ON THE TANGENT INDICATRIX OF SPECIAL VIVIANI'S CURVE AND ITS CORRESPONDING SMARANDACHE CURVES ACCORDING TO SABBAN FRAME

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The paper revisits the special Viviani's curve and introduces some special Smarandache curves according to Sabban frame. First, Frenet-Serret frame is obtained for the curve, second Saban frame is constructed by considering the tangent indicatrix. Then, the Smarandache curves are defined according to Saban frame. Finally, for each Smarandache curve, the geodesic curvatures are calculated and expressed with the principal curvatures of the special Viviani's curve.

Keywords: The Tangent Indicatrix, Viviani's Curve, Smarandache Curve, Sabban Frame

1. Introduction and Preliminaries

A Smarandache curve is defined such that if Frenet-Serret vectors are considered to be the position vector, then any curve drawn by that position vector is called as Smarandache curve [18]. This way of generating new curves are important in the field of differential geometry. By utilizing different frames, the construction of such Smarandache curves and their characteristics are studied in many research papers (see [1-5], [9-13]). Moreover, regarding spherical indicatrices, Koenderink defined Sabban frame and corresponding geodesic curvatures in [8]. Then, new Smarandache curves and their geodesic curvatures were discussed by using Sabban frame in [14-15, 17]. More recently, Saban frame according to the tangent indicatrix of helix curve has been examined in [16]

Let us recall some basic calculations. Let $\alpha : I \subset R \to R^3$ be a differentiable curve and denote the set of its Frenet-Serret frame as $\{T, N, B\}$. The calculation of the very famous Frenet-Serret formulas is given as following

$$T = \frac{\alpha'}{\|\alpha'\|}, N = B \wedge T, \quad B = \frac{\alpha' \wedge \alpha''}{\|\alpha' \wedge \alpha''\|}, \tag{1.1}$$

$$\kappa = \frac{\|\alpha' \wedge \alpha''\|}{\|\alpha'\|^3}, \quad \tau = \frac{\det(\alpha', \alpha'', \alpha''')}{\|\alpha' \wedge \alpha''\|^2} \tag{1.2}$$

where ' is for the derivative operator $\wedge$ is vector product sign, κ, is the curvature function, and τ is the torsion of the curve [6,7].

Moreover, let $\gamma = \gamma(s)$ be a unit vector with its arclength parameter s. Denote $\Gamma = \gamma'$ as the tangent vector of the curve on the unit sphere whose position vector γ, and compute another unit vector as $D = \gamma \wedge \Gamma$. Then the set of this new orthonormal system denoted $\{\gamma, \Gamma, D\}$ is defined as Sabban frame [8]. The Sabban formulas according to this frame are shown as below

$$\gamma' = \Gamma,\ \Gamma' = -\gamma + K_g D,\ \ D' = -K_g \Gamma \tag{1.3}$$

where

$$K_g = \langle \Gamma'\ D \rangle\ [8, 16]. \tag{1.4}$$

Viviani's curve, however, is defined as the intersection of a central sphere of radius 2r with a given cylinder of radius r. Thus, the common solution yields the following parametric equation for the curve as

$$\alpha(t) = r\left(1 + \cos t, \sin t, \sin\left(\frac{t}{2}\right)\right),\ t \in (-2\pi, 2\pi). \tag{1.5}$$

By using (1.2) the curvatures of the curve can be found as

$$\kappa(t) = \frac{\sqrt{3\cos t + 13}}{r(\cos t + 3)^{\frac{3}{2}}},\ \tau(t) = \frac{6\cos\left(\frac{t}{2}\right)}{r(3\cos t + 13)}\ [7]. \tag{1.6}$$

A better parametric representation for the curve can be obtained by the change of parameter as $t = 2s$, and by taking $r = —$. Thus, the new parametric form of the curve and its curvatures are given

$$\alpha(s) = \left(\cos^2 s, \cos s \sin s, \sin s\right), \tag{1.7}$$

$$\kappa(s) = \frac{\sqrt{3\cos^2 s + 5}}{(\cos^2 s + 1)\sqrt{\cos^2 s + 1}},\ \tau(s) = \frac{6\cos s}{3\cos^2 s + 5}, \tag{1.8}$$

By using (1.1) and (1.7) the Frenet vectors and formulas can be obtained as

$$T(s) = \frac{2(-\sin 2s, \cos 2s, \cos s)}{\sqrt{2\cos 2s + 6}},$$
$$N(s) = -\frac{(\cos 4s + 12\cos 2s + 3, \sin 4s + 12\sin 2s, 4\sin s)}{\sqrt{6\cos 4s + 88\cos 2s + 162}},$$
$$B(s) = \frac{(\sin 3s + 3\sin s, -\cos 3s - 3\cos s, 4)}{\sqrt{6\cos 2s + 26}}, \tag{1.9}$$

$$T'(s) = \eta\kappa N,\ N'(s) = \eta(-\kappa T + \tau B),\ B'(s) = -\eta\tau N,$$

where $\eta = \|\alpha'(s)\| = \sqrt{1+\cos^2 s}$. In addition, the following relation exists between the two curvatures given in (1.8)

$$\frac{\tau}{\kappa} = 6\cos s\left(\frac{\cos^2 s+1}{3\cos^2 s+5}\right)^{\frac{3}{2}}. \quad (1.10)$$

The following Fig.1 denotes the graph of Viviani's curve and its corresponding curvatures.

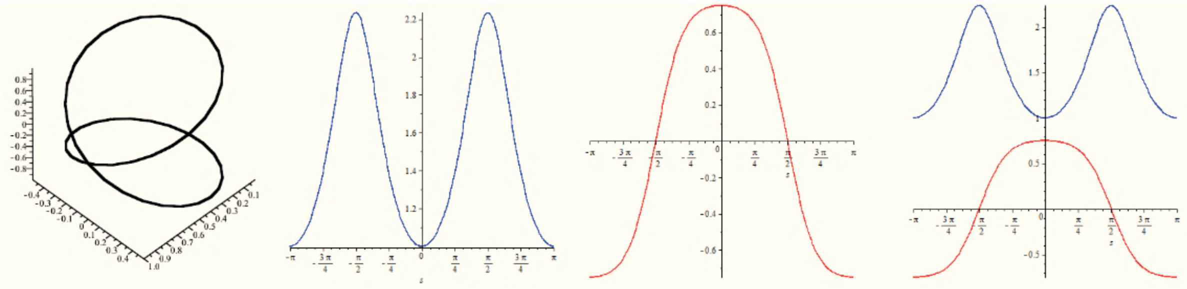

Fig. 1 Viviani's curve (black), its curvature κ (blue) and torsion τ (red)

2. On the tangent indicatrix of the Viviani's curve

In this main section of the paper, first the spherical tangent indicatrix curve drawn by the unit tangent vector of the Viviani's curve on the unit sphere is obtained. Then, the Frenet apparatus of the curve is calculated. Second, by using (1.3), the corresponding Sabban frame is provided. Besides, by assigning the vectors of Sabban frame as position vectors the special Smarandache curves are examined. Finally, each geodesic curvature of the new Smarandache curves is expressed by the principal curvatures of Viviani's curve.

Definition 2.1 The curve drawn by the unit tangent vector of the Viviani's curve on the unit sphere is called the tangent indicatrix curve.

Theorem 2.1 The set of Frenet vectors of the tangent indicatrix curve denoted by $\{T_T, N_T, B_T\}$ is given as following

$$T_T = N, \quad N_T = \frac{-\kappa T + \tau B}{\sqrt{\kappa^2+\tau^2}}, \quad B_T = \frac{\tau T + \kappa B}{\sqrt{\kappa^2+\tau^2}} . \quad (2.1)$$

Proof: From Definition 2.1, let $\alpha_T(s) = T(s)$. By taking derivatives and considering the last relation of (1.9), the following relations are obtained

$$\begin{aligned}
&\alpha_T' = T' = \eta\kappa N,\\
&\alpha_T'' = -\eta^2\kappa^2 T + (\eta\kappa)' N + \eta^2\kappa\tau B,\\
&\alpha_T''' = -\left(\left(\eta^2\kappa^2\right)' + \eta\kappa(\eta\kappa)'\right)T + \left(-\eta^3\kappa^3 + (\eta\kappa)'' - \eta^3\kappa\tau^2\right)N + \left(\left(\eta^2\kappa\tau\right)' + \eta\tau(\eta\kappa)'\right)B,\\
&\alpha_T' \wedge \alpha_T'' = \eta^3\kappa^2(\tau T + \kappa B),\\
&\left\|\alpha_T' \wedge \alpha_T''\right\| = \eta^3\kappa^2\sqrt{\kappa^2+\tau^2},\\
&\det\left(\alpha_T', \alpha_T'', \alpha_T'''\right) = \eta^5\kappa^3\left(\kappa\tau' - \kappa'\tau\right)
\end{aligned} \tag{2.2}$$

Thus, by using the system of equations given in (1.1) as $T_T = \dfrac{\alpha_T'}{\left\|\alpha_T'\right\|}$, $B_T = \dfrac{\alpha_T' \wedge \alpha_T''}{\left\|\alpha_T' \wedge \alpha_T''\right\|}$, $N_T = B_T \wedge T_T$, the Frenet elements of the tangent indicatrix is obtained which completes the proof.

Upon substituting (1.7) and (1.8) into (2.1), the following corollary can be expressed:

Corollary 2.1 The Frenet vectors of the tangent indicatrix curve of Viviani's curve are given as like below:

$$\begin{aligned}
T_T &= -\frac{(\cos 4s + 12\cos 2s + 3, \sin 4s + 12\sin 2s, 4\sin s)}{\sqrt{6\cos 4s + 88\cos 2s + 162}},\\
N_T &= \frac{-2(-\sin 2s, \cos 2s, \cos s)\left(3\cos^2 s + 5\right)^{\frac{3}{2}}}{\sqrt{2\cos 2s + 6}\sqrt{\left(3\cos^2 s+5\right)^2 + 36\cos^2 s\left(\cos^2 s + 1\right)^2}} + \frac{6\cos s\left(\cos^2 s + 1\right)^{\frac{3}{2}}(\sin 3s + 3\sin s, -\cos 3s - 3\cos s, 4)}{\sqrt{6\cos 2s + 26}\sqrt{\left(3\cos^2 s+5\right)^2 + 36\cos^2 s\left(\cos^2 s + 1\right)^2}},\\
B_T &= \frac{-12\cos s\left(\cos^2 s + 1\right)^{\frac{3}{2}}(-\sin 2s, \cos 2s, \cos s)}{\sqrt{6\cos 2s + 26}\sqrt{\left(3\cos^2 s+5\right)^2 + 36\cos^2 s\left(\cos^2 s + 1\right)^2}} + \frac{\left(3\cos^2 s + 5\right)^{\frac{3}{2}}(\sin 3s + 3\sin s, -\cos 3s - 3\cos s, 4)}{\sqrt{6\cos 2s + 26}\sqrt{\left(3\cos^2 s+5\right)^2 + 36\cos^2 s\left(\cos^2 s + 1\right)^2}}.
\end{aligned}$$

Theorem 2.2. The following relations hold for the curvature K_T and the torsion τ_T of the tangent indicatrix curve and the curvatures of the main curve as

$$K_T = \frac{\sqrt{\kappa^2+\tau^2}}{\kappa}, \quad \tau_T = \frac{\tau'\kappa - \kappa'\tau}{\eta\kappa\left(\kappa^2+\tau^2\right)}. \tag{2.3}$$

Proof: By substitution (2.2) into (1.2) the proof is completed.

Corollary 2.2 The curvatures of the tangent indicatrix of Viviani's curve can be given as

$$\begin{cases}
K_T = \dfrac{\sqrt{\left(3\cos^2 s + 5\right)^3 + 36\cos^2 s\left(\cos^2 s + 1\right)^3}}{\left(3\cos^2 s + 5\right)^{\frac{3}{2}}}\\[2ex]
\tau_T = \dfrac{\sqrt{\dfrac{3\cos^2 s + 5}{\cos^2 s + 1}}\left(6\cos s\left(\dfrac{\cos^2 s + 1}{3\cos^2 s + 5}\right)^{\frac{3}{2}}\right)'}{\left(3\cos^2 s + 5\right)^2 + 6\cos s\left(\cos^2 s + 1\right)^2}.
\end{cases} \tag{2.4}$$

Corollary 2.3 The ratio of the curvatures of the tangent indicatrix curve is

$$\frac{\tau_T}{\kappa_T}=\frac{\tau'\kappa-\kappa'\tau}{\eta\left(\kappa^2+\tau^2\right)^{\frac{3}{2}}} \tag{2.5}$$

and, specifically for Viviani's curve is given as

$$\frac{\tau_T}{\kappa_T}=\frac{\left(3\cos^2 s+5\right)^4\left(\cos^2 s+1\right)\left(6\cos s\left(\frac{\cos^2 s+1}{3\cos^2 s+5}\right)^{\frac{3}{2}}\right)'}{\left(\left(3\cos^2 s+5\right)^3+36\cos^2 s\left(\cos^2 s+1\right)^3\right)^{\frac{3}{2}}} \tag{2.6}$$

The following Fig.2 illustrates the graph of tangent indicatrix of Viviani's curve and its corresponding curvature functions

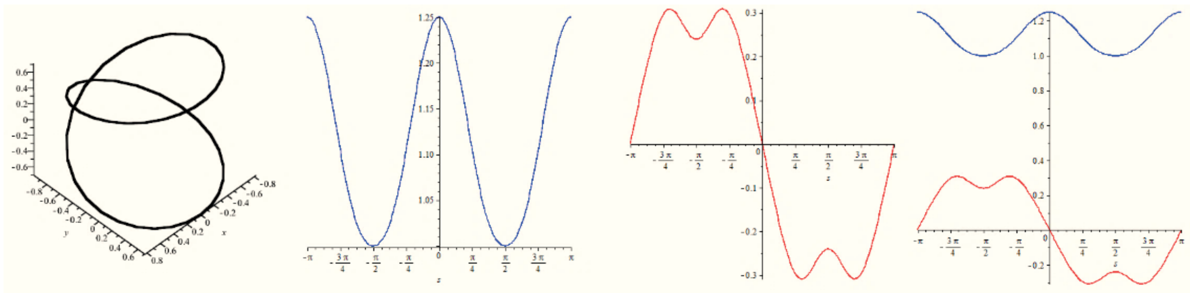

Fig. 2. The tangent idicatrix curve (black) and the curvature K_T (blue) and the torsion τ_T (red)

Even though the spherical indicatrix curves are obtained by the motion of unit vectors, they are not necessarily unit speed curves. For this reason, the Frenet formulas should be expressed in the general form as like relation (1.9) by the followings

$$\begin{aligned} T_T' &=\left\|\alpha_T'\right\|\kappa_T N_T, \\ N_T' &=\left\|\alpha_T'\right\|\left(-\kappa_T T_T+\tau_T B_T\right). \\ B_T' &=-\left\|\alpha_T'\right\|\tau_T N_T \end{aligned} \tag{2.7}$$

Let $\alpha_T\left(s\right)=T\left(s\right)$ be the tangent indicatrix curve of Viviani's curve and T_T be the tangent vector of α_T. Define $D_T = \;{}_T\wedge T_T$. Then, from (1.3) the vectors of Sabban frame are obtained as following

$$\begin{aligned} \alpha_T &=T=\frac{\left(-2\sin 2s,2\cos 2s,2\cos s\right)}{\sqrt{2\cos 2s+6}} \\ T_T &=\frac{\alpha_T'}{\left\|\alpha_T'\right\|}=N=-\frac{\left(\cos 4s+12\cos 2s+3,\sin 4s+12\sin 2s,4\sin s\right)}{\sqrt{162+88\cos 2s+6\cos 4s}} \\ D_T &=\alpha_T\wedge T_T=B=\frac{\begin{pmatrix}16\sin 2s\cos s+2\sin 4s\cos 2s+8\sin s, \\ -16\cos 2s\cos s-2\cos 4s\cos s-14\cos s,8\cos 2s+24\end{pmatrix}}{\sqrt{2\cos 2s+6}.\sqrt{162+88\cos 2s+6\cos 4s}} \end{aligned} \tag{2.8}$$

Besides, by recalling (1.4) Sabban formulas can be given as

$$\alpha_T{}' = \eta\kappa T_T,\ T_T{}' = -\eta\kappa\alpha_T + \eta\tau D_T,\ D_T{}' = -\eta\tau T_T. \tag{2.9}$$

Note that the geodesic curvature is $K_g{}^{\alpha_T} = \eta\tau$.

Definition 2.2. Let $\{\alpha_T, T_T, D_T\}$ denote the set of Sabban frame vectors for the tangent indicatrix of special Viviani's curve. Then, the curve drawn by the unified combination of the first two vectors in this set is called β_1 – Smarandache curve with the following parameterization

$$\beta_1(s) = \frac{1}{\sqrt{2}}(\alpha_T + T_T). \tag{2.10}$$

When substituted (2.8) into (2.10), the β_1 – Smarandache curve for Viviani's curve can be written as

$$\beta_1(s) = \frac{1}{\sqrt{2}}(T+N) = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{6\cos 4s + 88\cos 2s + 162}\,(-\sin 2s, \cos 2s, \cos s) - \sqrt{2\cos 2s + 6}\,(\cos 4s + 12\cos 2s + 3, \sin 4s + 12\sin 2s, 4\sin s)}{\sqrt{\cos 2s + 3}\cdot\sqrt{3\cos 4s + 44\cos 2s + 81}}\right) \text{ (see Fig.3).}$$

Theorem 2.2. The geodesic curvature $K_g{}^{\beta_1(s)}$ of β_1 – Smarandache curve is given by following

$$K_g{}^{\beta_1} = \frac{2(\tau'\kappa - \tau\kappa') + \eta\tau(2\kappa^2 + \tau^2)}{\sqrt{2}\sqrt{2\kappa^2 + \tau^2}}$$

Proof: By taking the derivative of the relation (2.10), the tangent vector T_{β_1} is found as

$$T_{\beta_1} = \frac{(-\kappa\alpha_T + \kappa T_T + \tau D_T)}{\sqrt{2\kappa^2 + \tau^2}}.$$

Next, the vector product of β_1 and T_{β_1} is

$$\beta_1 \wedge T_{\beta_1} = \frac{1}{\sqrt{2}\sqrt{2\kappa^2 + \tau^2}}(\tau\alpha_T - \tau T_T + 2\kappa D_T).$$

By taking the derivative of T_{β_1} as

$$T'_{\beta_1(s)} = -\left(\left(\frac{\kappa}{\sqrt{2\kappa^2+\tau^2}}\right)' + \frac{\eta\kappa^2}{\sqrt{2\kappa^2+\tau^2}}\right)\alpha_T + \left(\left(\frac{\kappa}{\sqrt{2\kappa^2+\tau^2}}\right)' - \frac{\eta(\kappa^2+\tau^2)}{\sqrt{2\kappa^2+\tau^2}}\right)T_T + \left(\left(\frac{\tau}{\sqrt{2\kappa^2+\tau^2}}\right)' + \frac{\eta\kappa\tau}{\sqrt{2\kappa^2+\tau^2}}\right)D_T$$

$$= \frac{-\tau(\kappa'\tau - \kappa\tau') - \eta\kappa^2(2\kappa^2+\tau^2)}{(2\kappa^2+\tau^2)^{\frac{3}{2}}}T + \frac{\tau(\kappa'\tau - \kappa\tau') - \eta(\kappa^2+\tau^2)(2\kappa^2+\tau^2)}{(2\kappa^2+\tau^2)^{\frac{3}{2}}}N + \frac{-2\kappa(\kappa'\tau - \kappa\tau') + \eta\kappa\tau(2\kappa^2+\tau^2)}{(2\kappa^2+\tau^2)^{\frac{3}{2}}}B$$

and using the relation (2.9), the proof is completed.

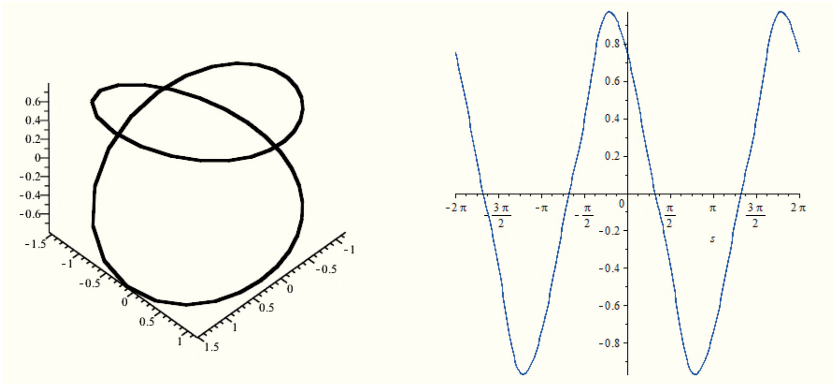

Fig. 3: The β_1- Smarandache curve (black) and its geodesic curvature $K_g^{\beta_1(s)}$ (blue), $s\in[-2\pi,2\pi]$

Definition 2.3 Let $\{\alpha_T, T_T, D_T\}$ denote the set of Sabban frame vectors for the tangent indicatrix of special Viviani's curve. Then, the curve drawn by the unified combination of the first and the last vectors in this set is called β_2- Smarandache curve with the following parameterization

$$\beta_2(s)=\frac{1}{\sqrt{2}}(\alpha_T+D_T). \tag{2.11}$$

When substituted (2.8) into (2.11), the β_2- Smarandache curve for Viviani's curve can be given as

$$\beta_2(s)=\frac{1}{\sqrt{2}}(T+B)=\frac{1}{\sqrt{2}}\left(\frac{\begin{array}{c}\sqrt{162+88\cos 2s+6\cos 4s}\,(-\sin 2s,\cos 2s,\cos s)\\ +\left(\begin{array}{l}8\sin 2s\cos s+\sin 4s\cos 2s+4\sin s,\\ -8\cos 2s\cos s-\cos 4s\cos s-7\cos s,4\cos 2s+12\end{array}\right)\end{array}}{\sqrt{\cos 2s+3}\sqrt{81+44\cos 2s+3\cos 4s}}\right)\text{(see Fig. 4).}$$

Theorem 2.3 The geodesic curvature $K_g^{\beta_2(s)}$ of β_2- Smarandache curve is given by following

$$K_g^{\beta_2}=\frac{\eta}{\sqrt{2}}(\tau-\kappa).$$

Proof: By taking the derivative of the relation (2.11), the tangent vector T_{β_2} is found as

$$T_{\beta_2}=T_T.$$

By the vector product of β_2 and T_{β_2}, the following relation is obtained

$$\beta_2\wedge T_{\beta_2}=\frac{1}{\sqrt{2}}(-\alpha_T+D_T)$$

By taking the derivative of T_{β_2} as

$$T_{\beta_2(s)}'=-\eta\kappa\alpha_T+\eta\tau D_T$$

and using the relation (2.9), the proof is completed.

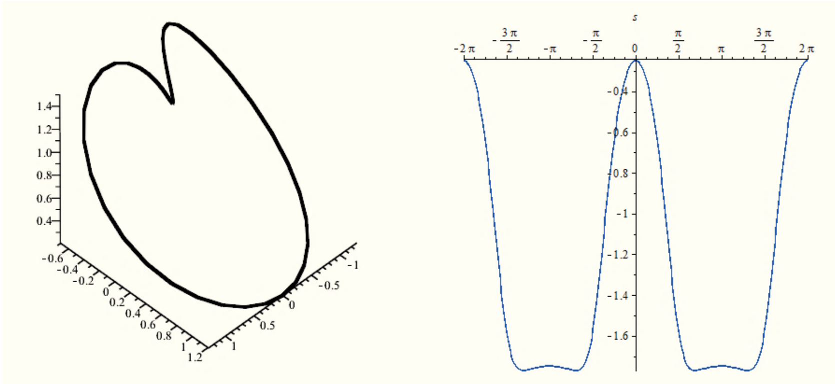

Fig. 4: The β_2- Smarandache curve (black) and its geodesic curvature $K_g^{\beta_2(s)}$ (blue), $s\in[-2\pi,2\pi]$

Definition 2.4 Let $\{\alpha_T, T_T, D_T\}$ denote the set of Sabban frame vectors for the tangent indicatrix of special Viviani's curve. Then, the curve drawn by the unified combination of the last two vectors in this set is called β_3-Smarandache curve with the following parameterization

$$\beta_3(s)=\frac{1}{\sqrt{2}}(T_T+D_T). \tag{2.12}$$

Upon substitution (2.8) into (2.11), the β_3- Smarandache curve for Viviani's curve can be given as

$$\beta_3(s)=\frac{1}{\sqrt{2}}(N+B)=\frac{1}{\sqrt{2}}\left(\begin{array}{c}\left(\begin{array}{l}16\sin 2s\cos s+2\sin 4s\cos 2s+8\sin s,\\ -16\cos 2s\cos s-2\cos 4s\cos s-14\cos s, 8\cos 2s+24\end{array}\right)\\ \dfrac{-\sqrt{2\cos 2s+6}\left(\cos 4s+12\cos 2s+3,\sin 4s+12\sin 2s,4\sin s\right)}{\sqrt{2\cos 2s+6}.\sqrt{162+88\cos 2s+6\cos 4s}}\end{array}\right) \text{(see Fig. 5).}$$

Theorem 2.4 The geodesic curvature $K_g^{\beta_3(s)}$ of β_3-Smarandache curve is given by following

$$K_g^{\beta_3}=\frac{2(\kappa\tau'-\kappa'\tau)+\eta\kappa(\kappa^2+2\tau^2)}{\sqrt{2}\sqrt{\kappa^2+2\tau^2}}.$$

Proof: By taking the derivative of the relation (2.12), the tangent vector T_{β_3} is found as

$$T_{\beta_3}=\frac{-\kappa\alpha_T-\tau T_T+\tau D_T}{\sqrt{\kappa^2+2\tau^2}}.$$

Moreover, the vector product of β_3 and T_{β_3} is

$$\beta_3\wedge T_{\beta_3}=\frac{1}{\sqrt{2}\sqrt{\kappa^2+2\tau^2}}(2\tau\alpha_T-\kappa T_T+\kappa D_T).$$

By taking the derivative of T_{β_3} as

$$T'_{\beta_3}=\left(-\left(\frac{\kappa}{\sqrt{\kappa^2+2\tau^2}}\right)'+\frac{\eta\kappa\tau}{\sqrt{\kappa^2+2\tau^2}}\right)\alpha_T-\left(\left(\frac{\tau}{\sqrt{\kappa^2+2\tau^2}}\right)'+\frac{\eta\left(\kappa^2+\tau^2\right)}{\sqrt{\kappa^2+2\tau^2}}\right)T_T+\left(\left(\frac{\tau}{\sqrt{\kappa^2+2\tau^2}}\right)'-\frac{\eta\tau^2}{\sqrt{\kappa^2+2\tau^2}}\right)D_T$$

$$=\frac{-2\tau\left(\kappa'\tau-\kappa\tau'\right)+\eta\kappa\tau\left(\kappa^2+2\tau^2\right)}{\left(\kappa^2+2\tau^2\right)^{\frac{3}{2}}}T+\frac{\kappa\left(\kappa'\tau-\kappa\tau'\right)-\eta\left(\kappa^2+\tau^2\right)\left(\kappa^2+2\tau^2\right)}{\left(\kappa^2+2\tau^2\right)^{\frac{3}{2}}}N-\frac{\kappa\left(\kappa'\tau-\kappa\tau'\right)+\eta\tau^2\left(\kappa^2+2\tau^2\right)}{\left(\kappa^2+2\tau^2\right)^{\frac{3}{2}}}B$$

and using the relation (2.9), the proof is completed.

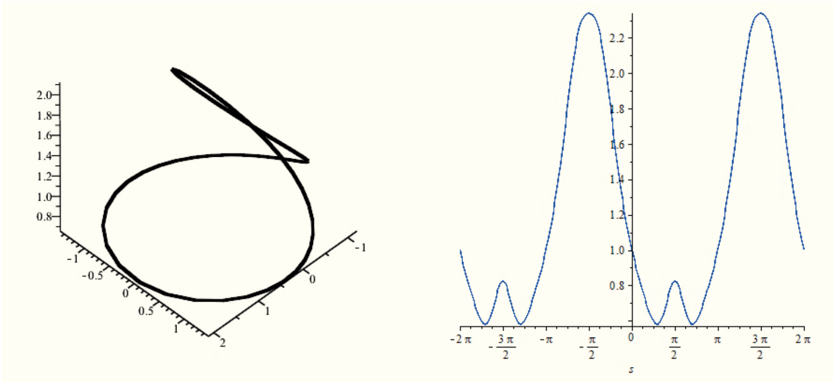

Fig. 5: The β_3- Smarandache curve (black) and its geodesic curvature $K_g^{\beta_3(s)}$ (blue) , $s\in[-2\pi,2\pi]$

Definition 2.5 Let $\{\alpha_T,T_T,D_T\}$ denote the set of Sabban frame vectors for the tangent indicatrix of special Viviani's curve. Then, the curve drawn by the unified combination of the all three vectors in this set is called β_4- Smarandache curve with the following parameterization

$$\beta_4(s)=\frac{1}{\sqrt{3}}\left(\alpha_T+T_T+D_T\right). \qquad (2.13)$$

By substituting (2.8) into (2.13), the β_4-Smarandache curve for Viviani's curve can be given as

$$\beta_4(s)=\frac{1}{\sqrt{3}}\left(T+N+B\right)$$

$$=\frac{1}{\sqrt{3}}\left(\begin{array}{c}\sqrt{162+88\cos 2s+6\cos 4s}\left(-2\sin 2s,2\cos 2s,2\cos s\right)\\ -\sqrt{2\cos 2s+6}\left(\cos 4s+12\cos 2s+3,\sin 4s+12\sin 2s,4\sin s\right)\\ \hline +\left(16\sin 2s\cos s+2\sin 4s\cos 2s+8\sin s,-16\cos 2s\cos s-2\cos 4s\cos s-14\cos s,8\cos 2s+24\right)\\ \sqrt{2\cos 2s+6}.\sqrt{162+88\cos 2s+6\cos 4s}\end{array}\right) \text{ (see Fig. 6).}$$

Theorem 2.5 The geodesic curvature $K_g^{\beta_4(s)}$ of β_4- Smarandache curve is given by following

$$K_g^{\beta_4} = \frac{1}{4\sqrt{3}}\left[\begin{array}{l}\left(\dfrac{(2\tau-\kappa)^2(\kappa\tau'-\kappa'\tau)-2\eta\kappa(\kappa-\tau)(2\tau-\kappa)(\kappa^2-\kappa\tau+\tau^2)}{(\kappa^2-\kappa\tau+\tau^2)^2}\right)\\ +\left(\dfrac{(\kappa+\tau)^2(\kappa\tau'-\kappa'\tau)+2\eta(\kappa+\tau)(\kappa^2+\tau^2)(\kappa^2-\kappa\tau+\tau^2)}{(\kappa^2-\kappa\tau+\tau^2)^2}\right)\\ +\left(\dfrac{(\tau-2\kappa)^2(\kappa\tau'-\kappa'\tau)-2\eta\tau(\tau-2\kappa)(\kappa-\tau)(\kappa^2-\kappa\tau+\tau^2)}{(\kappa^2-\kappa\tau+\tau^2)^2}\right)\end{array}\right]$$

$$= \frac{3(\kappa\tau'-\kappa'\tau)+2\eta\kappa(\kappa^2+\tau^2)}{2\sqrt{3}(\kappa^2-\kappa\tau+\tau^2)}.$$

Proof: By taking the derivative of the relation (2.13), the tangent vector T_{β_4} is found as

$$T_{\beta_4} = \frac{-\kappa\alpha_T+(\kappa-\tau)T_T+\tau D_T}{\sqrt{2}\sqrt{\kappa^2-\kappa\tau+\tau^2}}.$$

Besides, the vector product of β_4 and T_{β_4} is

$$\beta_4 \wedge T_{\beta_4} = \frac{(2\tau-\kappa)\alpha_T-(\kappa+\tau)T_T+(2\kappa-\tau)D_T}{\sqrt{6}\sqrt{\kappa^2-\kappa\tau+\tau^2}}.$$

By taking the derivative of T_{β_4} as

$$T'_{\beta_4(s)} = -\frac{1}{\sqrt{2}}\left(\left(\frac{\kappa}{\sqrt{\kappa^2-\kappa\tau+\tau^2}}\right)'+\frac{\eta\kappa(\kappa-\tau)}{\sqrt{\kappa^2-\kappa\tau+\tau^2}}\right)\alpha_T+\frac{1}{\sqrt{2}}\left(\left(\frac{(\kappa-\tau)}{\sqrt{\kappa^2-\kappa\tau+\tau^2}}\right)'-\frac{\eta(\kappa^2+\tau^2)}{\sqrt{\kappa^2-\kappa\tau+\tau^2}}\right)T_T$$
$$+\frac{1}{\sqrt{2}}\left(\left(\frac{\tau}{\sqrt{\kappa^2-\kappa\tau+\tau^2}}\right)'+\frac{\eta\tau(\kappa-\tau)}{\sqrt{\kappa^2-\kappa\tau+\tau^2}}\right)D_T,$$
$$=-\frac{1}{2\sqrt{2}}\left(\frac{(2\tau-\kappa)(\kappa'\tau-\kappa\tau')+2\eta\kappa(\kappa-\tau)(\kappa^2-\kappa\tau+\tau^2)}{(\kappa^2-\kappa\tau+\tau^2)^{\frac{3}{2}}}\right)T+\frac{1}{2\sqrt{2}}\left(\frac{(\kappa+\tau)(\kappa'\tau-\kappa\tau')-2\eta(\kappa^2+\tau^2)(\kappa^2-\kappa\tau+\tau^2)}{(\kappa^2-\kappa\tau+\tau^2)^{\frac{3}{2}}}\right)N$$
$$+\frac{1}{2\sqrt{2}}\left(\frac{(\tau-2\kappa)(\kappa'\tau-\kappa\tau')+2\eta\tau(\kappa-\tau)(\kappa^2-\kappa\tau+\tau^2)}{(\kappa^2-\kappa\tau+\tau^2)^{\frac{3}{2}}}\right)B$$

and using the relation (2.9), the proof is completed.

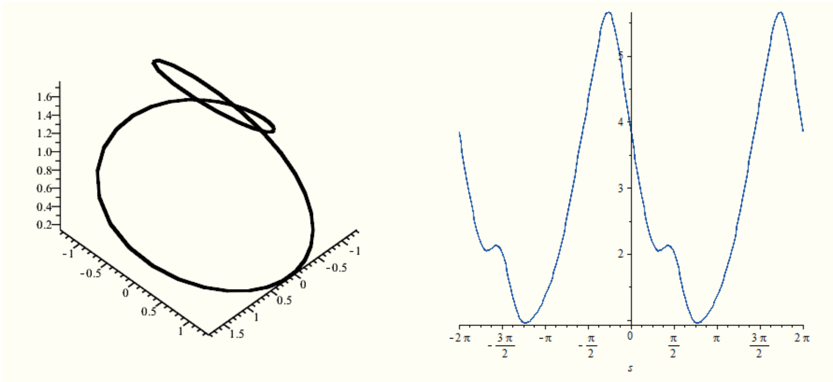

Fig. 6: The β_4 Smarandache curve (black) and its geodesic curvature $K_g^{\beta_4(s)}$ (blue), $s \in [-2\pi, 2\pi]$

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