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Smarandache Curves of a Spacelike Curve According to the Bishop Frame of Type-2

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Abstract: In this study, we introduce new Smarandache curves of a spacelike curve according to the Bishop frame of type-2 in E_1^3. Also, Smarandache breadth curves are defined according to this frame in Minkowski 3-space. A third order vectorial differential equation of position vector of Smarandache breadth curves has been obtained in Minkowski 3-space.

Key Words: Smarandache curves, the Bishop frame of type-2, Smarandache breadth curves, Minkowski 3-space.

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§1. Introduction

Bishop frame extended to study canal and tubular surfaces [1]. Rotating camera orientations relative to a stable forward-facing frame can be added by various techniques such as that of Hanson and Ma [2]. This special frame also extended to height functions on a space curve [3].

The construction of the Bishop frame is due to L. R. Bishop and the advantages of Bishop frame, and comparisons of Bishop frame with the Frenet frame in Euclidean 3-space were given by Bishop [4] and Hanson [5]. That is why he defined this frame that curvature may vanish at some points on the curve. That is, second derivative of the curve may be zero. In this situation, an alternative frame is needed for non continously differentiable curves on which Bishop (parallel transport frame) frame is well defined and constructed in Euclidean and its ambient spaces [6,7,8].

A regular curve in Euclidean 3-space, whose position vector is composed of Frenet frame vectors on another regular curve, is called a Smarandache curve. M. Turgut and S. Yılmaz have defined a special case of such curves and call it Smarandache TB_2 curves in the space E_1^4 ([9]) and Turgut also studied Smarandache breadth of pseudo null curves in E_1^4 ([10]). A.T.Ali has introduced some special Smarandache curves in the Euclidean space [11]. Moreover, special

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Smarandache curves have been investigated by using Bishop frame in Euclidean space [12]. Special Smarandache curves according to Sabban frame have been studied by [13]. Besides, some special Smarandache curves have been obtained in E_1^3 by [14].

Curves of constant breadth were introduced by L.Euler [15]. Some geometric properties of plane curves of constant breadth were given in [16]. And, in another work [17], these properties were studied in the Euclidean 3-space E^3. Moreover, M Fujivara [18] had obtained a problem to determine whether there exist space curve of constant breadth or not, and he defined breadth for space curves on a surface of constant breadth. In [19], these kind curves were studied in four dimensional Euclidean space E^4. In [20], Yılmaz introduced a new version of Bishop frame in E_1^3 and called it *Bishop frame of type-2* of regular curves by using common vector field as the binormal vector of Serret-Frenet frame. Also, some characterizations of spacelike curves were given according to the same frame by Yılmaz and Ünlütürk [21]. A regular curve more than 2 breadths in Minkowski 3-space is called a Smarandache breadth curve. In the light of this definition, we study special cases of Smarandache curves according to the new frame in E_1^3. We investigate position vector of simple closed spacelike curves and give some characterizations in case of constant breadth according to type-2 Bishop frame in E_1^3. Thus, we extend this classical topic in E^3 into spacelike curves of constant breadth in E_1^3, see [22] for details.

In this study, we introduce new Smarandache curves of a spacelike curve according to the Bishop frame of type-2 in E_1^3. Also, Smarandache breadth curves are defined according to this frame in Minkowski 3-space. A third order vectorial differential equation of position vector of Smarandache breadth curves has been obtained in Minkowski 3-space.

§2. Preliminaries

To meet the requirements in the next sections, here, the basic elements of the theory of curves in the Minkowski 3–space E_1^3 are briefly presented. There exists a vast literature on the subject including several monographs, for example [23,24].

The three dimensional Minkowski space E_1^3 is a real vector space R^3 endowed with the standard flat Lorentzian metric given by

$$\langle , \rangle_L = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of E_1^3. This metric is an indefinite one.

Let $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ be arbitrary vectors in E_1^3, the Lorentzian cross product of u and v is defined as

$$u \times v = -\det \begin{bmatrix} -i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}.$$

Recall that a vector $v \in E_1^3$ has one of three Lorentzian characters: it is a spacelike vector if $\langle v, v\rangle > 0$ or $v = 0$; timelike $\langle v, v\rangle < 0$ and null (lightlike) $\langle v, v\rangle = 0$ for $v \neq 0$. Similarly, an arbitrary curve $\delta = \delta(s)$ in E_1^3 can locally be spacelike, timelike or null (lightlike) if its velocity vector α' are ,respectively, spacelike, timelike or null (lightlike), for every $s \in I \subset \mathbb{R}$. The pseudo-norm of an arbitrary vector $a \in E_1^3$ is given by $\|a\| = \sqrt{|\langle a, a\rangle|}$. The curve $\alpha = \alpha(s)$ is called a unit speed curve if its velocity vector α' is unit one i.e., $\|\alpha'\| = 1$. For vectors $v, w \in E_1^3$, they are said to be orthogonal each other if and only if $\langle v, w\rangle = 0$. Denote by $\{T, N, B\}$ the moving Serret-Frenet frame along the curve $\alpha = \alpha(s)$ in the space E_1^3.

For an arbitrary spacelike curve $\alpha = \alpha(s)$ in E_1^3, the Serret-Frenet formulae are given as follows

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \gamma\kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \cdot \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{2.1}$$

where $\gamma = \mp 1$, and the functions κ and τ are, respectively, the first and second (torsion) curvature. $T(s) = \alpha'(s), N(s) = \dfrac{T'(s)}{\kappa(s)}, B(s) = T(s) \times N(s)$ and $\tau(s) = \dfrac{\det(\alpha', \alpha'', \alpha''')}{\kappa^2(s)}$.

If $\gamma = -1$, then $\alpha(s)$ is a spacelike curve with spacelike principal normal N and timelike binormal B, its Serret-Frenet invariants are given as

$$\kappa(s) = \sqrt{\langle T'(s), T'(s)\rangle} \text{ and } \tau(s) = -\langle N'(s), B(s)\rangle.$$

If $\gamma = 1$, then $\alpha(s)$ is a spacelike curve with timelike principal normal N and spacelike binormal B, also we obtain its Serret-Frenet invariants as

$$\kappa(s) = \sqrt{-\langle T'(s), T'(s)\rangle} \text{ and } \tau(s) = \langle N'(s), B(s)\rangle.$$

The Lorentzian sphere S_1^2 of radius $r > 0$ and with the center in the origin of the space E_1^3 is defined by

$$S_1^2(r) = \{p = (p_1, p_2, p_3) \in E_1^3 : \langle p, p\rangle = r^2\}.$$

Theorem 2.1 *Let $\alpha = \alpha(s)$ be a spacelike unit speed curve with a spacelike principal normal. If $\{\Omega_1, \Omega_2, B\}$ is an adapted frame, then we have*

$$\begin{bmatrix} \Omega_1' \\ \Omega_2' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & 0 & \xi_1 \\ 0 & 0 & -\xi_2 \\ -\xi_1 & -\xi_2 & 0 \end{bmatrix} \cdot \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ B \end{bmatrix} \tag{2.2}$$

Theorem 2.2 *Let $\{T, N, B\}$ and $\{\Omega_1, \Omega_2, B\}$ be Frenet and Bishop frames, respectively. There exists a relation between them as*

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} \sinh\theta(s) & \cosh\theta(s) & 0 \\ \cosh\theta(s) & \sinh\theta(s) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ B \end{bmatrix}, \tag{2.3}$$

where θ is the angle between the vectors N and Ω_1.

$$\xi_1 = \tau(s)\cosh\theta(s), \xi_2 = \tau(s)\sinh\theta(s).$$

The frame $\{\Omega_1, \Omega_2, B\}$ is properly oriented, and τ and $\theta(s) = \int_0^s \kappa(s)ds$ are polar coordinates for the curve $\alpha = \alpha(s)$. We shall call the set $\{\Omega_1, \Omega_2, B, \xi_1, \xi_2\}$ as type-2 Bishop invariants of the curve $\alpha = \alpha(s)$ in E_1^3.

§3. Smarandache Curves of a Spacelike Curve

In this section, we will characterize all types of Smarandache curves of spacelike curve $\alpha = \alpha(s)$ according to type-2 Bishop frame in Minkowski 3-space E_1^3.

3.1 $\Omega_1\Omega_2-$Smarandache Curves

Definition 3.1 *Let $\alpha = \alpha(s)$ be a unit speed regular curve in E_1^3 and $\{\Omega_1^\alpha, \Omega_2^\alpha, B_\alpha\}$ be its moving Bishop frame. $\Omega_1^\alpha\Omega_2^\alpha-$Smarandache curves are defined by*

$$\beta(s^*) = \tfrac{1}{\sqrt{2}}(\Omega_1^\alpha + \Omega_2^\alpha). \tag{3.1}$$

Now we can investigate Bishop invariants of $\Omega_1^\alpha\Omega_2^\alpha-$Smarandache curves of the curve $\alpha = \alpha(s)$. Differentiating (3.1) with respect to s gives

$$\dot{\beta} = \frac{d\beta}{ds}.\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}(\xi_1^\alpha - \xi_2^\alpha)B_\alpha, \tag{3.2}$$

and

$$T_\beta.\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}(\xi_1^\alpha - \xi_2^\alpha)B_\alpha,$$

where

$$\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\left|\xi_1^\alpha - \xi_2^\alpha\right|. \tag{3.3}$$

The tangent vector of the curve β can be written as follows

$$T_\beta = \beta_\alpha. \tag{3.4}$$

Differentiating (3.4) with respect to s, we obtain

$$\frac{dT_\beta}{ds^*}.\frac{ds^*}{ds} = -(\xi_1^\alpha\Omega_1^\alpha + \xi_2^\alpha\Omega_2^\alpha). \tag{3.5}$$

Substituting (3.3) into (3.5) gives

$$T'_\beta = -\frac{\sqrt{2}}{|\xi_1^\alpha - \xi_2^\alpha|}(\xi_1^\alpha\Omega_1^\alpha + \xi_2^\alpha\Omega_2^\alpha).$$

Then the first curvature and the principal normal vector field of β are, respectively, computed as

$$\left\|T'_\beta\right\| = \kappa_\beta = \frac{\sqrt{2}}{|\xi_1^\alpha - \xi_2^\alpha|}\sqrt{-(\xi_1^\alpha)^2 + (\xi_2^\alpha)^2},$$

and

$$N_\beta = \frac{-1}{\sqrt{-(\xi_1^\alpha)^2 + (\xi_2^\alpha)^2}}(\xi_1^\alpha \Omega_1^\alpha + \xi_2^\alpha \Omega_2^\alpha).$$

On the other hand, we express

$$B_\beta = \frac{-1}{\sqrt{-(\xi_1^\alpha)^2 + (\xi_2^\alpha)^2}}\begin{vmatrix} -\Omega_1^\alpha & \Omega_2^\alpha & \beta_\alpha \\ 0 & 0 & 1 \\ \xi_1^\alpha & \xi_2^\alpha & 0 \end{vmatrix},$$

So the binormal vector of β is computed as follows

$$B_\beta = \frac{-1}{\sqrt{-(\xi_1^\alpha)^2 + (\xi_2^\alpha)^2}}(\xi_2^\alpha \Omega_1^\alpha + \xi_1^\alpha \Omega_2^\alpha).$$

Differentiating (3.2) with respect to s in order to calculate the torsion of the curve β, we obtain

$$\ddot{\beta} = \frac{1}{\sqrt{2}}[-(\xi_1^\alpha + \xi_2^\alpha)\xi_1^\alpha \Omega_1^\alpha - (\xi_1^\alpha + \xi_2^\alpha)\xi_2^\alpha \Omega_2^\alpha + (\dot{\xi}_1^\alpha + \dot{\xi}_2^\alpha)B_\alpha],$$

and similarly

$$\begin{aligned}\dddot{\beta} = \tfrac{1}{\sqrt{2}}[&(-3\xi_1^\alpha \dot{\xi}_1^\alpha - 2\xi_1^\alpha \dot{\xi}_2^\alpha - \dot{\xi}_1^\alpha \xi_2^\alpha - \xi_1^\alpha \dot{\xi}_1^\alpha)\Omega_1^\alpha \\ &+(-2\dot{\xi}_1^\alpha \xi_2^\alpha - 2(\dot{\xi}_2^\alpha)^2 - \xi_1^\alpha \dot{\xi}_2^\alpha - \xi_2^\alpha \dot{\xi}_2^\alpha)\Omega_2^\alpha \\ &+(\ddot{\xi}_1^\alpha + \ddot{\xi}_2^\alpha - (\xi_1^\alpha)^3 - (\xi_1^\alpha)^2\xi_2^\alpha - \xi_1^\alpha(\xi_2^\alpha)^2 + (\xi_2^\alpha)^2)B_\alpha].\end{aligned}$$

The torsion of the curve β is found

$$\tau_\beta = \frac{1}{4\sqrt{2}}\frac{(\xi_1^\alpha - \xi_2^\alpha)^2}{(\xi_1^\alpha)^2 + (\xi_2^\alpha)^2}[(\xi_1^\alpha + \xi_2^\alpha)K_2(s) - (\xi_1^\alpha + \xi_2^\alpha)\xi_2^\alpha K_1(s)],$$

where

$$\begin{aligned}K_1(s) &= -3\xi_1^\alpha \dot{\xi}_1^\alpha - 2\xi_1^\alpha \dot{\xi}_2^\alpha - \dot{\xi}_1^\alpha \xi_2^\alpha - \xi_1^\alpha \dot{\xi}_1^\alpha, \\ K_2(s) &= -2\dot{\xi}_1^\alpha \xi_2^\alpha - 2(\dot{\xi}_2^\alpha)^2 - \xi_1^\alpha \dot{\xi}_2^\alpha - \xi_2^\alpha \dot{\xi}_2^\alpha, \\ K_3(s) &= \ddot{\xi}_1^\alpha + \ddot{\xi}_2^\alpha - (\xi_1^\alpha)^3 - (\xi_1^\alpha)^2\xi_2^\alpha - \xi_1^\alpha(\xi_2^\alpha)^2 + (\xi_2^\alpha)^2.\end{aligned}$$

3.2 $\Omega_1 B-$Smarandache Curves

Definition 3.2 *Let $\alpha = \alpha(s)$ be a unit speed regular curve in E_1^3 and $\{\Omega_1^\alpha, \Omega_2^\alpha, B_\alpha\}$ be its moving Bishop frame. $\Omega_1^\alpha B-$Smarandache curves are defined by*

$$\beta(s^*) = \frac{1}{\sqrt{2}}(\Omega_1^\alpha + B_\alpha). \tag{3.6}$$

Now we can investigate Bishop invariants of $\Pi_1^\alpha B_\alpha-$Smarandache curves of the curve $\alpha = \alpha(s)$. Differentiating (3.6) with respect to s, we have

$$\dot{\beta} = \frac{d\beta}{ds}.\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}(\xi_1^\alpha B_\alpha - \xi_1^\alpha \Omega_1^\alpha - \xi_2^\alpha \Omega_2^\alpha), \tag{3.7}$$

and

$$T_\beta.\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}(\xi_1^\alpha B_\alpha - \xi_1^\alpha \Omega_1^\alpha - \xi_2^\alpha \Omega_2^\alpha),$$

where

$$\frac{ds^*}{ds} = \frac{\xi_2^\alpha}{2}. \tag{3.8}$$

The tangent vector of the curve β can be written as follows

$$T_\beta = \frac{\sqrt{2}}{\xi_2^\alpha}(-\xi_1^\alpha \Omega_1^\alpha - \xi_2^\alpha \Omega_2^\alpha + \xi_1^\alpha B_\alpha). \tag{3.9}$$

Differentiating (3.9) with respect to s gives

$$\frac{dT_\beta}{ds^*}.\frac{ds^*}{ds} = \frac{\xi_2^\alpha}{\sqrt{2}}(L_1(s)\Omega_1^\alpha + L_2(s)\Omega_2^\alpha + L_3(s)B_\alpha), \tag{3.10}$$

where

$$L_1(s) = -\xi_1^\alpha - (\xi_1^\alpha)^2 + \frac{\xi_1^\alpha \dot{\xi}_2^\alpha}{\xi_2^\alpha}, L_2(s) = \xi_1^\alpha \xi_2^\alpha,$$
$$L_3(s) = -(\xi_1^\alpha)^2 + (\xi_2^\alpha)^2 + \dot{\xi}_1^\alpha - \frac{\xi_1^\alpha \dot{\xi}_2^\alpha}{\xi_2^\alpha}.$$

Substituting (3.8) into (3.10) gives

$$T'_\beta = \frac{2\sqrt{2}}{(\xi_2^\alpha)^2}(L_1(s)\Omega_1^\alpha + L_2(s)\Omega_2^\alpha + L_3(s)B_\alpha),$$

then the first curvature and the principal normal vector field of β are, respectively,

$$\left\| T'_\beta \right\| = \kappa_\beta = \frac{2\sqrt{2}}{(\xi_2^\alpha)^2}\sqrt{L_1^2(s) + L_2^2(s) - L_3^2(s)},$$

and

$$N_\beta = \frac{-1}{\sqrt{L_1^2(s) + L_2^2(s) - L_3^2(s)}}(L_1(s)\Omega_1^\alpha + L_2(s)\Omega_2^\alpha + L_3(s)B_\alpha).$$

On the other hand, we have

$$B_\beta = \frac{\sqrt{2}}{\xi_2^\alpha\sqrt{L_1^2(s) + L_2^2(s) - L_3^2(s)}}[(\xi_1^\alpha L_2(s) + \xi_2^\alpha L_3(s))\Omega_1^\alpha$$
$$+(\xi_1^\alpha L_1(s) + \xi_1^\alpha L_3(s))\Omega_2^\alpha + (\xi_2^\alpha L_1(s) - \xi_1^\alpha L_2(s))B_\alpha].$$

Differentiating (3.7) with respect to s in order to calculate the torsion of the curve β, we find

$$\ddot{\beta}=\frac{1}{\sqrt{2}}[-((\xi_1^\alpha)^2+\dot{\xi}_1^\alpha)\Omega_1^\alpha+(-\xi_1^\alpha\dot{\xi}_2^\alpha-\dot{\xi}_2^\alpha)\Omega_2^\alpha-(\ddot{\xi}_1^\alpha-(\xi_1^\alpha)^2+(\xi_2^\alpha)^2)B_\alpha],$$

and similarly

$$\dddot{\beta}=\frac{1}{\sqrt{2}}[(-2\xi_1^\alpha\dot{\xi}_1^\alpha-\ddot{\xi}_1^\alpha)\Omega_1^\alpha+(-\dot{\xi}_1^\alpha\dot{\xi}_2^\alpha-\ddot{\xi}_2^\alpha-\xi_1^\alpha\ddot{\xi}_2^\alpha)\Omega_2^\alpha$$
$$+(-(\xi_1^\alpha)^3-\xi_1^\alpha\dot{\xi}_1^\alpha+\xi_1^\alpha\dot{\xi}_2^\alpha\xi_2^\alpha-\xi_2^\alpha\dot{\xi}_2^\alpha)B_\alpha].$$

The torsion of the curve β is

$$\tau_\beta=\frac{(\varepsilon_2^\alpha)^4}{16\sqrt{2}}[-\xi_1^\alpha M_1(s)-\xi_2^\alpha M_2(s)-\xi_1^\alpha M_3(s)],$$

where

$$M_1(s)=-3\xi_1^\alpha\dot{\xi}_1^\alpha-2\xi_1^\alpha\dot{\xi}_2^\alpha-\dot{\xi}_1^\alpha\xi_2^\alpha-\xi_1^\alpha\dot{\xi}_1^\alpha,$$
$$M_2(s)=-2\dot{\xi}_1^\alpha\xi_2^\alpha-2(\dot{\xi}_2^\alpha)^2-\xi_1^\alpha\dot{\xi}_2^\alpha-\xi_2^\alpha\dot{\xi}_2^\alpha,$$
$$M_3(s)=\ddot{\xi}_1^\alpha+\ddot{\xi}_2^\alpha-(\xi_1^\alpha)^3-(\xi_1^\alpha)^2\xi_2^\alpha-\xi_1^\alpha(\xi_2^\alpha)^2+(\xi_2^\alpha)^2.$$

3.3 $\Omega_2 B-$Smarandache Curves

Definition 3.3 *Let $\alpha=\alpha(s)$ be a unit speed regular curve in E_1^3 and $\{\Omega_1^\alpha,\Omega_2^\alpha,B_\alpha\}$ be its moving Bishop frame. $\Omega_2^\alpha B-$Smarandache curves are defined by*

$$\beta(s^*)=\frac{1}{\sqrt{2}}(\Omega_2^\alpha+B_\alpha). \tag{3.11}$$

Now we can investigate Bishop invariants of $\Omega_1^\alpha B_\alpha-$Smarandache curves of the curve $\alpha=\alpha(s)$. Differentiating (3.11) with respect to s, we have

$$\dot{\beta}=\frac{d\beta}{ds}.\frac{ds^*}{ds}=\frac{1}{\sqrt{2}}(-\xi_2^\alpha B_\alpha-\xi_1^\alpha\Omega_1^\alpha-\xi_2^\alpha\Omega_2^\alpha), \tag{3.12}$$

and

$$T_\beta.\frac{ds^*}{ds}=\frac{1}{\sqrt{2}}(-\xi_1^\alpha\Omega_1^\alpha-\xi_2^\alpha\Omega_2^\alpha-\xi_2^\alpha B_\alpha),$$

where

$$\frac{ds^*}{ds}=\sqrt{\frac{2(\xi_2^\alpha)^2-(\xi_1^\alpha)^2}{2}}. \tag{3.13}$$

The tangent vector of the curve β can be written as follows

$$T_\beta=\frac{-\xi_1^\alpha\Omega_1^\alpha-\xi_2^\alpha\Omega_2^\alpha-\xi_2^\alpha B_\alpha}{\sqrt{2(\xi_2^\alpha)^2-(\xi_1^\alpha)^2}}. \tag{3.14}$$

Differentiating (3.14) with respect to s gives

$$\frac{dT_\beta}{ds^*}.\frac{ds^*}{ds}=(N_1(s)\Omega_1^\alpha+N_2(s)\Omega_2^\alpha+N_3(s)B_\alpha), \tag{3.15}$$

where

$$\begin{aligned}N_1(s) &= \tfrac{1}{2}(4\xi_2^\alpha\dot{\xi}_2^\alpha - 2\xi_1^\alpha\dot{\xi}_1^\alpha)(2(\xi_2^\alpha)^2-(\xi_1^\alpha)^2)^{\frac{-3}{2}}\xi_1^\alpha\\ &\quad -(2(\xi_2^\alpha)^2-(\xi_1^\alpha)^2)^{\frac{-1}{2}}\dot{\xi}_1^\alpha + (2(\xi_2^\alpha)^2-(\xi_1^\alpha)^2)^{\frac{-1}{2}}\xi_1^\alpha\xi_2^\alpha,\\ N_2(s) &= \tfrac{1}{2}(4\xi_2^\alpha\dot{\xi}_2^\alpha - 2\xi_1^\alpha\dot{\xi}_1^\alpha)(2(\xi_2^\alpha)^2-(\xi_1^\alpha)^2)^{\frac{-3}{2}}\xi_1^\alpha\\ &\quad -(2(\xi_2^\alpha)^2-(\xi_1^\alpha)^2)^{\frac{-1}{2}}\dot{\xi}_1^\alpha + (2(\xi_2^\alpha)^2-(\xi_1^\alpha)^2)(\xi_2^\alpha)^2,\\ N_3(s) &= \tfrac{1}{2}(4\xi_2^\alpha\dot{\xi}_2^\alpha - 2\xi_1^\alpha\dot{\xi}_1^\alpha)(2(\xi_2^\alpha)^2-(\xi_1^\alpha)^2)^{\frac{-3}{2}}\\ &\quad -(2(\xi_2^\alpha)^2-(\xi_1^\alpha)^2)^{\frac{-1}{2}}((\xi_2^\alpha)^2-(\xi_1^\alpha)^2-\dot{\xi}_2^\alpha).\end{aligned}$$

Substituting (3.13) into (3.15) gives

$$T_\beta' = \sqrt{\frac{2}{2(\xi_2^\alpha)^2-(\xi_1^\alpha)^2}}(N_1(s)\Omega_1^\alpha + N_2(s)\Omega_2^\alpha + N_3(s)B_\alpha),$$

then the first curvature and the principal normal vector field of β are, respectively, found as follows

$$\kappa_\beta = \left\|T_\beta'\right\| = \sqrt{\frac{2}{2(\xi_2^\alpha)^2-(\xi_1^\alpha)^2}}\sqrt{N_1^2(s)+N_2^2(s)+N_3^2(s)},$$

and

$$N_\beta = \frac{-1}{\sqrt{N_1^2(s)+N_2^2(s)+N_3^2(s)}}(N_1(s)\Omega_1^\alpha + N_2(s)\Omega_2^\alpha + L_3(s)B_\alpha). \tag{3.16}$$

On the other hand, we have

$$\begin{aligned}B_\beta = &\frac{1}{\sqrt{2(\xi_2^\alpha)^2-(\xi_1^\alpha)^2}\sqrt{N_1^2(s)+N_2^2(s)+N_3^2(s)}}[(-\xi_2^\alpha N_3(s)+\xi_2^\alpha N_2(s))\Omega_1^\alpha\\ &+(-\xi_2^\alpha N_3(s)+\xi_2^\alpha N_1(s))\Omega_2^\alpha + (\xi_1^\alpha N_2 - \xi_2^\alpha N_1(s))B_\alpha].\end{aligned} \tag{3.17}$$

Differentiating (3.12) with respect to s in order to calculate the torsion of the curve β, we obtain

$$\ddot{\beta} = \frac{1}{\sqrt{2}}[(\xi_2^\alpha\xi_1^\alpha + \dot{\xi}_1^\alpha)\Omega_1^\alpha + ((\xi_2^\alpha)^2 - \dot{\xi}_2^\alpha)\Omega_2^\alpha + (-\dot{\xi}_2^\alpha + \xi_2^\alpha - (\xi_1^\alpha)^2)B_\alpha],$$

and similarly

$$\begin{aligned}\dddot{\beta} = &\frac{1}{\sqrt{2}}[(2\dot{\xi}_2^\alpha\xi_1^\alpha + \xi_2^\alpha\dot{\xi}_1^\alpha - \ddot{\xi}_1^\alpha - \xi_2^\alpha\xi_1^\alpha + (\xi_1^\alpha)^3)\Omega_1^\alpha\\ &+(3\xi_2^\alpha\dot{\xi}_2^\alpha - \ddot{\xi}_2^\alpha - (\xi_2^\alpha)^2 - (\xi_1^\alpha)^2\xi_2^\alpha)\Omega_2^\alpha\\ &+((\xi_1^\alpha)^2\xi_2^\alpha - (\xi_2^\alpha)^3 + \xi_2^\alpha\dot{\xi}_2^\alpha - \ddot{\xi}_2^\alpha + \dot{\xi}_2^\alpha - 3\xi_1^\alpha\dot{\xi}_1^\alpha)B_\alpha].\end{aligned}$$

The torsion of the curve β is

$$\begin{aligned}\tau_\beta = &\frac{2(\xi_2^\alpha)^2-(\xi_1^\alpha)^2}{4\sqrt{2}}\{[P_3(s)((\xi_2^\alpha)^2-\dot{\xi}_2^\alpha) - P_2(s)(-\dot{\xi}_2^\alpha + \xi_2^\alpha - (\xi_1^\alpha)^2)]\xi_1^\alpha\\ &+[P_3(s)(\xi_2^\alpha\xi_1^\alpha - \dot{\xi}_1^\alpha) - P_1(s)(-\dot{\xi}_2^\alpha + \xi_2^\alpha - (\xi_1^\alpha)^2)]\xi_2^\alpha\\ &+[P_2(s)(\xi_2^\alpha\xi_1^\alpha - \dot{\xi}_1^\alpha) - P_1(s)((\xi_2^\alpha)^2 - \dot{\xi}_2^\alpha)]\xi_3^\alpha\},\end{aligned}$$

where

$$P_1(s) = 2\dot{\xi}_2^\alpha \xi_1^\alpha + \xi_2^\alpha \dot{\xi}_1^\alpha - \ddot{\xi}_1^\alpha - \xi_1^\alpha \dot{\xi}_1^\alpha + \xi_1^\alpha \xi_2^\alpha + (\xi_1^\alpha)^3,$$
$$P_2(s) = 3\dot{\xi}_2^\alpha \xi_2^\alpha - \ddot{\xi}_2^\alpha - (\xi_2^\alpha)^2 - (\xi_1^\alpha)^2 \xi_2^\alpha,$$
$$P_3(s) = (\xi_1^\alpha)^2 \xi_2^\alpha - (\xi_2^\alpha)^3 + \xi_2^\alpha \dot{\xi}_2^\alpha - \ddot{\xi}_2^\alpha + \dot{\xi}_2^\alpha - 3\dot{\xi}_1^\alpha \xi_1^\alpha.$$

3.4 $\Omega_1\Omega_2 B-$Smarandache Curves

Definition 3.4 *Let $\alpha = \alpha(s)$ be a unit speed regular curve in E_1^3 and $\{\Omega_1^\alpha, \Omega_2^\alpha, B_\alpha\}$ be its moving Bishop frame. $\Omega_1^\alpha \Omega_2^\alpha B-$Smarandache curves are defined by*

$$\beta(s^*) = \frac{1}{\sqrt{3}}(\Omega_1^\alpha + \Omega_2^\alpha + B_\alpha). \tag{3.18}$$

Now we can investigate Bishop invariants of $\Omega_1^\alpha \Omega_2^\alpha B-$Smarandache curves of the curve $\alpha = \alpha(s)$. Differentiating (3.18) with respect to s, we have

$$\dot{\beta} = \frac{d\beta}{ds}.\frac{ds^*}{ds} = \frac{1}{\sqrt{3}}(-\xi_1^\alpha \Omega_1^\alpha - \xi_2^\alpha \Omega_2^\alpha + (\xi_1^\alpha - \varepsilon_2^\alpha)B_\alpha), \tag{3.19}$$

and

$$T_\beta.\frac{ds^*}{ds} = \frac{1}{\sqrt{3}}(-\xi_1^\alpha \Omega_1^\alpha - \xi_2^\alpha \Omega_2^\alpha + (\xi_1^\alpha - \xi_2^\alpha)B_\alpha),$$

where

$$\frac{ds^*}{ds} = \sqrt{\frac{(\xi_1^\alpha - \xi_2^\alpha)^2 + (\xi_2^\alpha)^2 - (\xi_1^\alpha)^2}{3}}. \tag{3.20}$$

The tangent vector of the curve β is found as follows

$$T_\beta = \frac{1}{\sqrt{(\xi_1^\alpha - \xi_2^\alpha)^2 + (\xi_2^\alpha)^2 - (\xi_1^\alpha)^2}}(-\xi_1^\alpha \Omega_1^\alpha - \xi_2^\alpha \Omega_2^\alpha + (\xi_1^\alpha - \xi_2^\alpha)B_\alpha). \tag{3.21}$$

Differentiating (3.21) with respect to s, we find

$$\begin{aligned}\frac{dT_\beta}{ds^*}.\frac{ds^*}{ds} = &[-Q(s)\dot{\xi}_1^\alpha - Q(s)(\xi_1^\alpha)^2 + Q(s)\xi_1^\alpha \xi_2^\alpha - Q'(s)\xi_1^\alpha]\Omega_1^\alpha \\ &+[-Q(s)\dot{\xi}_2^\alpha - Q(s)\xi_1^\alpha \xi_2^\alpha + Q(s)(\xi_2^\alpha)^2 - Q'(s)\xi_2^\alpha]\Omega_2^\alpha \\ &+[Q(s)(\xi_1^\alpha - \xi_2^\alpha)'\text{-}Q(s)(\xi_1^\alpha)^2\text{+}Q'(s)(\xi_1^\alpha - \xi_2^\alpha)]B_\alpha,\end{aligned} \tag{3.22}$$

where

$$Q(s) = \frac{1}{\sqrt{(\xi_1^\alpha - \xi_2^\alpha)^2 + (\xi_2^\alpha)^2 - (\xi_1^\alpha)^2}}.$$

Substituting (3.20) into (3.22) by using (3.23) gives

$$T'_\beta = \frac{\sqrt{3}}{K(s)}(M_1(s)\Omega_1^\alpha + M_2(s)\Omega_2^\alpha + M_3(s)B_\alpha),$$

where

$$\begin{aligned}
R_1(s) &= -Q(s)\dot{\xi}_1^\alpha - Q(s)(\xi_1^\alpha)^2 + Q(s)\xi_1^\alpha\xi_2^\alpha - Q'(s)\xi_1^\alpha, \\
R_2(s) &= -Q(s)\dot{\xi}_2^\alpha - Q(s)\xi_1^\alpha\xi_2^\alpha + Q(s)(\xi_2^\alpha)^2 - Q'(s)\xi_2^\alpha, \\
R_3(s) &= Q(s)(\xi_1^\alpha - \xi_2^\alpha)' - Q(s)(\xi_1^\alpha)^2 + Q'(s)(\xi_1^\alpha - \xi_2^\alpha).
\end{aligned} \tag{3.23}$$

Then the first curvature and the principal normal vector field of β are, respectively, obtained as follows

$$\kappa_\beta = \left\|T'_\beta\right\| = \frac{\sqrt{3}}{K(s)}\sqrt{-R_1^2(s) + R_2^2(s) + R_3^2(s)},$$

and

$$\begin{aligned}
B_\beta = &\frac{-1}{K(s)\sqrt{-R_1^2(s) + R_2^2(s) + R_3^2(s)}}[(M_2(\xi_1^\alpha - \xi_2^\alpha) + M_3\xi_2^\alpha)\Omega_1^\alpha \\
&+(M_1(\xi_1^\alpha - \xi_2^\alpha) + M_3\xi_2^\alpha)\Omega_2^\alpha + (\xi_2^\alpha M_1(s) - \xi_1^\alpha M_2(s))B_\alpha].
\end{aligned} \tag{3.24}$$

Differentiating (3.19) with respect to s in order to calculate the torsion of the curve β, we obtain

$$\begin{aligned}
\ddot{\beta} = &\frac{1}{\sqrt{3}}[(-\dot{\xi}_1^\alpha - (\xi_1^\alpha)^2 + \xi_1^\alpha\xi_2^\alpha)\Omega_1^\alpha \\
&+(-\dot{\xi}_2^\alpha + (\xi_2^\alpha)^2 - \xi_1^\alpha\xi_2^\alpha + (\xi_2^\alpha)^2)\Omega_2^\alpha + (\dot{\xi}_1^\alpha - \dot{\xi}_2^\alpha - (\xi_1^\alpha)^2)B_\alpha],
\end{aligned}$$

and similarly

$$\begin{aligned}
\dddot{\beta} = &\frac{1}{\sqrt{3}}[(-\ddot{\xi}_1^\alpha - 2\xi_1^\alpha\dot{\xi}_1^\alpha + \dot{\xi}_1^\alpha\xi_2^\alpha + \xi_1^\alpha\dot{\xi}_2^\alpha)\Omega_1^\alpha \\
&+(-\ddot{\xi}_2^\alpha + 4\xi_2^\alpha\dot{\xi}_2^\alpha - \dot{\xi}_1^\alpha\xi_2^\alpha - 2\xi_1^\alpha\xi_2^\alpha)\Omega_2^\alpha \\
&+(\xi_2^\alpha\dot{\xi}_2^\alpha + \xi_1^\alpha(\xi_2^\alpha)^2 - (\xi_1^\alpha)^2 - (\xi_1^\alpha)^3 + (\xi_1^\alpha)^2\xi_2^\alpha)B_\alpha].
\end{aligned}$$

The torsion of the curve β is

$$\begin{aligned}
\tau_\beta = &\frac{1}{9}\frac{K^2(s)}{-M_1^2(s) + M_2^2(s) + M_3^2(s)}\{[Q_3(s)(-\dot{\xi}_2^\alpha + 2(\xi_2^\alpha)^2 - \xi_1^\alpha\xi_2^\alpha) \\
&-Q_2(s)(\xi_1^\alpha - \dot{\xi}_2^\alpha - \xi_1^\alpha\xi_2^\alpha)]\xi_1^\alpha + [Q_3(s)(-\dot{\xi}_1^\alpha - (\xi_1^\alpha)^2 + \xi_1^\alpha\xi_2^\alpha) \\
&-Q_1(s)(-\dot{\xi}_2^\alpha + 2(\xi_2^\alpha)^2 - \xi_1^\alpha\xi_2^\alpha)]\xi_2^\alpha - [Q_2(s)(-\xi_1^\alpha - (\xi_1^\alpha)^2 + \xi_1^\alpha\xi_2^\alpha) \\
&-Q_1(s)(-\xi_2^\alpha + 2(\xi_2^\alpha)^2 - \xi_1^\alpha\xi_2^\alpha)](\xi_1^\alpha - \xi_2^\alpha)\},
\end{aligned}$$

where

$$\begin{aligned}
Q_1(s) &= -\dot{\xi}_1^\alpha - (\xi_1^\alpha)^2 + \xi_1^\alpha\xi_2^\alpha, \\
Q_2(s) &= -\dot{\xi}_1^\alpha\xi_2^\alpha + 2\xi_2^\alpha\dot{\xi}_2^\alpha - \ddot{\xi}_2^\alpha, \\
Q_3(s) &= \xi_1^\alpha(\xi_2^\alpha)^2 - (\xi_1^\alpha)^2 - (\xi_1^\alpha)^3 + (\xi_1^\alpha)^2\xi_2^\alpha + \xi_2^\alpha\dot{\xi}_2^\alpha.
\end{aligned}$$

3.5 Example

Example 3.1 Next, let us consider the following unit speed curve $w = w(s)$ in E_1^3 as follows

$$w(s) = (s, \sqrt{2}\ln(\sec h(s)), \sqrt{2}\arctan(\sinh(s))). \tag{3.25}$$

It is rendered in Figure 1, as follows

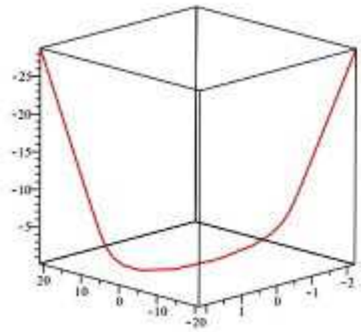

Figure 1

The curvature function and Serret-Frenet frame of the curve $w(s)$ is expressed as

$$\begin{aligned} T &= (1, -\sqrt{2}\tanh(s), \sqrt{2}\sec\mathrm{h}(s)), \\ N &= (0, -\sec\mathrm{h}(s), -\tanh(s)), \\ B &= (\sqrt{2}, -\tanh(s), \sec\mathrm{h}(s)), \end{aligned} \tag{3.26}$$

and

$$\kappa = \sqrt{2}\sec\mathrm{h}(s), \theta = \sqrt{2}\int_0^s \sec\mathrm{h}(s)ds = \sqrt{2}\arctan(\sinh(s)). \tag{3.27}$$

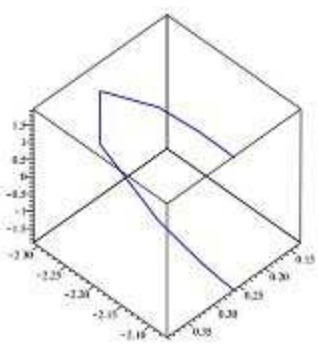

Figure 2 $\Omega_1\Omega_2$-Smarandache curve

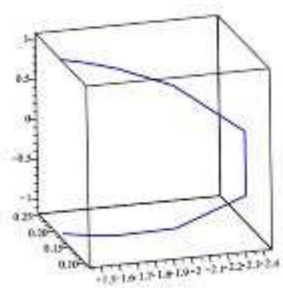

Figure 3 $\Omega_1 B$-Smarandache curve

Also the Bishop frame is computed as

$$\begin{gathered} \Omega_1 = (-\sinh\theta, -\sqrt{2}\sinh\theta\tanh(s) - \cosh\theta\sec\mathrm{h}(s), \\ -\sqrt{2}\sinh\theta\sec h(s) - \cosh\theta\tanh(s)), \end{gathered} \tag{3.28}$$

$$\begin{gathered} \Omega_2 = (\cosh\theta, -\sqrt{2}\cosh\theta\tanh(s) + \sinh\theta\sec\mathrm{h}(s), \\ \sqrt{2}\cosh\theta\sec h(s) - \sinh\theta\tanh(s)), \end{gathered} \tag{3.29}$$

$$B = (\sqrt{2}, -\tanh(s), \sec\mathrm{h}(s)). \tag{3.30}$$

Let us see the graphs which belong to all versions of Smarandache curves according to the Bishop frame in E_1^3.

The parametrizations and plottings of $\Omega_1\Omega_2, \Omega_1 B, \Omega_2 B$ and $\Omega_1\Omega_2 B-$ Smarandache curves are, respectively, given in Figures 2-5.

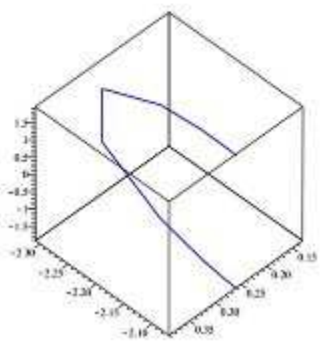

Figure 4 $\Omega_2 B$-Smarandache curve

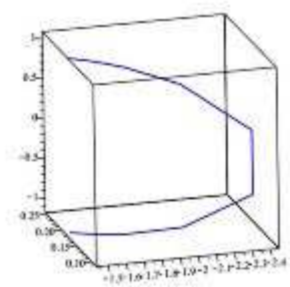

Figure 5 $\Omega_1\Omega_2 B$-Smarandache curve

§4. Smarandache Breadth Curves According to the Bishop Frame of Type-2 in E_1^3

A regular curve more than 2 breadths in Minkowski 3-space is called a Smarandache breadth curve.

Let $\alpha = \alpha(s)$ be a Smarandache breadth curve, and also suppose that $\alpha = \alpha(s)$ is a simple closed curve in E_1^3. This curve will be denoted by (C). The normal plane at every point P on the curve meets the curve at a single point Q other than P. We call the point Q as the opposite point of P

We consider a curve $\alpha^* = \alpha^*(s^*)$,in the class Γ, which has parallel tangents ζ and ζ^* at opposite directions at the opposite points α and α^* of the curve. A simple closed curve having parallel tangents in opposite directions at opposite points can be represented with respect to Bishop frame by the equation

$$\alpha^*(s^*) = \alpha(s) + \lambda\Omega_1 + \mu\Omega_2 + \eta B, \tag{4.1}$$

where $\lambda(s), \mu(s)$ and $\eta(s)$ are arbitrary functions, α and α^* are opposite points.

Differentiating both sides of (4.1) and considering Bishop equations, we have

$$\begin{aligned} \frac{d\alpha^*}{ds} = \Omega_1^* \frac{ds^*}{ds} = (\frac{d\lambda}{ds} - \eta\xi_1 + 1)\Omega_1 + (\frac{d\mu}{ds} - \eta\xi_2)\Omega_2 \\ +(\frac{d\eta}{ds} + \lambda\xi_1 - \mu\xi_2)B. \end{aligned} \tag{4.2}$$

Since $\Omega_1^* = -\Omega_1$, rewriting (4.2) we obtain respectively

$$\frac{d\lambda}{ds} = \eta\xi_1 - 1 - \frac{ds^*}{ds}, \quad \frac{d\mu}{ds} = \eta\xi_2, \quad \frac{d\eta}{ds} = -\lambda\xi_1 + \mu\xi_2. \tag{4.3}$$

If we call θ as the angle between the tangent of the curve (C) at the point $\alpha(s)$ with a given direction and consider $\frac{d\theta}{ds} = \tau$, (4.3) turns into the following form:

$$\frac{d\lambda}{d\theta} = \eta\frac{\xi_1}{\tau} - \frac{1}{\tau}(1 + \frac{ds^*}{ds}), \quad \frac{d\mu}{d\theta} = \eta\frac{\xi_2}{\tau}, \quad \frac{d\eta}{d\theta} = -\frac{\lambda}{\tau}\xi_1 + \frac{\mu}{\tau}\xi_2, \tag{4.4}$$

where $\frac{ds^*}{ds} = \frac{ds^*}{d\theta}.\frac{d\theta}{ds} = \frac{1}{\tau}\frac{ds^*}{d\theta}, 1 + \frac{ds^*}{ds} = f(\theta), \tau \neq 0.$

Using system (4.4), we have the following vectorial differential equation with respect to λ as follows

$$\begin{aligned}
&\frac{d^3\lambda}{d\theta^3} + \{\frac{\tau^2}{\xi_1^2\xi_2}f(\theta) - (\frac{\xi_1\xi_2}{\tau^3}(\frac{\xi_1}{\tau})')\frac{\tau}{\xi_1}\}\frac{d^2\lambda}{d\theta^2} + \{(\frac{\xi_1}{\tau})^2 \\
&+[\frac{\xi_1}{\tau^2}(\frac{\xi_1}{\tau})']'\,(1-\lambda)\,(\frac{\tau}{\xi_1})' - [\frac{\xi_1\xi_2}{\tau^3}(\frac{\xi_1}{\tau})]'(\frac{\tau}{\xi_1}) \\
&-[\frac{\xi_1\xi_2}{\tau^3}(\frac{\xi_1}{\tau})'] - [\frac{\xi_2^2}{\tau^3}(\frac{\xi_1}{\tau})'] + [\frac{1}{\xi_1}(\frac{\tau}{\xi_2})(\frac{\tau}{\xi_2})'f(\theta)]\}\frac{d\lambda}{d\theta} \\
&+\{[\frac{\xi_1\xi_2}{\tau^3}(\frac{\xi_1}{\tau})']'(\frac{\tau^2}{\xi_1\xi_2})(\frac{\tau}{\xi_1})'\}(\frac{d\lambda}{d\theta})^2 + \{\frac{1}{\xi_1} + \frac{\tau}{\xi_1}\}\frac{d\lambda}{d\theta}\frac{d^2\lambda}{d\theta^2} \\
&+\{(\frac{\xi_1}{\tau})'^2 + [\frac{\xi_1\xi_2}{\tau^3}(\frac{\xi_1}{\tau})']\frac{1}{\xi_1}f(\theta) + (\frac{\xi_1}{\xi_2} - 1)\frac{1}{\xi_1}f(\theta)\}\lambda \\
&+\{[(\frac{\xi_1}{\tau})^2 + \frac{1}{\xi_1}f(\theta)][\frac{\xi_1\xi_2}{\tau^3}(\frac{\xi_1}{\tau})']'[\frac{\tau}{\xi_2}(\frac{1}{\xi_1})'f(\theta) + \frac{\tau}{\xi_1\xi_2}f'(\theta) + \frac{\xi_1}{\xi_2}]\} \\
&+\{\frac{\xi_1\xi_2}{\tau^3}(\frac{\xi_1}{\tau})'[(\frac{1}{\xi_1})'f(\theta) - (\frac{1}{\xi_1})f'(\theta) + \frac{\xi_2}{\xi_1}\frac{1}{\tau}f(\theta)] \\
&+(\frac{1}{\tau})'f(\theta) + \frac{1}{\tau}f'(\theta)\} = 0.
\end{aligned} \tag{4.5}$$

The equation (4.5) is a characterization for α^*. If the distance between opposite points of (C) and (C^*) is constant, then we can write that

$$\|\alpha^* - \alpha\| = -\lambda^2 + \mu^2 + \eta^2 = l^2 = const., \tag{4.6}$$

hence, we write

$$-\lambda\frac{d\lambda}{d\theta} + \mu\frac{d\mu}{d\theta} + \eta\frac{d\eta}{d\theta} = 0. \tag{4.7}$$

Considering the system (4.4) together with (4.7), we obtain

$$\lambda f(\theta) = \eta(\frac{\xi_1}{\tau} - \tau\eta\xi_2 - \xi_1). \tag{4.8}$$

From system (4.4) we have

$$\eta = \frac{\tau}{\xi_1}\frac{d\lambda}{d\theta} + \frac{1}{\xi_1}f(\theta). \tag{4.9}$$

Substituting (4.9) into (4.8) gives

$$\lambda f(\theta) = (\tau \frac{d\lambda}{d\theta} + f(\theta))(\frac{\xi_1}{\tau} - \tau\eta\xi_2 - \xi_1)$$

or

$$\tau \frac{d\lambda}{d\theta} + f(\theta) = \frac{\lambda f(\theta)}{G(\theta)}, \tag{4.10}$$

where $G(\theta) = \frac{\xi_1}{\tau} - \tau\eta\xi_2 - \xi_1, \tau \neq 0$.

Thus we find

$$\lambda = \int_0^\theta \frac{f(\theta)}{\tau}(\tfrac{\lambda}{G(\lambda)} - 1)d\theta, \tag{4.11}$$

and also from $(4.4)_2$, (4.9) and $(4.4)_1$ we obtain

$$\mu = \int_0^\theta (\frac{d\lambda}{d\theta} + \frac{f(\theta)}{\tau})\xi_2 d\theta. \tag{4.12}$$

and

$$\eta = \int_0^\theta [\frac{\tau}{\xi_1}(\frac{d\lambda}{d\theta} + \frac{f(\theta)}{\tau^2})]d\theta. \tag{4.13}$$

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