On Mean Graphs

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Abstract: Let $G(V, E)$ be a graph with $p$ vertices and $q$ edges. For every assignment $f : V(G) \rightarrow \{0, 1, 2, 3, \ldots, q\}$, an induced edge labeling $f^* : E(G) \rightarrow \{1, 2, 3, \ldots, q\}$ is defined by

$$f^*(uv) = \begin{cases} 
\frac{f(u) + f(v)}{2} & \text{if } f(u) \text{ and } f(v) \text{ are of the same parity} \\
\frac{f(u) + f(v) + 1}{2} & \text{otherwise}
\end{cases}$$

for every edge $uv \in E(G)$. If $f^*(E) = \{1, 2, \ldots, q\}$, then we say that $f$ is a mean labeling of $G$. If a graph $G$ admits a mean labeling, then $G$ is called a mean graph. In this paper, we prove that the graphs double sided step ladder graph $2S(T_m)$, Jelly fish graph $J(m, n)$ for $|m - n| \leq 2$, $P_n(+N_m)$, $(P_2 \cup kK_1) + N_2$ for $k \geq 1$, the triangular belt graph $TB(\alpha)$, $TBL(n, \alpha, k, \beta)$, the edge $mC_n$ – snake, $m \geq 1, n \geq 3$ and $S_t(B(m)n)$ are mean graphs. Also we prove that the graph obtained by identifying an edge of two cycles $C_m$ and $C_n$ is a mean graph for $m, n \geq 3$.

Key Words: Smarandachely edge 2-labeling, mean graph, mean labeling, Jelly fish graph, triangular belt graph.

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§1. Introduction

Throughout this paper, by a graph we mean a finite, undirected, simple graph. Let $G(V, E)$ be a graph with $p$ vertices and $q$ edges. For notations and terminology we follow [1].

Path on $n$ vertices is denoted by $P_n$ and a cycle on $n$ vertices is denoted by $C_n$. $K_{1,m}$ is called a star and it is denoted by $S_m$. The bistar $B_{m,n}$ is the graph obtained from $K_2$ by identifying the center vertices of $K_{1,m}$ and $K_{1,n}$ at the end vertices of $K_2$ respectively. $B_{m,m}$ is often denoted by $B(m)$. The join of two graphs $G$ and $H$ is the graph obtained from $G \cup H$ by joining each vertex of $G$ with each vertex of $H$ by means of an edge and it is denoted by $G + H$. The edge $mC_n$ – snake is a graph obtained from $m$ copies of $C_n$ by identifying the edge $v_{k+1}v_{k+2}$ in each copy of $C_n$, $n$ is either $2k + 1$ or $2k$ with the edge $v_1v_2$ in the successive

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copy of $C_n$. The graph $P_n \times P_2$ is called a ladder. Let $P_{2n}$ be a path of length $2n - 1$ with $2n$ vertices $(1, 1), (1, 2), \ldots, (1, 2n)$ with $2n - 1$ edges $e_1, e_2, \ldots, e_{2n-1}$ where $e_i$ is the edge joining the vertices $(1, i)$ and $(1, i+1)$. On each edge $e_i$, for $i = 1, 2, \ldots, n$, we erect a ladder with $i + 1$ steps including the edge $e_i$ and on each edge $e_i$, for $i = n+1, n+2, \ldots, 2n - 1$, we erect a ladder with $2n + 1 - i$ steps including the edge $e_i$. The resultant graph is called double sided step ladder graph and is denoted by $2S(T_m)$, where $m = 2n$ denotes the number of vertices in the base.

A vertex labeling of $G$ is an assignment $f : V(G) \to \{0, 1, 2, \ldots, q\}$. For a vertex labeling $f$, the induced edge labeling $f^*$ is defined by

$$f^*(uv) = \begin{cases} \frac{f(u) + f(v)}{2} & \text{if } f(u) \text{ and } f(v) \text{ are of the same parity} \\ \frac{f(u) + f(v) + 1}{2} & \text{otherwise} \end{cases}$$

A vertex labeling $f$ is called a mean labeling of $G$ if its induced edge labeling $f^* : E(G) \to \{1, 2, \ldots, q\}$ is a bijection, that is, $f^*(E) = \{1, 2, \ldots, q\}$. If a graph $G$ has a mean labeling, then we say that $G$ is a mean graph. It is clear that a mean labeling is a Smarandachely edge 2-labeling of $G$.

A mean labeling of the Petersen graph is shown in Figure 1.

![Figure 1](image-url)

The concept of mean labeling was introduced and studied by S. Somasundaram and R. Ponraj [4]. Some new families of mean graphs are studied by S.K. Vaidya et al. [6], [7]. Further some more results on mean graphs are discussed in [2], [3], [5].

In this paper, we establish the meanness of the graphs double sided step ladder graph $2S(T_m)$, Jelly fish graph $J(m, n)$ for $|m - n| \leq 2$, $P_n (+)N_m$, $(P_2 \cup kK_1) + N_2$ for $k \geq 1$, the triangular belt graph $TB(\alpha)$, $TBL(n, \alpha, k, \beta)$, the edge $mC_n$-snake $m \geq 1, n \geq 3$ and $S_t(B(m)_{(n)})$. Also we prove that the graph obtained by identifying an edge of two cycles $C_m$ and $C_n$ is a mean graph for $m, n \geq 3$.

§2. Mean Graphs

**Theorem 2.1** The double sided step ladder graph $2S(T_m)$ is a mean graph where $m = 2n$ denotes the number of vertices in the base.
Proof Let $P_{2n}$ be a path of length $2n - 1$ with $2n$ vertices $(1,1), (1,2), \ldots, (1,2n)$ with $2n - 1$ edges, $e_1, e_2, \ldots , e_{2n-1}$ where $e_i$ is the edge joining the vertices $(1,i)$ and $(1,i+1)$. On each edge $e_i$, for $i = 1, 2, \ldots , n$, we erect a ladder with $i + 1$ steps including the edge $e_i$ and on each edge $e_i$, for $i = n + 1, n + 2, \ldots , 2n - 1$, we erect a ladder with $2n - 1 - i$ steps including the edge $e_i$.

The double sided step ladder graph $2S(T_m)$ has vertices denoted by $(1,1), (1,2), \ldots, (1,2n), (2,1), (2,2), \ldots, (2,2n), (3,1), (3,2), \ldots, (3,2n-1), (4,1), (4,2), \ldots, (4,2n-2), \ldots, (n+1,n), (n+1,n+1)$. In the ordered pair $(i,j)$, $i$ denotes the row (counted from bottom to top) and $j$ denotes the column (from left to right) in which the vertex occurs. Define $f : V(2S(T_m)) \rightarrow \{0,1,2,\ldots,q\}$ as follows:

\[
f(i,j) = (n + 1 - i)(2n - 2i + 3) + j - 1, \quad 1 \leq j \leq 2n, i = 1, 2
\]
\[
f(i,j) = (n + 1 - i)(2n - 2i + 3) + j + 1 - i, \quad i - 1 \leq j \leq 2n + 2 - i, 3 \leq i \leq n + 1.
\]

Then, $f$ is a mean labeling for the double sided step ladder graph $2S(T_m)$. Thus $2S(T_m)$ is a mean graph.

For example, a mean labeling of $2S(T_{10})$ is shown in Figure 2.

\[
\begin{array}{cccccccccccccccc}
\text{21} & \text{22} & \text{23} & \text{24} & \text{25} & \text{26} & \text{27} & \text{28} \\
\text{36} & \text{37} & \text{38} & \text{39} & \text{40} & \text{41} & \text{42} & \text{43} & \text{44} & \text{45} \\
\text{55} & \text{56} & \text{57} & \text{58} & \text{59} & \text{60} & \text{61} & \text{62} & \text{63} & \text{64} \\
\end{array}
\]

Figure 2

For integers $m, n \geq 0$ we consider the graph $J(m,n)$ with vertex set $V(J(m,n)) = \{u,v,x,y\} \cup \{x_1,x_2,\ldots,x_m\} \cup \{y_1,y_2,\ldots,y_n\}$ and edge set $E(J(m,n)) = \{(u,x),(u,v),(u,y), (v,x),(v,y)\} \cup \{(x_i,x): i = 1,2,\ldots,m\} \cup \{(y_i,y): i = 1,2,\ldots,n\}$. We will refer to $J(m,n)$ as a Jelly fish graph.

**Theorem 2.2** A Jelly fish graph $J(m,n)$ is a mean graph for $m, n \geq 0$ and $|m - n| \leq 2$.

**Proof** The proof is divided into cases following.

**Case 1** $m = n$. 

Define a labeling \( f : V(J(m,n)) \rightarrow \{0,1,2,\ldots,q = m+n+5\} \) as follows:

\[
f(u) = 2, \ f(y) = 0, \\
f(v) = m+n+4, \ f(x) = m+n+5, \\
f(x_i) = 4 + 2(i-1), \quad 1 \leq i \leq m \\
f(y_{n+1-i}) = 3 + 2(i-1), \quad 1 \leq i \leq n
\]

Then \( f \) provides a mean labeling.

**Case 2**  \( m = n + 1 \) or \( n + 2 \)

Define \( f : V(J(m,n)) \rightarrow \{0,1,2,\ldots,q = m+n+5\} \) as follows:

\[
f(u) = 2, \ f(v) = 2n+4, \ f(y) = 0, \\
f(x) = \begin{cases} 
  m+n+5 & \text{if } m = n + 1 \\
  m+n+4 & \text{if } m = n + 2 
\end{cases}
\\
f(x_i) = \begin{cases} 
  4 + 2(i-1), & 1 \leq i \leq n \\
  2n+5+2(i-(n+1)), & n+1 \leq i \leq m 
\end{cases} \\
f(y_{n+1-i}) = 3 + 2(i-1), \quad 1 \leq i \leq n
\]

Then \( f \) gives a mean labeling. Thus \( J(m,n) \) is a mean graph for \( m,n \geq 0 \) and \( |m-n| \leq 2. \quad \square \)

For example, a mean labeling of \( J(6,6) \) and \( J(9,7) \) are shown in Figure 3.

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![Figure 3](image-url)
Let $P_n(+)^N_m$ be the graph with $p = n + m$ and $q = 2m + n - 1$. $V(P_n(+)^N_m) = \{v_1, v_2, \ldots, v_n, y_1, y_2, \ldots, y_m\}$, where $V(P_n) = \{v_1, v_2, \ldots, v_n\}$, $V(N_m) = \{y_1, y_2, \ldots, y_m\}$ and

$$E(P_n(+)^N_m) = E(P_n) \cup \left\{(v_1, y_1), (v_1, y_2), \ldots, (v_1, y_m), (v_n, y_1), (v_n, y_2), \ldots, (v_n, y_m)\right\}$$

**Theorem 2.3** $P_n(+)^N_m$ is a mean graph for all $n, m \geq 1$.

*Proof* Let us define $f : V(P_n(+)^N_m) \rightarrow \{1, 2, 3, \ldots, 2m + n - 1\}$ as follows:

$$f(y_i) = 2i - 1, \quad 1 \leq i \leq m,$$

$$f(v_1) = 0,$$

$$f(v_i) = 2m + 1 + 2(i - 2), \quad 2 \leq i \leq \left\lceil \frac{n + 1}{2} \right\rceil$$

$$f(v_{n+1-i}) = 2m + 2 + 2(i - 1), \quad 1 \leq i \leq \left\lfloor \frac{n - 1}{2} \right\rfloor.$$  

Then, $f$ gives a mean labeling. Thus $P_n(+)^N_m$ is a mean graph for $n, m \geq 1$.  

For example, a mean labeling of $P_8(+)^N_5$ and $P_7(+)^N_6$ are shown in Figure 4.

![Image](image.png)

**Figure 4**

**Theorem 2.4** For $k \geq 1$, the planar graph $(P_2 \cup kK_1) + N_2$ is a mean graph.

*Proof* Let the vertex set of $P_2 \cup kK_1$ be $\{z_1, z_2, x_1, x_2, \ldots, x_k\}$ and $V(N_2) = \{y_1, y_2\}$. We have $q = 2k + 5$. Define a labeling $f : V((P_2 \cup kK_1) + N_2) \rightarrow \{1, 2, \ldots, 2k + 5\}$ by

$$f(y_1) = 0, \quad f(y_2) = 2k + 5, \quad f(z_1) = 2,$$

$$f(z_2) = 2k + 4,$$

$$f(x_i) = 4 + 2(i - 1), \quad 1 \leq i \leq k.$$
Then, \( f \) is a mean labeling and hence \((P_2 \cup kK_1) + N_2\) is a mean graph for \( k \geq 1 \).

For example, a mean labeling of \((P_2 \cup 5K_1) + N_2\) is shown in Figure 5.

Let \( S = \{\uparrow, \downarrow\} \) be the symbol representing the position of the block as given in Figure 6.

Let \( \alpha \) be a sequence of \( n \) symbols of \( S, \alpha \in S^n \). We will construct a graph by tiling \( n \) blocks side by side with their positions indicated by \( \alpha \). We will denote the resulting graph by \( TB(\alpha) \) and refer to it as a triangular belt.

For example, the triangular belts corresponding to sequences \( \alpha_1 = \{\uparrow\uparrow\}\), \( \alpha_2 = \{\downarrow\downarrow\} \) respectively are shown in Figure 7.
Theorem 2.5 A triangular belt $TB(\alpha)$ is a mean graph for any $\alpha$ in $S^n$ with the first and last block are being ↓ for all $n \geq 1$.

Proof Let $u_1, u_2, \ldots, u_n, u_{n+1}$ be the top vertices of the belt and $v_1, v_2, \ldots, v_n, v_{n+1}$ be the bottom vertices of the belt. The graph $TB(\alpha)$ has $2n + 2$ vertices and $4n + 1$ edges. Define $f : V(TB(\alpha)) \to \{0, 1, 2, \ldots, q = 4n + 1\}$ as follows:

\[
  f(u_i) = 4i, \quad 1 \leq i \leq n \\
f(u_{n+1}) = 4n + 1 \\
f(v_1) = 0 \\
f(v_i) = 2 + 4(i - 2), \quad 2 \leq i \leq n
\]

Then $f$ gives a mean labeling. Thus $TB(\alpha)$ is a mean graph for all $n \geq 1$. \hfill \Box

For example, a mean labeling of $TB(\alpha), TB(\beta)$ and $TB(\gamma)$ are shown in Figure 8.

![Figure 8](image_url)

Corollary 2.6 The graph $P_n^2$ is a mean graph.

Proof The graph $P_n^2$ is isomorphic to $TB(\downarrow, \downarrow, \ldots, \downarrow)$ or $TB(\uparrow, \uparrow, \ldots, \uparrow)$. Hence the result follows from Theorem 2.5. \hfill \Box

We now consider a class of planar graphs that are formed by amalgamation of triangular belts. For each $n \geq 1$ and $\alpha$ in $S^n$ $n$ blocks with the first and last block are ↓ we take the triangular belt $TB(\alpha)$ and the triangular belt $TB(\beta)$ in $S^k$ where $k > 0$.

We rotate $TB(\beta)$ by 90 degrees counter clockwise and amalgamate the last block with the first block of $TB(\alpha)$ by sharing an edge. The resulting graph is denoted by $TBL(n, \alpha, k, \beta)$, which has $2(nk + 1)$ vertices, $3(n + k) + 1$ edges with

\[
  V(TBL(n, \alpha, k, \beta)) = \{u_{1,1}, u_{1,2}, \ldots, u_{1,n+1}, u_{2,1}, u_{2,2}, \\
  \ldots, u_{2,n+1}, v_{3,1}, v_{3,2}, \ldots, v_{3,k-1}, v_{4,1}, v_{4,2}, \ldots, v_{4,k-1}\}.
\]
**Theorem 2.7** The graph $TBL(n, \alpha, k, \beta)$ is a mean graph for all $\alpha$ in $S^n$ with the first and last block are $\downarrow$ and $\beta$ in $S^k$ for all $k > 0$.

**Proof** Define $f : V(TBL(n, \alpha, k, \beta)) \to \{0, 1, \ldots, 3(n + k) + 1\}$ as follows:

- $f(u_{1,i}) = 4k + 4i$, \quad $1 \leq i \leq n$
- $f(u_{1,n+1}) = 4(n + k) + 1$
- $f(u_{2,1}) = 4k$
- $f(u_{2,i}) = 4k + 2 + 4(i - 2)$, \quad $2 \leq i \leq n + 1$
- $f(v_{3,i}) = 4i - 4$, \quad $1 \leq i \leq k$
- $f(v_{4,i}) = 4i - 2$, \quad $1 \leq i \leq k$

Then $f$ provides a mean labeling and hence $TBL(n, \alpha, k, \beta)$ is a mean graph. \( \square \)

For example, a mean labeling of $TBL(4, \downarrow, \uparrow, \uparrow, \uparrow, \downarrow, \uparrow, \uparrow, 2, \uparrow)$ and $TBL(5, \downarrow, \downarrow, \uparrow, \uparrow, \downarrow, \uparrow, \downarrow, \uparrow, \downarrow, \uparrow)$ is shown in Figure 9.
Theorem 2.8  The graph edge $mC_n$-snake, $m \geq 1, n \geq 3$ has a mean labeling.

Proof  Let $v_{1j}, v_{2j}, \ldots, v_{nj}$ be the vertices and $e_{1j}, e_{2j}, \ldots, e_{nj}$ be the edges of edge $mC_n$-snake for $1 \leq j \leq m$.

Case 1  $n$ is odd

Let $n = 2k + 1$ for some $k \in \mathbb{Z}^+$. Define a vertex labeling $f$ of edge $mC_n$-snake as follows:

$$f(v_{11}) = 0, \quad f(v_{21}) = 1$$
$$f(v_{1i}) = 2i - 2, \quad 3 \leq i \leq k + 1$$
$$f(v_{(k+1+i)1}) = n - 2(i - 1), \quad 1 \leq i \leq k$$
$$f(v_{12}) = f(v_{(k+2)1}), \quad f(v_{22}) = f(v_{(k+1)1}),$$
$$f(v_{2i}) = n + 4 + 2(i - 3), \quad 3 \leq i \leq k + 1$$
$$f(v_{(k+1+i)2}) = 2n - 2 - 2(i - 1), \quad 1 \leq i \leq k - 1$$
$$f(v_{1n}) = n + 2$$
$$f(v_{ij}) = f(v_{i-1j}) + 2n - 2, \quad 3 \leq j \leq m, \quad 1 \leq i \leq n.$$  

Then $f$ gives a mean labeling.

Case 2  $n$ is even

Let $n = 2k$ for some $k \in \mathbb{Z}^+$. Define a labeling $f$ of edge $mC_n$-snake as follows:

$$f(v_{11}) = 0, \quad f(v_{21}) = 1,$$
$$f(v_{1i}) = 2i - 2, \quad 3 \leq i \leq k + 1$$
$$f(v_{(k+1+i)1}) = n - 1 - 2(i - 1), \quad 1 \leq i \leq k - 1$$
$$f(v_{1n}) = f(v_{i-1}) + n - 1, \quad 2 \leq j \leq m, \quad 1 \leq i \leq n$$

Then $f$ is a mean labeling. Thus the graph edge $mC_n$-snake is a mean graph for $m \geq 1$ and $n \geq 3$. \hfill \Box

For example, a mean labeling of edge $4C_7$-snake and $5C_6$-snake are shown in Figure 10.
Theorem 2.9 Let $G'$ be a graph obtained by identifying an edge of two cycles $C_m$ and $C_n$. Then $G'$ is a mean graph for $m, n \geq 3$.

Proof Let us assume that $m \leq n$.

Case 1 $m$ is odd and $n$ is odd

Let $m = 2k + 1$, $k \geq 1$ and $n = 2l + 1$, $l \geq 1$. The $G'$ has $m + n - 2$ vertices and $m + n - 1$ edges. We denote the vertices of $G'$ as follows:

![Diagram](image)

**Figure 11**

Define $f : V(G') \rightarrow \{0, 1, 2, 3, \ldots, q = m + n - 1\}$ as follows:

$$
\begin{align*}
  f(v_1) &= 0, \\
  f(v_i) &= 2i - 1, \quad 2 \leq i \leq k + 1 \\
  f(v_i) &= m + 3 + 2(i - k - 2), \quad k + 2 \leq i \leq k + l \\
  f(v_i) &= m + n - 1 - 2(i - k - l - 1), \quad k + l + 1 \leq i \leq k + 2l \\
  f(v_i) &= m - 1 - 2(i - k - 2l - 1), \quad k + 2l + 1 \leq i \leq 2k + 2l
\end{align*}
$$

Then $f$ is a mean labeling.

Case 2 $m$ is odd and $n$ is even

Let $m = 2k + 1$, $k \geq 1$ and $n = 2l$, $l \geq 2$. Define $f : V(G') \rightarrow \{0, 1, 2, 3, \ldots, q = m + n - 1\}$ as follows:

$$
\begin{align*}
  f(v_1) &= 0, \\
  f(v_i) &= 2i - 1, \quad 2 \leq i \leq k + 1 \\
  f(v_i) &= m + 3 + 2(i - k - 2), \quad k + 2 \leq i \leq k + l \\
  f(v_i) &= m + n - 2 - 2(i - k - l - 1), \quad k + l + 1 \leq i \leq k + 2l - 1 \\
  f(v_i) &= m - 1 - 2(i - k - 2l), \quad k + 2l \leq i \leq 2k + 2l - 1
\end{align*}
$$

Then, $f$ gives a mean labeling.

Case 3 $m$ and $n$ are even
Let \( m = 2k, k \geq 2 \) and \( n = 2l, l \geq 2 \). Define \( f \) on the vertex set of \( G' \) as follows:

\[
\begin{align*}
    f(v_1) &= 0, \quad f(v_i) = 2i - 2, \quad 2 \leq i \leq k + 1 \\
    f(v_i) &= m + 3 + 2(i - k - 2), \quad k + 2 \leq i \leq k + l \\
    f(v_i) &= m + n - 2 - 2(i - k - l - 1), \quad k + l + 1 \leq i \leq k + 2l - 1 \\
    f(v_i) &= m - 1 - 2(i - k - 2l), \quad k + 2l \leq i \leq 2k + 2l - 2
\end{align*}
\]

Then, \( f \) is a mean labeling. Thus \( G' \) is a mean graph. \( \square \)

For example, a mean labeling of the graph \( G' \) obtained by identifying an edge of \( C_7 \) and \( C_{10} \) are shown in Figure 12.

\[ \text{Figure 12} \]

**Theorem 2.10** Let \( \{u_i v_i w_i u_i : 1 \leq i \leq n\} \) be a collection of \( n \) disjoint triangles. Let \( G \) be the graph obtained by joining \( w_i \) to \( u_{i+1}, 1 \leq i \leq n-1 \) and joining \( u_i \) to \( u_{i+1} \) and \( v_i+1, 1 \leq i \leq n-1 \). Then \( G \) is a mean graph.

**Proof** The graph \( G \) has \( 3n \) vertices and \( 6n - 3 \) edges respectively. We denote the vertices of \( G \) as in Figure 13.

\[ \text{Figure 13} \]

Define \( f : V(G) \to \{0, 1, 2, \ldots, 6n - 3\} \) as follows:

\[
\begin{align*}
    f(u_i) &= 6i - 4, 1 \leq i \leq n \\
    f(v_i) &= 6i - 6, 1 \leq i \leq n \\
    f(w_i) &= 6i - 3, 1 \leq i \leq n.
\end{align*}
\]

Then \( f \) gives a mean labeling and hence \( G \) is a mean graph. \( \square \)

For example, a mean labeling of \( G \) when \( n = 6 \) is shown Figure 14.
The graph obtained by attaching $m$ pendant vertices to each vertex of a path of length $2n - 1$ is denoted by $B(m(n))$. Dividing each edge of $B(m(n))$ by $t$ number of vertices, the resultant graph is denoted by $S_t(B(m(n)))$.

**Theorem 2.11** The $S_t(B(m(n)))$ is a mean graph for all $m, n, t \geq 1$.

**Proof** Let $v_1, v_2, \ldots, v_{2n}$ be the vertices of the path of length $2n - 1$ and $u_{i,1}, u_{i,2}, \ldots, u_{i,m}$ be the pendant vertices attached at $v_i, 1 \leq i \leq 2n$ in the graph $B(m(n))$. Each edge $v_i v_{i+1}, 1 \leq i \leq 2n - 1$, is subdivided by $t$ vertices $x_{i,1}, x_{i,2}, \ldots, x_{i,t}$ and each pendant edge $v_i u_{i,j}, 1 \leq i \leq 2n, 1 \leq j \leq m$ is subdivided by $t$ vertices $y_{i,j,1}, y_{i,j,2}, \ldots, y_{i,j,t}$.

The vertices and their labels of $S_t(B(m(n)))$ are shown in Figure 15.

Define $f : V(S_t(B(m(n)))) \to \{0, 1, 2, \ldots, (t + 1)(2mn + 2n - 1)\}$ as follows:

$$f(v_i) = \begin{cases} (t + 1)(m + 1)(i - 1) & \text{if } i \text{ is odd and } 1 \leq i \leq 2n - 1 \\ (t + 1)(m + 1)(i - 1) & \text{if } i \text{ is even and } 1 \leq i \leq 2n - 1 \end{cases}$$

$$f(x_{i,k}) = \begin{cases} (t + 1)[(m + 1)i + m - 1] + k & \text{if } i \text{ is odd, } 1 \leq i \leq 2n - 1 \text{ and } 1 \leq k \leq t \\ (t + 1)[(m + 1)i - 1] + k & \text{if } i \text{ is even, } 1 \leq i \leq 2n - 1 \text{ and } 1 \leq k \leq t \end{cases}$$

$$f(y_{i,j,k}) = \begin{cases} (t + 1)(m + 1)(i - 1) & \text{if } i \text{ is odd}, \\ +(2t + 2)(j - 1) + k, & 1 \leq i \leq 2n, 1 \leq j \leq m \text{ and } 1 \leq k \leq t \\ (t + 1)(m + 1)(i - 2) + 1 & \text{if } i \text{ is even,} \\ +(2t + 2)(j - 1) + k, & 1 \leq i \leq 2n, 1 \leq j \leq m \text{ and } 1 \leq k \leq t \end{cases}$$
and $f(u_{i,j}) = \begin{cases} 
(t + 1)(m + 1)(i - 1) + 1 & \text{if } i \text{ is odd,} \\
+(2t+2)(j-1), & 1 \leq i \leq 2n \text{ and } 1 \leq j \leq m \\
(t + 1)(m + 1)(i - 2) + 2 & \text{if } i \text{ is even,} \\
+(2t+2)(j-1), & 1 \leq i \leq 2n \text{ and } 1 \leq j \leq m. 
\end{cases}$

Then, $f$ is a mean labeling. Thus $S_t(B(m)(n))$ is a mean graph. \hfill \Box

For example, a mean labeling of $S_3(B(4)(2))$ is shown in Figure 16.

![Figure 16](image)

References