Abstract. In this paper we introduce the notions of \( n \)-fold \( BL \)-Smarandache positive implicateive filter and \( n \)-fold \( BL \)-Smarandache implicateve filter in Smarandache residuated lattices and study the relations among them. And we also introduce the notions of \( n \)-fold Smarandache positive implicateve \( BL \)-residuated lattice, \( n \)-fold Smarandache implicateve \( BL \)-residuated lattice and investigate its properties.

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Keywords: Smarandache residuated lattice, \( n \)-fold Smarandache (positive) implicateve filter, \( n \)-fold Smarandache (positive) implicateve residuated lattice.

1. Introduction

\( BL \)-algebras (basic logic algebras) are the algebraic structures for Hájek basic logic [3], in order to investigate many valued logic by algebraic means. Residuated lattices play an important role in the study of fuzzy logic and filters are basic concepts in residuated lattices and other algebraic structures. A Smarandache structure on a set \( L \) means a weak structure \( W \) on \( L \) such that there exists a proper subset \( B \) of \( L \) which is embedded with a strong structure \( S \). In [9], Vasantha Kandasamy studied the concept of Smarandache groupoids, subgroupoids, ideal of groupoids and strong Bol groupoids and obtained many interesting results about them. It will be very interesting to study the Smarandache structure in these algebraic structures. A \( BL \)-algebra is a weaker structure than residuated lattice, then we can consider in any residuated lattice a weaker structure as \( BL \)-algebra.

The concept of Smarandache residuated lattice, Smarandache (positive) implicative filters and Smarandache fantastic filters defined in [1]. In [4, 8] the authors defined the notion of \( n \)-fold (positive) implicative filters, \( n \)-fold fantastic filters, \( n \)-fold obstinate filters in \( BL \)-algebras and studied the
relation among many type of n-fold filters in BL-algebra. The aim of this paper is to extend this research to Smarandache residuated lattices.

2. Preliminaries

[2] A residuated lattice is an algebra \( L = (L, \land, \lor, \circ, \rightarrow, 0, 1) \) of type \((2, 2, 2, 2, 0, 0)\) equipped with an order \( \leq \) satisfying the following:

\begin{itemize}
  \item [(LR1)] \((L, \land, \lor, 0, 1)\) is a bounded lattice,
  \item [(LR2)] \((L, \circ, 1)\) is a commutative ordered monoid,
  \item [(LR3)] \(\circ\) and \(\rightarrow\) form an adjoint pair i.e, \(c \leq a \rightarrow b\) if and only if \(a \circ c \leq b\) for all \(a,b,c \in L\).
\end{itemize}

A BL-algebra is a residuated lattice \(L\) if satisfying the following identity, for all \(a,b \in L\):

\begin{itemize}
  \item [(BL1)] \((a \rightarrow b) \lor (b \rightarrow a) = 1\),
  \item [(BL2)] \(a \land b = a \circ (a \rightarrow b)\).
\end{itemize}

**Theorem 2.1.** \([2, 5, 6, 7]\) Let \(L\) be a residuated lattice. Then the following properties hold, for all \(x, y, z \in L\):

\begin{itemize}
  \item [(lr1)] \(1 \rightarrow x = x, x \rightarrow x = 1\),
  \item [(lr2)] \(x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y) \leq z \rightarrow (x \rightarrow y)\),
  \item [(lr3)] \(x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)\) and \((x \rightarrow y) \circ (y \rightarrow z) \leq x \rightarrow z\),
  \item [(lr4)] \(x \leq y \iff x \rightarrow y = 1, x \leq y \rightarrow x\),
  \item [(lr5)] \(x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z) = (x \circ y) \rightarrow z\),
  \item [(lr6)] \(x \circ (x \rightarrow y) \leq y, x \leq y \rightarrow (x \circ y)\) and \(y \leq (y \rightarrow x) \rightarrow x\),
  \item [(lr7)] If \(x \leq y\), then \(y \rightarrow z \leq x \rightarrow z, z \rightarrow x \leq z \rightarrow y\) and \(y^* \leq x^*\),
  \item [(lr8)] \(x \leq y\) and \(z \leq w\) then \(x \circ z \leq y \circ w\),
  \item [(lr9)] \(x \leq x^{**}, x^{***} = x^*\),
  \item [(lr10)] \(x^* \circ y^* \leq (x \circ y)^*\) (so, \((x^*)^n \leq (x^n)^*\) for every \(n \geq 1\)),
  \item [(lr11)] \(x^{**} \circ y^{**} \leq (x \circ y)^{**}\) (so, \((x^{**})^n \leq (x^n)^{**}\) for every \(n \geq 1\)).
\end{itemize}

The following definitions and theorems are stated from \([1]\).

A Smarandache \(B_L\)-residuated lattice is a residuated lattice \(L\) in which there exists a proper subset \(B\) of \(L\) such that:

1. \(0, 1 \in B\) and \(|B| > 2\),
2. \(B\) is a \(B_L\)-algebra under the operations of \(L\).

From now on \(L_B = (L, \land, \lor, \circ, \rightarrow, 0, 1)\) is a Smarandache \(B_L\)-residuated lattice and \(B = (B, \land, \lor, \circ, \rightarrow, 0, 1)\) is a BL-algebra unless otherwise specified.

**Definition 2.2.** A nonempty subset \(F\) of \(L_B\) is called a

(i) Smarandache deductive system of \(L_B\) related to \(B\) (or briefly \(B_L\)-Smarandache deductive system of \(L_B\)) if it satisfies:

\begin{itemize}
  \item [(DB1)] \(1 \in F\),
  \item [(DB2)] If \(x \in F, y \in B\) and \(x \rightarrow y \in F\) then \(y \in F\).
\end{itemize}

(ii) Smarandache filter of \(L_B\) related to \(B\) (or briefly \(B_L\)-Smarandache filter of \(L_B\)) if satisfies:

\begin{itemize}
  \item [(FB1)] if \(x, y \in F\), then \(x \circ y \in F\),
\end{itemize}
Let $F$ be a $B_L$-Smarandache filter of $L_B$, then $F$ is a $B_L$-Smarandache deductive system of $L_B$.

Let $F$ be a $B_L$-Smarandache deductive system of $L_B$. If $F \subseteq B$, then $F$ is a $B_L$-Smarandache filter of $L_B$.

Definition 2.4. Let $F$ be a subset of $L_B$ and $1 \in F$. $\diamondsuit F$ is called a Smarandache implicative filter of $L_B$ related to $B$ (or briefly $B_L$-Smarandache implicative filter of $L_B$) if $z \in F$, $x, y \in B$ and $z \rightarrow ((x \rightarrow y) \rightarrow x) \in F$ then $x \in F$, $\diamondsuit F$ is called a Smarandache positive implicative filter of $L_B$ related to $B$ (or briefly $B_L$-Smarandache positive implicative filter of $L_B$) if $x, y, z \in B$, $z \rightarrow (x \rightarrow y) \in F$ and $z \rightarrow x \in F$, then $z \rightarrow y \in F$.

Now, unless mentioned otherwise, $n \geq 1$ will be an integer.

3. $n$-Fold $B_L$-Smarandache (Positive) Implicative Filters

Definition 3.1. A subset $F$ of $L_B$ is called an $n$-fold Smarandache positive implicative filter of $L_B$ related to $B$ (or briefly $n$-fold $B_L$-Smarandache positive implicative filter of $L_B$) if it satisfies in the following conditions:

(i) $1 \in F$,
(ii) for all $x, y, z \in B$, if $x^n \rightarrow (y \rightarrow z) \in F$ and $x^n \rightarrow y \in F$, then $x^n \rightarrow z \in F$.

Example 3.2. Let $L = \{0, a, b, c, d, 1\}$ be a residuated lattice such that $0 < a < c < d < 1$ and $0 < b < c < d < 1$. We define

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We can see that $L = (L, \land, \lor, \odot, \rightarrow, 0, 1)$ is a residuated lattice, in which $B = \{0, a, c, 1\}$ is a $BL$-algebra which properly contained in $L$. Then $L$ is a Smarandache $B_L$-residuated lattice. $F = \{d, 1\}$ is an $n$-fold $B_L$-Smarandache positive implicative filter of $L_B$, for $n \geq 2$. But $F$ is not 1-fold $B_L$-Smarandache positive implicative filter, since $a \rightarrow (a \rightarrow a^2) = 1 \in F$ and $a \rightarrow a = 1 \in F$, while $a \rightarrow a^2 = c \notin F$.

Theorem 3.3. $n$-fold $B_L$-Smarandache positive implicative filters are $B_L$-Smarandache deductive systems.
\textbf{Proof.} Let $F$ be an $n$-fold $B_L$-Smarandache positive implicative filter of $L_B$. Suppose $z \in B$, such that $y, y \rightarrow z \in F$. We have $1^n \rightarrow y, 1^n \rightarrow (y \rightarrow z) \in F$, these imply $z = 1^n \rightarrow z \in F$. Hence $F$ is a $B_L$-Smarandache deductive system of $L_B$.

\textbf{Theorem 3.4.} Let $F$ be a $B_L$-Smarandache deductive system of $L_B$. Then for all $x, y, z \in B$, the following conditions are equivalent:

(i) $F$ is an $n$-fold $B_L$-Smarandache positive implicative filter of $L_B$,
(ii) $x^n \rightarrow x^{2n} \in F$, for all $n \in N$,
(iii) if $x^{n+1} \rightarrow y \in F$, then $x^n \rightarrow y \in F$, for all $n \in N$,
(iv) if $x^n \rightarrow (y \rightarrow z) \in F$, then $(x^n \rightarrow y) \rightarrow (x^n \rightarrow z) \in F$, for all $n \in N$,
(v) $A_n = \{ b \in B : a^n \rightarrow b \in F \}$ is a $B_L$-Smarandache deductive system of $L_B$, for any $a \in B$.

\textbf{Proof.} (i) \Rightarrow (ii) We have $x^n \rightarrow (x^n \rightarrow x^{2n}) = x^n \circ x^n \rightarrow x^{2n} = x^{2n} = 1 \in F$ and $x^n \rightarrow x^n = 1 \in F$. Since $F$ is an $n$-fold $B_L$-Smarandache positive implicative filter of $L_B$, we get $x^n \rightarrow x^{2n} \in F$.

(ii) \Rightarrow (i) Let $x^n \rightarrow (y \rightarrow z) \in F$ and $x^n \rightarrow y \in F$. By Theorem 2.1 we have $x^n \circ (x^n \rightarrow (y \rightarrow z)) \leq y \rightarrow z$ and $x^n \circ (x^n \rightarrow y) \leq y$. Hence by (LR$_8$), $(x^n \circ (x^n \rightarrow (y \rightarrow z))) \circ (x^n \circ (x^n \rightarrow y)) \leq y \circ (y \rightarrow z)$. Then by (LR$_6$), we have $(x^n \rightarrow (y \rightarrow z)) \circ (x^n \rightarrow y) \circ x^{2n} \leq z$. So by (LR$_3$), we get $(x^n \rightarrow (y \rightarrow z)) \circ (x^n \rightarrow y) \leq x^{2n} \rightarrow z$, (I). Since $x^n \rightarrow (y \rightarrow z) \in F$ and $x^n \rightarrow y \in F$, then we get $(x^n \rightarrow (y \rightarrow z)) \circ (x^n \rightarrow y) \in F$. Hence by (I) we get $x^{2n} \rightarrow z \in F$. By Theorem 2.1, we have $x^n \rightarrow x^{2n} \leq (x^{2n} \rightarrow z) \rightarrow (x^n \rightarrow z)$. So by hypothesis we get $(x^{2n} \rightarrow z) \rightarrow (x^n \rightarrow z) \in F$. Then by the fact that $x^{2n} \rightarrow z \in F$, we obtain $x^n \rightarrow z \in F$. Therefore $F$ is an $n$-fold $B_L$-Smarandache positive implicative filter of $L_B$.

(i) \Rightarrow (iii) By Theorem 2.1 we have $x^n \rightarrow (x \rightarrow y) = x^{n+1} \rightarrow y \in F$ and $x^n \rightarrow x = 1 \in F$. So by (i) we get $x^n \rightarrow y \in F$.

(iii) \Rightarrow (ii) We have $x^{n+1} \rightarrow (x^{n-1} \rightarrow x^{2n}) = x^{2n} \rightarrow x^{2n} = 1 \in F$. From this and the fact that (iii) holds, we also have $x^n \rightarrow (x^n \rightarrow x^{2n}) \in F$. But $x^{n+1} \rightarrow (x^{n-2} \rightarrow x^{2n}) = x^n \rightarrow (x^{n-1} \rightarrow x^{2n}) \in F$. From this and the fact that (iii) holds, we have $x^n \rightarrow (x^{n-2} \rightarrow x^{2n}) \in F$. By repeating the process $n$ times, we get $x^n \rightarrow (x^n \rightarrow x^{2n}) = x^n \rightarrow x^n \in F$.

(ii) \Rightarrow (iv) Assume that $x^n \rightarrow (y \rightarrow z) \in F$. By Theorem 2.1 we have $y \rightarrow z \leq (x^n \rightarrow y) \rightarrow (x^n \rightarrow z)$ and

\begin{align*}
   x^n \rightarrow (y \rightarrow z) & \leq x^n \rightarrow ((x^n \rightarrow y) \rightarrow (x^n \rightarrow z)); \\
   & = x^n \rightarrow (x^n \rightarrow ((x^n \rightarrow y) \rightarrow z)); \\
   & = x^{2n} \rightarrow ((x^n \rightarrow y) \rightarrow z).
\end{align*}

Hence $x^{2n} \rightarrow ((x^n \rightarrow y) \rightarrow z) \in F$. We get

\begin{align*}
   x^{2n} \rightarrow ((x^n \rightarrow y) \rightarrow z) & \leq (x^n \rightarrow x^{2n}) \rightarrow (x^n \rightarrow ((x^n \rightarrow y) \rightarrow z)); \\
   & = (x^n \rightarrow x^{2n}) \rightarrow ((x^n \rightarrow y) \rightarrow (x^n \rightarrow z)).
\end{align*}
Any Proposition 3.5. Smarandache positive implicative filter of $L$, $y, z$ we get an $(x, y, z, x, y, z)$ ∈ $F$. Then by (ii) we get $(x^n → y) → (x^n → z) ∈ F$.

(iii) Let $(x^n → y) → (x^n → z) ∈ F$, since $a^n → 1 = 1 ∈ F$, we have $1 ∈ A_a$. Let $x, x → y ∈ A_a$. Then $a^n → x ∈ F$ and $a^n → (x → y) ∈ F$. Since $F$ is an $n$-fold $B_L$-Smarandache positive implicative filter of $L_B$, $a^n → y ∈ F$, hence $y ∈ A_a$. Therefore $A_a$ is a $B_L$-Smarandache deductive system of $L_B$.

Example 3.8. (i) Let $L = \{0, a, b, c, d, 1\}$ be a residuated lattice such that $0 < b < a < 1$ and $0 < d < a, c < 1$. We define

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Then \((L, \land, \lor, \circ, \rightarrow, 0, 1)\) is a residuated lattice, in which \(B = \{0, b, c, 1\}\) is a BL-algebra which properly contained in \(L\). Then \(L\) is a Smarandache \(B_L\)-residuated lattice. Clearly \(L\) is an \(n\)-fold Smarandache positive implicative \(B_L\)-residuated lattice.

(ii) Consider the Smarandache \(B_L\)-residuated lattice \(L\) in Example 3.2. \(L_B\) is not a 1-fold Smarandache positive implicative \(B_L\)-residuated lattice, since \(a \in B\) and \(0 = a^{1+1} \neq a^1\).

**Corollary 3.9.** Let \(L_B\) be an \(n\)-fold Smarandache positive implicative \(B_L\)-residuated lattice. The following conditions are equivalent:

(i) \(F\) is an \(n\)-fold \(B_L\)-Smarandache positive implicative filter,

(ii) \(F\) is a \(B_L\)-Smarandache deductive system.

**Proof.** (i) \(\Rightarrow\) (ii) By Theorem 3.3, the proof is clear.

(ii) \(\Rightarrow\) (i) By Definition 3.7 and Theorem 3.4 ((iii) \(\Rightarrow\) (i)), the proof is clear.

\(\Box\)

**Theorem 3.10.** Let \(F\) be a \(B_L\)-Smarandache deductive system of \(L_B\). The following conditions are equivalent:

(i) \(L_B\) is an \(n\)-fold Smarandache positive implicative \(B_L\)-residuated lattice,

(ii) every \(B_L\)-Smarandache deductive system of \(L_B\) is an \(n\)-fold \(B_L\)-Smarandache positive implicative filter of \(L_B\),

(iii) \(\{1\}\) is an \(n\)-fold \(B_L\)-Smarandache positive implicative filter of \(L_B\),

(iv) \(x^n = x^{2n}\), for all \(x \in B\).

**Proof.**

(i) \(\Rightarrow\) (ii) Follows from Corollary 3.9.

(ii) \(\Rightarrow\) (iii) It is clear.

(iii) \(\Rightarrow\) (iv) Assume that \(\{1\}\) is an \(n\)-fold \(B_L\)-Smarandache positive implicative filter of \(L_B\). So from Theorem 3.4, we have \(x^n \rightarrow x^{2n} = 1\), for all \(x \in B\). Then by Theorem 2.1, \(x^n = x^{2n}\) for all \(x \in B\).

(iv) \(\Rightarrow\) (i) Let \(x^n = x^{2n}\), for all \(x \in B\). Then \(x^n \rightarrow x^{2n} = 1 \in \{1\}\), for all \(x \in B\). Then by Theorem 3.4, \(\{1\}\) is an \(n\)-fold \(B_L\)-Smarandache positive implicative filter of \(L_B\). Since \(x^n \rightarrow (x^n \rightarrow x^{n+1}) = x^{2n} \rightarrow x^{n+1} = 1 \in \{1\}\) and \(x^n \rightarrow x^n = 1 \in \{1\}\), we get \(x^n \rightarrow x^{n+1} \in \{1\}\), that is \(x^{n+1} = x^n\), for all \(x \in B\). Therefore \(L_B\) is an \(n\)-fold Smarandache positive implicative \(B_L\)-residuated lattice.

\(\Box\)

**Corollary 3.11.** Let \(F\) be a proper \(B_L\)-Smarandache deductive system of \(L_B\). Then the following statements are equivalent:

(i) \(F\) is an \(n\)-fold \(B_L\)-Smarandache positive implicative filter (in short \(n\)-fold \(B_L\)-SPIF),

(ii) \(L_B/F\) is an \(n\)-fold Smarandache positive \(B_{L/F}\)-residuated lattice (in short \(n\)-fold \(B_{L/F}\)-SPRL).
Proof. Let $F$ be a $B_L$-Smarandache deductive system of $L_B$. By Theorems 3.4 and 3.10 we get:

$F$ is an $n$-fold $B_L$-SPIF $\iff x^n \rightarrow x^{2n} \in F, \forall x \in B,$

$\iff (x^n \rightarrow x^{2n})/F = 1/F, \forall x/F \in B/F,$

$\iff (x/F)^n \rightarrow (x/F)^{2n} = 1/F, \forall x/F \in B/F,$

$\iff (x/F)^n \leq (x/F)^{2n}, \forall x/F \in B/F,$

$\iff (x/F)^n = (x/F)^{2n}, \forall x/F \in B/F,$

$\iff L_B/F$ is an $n$-fold $B_{L/F}$-SPIRL.

□

Definition 3.12. A subset $F$ of $L_B$ is called an $n$-fold Smarandache implicative filter of $L_B$ related to $B$ (or briefly $n$-fold $B_L$-Smarandache implicative filter of $L_B$) if it satisfies in the following conditions:

(i) $1 \in F$,

(ii) For all $y, z \in B$ and $x \in F$, if $x \rightarrow ((y^n \rightarrow z) \rightarrow y) \in F$ then $y \in F$.

Example 3.13. (i) In Example 3.2, $F = \{d, 1\}$ is an $n$-fold $B_L$-Smarandache implicative filter, for $n \geq 2$.

(ii) Let $L = \{0, a, b, c, 1\}$ be a residuated lattice such that $a, b, c$ are incomparable. We define

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We can see that $L = (L, \land, \lor, \odot, \rightarrow, 0, 1)$ is a residuated lattice, in which $B = \{0, a, 1\}$ is a $BL$-algebra which properly contained in $L$. Then $L$ is a Smarandache $B_L$-residuated lattice. $F = \{b, 1\}$ is not an $n$-fold $B_L$-Smarandache implicative filter, since $b, b \rightarrow ((a^n \rightarrow 0) \rightarrow a) = 1 \in F$ but $a \notin F, (0, a \in B)$.

Theorem 3.14. Every $n$-fold $B_L$-Smarandache implicative filter is a $B_L$-Smarandache deductive system.

Proof. Let $F$ be an $n$-fold $B_L$-Smarandache implicative filter of $L_B$. Let $x \in F, y \in B$ and $x \rightarrow y \in F$. We have $x \rightarrow ((y^n \rightarrow 1) \rightarrow y) = x \rightarrow y \in F$. Since $F$ is an $n$-fold $B_L$-Smarandache implicative filter and $x \in F$, we get $y \in F$. So $F$ is a $B_L$-Smarandache deductive system of $L_B$.

□

Theorem 3.15. The following conditions are equivalent for $B_L$-Smarandache deductive system $F$ of $L_B$.

(i) $F$ is an $n$-fold $B_L$-Smarandache implicative filter,
(ii) for all \( x, y \in B \), \( (x^n \rightarrow y) \rightarrow x \in F \) implies \( x \in F \),

(iii) for all \( x \in B \), \( (x^n)^* \rightarrow x \in F \) implies \( x \in F \),

(iv) for all \( x \in B \), \( x \lor (x^n)^* \in F \).

**Proof.** (i) \( \Rightarrow \) (ii) and (ii) \( \Rightarrow \) (iii) are clear, by Definition 3.12.

(iii) \( \Rightarrow \) (i) Suppose that \( x \rightarrow ((y^n \rightarrow z) \rightarrow y) \in F \) and \( x \in F \). Then \( (y^n \rightarrow z) \rightarrow y \in F \). By Theorem 2.1 we have \( 0 \leq z \) then \( y^n \rightarrow 0 \leq y^n \rightarrow z \) and so \( (y^n \rightarrow z) \rightarrow y \leq (y^n \rightarrow 0) \rightarrow y \). Hence \( (y^n \rightarrow 0) \rightarrow y \in F \). We apply the hypothesis and obtain \( y \in F \). Therefore \( F \) is an \( n \)-fold \( B_L \)-Smarandache implicative filter.

(iii) \( \Rightarrow \) (iv) We know \( x \leq x \lor (x^n)^* \), then \( x^n \leq (x \lor (x^n)^*)^n \) and then \( ((x \lor (x^n)^*))^n \leq (x^n)^* \lor x \). Hence \( ((x \lor (x^n)^*))^n \rightarrow (x \lor (x^n)^*) = 1 \in F \). Hence by (iii) we get \( x \lor (x^n)^* \in F \).

(iv) \( \Rightarrow \) (iii) Assume that for all \( x \in B \), \( (x^n)^* \rightarrow x \in F \). By Theorem 2.1 we have \( x \lor (x^n)^* \leq ((x^n)^* \rightarrow x) \rightarrow x \) so by (iv), \( ((x^n)^* \rightarrow x) \rightarrow x \in F \). By using the fact that \( (x^n)^* \rightarrow x \in F \), we get \( x \in F \).

**Theorem 3.16.** Let \( F \) be an \( n \)-fold \( B_L \)-Smarandache implicative filter of \( L_B \). Then \( (x^n \rightarrow y) \rightarrow y \in F \) implies \( (y \rightarrow x) \rightarrow x \in F \), for all \( x, y \in B \).

**Proof.** Let \( F \) be an \( n \)-fold \( B_L \)-Smarandache implicative filter and \( (x^n \rightarrow y) \rightarrow y \in F \). We have \( x \leq (y \rightarrow x) \rightarrow x \) then by Theorem 2.1 we get \( x^n \leq ((y \rightarrow x) \rightarrow x)^n \), for all \( n \in N \). And so \( ((y \rightarrow x) \rightarrow x)^n \rightarrow y \leq x^n \rightarrow y \).

So we have

\[
(x^n \rightarrow y) \rightarrow y \leq (y \rightarrow x) \rightarrow ((x^n \rightarrow y) \rightarrow y) = (x^n \rightarrow y) \rightarrow ((y \rightarrow x) \rightarrow x) \leq (((y \rightarrow x) \rightarrow x))^n \rightarrow y) \rightarrow ((y \rightarrow x) \rightarrow x).
\]

Thus \( (((y \rightarrow x) \rightarrow x))^n \rightarrow y) \rightarrow ((y \rightarrow x) \rightarrow x) \in F \). By the fact that \( F \) is an \( n \)-fold \( B_L \)-Smarandache implicative filter of \( L_B \), we get \( (y \rightarrow x) \rightarrow x \in F \).

By Theorem 3.15 \( (i) \Leftrightarrow (iv) \) we have the following theorem.

**Theorem 3.17.** Let \( F \) and \( G \) be two \( B_L \)-Smarandache deductive systems of \( L_B \) such that \( F \subseteq G \). If \( F \) is an \( n \)-fold \( B_L \)-Smarandache implicative filter of \( L_B \), then so is \( G \).

**Theorem 3.18.** Let every Smarandache deductive system be a Smarandache filter. Then every \( n \)-fold Smarandache implicative filter is \( n \)-fold Smarandache positive implicative filter.

**Proof.** Let \( F \) be an \( n \)-fold \( B_L \)-Smarandache implicative filter of \( L_B \) and \( x^{n+1} \rightarrow y \in F \), where \( x, y \in B \). By Theorem 2.1 we have the following
(x^{n+1} \rightarrow y)^n \rightarrow (x^n \rightarrow y) = ((x^{n+1} \rightarrow y)^{n-1} \circ (x^{n+1} \rightarrow y)) \rightarrow (x^n \rightarrow y)
= ((x^{n+1} \rightarrow y)^{n-1} \rightarrow ((x^{n+1} \rightarrow y) \rightarrow (x^n \rightarrow y)))
= ((x^{n+1} \rightarrow y)^{n-1} \rightarrow ((x^{n+1} \rightarrow y) \rightarrow (x^n \rightarrow y)))
= ((x^{n+1} \rightarrow y)^{n-1} \rightarrow (x^{n-1} \rightarrow ((x^{n+1} \rightarrow y) \rightarrow (x^n \rightarrow y))))
= ((x^{n+1} \rightarrow y)^{n-1} \rightarrow (x^{n-1} \rightarrow ((x \rightarrow (x^n \rightarrow y)) \rightarrow (x^n \rightarrow y)))).

So (x^{n+1} \rightarrow y)^n \rightarrow (x^n \rightarrow y) = ((x^{n+1} \rightarrow y)^{n-1} \rightarrow (x^{n-1} \rightarrow ((x \rightarrow (x^n \rightarrow y)) \rightarrow (x^n \rightarrow y)))). (I) We have (x^n \rightarrow y) \rightarrow y \leq ((x \rightarrow (x^n \rightarrow y)) \rightarrow (x \rightarrow y)). By (lr_7) we get ((x^{n+1} \rightarrow y)^{n-1} \rightarrow (x^{n-1} \rightarrow ((x \rightarrow (x^n \rightarrow y)) \rightarrow (x^n \rightarrow y))) \leq ((x^{n+1} \rightarrow y)^{n-1} \rightarrow (x^{n-1} \rightarrow ((x \rightarrow (x^n \rightarrow y)) \rightarrow (x^n \rightarrow y))). So by (I) we have ((x^{n+1} \rightarrow y)^{n-1} \rightarrow (x^{n-1} \rightarrow ((x \rightarrow (x^n \rightarrow y)) \rightarrow (x^n \rightarrow y))) \leq (x^{n+1} \rightarrow y)^n \rightarrow (x^n \rightarrow y). (II) By Theorem 2.1 we have

((x^{n+1} \rightarrow y)^{n-1} \rightarrow (x^{n-1} \rightarrow ((x^n \rightarrow y) \rightarrow y))) =
((x^{n+1} \rightarrow y)^{n-1} \rightarrow ((x^n \rightarrow y) \rightarrow (x^{n-1} \rightarrow y))) =
((x^n \rightarrow y) \rightarrow ((x^{n+1} \rightarrow y)^{n-1} \rightarrow (x^{n-1} \rightarrow y))).

Hence by (II) we obtain (x^n \rightarrow y) \rightarrow ((x^{n+1} \rightarrow y)^{n-1} \rightarrow (x^{n-1} \rightarrow y)) \leq (x^{n+1} \rightarrow y)^n \rightarrow (x^n \rightarrow y). (III) We have

(x^n \rightarrow y) \circ (x^{n+1} \rightarrow y)^{n-1} \circ x^{n-1} =
(x^n \rightarrow y) \circ (x^{n+1} \rightarrow y)^{n-2} \circ (x^{n+1} \rightarrow y) \circ x^{n-2} \circ x =
(x^n \rightarrow y) \circ (x^{n+1} \rightarrow y)^{n-2} \circ x^{n-2} \circ x \circ (x^{n+1} \rightarrow y). (IV)

Also we have x \circ (x^{n+1} \rightarrow y) = x \circ (x \rightarrow (x^n \rightarrow y)) \leq x \rightarrow y. Therefore by (IV) we obtain (x^n \rightarrow y) \circ (x^{n+1} \rightarrow y)^{n-2} \circ x^{n-2} \circ x \circ (x^{n+1} \rightarrow y) \leq (x^n \rightarrow y) \circ (x^{n+1} \rightarrow y)^{n-2} \circ x^{n-2} \circ (x^n \rightarrow y). So we have

(x^n \rightarrow y) \circ (x^{n+1} \rightarrow y)^{n-1} \circ x^{n-1} \leq (x^n \rightarrow y)^2 \circ (x^{n+1} \rightarrow y)^{n-2} \circ x^{n-2}.

Then by (lr_7) we get ((x^n \rightarrow y)^2 \circ (x^{n+1} \rightarrow y)^{n-2} \circ x^{n-2}) \rightarrow y \leq ((x^n \rightarrow y)^1 \circ (x^{n+1} \rightarrow y)^{n-1} \circ x^{n-1}) \rightarrow y. So by (lr_5) we obtain

((x^n \rightarrow y)^2 \circ (x^{n+1} \rightarrow y)^{n-2} \circ x^{n-2}) \rightarrow y \leq ((x^n \rightarrow y) \circ (x^{n+1} \rightarrow y)^{n-1} \rightarrow (x^{n-1} \rightarrow y)). Thus by (lr_5) we get (x^n \rightarrow y)^2 \rightarrow ((x^{n+1} \rightarrow y)^{n-2} \circ x^{n-2} \circ (x^n \rightarrow y)) \leq (x^n \rightarrow y) \rightarrow (x^{n+1} \rightarrow y)^{n-1} \rightarrow (x^{n-1} \rightarrow y)). So by (III) we have

(x^n \rightarrow y)^2 \rightarrow ((x^{n+1} \rightarrow y)^{n-2} \circ x^{n-2} \circ (x^n \rightarrow y)) \leq (x^{n+1} \rightarrow y)^n \rightarrow (x^n \rightarrow y).

By repeating n times, we obtain (x^n \rightarrow y)^n \rightarrow ((x^{n+1} \rightarrow y)^0 \rightarrow (x^0 \rightarrow y)) \leq (x^{n+1} \rightarrow y)^n \rightarrow (x^n \rightarrow y). This implies (x^n \rightarrow y)^n \rightarrow (1 \rightarrow (1 \rightarrow y)) \leq (x^{n+1} \rightarrow y)^n \rightarrow (x^n \rightarrow y) \rightarrow (x^n \rightarrow y). Hence (x^n \rightarrow y)^n \rightarrow y \leq (x^{n+1} \rightarrow y)^n \rightarrow (x^n \rightarrow y). Then ((x^n \rightarrow y)^n \rightarrow y) \rightarrow ((x^{n+1} \rightarrow y)^n \rightarrow (x^n \rightarrow y)) = 1. Hence by (lr_5), we have ((x^{n+1} \rightarrow y)^n \rightarrow ((x^n \rightarrow y)^n \rightarrow (x^n \rightarrow y)) = 1 \in F, (V). Since x^{n+1} \rightarrow y \in F and F is a B_L-Smarandache filter, we have (x^{n+1} \rightarrow y)^n \in F. By (V), we have ((x^n \rightarrow y)^n \rightarrow y) \rightarrow (x^n \rightarrow y) \in F. Then by Theorem 3.15 ((i) \Rightarrow (ii)), we get x^n \rightarrow y \in F and so by Theorem 3.4, F is an n-fold B_L-Smarandache positive implicative filter.
In the following example we show that the converse of Theorem 3.18 is not true in general.

**Example 3.19.** In Example 3.13 (ii), $F = \{b, 1\}$ is an $n$-fold $B_L$-Smarandache positive implicative filter, while it is not an $n$-fold $B_L$-Smarandache implicative filter.

**Proposition 3.20.** Every $n$-fold $B_L$-Smarandache implicative filter is an $(n + 1)$-fold $B_L$-Smarandache implicative filter.

**Proof.** Let $F$ be an $n$-fold $B_L$-Smarandache implicative filter of $L_B$ and $(x^{n+1})^* \rightarrow x \in F$, where $x \in B$. By Theorem 2.1, $(x^n)^* \leq (x^{n+1})^*$ then $(x^{n+1})^* \rightarrow x \leq (x^n)^* \rightarrow x$. Then we get $(x^n)^* \rightarrow x \in F$. Since $F$ is an $n$-fold $B_L$-Smarandache implicative filter of $L_B$, by Theorem 3.15, we obtain $x \in F$. So by Theorem 3.15, $F$ is an $(n + 1)$-fold $B_L$-Smarandache implicative filter of $L_B$. □

**Definition 3.21.** A Smarandache residuated lattice $L_B$ is called $n$-fold Smarandache implicative $B_L$-residuated lattice if it satisfies $(x^n)^* \rightarrow x = x$, for each $x \in B$.

**Example 3.22.** In Example 3.2, $L_B$ is a 2-fold Smarandache implicative $B_L$-residuated lattice and it is not a 1-fold Smarandache implicative $B_L$-residuated lattice. Since $(a^1)^* \rightarrow a = c \neq a$.

**Lemma 3.23.** (i) 1-fold Smarandache implicative $B_L$-residuated lattices are $n$-fold Smarandache implicative $B_L$-residuated lattice.

(ii) $n$-fold Smarandache implicative $B_L$-residuated lattices are $(n+1)$-fold Smarandache implicative $B_L$-residuated lattice.

**Proof.** (i) Let $L_B$ be a 1-fold Smarandache implicative $B_L$-residuated lattice. Then $x^* \rightarrow x = x$ for each $x \in B$. By Theorem 2.1 we have $(x^n)^* \rightarrow x \leq x^* \rightarrow x$. Hence $(x^n)^* \rightarrow x \leq x$ and so $(x^n)^* \rightarrow x = x$, for each $x \in B$. Therefore $L_B$ is an $n$-fold Smarandache implicative $B_L$-residuated lattice.

(ii) Let $L_B$ be an $n$-fold Smarandache implicative $B_L$-residuated lattice. Then $(x^n)^* \rightarrow x = x$ for each $x \in B$. By Theorem 2.1 we have $(x^{n+1})^* \rightarrow x \leq (x^n)^* \rightarrow x$. Hence $(x^{n+1})^* \rightarrow x \leq x$ and so $(x^{n+1})^* \rightarrow x = x$, for each $x \in B$. Therefore $L_B$ is an $(n + 1)$-fold Smarandache implicative $B_L$-residuated lattice. □

**Proposition 3.24.** If $L_B$ is an $n$-fold Smarandache implicative $B_L$-residuated lattice. Then the following statements are equivalent:

(i) $F$ is an $n$-fold $B_L$-Smarandache implicative filter.

(ii) $F$ is a $B_L$-Smarandache deductive system.

**Proof.** By Theorem 3.14 and Theorem 3.15 (iii) $\Rightarrow$ (i), the proofs are clear. □
Proposition 3.25. The following conditions are equivalent:

(i) $L_B$ is an $n$-fold Smarandache implicative $B_L$-residuated lattice.
(ii) Every $n$-fold $B_L$-Smarandache deductive system $F$ of $L_B$ is an $n$-fold $B_L$-Smarandache implicative filter of $L_B$.
(iii) $(x^n \to y) \to x = x$, for all $x, y \in B$.
(iv) $\{1\}$ is an $n$-fold $B_L$-Smarandache implicative filter.

Proof. (i) $\Rightarrow$ (ii) Follows from Proposition 3.24.
(ii) $\Rightarrow$ (iv) The proof is easy.
(iii) $\Rightarrow$ (iv) The proof is obvious.
(i) $\Rightarrow$ (iii) By hypothesis (i) and Definition 3.21, we have $(x^n)^* \to x = x$, for all $x \in B$. So by Theorem 2.1, we have $0 \leq y$ then $(x^n \to y) \to x \leq (x^n)^* \to x = x$. Therefore $(x^n \to y) \to x = x$.
(iv) $\Rightarrow$ (i) Let $\{1\}$ be an $n$-fold $B_L$-Smarandache implicative filter. By Theorem 3.15, for all $x \in B$, $x \vee (x^n)^* = 1$. By Theorem 2.1, $x \vee (x^n)^* \leq ((x^n)^* \to x) \to x$. Hence $((x^n)^* \to x) \to x = 1$ or equivalently $(x^n)^* \to x \leq x$. Hence by $(lr_4)$ we have $(x^n)^* \to x = x$, for all $x \in B$. So the proof is complete.

Proposition 3.26. $n$-fold Smarandache implicative $B_L$-residuated lattices are $n$-fold Smarandache positive implicative $B_L$-residuated lattices.

Proof. Let $L_B$ be an $n$-fold Smarandache implicative $B_L$-residuated lattice. Then $(x^n)^* \to x = x$ for each $x \in B$. Hence every $B_L$-Smarandache deductive system is an $n$-fold $B_L$-Smarandache implicative filter. So $\{1\}$ is an $n$-fold $B_L$-Smarandache implicative filter. We know $\{1\}$ is a $B_L$-Smarandache filter, hence by Theorem 3.18, $\{1\}$ is an $n$-fold $B_L$-Smarandache positive implicative filter. Since $\{1\}$ is a Smarandache deductive system, then by Theorem 3.6, every deductive system is an $n$-fold $B_L$-Smarandache positive implicative filter. Hence by Theorem 3.10, $L_B$ is an $n$-fold Smarandache positive implicative $B_L$-residuated lattice.

In the following example we show that the converse of above proposition is not true in general.

Example 3.27. Consider $L_B$ in Example 3.13(ii). Clearly $L_B$ is an $n$-fold Smarandache positive implicative $B_L$-residuated lattices, while it is not an $n$-fold Smarandache implicative $B_L$-residuated lattice, since $(a^n)^* \to a \neq a$.

Proposition 3.28. Let $F$ be a proper $B_L$-Smarandache deductive system of $L_B$. Then the following statements are equivalent:

(i) $F$ is an $n$-fold $B_L$-Smarandache implicative filter of $L_B$.
(ii) $L_B/F$ is an $n$-fold Smarandache implicative $B_{L/F}$-residuated lattice.

Proof. (i) $\Rightarrow$ (ii) Let $F$ be an $n$-fold $B_L$-Smarandache implicative filter of $L_B$. By Proposition 3.25, it is enough show that $\{1/F\}$ is an $n$-fold $B_{L/F}$-Smarandache implicative filter of $L_B/F$. Let $((x/F)^n)^* \to x/F \in \{1/F\}$, for all $x/F \in B/F$. Then $((x^n)^* \to x)/F = 1/F$, so $(x^n)^* \to x \in F$. Since $F$ is an $n$-fold $B_L$-Smarandache implicative filter we get $x \in F$, for all $x \in B$. 

And so $x/F = 1/F \in \{1/F\}$, for all $x/F \in B/F$, i.e. $\{1/F\}$ is an $n$-fold $B_{L/F}$-Smarandache implicative filter of $L_{B/F}$.

(ii) $\Rightarrow$ (i): Let $L_{B/F}$ be an $n$-fold Smarandache implicative $B_{L/F}$-residuated lattice and $(x^n)^* \rightarrow x \in F$, for all $x \in B$. We get $((x^n)^* \rightarrow x)/F = 1/F$, for all $x/F \in B/F$, this implies $((x/F)^n)^* \rightarrow x/F \in \{1/F\}$. Since $L_{B/F}$ is an $n$-fold Smarandache implicative $B_{L}$-residuated lattice, by Proposition 3.25, $\{1/F\}$ is an $n$-fold $B_{L/F}$-Smarandache implicative filter of $L_{B/F}$. Hence $x/F \in \{1/F\}$ or equivalently $x \in F$. Then by Theorem 3.15, $F$ is an $n$-fold $B_{L}$-Smarandache implicative filter of $L_{B}$. $\Box$

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