Long Dominating Cycles in Graphs

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Abstract: Let $G$ be a connected graph of order $n$, and $NC^2(G)$ denote $\min\{|N(u)\cup N(v)| : \text{dist}(u,v) = 2\}$, where $\text{dist}(u,v)$ is the distance between $u$ and $v$ in $G$. A cycle $C$ in $G$ is called a dominating cycle, if $V(G)\setminus V(C)$ is an independent set in $G$. In this paper, we prove that if $G$ contains a dominating cycle and $\delta \geq 2$, then $G$ contains a dominating cycle of length at least $\min\{n, 2NC^2(G) - 1\}$ and give a family of graphs showing our result is sharp, which proves a conjecture of R. Shen and F. Tian, also related with the cyclic structures of algebraically Smarandache multi-spaces.

Key words: Dominating cycle, neighborhood union, distance.

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§1. Introduction

All graphs considered in this paper will be finite and simple. We use Bondy & Murty [1] for terminology and notations not defined here.

Let $G = (V,E)$ be a graph of order $n$ and $C$ be a cycle in $G$. $C$ is called a dominating cycle, or briefly a D-cycle, if $V(G)\setminus V(C)$ is an independent set in $G$. For a vertex $v$ in $G$, the neighborhood of $v$ is denoted by $N(v)$, and the degree of $v$ is denoted by $d(v)$. For two subsets $S$ and $T$ of $V(G)$, we set $N_T(S) = \{v \in T : N(v) \cap S \neq \emptyset\}$. We write $N(u,v)$ instead of $N_{V(G)}(\{u,v\})$ for any $u,v \in V(G)$. If $F$ and $H$ are two subgraphs of $G$, we also write $N_F(H)$ instead of $N_{V(F)}(V(H))$. In the case $F = G$, if no ambiguity can arise, we usually omit the subscript $G$ of $N_G(H)$. We denote by $G[S]$ the subgraph of $G$ induced by any subset $S$ of $V(G)$.

For a connected graph $G$ and $u,v \in V(G)$, we define the distance between $u$ and $v$ in $G$, denoted by $\text{dist}(u,v)$, as the minimum value of the lengths of all paths joining $u$ and $v$ in $G$. If $G$ is non-complete, let $NC(G)$ denote $\min\{|N(u,v)| : uv \notin E(G)\}$ and $NC^2(G)$ denote $\min\{|N(u,v)| : \text{dist}(u,v) = 2\}$; if $G$ is complete, we set $NC(G) = n - 1$ and $NC^2(G) = n - 1$.

In [2], Broersma and Veldman gave the following result.

Theorem 1([2]) If $G$ is a 2-connected graph of order $n$ and $G$ contains a D-cycle, then $G$ has a D-cycle of length at least $\min\{n, 2NC(G)\}$ unless $G$ is the Petersen graph.

For given positive integers $n_1,n_2$ and $n_3$, let $K(n_1,n_2,n_3)$ denote the set of all graphs
of order \(n_1 + n_2 + n_3\) consisting of three disjoint complete graphs of order \(n_1, n_2\) and \(n_3\), respectively. For any integer \(p \geq 3\), let \(\mathcal{J}_1^*\) (resp. \(\mathcal{J}_2^*\)) denote the family of all graphs of order \(2p + 3\) (resp. \(2p + 4\)) which can be obtained from a graph \(H\) in \(K(3, p, p)\) (resp. \(K(3, p, p + 1)\)) by adding the edges of two triangles between two disjoint triples of vertices, each containing one vertex of each component of \(H\). Let \(\mathcal{J}_1 = \{G : G\) is a spanning subgraph of some graph in \(\mathcal{J}_1^*\}\) and \(\mathcal{J}_2 = \{G : G\) is a spanning subgraph of some graph in \(\mathcal{J}_2^*\}\). In [5], Tian and Zhang got the following result.

**Theorem 2** ([5]) If \(G\) is a 2-connected graph of order \(n\) such that every longest cycle in \(G\) is a D-cycle, then \(G\) contains a D-cycle of length at least \(\min\{n, 2NC2(G)\}\) unless \(G\) is the Petersen graph or \(G \in \mathcal{J}_1 \cup \mathcal{J}_2\).

In [4], Shen and Tian weakened the conditions of Theorem 2 and obtained the following theorem.

**Theorem 3** ([4]) If \(G\) contains a D-cycle and \(\delta \geq 2\), then \(G\) contains a D-cycle of length at least \(\min\{n, 2NC2(G) - 3\}\).

**Theorem 4** ([6]) If \(G\) contains a D-cycle and \(\delta \geq 2\), then \(G\) contains a D-cycle of length at least \(\min\{n, 2NC2(G) - 2\}\).

In [4], Shen and Tian believed the followings are true.

**Conjecture 1** If \(G\) satisfies the conditions of Theorem 3, then \(G\) contains a D-cycle of length at least \(\min\{n, 2NC2(G) - \epsilon(n)\}\), where \(\epsilon(n) = 1\) if \(n\) is even, and \(\epsilon(n) = 2\) if \(n\) is odd.

**Conjecture 2** If \(G\) contains a D-cycle and \(\delta \geq 2\), then \(G\) contains a D-cycle of length at least \(\min\{n, 2NC2(G)\}\) unless \(G\) is one of the exceptional graphs listed in Theorem 2. And the complete bipartite graphs \(K_{m,m+q}\) \((q \geq 1)\) show that the bound \(2NC2(G)\) is sharp.

In this paper, we prove the following result, which solves Conjecture 1 due to Shen and Tian, also related with the cyclic structures of algebraically Smarandache multi-spaces (see [3] for details).

**Theorem 5** If \(G\) contains a D-cycle and \(\delta \geq 2\), then \(G\) contains a D-cycle of length at least \(\min\{n, 2NC2(G) - 1\}\) unless \(G \in \mathcal{J}_1\).

**Remark** The Petersen graph shows that our bound \(2NC2(G) - 1\) is sharp.

§2. **Proof of Theorem 5**

In order to prove Theorem 5, we introduce some additional notations.

Let \(C\) be a cycle in \(G\). We denote by \(\overrightarrow{C}\) the cycle \(C\) with a given orientation. If \(u, v \in V(C)\), then \(u\overrightarrow{C}v\) denotes the consecutive vertices on \(C\) from \(u\) to \(v\) in the direction specified by \(\overrightarrow{C}\). The same vertices, in reverse order, are given by \(v\overrightarrow{C}u\). We will consider \(u\overrightarrow{C}v\) and \(v\overrightarrow{C}u\) both as paths and as vertex sets. We use \(u^+\) to denote the successor of \(u\) on \(\overrightarrow{C}\) and \(u^-\) to
denote its predecessor. We write $u^{+2} := (u^+)^+ + u^{-2} := (u^-)^-$, etc. If $A \subseteq V(G)$, then $A^+ = \{ v^+ : v \in A \}$ and $A^- = \{ v^- : v \in A \}$. For any subset $S$ of $V(G)$, we write $N^+(S)$ and $N^-(S)$ instead of $(N(S))^+$ and $(N(S))^-$, respectively.

Let $G$ be a graph satisfying the conditions of Theorem 4, i.e. $G$ contains a D-cycle and $\delta \geq 2$. Throughout, we suppose that

1. $G$ is non-hamiltonian and $C$ is a longest D-cycle in $G$,
2. $|V(C)| \leq 2NC2(G) - 2$,
3. $R = G \setminus V(C)$ and $x \in R$, such that $d(x)$ is as large as possible.

First of all, we prove some claims.

By the maximality of $C$ and the definition of D-cycle, we have

**Claim 1** \( N(x) \subseteq V(C) \).

**Claim 2** \( N(x) \cap N^+(x) = N(x) \cap N^-(x) = \emptyset \).

Let $v_1, v_2, \ldots, v_k$ be the vertices of $N(x)$, in cyclic order around $C$. Then $k \geq 2$ since $\delta \geq 2$. For any $i \in \{1, 2, \ldots, k\}$, we have $v^{+}_i \neq v_{i+1}$ (indices taken modulo $k$) by Claim 2. Let $u_i = v^{+}_i$, $w_i = v_{i+1}^- \ (\text{indices taken modulo } k)$, $T_i = u_i \overrightarrow{C} w_i$, $t_i = |T_i|$.

**Claim 3** \( N_R(v_1) \cap N_R(v_2) = \emptyset \), if $y_1, y_2 \in N^+(x)$ or $y_1, y_2 \in N^-(x)$. In particular, $N^+(x) \cap N(u_i) = N^-(x) \cap N(w_i) = \emptyset$.

For any $i, j \in \{1, 2, \ldots, k\} (i \neq j)$, we also have the following Claims.

**Claim 4** Each of the followings does not hold:

1. There are two paths $P_1[w_j, z]$ and $P_2[u_i, z^-] \ (z \in v^{+}_{j+1} \overrightarrow{C} v_i)$ of length at most two that are internally disjoint from $C$ and each other;
2. There are two paths $P_1[w_j, z]$ and $P_2[u_i, z^+] \ (z \in v^{+}_{j+1} \overrightarrow{C} v_i)$ of length at most two that are internally disjoint from $C$ and each other;
3. There are two paths $P_1[u_i, z]$ and $P_2[u_j, z^+] \ (z \in u^+_i \overrightarrow{C} v_j)$ of length at most two that are internally disjoint from $C$ and each other, and similarly for $P_1[u_i, z]$ and $P_2[u_j, z^-] \ (z \in u^+_i \overrightarrow{C} v_j)$.

**Claim 5** For any \( v \in V(G) \), we have $d_R(v) \leq 1$.

If not, then by Claim 1, there exists a vertex, say $v$, in $C$ such that $d_R(v) > 1$. Let $x_1, x_2 \in N_R(v)$, then $|N(x_1, x_2)| \geq NC2(G)$.

First, we prove that $|N(x_1, x_2) \cap N^+(x_1, x_2)| \leq 2$. Otherwise, let $y_1, y_2$ and $y_3$ be three distinct vertices in $N(x_1, x_2) \cap N^+(x_1, x_2)$. By Claim 2, we know $y_i \in N(x_1) \cap N^+(x_2)$ or $y_i \in N(x_2) \cap N^+(x_1)$ for any $i \in \{1, 2, 3\}$. Thus, there must exist $i$ and $j$ ($i \neq j, i, j \in \{1, 2, 3\}$) such that $y_i, y_j \in N(x_1) \cap N^+(x_2)$ or $y_i, y_j \in N(x_2) \cap N^+(x_1)$. In either case, it contradicts Claim 3. So we have that $|N(x_1, x_2) \cap N^+(x_1, x_2)| \leq 2$. 


Now we have
\[ |V(C)| \geq |N(x_1, x_2) \cup N^+(x_1, x_2)| \]
\[ \geq 2|N(x_1, x_2)| - 2 \]
\[ \geq 2NC2(G) - 2, \]
so \( V(C) = N(x_1, x_2) \cup N^+(x_1, x_2) \) by assumption on \( |V(C)| \), and in particular, \( N(x_1, x_2) \cap N^+(x_1, x_2) = \{y_1, y_2\} \). Therefore \( y_1 \in N(x_1) \cap N^+(x_2) \) and \( y_2 \in N^+(x_1) \cap N(x_2) \).

Now, we prove that \( d_R(v^+) \leq 1, d_R(v^-) \leq 1 \). If not, suppose \( d_R(v^-) > 1 \), let \( z_1, z_2 \in N_R(v^-) \), by Claim 1 and \( V(C) = N(x_1, x_2) \cup N^+(x_1, x_2), N(z_1, z_2) \subseteq N^+(x_1, x_2) \), so we have \( x_1 (or x_2) \in N(v^-) \). Using a similar argument as above, we have \( z_1 (or z_2) \in N(v^-) \), which contradicts Claim 3. Thus, we have \( d_R(v^-) \leq 1 \); similarly, \( d_R(v^+) \leq 1 \).

Now, we consider \( (v_2, v^-) \cap N^-(x_1, v^+) \). Since \( dist(x_2, v^-) = \text{dist}(x_1, v^+) = 2 \) and \( |N(x_2, v^-)| \geq NC2(G), \left| N^-(x_1, v^+) \right| = |N(x_1, v^+)| \geq NC2(G) \). We prove that \( |N_C(x_2, v^-) \cap N_C^-(x_1, v^+)| \leq 1 \). Let \( z \in \{N_C(x_2, v^-) \cap N_C^-(x_1, v^+)\} \setminus \{y_1\} \).

We consider following cases.

(i) Let \( z \in y_1^+ \overline{C} y_2^{-2} \), if \( xz \in E(G) \) and \( xz \in E(G) \), or \( xz \in E(G) \) and \( vz \in E(G) \), each case contradicts Claim 3; if \( vz \in E(G) \) and \( vz \in E(G) \), then \( C' = x_1 y_2 \overline{C} z^+ v^+ \overline{C} zv^- \overline{C} y_2 x_2 x_1 \) is a D-cycle longer than \( C \), a contradiction.

(ii) Let \( z \in y_2^+ \overline{C} y_1^{-2} \), if \( xz \in E(G) \) and \( xz \in E(G) \), or \( xz \in E(G) \) and \( vz \in E(G) \), each case contradicts Claim 3; if \( vz \in E(G) \) and \( vz \in E(G) \), it contradicts Claim 3; if \( vz \in E(G) \) and \( vz \in E(G) \), then \( C' = x_1 y_2 \overline{C} z^+ v^+ \overline{C} zv^- \overline{C} y_2 x_2 x_1 \) is a D-cycle longer than \( C \), for \( z \in v \overline{C} y_2^{-1} \); and \( C' = x_1 y_2 \overline{C} z^+ v^+ \overline{C} zv^- \overline{C} y_2 x_2 x_1 \) is a D-cycle longer than \( C \) for \( z \in y_2^{-1} \).

So, we have \( |N_C(x_2, v^-) \cap N_C^-(x_1, v^+)| \leq 1 \). Moreover, \( y_1, y_2 \notin N(x_2, v^-) \cup N^-(x_1, v^+) \). Otherwise, if \( y_1 \in N(v^-) \), then \( C' = x_1 y_2 \overline{C} v^- \overline{C} y_2 x_2 y_1 \overline{C} \overline{C} x_1 \) is a D-cycle longer than \( C \).

By Claim 2, \( y_1 \notin N(x_2) \cup N^-(x_1, v^+) \), so we have \( y_1 \notin N(x_2, v^-) \cup N^-(x_1, v^+) \). By Claims 1 and 3 we have \( y_2 \notin N(x_2, v^-) \cup N^-(x_1, v^+) \). Thus, we have

\[ |V(C)| \geq |N_C(x_2, v^-) \cup N_C^-(x_1, v^+)| + 2 \]
\[ \geq |N_C(x_2, v^-)| + |N_C^-(x_1, v^+)| - 1 + 2 \]
\[ = |N(x_2, v^-)| \setminus N_R(x_2, v^-) + |N(x_1, v^+)| \setminus N_R(x_1, v^+) + 1 \]
\[ \geq 2NC2(G) - 2 + 1 \]
\[ = 2NC2(G) - 1, \]
a contradiction with \( |V(C)| \leq 2NC2(G) - 2 \). So, we have \( d_R(v) \leq 1 \), for any \( v \in V(G) \).

Claim 6 \( t_i \geq 2 \).

If \( t_i = 1 \) for all \( i \), then \( N_R(u_i) = \emptyset \) for all \( i \) (if not, let \( z \in N_R(u_i) \) for some \( i \), by Claim 1 and Claim 5 \( N(z) \subseteq V(C) \) and \( u_i z \in E(G) \) for some \( j \), then \( z \in N_R(u_i) \cap N_R(u_j) \), a contradiction). Then \( N(u_i) \cap N^+(u_i) = \emptyset \) (otherwise, \( y \in N(u_i) \cap N^+(u_i) \), then \( C' = \overline{C} \).
$x_{i+1}^{v} C y u_1 y C v x$ is a $D$-cycle longer than $C$). Moreover, we have $N(x) \cap N^+(x) = \emptyset$ by Claim 2, $N^+(x) \cap N(u_i) = N^+(u_i) \cap N(x) = \emptyset$ by Claim 3. Hence, $N(x, u_i) \cap N^+(x, u_i) = \emptyset$. So we have

$$|V(C)| \geq |N(x, u_i) \cup N^+(x, u_i)| \geq 2|N(x, u_i)| \geq 2NC2(G),$$

a contradiction. So we may assume $t_i = 1$ for some $i$, without loss of generality, suppose $t_1 = 1$ and $N_R(w_k) \neq \emptyset$. Let $y \in N_R(w_k)$, choose $y_1 \in N(y)$ such that $N(y) \cap (y_1 C w_k) = \emptyset$. Using a similar argument as above and $d_R(u_1) \leq 1$, by Claim 5, we have

$$|V(C)| = |N_C(x, u_1) \cup N_C^+(x, u_1)| \geq 2NC2(G) - 2.$$

So $V(C) = N_C(x, u_1) \cup N_C^+(x, u_1)$. Similarly, we know that $V(C) = N_C(x, u_1) \cup N_C^-(x, u_1)$. Moreover, $u_1 w_k \in E(G)$. If $|y_1 C w_k| = 1$, then $C' = x_{i+1}^{v} C y_1 y w_k w_k u_1 x$ is a $D$-cycle longer than $C$, a contradiction. So we may assume that $|y_1 C w_k| \geq 2$.

Now, we consider $N_C(y, y_1) \cup N_C^-(x, u_1)$. Since $dist(y, y_1) = dist(x, u_2) = 2$, $|N(y, y_1)| \geq NC2(G), |N^-(x, u_1)| = |N(x, u_1)| \geq NC2(G)$. Moreover, we have $v_1, v_2 \notin N_C(y, y_1) \cup N_C^-(x, u_1)$ and $N_C(y, y_1) \cap N_C^-(x, u_1) \subseteq \{w_k\}$. In fact, $v_1 \notin N(y, y_1)$ by Claims 3 and 5, if $v_1 \in N^-(x, u_1)$, then $v_1 x \in E(G)$ or $v_1 u_1 \in E(G)$, which contradicts to Claims 2 and 3. So $v_1 \notin N_C(y, y_1) \cup N_C^-(x, u_1)$ if $v_2 \in N_C(y, y_1)$, then $v_2 y_1 \in E(G)$ by Claim 5, which contradicts to Claim 4. If $v_2 \in N_C(x, u_1)$ then $v_2 \in N(x, u_1)$, which contradicts to Claims 2 and 3. So $v_2 \notin N_C(y, y_1) \cup N_C^-(x, u_1)$. Suppose $z \in N_C(y, y_1) \cap N_C^-(x, u_1) \backslash \{w_k\}$. Now, we consider the following cases.

(i) $z \in v_2 C y_1$. If $yz \in E(G)$ and $xz \in E(G)$, then, it contradicts to Claim 3. Put

$$C' = \begin{cases} yz C v_2 x v_1 u_1 z C w_k y & \text{ if } yz \in E(G) \text{ and } u_1 z \in E(G); \\ xz C y_1 y w_k y z C v_1 x & \text{ if } y_1 z \in E(G) \text{ and } xz \in E(G); \\ x_{i+1}^{v} C y_1 z v_1 w_k y_1 z u_1 v_1 x & \text{ if } y_1 z \in E(G) \text{ and } u_1 z \in E(G). \end{cases}$$

(ii) $z \in y_1 C w_k$, then $z \in N(y_1)$ since $N(y) \cap (y_1 C w_k) = \emptyset$. Let $yz_1 \in E(G)$ and $z \in N_C(x, u_1)$. Since $V(C) = N_C(x, u_1) \cup N_C^-(x, u_1)$. So $y_1 \in N_C(x, u_1) \cup N_C(x, u_1)$. If $u_1 y_1 \in E(G)$ then $C' = x_{i+1}^{v} C y_1 y w_k y_1 z u_1 v_1 x$ is a $D$-cycle longer than $C$, a contradiction; if $x y_1 \in E(G)$, then it contradicts with Claim 3. Then, $y_1 \in N^-(x, u_1)$. If $xz \in E(G)$ and $y_1 z x \in E(G)$, then it contradicts to Claim 3. Put

$$C' = \begin{cases} x y_1 z y_{i+1}^{2} C u_1 z C v_1 x & \text{ if } y_1 z \in E(G) \text{ and } xz \in E(G); \\ x_{i+1}^{v} C y_1 z C u_1 z C x & \text{ if } x y_1 \in E(G) \text{ and } xz \in E(G); \\ x_{i+1}^{v} C y_1 z C u_1 z C v_1 x & \text{ if } y_1 z \in E(G) \text{ and } xz \in E(G). \end{cases}$$

In any cases, $C'$ is a $D$-cycle longer than $C$, a contradiction. Therefore, $v_1, v_2 \notin N_C(y, y_1) \cup N_C^-(x, u_1), N_C(y, y_1) \cap N_C^-(x, u_1) \subseteq \{w_k\}$. Hence, we have

$$|V(C)| \geq |N_C(y, y_1) \cup N_C^-(x, u_1)| + 2$$

$$\geq |N_C(y, y_1)| + |N_C^-(x, u_1)| - 1 + 2$$

$$= |N(y, y_1) \backslash N_R(y, y_1)| + |N(x_1, u_1) \backslash N_R(x_1, u_1)| + 1$$

$$\geq 2NC2(G) - 2 + 1$$

$$= 2NC2(G) - 1,$$
a contradiction with $|V(C)| \leq 2NC2(G) - 2$.

**Claim 7** If $\bigcup_{i=1}^{k} N_R(y_i) \neq \emptyset$, then $N_R(y_i) \neq \emptyset$ for all $i \in \{1, 2, \ldots, k\}$, where $y_i = w_i$ (respectively).

If not, without loss of generality, we assume that $N_R(u_1) \neq \emptyset$ and $N_R(u_k) = \emptyset$. Suppose $x_1 \in N_R(u_1)$ and $y \in N(x_1) \ (y \neq u_1)$. Then $dist(x_1, y^+) = dist(x_1, y^-) = 2$ and $|N(x_1, y^+)| \geq NC2(G)$, $|N(x_1, y^-)| \geq NC2(G)$.

**Case 1** $N(x_1) \cap (u_1^+ C v_k) = \emptyset$.

If not, we may choose $y_1, y \in N(x_1) \cap (u_1^+ C v_k)$, such that $N(x_1) \cap (u_1^+ C y^-) = \emptyset$. We define a mapping $f$ on $V(C)$ as follows:

$$f(v) = \begin{cases} 
   v^- & \text{if } v \in u_k C y^-; \\
   v^+ & \text{if } v \in y C w_{k-1}; \\
   y^- & \text{if } v = v_k.
\end{cases}$$

Then $|f(N_C(x, u_k))| = |N_C(x, u_k)| = |N(x, u_k)| \geq NC2(G)$ by Claim 1 and the assumption $N_R(u_k) = \emptyset$. Moreover, we have $f(N_C(x, u_k)) \cap N(x_1, y^-) \subseteq \{w_k, u_1\}$. In fact, suppose that $z \in f(N_C(x, u_k)) \cap N(x_1, y^-) \setminus \{w_k, u_1\}$. Obviously, $z \neq v_1, y^-$ by Claims 2 and 4. Now we consider the following cases.

(i) If $z \in u_k C w_k$, then $z \in N_C(u_k)$ since $N(x) \cap (u_k C w_k) = \emptyset$. Put

$$C' = \begin{cases} 
   u_k z^+ C v_1 x v_k C u_1 x z C w_k & \text{if } x z \in E(G); \\
   u_k z^+ C v_1 x v_k C y_1 u_1 C y^- z C w_k & \text{if } y z \in E(G).
\end{cases}$$

(ii) If $z \in u_1^+ C y^-$, then $z \in y E(G)$ since $N(x_1) \cap (u_1^+ C y^-) = \emptyset$. Put

$$C' = \begin{cases} 
   u_1 C z y^- C z^+ x v_1 C y x_1 u_1 & \text{if } x z \in E(G); \\
   u_1 C z y^- C z^+ u_k C v_1 x v_k C y x_1 u_1 & \text{if } u_k z \in E(G).
\end{cases}$$

(iii) If $z \in y^+ C v_k$, we put

$$C' = \begin{cases} 
   u_1 C z^- x v_1 C z x_1 u_1 & \text{if } x z \in E(G) \text{ and } x z \in E(G); \\
   u_1 C y^- z C v_1 x z^- C y x_1 u_1 & \text{if } x z \in E(G) \text{ and } y z \in E(G); \\
   u_1 C z^- u_k C v_1 x v_k C z x_1 u_1 & \text{if } u_k z \in E(G) \text{ and } x z \in E(G); \\
   u_1 C y^- z C v_1 x v_k C u_k z^- C y x_1 u_1 & \text{if } u_k z \in E(G) \text{ and } y z \in E(G).
\end{cases}$$

In any case, $C'$ is a D-cycle longer than $C$, a contradiction. Therefore, we have $f(N_C(x, u_k)) \cap N(x_1, y^-) \subseteq \{w_k, u_1\}$. By Claims 2 and 4, we have $u_1 \notin N(x, u_k)$ and $v_1 \notin N(x_1, y^-)$. Then $u_1 \notin f(N_C(x, u_k)) \cup N(x_1, y^-)$. Hence, by Claim 6 we have

$$|V(C)| \geq |f(N_C(x, u_k)) \cup N_C(x_1, y^-)| + 1$$

$$\geq |f(N_C(x, u_k))| + |N_C(x_1, y^-)| - 2 + 1$$

$$\geq 2NC2(G) - 2.$$
So, we have $V(C) = N_C(x_1, y^-) \cup f(N_C(x, u_k)) \cup \{v_i\}, N_C(x_1, y^-) \cap f(N_C(x, u_k)) = \{w_k, u_1\}$. Hence, $y^-w_k \in E(G)$ and $u_ku_1^+ \in E(G)$ since $t_i \geq 2$.

Now, we prove that $N_R(y^-) = \emptyset$. If not, there exist $y_1 \in N_R(y^-), z \in N_C(y_1) (z \neq y^-)$ by Claim 1 and $\delta \geq 2$.

**Subcase 1** $N(y_1) \cap (u_1^+Cy^-2) = \emptyset$.

If not, we choose $z \in N(y_1)$, such that $N(y_1) \cap (z^+Cy^-2) = \emptyset$. Therefore we can define a mapping $f_1$ on $V(C)$ as follows:

$$f_1(v) = \begin{cases} 
  v^- & \text{if } v \in u_k^+Cy^+; \\
  v^+ & \text{if } v \in z^+Cy^-k-1; \\
  z+2 & \text{if } v = u_k; \\
  z^- & \text{if } v = u_k.
\end{cases}$$

Using an argument as above, we have $|f_1(N_C(x, u_k)| \geq NC2(G)$. Moreover, we have $z^+, v_1, y \notin N_C(y_1, z^+) \cup f_1(N_C(x, u_k))$ and $N_C(y_1, z^+) \cap f_1(N_C(x, u_k)) \subseteq \{z^+2, y^-, w_k\}$. Clearly, $z^+ \notin N_C(y_1, z^+)$. If $z^+ \in f_1(N_C(x, u_k))$, then $u_k \in N_C(x, u_k)$, a contradiction. $y_1v_1 \notin E(G)$ by Claim 5. If $v_1z^+ \in E(G)$, since $y, z^+ \in N^+(y_1)$, the two paths $yx_1u_1$ and $z^+v_1$ contradict with Claim 4; By Claims 2 and 4, we have $y \notin N(y_1, z^+)$, if $y \in f_1(N_C(x, u_k))$ then $y^- \in N_C(x, u_k)$, by Claim 3 $y^- \notin N(x)$, so $y^- \in (u_k)$, then $C' = xyzkCyx_1u_1Cy^-u_kCyv_1x$ is a cycle longer than $C$, a contradiction. So we have $z^+, v_1, y \notin N_C(y_1, z^+) \cup f_1(N_C(x, u_k))$. Suppose $s \in N_C(y_1, z^+) \cup f_1(N_C(x, u_k)) \{z^+2, y^-, w_k\}$.

Now, we consider the following cases.

(i) $s \in y^+Cy_k$. If $y_1s \in E(G)$ and $xs^- \in E(G)$ then it contradicts with Claim 4. We put

$$C' = \begin{cases} 
  xvkCyx_1u_1Cy^-yCy^+u_kCyv_1x & \text{if } y_1s, u_k s^- \in E(G); \\
  xs^-Cyx_1u_1Cy^+yCy^+zCy^-zCy^+v_1x & \text{if } z^+, xs^- \in E(G); \\
  xvkCyx_1u_1Cy^+yCy^+zCy^-xCy^-u_kCyv_1x & \text{if } z^+, u_k s^- \in E(G).
\end{cases}$$

(ii) $s \in u_kCyw_k-1$. We have $s \in N^-(u_k)$ since $N(x) \cap (u_kCyw_k) = \emptyset$. Put

$$C' = \begin{cases} 
  xvkCyx_1u_1Cy^-yCy^+u_kCyv_1x & \text{if } y_1s, u_k s^+ \in E(G); \\
  xvkCyx_1u_1Cy^+yCy^+zCy^+sCy^+u_kCyv_1x & \text{if } z^+, u_k s^+ \in E(G).
\end{cases}$$

(iii) $s \in u_1Cy^-2$. If $y_1s, xs^+ \in E(G)$ then contradicts to Claim 4. If $y_1s, u_k s^+ \in E(G)$, then

$$C' = xvkCyx_1u_1Cy^-yCy^+sCy^+u_kCyv_1x$$

is a cycle longer than $C$, a contradiction. If $s \in z^+Cy^-$, we put

$$C' = \begin{cases} 
  xs^-Cy^+zCy^+yCy^+u_kCyv_1x & \text{if } z^+, s^-x \in E(G); \\
  xvkCyx_1u_1Cy^+yCy^+zCy^+sCy^+u_kCyv_1x & \text{if } z^+, s^-u_k \in E(G).
\end{cases}$$
If \( s \in u_1 C z \), we put
\[
C' = \begin{cases} 
  x s^+ C y_1 y y^+ C z^+ s s C u_1 x_1 y C v_1 x & \text{if } z^+ s, x s^+ \in E(G); \\
  x v_k C y x_1 u_1 C s z^+ C y^+ y_1 z C s^+ s u_k C v_1 x & \text{if } z^+ s, u_k s^+ \in E(G).
\end{cases}
\]

In any cases, \( C' \) is a \( D \)-cycle longer than \( C \), a contradiction. Hence, by Claim 5 we have
\[
|V(C)| \geq |f_1(N_C(x, u_k)) \cup N_C(y_1, z^+)| + 3 \\
\geq |f_1(N_C(x, u_k))| + |N_C(y_1, z^+)| - 3 + 3 \\
\geq 2NC2(G) - 1,
\]
a contradiction. So \( N(y_1) \cap (u_1 C y^{-2}) = \emptyset \).

**Subcase 2** \( N(y_1) \cap (y C v_k) = \emptyset \).

If not, we may choose \( z \in N(y_1) \cap (y C v_k) \), such that \( N(y_1) \cap (y C z^-) = \emptyset \). Therefore, we can define a mapping \( f_2 \) on \( V(C) \) as follows:
\[
f_2(v) = \begin{cases} 
  v^+ & \text{if } v \in u_1 C y^{-2} \cup z^{-} C w_{k-1} \\
  v^- & \text{if } v \in y^+ C z^{-2} \cup u_k^+ C v_1 \\
  z^- & \text{if } v = v_k \\
  v_1 & \text{if } v = u_k \\
  z^{-2} & \text{if } v = y \\
  u_1 & \text{if } v = y^- 
\end{cases}
\]

Using a similar argument as above, we have \( |f_2(N_C(x, u_k))| \geq NC2(G) \). We consider \( N_C(y_1, z^-) \cup f_2(N_C(x, u_k)) \), then \( v_1, u_1^+ \notin N_C(y_1, z^-) \cup f_2(N_C(x, u_k)) \), and \( N_C(y_1, z^-) \cap f_2(N_C(x, u_k)) \subseteq \{y^-, w_k\} \). In fact, \( v_1 \notin N(y_1, z^-) \) by Claims 4, 5; if \( v_1 \in f_2(N(x, u_k)) \) then \( u_k \in N(x, u_k) \), a contradiction; if \( u_1^+ \in N(z^-) \), then the paths \( y z_1 u_1 \) and \( z^- u_1^+ \) contradict with Claim 5; if \( u_1^+ \in f_2(N_C(x, u_k)) \), then \( u_1 \in N(x, u_k) \), a contradiction. So we have \( v_1, u_1^+, \notin N_C(y_1, z^-) \cup f_2(N_C(x, u_k)) \). For \( s \in N_C(y_1, z^-) \cap f_2(N_C(x, u_k)) \setminus \{y^-, w_k\} \), we consider the following cases.

(i) \( s \in u_1 C y \). We have \( s \in N(z^-) \) since \( N(y_1) \cap (u_1 C y^{-2}) = \emptyset \). Put
\[
C' = \begin{cases} 
  x s^{-1} C u_1 x_1 y C z^+ s s C y^{-1} z C v_1 x & \text{if } s^{-} x \in E(G); \\
  x v_k C y z_1 y C sz C y x_1 u_1 C s u_k C v_1 x & \text{if } s^{-} u_k \in E(G).
\end{cases}
\]

(ii) \( s \in u_k C v_1 \), then \( s^+ \in N(u_k) \) since \( N(x) \cap (u_k C w_k) = \emptyset \). Put
\[
C' = \begin{cases} 
  x v_k C y z_1 y C u_1 x_1 y C z^+ C u_k s^+ C v_1 x & \text{if } z^{-} s \in E(G); \\
  x v_k C y x_1 u_1 C y^{-} y_1 s C u_k s^+ C v_1 x & \text{if } y_1 s \in E(G).
\end{cases}
\]

(iii) \( s \in y C z^- \), then we have \( s \in N(z^-) \) since \( N(y_1) \cap (y C z^-) = \emptyset \). Put
\[
C' = \begin{cases} 
  x y C z s^{-1} C s^+ C z y_1 C y C u_1 x_1 & \text{if } x s^+ \in E(G); \\
  x v_k C y z_1 y C u_1 x_1 y C s z^+ s u_k C v_1 x & \text{if } u_k s^+ \in E(G).
\end{cases}
\]
(iv) If \( s \in z^{-C}v_k \). If \( y_1s, xs^- \in E(G) \) then it contradicts to Claim 4. We put
\[
C' = \begin{cases} 
    xy_kC^{-sy_1y}C^{-u_1x_1y}C^{-s}ukCv_1x & \text{if } y_1s, u_ks^- \in E(G); \\
    xs^-C^{-zy_1y}C^{-u_1x_1y}C^{-z}sv_1x & \text{if } z^-s, s^-x \in E(G); \\
    xv_kC^{-sz}C^{-y_1x_1u_1C^{-y}y_1zC^{-s}ukCv_1x} & \text{if } z^-s, s^-uk \in E(G).
\end{cases}
\]

In any cases, \( C' \) is a \( D \)-cycle longer than \( C \), a contradiction. Therefore, we have \( v_1, u_1^+ \notin N_C(y_1, z^-) \cup f_2(N_C(x, u_k)) \), and \( N_C(y_1, z^-) \cap f_2(N_C(x, u_k)) \subseteq \{y^-, w_k\} \). So
\[
|V(C)| \geq |N_C(y_1, z^-) \cup f_2(N_C(x, u_k))| + 2 \\
\geq |N_C(y_1, z^-)| + |N_C(x, u_k)| - 2 + 2 \\
\geq 2NC2(G) - 1,
\]
a contradiction with \( |V(C)| \leq 2NC2(G) - 2 \). Hence, \( N(y_1) \setminus \{y^-\} \subseteq (u_kC)u_1 \).

**Subcase 3** \( N(y_1) \cap (u_kC)u_1 = \emptyset \).

If not, we may choose \( z \in N(y_1) \cap (u_kC)u_1 \), such that \( N(y_1) \cap (z^+C)u_1 = \emptyset \). We define a mapping \( f_3 \) on \( V(C) \) as follows:
\[
f_3(v) = \begin{cases} 
    v^- & \text{if } v \in y^+Cv_k \cup u_1^+Cz^+; \\
    v^+ & \text{if } v \in z^+C^{-y}y^2; \\
    z^+ & \text{if } v = u_k; \\
    v_k & \text{if } v = y; \\
    z^+2 & \text{if } v = y^-.
\end{cases}
\]

Using a similar argument as above, we have \( |f_3(N_C(x, u_k))| \geq NC2(G) \). Moreover, \( z^+, u_1^+ \notin N_C(y_1, z^+) \cup f_3(N_C(x, u_k)), N_C(y_1, z^+) \cap f_3(N_C(x, u_k)) \subseteq \{y^-, w_k\} \). In fact, clearly, \( z^+, u_1^+ \notin N_C(y_1, z^+) \), if \( z^+ \in f_3(N_C(x, u_k)) \), then \( u_k \in N_C(x, u_k) \), a contradiction; if \( u_1^+ \notin N_C(y_1, z^+) \), then \( u_1^+ \in N(z^+) \) since \( N_C(y_1) \cap (y^{-2}C)u_k = \emptyset \), so \( C' = x_1yC^zy_1y^{-2}C^{-1}u_1^+z^+C^{-1}u_1x_1 \) is a \( D \)-cycle longer than \( C \), a contradiction; so we have \( z^+, u_1^+ \notin N_C(y_1, z^+) \cup f_3(N_C(x, u_k)) \). Suppose \( s \in N(y_1, z^+) \cap f_3(N_C(x, u_k)) \setminus \{y^-, w_k\} \). Now, we consider the following cases.

(i) If \( s \in v_kCz^+ \), then We have \( s^+u_k \in E(G) \) since \( N(x) \cap (u_kC)w_k = \emptyset \). Put
\[
C' = \begin{cases} 
    xvy_1u_1C^{-y}y_1sC^{-1}u_1sC^{-1}v_1x & \text{if } y_1s \in E(G); \\
    xv_kC^{-y}y_1zC^{-1}sC^{-1}u_1C^{-1}C^{-1}v_1x & \text{if } zs \in E(G).
\end{cases}
\]

(ii) If \( s \in z^+Cw_k^- \), then we have \( s^-u_k, sz^+ \in E(G) \) since \( N(x) \cap (u_kC)w_k = N(y_1) \cap (z^+C)u_1 = \emptyset \). Put
\[
C' = xvy_1u_1C^{-y}y_1zC^{-1}sC^{-1}u_1sC^{-1}v_1x
\]

(iii) If \( s \in u_1^+C^{-y}y^-2 \), then we have \( sz^+ \in E(G) \) since \( N(y_1) \cap (u_1^+C^{-y}y^-2) = \emptyset \). Put
A contradiction with \( N(y_1) \cap (y \overrightarrow{C} v_k) = \emptyset \). Put

\[
C' = \begin{cases} 
  x^{-1} \overrightarrow{C} u_1 x_1 y^{-1} \overrightarrow{C} y_1 y^{-1} \overrightarrow{C} z_1 z_1 v^{-1} x & \text{if } x^{-1} \in E(G); \\
  x_{v_k} \overrightarrow{C} y_1 x_1 u_1 \overrightarrow{C} y_1 y^{-1} \overrightarrow{C} z_1 z_1 v^{-1} x & \text{if } \ u_k^{-1} \in E(G).
\end{cases}
\]

(iv) If \( s \in y \overrightarrow{C} v_k \), then we have \( z^{-1} \in E(G) \) since \( N(y_1) \cap (y \overrightarrow{C} v_k) = \emptyset \). Put

\[
C' = \begin{cases} 
  x^{-1} \overrightarrow{C} z_1 y_1 y^{-1} \overrightarrow{C} u_1 x_1 y^{-1} \overrightarrow{C} z_1 z_1 v^{-1} x & \text{if } x^{-1} \in E(G); \\
  x_{v_k} \overrightarrow{C} s \overrightarrow{C} u_1 \overrightarrow{C} z_1 y_1 y^{-1} \overrightarrow{C} u_1 x_1 y^{-1} \overrightarrow{C} z_1 z_1 v^{-1} x & \text{if } \ u_k s^{-1} \in E(G).
\end{cases}
\]

In any cases, \( C' \) is a \( D \)-cycle longer than \( C \), a contradiction. Therefore we have \( N_C(y_1, z^+) \cap f_3(N_C(x, u_k)) \subseteq \{y^{-1}, w_k\} \). So we have

\[
|V(C)| \geq |N_C(y_1, z^+) \cup f_3(N_C(x, u_k))| + 2 \\
\geq |N_C(y_1, z^+) + |N_C(x, u_k)| - 2 + 2 \\
\geq 2NC2(G) - 1,
\]

a contradiction with \( |V(C)| \leq 2NC2(G) - 2 \). Hence, \( N(y_1) \cap (u_{v_1} \overrightarrow{C} v_1) = \emptyset \).

Thus, \( N(y_1) = \{y^{-1}\} \), which contradicts to \( \delta \geq 2 \). Therefore, we know that \( N_R(y^{-1}) = \emptyset \).

So we have

\[
|V(C)| \geq |f(N_C(x, u_k)) \cup N_C(x_1, y^{-1})| + 1 \\
\geq |f(N_C(x, u_k))| + |N_C(x_1, y^{-1})| - 2 + 1 \\
= |N(x, u_k) \setminus N_R(x, u_k)| + |N(x, y^{-1}) \setminus N_R(x_1, y^{-1})| - 1 \\
= |N(x, u_k)| + |N(x_1, y^{-1})| - 1 \\
\geq 2NC2(G) - 1,
\]

a contradiction. So we have \( N(x_1) \cap (u_1 \overrightarrow{C} v_1) = \emptyset \), hence, \( N(x_1) \subseteq u_k \overrightarrow{C} u_1 \).

Case 2 \( N(x_1) \cap (u_1 \overrightarrow{C} v_1) = \emptyset \).

Otherwise, since \( v_1 x_1 \notin E(G) \), we can choose \( y, y \in u_k \overrightarrow{C} w_k \), such that \( N(x_1) \cap (y \overrightarrow{C} v_1) = \emptyset \). Therefore, we can define a mapping \( g \) on \( V(C) \) as follows:

\[
g(v) = \begin{cases} 
  v^{-1} & \text{if } v \in u_1 \overrightarrow{C} y; \\
  v^+ & \text{if } v \in y \overrightarrow{C} w_k; \\
  y^+ & \text{if } v = u_1; \\
  y & \text{if } v = v_1.
\end{cases}
\]

Using a similar argument as before, we have \( |g(N_C(x, u_k))| \geq NC2(G), y^+ \notin g(N_C(x, u_k)) \cup N(x_1, y^+) \) and \( g(N_C(x, u_k)) \cap N(x_1, y^+) \subseteq \{u_1\} \). Hence, by Claim 6 we have

\[
|V(C)| \geq |g(N_C(x, u_k)) \cup N(x_1, y^+)| + 1 \\
\geq |g(N_C(x, u_k))| + |N(x_1, y^+)| - 1 + 1 \\
\geq 2NC2(G) - 1,
\]
a contradiction. So \( N(x_1) \cap (u_2 \overline{C} v_1) = \emptyset \). Then \( N(x_1) = \{u_1\} \), which contradicts to \( \delta \geq 2 \).

**Claim 8** If \( x_1 \in N_R(u_1) \) and \( N(x_1) \cap (u_1^+ \overline{C} v_k) \neq \emptyset \), then \( \{|u_ku_1^+, y^{-}w_k\} \cap E(G) = \emptyset \) for \( y \in N(x_1) \cap (u_1^+ \overline{C} v_k) \) with \( N(x_1) \cap (u_1^+ \overline{C} y^{-}) = \emptyset \).

First we have \( d(x_1, y^{-}) = 2 \) and \( |N(x_1, y^{-})| \geq NC2(G) \). Let \( u_ku_1^+ \notin E(G) \). Now we define a mapping \( f \) on \( V(C) \) as follows:

\[
 f(v) = \begin{cases} 
 v^- & \text{if } v \in u_k^+ \overline{C} v_1 \cup u_1^+ \overline{C} y^-; \\
 v^+ & \text{if } v \in y \overline{C} w_{k-1}; \\
 y^- & \text{if } v = u_k; \\
 y & \text{if } v = v_k; \\
 u_1 & \text{if } v = u_k^+; \\
 u_1 & \text{if } v = u_1^+; \\
 u_k & \text{if } v = u_1.
\end{cases}
\]

Then \( |f(N_C(x, u_k))| = |N_C(x, u_k)| \geq NC2(G) - 1 \) by Claim 5. Moreover using a similar argument as in Claim 7, we have \( f(N_C(x, u_k)) \cap N(x_1, y^-) \subseteq \{w_k, u_1, y\} \). But we have \( y^-, v_1, u_k \notin f(N_C(x, u_k)) \cup N(x_1, y^-) \) by the choice of \( y \) Claims 2 and 4, respectively. Therefore, by Claim 5 we have

\[
 |V(C)| \geq |f(N_C(x, u_k)) \cup N_C(x_1, y^-)| + 3 \\
 \geq |f(N_C(x, u_k))| + |N_C(x_1, y^-)| - 3 + 3 \\
 \geq 2NC2(G) - 2.
\]

So \( V(C) = f(N_C(x, u_k)) \cup N_C(x_1, y^-) \cup \{v_1, y^-, u_k\} \) by the assumption on \( |V(C)| \), and in particular, \( f(N_C(x, u_k)) \cap N_C(x_1, y^-) = \{w_k, u_1, y\} \). Therefore, \( y^{-}w_k \in E(G) \). Using a similar argument as above, we have if \( y^{-}w_k \notin E(G) \), then \( u_ku_1^+ \in E(G) \).

**Claim 9** There exists a vertex \( x \) with \( x \notin V(C) \) such that \( N_R(u_i) = N_R(w_i) = \emptyset \).

We only prove \( N_R(u_i) = \emptyset \). If not, we may choose \( x \notin V(C) \) such that \( \min \{t_i\} \) is as small as possible. By Claim 7, without loss of generality, suppose that \( t_k = \min \{t_i\} \) for the vertex \( x \). Let \( x_1 \in N_R(u_1), x_2 \in N_R(u_k) \). By Claims 2 and 3, \( x_1 \neq x_1, x_2; x_1 \neq x_2 \). And by Claim 5 and the choice of \( x \), we have \( N(x_1) \cap (u_k \overline{C} v_1) = \emptyset \), for \( i = 1, 2 \). Since \( \delta \geq 2 \), \( N(x_1) \cap (u_1^+ \overline{C} v_k) \neq \emptyset \). Choose \( y \in N(x_1) \cap (u_k \overline{C} v_k) \) such that \( N(x_1) \cap (u_1^+ \overline{C} y^-) = \emptyset \), then \( d(x_1, y^-) = 2 \) and \( |N(x_1, y^-)| \geq NC2(G) \). By Claim 8, we have \( u_ku_1^+ \) or \( y^-w_k \in E(G) \).

First we prove that \( N(x_2) \cap (y \overline{C} v_k) = \emptyset \). If not, we may choose \( z \in y^+ \overline{C} v_k \) such that \( N(x_2) \cap (z^+ \overline{C} v_k) = \emptyset \) by Claim 5. Then \( d(x_2, z^+) = 2 \) and \( |N(x_2, z^+)| \geq NC2(G) \). Now we define a mapping \( f \) on \( V(C) \) as follows:
Then \(|f(N_C(x_2, z^+))| = |N_C(x_2, z^+)| \geq NC2(G) - 1\) by Claim 5. Moreover using a similar argument as in Claim 7, we have \(f(N_C(x_2, z^+)) \cap N(x_1, y^-) \subseteq \{u_1, y\}\). But \(y^-, v_k, v_1 \notin f(N_C(x_2, z^+)) \cup N(x_1, y^-)\), otherwise, \(u_1z^+ \in E(G)\) or \(y^-v_k \in E(G)\) or \(z^+w_k \in E(G)\) by Claim 5, and hence the D-cycle

\[
C' = \begin{cases} 
    u_1z^+ \in E(G) & \text{if } u_1z^+ \in E(G); \\
    u_1x_1y^-v_ku_1^{-1}ukw_k & \text{if } y^-v_k \in E(G); \\
    u_1x_1z^-w_ku_kx_2z^-v_k & \text{if } z^+w_k \in E(G).
\end{cases}
\]

is longer than \(C\), a contradiction. Therefore, by Claim 5 we have

\[
|V(C)| \geq |f(N_C(x_2, z^+)) \cup N_C(x_1, y^-)| + 3 \\
\geq |f(N_C(x_2, z^+))| + |N_C(x_1, y^-)| - 2 + 3 \\
\geq 2NC2(G) - 1,
\]

which contradicts to that \(|V(C)| \leq 2NC2(G) - 2\). So we have \(N(x_2) \cap (y^-v_k) = \emptyset\). Hence \(N(x_2)\setminus (u_1^{-1}Cy^-) \cup \{u_k\}\).

Now, we prove that \(N(x_2) \cap (u_1^{-1}Cy^-) = \emptyset\). In fact, we may choose \(z \in u_1^{-1}Cy^-\) with \(z \in N(x_2)\) such that \(N(x_2) \cap (u_1^{-1}Cy^-) = \emptyset\). (Since \(x_2y^- \notin E(G)\), otherwise, \(C' = u_1^{-1}Cy^-x_2u_kCyv_kCyx_1u_1\) is a D-cycle longer than \(C\), a contradiction.) Then \(d(x_2, y^-) = 2\) and \(|N(x_2, z^-)| \geq NC2(G)\). We define a mapping \(g\) on \(V(C)\) as follows:

\[
g(v) = \begin{cases} 
    v^- & \text{if } v \in z^+v_k; \\
    v^+ & \text{if } v \in u_kCy^-z^-; \\
    v_k & \text{if } v = z; \\
    u_k & \text{if } v = z^-.
\end{cases}
\]

Then we have \(|g(N_C(x_2, z^-))| \geq NC2(G) - 1\) by Claim 5. Moreover using a similar argument as in Claim 7, we have \(g(N_C(x_2, z^-)) \cap N(x_1, y^-) \subseteq \{u_1\}\). But \(v_1, u_k \notin g(N_C(x_2, z^-)) \cup N(x_1, y^-)\), otherwise since \(u_k \notin g(N_C(x_2, z^-)) \cup N(x_1, y^-)\), \(w_kz^- \in E(G)\) by Claims 2 and 4, and hence the D-cycle \(u_1Cy^-w_kCy^{-1}u_kxzCy^{-2}v_kx_1u_1\) is longer than \(C\), a contradiction. Therefore, by Claim 5 we have
\[ |V(C)| \geq |g(N_C(x_2, z^-)) \cap N(x_1, y^-)| + 2 \]
\[ \geq |g(N_C(x_2, z^-))| + |N(x_1, y^-)| - 1 + 2 \]
\[ \geq 2NC2(G) - 1, \]

which contradicts to that \(|V(C)| \leq 2NC2(G) - 2\). So we have \(N(x_2) \cap (u^+_1 \overrightarrow{C} y^-) = \emptyset\).

Therefore, \(N(x_2) = \{u_k\}\), which contradicts to \(\delta \geq 2\).

**Claim 10** For any \(x \not\in V(C), t_i \geq 3\).

Otherwise, there exists a vertex \(x, x \not\in V(C)\), such that \(\min\{t_i\} = 2\) by Claim 6. Note that the choice of the vertex \(x\) in Claim 9, we have \(N_R(u_i) = N_R(u_i) = \emptyset\) for the vertex \(x\). Without loss of generality, suppose \(t_1 = 2\), then \(N_C(u_1) \cap N_C(w_1) = \{u_1\}\) by Claim 4, \(N(x) \cap N^+(x) = \emptyset\) by Claim 2, and \(N_C(u_1) \cap N(x) = N^-(x) \cap N_C(w_1) = \emptyset\) by Claim 3. Hence, \(N_C^-(x, u_1) \cap N_C(x, w_1) = \{u_1\}\). We also have \(|N_C(x, u_1)| \geq NC2(G)\) and \(|N_C(x, w_1)| \geq NC2(G)\) since \(d(x, u_1) = d(x, w_1) = 2\). Then

\[ |V(C)| \geq |N_C^-(x, u_1) \cup N_C(x, w_1)| \]
\[ \geq |N_C^-(x, u_1)| + |N_C(x, w_1)| - 1 \]
\[ \geq 2NC2(G) - 1, \]

which contradicts to that \(|V(C)| \leq 2NC2(G) - 2\).

By Claim 10, we have \(|V(C)| = k + \sum_{i=1}^{k} t_i \geq 4k\). Thus we get the following.

**Claim 11** For any \(x, x \not\in V(C),\)

\[ d(x) \leq \frac{|V(C)|}{4} \leq \frac{2NC2(G) - 2}{4} = (NC2(G) - 1)/2. \]

**Claim 12** \(u^+_i u_j \not\in E(G)\), for the vertex \(x\) as in Claim 9.

In fact, if \(u^+_i u_j \in E(G)\), then the cycle \(u^+_i \overrightarrow{C} v_j x_i \overrightarrow{C} u_j u^+_i\) is a longest D-cycle not containing \(u_i\), by Claim 9. Thus \(d(u_i) \leq (NC2(G) - 1)/2\) by Claim 11. So we have

\[ NC2(G) \leq |N(x, u_i)| \leq d(x) + d(u_i) \leq NC2(G) - 1, \]

a contradiction. We choose \(x\) as in Claim 9, and define a mapping \(f\) on \(V(C)\) as follows:

\[ f(v) = \begin{cases} 
       v^+ & \text{if } v \in u_1 \overrightarrow{C} u_k; \\
       v^- & \text{if } v \in u^+_k \overrightarrow{C} v_1; \\
       u_1 & \text{if } v = v_k; \\
       v_1 & \text{if } v = u_k.
\end{cases} \]

Then \(|f(N_C(x, u_k))| \geq NC2(G)\) and \(|N_C(x, u_1)| \geq NC2(G)\) by Claim 10. Moreover, we have \(f(N_C(x, u_k)) \cap N_C(x, u_1) \{v_2, v_3, \ldots, v_k, w_k\}\). By Claims 2, 4, and 12, we also have \(u^+_2, u^+_3, \ldots, u^+_k \not\in f(N_C(x, u_k)) \cup N_C(x, u_1)\). Therefore, we have
\[ |V(C)| \geq |f(N_C(x, u_k)) \cup N_C(x, u_1)| + k - 2 \]
\[ \geq |f(N_C(x, u_k))| + |N_C(x, u_1)| - k + k - 2 \]
\[ \geq 2NC2(G) - 2. \]

So
\[ V(C) = f(N_C(x, u_k)) \cup N_C(x, u_1) \cup \{u_2^+, u_3^+, \ldots, u_k^+\} \]
by the assumption on \(|V(C)|\), and in particular,
\[ f(N_C(x, u_k)) \cap N_C(x, u_1) = \{v_2, v_3, \ldots, v_k, w_k\}. \]

Then \(u_1w_k, u_kw_{k-1} \in E(G)\).

**Claim 13** \( k = 2. \)

If there exists \(v \in V(C) \setminus \{v_1, v_k\} \), by partition of \(V(C)\), we have \(v^+ \in f(N_C(x, u_k)) \cup N_C(x, u_1) \cup \{u_2^+, u_3^+, \ldots, u_k^+\}\). If \(v^+ \in N_C(x, u_1)\), then \(v^+u_1 \in E(G)\), and the cycle \(u_1v^+Cv_1xv^+Cu_1\) is a D-cycle not containing \(v^+\) by Claim 9. Thus \(d(v^+) \leq (NC2(G) - 1)/2\) by Claim 11. So we have
\[ NC2(G) \leq |N(x, v^+)| \leq d(x) + d(v^+) \leq NC2(G) - 1, \]
a contradiction. So \(v^+ \in N(x, u_k)\), which contradicts to Claims 2,3. Hence we have \(k = 2\).

**Claim 14** Each of the followings does not hold:

1. There is \(u \in u_1Cv_2\), such that \(u^+u_1 \in E(G)\) and \(u^−u_2 \in E(G)\).
2. There is \(u \in u_2Cv_1\), such that \(u^-u_1 \in E(G)\) and \(u^+u_2 \in E(G)\).
3. There is \(u \in u_2Cv_1\), such that \(u^+u_1 \in E(G)\) and \(u^-u_2 \in E(G)\).
4. There is \(u \in u_1Cv_2\), such that \(u^-u_2 \in E(G)\) and \(u^-u_1 \in E(G)\).

If not, suppose there is \(u \in u_1Cv_2\), such that \(u^+u_1 \in E(G)\) and \(u^-u_2 \in E(G)\). We define a mapping \(h\) on \(V(C)\) as follows:
\[
h(v) = \begin{cases} 
v^+ & \text{if } v \in u_1Cv_2 \cup u^+Cw_1; 
v^- & \text{if } v \in u_2^+Cv_1; 
u^+ & \text{if } v = v_2; 
v_1 & \text{if } v = u_2; 
u_1 & \text{if } v = u; 
u & \text{if } v = u_2^+. \end{cases}
\]

Then \(|h(N_C(x, u_2))| \geq NC2(G)\) and \(|N_C(x, u_1)| \geq NC2(G)\). Moreover we have \(u_1 \notin N(x, u_1) \cup h(N(x, u_2))\), and \(N(x, u_1) \cap h(N(x, u_2)) \subseteq \{v_2, u^+\}\). In fact, clearly \(u_1 \notin N(x, u_1)\), if \(u_1 \in h(N(x, u_2))\), then \(u \in N(x, u_2)\), a contradiction. Let \(s \in N(x, u_1) \cap h(N(x, u_2)) \setminus \{v_2, u^+\}\), if \(s \in u_1^+Cv_2 \cap N(x, u_1) \cap h(N(x, u_2)) \setminus \{v_2, u^+\}\) then \(su_1 \in E(G)\) and \(s^-u_2 \in E(G)\); or if
s \in u_2Cw_2 \cap N(x,u_1) \cap h(N(x,u_2))$, then $su_1 \in E(G)$ and $s^+u_2 \in E(G)$, both cases contradict to Claim 3. So $u_1 \notin N(x,u_1) \cup h(N(x,u_2))$, $N(x,u_1) \cap h(N(x,u_2)) \subseteq \{v_2, u^+ \}$. Hence

$$|V(C)| \geq |h(N_C(x,u_2)) \cup N_C(x,u_1)| + 1 \geq |h(N_C(x,u_2))| + |N_C(x,u_1)| - 2 + 1 \geq 2NC2(G) - 1,$$

a contradiction. Similarly, (2), (3) and (4) are true.

**Claim 15** $N(u_2) \cap (u_1Cw_1) = N(u_1) \cap (u_2Cw_2) = \emptyset$.

If not, we may choose $z \in N(u_2) \cap (u_1Cw_1)$, such that $N(u_2) \cap (u_1Cz^-) = \emptyset$. Then $u_1z \in E(G)$ (if not, $u_1z \notin E(G)$ then $u_2z^- \in E(G)$ by partition of $V(G)$, which contradicts the choice of $z$) and $N(u_1) \cap (z^+Cw_1) = \emptyset$ (if not, we may choose $s \in N(u_1) \cap (z^+Cw_1)$, such that $N(u_1) \cap (z^+Cw_1) = \emptyset$ since $z^+u_1 \notin E(G)$. So $s^{-2}u_2 \notin E(G)$. Which contradicts Claim 14) Moreover $u_1Cz \subseteq N(u_1)$, and $zCv_2 \subseteq N(u_2)$. Similarly, we have $y \in u_2Cw_2$, such that $u_2y, u_1y \in E(G)$ and $N(u_1) \cap (u_2Cy^-) = N(u_2) \cap (y^+Cw_2) = \emptyset$, $yCv_1 \subseteq N(u_1)$ and $u_2Cv_2 \subseteq N(u_2)$.

Now we define a mapping $g$ on $V(C)$ as follows:

$$g(v) = \begin{cases} 
v^+ & \text{if } v \in v_2Cw_2; \\
v^- & \text{if } v \in u_1Cw_1; \\
v_2 & \text{if } v = w_2; \\
v_1 & \text{if } v = v_1. \end{cases}$$

Using similar argument as above, consider $N(x,u_1) \cup g(N(x,u_2))$, there exists $u \in V(C)$, such that $u_1u, u_2u \in E(G)$. Without loss generality, we may assume $u \in u_1Cw_1$. Moreover then $N(u_2) \cap (u_1Cw_1) = N(u_2) \cap (v_1Cw^-) = \emptyset$, and $v_1Cw_2 \subseteq N(u_2)$, $uCv_2 \subseteq N(u_1)$. Let $u \neq z$. If $u \in zCw_1^-$, $u^-u_2 \in E(G)$ by partition of $V(C)$ since $u_1 \notin E(G)$, which contradicts to Claim 4; if $u \in u_1Cz$, then $C' = xv_2w_1uCw_1^-u_2Cw_2w^-Cv_1x$ is a $D$-cycle longer than $C$, a contradiction. If $u = z$, since $z^+u_1 \notin E(G)$, $z^+u_2 \in E(G)$ by partition of $V(C)$, which contradicts to Claim 4. Hence $N(u_2) \cap (u_1Cw_1) = \emptyset$. Similarly $N(u_1) \cap (u_2Cw_1) = \emptyset$.

By Claim 15 we have

**Claim 16** If there exists $z \in v_1Cv_2$, such that $u_2z \in E(G)$, then $u_1z \in E(G)$ and $u_1Cz \subseteq N(u_1)$, $zCv_1 \subseteq N(u_2)$. Similarly if there exists $z \in v_2Cv_1$, such that $u_2z \in E(G)$, then $u_1z \in E(G)$ and $u_2Cz \subseteq N(u_2)$, $zCv_2 \subseteq N(u_1)$.

**Proof of Theorem 5**

Now we are going to complete the proof of Theorem 5. We choose $x$ as in Claim 9. By Claim 13, we know that $k = 2$.

First we prove that there exists $u \in V(C)$ such that $u_1, u_2 \in N(u)$. If there is not any $u \in V(C) \setminus \{v_2, v_1, u_2^+ \}$ such that $u_2u \notin E(G)$, then $u_1u \in E(G)$ (if not, $u_1u^+ \in E(G)$ by
partition of $V(C)$. If $u_1w_1 \notin E(G)$ then $u_2w_1^{-} \in E(G)$, so we have $u_1, u_2 \in N(w_1^{-})$; if there is $u \in V(C)$, such that $u_2w_1 \in E(G)$ then, by Claim 16, $u_1w_1 \in E(G)$, hence $u_1, u_2 \in N(u)$.

By Claim 16, clearly, there are not $z \in u_1w_1^{-}, y \in u_2w_1^{-}$, such that $yz \in E(G)$.

So we have $G \in \mathcal{J}_1$. The proof of Theorem 5 is finished.

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**References**


