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Smarandache Curves In Terms of Sabban Frame of Fixed Pole Curve

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abstract: In this paper, we study the special Smarandache curve in terms of Sabban frame of Fixed Pole curve and we give some characterization of Smarandache curves. Besides, we illustrate examples of our results.

Key Words: Smarandache Curves, Sabban Frame, Geodesic Curvature, Fixed Pole Curve

Contents

1. Introduction

A regular curve in Minkowski space-time, whose position vector is composed by Frenet frame vectors on another regular curve, is called a Smarandache curve [12]. Special Smarandache curves have been studied by some authors . Ahmad T.Ali studied some special Smarandache curves in the Euclidean space.He studied Frenet-Serret invariants of a special case [1]. M. Çetin , Y. Tunçer and K. Karacan investigated special smarandache curves according to Bishop frame in Euclidean 3-Space and they gave some differential goematric properties of Smarandache curves [5]. Şenyurt and Çalışkan investigated special Smarandache curves in terms of Sabban frame of spherical indicatrix curves and they gave some characterization of Smarandache curves, [3].Also, in their other work, when the unit Darboux vector of the partner curve of Mannheim curve were taken as the position vectors, the curvature and the torsion of Smarandache curve were calculated. These values were expressed depending upon the Mannheim curve, [4]. They defined NC-Smarandache curve, then they calculated the curvature and torsion of NB and TNB- Smarandache curves together with NC-Smarandache curve, [10]. Ö.

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Bektaş and S. Yüce studied some special smarandache curves according to Darboux Frame in E^3 [2]. M. Turgut and S. Yılmaz studied a special case of such curves and called it smarandache TB_2 curves in the space E_1^4 [12]. K. Taşköprü , M. Tosun studied special Smarandache curves according to Sabban frame on S^2 [11].

In this paper, the special smarandache curves such as $CT_C, T_C(C \wedge T_C), CT_C(C \wedge T_C)$ created by Sabban frame , $\{C, T_C, C \wedge T_C\}$, that belongs to fixed pole of a α curve are defined. Besides, we have found some results.

2. Preliminaries

The Euclidean 3-space E^3 be inner product given by

$$\langle , \rangle = x_1^2 + x_2^3 + x_3^2$$

where $(x_1, x_2, x_3) \in E^3$. Let $\alpha : I \to E^3$ be a unit speed curve denote by $\{T, N, B\}$ the moving Frenet frame . For an arbitrary curve $\alpha \in E^3$, with first and second curvature, κ and τ respectively, the Frenet formulae is given by [6]

$$\begin{cases} T' = \kappa N \\ N' = -\kappa T + \tau B \\ B' = -\tau N. \end{cases} \tag{2.1}$$

Accordingly, the spherical indicatrix curves of Frenet vectors are (T), (N) and (B) respectively.These equations of curves are given by [7], [9]

$$\begin{cases} \alpha_T(s) = T(s) \\ \alpha_N(s) = N(s) \\ \alpha_B(s) = B(s) \end{cases} \tag{2.2}$$

Let $\gamma : I \to S^2$ be a unit speed spherical curve. We denote s as the arc-length parameter of γ. Let us denote by

$$\begin{cases} \gamma(s) = \gamma(s) \\ t(s) = \gamma'(s) \\ d(s) = \gamma(s) \wedge t(s). \end{cases} \tag{2.3}$$

We call $t(s)$ a unit tangent vector of γ. $\{\gamma, t, d\}$ frame is called the Sabban frame of γ on S^2 . Then we have the following spherical Frenet formulae of γ :

$$\begin{cases} \gamma' = t \\ t' = -\gamma + \kappa_g d \\ d' = -\kappa_g t \end{cases} \tag{2.4}$$

where is called the geodesic curvature of κ_g on S^2 and

$$\kappa_g = \langle t', d \rangle, \quad [8]. \tag{2.5}$$

3. Smarandache Curves According to Sabban Frame of Fixed Pole Curve

In this section, we investigate Smarandache curves according to the Sabban frame of fixed pole curve (C). Let $\alpha_C(s) = C(s)$ be a unit speed regular spherical curves on S^2.We denote s_C as the arc-lenght parameter of fixed pole curve (C)

$$\alpha_C(s) = C(s) \tag{3.1}$$

Differentiating (3.1) , we have

$$\frac{d\alpha_C}{ds_C}\frac{ds_C}{ds} = C'(s)$$

and

$$T_C\frac{ds_C}{ds} = \varphi' \cos\varphi T - \varphi' \sin\varphi B \tag{3.2}$$

From the equation (3.2)

$$T_C = \cos\varphi T - \sin\varphi B$$

and

$$C \wedge T_C = N$$

From the equation (2.3)

$$\begin{cases} C(s) = C(s) \\ T_C(s) = \cos\varphi T - \sin\varphi B \\ (C \wedge T_C)(s) = N(s) \end{cases}$$

is called the Sabban frame of fixed pole curve (C) .From the equation (2.5)

$$\kappa_g = \langle T_C', C \wedge T_C\rangle \Longrightarrow \kappa_g = \frac{\|W\|}{\varphi'}$$

Then from the equation (2.4) we have the following spherical Frenet formulae of (C):

$$\begin{cases} C' = T_C \\ T_C{}' = -C + \frac{\|W\|}{\varphi'}(C \wedge T_C) \\ (C \wedge T_C)' = -\frac{\|W\|}{\varphi'}T_C \end{cases} \tag{3.3}$$

3.1. CT_C-Smarandache Curves

Definition 3.1. *Let S^2 be a unit sphere in E^3 and suppose that the unit speed regular curve $\alpha_C(s) = C(s)$ lying fully on S^2. In this case, CT_C - Smarandache curve can be defined by*

$$\psi(s^*) = \frac{1}{\sqrt{2}}(C + T_C). \tag{3.4}$$

Now we can compute Sabban invariants of CT_C - Smarandache curves. Differentiating (3.4) , we have

$$T_\psi \frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\big((\cos\varphi - \sin\varphi)T + \frac{\|W\|}{\varphi'}N - (\cos\varphi + \sin\varphi)B\big),$$

where

$$\frac{ds^*}{ds} = \sqrt{\frac{2 + \left(\frac{\|W\|}{\varphi'}\right)^2}{2}}. \tag{3.5}$$

Thus, the tangent vector of curve ψ is to be

$$T_\psi = \frac{1}{\sqrt{2 + \left(\frac{\|W\|}{\varphi'}\right)^2}}\big((\cos\varphi - \sin\varphi)T + \frac{\|W\|}{\varphi'}N - (\cos\varphi + \sin\varphi)B\big). \tag{3.6}$$

Differentiating (3.6), we get

$$T'_\psi \frac{ds^*}{ds} = \frac{1}{(2 + (\frac{\|W\|}{\varphi'})^2)^{\frac{3}{2}}}(\lambda_1 T + \lambda_2 N + \lambda_3 B) \tag{3.7}$$

where

$$\lambda_1 = -2\varphi'(\sin\varphi + \cos\varphi) - \kappa\big(2\tfrac{\|W\|}{\varphi'} + \big(\tfrac{\|W\|}{\varphi'}\big)^3\big) - \tfrac{\|W\|}{\varphi'}^2(\sin\varphi + \cos\varphi) -$$
$$\tfrac{\|W\|}{\varphi'}\big(\tfrac{\|W\|}{\varphi'}\big)'(\cos\varphi - \sin\varphi)$$
$$\lambda_2 = \big(2 + \big(\tfrac{\|W\|}{\varphi'}\big)^2\big)\big(\kappa(\cos\varphi - \sin\varphi) + \tau(\cos\varphi + \sin\varphi)\big) + 2\big(\tfrac{\|W\|}{\varphi'}\big)'$$
$$\lambda_3 = \tau\tfrac{\|W\|}{\varphi'}\big(2 + \big(\tfrac{\|W\|}{\varphi'}\big)^2\big) - 2\varphi'(\cos\varphi - \sin\varphi) - \tfrac{\|W\|}{\varphi'}^2(\cos\varphi - \sin\varphi) +$$
$$\tfrac{\|W\|}{\varphi'}\big(\tfrac{\|W\|}{\varphi'}\big)'(\cos\varphi + \sin\varphi).$$

Substituting the equation (3.5) into equation (3.7) , we reach

$$T'_\psi = \frac{\sqrt{2}}{\big(2 + \big(\frac{\|W\|}{\varphi'}\big)^2\big)^2}(\lambda_1 T + \lambda_2 N + \lambda_3 B). \tag{3.8}$$

Considering the equations (3.4) and (3.6) , it easily seen that

$$\begin{aligned}(C \wedge T_C)_\psi \quad = \quad & \frac{1}{\sqrt{4 + 2\big(\frac{\|W\|}{\varphi'}\big)^2}}\Big(- \frac{\|W\|}{\varphi'}(\cos\varphi - \sin\varphi)T + \\ & +2N + (\frac{\|W\|}{\varphi'}(\cos\varphi + \sin\varphi))B\Big)\end{aligned} \tag{3.9}$$

From the equation (3.8) and (3.9) , the geodesic curvature of $\psi(s^*)$ is

$$\begin{aligned}\kappa_g{}^{\psi} &= \langle T'_\psi, (C\wedge T_C)_\psi\rangle \\ &= \frac{1}{\left(2+\left(\frac{\|W\|}{\varphi'}\right)^2\right)^{\frac{3}{2}}}\left(-\frac{\|W\|}{\varphi'}(\cos\varphi-\sin\varphi)\lambda_1+2\lambda_2+\frac{\|W\|}{\varphi'}(\cos\varphi+\sin\varphi)\lambda_3\right).\end{aligned}$$

3.2. $T_C(C\wedge T_C)$-Smarandache Curves

Definition 3.2. *Let S^2 be a unit sphere in E^3 and suppose that the unit speed regular curve $\alpha_C(s)=C(s)$ lying fully on S^2. In this case, $T_C(C\wedge T_C)$ - Smarandache curve can be defined by*

$$\psi(s^*)=\frac{1}{\sqrt{2}}(T_C+C\wedge T_C). \tag{3.10}$$

Now we can compute Sabban invariants of $T_C(C\wedge T_C)$ - Smarandache curves. Differentiating (3.10), we have

$$T_\psi\frac{ds^*}{ds}=\frac{1}{\sqrt{2}}\left((-\sin\varphi-\frac{\|W\|}{\varphi'}\cos\varphi)T+\frac{\|W\|}{\varphi'}N+(\frac{\|W\|}{\varphi'}\sin\varphi-\cos\varphi)B\right)$$

where

$$\frac{ds^*}{ds}=\sqrt{\frac{1+2\left(\frac{\|W\|}{\varphi'}\right)^2}{2}}. \tag{3.11}$$

In that case, the tangent vector of curve ψ is as follows

$$\begin{aligned}T_\psi &= \frac{1}{\sqrt{1+2\left(\frac{\|W\|}{\varphi'}\right)^2}}\Big((-\sin\varphi-\frac{\|W\|}{\varphi'}\cos\varphi)T+ \\ &\quad +\frac{\|W\|}{\varphi'}N+(\frac{\|W\|}{\varphi'}\sin\varphi-\cos\varphi)B\Big)\end{aligned} \tag{3.12}$$

Differentiating (3.12), it is obtained that

$$T'_\psi\frac{ds^*}{ds}=\frac{1}{\left(1+2\left(\frac{\|W\|}{\varphi'}\right)^2\right)^{\frac{3}{2}}}(\lambda_1T+\lambda_2N+\lambda_3B) \tag{3.13}$$

where

$\lambda_1 = -\varphi' \cos\varphi + \|W\|(\sin\varphi + 2\frac{\|W\|}{\varphi'}\cos\varphi - \frac{\kappa}{\varphi'} + 2\big(\frac{\|W\|}{\varphi'}\big)^2 \sin\varphi - 2\kappa\frac{\|W\|^2}{\varphi'^3}) -$

$\big(\frac{\|W\|}{\varphi'}\big)'(\cos\varphi - 2\frac{\|W\|}{\varphi'}\sin\varphi)$

$\lambda_2 = \kappa(-\sin\varphi - \frac{\|W\|}{\varphi'}\cos\varphi + 2\big(\frac{\|W\|}{\varphi'}\big)^2(-\sin\varphi - \frac{\|W\|}{\varphi'}\cos\varphi))-$

$\tau(\frac{\|W\|}{\varphi'}(\sin\varphi - \cos\varphi) - 2\big(\frac{\|W\|}{\varphi'}\big)^3(\sin\varphi - \cos\varphi)) + \big(\frac{\|W\|}{\varphi'}\big)'$

$\lambda_3 = \varphi'\sin\varphi + \frac{\|W\|}{\varphi'}(\tau + \varphi'\cos\varphi + 2\tau\big(\frac{\|W\|}{\varphi'}\big)^2 + 2\big(\frac{\|W\|}{\varphi'}\big)^2\varphi'\cos\varphi + 2\|W\|\sin\varphi) +$

$\big(\frac{\|W\|}{\varphi'}\big)'(\sin\varphi + 2\frac{\|W\|}{\varphi'}\cos\varphi).$

Substituting the equation (3.11) into equation (3.13) , we get

$$T'_\psi = \frac{\sqrt{2}}{\Big(1 + 2\big(\frac{\|W\|}{\varphi'}\big)^2\Big)^2}(\lambda_1 T + \lambda_2 N + \lambda_3 B) \tag{3.14}$$

Using the equations (3.10) and (3.12) , we easily find

$$\begin{aligned}(C \wedge T_C)_\psi \quad &= \quad \frac{1}{\sqrt{2 + 4\big(\frac{\|W\|}{\varphi'}\big)^2}}\Big((2\frac{\|W\|}{\varphi'}\sin\varphi - \cos\varphi)T + \\ &\quad +N + (2\frac{\|W\|}{\varphi'}\cos\varphi + \sin\varphi)B\Big)\end{aligned} \tag{3.15}$$

So, the geodesic curvature of $\psi(s^*)$ is as follows

$$\begin{aligned}{\kappa_g}^\psi &= \langle T'_\psi, (C \wedge T_C)_\psi \rangle \\ &= \frac{1}{\Big(1+2\big(\frac{\|W\|}{\varphi'}\big)^2\Big)^{\frac{3}{2}}}((2\frac{\|W\|}{\varphi'}\sin\varphi - \cos\varphi)\lambda_1 + \lambda_2 + (2\frac{\|W\|}{\varphi'}\cos\varphi + \sin\varphi)\lambda_3).\end{aligned}$$

3.3. $CT_C(C \wedge T_C)$-Smarandache Curves

Definition 3.3. *Let S^2 be a unit sphere in E^3 and suppose that the unit speed regular curve $\alpha_C(s) = C(s)$ lying fully on S^2. In this case, $CT_C(C \wedge T_C)$ - Smarandache curve can be defined by*

$$\psi(s^*) = \frac{1}{\sqrt{3}}(C + T_C + C \wedge T_C). \tag{3.16}$$

Let us calculate Sabban invariants of $CT_C(C \wedge T_C)$ - Smarandache curves. Differentiating (3.16), we have

$$T_\psi \frac{ds^*}{ds} = \frac{1}{\sqrt{3}}\Big((\cos\varphi - \sin\varphi - \frac{\|W\|}{\varphi'}\cos\varphi)T + \\ +\frac{\|W\|}{\varphi'}N + (-\sin\varphi - \cos\varphi + \frac{\|W\|}{\varphi'}\sin\varphi)B\Big)$$

where

$$\frac{ds^*}{ds} = \sqrt{\frac{2(1 - \frac{\|W\|}{\varphi'} + (\frac{\|W\|}{\varphi'})^2)}{3}}. \tag{3.17}$$

Thus, the tangent vector of curve ψ is

$$T_\psi = \frac{1}{\sqrt{2\Big(1 - \frac{\|W\|}{\varphi'} + (\frac{\|W\|}{\varphi'})^2\Big)}}\Big((\cos\varphi - \sin\varphi - \frac{\|W\|}{\varphi'}\cos\varphi)T \\ +\frac{\|W\|}{\varphi'}N + (-\sin\varphi - \cos\varphi + \frac{\|W\|}{\varphi'}\sin\varphi)B\Big) \tag{3.18}$$

Differentiating (3.18), it is obtained that

$$T'_\psi \frac{ds^*}{ds} = \frac{1}{2\sqrt{2}\big(1 - \frac{\|W\|}{\varphi'} + (\frac{\|W\|}{\varphi'})^2\big)^{\frac{3}{2}}}(\lambda_1 T + \lambda_2 N + \lambda_3 B) \tag{3.19}$$

where

$$\lambda_1 = \big(1 - \tfrac{\|W\|}{\varphi'} + (\tfrac{\|W\|}{\varphi'})^2\big)\big(-2\varphi'(\sin\varphi + \cos\varphi) + 2\|W\|\sin\varphi - 2\kappa\tfrac{\|W\|}{\varphi'}\big) + \\ \tfrac{\|W\|}{\varphi'}(\cos\varphi - \sin\varphi) + 2\big(\tfrac{\|W\|}{\varphi'}\big)'(-\cos\varphi + \tfrac{\|W\|}{\varphi'}\sin\varphi)$$

$$\lambda_2 = \big(1 - \tfrac{\|W\|}{\varphi'} + (\tfrac{\|W\|}{\varphi'})^2\big)\big(2\kappa\big(\cos\varphi - \sin\varphi - \tfrac{\|W\|}{\varphi'}\cos\varphi\big) + \\ 2\tau\big(\sin\varphi + \cos\varphi - \tfrac{\|W\|}{\varphi'}\cos\varphi\big)\big) + 2\big(\tfrac{\|W\|}{\varphi'}\big)'\big(1 - \tfrac{\|W\|}{\varphi'}\big) + \big(\tfrac{\|W\|}{\varphi'}\big)^2$$

$$\lambda_3 = \big(1 - \tfrac{\|W\|}{\varphi'} + (\tfrac{\|W\|}{\varphi'})^2\big)\big(2\tau\tfrac{\|W\|}{\varphi'} + 2\varphi'(\sin\varphi - \cos\varphi) + 2\|W\|\cos\varphi\big) + \\ \big(\tfrac{\|W\|}{\varphi'}\big)^2\sin\varphi + 2\big(\tfrac{\|W\|}{\varphi'}\big)'(\sin\varphi + \tfrac{\|W\|}{\varphi'}\cos\varphi) - \tfrac{\|W\|}{\varphi'}(\sin\varphi + \cos\varphi).$$

Substituting the equation (3.17) into equation (3.19) , we reach

$$T'_\psi = \frac{\sqrt{3}}{4\Big(1 - \frac{\|W\|}{\varphi'} + (\frac{\|W\|}{\varphi'})^2\Big)^2}(\lambda_1 T + \lambda_2 N + \lambda_3 B). \tag{3.20}$$

Using the equations (3.16) and (3.18) , we have

$$
\begin{aligned}
(C\wedge T_C)_\psi \;=\; & \frac{1}{\sqrt{6}\sqrt{1-\frac{\|W\|}{\varphi'}+\left(\frac{\|W\|}{\varphi'}\right)^2}}\Big((2\frac{\|W\|}{\varphi'}\sin\varphi-\frac{\|W\|}{\varphi'}\cos\varphi \\
& -\cos\varphi)T+(1-\frac{\|W\|}{\varphi'})N+(\sin\varphi+2\frac{\|W\|}{\varphi'}\cos\varphi \\
& +\frac{\|W\|}{\varphi'}\sin\varphi)B\Big)
\end{aligned} \tag{3.21}
$$

From the equation (3.20) and (3.21), the geodesic curvature of $\psi(s^*)$ is

$$
\begin{aligned}
\kappa_g^\psi \;=\; & \langle T'_\psi,(C\wedge T_C)_\psi\rangle=\frac{1}{4\sqrt{2}(1-\frac{\|W\|}{\varphi'}+(\frac{\|W\|}{\varphi'})^2)^{\frac{3}{2}}}[\lambda_1(2\frac{\|W\|}{\varphi'}\sin\varphi- \\
& \frac{\|W\|}{\varphi'}\cos\varphi-\cos\varphi)+\lambda_2(1-\frac{\|W\|}{\varphi'})+ \\
& \lambda_3(\sin\varphi+2\frac{\|W\|}{\varphi'}\cos\varphi+\frac{\|W\|}{\varphi'}\sin\varphi)].
\end{aligned}
$$

3.4. Example

Let us consider the unit speed spherical curve:

$$\alpha(s)=\{\frac{9}{208}\sin 16s-\frac{1}{117}\sin 36s,-\frac{9}{208}\cos 16s+\frac{1}{117}\cos 36s,\frac{6}{65}\sin 10s\}.$$

It is rendered in Figure 1.

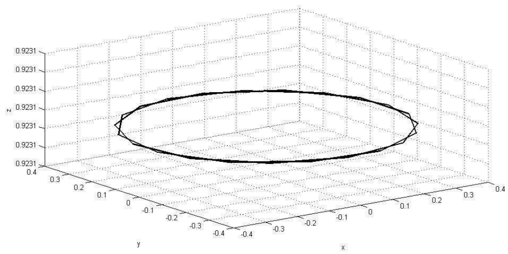

Figure 1: *Fixed Pole curve* (T)

In terms of definitions, we obtain Smarandache curves according to Sabban frame on S^2, see Figures 2 - 4.

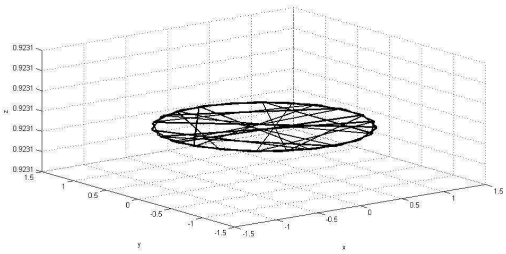

Figure 2: CT_C - Smarandache Curve

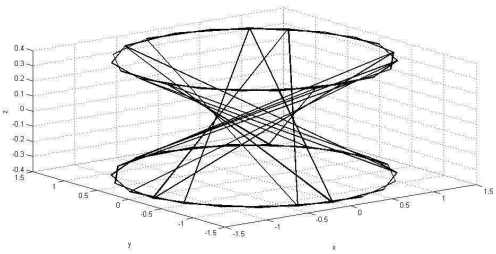

Figure 3: $T_C(C \wedge T_C)$ - Smarandache Curve

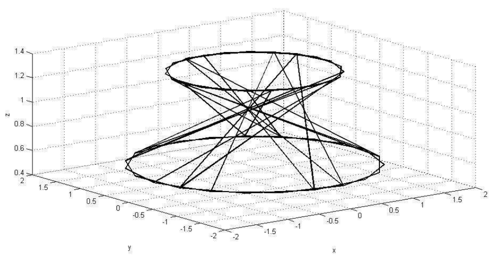

Figure 4: $CT_C(C \wedge T_C)$ - Smarandache Curve

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