A CLASS OF RECURSIVE SETS

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In this article one builds a class of recursive sets, one establishes properties of these sets and one proposes applications. This article widens some results of [1].

1) Definitions, properties.

One calls recursive sets the sets of elements which are built in a recursive manner: let $T$ be a set of elements and $f_i$ for $i$ between 1 and $s$, of operations $n_i$, such that $f_i : T^n \rightarrow T$. Let’s build by recurrence the set $M$ included in $T$ and such that:

(Def. 1) 1°) certain elements $a_1,...,a_n$ of $T$, belong to $M$.

2°) if $(\alpha_{i_1},...,\alpha_{i_n})$ belong to $M$, then $f_i(\alpha_{i_1},...,\alpha_{i_n})$ belong to $M$ for all $i \in \{1,2,...,s\}$.

3°) each element of $M$ is obtained by applying a number finite of times the rules 1° or 2°.

We will prove several proprieties of these sets $M$, which will result from the manner in which they were defined. The set $M$ is the representative of a class of recursive sets because in the rules 1° and 2°, by particularizing the elements $a_1,...,a_n$ respectively $f_1,...,f_s$ one obtains different sets.

Remark 1: To obtain an element of $M$, it is necessary to apply initially the rule 1.

(Def. 2) The elements of $M$ are called elements $M$-recursive.

(Def. 3) One calls order of an element $a$ of $M$ the smallest natural $p \geq 1$ which has the propriety that $a$ is obtained by applying $p$ times the rule 1° or 2°.

One notes $M_p$ the set which contains all the elements of order $p$ of $M$. It is obvious that $M_1 = \{a_1,...,a_n\}$.

$$M_2 = \bigcup_{i=1}^{s} \left( \bigcup_{(a_{i_1},...,a_{i_n}) \in M_1^n} f_i(\alpha_{i_1},...,\alpha_{i_n}) \right) \setminus M_1.$$ 

One withdraws $M_1$ because it is possible that $f_j(a_{j_1},...,a_{j_n}) = a_i$ which belongs to $M_1$, and thus does not belong to $M_2$.

One proves that for $k \geq 1$ one has:
\[ M_{k+1} = \bigcup_{i=1}^{r} \left( \bigcup_{(a_1, \ldots, a_n) \in \prod_{i}^{(i)}} f_i(\alpha_{i_1}, \ldots, \alpha_{i_n}) \right) \setminus \bigcup_{h=1}^{k} M_h \]

where each

\[ \prod_{k}^{(i)} = \left\{ (\alpha_{i_1}, \ldots, \alpha_{i_n}) \mid \alpha_{i_j} \in M_{q_j}, \ j \in \{1, 2, \ldots, n_i\}; \ 1 \leq q_j \leq k \right\} \]

The sets \( M_p, \ p \in \mathbb{N}^* \), form a partition of the set \( M \).

**Theorem 1:**

\[ M = \bigcup_{p \in \mathbb{N}^*} M_p, \text{ where } \mathbb{N}^* = \{1, 2, 3, \ldots\}. \]

**Proof:**

From the rule 1° it results that \( M_1 \subseteq M \).

One supposes that this propriety is true for values which are less than \( p \). It results that \( M_p \subseteq M \), because \( M_p \) is obtained by applying the rule 2° to the elements of \( \bigcup_{i=1}^{p-1} M_i \).

Thus \( \bigcup_{p \in \mathbb{N}^*} M_p \subseteq M \). Reciprocally, one has the inclusion in the contrary sense in accordance with the rule 3°.

**Theorem 2:** The set \( M \) is the smallest set, which has the properties 1° and 2°.

**Proof:**

Let \( R \) be the smallest set having properties 1° and 2°. One will prove that this set is unique.

Let’s suppose that there exists another set \( R' \) having properties 1° and 2°, which is the smallest. Because \( R \) is the smallest set having these proprieties, and because \( R' \) has these properties also, it results that \( R \subseteq R' \); of an analogue manner, we have \( R' \subseteq R \): therefore \( R = R' \).

It is evident that \( M' \subseteq R \). One supposes that \( M_i \subseteq R \) for \( 1 \leq i < p \). Then (rule 3°), and taking in consideration the fact that each element of \( M_p \) is obtained by applying rule 2° to certain elements of \( M_i \), \( 1 \leq i < p \), it results that \( M_p \subseteq R \). Therefore \( \bigcup_{p \in \mathbb{N}^*} M_p \subseteq R \) (\( p \in \mathbb{N}^* \)), thus \( M \subseteq R \). And because \( R \) is unique, \( M = R \).

**Remark 2.** The theorem 2 replaces the rule 3° of the recursive definition of the set \( M \) by: "\( M \) is the smallest set that satisfies proprieties 1° and 2°."

**Theorem 3:** \( M \) is the intersection of all the sets of \( T \) which satisfy conditions 1° and 2°.

**Proof:**
Let \( T_{12} \) be the family of all sets of \( T \) satisfying the conditions 1° and 2°. We note \( I = \bigcap_{A \in T_{12}} A \).

\( I \) has the properties 1° and 2° because:
1) For all \( i \in \{1,2,...,n\} \), \( a_i \in I \), because \( a_i \in A \) for all \( A \) of \( T_{12} \).
2) If \( \alpha_{i_1},...,\alpha_{i_n} \in I \), it results that \( \alpha_{i_1},...,\alpha_{i_n} \) belong to \( A \) that is \( A \) of \( T_{12} \).

Therefore,
\[ \forall i \in \{1,2,...,s\}, \ f_i(\alpha_{i_1},...,\alpha_{i_n}) \in A \] which is \( A \) of \( T_{12} \), therefore \( f_i(\alpha_{i_1},...,\alpha_{i_n}) \in I \)

for all \( i \) from \( \{1,2,...,s\} \).

From theorem 2 it results that \( M \subseteq I \).
Because \( M \) satisfies the conditions 1° and 2°, it results that \( M \in T_{12} \), from which \( I \subseteq M \). Therefore \( M = I \)

(Def. 4) A set \( A \subseteq I \) is called closed for the operation \( f_{b_0} \) if and only if for all \( \alpha_{b_1},...,\alpha_{b_{n_0}} \) of \( A \), one has \( f_{b_0}(\alpha_{b_1},...,\alpha_{b_{n_0}}) \) belong to \( A \).

(Def. 5) A set \( A \subseteq T \) is called \( M \)-recursively closed if and only if:
1) \( \{a_1,...,a_n\} \subseteq A \).
2) \( A \) is closed in respect to operations \( f_1,...,f_s \).

With these definitions, the precedent theorems become:

**Theorem 2':** The set \( M \) is the smallest \( M \)-recursively closed set.

**Theorem 3':** \( M \) is the intersection of all \( M \)-recursively closed sets.

(Def. 6) The system of elements \( \langle \alpha_1,...,\alpha_m \rangle, \ m \geq 1 \) and \( \alpha_i \in T \) for \( i \in \{1,2,...,m\} \), constitute a \( M \)-recursive description for the element \( \alpha \), if \( \alpha_m = \alpha \) and that each \( \alpha_i \) (\( i \in \{1,2,...,m\} \)) satisfies at least one of the proprieties:
1) \( \alpha_i \in \{a_1,...,a_n\} \).
2) \( \alpha_i \) is obtained starting with the elements which precede it in the system by applying the functions \( f_j, \ 1 \leq j \leq s \) defined by property 2° of (Def. 1).

(Def. 7) The number \( m \) of this system is called the length of the \( M \)-recursive description for the element \( \alpha \).

**Remark 3:** If the element \( \alpha \) admits a \( M \)-recursive description, then it admits an infinity of such descriptions.

Indeed, if \( \langle \alpha_1,...,\alpha_m \rangle \) is a \( M \)-recursive description of \( \alpha \) then \( \langle a_{1},...,a_{h},\alpha_{1},...,\alpha_{m} \rangle \) is also a \( M \)-recursive description for \( \alpha \), \( h \) being able to take all values from \( \mathbb{N} \).
**Theorem 4:** The set $M$ is identical with the set of all elements of $T$ which admit a $M$-recursive description.

**Proof:** Let $D$ be the set of all elements, which admit a $M$-recursive description. We will prove by recurrence that $M_p \subseteq D$ for all $p$ of $\mathbb{N}^*$.

For $p=1$ we have: $M_1 = \{a_1,\ldots,a_n\}$, and the $a_j$, $1 \leq j \leq n$, having as $M$-recursive description: $\langle a_j \rangle$. Thus $M_1 \subseteq D$. Let’s suppose that the property is true for the values smaller than $p$. $M_p$ is obtained by applying the rule $2^\circ$ to the elements of $\bigcup_{i=1}^{p-1} M_i$; $\alpha \in M_p$ implies that $\alpha \in f_j(\alpha_{i_1},\ldots,\alpha_{i_{n_1}})$ and $\alpha_{i_j} \in M_{h_j}$ for $h_j < p$ and $1 \leq j \leq n_i$.

But $a_j$, $1 \leq j \leq n_i$, admits $M$-recursive descriptions according to the hypothesis of recurrence, let’s have $\langle \beta_{j_1},\ldots,\beta_{j_{n_j}} \rangle$. Then $\langle \beta_{j_1},\ldots,\beta_{j_{n_1}},\beta_{2j_{1}},\ldots,\beta_{2j_{2}},\ldots,\beta_{n_{j_1}},\ldots,\beta_{n_{j_{n_j}}},\alpha \rangle$ constitute a $M$-recursive description for the element $\alpha$. Therefore if $\alpha$ belongs to $D$, then $M_p \subseteq D$ which is $M = \bigcup_{p \in \mathbb{N}} M_p \subseteq D$.

Reciprocally, let $x$ belong to $D$. It admits a $M$-recursive description $\langle b_1,\ldots,b_t \rangle$ with $b_1 = x$. It results by recurrence by the length of the $M$-recursive description of the element $x$, that $x \in M$. For $t = 1$ we have $\langle b_1 \rangle$, $b_1 = x$ and $b_1 \in \{a_1,\ldots,a_n\} \subseteq M$. One supposes that all elements $y$ of $D$ which admit a $M$-recursive description of a length inferior to $t$ belong to $M$. Let $x \in D$ be described by a system of length $t$: $\langle b_1,\ldots,b_t \rangle$, $b_t = x$. Then $x \in \{a_1,\ldots,a_n\} \subseteq M$, where $x$ is obtained by applying the rule $2^\circ$ to the elements which precede it in the system: $b_1,\ldots,b_{t-1}$. But these elements admit the $M$-recursive descriptions of length which is smaller that $t$: $\langle b_1 \rangle,\langle b_1,b_2 \rangle,\ldots,\langle b_1,\ldots,b_{t-1} \rangle$. According to the hypothesis of the recurrence, $b_1,\ldots,b_{t-1}$ belong to $M$. Therefore $b_t$ belongs also to $M$. It results that $M \equiv D$.

**Theorem 5:** Let $b_1,\ldots,b_q$ be elements of $T$, which are obtained from the elements $a_1,\ldots,a_n$ by applying a finite number of times the operations $f_1,f_2,\ldots, f_s$. Then $M$ can be defined recursively in the following mode:

1) Certain elements $a_1,\ldots,a_n,b_1,\ldots,b_q$ of $T$ belong to $M$.

2) $M$ is closed for the applications $f_i$, with $i \in \{1,2,\ldots,s\}$.

3) Each element of $M$ is obtained by applying a finite number of times the rules (1) or (2) which precede.

**Proof:** evident. Because $b_1,\ldots,b_q$ belong to $T$, and are obtained starting with the elements $a_1,\ldots,a_n$ of $M$ by applying a finite number of times the operations $f_i$, it results that $b_1,\ldots,b_q$ belong to $M$. 

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Theorem 6: Let’s have $g_j$, $1 \leq j \leq r$, of the operations $n_j$, where $g_j : T^{n_j} \rightarrow T$ such that $M$ to be closed in rapport to these operations. Then $M$ can be recursively defined in the following manner:

1) Certain elements $a_1, ..., a_n$ de $T$ belong to $M$.
2) $M$ is closed for the operations $f_i$, $i \in \{1,2,...,s\}$ and $g_j$, $j \in \{1,2,...,r\}$.
3) Each element of $M$ is obtained by applying a finite number of times the precedent rules.

Proof is simple: Because $M$ is closed for the operations $g_j$ (with $j \in \{1,2,...,r\}$), one has, that for any $\alpha_{j_1},...,\alpha_{j_n}$ from $M$, $g_j(\alpha_{j_1},...,\alpha_{j_n}) \in M$ for all $j \in \{1,2,...,r\}$.

From the theorems 5 and 6 it results:

Theorem 7: The set $M$ can be recursively defined in the following manner:

1) Certain elements $a_1, ..., a_n, b_1, ..., b_q$ of $T$ belong to $M$.
2) $M$ is closed for the operations $f_i$ ($i \in \{1,2,...,s\}$) and for the operations $g_j$ ($j \in \{1,2,...,r\}$) previously defined.
3) Each element of $M$ is defined by applying a finite number of times the previous 2 rules.

(Def. 8) The operation $f_i$ conserves the property $P$ iff for any elements $\alpha_{i_1},...,\alpha_{i_n}$ having the property $P$, $f_i(\alpha_{i_1},...,\alpha_{i_n})$ has the property $P$.

Theorem 8: If $a_1, ..., a_n$ have the property $P$, and if the functions $f_1, ..., f_s$ preserve this property, then all elements of $M$ have the property $P$.

Proof:

$$M = \bigcup_{p \in \mathbb{N}} M_p.$$ The elements of $M_1$ have the property $P$.

Let’s suppose that the elements of $M_i$ for $i < p$ have the property $P$. Then the elements of $M_p$ also have this property because $M_p$ is obtained by applying the operations $f_1, f_2, ..., f_s$ to the elements of: $\bigcup_{i=1}^{p} M_i$, elements which have the property $P$.

Therefore, for any $p$ of $\mathbb{N}$, the elements of $M_p$ have the property $P$.

Thus all elements of $M$ have it.

Corollary 1: Let’s have the property $P$: ”$x$ can be represented in the form $F(x)$”.

If $a_1, ..., a_n$ can be represented in the form $F(a_1), ..., F(a_n)$, and if $f_1, ..., f_s$ maintains the property $P$, then all elements $\alpha$ of $M$ can be represented in the form $F(\alpha)$.

Remark. One can find more other equivalent def. of $M$.

2) APPLICATIONS, EXAMPLES.
In applications, certain general notions like: $M$ - recursive element, $M$ -recursive description, $M$ - recursive closed set will be replaced by the attributes which characterize the set $M$. For example in the theory of recursive functions, one finds notions like: recursive primitive functions, primitive recursive description, primitively recursive closed sets. In this case “ $M$ ” has been replaced by the attribute “primitive” which characterizes this class of functions, but it can be replaced by the attributes ”general”, ”partial”.

By particularizing the rules $1^o$ and $2^o$ of the def. 1, one obtains several interesting sets:

**Example 1:** (see [2], pp. 120-122, problem 7.97).

**Example 2:** The set of terms of a sequence defined by a recurring relation constitutes a recursive set.

Let’s consider the sequence: $a_{n+k} = f(a_n,a_{n+1},...,a_{n+k-1})$ for all $n$ of $\mathbb{N}^+$, with $a_i = a_0^i$, $1 \leq i \leq k$. One will recursively construct the set $A = \{a_m\}_{m \in \mathbb{N}^+}$ and one will define in the same time the position of an element in the set $A$:

1°) $a_1^0,...,a_k^0$ belong to $A$, and each $a_i^0$ ($1 \leq i \leq k$) occupies the position $i$ in the set $A$;

2°) if $a_n,a_{n+1},...,a_{n+k-1}$ belong to $A$, and each $a_j$ for $n \leq j \leq n+k-1$ occupies the position $j$ in the set $A$, then $f(a_n,a_{n+1},...,a_{n+k-1})$ belongs to $A$ and occupies the position $n+k$ in the set $A$.

3°) each element of $B$ is obtained by applying a finite number of times the rules $1^o$ or $2^o$.

**Example 3:** Let $G = \{e,a^1,a^2,...,a^p\}$ be a cyclic group generated by the element $a$. Then $(G, \cdot)$ can be recursively defined in the following manner:

1°) $a$ belongs to $G$.

2°) if $b$ and $c$ belong to $G$ then $b \cdot c$ belongs to $G$.

3°) each element of $G$ is obtained by applying a finite number of times the rules 1 or 2.

**Example 4:** Each finite set $ML = \{x_1,x_2,...,x_n\}$ can be recursively defined (with $ML \subseteq T$):

1°) The elements $x_1,x_2,...,x_n$ of $T$ belong to $ML$.

2°) If $a$ belongs to $ML$, then $f(a)$ belongs to $ML$, where $f : T \rightarrow T$ such that $f(x) = x$;

3°) Each element of $ML$ is obtained by applying a finite number of times the rules $1^o$ or $2^o$.

**Example 5:** Let $L$ be a vectorial space on the commutative corps $K$ and $\{x_1,...,x_m\}$ be a base of $L$. Then $L$, can be recursively defined in the following manner:

1°) $x_1,...,x_m$ belong to $L$;

2°) if $x,y$ belong to $L$ and if $a$ belongs to $K$, then $x \perp y$ $y$ belong to $L$ and $a \ast x$ belongs to $L$;

3°) each element of $L$ is recursively obtained by applying a finite number of times the rules $1^o$ or $2^o$. 

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The operators \( \perp \) and \( \ast \) are respectively the internal and external operators of the vectorial space \( L \).

**Example 6:** Let \( X \) be an \( A \)-module, and \( M \subset X \) (\( M \neq \emptyset \)), with \( M = \{ x_i \}_{i \in I} \). The sub-module generated by \( M \) is:

\[
\langle M \rangle = \left\{ x \in X / x = a_1 x_1 + \ldots + a_n x_n , \ a_i \in A, \ x_i \in M, \ i \in \{ 1, \ldots , n \} \right\}
\]

can be recursively defined in the following way:

1°) for all \( i \) of \( \{ 1, 2, \ldots , n \} \), \( \{ 1, 2, \ldots , n \} \) \( x_i \in \langle M \rangle \);

2°) if \( x \) and \( y \) belong to \( \langle M \rangle \) and \( a \) belongs to \( A \), then \( x + y \) belongs to \( \langle M \rangle \), and \( ax \) also;

3°) each element of \( \langle M \rangle \) is obtained by applying a finite number of times the rules 1° or 2°.

In accordance to the paragraph 1 of this article, \( \langle M \rangle \) is the smallest sub-set of \( X \) that verifies the conditions 1° and 2°, that is \( \langle M \rangle \) is the smallest sub-module of \( X \) that includes \( M \). \( \langle M \rangle \) is also the intersection of all the subsets of \( X \) that verify the conditions 1° and 2°, that is \( \langle M \rangle \) is the intersection of all sub-modules of \( X \) that contain \( M \). One also directly refines some classic results from algebra.

One can also talk about sub-groups or ideal generated by a set: one also obtains some important applications in algebra.

**Example 7:** One also obtains like an application the theory of formal languages, because, like it was mentioned, each regular language (linear at right) is a regular set and reciprocally. But a regular set on an alphabet \( \Sigma = \{ a_1, \ldots , a_n \} \) can be recursively defined in the following way:

1°) \( \emptyset, \{ \epsilon \}, \{ a_1 \}, \ldots , \{ a_n \} \) belong to \( R \).

2°) if \( P \) and \( Q \) belong to \( R \), then \( P \cup Q \), \( PQ \), and \( P^* \) belong to \( R \), with \( P \cup Q = \{ x / x \in P \ or \ x \in Q \} ; \ PQ = \{ xy / x \in P \ and \ y \in Q \} \), and \( P^* = \bigcup_{n=0}^{\infty} P^n \) with \( P^n = P \cdot P \cdot \ldots \cdot P \) and, by convention, \( P^0 = \{ \epsilon \} \).

3°) Nothing else belongs to \( R \) other than those which are obtained by using 1° or 2°.

From which many properties of this class of languages with applications to the programming languages will result.

**REFERENCES:**
