Star Chromatic and Defining Number of Graphs

D.A. Mojdeh1, H. Abdollahzadeh Ahangar2, F. Chooopani1, and F. Zeinali1

1. Department of Mathematics, University of Tafresh, Tafresh, Iran
2. Department of Basic Science, Babol University of Technology, Babol, I.R. Iran

E-mail: damojdeh@tafreshu.ac.ir, ha.ahangar@nit.ac.ir

Abstract: Let $u$ and $v$ be adjacent vertices in $G$. If we assign colors to $N[v]$ and $N[u]$ such that the assignment colors to $N[v]$ are different with the assignment colors to $N[u]$, then this colorings is said to be vertex star colorings. In this paper we initiate the study of the star chromatic number and star defining number.

Key Words: Star coloring, star chromatic number, star defining number, Smarandachely $\Lambda$-coloring.

AMS(2010): 05C15

§1. Introduction

In the whole paper, $G$ is a simple graph with vertex set $V(G)$ and edge set $E(G)$ (briefly $V$ and $E$). For every vertex $v \in V$, the open neighborhood $N(v)$ is the set $\{u \in V \mid uv \in E\}$ and its closed neighborhood is the set $N[v] = N(v) \cup \{v\}$. The open neighborhood of a set $S \subseteq V$ is the set $N(S) = \bigcup_{v \in S} N(v)$, and the closed neighborhood of $S$ is the set $N[S] = N(S) \cup S$. We use [9] for terminology and notation which are not defined here.

Let $\Lambda$ be a subgraph of a graph $G$. A Smarandachely $\Lambda$-coloring $\varphi_{\Lambda}|_{V(G)} : \mathcal{C} \to V(G)$ of a graph $G$ by colors in $\mathcal{C}$ is a mapping $\varphi_{\Lambda} : \mathcal{C} \to V(G) \cup E(G)$ such that $\varphi(u) \neq \varphi(v)$ if $u$ and $v$ are vertices of a subgraph isomorphic to $\Lambda$ in $G$. Particularly, if $\Lambda = G$, such a coloring is called a $k$-coloring of $G$. A graph is $k$-colorable if it has a proper $k$-coloring. The chromatic number $\chi(G)$ is the least $k$ such that $G$ is $k$-colorable. Let $\chi(G) \leq k \leq |V(G)|$. A set $S \subseteq V(G)$ with an assignment of colors to them is called a defining set of the vertex coloring of $G$ if there exists a unique extension of $S$ to a $k$-coloring of $G$. A defining set with minimum cardinality is called a minimum defining set and its cardinality is the defining number, denoted by $d(G, k)$, for more see [1, 3, 4, 5, 6, 7].

In this note we introduce vertex star coloring of graphs as follows:

If $u$ and $v$ are arbitrary adjacent vertices in $G$, then the set of colors that we assign to $N[v]$ is different with the set of colors that assign to $N[u]$. We call this vertex coloring as vertex star coloring. It is obvious that vertex star coloring does not include the family of graphs with

1Received October 28, 2013, Accepted March 6, 2014.
2Corresponding author: H.A. Ahangar
following property:

\[ \exists u, v \in V(G) \text{ with } N[v] = N[u], \text{ for which } uv \in E(G). \]

The chromatic number and defining number of vertex star coloring are called the star chromatic number \((\chi^*)\) and star defining number \((d^*)\), respectively.

We make the following observations:

**Observation 1** For every connected graph \(G\) of order \(n \geq 3\), \(\chi^*(G) \geq 3\).

**Observation 2** If \(\chi^*(G) = 3\), then \(|f(N[v])| = 2, |f(N[u])| = 3\) for every two adjacent vertices \(u, v \in V(G)\) for which \(f\) is a star coloring function.

Our purpose in this paper is to initiate the study of the star chromatic number and the star defining number \((d^*)\) of cycles, paths and complete bipartite, hyper cube and Cartesian product \(P_n \times P_m\) graphs.

### §2. Star Chromatic Numbers

In this section the star chromatic number of cycle, path, complete bipartite and Cartesian product \(P_n \times P_m\) graphs are studied.

First, we present a general result as follows:

**Proposition 3** Let \(G\) be a graph. Then \(\chi^*(G) > \chi(G)\).

**Proof** On the one hand, \(\chi^*(G) \geq \chi(G)\). On the other hand, it is enough to show that \(\chi^*(G) \neq \chi(G)\). Suppose to the contrary. First, we increasingly order vertices of \(G\) and color the vertex with the least index by 1. Now, we color the remaining vertices by this manner, i.e: for the next uncolored vertex, we assign an unused color on its neighbors or a new color if be necessary (Greedy algorithm). Hence, a vertex color by \(\chi(G)\) such that its neighbors colored by \(\{1, 2, \cdots, \chi(G) - 1\}\). And a vertex color by \(\chi(G) - 1\) such that its neighbors colored by \(\{1, 2, \cdots, \chi(G) - 2\}\). Without loss of generality, we may assume that \(u\) and \(v\) are two vertices which colored by \(\chi(G) - 1\) and \(\chi(G)\). It follows that the set \(\{1, 2, \cdots, \chi(G)\}\) is the used colors on \(u\) and its neighbors, and on the vertex \(v\) and its neighbors, a contradiction. \(\square\)

**Proposition 4** (i) \(\chi^*(C_n) = 3\) where \(n = 4m\).

(ii) \(\chi^*(C_n) = 4\) where \(n = 4m + 2\).

**Proof** (i) Consider the star coloring function \(f\) as follows:

\[
 f(v_i) = \begin{cases} 
 2 & i \text{ is odd}, \\
 1 & i = 4t + 2, \\
 3 & i = 4t. 
\end{cases}
\]

It implies that \(\chi^*(G) \leq 3\). Hence, by Proposition 3 the desired result follows.

(ii) Define the star coloring function \(f\) as follows:
Let \( f(v_i) = \begin{cases} 
2 & \text{i is odd and } i \neq 4m + 1, \\
3 & i = 4t + 2 \text{ and } i \leq 4m, \\
1 & i = 4t, 4m + 2, \\
4 & i = 4m + 1.
\end{cases} \)

It follows that \( \chi^*(G) \leq 4 \). Now, we show that \( \chi^*(G) \geq 4 \). It is easy to check that for any four consecutive vertices in \( C_n \), namely \( v_i, v_{i+1}, v_{i+2}, v_{i+3} \), we have \( f(v_i) \neq f(v_{i+3}) \). Otherwise, a contradiction. Moreover, we must use 3 different colors on any four consecutive vertices. Using the star coloring function \( f \) in the proof of Part (i), which implies that the vertex \( v_{n-1} \) cannot be colored by 2. The set of the colors of \( v_{4m+1} \) and its neighbors will be the same as the ones of \( v_{4m+2} \) and its neighbors. Thus, it can be colored by 4. Hence the desired result follows.

Now, we continue the study of the star chromatic numbers on odd cycle.

**Proposition 5** \( \chi^*(C_n) = 4 \) where \( n(\neq 5,7) \) is an odd integer.

**Proof** For \( n = 5 \), the star coloring function of \( C_5 \) can be defined as follows: \( f(v_1) = 1, f(v_2) = 3, f(v_3) = 2, f(v_4) = 4, f(v_5) = 5. \)

For \( n = 7 \), the star coloring function of \( C_7 \) can be defined as follows: \( f(v_1) = 1, f(v_2) = 2, f(v_3) = 1, f(v_4) = 3, f(v_5) = 4, f(v_6) = 3, f(v_7) = 5. \)

Let \( n-1 = 6t + 4 \). Consider the star coloring function \( f \) as follows:

\[
\begin{align*}
f(v_i) &= \begin{cases} 
3 & i = 6t + 2, t \geq 1 \text{ and } i = 1, 3, \\
4 & i = 6t + 4, \\
2 & i = 6t, 2, \\
1 & i = n \text{ and } i \text{ is odd and } i \neq 1, 3.
\end{cases}
\end{align*}
\]

Let \( n-1 = 6t \). Consider the star coloring function \( f \) as follows:

\[
\begin{align*}
f(v_i) &= \begin{cases} 
3 & i = 6t + 2, n, \\
4 & i = 6t + 4, n - 1, \\
2 & i = 6t \text{ and } i = 1, n - 3, \\
1 & i \text{ is odd and } i \neq 1, n.
\end{cases}
\end{align*}
\]

Let \( n-1 = 6t + 2, n > 9 \). Consider the star coloring function \( f \) as follows:

\[
\begin{align*}
f(v_i) &= \begin{cases} 
3 & i = 6t + 2, t \geq 1 \text{ and } i = 1, 3 \\
4 & i = 6t + 4, n - 1, \\
2 & i = 6t \text{ and } i = 6t, 2, \\
1 & i \text{ is odd and } i \neq 1, 3.
\end{cases}
\end{align*}
\]

Hence, by Proposition 3 and the fact that \( \chi(C_n) = 3 \) for which \( n \) is an odd integer, we get that \( \chi^*(G) = 4. \)

**Proposition 6** (i) \( \chi^*(P_n) = 3 \) where \( n \) is an odd integer.
(ii) \( \chi^*(P_n) = 4 \) where \( n \geq 4 \) is an even integer.

**Proof** (i) Define the the star coloring function \( f \) as follows:

\[
f(v_i) = \begin{cases} 
2 & i = 2t, \\
1 & i = 4t + 1, \\
3 & i = 4t + 3.
\end{cases}
\]

This completes the proof.

(ii) Using a same fashion star coloring function \( f \) in Part (i), but \( f(v_{n=2m}) = 4 \). It follows that \( \chi^*(P_{n=2m}) \leq 4 \). Now, we consider two cases as follows.

**Case 1** If \( m = 2t \), then, according to the star coloring function \( f \), let \( f(v_{2m-1}) = 3 \). It follows that the vertex \( v_{2m} \) cannot be colored by 2 or 3. Color the vertex \( v_{n-1} \) by 3, so the vertex \( v_n \) cannot be colored by 1, 2 and 3. Thus, it can be colored by 4. Hence the result holds.

**Case 2** If \( m = 2t + 1 \), In the same manner in Case 1 settle this case.

**Proposition 7** \( \chi^*(K_{m,n}) = 3 \).

**Proof** Let \( X = \{x_1, \ldots, x_m\} \) and \( Y = \{y_1, \ldots, y_n\} \) be partite sets of \( K_{m,n} \). On the one hand, we may define the star coloring function \( f \) as follows: \( f(v_i) = 1 \) (1 \( \leq i \leq m \)), \( f(u_j) = 2 \) (1 \( \leq j \leq n-1 \)), \( f(u_n) = 3 \). Thus \( \chi^*(K_{m,n}) \leq 3 \). On the other hand, if we use two colors on vertices of complete bipartite graphs, we imply that \( N[u] = N[v] \) for every vertex \( u \in X \) and \( v \in Y \). So \( \chi^*(K_{m,n}) \geq 3 \). Hence the result holds.

**Theorem 8** \( \chi^*(P_n \times P_m) = 3 \).

**Proof** Let \( v_{ij} \) be the vertex in \( i \)th row and \( j \)th column. Define the star coloring function \( c^* \) as follows:

\[
c^*(v_{ij}) = \begin{cases} 
2 & j \equiv 2 \ (mod \ 4) \text{ and } i \text{ is odd or } j \equiv 3 \ (mod \ 4) \text{ and } i \text{ is even}, \\
3 & j \equiv 0 \ (mod \ 4) \text{ and } i \text{ is odd or } j \equiv 1 \ (mod \ 4) \text{ and } i \text{ is even}, \\
1 & \text{o.w.}
\end{cases}
\]

Hence the result holds.

The following observation has straightforward proof.

**Observation 9** \( \chi^*(Q_k) = 3 \).

§3. Star Defining Numbers

**Proposition 10** \( d^*(C_n, \chi^*) = 2 \) where \( n = 4m \).

**Proof** Let \( S = \{v_1, v_3\} \) and define the star coloring function \( f \) on \( S \) as follows: \( f(v_1) = 1 \), \( f(v_3) = 3 \). It is easy to check that the remaining vertices are forced to get one color which implies that \( d^*(C_{n=4m}, \chi^*) \leq 2 \).
On the other side, it is well-known that $d^*(C_{n=4k},\chi^*) \geq \chi^*(G) - 1 = 2$. This completes the proof. \hfill \Box

Now, the star defining numbers of odd paths are studied.

**Proposition 11**

(i) $d^*(P_n, \chi^*) \leq m - 1$ where $n = 2m$.

(ii) $d^*(P_n, \chi^*) = 2$ where $n = 2m + 1$.

**Proof**

(i) We define $S = \{v_i | i = 3t+1$ and $t(> 0) t$ is even $\} \cup \{v_i | i = 3t, t = 1$ and $t(\geq 3)$ is odd $\} \cup \{v_i | i = 3t + 2$ and $t$ is odd $\}$ with

$$f(v_i) = \begin{cases} 
2 & i = 3t + 1$ and $t \geq 3$ and $t$ is odd, \\
4 & i = 3t + 1$ and $t > 0$ and $t$ is even, \\
3 & i = 3t + 2$ and $t$ is odd.
\end{cases}$$

(ii) Define $S = \{v_1, v_2\}$ with $f(v_1) = 1$, $f(v_2) = 2$. The rest of vertices orderly get colors from $v_3$, $v_4$, · · ·, $v_{2n+1}$. We know that for every graph $G$, $d^*(G, \chi^*) \geq \chi^* - 1$. Therefore $d^*(P_n, \chi^*) = 2$ where $n = 2m + 1$. \hfill \Box

**Proposition 12** $d^*(K_{1,n}, \chi^*) = n$.

**Proof** Let $X = \{x_1\}$ and $Y = \{y_1, \ldots, y_n\}$ be partite sets of $K_{1,n}$. Define $S = Y$ with $f(y_i) = 3$ $(1 \leq i \leq n - 1)$, $f(y_n) = 2$. So $f(x_1) = 1$. Thus, $d^*(K_{1,n}, \chi^*) \leq n$.

Now, we show that $d^*(K_{1,n}, \chi^*) \geq n$. It is easy to check that if we use two colors on $n - 1$ vertices of $Y$, thus one can obtain two different colorings. Hence, $d^*(K_{1,n}, \chi^*) = n$. \hfill \Box

**Proposition 13** $d^*(K_{m,n}, \chi^*) = m$ where $1 < m \leq n$.

**Proof** Let $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$ be partite sets of $K_{m,n}$. We define $S = \{x_1, x_2, \ldots, x_m\}$ with $f(x_i) = 3$ $(1 \leq i \leq m - 1)$, $f(x_m) = 2$ and get the result $f(y_j) = 1$ $(1 \leq j \leq n)$.

Now, we show that $d^*(K_{m,n}, \chi^*) = 3$ $\geq m$. Suppose that we color $m - 1$ vertices of $X$ by two colors, then the remaining vertex of $X$ can be colored by two different colors, a contradiction. Hence the result. \hfill \Box

**Proposition 14** If $G = K_{m,n}$, $m \leq n$ and $m > 1$ then

$$d^*(K_{m,n}, c \geq \chi^* + 1) = \begin{cases} 
m & c \leq m, \\
m + n & c > max\{m, n\}, \\
n & m < c \leq n.
\end{cases}$$

**Proof** The same used manner in Propositions 12 and 13 settles the stated result. \hfill \Box

**Proposition 15**

(i) $d^*(P_3 \times P_3) = d^*(P_3 \times P_4) = d^*(P_3 \times P_5) = 2$.

(ii) $d^*(P_2 \times P_3) = d^*(P_2 \times P_4) = d^*(P_2 \times P_5) = 2$.

**Proof** We know that $d^*(P_n \times P_m) \geq \chi^*(P_n \times P_m) - 1 = 3 - 1 = 2$. It is enough to
present a star defining set of size 2 for each of these graphs. Define the star defining sets of
$P_2 \times P_3, P_2 \times P_4, P_2 \times P_5, P_3 \times P_3, P_3 \times P_4, P_3 \times P_5$, as follows:

\[
\begin{bmatrix}
* & 2 & * & \ldots \\
3 & * & * & \\
\end{bmatrix},
\begin{bmatrix}
* & * & * & * \\
2 & 3 & * & \\
\end{bmatrix},
\begin{bmatrix}
* & * & * & * \\
* & 2 & * & 3 \\
\end{bmatrix},
\begin{bmatrix}
* & 2 & * \\
3 & * & * \\
\end{bmatrix},
\begin{bmatrix}
* & * & * & * \\
* & 3 & 2 & \\
\end{bmatrix},
\begin{bmatrix}
* & * & * & * \\
3 & 2 & * & \\
\end{bmatrix}.
\]

**Theorem 16** If $n$ is an even integer and $n/2 \times \lfloor m/2 \rfloor \neq 1$, then $d^*(P_n \times P_m) \leq n/2 \times \lfloor m/2 \rfloor$.

**Proof** In the following table, a star defining set of size $n/2 \times \lfloor m/2 \rfloor$ is presented.

\[
\begin{bmatrix}
* & 2 & * & 2 & \ldots \\
* & * & * & * & \ldots \\
* & 3 & 3 & * & \ldots \\
* & * & * & * & \ldots \\
* & 2 & 2 & * & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
* & a & a & * & \ldots \\
* & * & * & * & \ldots \\
\end{bmatrix}
\]

if $n = 4k + 2$, then $a = 2$, and if $n = 4k$, then $a = 3$. \(\Box\)

**Conjecture 17** If $n$ is an even number and $n/2 \times \lfloor m/2 \rfloor \neq 1$, then $d^*(P_n \times P_m) = n/2 \times \lfloor m/2 \rfloor$.

**Theorem 18** If $m(k + 1) \geq 4$, then $d^*(P_{2k+1} \times P_{2m+1}, \chi^*) \leq m(k + 1) - 2$.

**Proof** In the following table, a star defining set of size $m(k + 1) - 2$ is shown.

\[
\begin{bmatrix}
* & 2 & * & \ldots & 2 & * & 2 \\
* & * & * & \ldots & * & * & * \\
* & 3 & 3 & \ldots & 3 & 3 & * \\
* & * & * & \ldots & * & * & * \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
* & * & 3 & \ldots & 3 & * & * \\
\end{bmatrix}
\]

So, the star defining number is less or equal to this value. \(\Box\)

**Conjecture 19** If $m(k + 1) \geq 4$ and $k \leq m$, then $d^*(P_{2k+1} \times P_{2m+1}, \chi^*) = m(k + 1) - 2$. 
**Theorem 20** If \( k \geq 2 \), then \( d^*(Q_k, 3) = 2^{k-2} + 1 \).

**Proof** First, we show that \( d^*(Q_k, \chi^*) \leq 2^{k-2} + 1 \). It is well-known that each \( Q_k \) is \( 2^{k-3} \) copies of \( Q_3 \). We label the vertices of \( Q_3 \) as the following figure:

We define the star defining set as the following matrix for which \( i \)th row is dependent to the vertices of \( i \)th copy of \( Q_3 \) in \( Q_k \). Note that at the defining set of \( Q_k \), just one vertex gets color \( i \) and the remaining vertices get color \( j \).

For \( Q_3 \): \[
\begin{bmatrix}
i & * & j & * & * & j & * & * \\
\end{bmatrix}
\]

For \( Q_4 \): \[
\begin{bmatrix}
i & * & j & * & * & j & * & * \\
* & j & * & * & * & j & * & *
\end{bmatrix}
\]

For \( Q_5 \): \[
\begin{bmatrix}
i & * & j & * & * & j & * & * \\
* & j & * & * & * & j & * & *
\end{bmatrix}
\]

For \( Q_6 \): \[
\begin{bmatrix}
i & * & j & * & * & j & * & * \\
* & j & * & * & * & j & * & *
\end{bmatrix}
\]
We know that $Q_k$ is constructed by two copies of $Q_{k-1}$. Therefore, we may give a star defining set in general form for the graph as follows: We assign for the first copy as above. For the next copy; if in a row of the first copy we define $\ast \ast j \ast \ast j \ast \ast$, we may define in the symmetric row of the new copy as $\ast j \ast \ast \ast j \ast$, and if in the first copy we define $\ast j \ast \ast \ast j \ast$, we may define in the symmetric row of the next copy we define $\ast \ast j \ast \ast \ast j$. Note that in the first row we have $i \ast j \ast \ast j \ast \ast$ but for the its symmetric row in the new copy we define as $\ast j \ast \ast \ast j \ast \ast$.

Now, we show that $d^*(Q_k, \chi^*) \geq 2^{k-2} + 1$. If $k = 2$, it is obvious. For completing of the proof, first we show that in each $Q_3$ of $Q_k$ which colored by three colors $i, j, k$. Then we have just one way to color of each $Q_3$. Let $c(i)$ be the set of vertices with color $i$. It is easy to check that $|c(i)| = 1, |c(j)| = 1$ or $|c(k)| = 1$ is not possible. Because, we cannot find a proper star coloring for $Q_k$. Now, let $|c(i)| = 2$. We have two cases: (a): $|c(j)| = |c(k)| = 3$. By simple verification one can see that this cases also cannot be holden. (b): $|c(j)| = 2$ and $|c(k)| = 4$ (or symmetrically $|c(k)| = 2$ and $|c(j)| = 4$). Hence, we may color the graphs $Q_3, Q_4, Q_5$ and $Q_6$ as follows, respectively.

$$Q_3: \begin{bmatrix} i & k & j & k & j & k & i \end{bmatrix}.$$

$$Q_4: \begin{bmatrix} i & k & j & k & j & k & i \\
                    k & j & k & i & j & k & k \end{bmatrix}.$$

$$Q_5: \begin{bmatrix} i & k & j & k & j & k & i \\
                    k & j & k & i & j & k & k \\
                    k & j & k & i & j & k & k \\
                    i & k & j & k & j & k & i \end{bmatrix}.$$

$$Q_6: \begin{bmatrix} i & k & j & k & j & k & i \\
                    k & j & k & i & j & k & k \\
                    k & j & k & i & j & k & k \\
                    i & k & j & k & j & k & i \\
                    k & j & k & i & j & k & k \\
                    i & k & j & k & j & k & i \\
                    i & k & j & k & j & k & i \\
                    k & j & k & i & j & k & k \end{bmatrix}.$$

To color of the graph $Q_k$ with $k \geq 5$, we should color it by the above method, otherwise we cannot find a proper star coloring for the graph. We may also replace color 2 with 3, and conversely to find a new proper star coloring of $Q_k$. Let $S$ be a defining set of $Q_k$. It is so easy that $|S| \geq 3$ for $Q_3$. It is well-known that the graph $Q_k$ with $k \geq 3$ containing of $2^{k-3}$ copies of $Q_3$. Simple verification shows that there exist no copy $Q_3$ of $Q_k$ such that $S \cap V(Q_3) = 1$. Because, it is possible to assign at least two star coloring functions. It follows that $S \cap V(Q_i^j) \geq 2$ where $2 \leq i \leq 2^{k-3}$. Hence, the desired result follows. \(\square\)
References


