. A function $f : (X,\ \tau) \to (Y,\ \eta)$ is called slightly regular fuzzy continuous (srfc, for short) if for each $\lambda \in I^X,\ \mu \in I^Y$ and $r \in I_0$ such that μ is an r-frco set and $f(\lambda) \leq \mu$, there exists r-fro set $\nu \in I^X,\ r \in I_0$, $\lambda \leq \nu$ and $f(\nu) \leq \mu$.*

Proposition 3.1. *Let $(X,\ \tau)$ and $(Y,\ \eta)$ be fts's. For the function $f : (X,\ \tau) \to (Y,\ \eta)$, the following statements are equivalent:*

(1) f is srfc function.

(2) $f^{-1}(\nu)$ is an r-fro set for each $\nu \in I^Y,\ r \in I_0$ such that ν is r-frco set.

(3) $f^{-1}(\nu)$ is an r-frc set for each $\nu \in I^Y,\ r \in I_0$ such that ν is r-frco set.

(4) $f^{-1}(\nu)$ *is an* r*-frco set for each* $\nu \in I^Y,\ r \in I_0$ *such that* ν *is* r*-frco set.*

(5) For each fuzzy set $\lambda \in I^X, r \in I_0$ *and for every fuzzy net* $\{S_n : n \in D\}$ *which converges to* λ*, the fuzzy net* $\{f(S_n) : n \in D\}$ *is eventually in each* r*-frco set* μ *with* $f(\lambda) \leq \mu$*.*

Proof. (1)⇒(2): Let $\nu \in I^Y, r \in I_0$ such that ν is r-frco set and let $\lambda \in I^X$ such that $\lambda \leq f^{-1}(\nu)$. Since ν is an r-frco set with $f(\lambda) \leq \nu$. By (1), there exists r-fro set $\mu \in I^X,\ r \in I_0$, $\lambda \leq \mu$ and $f(\mu) \leq \nu$. Hence $f^{-1}(\nu)$ is an r-fro set.

(2)⇒(3): Let $\nu \in I^Y, r \in I_0$ such that ν is r-frco set, then $\overline{1} - \nu$ is r-frco. By (2), $f^{-1}(\overline{1} - \nu) = \overline{1} - f^{-1}(\nu)$ is r-fro set in X, thus $f^{-1}(\nu)$ is r-frc set in X.

(3)⇒(4): It is obvious from (2) and (3).

(4)⇒(5): Let $\{S_n : n \in D\}$ be a fuzzy net converges to the r-frco set $\lambda \in I^X$ and let $\mu \in I^Y$ be an r-frco set such that $f(\lambda) \leq \mu$. By using (3), there exist an r-frco set $\nu \in I^X,\ r \in I_0$ such that $\lambda \leq \nu$ and $f(\nu) \leq \mu$. Since the fuzzy net $\{S_n : n \in D\}$ converges to λ, $S_n \leq \lambda \leq \nu$. Now $S_n \leq \lambda \leq \nu$. Thus $f(S_n) \leq f(\nu) \leq \mu$. Hence $\{f(S_n) : n \in D\}$ is eventually in each r-frco set μ.

(5)⇒(1): Suppose that f is not srfc function. Then for every $\lambda \in I^X,\ \mu \in I^Y,\ r \in I_0$ such that μ is an r-frco set and $f(\lambda) \leq \mu$, there does not exist r-fro set $\nu \in I^X$ such that $\lambda \leq \nu$ and $f(\nu) \leq \mu$. Hence $f(S_n) \leq \mu$. That is, the fuzzy net $\{f(S_n) : n \in D\}$ is not eventually in an r-frco set μ with $f(\lambda) \leq \mu$, which is a contradiction. Hence f is srfc function. □

Proposition 3.2. *Let* $(X,\ \tau_1)$*,* $(Y,\ \tau_2)$ *and* $(Z,\ \tau_3)$ *be fts's. For the function* $f : (X,\ \tau_1) \to (Y,\ \tau_2)$ *and* $g : (Y,\ \tau_2) \to (Z,\ \tau_3)$*, the following statements are satisfied:*

(1) If f *and* g *are srfc functions, then so is* $g \circ f$*.*

(2) If f *is a surjective fuzzy regular irresolute, fuzzy regular open function and* g *be any function, then* $g \circ f$ *is srfc function iff* g *is srfc.*

Proof. (1): Clear.

(2): Suppose that $g \circ f$ is srfc function, $\lambda \in I^Z,\ r \in I_0$ such that λ is an r-frco set. By using Proposition (2), $f^{-1}(g^{-1}(\lambda)) = (g \circ f)^{-1}(\lambda)$ is an r-fro set in I^X. Since f is fuzzy regular open, $g^{-1}(\lambda) = f(f^{-1}(g^{-1}(\lambda)))$ is an r-fro set. Therefore by Proposition , g is srfc function.

Conversely, let $\nu \in I^Z,\ r \in I_0$ such that ν an r-frco set. Since g is srfc function, $g^{-1}(\nu)$ is an r-fro set in I^Y and f is fuzzy regular irresolute function, $f^{-1}(g^{-1}(\nu)) = (g \circ f)^{-1}(\nu)$ is an r-fro set in I^X. Therefore by Proposition , $g \circ f$ is srfc function. □

Definition 3.2. *Let* $(X,\ \tau)$ *is said to be an* r*-fuzzy regular connected iff* $\overline{0}$ *and* $\overline{1}$ *are the only fuzzy sets which are both* r*-fro and* r*-frc.*

Proposition 3.3. *Let* $(X,\ \tau)$ *and* $(Y,\ \eta)$ *be fts's, and let* $f : (X,\ \tau) \to (Y,\ \eta)$ *be a function. If* $(Y,\ \eta)$ *is an* r*-fuzzy regular connected, then* f *is srfc function.*

Proof. Let $(Y,\ \eta)$ be an r-fuzzy regular connected space. Then $\overline{0}$ and $\overline{1}$ are the only r-frco sets. Since $f^{-1}(\overline{0})$ and $f^{-1}(\overline{1})$ are both r-fro in I^X. Hence by Proposition , f is srfc function. □

Proposition 3.4. *Let* $(X,\ \tau)$ *and* $(Y,\ \eta)$ *be fts's, and let* $f : (X,\ \tau) \to (Y,\ \eta)$ *be srfc function. If* $(X,\ \tau)$ *is an* r*-fuzzy regular connected, then so is* $(Y,\ \eta)$*.*

Proof. Suppose that $(Y,\ \eta)$ be an r-fuzzy regular disconnected space and $\nu \in I^Y - \{\overline{0}, \overline{1}\}$ be an r-frco set. Since f is srfc function, $f^{-1}(\nu)$ is an r-frco set which is contradiction. Hence $(Y,\ \eta)$ is an r-fuzzy regular connected. □

§4. Slightly generalized regular fuzzy continuous functions

Definition 4.1. *Let $(X,\ \tau)$ and $(Y,\ \eta)$ be fts's. A function $f : (X,\ \tau) \to (Y,\ \eta)$ is called:*

(1) almost ∗-generalized regular fuzzy continuous (a∗-grfc, for short) if for each $\lambda \in I^X,\ \mu \in I^Y,\ r \in I_0$ such that $\eta(\mu) \geq r$ and $f(\lambda) \leq \mu$, there exists an r-grfo set $\nu \in I^X$ such that $\lambda \leq \nu$ and $f(\nu) \leq I_\eta(C_\eta(\mu, r), r)$.

(2) θ∗-generalized regular fuzzy continuous (θ∗-grfc, for short) if for each $\lambda \in I^X,\ \mu \in I^Y,\ r \in I_0$ such that $\eta(\mu) \geq r$ and $f(\lambda) \leq \mu$, there exists an r-grfo set $\nu \in I^X$ such that $\lambda \leq \nu$ and $f(C_\tau(\nu, r)) \leq C_\eta(\mu, r)$.

(3) weakly∗-generalized regular fuzzy continuous (w∗-grfc, for short) if for each $\lambda \in I^X,\ \mu \in I^Y,\ r \in I_0$ such that $\eta(\mu) \geq r$ and $f(\lambda) \leq \mu$, there exists an r-grfo set $\nu \in I^X$ such that $\lambda \leq \nu$ and $f(\nu) \leq C_\eta(\mu, r)$.

(4) slightly generalized regular fuzzy continuous (sgrfc, for short) if for each $\lambda \in I^X,\ \mu \in I^Y,\ r \in I_0$ such that μ is an r-frco set and $f(\lambda) \leq \mu$, there exists an r-grfo set $\nu \in I^X$ such that $\lambda \leq \nu$ and $f(\nu) \leq \mu$.

Remark 4.1.

(1) Every a∗-grfc function is θ∗-grfc function.

(2) Every θ∗-grfc function is sgrfc (resp. w∗-grfc) function.

(3) Every sgrfc function is w∗-grfc function.

The above Definition and Remark show the following implication is true but the reverse implication is not true in general.

Example 4.1. *Let $X = Y = \{a,\ b,\ c\}$ and $f : (X,\ \tau) \to (Y,\ \eta)$ be the identity function. Define $\lambda,\ \delta \in I^X,\ \mu \in I^Y$ as follows: $\lambda(a) = 0.3,\ \lambda(b) = 0.4,\ \lambda(c) = 0.5$; $\mu(a) = 0.3,\ \mu(b) = 0.4,\ \mu(c) = 0.5$; $\delta(a) = 0.4,\ \delta(b) = 0.4,\ \delta(c) = 0.5$. We define a fuzzy topologies τ and η as follows:*

$$\tau(\lambda) = \begin{cases} 1 & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ \frac{1}{2} & \text{if } \lambda = \lambda, \\ 0 & \text{otherwise}, \end{cases} \qquad \eta(\lambda) = \begin{cases} 1 & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ \frac{1}{2} & \text{if } \lambda = \mu, \\ 0 & \text{otherwise}, \end{cases}$$

For $r = 1/2$, then f is $\theta*$-grfc function but not $a*$-grfc, because $\lambda \in I^X$, $\mu \in I^Y$, $r \in I_0$ such that $\eta(\mu) \geq r$ and $f(\lambda) \leq \mu$, there exists an r-grfo set $\delta \in I^X$ and $\lambda \leq \delta$ such that $f(C_\tau(\delta, r)) \leq C_\eta(\mu, r)$ but $f(\delta) \nleq I_\eta(C_\eta(\mu, r), r)$.

Example 4.2. *Let $X = Y = \{a, b, c\}$ and $f : (X, \tau) \to (Y, \eta)$ be the identity function. Define λ, $\delta \in I^X$, $\mu \in I^Y$ as follows: $\lambda(a) = 0.3$, $\lambda(b) = 0.4$, $\lambda(c) = 0.5$; $\mu(a) = 0.5$, $\mu(b) = 0.5$, $\mu(c) = 0.5$; $\delta(a) = 0.4$, $\delta(b) = 0.4$, $\delta(c) = 0.5$. We define a fuzzy topologies τ and η as follows:*

$$\tau(\lambda) = \begin{cases} 1 & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ \frac{1}{2} & \text{if } \lambda = \lambda, \\ 0 & \text{otherwise}, \end{cases} \qquad \eta(\lambda) = \begin{cases} 1 & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ \frac{1}{2} & \text{if } \lambda = \mu, \\ 0 & \text{otherwise}, \end{cases}$$

For $r = 1/2$, then f is $w*$-grfc function but not $\theta*$-grfc, because $\lambda \in I^X$, $\mu \in I^Y$, $r \in I_0$ such that $\eta(\mu) \geq r$ and $f(\lambda) \leq \mu$, there exists an r-grfo set $\delta \in I^X$ and $\lambda \leq \delta$ such that $f(\delta) \leq C_\eta(\mu, r)$ but $f(C_\tau(\delta, r)) \nleq C_\eta(\mu, r)$.

Example 4.3. *In Example , f is sgrfc function but not $\theta*$-grfc.*

Example 4.4. *Let $X = Y = \{a, b, c\}$ and $f : (X, \tau) \to (Y, \eta)$ be the identity function. Define λ, $\delta \in I^X$, $\mu \in I^Y$ as follows: $\lambda(a) = 0.3$, $\lambda(b) = 0.4$, $\lambda(c) = 0.5$; $\mu(a) = 0.4$, $\mu(b) = 0.6$, $\mu(c) = 0.5$; $\delta(a) = 0.4$, $\delta(b) = 0.4$, $\delta(c) = 0.5$. We define a fuzzy topologies τ and η as follows:*

$$\tau(\lambda) = \begin{cases} 1 & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ \frac{1}{2} & \text{if } \lambda = \lambda, \\ 0 & \text{otherwise}, \end{cases} \qquad \eta(\lambda) = \begin{cases} 1 & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ \frac{1}{2} & \text{if } \lambda = \mu, \\ 0 & \text{otherwise}, \end{cases}$$

For $r = 1/2$, then f is $w*$grfc function but not sgrfc, because $\lambda \in I^X$, $\mu \in I^Y$, $r \in I_0$ such that $\eta(\mu) \geq r$ and $f(\lambda) \leq \mu$, there exists an r-grfo set $\delta \in I^X$ and $\lambda \leq \delta$ such that $f(\delta) \leq C_\eta(\mu, r)$ but μ is not r-frco.

Proposition 4.1. *Let (X, τ) and (Y, η) be fts's. For the function $f : (X, \tau) \to (Y, \eta)$, the following statements are equivalent:*

(1) f is sgrfc function.

(2) $f^{-1}(\nu)$ is an r-grfo set for each $\nu \in I^Y, r \in I_0$ such that ν is r-grfco set.

(3) $f^{-1}(\nu)$ is an r-grfc set for each $\nu \in I^Y$, $r \in I_0$ such that ν is r-grfco set.

(4) $f^{-1}(\nu)$ is an r-grfco set for each $\nu \in I^Y, r \in I_0$ such that ν is r-grfco set.

(5) For each fuzzy set $\lambda \in I^X$, $r \in I_0$ and for every fuzzy net $\{S_n : n \in D\}$ with converges to λ, the fuzzy net $\{f(S_n) : n \in D\}$ is eventually in each r-grfco set μ with $f(\lambda) \leq \mu$.

Proof. (1)⇒(2): Let $\nu \in I^Y$, $r \in I_0$ such that ν is r-grfco set and let $\lambda \in I^X$ such that $\lambda \leq f^{-1}(\nu)$. Since ν is an r-grfco set with $f(\lambda) \leq \nu$. By (1), there exists $\mu \in I^X$ such that μ is an r-grfo, $\lambda \leq \mu$ and $f(\mu) \leq \nu$. Hence $f^{-1}(\nu)$ is an r-grfo set.

(2)⇒(3): Let $\nu \in I^Y, r \in I_0$ such that ν is r-grfco set, then $\overline{1} - \nu$ is r-grfco. By (2), $f^{-1}(\overline{1} - \nu) = \overline{1} - f^{-1}(\nu)$ is r-grfo set in X, thus $f^{-1}(\nu)$ is r-grfc set in X.

(3)⇒(4): It is obvious from (2) and (3).

(4)⇒(5): Let $\{S_n : n \in D\}$ be a fuzzy net converges to the r-grfco set $\lambda \in I^X$ and let $\mu \in I^Y$ be an r-grfco set such that $f(\lambda) \leq \mu$. By using (3), there exist an r-grfo set $\nu \in I^X$ such that $\lambda \leq \mu$ and $f(\nu) \leq \mu$. Since the fuzzy net $\{S_n : n \in D\}$ converges to λ, $S_n \leq \lambda \leq \nu$. Thus $\{f(S_n) : n \in D\}$ is eventually in each r-grfco set μ.

(5)⇒(1): Suppose that f is not sgrfc function. Then for every $\lambda \in I^X,\ \mu \in I^Y,\ r \in I_0$ such that μ is an r-grfo set and $f(\lambda) \leq \mu$, there does not exist $\nu \in I_X$ such that $\lambda \leq \nu$ and $f(\nu) \leq \mu$. Hence $f(S_n) \leq \mu$. That is the fuzzy net $\{f(S_n) : n \in D\}$ is not eventually in an r-grfco set μ with $f(\lambda) \leq \mu$, which is a contradiction. Hence f is sgrfc function. □

Proposition 4.2. *Let $(X,\ \tau_1)$, $(Y,\ \tau_2)$ and $(Z,\ \tau_3)$ be fts's. For the function $f : (X,\ \tau_1) \to (Y,\ \tau_2)$ and $g : (Y,\ \tau_2) \to (Z,\ \tau_3)$, the following statements are satisfied:*

(1) If f and g are sgrfc functions, then so is $g \circ f$.

(2) If f is a surjective grfi, grfo function and g be any function, then $g \circ f$ is sgrfc function iff g is sgrfc.

Proof. (1): is clear.

(2): Suppose that $g \circ f$ is sgrfc function, $\lambda \in I^Z$ is an r-grfco set. By using Proposition (2), $f^{-1}(g^{-1}(\nu)) = (g \circ f)^{-1}(\nu)$ is an r-grfo set in I^X. Since f is grfo, $g^{-1}(\lambda) = f(f^{-1}(g^{-1}(\lambda)))$ is an r-grfo set. Therefore by Proposition , g is sgrfc function.

Conversely, let $\nu \in I^Z$ be an r-grfco set where $r \in I_0$. Since g is sgrfc function, $g^{-1}(\nu)$ is an r-grfo set $\in I^Y$ and f is grfi function, $f^{-1}(g^{-1}(\nu)) = (g \circ f)^{-1}(\nu)$ is an r-grfo set $\in I^X$. Therefore by Proposition , $g \circ f$ is sgrfc function. □

Definition 4.2. *A fts $(X,\ \tau)$ is said to be an r-generalized regular fuzzy connected iff $\overline{0}$ and $\overline{1}$ are the only fuzzy sets which are both r-grfo and r-grfc.*

Proposition 4.3. *Let $(X,\ \tau)$ and $(Y,\ \eta)$ be fts's, and let $f : (X,\ \tau) \to (Y,\ \eta)$ be a function. If $(Y,\ \eta)$ is an r-generalized regular fuzzy connected, then f is sgrfc function.*

Proof. Let $(Y,\ \eta)$ be an r-generalized regular fuzzy connected space. Then $\overline{0}$ and $\overline{1}$ are the only r-grfco sets. Since $f^{-1}(\overline{0})$ and $f^{-1}(\overline{1})$ are both r-grfo in I^X. Hence by Proposition , f is sgrfc function. □

Proposition 4.4. *Let $(X,\ \tau)$ and $(Y,\ \eta)$ be fts's, and let $f : (X,\ \tau) \to (Y,\ \eta)$ be sgrfc function. If $(X,\ \tau)$ is an r-generalized regular fuzzy connected, then so is $(Y,\ \eta)$.*

Proof. Suppose that $(Y,\ \eta)$ be an r-generalized regular fuzzy disconnected space and $\nu \in I^Y - \{\overline{0}, \overline{1}\}$ be an r-grfco set. Since $f^{-1}(\nu)$ is an r-grfco set which is contradiction. Hence $(Y,\ \eta)$ is an r-generalized regular fuzzy connected. □

Definition 4.3. *A fts $(X,\ \tau)$ is said to be an r-generalized regular fuzzy extremely disconnected if $GRC_\tau(\lambda, r)$ is an r-grfo set for each $\lambda \in I^X, r \in I_0$ such that λ is an r-grfo set.*

Proposition 4.5. *Let $(X,\ \tau)$ and $(Y,\ \eta)$ be fts's. If $f : (X,\ \tau) \to (Y,\ \eta)$ be sgrfc function and $(Y,\ \eta)$ is an r-generalized regular fuzzy extremely disconnected, then f is a∗grfc function.*

Proof. $\lambda \in I^X, \mu \in I^Y, r \in I_0$ such that λ and μ are r-grfo sets. Since $(Y,\ \eta)$ is an r-generalized regular fuzzy extremely disconnected, $GRC_\eta(\mu, r)$ is an r-grfco set. Now, $f(\lambda) \le GRC_\eta(\mu, r)$ and since f is sgrfc function, there exists an r-grfo set $\nu \in I^X$ such that $\lambda \le \nu$ and $f(\nu) \le C_\eta(\mu, r)$. Therefore, f is a$*$gfc function. □

§5. Somewhat slightly generalized regular fuzzy continuous and open functions

Definition 5.1. *Let $(X,\ \tau)$ and $(Y,\ \eta)$ be fts's. A function $f : (X,\ \tau) \to (Y,\ \eta)$ is called somewhat slightly generalized regular fuzzy continuous (swsgrfc, for short) if for each $\lambda \in I^X$, $\mu \in I^Y$ and $r \in I_0$ such that $f^{-1}(\mu) \ne \overline{0}$ and $f(\lambda) \le \mu$, there exists an r-grfo set $\overline{0} \ne \nu \in I^X$ such that $\lambda \le \nu$ and $\nu \le f^{-1}(\mu)$.*

Remark 5.1.

(1) Evrey srf-continuous function is sgrf-continuous.

(2) Evrey srf-continuous (resp. sgrf-continuous) function is swsgrf-continuous.

The above Definitions , (4), and Remark show the following implication is true but the reverse implication is not true in general.

Example 5.1. *In Example , for $r = 1/2$, then f is sgrfc function but not srfc, because $\lambda \in I^X$, $\mu \in I^Y$, $r \in I_0$, such that $\eta(\mu) \ge r$ and $f(\lambda) \le \mu$, there exists an r-grfo set $\delta \in I^X$ and $\lambda \le \delta$ such that $f(\delta) \le \mu$ but δ is not r-fro set.*

Example 5.2. *In Example , for $r = 1/2$, then f is swsgrfc function but not sgrfc, because $\lambda \in I^X$, $\mu \in I^Y$, $r \in I_0$ such that $\eta(\mu) \ge r$ and $f(\lambda) \le \mu$, there exists an r-grfo set $\delta \in I^X$ and $\lambda \le \delta$ such that $f(\delta) \le \mu$ but μ is not r-frco.*

Example 5.3. *In Example , f is swsgrfc function but not srfc.*

Definition 5.2. *A fuzzy set λ in a fts $(X,\ \tau)$ is called r-generalized regular fuzzy dense (resp. r-fuzzy regular dense) set if there exists no r-grfc (resp. r-frco) set $\mu \in I^X$, $r \in I_0$ such that $\lambda < \mu < \overline{1}$.*

Example 5.4. *Let $X = \{a,\ b\}$. Define $\lambda,\ \mu \in I^X$ as follows: $\mu(a) = 0.9$, $\mu(b) = 0.9$. We define a fuzzy topology τ as follows:*

$$\tau(\lambda) = \begin{cases} 1 & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ \frac{1}{3} & \text{if } \lambda = \mu, \\ 0 & \text{otherwise.} \end{cases}$$

So, if $\lambda(a) = 0.9,\ \lambda(b) = 0.8$, then there exists no $1/3$-grfc set μ in I^X such that $\lambda < \mu < \overline{1}$. Therefore, λ is an $1/3$-generalized regular fuzzy dense set in I^X.

Example 5.5. *In Example , if $\lambda(a) = 0.8,\ \lambda(b) = 0.9$, then there exists no $1/3$-frco set μ in I^X such that $\lambda < \mu < \overline{1}$. Therefore, λ is an $1/3$-fuzzy regular dense set in I^X.*

Definition 5.3. *Let $(X,\ \tau)$ be a fts. For a fuzzy set $\lambda \in I^X,\ r \in I_0$, I^r_τ and C^r_τ are defined as follows:*

(1) $I^r_\tau = \bigvee\{\mu \in I^X \mid \mu \leq \lambda$ and μ is r-frco$\}$,

(2) $C^r_\tau = \bigwedge\{\mu \in I^X \mid \lambda \leq \mu$ and μ is r-frco$\}$.

Proposition 5.1. *Let $(X,\ \tau)$ and $(Y,\ \eta)$ be fts's, and let $f : (X,\ \tau) \to (Y,\ \eta)$ be any function. Then the following are equivalent:*

(1) f is swsgrfc function.

(2) If λ is an r-frco set such that $f^{-1}(\lambda) \neq \overline{1}$ and $\lambda \leq f(\overline{1} - \nu)$, for each $\nu \in I^X,\ r \in I_0$ then there exists an r-grfc set $\mu \leq \overline{1} - \nu \in I^X$ such that $\mu \geq f^{-1}(\lambda)$.

(3) If λ is r-generalized regular fuzzy dense set in I^X, then $f(\lambda)$ is r-fuzzy regular dense set in I^Y such that every r-frco set $\mu \leq f(\overline{1} - \nu)$, for each $\nu \in I^X$ and $r \in I_0$.

Proof. (1)⇒(2) Suppose f is swsgrfc function, and let λ be any r-frco set in I^Y such that $f^{-1}(\lambda) \neq \overline{1}$ and $\lambda \leq f(\overline{1} - \nu)$, for each $\nu \in I^X,\ r \in I_0$. Then, $\overline{1} - \lambda$ is r-frco in I^Y such that $f^{-1}(\overline{1} - \lambda) \neq \overline{0}$ and $f(\nu) \leq \overline{1} - \lambda$. Then by the hypothesis, there exists an r-grfo set $\overline{0} \neq \alpha \in I^X,\ r \in I_0$ such that $\nu \leq \alpha$ and $\alpha \leq f^{-1}(\overline{1} - \lambda)$. That is, $\overline{1} - \alpha$ is an r-grfc set and $\overline{1} - \alpha \geq \overline{1} - f^{-1}(\overline{1} - \lambda) = f^{-1}(\lambda)$. Put $\overline{1} - \alpha = \mu$. Then μ is an r-grfc set in I^X such that $\mu \geq f^{-1}(\lambda)$.

(2)⇒(3) Let λ be an r-generalized regular fuzzy dense set in I^X, and suppose that $f(\lambda)$ is not a fuzzy regular dense set in I^Y, such that each r-frco set $\mu \leq f(\overline{1} - \nu)$, for each $\nu \in I^X,\ r \in I_0$. Then, there exists an r-frco set $\alpha \in I^Y$ such that $f(\lambda) < \alpha < \overline{1}$, since $\alpha < \overline{1}$, $f^{-1}(\alpha) \neq \overline{1}$.

Now, α is an r-frco set such that $f^{-1}(\alpha) \neq \overline{1}$ and $f(\overline{1} - \nu) \geq \alpha$, for each $\nu \in I^X,\ r \in I_0$. Then by the hypothesis, there exists an r-grfc set $\gamma \leq \overline{1} - \nu \in I^X$ such that $\gamma \geq f^{-1}(\alpha)$. But $f^{-1}(\alpha) > f^{-1}(f(\lambda)) = \lambda$. That is, $\gamma \geq \lambda$. Therefore, there exists an r-grfc set $\gamma \in I^X, r \in I_0$ such that $\gamma \geq \lambda$, which is a contradiction. Therefore, $f(\lambda)$ is an r-fuzzy regular dense set in I^Y such that $\gamma \leq f(\overline{1} - \nu)$, for each $\nu \in I^X$ and r-frco set $\gamma \in I^Y$.

(3)⇒(1) Let λ be an r-frco set such that $f^{-1}(\lambda) \neq \overline{0}$ and $f(\nu) \leq \lambda$, for each $\nu \in I^X, r \in I_0$. Then, $\lambda \neq \overline{0}$. Now, suppose that $\nu \leq \alpha$ and $GRI_\tau(f^{-1}(\lambda), r) = \overline{0} \in I^X$. Then, $GRC_\tau(\overline{1} - f^{-1}(\lambda), r) = \overline{1} \in I^X$.

That is, $\overline{1} - f^{-1}(\lambda)$ is an r-generalized regular fuzzy dense in I^X. Then by (3), $f(\overline{1} - f^{-1}(\lambda))$ is an r-fuzzy regular dense set such that there exists an r-frco set $\mu \leq f(\overline{1} - \nu)$, for each $\nu \in I^X, r \in I_0$. But $f(\overline{1} - f^{-1}(\lambda)) = f(f^{-1}(\overline{1} - \lambda)) \leq \overline{1} - \lambda < \overline{1}$, since $\overline{1} - \lambda$ is an r-frco and $f(\overline{1} - f^{-1}(\lambda)) \leq \overline{1} - \lambda$, $RC_\tau(f(\overline{1} - f^{-1}(\lambda)), r) \leq \overline{1} - \lambda$. That is, $\overline{1} - \lambda \geq \overline{1} \Rightarrow \lambda = \overline{0}$, which is a contradiction, since $\lambda \neq \overline{0}$. Therefore, $\nu \leq \alpha$ and $GRI_\tau(f^{-1}(\lambda), r) \neq \overline{0}$. So f is swsgfc. □

Definition 5.4. *Let (X, τ) and (Y, η) be fts's. A function $f : (X, \tau) \to (Y, \eta)$ is called*

(1) slightly generalized regular fuzzy open (briefly, sgrfo) if for each r-grfo set $\lambda \in I^X$ and each $\mu \in I^X,\ r \in I_0$ such that $\lambda \leq \mu$, $f(\lambda)$ is an r-frco set in I^Y and $f(\lambda) \leq f(\mu)$,

(2) somewhat generalized regular fuzzy open (briefly, swgrfo) if for each r-grfo set $\overline{0} \neq \lambda \in I^X, r \in I_0$ there exists an r-grfo set in $\overline{0} \neq \mu \in I^Y$ such that $f(\lambda) \geq \mu$,

(3) somewhat slightly generalized regular fuzzy open (briefly, swsgrfo) if for each r-grfo set $\overline{0} \neq \lambda \in I^X$ such that $\lambda \leq \nu$ and for each $\nu \in I^X, r \in I_0$, there exists an r-frco set $\overline{0} \neq \mu \in I^Y,\ \mu \leq f(\nu)$ such that $f(\lambda) \geq \mu$.

That is, $I^r_\tau(f(\lambda), r) \neq \overline{0}$, and there exists an r-frco set μ such that $f(\nu) \geq \mu$ and $\lambda \leq \nu$, for each $\nu \in I^X,\ r \in I_0$.

Remark 5.2. *Evrey sgrfo (resp. swgrfo) function is swsgrfo function but the converse is not true in general as shown by the following example.*

Example 5.6. *In Example , f is swsgrfo function but not sgrfo, since for each r-grfo set $\lambda \in I^X$ and each $\nu \in I^X,\ r \in I_0$ such that $\lambda \leq \mu$, $f(\lambda)$ is not r-frco in I^Y and $f(\lambda) \leq f(\mu)$.*

Example 5.7. *Let $X = Y = \{a,\ b,\ c\}$ and $f : (X,\ \tau) \to (Y,\ \eta)$ be the function. Define $\lambda,\ \lambda_1,\ \nu \in I^X,\ \lambda_2,\ \mu \in I^Y$ as follows: $\lambda_1(a) = 0.5,\ \lambda_1(b) = 0.5,\ \lambda_1(c) = 0.5;$ $\lambda_2(a) = 0.5,\ \lambda_2(b) = 0.5,\ \lambda_2(c) = 0.5;$ $\lambda(a) = 0.5,\ \lambda(b) = 0.6,\ \lambda(c) = 0.5;$ $\mu(a) = 0.5,\ \mu(b) = 0.5,\ \mu(c) = 0.5;$ $\nu(a) = 0.7,\ \nu(b) = 0.6,\ \nu(c) = 0.5;$ $\delta(a) = 0.5,\ \delta(b) = 0.6,\ \delta(c) = 0.6$. We define a fuzzy topologies τ and η as follows:*

$$\tau(\lambda) = \begin{cases} 1 & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_1, \\ 0 & \text{otherwise}, \end{cases} \qquad \eta(\lambda) = \begin{cases} 1 & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ \frac{1}{2} & \text{if } \lambda = \lambda_2, \\ 0 & \text{otherwise}. \end{cases}$$

For $r = 1/2$, then f is swsgrfo function but not swgrfo, because for each r-grfo set $\overline{0} \neq \lambda \in I^X,\ r \in I_0$ such that $\lambda \leq \nu$ for each $\nu \in I^X$, there exists an r-frco set $\overline{0} \neq \mu \in I^Y,\ \mu \leq f(\nu)$ such that $f(\lambda) \geq \mu$ but for each r-grfo set $\overline{0} \neq \lambda \in I^X,\ r \in I_0$ there exists an r-grfo set $\overline{0} \neq \delta \in I^Y$, such that $f(\lambda) \ngeq \mu$.

Proposition 5.2. *Let $(X,\ \tau_1)$, $(Y,\ \tau_2)$ and $(Z,\ \tau_3)$ be fts's. If $f : (X,\ \tau_1) \to (Y,\ \tau_2)$ and $g : (Y,\ \tau_2) \to (Z,\ \tau_3)$ are swsgrfo functions, then $g \circ f : (X,\ \tau_1) \to (Z,\ \tau_3)$ are swsgrfo function.*

Proof. Let $\overline{0} \neq \lambda \in I^X$ be an r-grfo set $r \in I_0$ such that $\lambda \leq \mu$, for each fuzzy set $\mu \in I^X,\ r \in I_0$. Since f is swsgrfo, then there exists an r-frco set $\overline{0} \neq \nu \in I^Y$, and $f(\mu) \geq \nu$ such that $f(\lambda) \geq \nu$.

Now, $GRI_{\tau_2}(f(\lambda), r)$ is an r-grfo in I^Y such that
$GRI_{\tau_2}(f(\lambda), r) \neq \overline{0}$, $GRI_{\tau_2}(f(\lambda), r) \leq f(\mu)$, for each $f(\mu) \in I^Y$.

Since g is swsgrfo, then there exists an r-frco set $\overline{0} \neq \gamma \in I^Z$ and $\gamma \leq g(f(\mu))$ such that $\gamma \leq g(GRI_{\tau_2}(f(\lambda), r))$. But $g(GRI_{\tau_2}(f(\lambda), r)) \leq g(f(\lambda))$. Thus, there exists an r-frco set $\overline{0} \neq \gamma \in I^Z$ and $(g \circ f)(\mu) \geq \gamma$, such that $(g \circ f)(\lambda) \geq \gamma$. Therefore, $g \circ f$ is swsgrfo. □

Proposition 5.3. *Let $(X,\ \tau)$ and $(Y,\ \eta)$ be fts's, and let $f : (X,\ \tau) \to (Y,\ \eta)$ be a bijective function. Then the following are equivalent:*

(1) f is swsgrfo function.

(2) If λ is an r-grfc set in I^X such that $f(\lambda) \neq \overline{1}$ and $\lambda \geq \nu$ for each $\nu \in I^X$, then there exists an r-frco set $\mu \in I^Y$, $\mu \neq \overline{1}$ and $f(\nu) \leq \mu$ such that $f(\lambda) \leq \mu$.

Proof. (1)⇒(2) let λ be an r-grfc set in I^X such that $f(\lambda) \neq \overline{1}$ and $\lambda \geq \nu$, for each $\nu \in I^X$, $r \in I_0$. Then, $\overline{1} - \lambda$ is an r-grfo set in I^X such that $f(\overline{1} - \lambda) \neq \overline{0}$ and $\overline{1} - \lambda \leq \overline{1} - \nu$, for each $\nu \in I^X$. So $\overline{1} - \lambda \neq \overline{0}$. Since f is a swsgrfo, then there exists an r-frco set $\overline{0} \neq \delta \in I^Y$ and $f(\overline{1} - \nu) \geq \delta$ such that $f(\overline{1} - \lambda) \geq \delta$.

Now, $\overline{1} - \delta$ is an r-frco set in I^Y such that $\overline{1} - \delta \neq \overline{1}$ and $\overline{1} - \delta \geq f(\nu)$ such that $\overline{1} - \delta \geq f(\lambda)$. Take $\overline{1} - \delta = \mu$, so (2) is proved.

(2)⇒(1) Let $\lambda \neq \overline{0}$ be any r-grfo set in I^X such that $\lambda \leq \nu$, for each $\nu \in I^X$. Then, $\overline{1} - \lambda$ is an r-grfc set in I^X such that $\overline{1} - \lambda \neq \overline{1}$ and $\overline{1} - \lambda \geq \overline{1} - \nu$ for each $\nu \in I^X$, $r \in I_0$. Now, $f(\overline{1} - \lambda) = \overline{1} - f(\lambda) \neq \overline{1}$. For, if $\overline{1} - f(\lambda) = \overline{1}$, then $f(\lambda) = \overline{0} \Rightarrow \lambda = \overline{0}$.

Hence by the hypothesis, there exists an r-frco set $\mu \in I^Y$, $\overline{1} \neq \mu \geq f(\overline{1} - \nu)$, such that $f(\overline{1} - \lambda) \leq \mu$. That is $\overline{0} \neq \overline{1} - \mu \leq f(\nu)$, such that $\overline{1} - \mu \leq f(\lambda)$. Let $\overline{1} - \mu = \gamma$. Then, $\gamma \neq \overline{0}$ is an r-frco set in I^Y such that $f(\nu) \geq \gamma$ and $f(\lambda) \geq \gamma$. Therefore, f is swsgrfo function. □

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On fuzzy upper and lower e-continuous multifunctions

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Abstract In this paper, we introduce the concepts of fuzzy upper and fuzzy lower e-continuous multifunction, fuzzy upper and fuzzy lower e-irresolute multifunction on fuzzy topological spaces in $\hat{S}$ostak sense. Several characterizations and properties of these fuzzy upper (resp. fuzzy lower) e-continuous, fuzzy upper (resp. lower) e-irresolute multifunctions are presented and their mutual relationships are established in L-fuzzy topological spaces. Later, composition and union between these multifunctions have been studied.

Keywords fuzzy upper (resp. fuzzy lower) e-continuous multifunction, fuzzy upper (resp. lower) e-irresolute multifunction.

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§1. Introduction and preliminaries

Kubiak [15] and $\hat{S}$ostak [23] introduced the notion of (L-)fuzzy topological space as a generalization of L-topological spaces (originally called (L-) fuzzy topological spaces by Chang [6] and Goguen [8]. It is the grade of openness of an L-fuzzy set. A general approach to the study of topological type structures on fuzzy powersets was developed in [[9]- [11], [15], [16], [23]- [25]].

Berge [5] introduced the concept multimapping $F : X \multimap Y$ where X and Y are topological spaces and Popa [21, 22] introduced the notion of irresolute multimapping. After Chang introduced the concept of fuzzy topology [6], continuity of multifunctions in fuzzy topological spaces have been defined and studied by many authors from different view points (eg. see [3], [4], [18]-[20]). Tsiporkova et al., [27, 28] introduced the continuity of fuzzy multivalued mappings in the Chang's fuzzy topology [6]. Later, Abbas et al., [1] introduced the concepts of fuzzy upper and fuzzy lower semi-continuous multifunctions in L-fuzzy topological spaces. Recently, Sobana [26] and Vadivel [29] introduced r-feo sets (r-fec) sets, fuzzy e-continuity, fuzzy e-openness, fuzzy

e-closedness and r-fuzzy e-irresolute in a smooth topological space.

In this paper, we introduce the concepts of fuzzy upper and fuzzy lower e-continuous multifunction, fuzzy upper and fuzzy lower e-irresolute multifunction on fuzzy topological spaces in $\hat{S}$ostak sense. Several characterizations and properties of these multifunctions are presented and their mutual relationships are established in L-fuzzy topological spaces. Later, composition and union between these multifunctions have been studied.

Throughout this paper, nonempty sets will be denoted by X, Y etc., $L = [0,\ 1]$ and $L_0 = (0,\ 1]$. The family of all fuzzy sets in X is denoted by L^X. The complement of an L-fuzzy set λ is denoted by λ^c. This symbol $\multimap$ for a multifunction.

For $\alpha \in L$, $\overline{\alpha}(x) = \alpha$ for all $x \in X$. A fuzzy point x_t for $t \in L_0$ is an element of L^X such that $x_t(y) = \begin{cases} t & \text{if } y = x \\ 0 & \text{if } y \neq x. \end{cases}$ The family of all fuzzy points in X is denoted by $Pt(X)$. A fuzzy point $x_t \in \lambda$ iff $t \leq \lambda(x)$.

All other notations are standard notations of L-fuzzy set theory.

Definition 1.1. [1] *Let $F : X \multimap Y$, then F is called a fuzzy multifunction (FM, for short) if and only if $F(x) \in L^Y$ for each $x \in X$. The degree of membership of y in $F(x)$ is denoted by $F(x)(y) = G_F(x,\ y)$ for any $(x,\ y) \in X \times Y$. The domain of F, denoted by domain(F) and the range of F, denoted by $rng(F)$, for any $x \in X$ and $y \in Y$, are defined by :*

$$dom(F)(x) = \bigvee_{y\in Y} G_F(x,\ y) \quad and \quad rng(F)(y) = \bigvee_{x\in X} G_F(x,\ y).$$

Definition 1.2. [1] *Let $F : X \multimap Y$ be a FM. Then F is called:*

(i) Normalized iff for each $x \in X$, there exixts $y_0 \in Y$ such that $G_F(x,\ y_0) = \overline{1}$.

(ii) A crisp iff $G_F(x,\ y) = \overline{1}$ for each $x \in X$ and $y \in Y$.

Definition 1.3. [1] *Let $F : X \multimap Y$ be a FM. Then*

(i) The image of $\lambda \in L^X$ is an L-fuzzy set $F(\lambda) \in L^Y$ defined by

$$F(\lambda)(y) = \bigvee_{x\in X} [G_F(x,\ y) \wedge \lambda(x)].$$

(ii) The lower inverse of $\mu \in L^Y$ is an L-fuzzy set $F^l(\mu) \in L^X$ defined by

$$F^l(\mu)(x) = \bigvee_{y\in Y} [G_F(x,\ y) \wedge \mu(y)].$$

(iii) The upper inverse of $\mu \in L^Y$ is an L-fuzzy set $F^u(\mu) \in L^X$ defined by

$$F^u(\mu)(x) = \bigwedge_{y\in Y} [G^c_F(x,\ y) \vee \mu(y)].$$

Theorem 1.1. [1] *Let $F : X \multimap Y$ be a FM. Then*

(i) $F(\lambda_1) \leq F(\lambda_2)$ if $\lambda_1 \leq \lambda_2$.

(ii) $F^l(\mu_1) \leq F^l(\mu_2)$ *and* $F^u(\mu_1) \leq F^u(\mu_2)$, *if* $\mu_1 \leq \mu_2$.

(iii) $F^u(\mu) \leq F^l(\mu)$, *if* F *is normalized.*

(iv) $(F(\lambda))^c \leq F(\lambda^c)$, *if* F *is surjective.*

(v) $(F^l(\mu))^c \leq F^l(\mu^c)$, *if* F *is normalized.*

(vi) $F^l(\overline{1} - \mu) = \overline{1} - F^u(\mu)$ *and* $F^u(\overline{1} - \mu) = \overline{1} - F^l(\mu)$.

(vii) $F(F^u(\mu)) \leq \mu$ *if* F *is a crisp.*

(viii) $F^u(F(\lambda)) \geq \lambda$ *if* F *is a crisp.*

Definition 1.4. [1] *Let* $F : X \multimap Y$ *and* $H : Y \multimap Z$ *be two FM. Then the composition* $H \circ F$ *is defined by*

$$((H \circ F)(x))(z) = \bigvee_{y \in Y} [G_F(x,\ y) \wedge G_H(y,\ z)].$$

Theorem 1.2. [1] *Let* $F : X \multimap Y$ *and* $H : Y \multimap Z$ *be FM. Then we have the following*

(i) $(H \circ F) = F(H)$.

(ii) $(H \circ F)^u = F^u(H^u)$.

(iii) $(H \circ F)^l = F^l(H^l)$.

Theorem 1.3. [1] *Let* $F_i : X \multimap Y$ *be a FM. Then we have the following*

(i) $(\bigcup_{i\in\Gamma} F_i)(\lambda) = \bigvee_{i\in\Gamma} F_i(\lambda)$.

(ii) $(\bigcup_{i\in\Gamma} F_i)^l(\mu) = \bigvee_{i\in\Gamma} F_i^l(\mu)$.

(iii) $(\bigcup_{i\in\Gamma} F_i)^u(\mu) = \bigwedge_{i\in\Gamma} F_i^u(\mu)$.

Definition 1.5. [11, 15, 17, 23] *An L-fuzzy topological space (L-fts, in short) is a pair* $(X,\ \tau)$, *where* X *is a nonempty set and* $\tau : L^X \to L$ *is a mapping satisfying the following properties.*

(1) $\tau(\overline{0}) = \tau(\overline{1}) = 1$,

(2) $\tau(\mu_1 \wedge \mu_2) \geq \tau(\mu_1) \wedge \tau(\mu_2)$, *for any* $\mu_1,\ \mu_2 \in I^X$.

(3) $\tau(\bigvee_{i\in\Gamma} \mu_i) \geq \bigwedge_{i\in\Gamma} \tau(\mu_i)$, *for any* $\{\mu_i\}_{i\in\Gamma} \subset I^X$,

Then τ is called an L-fuzzy topology on X. For every $\lambda \in L^X$, $\tau(\lambda)$ is called the degree of openness of the L-fuzzy set λ.

A mapping $f : (X,\ \tau) \to (Y,\ \eta)$ is said to be continuous with respect to L-fuzzy topologies τ and η iff $\tau(f^{-1}(\mu)) \geq \eta(\mu)$ for each $\mu \in L^Y$.

Theorem 1.4. [7,13,14,17] *Let* $(X,\ \tau)$ *be a an L-fts. Then for each* $\lambda \in L^X,\ r \in L_0$, *we define L-fuzzy operators* C_τ *and* $I_\tau : L^X \times L_0 \to L^X$ *as follows:*

$C_\tau(\lambda,\ r) = \bigwedge\{\mu \in L^X : \lambda \leq \mu,\ \tau(\overline{1}-\mu) \geq r\}$.
$I_\tau(\lambda,\ r) = \bigvee\{\mu \in L^X : \lambda \geq \mu,\ \tau(\mu) \geq r\}$.

For $\lambda,\ \mu \in L^X$ and $r,\ s \in L_0$, the operator C_τ satisfies the following conditions:

(1) $C_\tau(\overline{0}, r) = \overline{0}$,

(2) $\lambda \leq C_\tau(\lambda,\ r)$,

(3) $C_\tau(\lambda,\ r) \vee C_\tau(\mu,\ r) = C_\tau(\lambda \vee \mu,\ r)$,

(4) $C_\tau(C_\tau(\lambda,\ r),\ r) = C_\tau(\lambda,\ r)$,

(5) $C_\tau(\lambda,\ r) = \lambda$ *iff* $\tau(\lambda^c) \geq r$.

(6) $C_\tau(\lambda^c,\ r) = (I_\tau(\lambda,\ r))^c$ *and* $I_\tau(\lambda^c,\ r) = (C_\tau(\lambda,\ r))^c$.

Definition 1.6. [1] *Let $F : X \multimap Y$ be a FM between two L-fts's $(X,\ \tau)$, $(Y,\ \eta)$ and $r \in L_0$. Then F is called:*

(i) Fuzzy upper semi continuous (or Fuzzy upper) (in short, FUS (or FU)-continuous) at a L-fuzzy point $x_t \in dom(F)$ iff $x_t \in F^u(\mu)$ for each $\mu \in L^Y$ and $\eta(\mu) \geq r$, there exists $\lambda \in L^X$, $\tau(\lambda) \geq r$ and $x_t \in \lambda$ such that $\lambda \wedge dom(F) \leq F^u(\mu)$. F is FU-continuous iff it is FU-continuous at every $x_t \in dom(F)$.

(ii) Fuzzy lower semi continuous (or Fuzzy lower) (in short, FLS (or FL)-continuous) at a L-fuzzy point $x_t \in dom(F)$ iff $x_t \in F^l(\mu)$ for each $\mu \in L^Y$ and $\eta(\mu) \geq r$, there exists $\lambda \in L^X$, $\tau(\lambda) \geq r$ and $x_t \in \lambda$ such that $\lambda \leq F^l(\mu)$. F is FL-continuous iff it is FL-continuous at every $x_t \in dom(F)$.

(iii) Fuzzy continuous if it is FU-continuous and FL-continuous.

Theorem 1.5. [1] *Let $F : X \multimap Y$ be a fuzzy multifunction between two L-fts's $(X,\ \tau)$ and $(Y,\ \eta)$. Let $\mu \in L^Y$. Then we have the following*

(1) F is FL-continuous iff $\tau(F^l(\mu)) \geq \eta(\mu)$.

(2) If F is normlized, then F is FU-continuous iff $\tau(F^u(\mu)) \geq \eta(\mu)$.

(3) F is FL-continuous iff $\tau(\overline{1} - F^u(\mu)) \geq \eta(\overline{1} - \mu)$.

(4) If F is normalized, then F is FU-continuous iff $\tau(\overline{1} - F^l(\mu)) \geq \eta(\overline{1} - \mu)$.

Remark 1.1 [4,30] *Let $(X,\ \tau)$ and $(Y,\ \eta)$ be a fts's. The fuzzy sets of the form $\lambda \times \mu$ with $\tau(\lambda) \geq r$ and $\eta(\mu) \geq r$ form a basis for the product fuzzy topology $\tau \times \eta$ on $X \times Y$, where for any $(x,\ y) \in X \times Y$, $(\lambda \times \mu)(x,\ y) = min\{\lambda(x),\ \mu(y)\}$.*

Definition 1.7. [4,19] *Let $F : X \multimap Y$ be a FM between two fts's $(X,\ \tau)$ and $(Y,\ \eta)$. The graph fuzzy multifunction $G_f : X \to X \times Y$ of F is defined as $G_f(x) = x_1 \times F(x)$, for every $x \in X$.*

Definition 1.8. [12] *Let $(X,\ \tau)$ be a fts. For $\lambda,\ \mu \in I^X$ and $r \in I_0$, λ is called r-fuzzy regular open (for short, r-fro) (resp. r-fuzzy regular closed (for short, r-frc)) if $\lambda = I_\tau(C_\tau(\lambda, r), r)$ (resp. $\lambda = C_\tau(I_\tau(\lambda, r), r)$).*

Definition 1.9. [12] *Let $(X,\ \tau)$ be a fts. Then for each $\mu \in I^X$, $x_t \in P_t(X)$ and $r \in I_0$,* *(i)* μ *is called r-open* Q_τ*-neighbourhood of* x_t *if* $x_t q \mu$ *with* $\tau(\mu) \geq r$.

(ii) μ *is called r-open* R_τ*-neighbourhood of* x_t *if* $x_t q \mu$ *with* $\mu = I_\tau(C_\tau(\mu,\ r),\ r)$.

We denoted

$$Q_\tau(x_t,\ r) = \{\mu \in I^X : x_t q\mu,\ \tau(\mu) \geq r\},$$

$$R_\tau(x_t,\ r) = \{\mu \in I^X : x_t q\mu,\ \mu = I_\tau(C_\tau(\mu,\ r),\ r)\}.$$

Definition 1.10. [12] *Let* $(X,\ \tau)$ *be a fts. Then for each* $\lambda \in I^X,\ x_t \in P_t(X)$ *and* $r \in I_0$,

(i) x_t *is called r-*τ *cluster point of* λ *if for every* $\mu \in Q_\tau(x_t,\ r)$, *we have* $\mu q \lambda$.

(ii) x_t *is called r-*δ *cluster point of* λ *if for every* $\mu \in R_\tau(x_t,\ r)$, *we have* $\mu q \lambda$.

(iii) *An* δ*-closure operator is a mapping* $DC_\tau : I^X \times I \to I^X$ *defined as follows:*
$\delta C_\tau(\lambda,\ r)$ *or* $DC_\tau(\lambda,\ r) = \bigvee\{x_t \in P_t(X) : x_t$ *is r-*δ*-cluster point of* $\lambda\}$.
Equivalently, $\delta C_\tau(\lambda,\ r) = \bigwedge\{\mu \in I^X : \mu \geq \lambda,\ \mu$ *is a r-frc set*$\}$ *and*
$\delta I_\tau(\lambda,\ r) = \bigvee\{\mu \in I^X : \mu \leq \lambda,\ \mu$ *is a r-fro set*$\}$.

Definition 1.11. [12] *Let* $(X,\ \tau)$ *be a fuzzy topological space. For* $\lambda \in I^X$ *and* $r \in I_0$, λ *is called r-fuzzy* δ*-closed iff* $\lambda = \delta C_\tau(\lambda,\ r)$ *or* $DC_\tau(\lambda,\ r)$.

Definition 1.12. [26] *Let* $(X,\ \tau)$ *be a an L-fts. Then for each* $\lambda,\ \mu \in L^X,\ r \in L_0$. *Then* λ *is called*

(1) λ *is called an r-fuzzy e-open (briefly, r-feo) set if* $\lambda \leq C_\tau(\delta_\tau(\lambda,\ r),\ r) \vee I_\tau(\delta C_\tau(\lambda,\ r),\ r)$.

(2) λ *is called an r-fuzzy e-closed (briefly, r-feo) set if* $C_\tau(\delta I_\tau(\lambda,\ r),\ r) \wedge I_\tau(\delta C_\tau(\lambda,\ r),\ r) \leq \lambda$.

Definition 1.13. [26] *Let* $(X,\ \tau)$ *be an L-fts. Then for each* $\lambda,\ \mu \in L^X,\ r \in L_0$. *Then* λ *is called*

(i) $eI_\tau(\lambda,\ r) = \bigvee\{\mu \in I^X : \mu \leq \lambda,\ \mu$ *is a r-feo set* $\}$ *is called the r-fuzzy e-interior of* λ.

(ii) $eC_\tau(\lambda,\ r) = \bigwedge\{\mu \in I^X : \mu \geq \lambda,\ \mu$ *is a r-fec set* $\}$ *is called the r-fuzzy e-closure of* λ.

§2. Fuzzy upper and lower e-continuous multifunctions

Definition 2.1. *Let* $F : X \multimap Y$ *be a FM between two L-fts's* $(X,\ \tau)$, $(Y,\ \eta)$ *and* $r \in L_0$. *Then F is called:*

(i) *Fuzzy upper e-continuous (FUe-continuous, in short) at any L-fuzzy point* $x_t \in dom(F)$ *iff* $x_t \in F^u(\mu)$ *for each* $\mu \in L^Y$ *and* $\eta(\mu) \geq r$ *there exists r-feo set,* $\lambda \in L^X$ *and* $x_t \in \lambda$ *such that* $\lambda \wedge dom(F) \leq F^u(\mu)$.

(ii) *Fuzzy lower e-continuous (FLe-continuous, in short) at any L-fuzzy point* $x_t \in dom(F)$ *iff* $x_t \in F^l(\mu)$ *for each* $\mu \in L^Y$ *and* $\eta(\mu) \geq r$ *there exists r-feo set,* $\lambda \in L^X$ *and* $x_t \in \lambda$ *such that* $\lambda \leq F^l(\mu)$.

(iii) *FUe-continuous (resp. FLe-continuous) iff it is FUe-continuous (resp. FLe-continuous) at every* $x_t \in dom(F)$.

Remark 2.1 *Let F be a FM between two L-fts's $(X,\ \tau)$ and $(Y,\ \eta)$. For the mapping $F : X \multimap Y$, the following statements are valid:*

(1) FU-continuous $\Rightarrow$ FUe-continuous.

(2) FL-continuous $\Rightarrow$ FLe-continuous.

The converse of the above Remark 2.1 need not be true as shown by the following examples.

Example 2.1 *Let $X = \{x_1,\ x_2\}$, $Y = \{y_1,\ y_2,\ y_3\}$ and $F : X \multimap Y$ be a FM defined by $G_F(x_1,\ y_1) = 0.8$, $G_F(x_1,\ y_2) = 0.9$, $G_F(x_1,\ y_3) = 0.8$, $G_F(x_2,\ y_1) = \overline{1}$, $G_F(x_2,\ y_2) = 0.7$, and $G_F(x_2,\ y_3) = 0.9$. Let λ_1 and λ_2 be a fuzzy subsets of X be defined as $\lambda_1(x_1) = 0.3$, $\lambda_1(x_2) = 0.1$ and $\lambda_2(x_1) = 0.7$, $\lambda_2(x_2) = 0.7$ and μ be a fuzzy subset of Y defined as $\mu(y_1) = 0.7$, $\mu(y_2) = 0.9$, $\mu(y_3) = 0.8$. We assume that $\overline{1} = 1$ and $\overline{0} = 0$. Define L-fuzzy topologies $\tau : L^X \to L$ and $\eta : L^Y \to L$ as follows:*

$$\tau(\lambda) = \begin{cases} 1, & \textit{if } \lambda = \overline{0} \textit{ or } \overline{1}\ , \\ \frac{1}{2}, & \textit{if } \lambda = \lambda_1, \\ 0, & \textit{otherwise}, \end{cases} \qquad \eta(\mu) = \begin{cases} 1, & \textit{if } \mu = \overline{0} \textit{ or } \overline{1}\ , \\ \frac{1}{2}, & \textit{if } \mu = \mu, \\ 0, & \textit{otherwise}. \end{cases}$$

are fuzzy topologies on X and Y. For $r=\frac{1}{2}$, as μ is $\frac{1}{2}$-fuzzy open in Y and $F^u(\mu) = \lambda_2$ is $\frac{1}{2}$-feo set in X. Then $F : X \multimap Y$ is FUe-continuous. But F is not FU-continuous, because μ is $\frac{1}{2}$-fuzzy open in Y and $F^u(\mu) = \lambda_2$ is not $\frac{1}{2}$-fuzzy open set in X.

Example 2.2 *Let $X = \{x_1,\ x_2\}$, $Y = \{y_1,\ y_2,\ y_3\}$ and $F : X \multimap Y$ be a FM defined by $G_F(x_1,\ y_1) = 0.2$, $G_F(x_1,\ y_2) = \overline{1}$, $G_F(x_1,\ y_3) = \overline{0}$, $G_F(x_2,\ y_1) = 0.5$, $G_F(x_2,\ y_2) = \overline{0}$, and $G_F(x_2,\ y_3) = 0.3$. Let λ_1 and λ_2 be a fuzzy subsets of X be defined as $\lambda_1(x_1) = 0.4$, $\lambda_1(x_2) = 0.3$; $\lambda_2(x_1) = 0.9$, $\lambda_2(x_2) = 0.5$ and μ be a fuzzy subset of Y defined as $\mu(y_1) = 0.6$, $\mu(y_2) = 0.9$, $\mu(y_3) = \overline{0}$. We assume that $\overline{1} = 1$ and $\overline{0} = 0$. Define L-fuzzy topologies $\tau : L^X \to L$ and $\eta : L^Y \to L$ as follows:*

$$\tau(\lambda) = \begin{cases} 1, & \textit{if } \lambda = \overline{0} \textit{ or } \overline{1}\ , \\ \frac{1}{2}, & \textit{if } \lambda = \lambda_1, \\ 0, & \textit{otherwise}, \end{cases} \qquad \eta(\mu) = \begin{cases} 1, & \textit{if } \mu = \overline{0} \textit{ or } \overline{1}\ , \\ \frac{1}{2}, & \textit{if } \mu = \mu, \\ 0, & \textit{otherwise}. \end{cases}$$

are fuzzy topologies on X and Y. For $r=\frac{1}{2}$, as μ is $\frac{1}{2}$-fuzzy open set in Y and $F^l(\mu) = \lambda_2$ is $\frac{1}{2}$-feo set in X. Then $F : X \multimap Y$ is FLe-continuous. But F is not FL-continuous, because μ is $\frac{1}{2}$-fuzzy open in Y and $F^l(\mu) = \lambda_2$ is not $\frac{1}{2}$-fuzzy open set in X.

Proposition 2.1 *If F is normalized, then F is FUe-continuous at an L-fuzzy point $x_t \in dom(F)$ iff $x_t \in F^u(\mu)$ for each $\mu \in L^Y$ and $\eta(\mu) \geq r$ there exists $\lambda \in L^X$, λ is r-feo set and $x_t \in \lambda$ such that $\lambda \leq F^u(\mu)$.*

Theorem 2.1 *Let $F : X \multimap Y$ be a FM between two L-fts's $(X,\ \tau)$, $(Y,\ \eta)$ and $\mu \in L^Y$, then the following are equivalent:*

(i) F is FLe-continuous.

(ii) $F^l(\mu)$ is r-feo set, for any $\eta(\mu) \geq r$.

(iii) $F^u(\mu)$ *is* r-fec *set, for any* $\eta(\overline{1}-\mu)\geq r$.

(iv) $eC_\tau(F^u(\mu),r)\leq F^u(C_\eta(\mu,r))$, *for any* $\mu\in L^Y$.

(v) $C_\tau(\delta I_\tau(F^u(\mu),r),r)\wedge I_\tau(\delta C_\tau(F^u(\mu),r),r)\leq F^u(C_\eta(\mu,r))$, *for any* $\mu\in L^Y$.

Proof. (i) $\Rightarrow$ (ii): Let $x_t\in dom(F),\ \mu\in L^Y,\ \eta(\mu)\geq r$ and $x_t\in F^l(\mu)$ then, there exist $\lambda\in L^X$, λ is r-feo set and $x_t\in\lambda$ such that $\lambda\leq F^l(\mu)$ and hence $x_t\in eI_\tau(F^l(\mu),r)$. Therefore, we obtain $F^l(\mu)\leq eI_\tau(F^l(\mu),r)$. Thus $F^l(\mu)$ is r-feo set.

(ii) $\Rightarrow$ (iii): Let $\mu\in L^Y$ and $\eta(\overline{1}-\mu)\geq r$ hence by (ii), $F^l(\overline{1}-\mu)=\overline{1}-F^u(\mu)$ is r-feo. Then $F^u(\mu)$ is r-fec.

(iii) $\Rightarrow$ (iv): Let $\mu\in L^Y$ hence by (iii), $F^u(C_\eta(\mu,r))$ is r-fec. Then we obtain

$$eC_\tau(F^u(\mu),\ r)\leq F^u(C_\eta(\mu,\ r)).$$

(iv) $\Rightarrow$ (v): Let $\mu\in L^Y$ hence by (iv), we obtain

$$C_\tau(\delta I_\tau(F^u(\mu),r),r)\wedge I_\tau(\delta C_\tau(F^u(\mu),r),r)\leq eC_\tau(F^u(\mu),r)\leq F^u(C_\eta(\mu,r)).$$

(v) $\Rightarrow$ (ii): Let $\mu\in L^Y$, $\eta(\mu)\geq r$, hence by (v), we have

$$\begin{aligned}\overline{1}-F^l(\mu)&=F^u(\overline{1}-\mu)\\&\geq C_\tau(\delta I_\tau(F^u(\overline{1}-\mu),r),r)\wedge I_\tau(\delta C_\tau(F^u(\overline{1}-\mu),r),r)\\&=C_\tau(\delta I_\tau(\overline{1}-F^l(\mu),r),r)\wedge I_\tau(\delta C_\tau(\overline{1}-F^l(\mu),r),r)\\&=\overline{1}-[C_\tau(\delta I_\tau(F^l(\mu),r),r)\vee I_\tau(\delta C_\tau(F^l(\mu),r),r)]\\F^l(\mu)&\leq C_\tau(\delta I_\tau(F^l(\mu),r),r)\vee I_\tau(\delta C_\tau(F^l(\mu),r),r).\end{aligned}$$

Hence, $F^l(\mu)$ is r-feo.

(ii) $\Rightarrow$ (i): Let $x_t\in dom(F),\ \mu\in L^Y,\ \eta(\mu)\geq r$, with $x_t\in F^l(\mu)$ we have by (ii), $F^l(\mu)$ is r-feo-set. Let $F^l(\mu)=\lambda$(say), then there exists $\lambda\in L^X$, λ is r-feo-set and $x_t\in\lambda$ such that $\lambda\leq F^l(\mu)$. Thus F is FLe-continuous. □

Theorem 2.2 *Let* $F:X\multimap Y$ *be a FM and normalized between two L-fts's* $(X,\ \tau)$, $(Y,\ \eta)$ *and* $\mu\in L^Y$, *then the following are equivalent:*

(i) F *is* FUe*-continuous.*

(ii) $F^u(\mu)$ *is* r-feo *set, for any* $\eta(\mu)\geq r$.

(iii) $F^l(\mu)$ *is* r-fec *set, for any* $\eta(\overline{1}-\mu)\geq r$.

(iv) $eC_\tau(F^l(\mu),\ r)\leq F^l(C_\eta(\mu,\ r))$, *for any* $\mu\in L^Y$.

(v) $C_\tau(\delta I_\tau(F^l(\mu),\ r),\ r)\wedge I_\tau(\delta C_\tau(F^l(\mu),\ r),\ r)\leq F^l(C_\eta(\mu,\ r))$, *for any* $\mu\in L^Y$.

Proof. This can be proved in a similar way as Theorem 2.1. □

Corollary 2.1 *Let* $F:X\multimap Y$ *be a FM between two fts's* $(X,\ \tau)$, $(Y,\ \eta)$ *and* $\mu\in L^Y$. *Then we have the following:*

(i) If F *is normalized, then* F *is* FUe*-continuous at* x_t *iff* $x_t\in r$-feo *set of* $F^u(\mu)$, *for each* $\eta(\mu)\geq r$ *and* $x_t\in F^u(\mu)$.

(ii) F is FLe-continuous at x_t iff $x_t \in r$-feo set of $F^l(\mu)$, for each $\eta(\mu) \geq r$ and $x_t \in F^l(\mu)$.

Remark 2.2 *Let $F : X \multimap Y$ be a FM between two fts's $(X,\ \tau)$, $(Y,\ \eta)$ and $\mu \in L^Y$. Then we will show that if F is FUe-continuous and not normalized then $x_t \notin r$-feo set of $F^u(\mu)$, for each $\eta(\mu) \geq r$, by the following example.*

Example 2.3 *Let $X = \{x_1,\ x_2\}$, $Y = \{y_1,\ y_2,\ y_3\}$ and $F : X \multimap Y$ be a FM defined by $G_F(x_1,\ y_1) = 0.1$, $G_F(x_1,\ y_2) = 0.6$, $G_F(x_1,\ y_3) = \overline{0}$, $G_F(x_2,\ y_1) = 0.7$, $G_F(x_2,\ y_2) = \overline{0}$, and $G_F(x_2,\ y_3) = 0.7$. Let λ_1 and λ_2 be a fuzzy subsets of X be defined as $\lambda_1(x_1) = 0.5$, $\lambda_1(x_2) = 0.5$ and $\lambda_2(x_1) = 0.6$, $\lambda_2(x_2) = 0.6$ and μ be a fuzzy subset of Y defined as $\mu(y_1) = 0.6$, $\mu(y_2) = 0.6$, $\mu(y_3) = 0.6$. We assume that $\overline{1} = 1$ and $\overline{0} = 0$. Define L-fuzzy topologies $\tau : L^X \to L$ and $\eta : L^Y \to L$ as follows:*

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda = \overline{0} \text{ or } \overline{1}\ , \\ \frac{1}{2}, & \text{if } \lambda = \lambda_1, \\ 0, & \text{otherwise}, \end{cases} \qquad \eta(\mu) = \begin{cases} 1, & \text{if } \mu = \overline{0} \text{ or } \overline{1}\ , \\ \frac{1}{2}, & \text{if } \mu = \mu, \\ 0, & \text{otherwise.} \end{cases}$$

are fuzzy topologies on X and Y. Since $dom(F)(x) = \bigvee_{y\in Y} G_F(x,y)$, i.e $x^1_{0.6} \in dom(F)$ and $x^2_{0.7} \in dom(F)$. From Definition 2.3, we have

$F^u(0.6)(x_1) = 0.6, \qquad F^u(\overline{0})(x_1) = 0.4, \qquad F^u(\overline{1})(x_1) = \overline{1}$

$F^u(0.6)(x_2) = 0.6,, \qquad F^u(\overline{0})(x_2) = 0.3, \qquad F^u(\overline{1})(x_2) = \overline{1}.$

For $r=\frac{1}{2}$, as μ is $\frac{1}{2}$-fuzzy open in Y and $F^u(\mu) = \lambda_2$ is $\frac{1}{2}$-feo set in X. Then
(i) F is FUe-continuous.
(ii) F is not normalized.
(iii) The fuzzy point x_t with $x^2_{0.7} \notin \lambda_2$ where $F^u(\mu) = \lambda_2$ is r-feo set and $\eta(\mu) \geq \frac{1}{2}$.

Theorem 2.3 *Let $\{F_i\}_{i\in\Gamma}$ be a family of FLe-continuous between two fts's $(X,\ \tau)$ and $(Y,\ \eta)$. Then $\bigcup_{i\in\Gamma} F_i$ is FLe-continuous.*

Proof. Let $\mu \in L^Y$, then $(\bigcup_{i\in\Gamma} F_i)^l(\mu) = \bigvee_{i\in\Gamma} ({F_i}^l(\mu))$ by, Theorem 1.3 (ii). Since $\{F_i\}_{i\in\Gamma}$ is a family of FLe-continuous between two fts's $(X,\ \tau)$ and $(Y,\ \eta)$, then ${F_i}^l(\mu)$ is r-feo for any $\eta(\mu) \geq r$. Then we have $(\bigcup_{i\in\Gamma} F_i)^l(\mu) = \bigvee_{i\in\Gamma}({F_i}^l(\mu))$ is r-feo for any $\eta(\mu) \geq r$. Hence $\bigcup_{i\in\Gamma} F_i$ is FLe-continuous. □

Theorem 2.4 *Let $\{F_i\}_{i\in\Gamma}$ be a family of normalized FUe-continuous between two fts's $(X,\ \tau)$ and $(Y,\ \eta)$. Then $F_1 \bigcup F_2$ is FUe-continuous.*

Proof. Let $\mu \in L^Y$, then

$$(F_1 \cup F_2)^u(\mu) = {F_1}^u(\mu) \wedge {F_2}^u(\mu)$$

by, Theorem 1.3 (iii). Since $\{F_i\}_{i\in\Gamma}$ is a family of normalized FUe-continuous between two fts's $(X,\ \tau)$ and $(Y,\ \eta)$, then $({F_i}^u(\mu))$ if r-feo, for any $\eta(\mu) \geq r$ for each $i \in \{1, 2\}$. Then for each $\mu \in L^Y$, we have $(F_1 \cup F_2)^u(\mu) = {F_1}^u(\mu) \wedge {F_2}^u(\mu)$ is r-feo set for any $\eta(\mu) \geq r$. Hence $F_1 \cup F_2$ is FUe-continuous. □

Definition 2.2 *A fuzzy set λ in a fts $(X,\ \tau)$ is called r-fuzzy e-compact iff every family in $\{\mu : \mu$ is r-feo, $\mu \in L^X$ and $r \in L\}$ covering λ has a finite subcover.*

Definition 2.3 *Let $F : X \multimap Y$ be a FM between two fts's $(X,\ \tau)$, $(Y,\ \eta)$ and $r \in L_0$. Then F is called fuzzy e-compact valued iff $F(x_t)$ is r-fuzzy e-compact for each $x_t \in dom(F)$.*

Theorem 2.5 *Let $F : X \multimap Y$ be a crisp FUe-continuous and e-compact valued between two fts's $(X,\ \tau)$ and $(Y,\ \eta)$. Then the direct image of a r-fuzzy e-compact in X under F is also r-fuzzy e-compact.*

Proof. Let λ be r-fuzzy e-compact set in X and $\{\gamma_i :\ \gamma_i$ is r-feo set in Y, $i \in \Gamma\}$ be a family of covering of $F(\lambda)$. i.e. $F(\lambda) \leq \bigvee_{i\in\Gamma} \gamma_i$. Since $\lambda = \bigvee_{x_t\in\lambda} x_t$, we have

$$F(\lambda) = F(\bigvee_{x_t\in\lambda} x_t) = \bigvee_{x_t\in\lambda} F(x_t) \leq \bigvee_{i\in\Gamma} \gamma_i.$$

It follows that for each $x_t \in \lambda,\ F(x_t) \leq \bigvee_{i\in\Gamma} \gamma_i$. Since F is r-fuzzy e-compact valued, then there exists finite subset Γ_{x_t} of Γ such that $F(x_t) \leq \bigvee_{n\in\Gamma_{x_t}} \gamma_n = \gamma_{x_t}$. By Theorem 1.1 (viii), we have

$$x_t \leq F^u(F(x_t)) \leq F^u(\gamma_{x_t}) \text{ and } \lambda = \bigvee_{x_t\in\lambda} x_t = \bigvee_{x_t\in\lambda} F^u(\gamma_{x_t}).$$

Since, $\eta(\gamma_{x_t}) \geq r$, then from Theorem 2.2, we have $F^u(\gamma_{x_t})$ is r-feo-set. Hence $\{F^u(\gamma_{x_t}) : F^u(\gamma_{x_t})$ is r-feo-set, $\ x_t \in \lambda\}$ is a family covering the set λ. Since λ is r-fuzzy e-compact, then there exists finite index set N such that $\lambda \leq \bigvee_{n\in N} F^u(\gamma_{x_{t_n}})$. From Theorem 1.1 (vii), we have

$$F(\lambda) \leq F(\bigvee_{n\in N} F^u(\gamma_{x_{t_n}})) = \bigvee_{n\in N} F(F^u(\gamma_{x_{t_n}})) \leq \bigvee_{n\in N} \gamma_{x_{t_n}}.$$

Then $F(\lambda)$ is r-fuzzy e-compact. □

Theorem 2.6 *Let $F : X \multimap Y$ and $H : Y \multimap Z$ be two FM's and let $(X,\ \tau)$, $(Y,\ \eta)$ and $(Z,\ \delta)$ be three fts's. Then we have the following:*

(i) If F and H are normalized, FUe-continuous, then $H \circ F$ is FUe-continuous.

(ii) If F and H are FLe-continuous, then $H \circ F$ is FLe-continuous.

Proof. (i) Let F and H are normalized, FUe-continuous and $\nu \in L^Z$. Then from Theorem 1.2, we have $(H \circ F)^u(\nu) = F^u(H^u(\nu))$ is r-feo with $\nu(H^u(\nu)) \geq \delta(\nu)$. Thus $H \circ F$ is FUe-continuous.

(ii) Similar of (i). □

Theorem 2.7 *Let $F : X \multimap Y$ and $H : Y \multimap Z$ be two FM's and let $(X,\ \tau)$, $(Y,\ \eta)$ and $(Z,\ \delta)$ be three L-fts's. If F is FLe-continuous and H is FL-continuous, then $H \circ F$ is FLe-continuous.*

Proof. Let $\nu \in L^Z,\ \delta(\nu) \geq r$. Since H is FL-continuous, then by Theorem 1.5, $H^l(\nu)$ is r-fuzzy open set in Y. Also, F is FLe-irresolute implies $F^l(H^l(\nu))$ is r-feo set in X. Hence, we have $(H \circ F)^l(\nu) = F^l(H^l(\nu))$ is r-feo. Thus $H \circ F$ is FLe-continuous. □

Theorem 2.8 *Let $F : X \multimap Y$ and $H : Y \multimap Z$ be two FM's and let $(X,\ \tau)$, $(Y,\ \eta)$ and $(Z,\ \delta)$ be three L-fts's. If F and H are normalized, F is FUe-continuous and H is FU-continuous, then $H \circ F$ is FUe-continuous.*

Proof. This can be proved in a similar way as Theorem 2.7. □

Theorem 2.9 *Let $F : X \multimap Y$ be a FM between two fts's $(X,\ \tau)$ and $(Y,\ \eta)$. If G_f is FLe-continuous, then F is FLe-continuous.*

Proof. For the fuzzy sets $\rho \in L^X$, $\tau(\rho) \geq r$, $\nu \in L^Y$ and $\eta(\nu) \geq r$, we take, $(\rho \times \nu)(x,\ y) = \begin{cases} 0, & \text{if } x \notin \rho, \\ \nu(y), & \text{if } x \in \rho. \end{cases}$

Let $x_t \in dom(F)$, $\mu \in L^Y$ and $\eta(\mu) \geq r$ with $x_t \in F^l(\mu)$, then we have $x_t \in G^l_f(X \times \mu)$ and $\eta(X \times \mu) \geq r$. Since G_f is FLe-continuous, it follows that there exists $\lambda \in L^X$, λ is r-feo and $x_t \in \lambda$ such that $\lambda \leq G^l_f(X \times \mu)$. From here, we obtain that $\lambda \leq F^l(\mu)$. Thus F is FLe-continuous. □

Theorem 2.10 *Let $F : X \multimap Y$ be a FM between two fts's $(X,\ \tau)$ and $(Y,\ \eta)$. If G_f is FUe-continuous, then F is FUe-continuous.*

Proof. This can be proved in a similar way as Theorem 2.9. □

Theorem 2.11 *Let $(X,\ \tau)$ and $(X_i,\ \tau_i)$ be L-fts's $(i \in I)$. If a FM $F : X \multimap \Pi_{i\in I}X_i$ is FLe-continuous (where $\Pi_{i\in I}X_i$ is the product space), then $P_i \circ F$ is FLe-continuous for each $i \in I$, where $P_i : \Pi_{i\in I}X_i \to X_i$ is the projection multifunction which is defined by $P_i(x_i) = \{x_i\}$ for each $i \in I$.*

Proof. Let $\mu_{i_0} \in L^{X_{i_0}}$ and $\tau_i(\mu_{i_0}) \geq r$. Then

$$(P_{i_0} \circ F)^l(\mu_{i_0}) = F^l(P^l_{i_0}(\mu_{i_0})) = F^l(\mu_{i_0} \times \Pi_{i\neq i_0}X_i).$$

Since F is FLe-continuous and $\tau_i(\mu_{i_0} \times \Pi_{i\neq i_0}X_i) \geq r$, it follows that $F^l(\mu_{i_0} \times \Pi_{i\neq i_0}X_i)$ is r-feo set. Then $P_i \circ F$ is an FLe-continuous. □

We state the following result without proof in view of the above theorem.

Theorem 2.12 *Let $(X,\ \tau)$ and $(X_i,\ \tau_i)$ be L-fts's $(i \in I)$. If a FM $F : X \multimap \Pi_{i\in I}X_i$ is FUe-continuous (where $\Pi_{i\in I}X_i$ is the product space), then $P_i \circ F$ is FUe-continuous for each $i \in I$, where $P_i : \Pi_{i\in I}X_i \to X_i$ is the projection multifunction which is defined by $P_i(x_i) = \{x_i\}$ for each $i \in I$.*

Theorem 2.13 *Let $(X_i,\ \tau_i)$ and $(Y_i,\ \eta_i)$ be L-fts's and $F_i : X_i \multimap Y_i$ be a FM for each $i \in I$. Suppose that $F : \Pi_{i\in I}X_i \multimap \Pi_{i\in I}Y_i$ is defined by $F(x_i) = \Pi_{i\in I}F_i(x_i)$. If F is FLe-continuous, then F_i is FLe-continuous for each $i \in I$.*

Proof. Let $\mu_i \in L^{Y_i}$ and $\eta_i(\mu_i) \geq r$. Then $\eta_i(\mu_i \times \Pi_{i\neq j}Y_j) \geq r$. Since F is FLe-continuous, it follows that $F^l(\mu_i \times \Pi_{i\neq j}Y_j) = F^l(\mu_i) \times \Pi_{i\neq j}X_j$ is r-feo. Consequently, we obtain that $F^l(\mu_i)$ is $r\text{-}feo$ for each $i \in I$. Thus, F_i is FLe-continuous. □

We state the following result without proof in view of above theorem.

Theorem 2.14 *Let $(X_i,\ \tau_i)$ and $(Y_i,\ \eta_i)$ be L-fts's and $F_i : X_i \multimap Y_i$ be a FM for each $i \in I$. Suppose that $F : \Pi_{i\in I}X_i \multimap \Pi_{i\in I}Y_i$ is defined by $F(x_i) = \Pi_{i\in I}F_i(x_i)$. If F is FUe-continuous, then F_i is FUe-continuous for each $i \in I$.*

§3. Fuzzy upper and lower e-irresolute multifunctions

Definition 3.1 *Let $F : X \multimap Y$ be a FM between two L-fts's $(X,\ \tau)$, $(Y,\ \eta)$ and $r \in L_0$. Then F is called:*

(i) Fuzzy upper e-irresolute (FUe-irresolute, in short) at an L-fuzzy point $x_t \in dom(F)$ iff $x_t \in F^u(\mu)$ for each $\mu \in L^Y$ and μ is r-feo, there exists $\lambda \in L^X$, λ is r-feo and $x_t \in \lambda$ such that $\lambda \wedge dom(F) \leq F^u(\mu)$.

(ii) Fuzzy lower e-irresolute (FLe-irresolute, in short) at an L-fuzzy point $x_t \in dom(F)$ iff $x_t \in F^l(\mu)$ for each $\mu \in L^Y$ and μ is r-feo, there exists $\lambda \in L^X$, λ is r-feo and $x_t \in \lambda$ such that $\lambda \leq F^l(\mu)$.

(iii) FUe-irresolute (resp. FLe-irresolute) iff it is FUe-irresolute (resp. FLe-irresolute) at every $x_t \in dom(F)$.

Example 3.1 *Let $X = \{x_1,\ x_2\}$, $Y = \{y_1,\ y_2,\ y_3\}$ and $F : X \multimap Y$ be a FM defined by $G_F(x_1,\ y_1) = 0.1$, $G_F(x_1,\ y_2) = \overline{1}$, $G_F(x_1,\ y_3) = \overline{0}$, $G_F(x_2,\ y_1) = 0.5$, $G_F(x_2,\ y_2) = \overline{0}$, and $G_F(x_2,\ y_3) = \overline{1}$. Let λ_1 and λ_2 be a fuzzy subsets of X be defined as $\lambda(x_1) = 0.5$, $\lambda(x_2) = 0.5$: μ_1 and μ_2 be a fuzzy subsets of Y defined as $\mu_1(y_1) = 0.5$, $\mu_1(y_2) = 0.5$, $\mu_1(y_3) = 0.5$ and $\mu_2(y_1) = 0.4$, $\mu_2(y_2) = 0.4$, $\mu_2(y_3) = 0.4$. We assume that $\overline{1} = 1$ and $\overline{0} = 0$. Define L-fuzzy topologies $\tau : L^X \to L$ and $\eta : L^Y \to L$ as follows:*

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda = \overline{0} \text{ or } \overline{1}\ , \\ \frac{1}{2}, & \text{if } \lambda = \lambda, \\ 0, & \text{otherwise,} \end{cases} \qquad \eta(\mu) = \begin{cases} 1, & \text{if } \mu = \overline{0} \text{ or } \overline{1}\ , \\ \frac{1}{2}, & \text{if } \mu = \mu_1, \\ 0, & \text{otherwise.} \end{cases}$$

are fuzzy topologies on X and Y. For $r = \frac{1}{2}$, then F is FUe-irresolute and FLe-irresolute.

Proposition 3.1 *F is normalized implies F is FUe-irresolute at $x_t \in dom(F)$ iff $x_t \in F^u(\mu)$ for each $\mu \in L^Y$ and μ is r-feo, there exists $\lambda \in L^X$, λ is r-feo and $x_t \in \lambda$ such that $\lambda \leq F^u(\mu)$.*

Remark 3.1 *Let F be a FM between two L-fts's $(X,\ \tau)$ and $(Y,\ \eta)$. For the mapping $F : X \multimap Y$, the following statements are valid:*

(1) FUe-irresolute $\Rightarrow$ FUe-continuous.

(2) FLe-irresolute $\Rightarrow$ FLe-continuous.

In general, the converse of Remark 3.1 need not be true from the following examples.

Example 3.2 *Let $X = \{x_1, x_2\}$, $Y = \{y_1,\ y_2,\ y_3\}$ and $F : X \multimap Y$ be a FM defined by $G_F(x_1,\ y_1) = 0.8$, $G_F(x_1,\ y_2) = 0.9$, $G_F(x_1,\ y_3) = 0.8$, $G_F(x_2,\ y_1) = \overline{1}$, $G_F(x_2,\ y_2) = 0.7$,* *and* $G_F(x_2,\ y_3) = 0.9$. *Let* λ_1 *and* λ_2 *be a fuzzy subset of* X *be defined as* $\lambda_1(x_1) = 0.3$, $\lambda_1(x_2) = 0.1$; $\lambda_2(x_1) = 0.1$, $\lambda_2(x_2) = 0.2$ *and* μ_1 *and* μ_2 *be a fuzzy subsets of* Y *defined as* $\mu_1(y_1) = 0.7$, $\mu_1(y_2) = 0.9$, $\mu_1(y_3) = 0.8$ *and* $\mu_2(y_1) = 0.3$, $\mu_2(y_2) = 0.1$, $\mu_2(y_3) = 0.2$ *We assume that* $\overline{1} = 1$ *and* $\overline{0} = 0$. *Define L-fuzzy topologies* $\tau : L^X \to L$ *and* $\eta : L^Y \to L$ *as follows:*

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda = \overline{0} \text{ or } \overline{1}\ , \\ \frac{1}{2}, & \text{if } \lambda = \lambda_1, \\ 0, & \text{otherwise}, \end{cases} \qquad \eta(\mu) = \begin{cases} 1, & \text{if } \mu = \overline{0} \text{ or } \overline{1}\ , \\ \frac{1}{2}, & \text{if } \mu = \mu_1, \\ 0, & \text{otherwise}. \end{cases}$$

are fuzzy topologies on X *and* Y. *For* $r = \frac{1}{2}$, *then* $F : X \multimap Y$ *is FUe-continuous but not FUe-irresolute because* μ_2 *is* $\frac{1}{2}$*-feo in* $(Y,\ \eta)$, $F^u(\mu_2) = \lambda_2$ *is not* $\frac{1}{2}$*-feo set in* (X, τ).

Example 3.3 *Let* $X = \{x_1, x_2\}$, $Y = \{y_1,\ y_2,\ y_3\}$ *and* $F : X \multimap Y$ *be a FM defined by* $G_F(x_1,\ y_1) = 0.2$, $G_F(x_1,\ y_2) = \overline{1}$, $G_F(x_1,\ y_3) = \overline{0}$, $G_F(x_2,\ y_1) = 0.5$, $G_F(x_2,\ y_2) = \overline{0}$, *and* $G_F(x_2,\ y_3) = 0.3$. *Let* λ_1 *and* λ_2 *be a fuzzy subsets of* X *be defined as* $\lambda_1(x_1) = 0.4$, $\lambda_1(x_2) = 0.3$; $\lambda_2(x_1) = 0.2$, $\lambda_2(x_2) = 0.4$, μ_1 *and* μ_2 *be a fuzzy subsets of* Y *defined as* $\mu_1(y_1) = 0.6$, $\mu_1(y_2) = 0.9$, $\mu_1(y_3) = \overline{0}$; $\mu_2(y_1) = 0.4$, $\mu_2(y_2) = 0.1$, $\mu_2(y_3) = \overline{1}$. *We assume that* $\overline{1} = 1$ *and* $\overline{0} = 0$. *Define L-fuzzy topologies* $\tau : L^X \to L$ *and* $\eta : L^Y \to L$ *as follows:*

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda = \overline{0} \text{ or } \overline{1}\ , \\ \frac{1}{2}, & \text{if } \lambda = \lambda_1, \\ 0, & \text{otherwise}, \end{cases} \qquad \eta(\mu) = \begin{cases} 1, & \text{if } \mu = \overline{0} \text{ or } \overline{1}\ , \\ \frac{1}{2}, & \text{if } \mu = \mu_1, \\ 0, & \text{otherwise}. \end{cases}$$

are fuzzy topologies on X *and* Y. *For* $r=\frac{1}{2}$, *then* $F : X \multimap Y$ *is FLe-continuous but not FLe-irresolute because* μ_2 *is* $\frac{1}{2}$*-feo in* $(Y,\ \eta)$, $F^l(\mu_2) = \lambda_2$, *is not* $\frac{1}{2}$*-feo set in* $(X,\ \tau)$.

Theorem 3.1 *Let* $F : X \multimap Y$ *be a FM between two L-fts's* $(X,\ \tau)$, $(Y,\ \eta)$ *and* $\mu \in L^Y$, *then the following are equivalent:*

(i) F *is FLe-irresolute.*

(ii) $F^l(\mu)$ *is* r*-feo set, for any* μ *is* r*-feo.*

(iii) $F^u(\mu)$ *is* r*-fec set, for any* μ *is* r*-fec.*

(iv) $eC_\tau(F^u(\mu), r) \leq F^u(eC_\eta(\mu, r))$, *for any* $\mu \in L^Y$.

(v) $C_\tau(\delta I_\tau(F^u(\mu), r), r) \wedge I_\tau(\delta C_\tau(F^u(\mu), r), r) \leq F^u(eC_\eta(\mu, r))$, *for any* $\mu \in L^Y$.

Proof. (i) $\Rightarrow$ (ii): Let $x_t \in dom(F)$, $\mu \in L^Y$, μ is r-feo and $x_t \in F^l(\mu)$ then, there exist $\lambda \in L^X$, λ is r-feo set and $x_t \in \lambda$ such that $\lambda \leq F^l(\mu)$ thus $x_t \in eI_\tau(F^l(\mu), r)$. Therefore, we obtain $F^l(\mu) \leq eI_\tau(F^l(\mu), r)$. Thus $F^l(\mu)$ is r-feo set.

(ii) $\Rightarrow$ (iii): Let $\mu \in L^Y$ and μ is r-fec. Hence by (ii), $F^l(\overline{1} - \mu) = \overline{1} - F^u(\mu)$ is r-feo. Then $F^u(\mu)$ is r-fec.

(iii) $\Rightarrow$ (iv): Let $\mu \in L^Y$ hence by (iii), $F^u(eC_\eta(\mu, r))$ is r-fec. Then we obtain

$$eC_\tau(F^u(\mu), r) \leq F^u(eC_\eta(\mu, r)).$$

(iv) $\Rightarrow$ (v): Let $\mu \in L^Y$ hence by (iv), we obtain

$$C_\tau(\delta I_\tau(F^u(\mu), r), r) \wedge I_\tau(\delta C_\tau(F^u(\mu), r), r) \leq eC_\tau(F^u(\mu), r) \leq F^u(eC_\eta(\mu, r)).$$

(v) $\Rightarrow$ (ii): Let $\mu \in L^Y$ and μ is r-feo. Hence by (v), we have

$$\begin{aligned} \overline{1} - F^l(\mu) &= F^u(\overline{1} - \mu) \\ &\geq C_\tau(\delta I_\tau(F^u(\overline{1} - \mu), r), r) \wedge I_\tau(\delta C_\tau(F^u(\overline{1} - \mu), r), r) \\ &= C_\tau(\delta I_\tau(\overline{1} - F^l(\mu), r), r) \wedge I_\tau(\delta C_\tau(\overline{1} - F^l(\mu), r), r) \\ &= \overline{1} - [C_\tau(\delta I_\tau(F^l(\mu), r), r) \vee I_\tau(\delta C_\tau(F^l(\mu), r), r)] \\ F^l(\mu) &\leq C_\tau(\delta I_\tau(F^l(\mu), r), r) \vee I_\tau(\delta C_\tau(F^l(\mu), r), r). \end{aligned}$$

Hence, $F^l(\mu)$ is r-feo.

(ii) $\Rightarrow$ (i): Let $x_t \in dom(F)$, $\mu \in L^Y$ and μ is r-feo, with $x_t \in F^l(\mu)$ we have by (ii), $F^l(\mu) = \lambda$(say) is r-feo, then there exists $\lambda \in L^X$, λ is r-feo-set and $x_t \in \lambda$ such that $\lambda \leq F^l(\mu)$. Thus F is FLe-irreesolute. □

Theorem 3.2 *Let $F : X \multimap Y$ be a FM and normalized between two L-fts's (X, τ), (Y, η) and $\mu \in L^Y$, then the following are equivalent:*

(i) F is FUe-irresolute.

(ii) $F^u(\mu)$ is r-feo set, for any μ is r-feo.

(iii) $F^l(\mu)$ is r-fec set, μ is r-fec.

(iv) $eC_\tau(F^l(\mu), r) \leq F^l(eC_\eta(\mu, r))$, for any $\mu \in L^Y$.

(v) $C_\tau(\delta I_\tau(F^l(\mu), r), r) \wedge I_\tau(\delta C_\tau(F^l(\mu), r), r) \leq F^l(eC_\eta(\mu, r))$, for any $\mu \in L^Y$.

Proof. This can be proved in a similar way as Theorem 3.1. □

Corollary 3.1 *Let $F : X \multimap Y$ be a FM between two fts's (X, τ), (Y, η) and $\mu \in L^Y$. Then we have the following:*

(i) If F is normalized, then F is FUe-irresolute at a fuzzy point x_t iff $x_t \in r$-feo set of $F^u(\mu)$, for each μ is r-feo and $x_t \in F^u(\mu)$.

(ii) F is FLe-irresolute at a fuzzy point x_t iff $x_t \in r$-feo set of $F^l(\mu)$, for each μ is r-feo and $x_t \in F^l(\mu)$.

Theorem 3.3 *Let $\{F_i\}_{i\in\Gamma}$ be a family of FLe-irresolute between two fts's (X, τ) and (Y, η). Then $\bigcup\limits_{i\in\Gamma} F_i$ is FLe-irresolute.*

Proof. Let $\mu \in L^Y$, then $(\bigcup\limits_{i\in\Gamma} F_i)^l(\mu) = \bigvee\limits_{i\in\Gamma}(F_i^l(\mu))$ by, Theorem 1.3 (ii). Since $\{F_i\}_{i\in\Gamma}$ is a family of FLe-irresolute between two fts's (X, τ) and (Y, η), then $F_i^l(\mu)$ is r-feo for any μ is r-feo. Then we have $(\bigcup\limits_{i\in\Gamma} F_i)^l(\mu) = \bigvee_{i\in\Gamma}(F_i^l(\mu))$ is r-feo for any μ is r-feo. Hence $\bigcup\limits_{i\in\Gamma} F_i$ is FLe-irresolute. □

Theorem 3.4 *Let $\{F_i\}_{i\in\Gamma}$ be a family of normalized FUe-irresolute between two fts's (X, τ) and (Y, η). Then $F_1 \bigcup F_2$ is FUe-irresolute.*

Proof. Let $\mu \in L^Y$, then

$$(F_1 \cup F_2)^u(\mu) = {F_1}^u(\mu) \wedge {F_2}^u(\mu)$$

by, Theorem 1.3 (iii). Since $\{F_i\}_{i\in\Gamma}$ is a family of normalized FUe-irresolute between two fts's $(X,\ \tau)$ and $(Y,\ \eta)$, then ${F_i}^u(\mu)$ is r-feo, for any μ is r-feo, for each $i \in \{1, 2\}$. Then for each $\mu \in L^Y$, we have $(F_1 \cup F_2)^u(\mu) = {F_1}^u(\mu) \wedge {F_2}^u(\mu)$ is r-feo, for any μ is r-feo set. Hence $F_1 \cup F_2$ is FUe-irresolute. □

Theorem 3.5 *Let $F : X \multimap Y$ be a crisp FUe-irresolute and e-compact valued between two fts's $(X,\ \tau)$ and $(Y,\ \eta)$. Then the direct image of a r-fuzzy e-compact in X under F is also r-fuzzy e-compact.*

Proof. Let λ be r-fuzzy e-compact set in X and $\{\gamma_i : \gamma_i$ is r-feo set in Y, $i \in \Gamma\}$ be a family of covering of $F(\lambda)$. i.e. $F(\lambda) \leq \bigvee_{i\in\Gamma} \gamma_i$. Since $\lambda = \bigvee_{x_t\in\lambda} x_t$, we have

$$F(\lambda) = F(\bigvee_{x_t\in\lambda} x_t) = \bigvee_{x_t\in\lambda} F(x_t) \leq \bigvee_{i\in\Gamma} \gamma_i.$$

It follows that for each $x_t \in \lambda$, $F(x_t) \leq \bigvee_{i\in\Gamma} \gamma_i$. Since F is fuzzy e-compact valued, then there exists finite subset Γ_{x_t} of Γ such that $F(x_t) \leq \bigvee_{n\in\Gamma_{x_t}} \gamma_n = \gamma_{x_t}$. By Theorem 1.1 (viii), we have,

$$x_t \leq F^u(F(x_t)) \leq F^u(\gamma_{x_t}) \text{ and } \lambda = \bigvee_{x_t\in\lambda} x_t = \bigvee_{x_t\in\lambda} F^u(\gamma_{x_t}).$$

Since, γ_{x_t} is r-feo, then from Theorem 2.2, we have $F^u(\gamma_{x_t})$ is r-feo. Hence $\{F^u(\gamma_{x_t}) : F^u(\gamma_{x_t})$ is r-feo, $x_t \in \lambda\}$ is a family covering the set λ. Since λ is r-fuzzy e-compact, then there exists finite index set N such that $\lambda \leq \bigvee_{n\in N} F^u(\gamma_{x_{t_n}})$. From Theorem 1.1 (vii), we have

$$F(\lambda) \leq F(\bigvee_{n\in N} F^u(\gamma_{x_{t_n}})) = \bigvee_{n\in N} F(F^u(\gamma_{x_{t_n}})) \leq \bigvee_{n\in N} \gamma_{x_{t_n}}.$$

Then $F(\lambda)$ is r-fuzzy e-compact. □

Theorem 3.6 *Let $F : X \multimap Y$ and $H : Y \multimap Z$ be two FM's and let $(X,\ \tau)$, $(Y,\ \eta)$ and $(Z,\ \delta)$ be three L-fts's. If F is FLe-irresolute and H is FLe-irresolute, then $H \circ F$ is FLe-irresolute.*

Proof. Let $\nu \in L^Z$, ν is r-feo. Since H is FLe-irresolute, then by Theorem 3.1, $H^l(\nu)$ is r-feo set in Y. Also, F is FLe-irresolute implies $F^l(H^l(\nu))$ is r-feo set in X. Hence, we have $(H \circ F)^l(\nu) = F^l(H^l(\nu))$ is r-feo. Thus $H \circ F$ is FLe-irresolute. □

Theorem 3.7. *Let $F : X \multimap Y$ and $H : Y \multimap Z$ be two FM's and let $(X,\ \tau)$, $(Y,\ \eta)$ and $(Z,\ \delta)$ be three L-fts's. If F is FUe-irresolute and H is FUe-irresolute, then $H \circ F$ is FUe-irresolute.*

Proof. This can be proved in a similar way as Theorem 3.6. □

Theorem 3.8. *Let $F : X \multimap Y$ and $H : Y \multimap Z$ be two FM's and let $(X,\ \tau)$, $(Y,\ \eta)$ and $(Z,\ \delta)$ be three L-fts's. If F is FLe-irresolute and H is FLe-continuous, then $H \circ F$ is FLe-continuous.*

Proof. Let $\nu \in L^Z,\ \delta(\nu) \geq r$. Since H is FLe-continuous, then by Theorem 2.1, $H^l(\nu)$ is r-feo set in Y. Also, F is FLe-irresolute implies $F^l(H^l(\nu))$ is r-feo set in X. Hence, we have $(H \circ F)^l(\nu) = F^l(H^l(\nu))$ is r-feo. Thus $H \circ F$ is FLe-continuous. □

Theorem 3.9. *Let $F : X \multimap Y$ and $H : Y \multimap Z$ be two FM's and let $(X,\ \tau)$, $(Y,\ \eta)$ and $(Z,\ \delta)$ be three L-fts's. If F and H are normalized, F is FUe-irresolute and H is FUe-continuous, then $H \circ F$ is FUe-continuous.*

Proof. This can be proved in a similar way as Theorem 3.8. □

Theorem 3.10. *Let $F : X \multimap Y$ and $H : Y \multimap Z$ be two FM's and let $(X,\ \tau)$, $(Y,\ \eta)$ and $(Z,\ \delta)$ be three L-fts's. If F is normalized and FUe-irresolute and H is FLe-continuous, then $H \circ F$ is FUe-continuous.*

Proof. Let $\nu \in L^Z,\ \delta(\nu) \geq r$. Since H is FLe-continuous, then from Theorem 2.1, $H^l(\nu)$ is r-feo set in Y. Also, F is normalized and FUe-irresolute implies $F^u(H^l(\nu))$ is r-feo set in X by, Theorem 3.2. Hence, we have $(H \circ F)^u(\nu) = F^u(H^l(\nu))$ is r-feo. Thus $H \circ F$ is FUe-continuous. □

Theorem 3.11. *Let $F : X \multimap Y$ and $H : Y \multimap Z$ be two FM's and let $(X,\ \tau)$, $(Y,\ \eta)$ and $(Z,\ \delta)$ be three L-fts's. If H is normalized and FUe-continuous, F is FLe-irresolute, then $H \circ F$ is FLe-continuous.*

Proof. Let $\nu \in L^Z,\ \delta(\nu) \geq r$. Since H is FUe-continuous, then $H^u(\nu)$ is r-feo set in Y. Also, F is FLe-irresolute implies $F^l(H^u(\nu))$ is r-feo set in X by, Theorem 3.1. Hence, we have $(H \circ F)^l(\nu) = F^l(H^u(\nu))$ is r-feo. Thus $H \circ F$ is FLe-continuous. □

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An integral representation of a subclass of analytic functions

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Abstract In the present paper, we study the class $R_p(\gamma,\ \alpha)$ given as

$$R_p(\gamma,\ \alpha) = \left\{ f \in \mathcal{A}_p : \Re\left((1-\alpha)\frac{I_p(n,\ \lambda)f(z)}{z^p} + \alpha\frac{I_p(n+1,\ \lambda)f(z)}{z^p} \right) > \gamma,\ z \in \mathbb{E} \right\}.$$

We find the integral representation of $I_p(n,\ \lambda)f(z)$ as a sufficient condition for $f \in \mathcal{A}_p$ to be a member of the class $R_p(\gamma,\ \alpha)$. The results of some known classes in this direction appear as particular cases of our main result.

Keywords multivalent function, analytic function, multiplier transformation, extreme points.
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§1. Introduction

Let $\mathcal{A}$ be the class of functions f, analytic in the open disk $\mathbb{E} = \{z : |z| < 1\}$ in the complex plane $\mathbb{C}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$. Then $f \in \mathcal{A}$ has the Taylor series expansion

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k,\ p \in \mathbb{N} = \{1,\ 2,\ 3.\dots\},$$

analytic and multivalent in the open disk $\mathbb{E}$. Note that $\mathcal{A}_1 = \mathcal{A}$. For $f \in \mathcal{A}_p$, define a multiplier transformation $I_p(n,\ \lambda)f(z)$, as follows:

$$I_p(n,\ \lambda)f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{k+\lambda}{p+\lambda}\right)^n a_k z^k,\ (\lambda \ge 0,\ n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}),$$

The special case $I_1(n,\ 0)$ of the above defined operator is the well-known Sălăgean [9] derivative operator D^n, defined for $f \in \mathcal{A}$ as given below:

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k.$$

Singh et al. [11], Krzyz [5] and Chichra [2] studied the class $R(\beta),\ \beta < 1$, defined as:

$$R(\beta) = \{f \in \mathcal{A} : \Re(f'(z) + zf''(z)) > \beta,\ z \in \mathbb{E}\},$$

where β is given by

$$\beta_{\mathcal{S}} = \inf\{\beta : R(\beta) \subset \mathcal{S}\}$$

and

$$\beta_{\mathcal{S}^*} = \inf\{\beta : R(\beta) \subset \mathcal{S}^*\}.$$

Later on, Singh et al. [12] showed that $\beta_{\mathcal{S}^*} \le -\frac{1}{4}$ which was further improved by Ali [1]. Gao [3] and Silverman [10] proved independently and obtained $\beta_{\mathcal{S}^*} \le \frac{6-\pi^2}{24-\pi^2}$. In 2007, Gao et al. [4] studied the following subclass of $\mathcal{A}$:

$$R(\beta,\ \alpha) = \{f \in \mathcal{A} : \Re(f'(z) + \alpha z f''(z)) > \beta,\ z \in \mathbb{E}\},$$

where $\beta < 1,\ \alpha > 0$. They determined the extreme points of $R(\beta,\ \alpha)$ and obtain sharp bounds for $\Re(f'(z))$ and $\Re(f(z)/z)$. They also determined the number $\beta(\alpha)$ such that $R(\beta,\ \alpha) \subset \mathcal{S}^*$, for certain fixed number α in $[1,\ \infty)$. Recently, Wang et al. [13] studied the class $Q(\alpha,\ \beta,\ \gamma)$ defined as:

$$Q(\alpha,\ \beta,\ \gamma) = \{f \in \mathcal{A} : \Re[\alpha(f(z)/z) + \beta f'(z)] > \gamma,\ (\alpha,\ \beta) > 0,\ 0 \le \gamma < \alpha + \beta \le 1;\ z \in \mathbb{E}\}.$$

They provided the extreme points and radius of univalence for the members of this class. In the present paper, we study the following subclass $R_p(\gamma,\ \alpha)$ of $\mathcal{A}_p$ involving multivalent functions :

$$R_p(\gamma,\ \alpha) = \left\{ f \in \mathcal{A}_p : \Re\left((1-\alpha)\frac{I_p(n,\ \lambda)f(z)}{z^p} + \alpha \frac{I_p(n+1,\ \lambda)f(z)}{z^p} \right) > \gamma,\ z \in \mathbb{E} \right\},$$

where $\alpha > 0$ and $0 \le \gamma < \alpha \le 1$.

§2. Main Result

Theorem 2.1 *A function $f \in \mathcal{A}_p$ is in $R_p(\gamma,\ \alpha)$ if and only if $I_p(n,\ \lambda)f(z)$ can be expressed as*

$$I_p(n,\ \lambda)f(z) = \int_{|x|=1} \left[(2\gamma - 1)z^p + (2 - 2\gamma) \sum_{m=0}^{\infty} \frac{x^m z^{m+p}}{m\beta + 1} \right] d\mu(x) \qquad (1)$$

where $\mu(x)$ is the probability measure defined on the $X = \{x : |x| = 1\}$. For fixed β and α, $R_p(\gamma,\ \alpha)$ and the probability measures $\{\mu\}$ defined on X are one-to-one by the expression (1).

Proof. Let $u(z) = \frac{I_p(n,\ \lambda)f(z)}{z^p}$.
Differentiating lograthmically, we have :

$$\frac{zu'(z)}{u(z)} = \frac{z(I_p(n,\ \lambda)f(z))'}{Ip(n,\ \lambda)f(z)} - p. \tag{2}$$

In the view of relation

$$zI_p^{'}(n,\ \lambda)f(z) = (p+\lambda)I_p(n+1,\ \lambda)f(z) - \lambda I_p(n,\ \lambda)f(z),$$

(2) becomes

$$\frac{zu'(z)}{(p+\lambda)u(z)} + 1 = \frac{I_p(n+1,\ \lambda)f(z))}{Ip(n,\ \lambda)f(z)}.$$

Hence

$$\frac{I_p(n+1,\ \lambda)f(z)}{Ip(n,\ \lambda)f(z)} = u(z) + \frac{1}{p+\lambda}zu'(z).$$

Now

$$(1-\alpha)\frac{I_p(n,\ \lambda)f(z)}{z^p} + \alpha\frac{I_p(n+1,\ \lambda)f(z)}{z^p} = u(z) + \beta zu'(z), \tag{3}$$

where $\beta = \frac{\alpha}{p+\lambda}$. Since $f \in R_p(\gamma,\ \alpha)$, therefore

$$\Re(u(z) + \beta zu'(z)) > \gamma.$$

Let $\mathcal{P}$ denote the normalized class of analytic functions which have positive real part. Therefore $f \in R_p(\gamma,\ \alpha)$ if and only if

$$\frac{u(z) + \beta zu'(z) - \gamma}{1-\gamma} \in \mathcal{P},\ u(0) = 1.$$

By Herglotz expression of functions in $\mathcal{P}$, we have

$$\frac{u(z) + \beta zu'(z) - \gamma}{1-\gamma} = \int_{|x|=1} \frac{1+xz}{1-xz} d\mu(x),$$

which is equivalent to

$$\frac{1}{\beta}u(z) + zu'(z) = \frac{1}{\beta}\int_{|x|=1} \frac{1+(1-2\gamma)xz}{1-xz} d\mu(x).$$

Therefore

$$z^{-\frac{1}{\beta}} \int_0^z \left(\frac{1}{\beta}u(\zeta) + \zeta u'(z)\right) \zeta^{\frac{1}{\beta}-1} d\zeta = \frac{1}{\beta}\int_{|x|=1} \left(z^{-\frac{1}{\beta}} \int_0^z \frac{1+(1-2\gamma)x\zeta}{1-x\zeta}\zeta^{\frac{1}{\beta}-1} d\zeta\right) d\mu(x)$$

i.e.

$$u(z) = \int_{|x|=1} \left((2\gamma-1) + (2-2\gamma)\sum_{m=0}^{\infty} \frac{(xz)^m}{m\beta+1}\right) d\mu(x),$$

which is equivalent to

$$I_p(n,\ \lambda)f(z) = \int_{|x|=1} \left((2\gamma-1)z^p + (2-2\gamma)\sum_{m=0}^{\infty} \frac{x^m z^{m+p}}{m\beta+1}\right) d\mu(x),$$

Since the probability measures $\{\mu\}$ and the class $\mathcal{P}$ as well as class $\mathcal{P}$ and $R_p(\gamma,\ \alpha)$ are one-to-one, so the second part of the theorem is true and can be proved by deduction. This completes the proof of Theorem 2.1. □

Corollary 2.2 *The extreme points of the class $R_p(\gamma,\ \alpha)$ are*

$$I_p(n,\ \lambda)f_x(z) = (2\gamma-1)z^p + (2-2\gamma)\sum_{m=0}^{\infty}\frac{x^m z^{m+p}}{m\beta+1},\ |x|=1. \tag{4}$$

Proof. Using the notation $I_p(n,\ \lambda)f_x(z)$, equation (1) can be written as

$$I_p(n,\ \lambda)f_\mu(z) = \int_{|x|=1} I_p(n,\ \lambda)f_x(z)d\mu(x).$$

By Theorem 2.1, the map $\mu \to f_\mu$ is one-to-one, so the proof follows. □

For $p=1$ and $n=0=\lambda$ in the Theorem 2.1, we get:

Corollary 2.3 *For $f \in R_1(\gamma,\ \alpha)$, where $\alpha > 0$ and $0 \le \gamma < \alpha \le 1$,*

$$\Re\left((1-\beta)\frac{f(z)}{z} + \beta f'(z)\right) > \gamma,\ \beta = \alpha.$$

therefore

$$I_1(0,\ 0)f(z) = f(z) = \int_{|x|=1}\left((2\gamma-1)z + (2-2\gamma)\sum_{m=0}^{\infty}\frac{x^m z^{m+1}}{m\beta+1}\right)d\mu(x).$$

For $\alpha = 1-\beta$, the above expression obtained by Wang et al. [13]. Saitoh [8] and Owa [6, 7] discussed the related properties of $Q(1-\beta,\ \beta,\gamma) = R_1(\gamma,\ 1-\beta)$.

Selecting $p=1=n$ and $\lambda=0$ in Theorem 2.1, we have the following result:

Corollary 2.4 *If $f \in R_1(\gamma,\ \alpha)$, where $\alpha > 0$ and $0 \le \gamma < \alpha \le 1$, satisfies*

$$\Re\left((1-\alpha)\frac{I_1(1,\ 0)f(z)}{z} + \alpha\frac{I_1(2,\ 0)f(z)}{z}\right) = \Re(f'(z) + \beta f''(z)) > \gamma,$$

where $\beta = \alpha > 0$, then

$$I_1(1,\ 0)f(z) = zf'(z) = \int_{|x|=1}\left((2\gamma-1)z + (2-2\gamma)\sum_{m=0}^{\infty}\frac{x^m z^{m+1}}{m\beta+1}\right)d\mu(x)$$

which on further simplification gives

$$f(z) = \int_{|x|=1}\left((2\gamma-1)z + (2-2\gamma)\bar{x}\sum_{m=0}^{\infty}\frac{(xz)^{m+1}}{(m+1)(m\beta+1)}\right)d\mu(x).$$

This result was obtained by Gao et al. [4]. If we select $\beta=1$ in the above result, we get:

Corollary 2.5 *If $f \in R_1(\gamma,\ 1)$, where $\alpha=\beta=1$ and $0 \le \gamma < 1$, then*

$$f(z) = \int_{|x|=1}\left((2\gamma-1)z + (2-2\gamma)\bar{x}\sum_{m=0}^{\infty}\frac{(xz)^{m+1}}{(m+1)^2}\right)d\mu(x)$$

$$= \int_{|x|=1} \left(\int_0^z \frac{(2\gamma - 1)\zeta + (2\gamma - 2)\bar{x}\log(1 - x\zeta)}{\zeta} d\zeta \right) d\mu(x),$$

this result was also obtained by Silverman [10].

Selecting $p = 1,\ n = 2$ and $\lambda = 0$, we get the following result from Theorem 2.1:

Corollary 2.6 *If $f \in R_1(\gamma,\ \alpha),\ \alpha > 0$ and $0 \leq \gamma < \alpha \leq 1$, satisfies*

$$\Re\left((1-\alpha)\frac{I_1(2,\ 0)f(z)}{z} + \alpha\frac{I_1(3,\ 0)f(z)}{z} \right) = \Re(f'(z) + (1+2\beta)zf''(z)) + \beta z^2 f'''(z)) > \gamma,$$

where $\beta = \alpha > 0,\ 0 \leq \gamma < 1$, then

$$I_1(2,\ 0)f(z) = zf'(z) + z^2 f''(z) = \int_{|x|=1} \left((2\gamma - 1)z + (2 - 2\gamma) \sum_{m=0}^{\infty} \frac{x^m z^{m+1}}{m\beta + 1} \right) d\mu(x)$$

Further, we get

$$zf'(z) = \int_{|x|=1} \left((2\gamma - 1)z + (2 - 2\gamma) \sum_{m=0}^{\infty} \frac{x^m z^{m+1}}{(m+1)(m\beta + 1)} \right) d\mu(x).$$

Hence

$$f(z) = \int_{|x|=1} \left((2\gamma - 1)z + (2 - 2\gamma)\bar{x} \sum_{m=0}^{\infty} \frac{(xz)^{m+1}}{(m+1)^2(m\beta + 1)} \right) d\mu(x). \tag{5}$$

Corollary 2.7 *If $f \in \mathcal{A}$ and*

$$\Re(f'(z) + (1+2\beta)zf''(z)) + \beta z^2 f'''(z)) > \gamma,$$

where $\beta > 0,\ 0 \leq \gamma < 1$, then extreme points of this class are given by (5) as

$$f_x(z) = (2\gamma - 1)z + (2 - 2\gamma)\bar{x} \sum_{m=0}^{\infty} \frac{(xz)^{m+1}}{(m+1)^2(m\beta + 1)},\ |x| = 1. \tag{6}$$

Corollary 2.8 *If $f \in \mathcal{A}$ and*

$$\Re(f'(z) + (1+2\beta)zf''(z)) + \beta z^2 f'''(z)) > \gamma,$$

where $\beta > 0,\ 0 \leq \gamma < 1$, then

$$|a_n| \leq \frac{2(1-\gamma)}{m^2(\beta(m-1)+1)},\ m \geq 2.$$

The result is sharp.

Proof. The coefficient bounds are maximized at an extreme point.Thus from (6), $f_x(z)$ can be expressed as

$$f_x(z) = z + 2(1-\gamma) \sum_{m=2}^{\infty} \frac{x^{m-1} z^m}{m^2(\beta(m-1)+1)},\ |x| = 1.$$

and hence the result follows. □

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