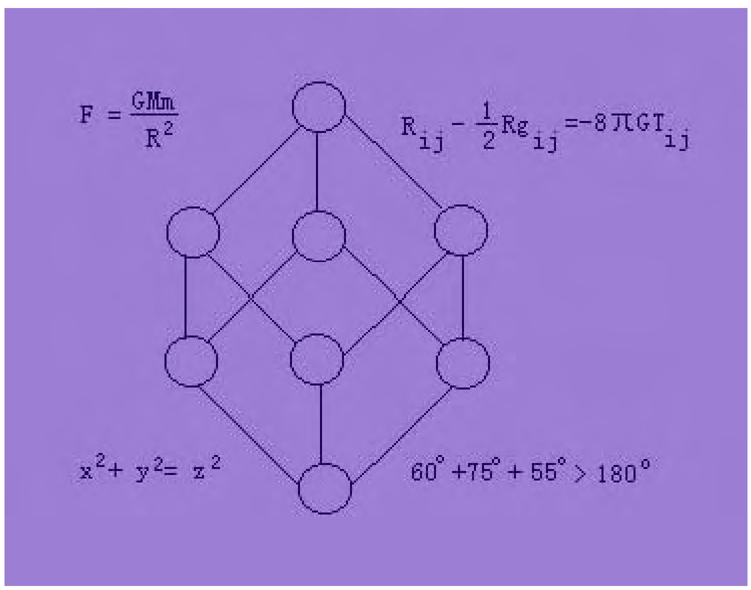
LINFAN MAO

SMARANDACHE MULTI-SPACE THEORY

(Partially post-doctoral research for the Chinese Academy of Sciences)



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Phoenix, AZ 2006

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Preface

A Smarandache multi-space is a union of n different spaces equipped with some different structures for an integer $n \geq 2$, which can be used both for discrete or connected spaces, particularly for geometries and spacetimes in theoretical physics.

We are used to the idea that our space has three dimensions: *length, breadth* and *height*; with time providing the fourth dimension of spacetime by Einstein. In the string or superstring theories, we encounter 10 dimensions. However, *we do not even know what the right degree of freedom is*, as Witten said. In 21th century, theoretical physicists believe the 11-dimensional M-theory is the best candidate for the *Theory of Everything*, i.e., a fundamental united theory of all physical phenomena, but the bottleneck is that 21st century mathematics has not yet been discovered. Today, we think the *Smarandache multi-space theory* is the best candidate for 21st century mathematics and a new revolution for mathematics has come. Although it is important, only a few books can be found in the libraries research about these spaces and their relations with classical mathematics and theoretical physics. The purpose of this book is to survey this theory, also to establish its relation with physics and cosmology. Many results and materials included in this book are now inspired by the Smarandache's notion.

Now we outline the content of this book. Three parts are included in this book altogether.

Part one consists of Chapters 1 and 2 except the Section 1.4, may be called *Algebraic multi-spaces*. In Chapter 1, we introduce various algebraic Smarandache multi-spaces including those such as multi-groups, multi-rings, multi-vector spaces, multi-metric spaces, multi-operation systems and multi-manifolds and get some elementary results on these multi-spaces.

Chapter 2 concentrates on multi-spaces on graphs. In an algebraic view, each multi-space is a directed graph and each graph is also a multi-space. Many new conceptions are introduced in this chapter, such as multi-voltage graphs, graphs in an n-manifold, multi-embeddings in an n-manifold, Cayley graphs of a multi-group and many results in graph theory, combinatorial map theory are generalized. These new conceptions and results enlarge the research fields of combinatorics. A graph phase is introduced and discussed in Section 2.5, which is a generalization of a particle and can be used to construct a model of p-branes in Chapter 6.

The second part of this book consists of Chapters 3-5 including the Section 1.4 concentrates on these *multi-metric spaces*, particularly, these *Smarandache geometries.* In fact, nearly all geometries, such as *Finsler geometry*, *Riemann geometry*, *Euclid geometry* and *Lobachevshy-Bolyai-Gauss geometry* are particular case of Smarandache geometries. In Chapter 3, we introduce a new kind geometry, i.e., *map geometries*, which is a generalization of Iseri's s-manifolds. By applying map geometries with or without boundary, *paradoxist geometries, non-geometries, counter-projective geometries* and *anti-geometries* are constructed. The enumeration of map geometries with or without boundary underlying a graph are also gotten in this chapter.

Chapter 4 is on planar map geometries. Those fundamental elements such as points, lines, polygons, circles and line bundles are discussed. Since we can investigate planar map geometries by means of Euclid plane geometry, some conceptions for general map geometries are extend, which enables us to get some interesting results. Those measures such as angles, curvatures, areas are also discussed.

Chapter 5 is a generalization of these planar map geometries. We introduce the conception of *pseudo-planes*, which can be seen as the limitation case of planar map geometries when the diameter of each face tends to zero but more general than planar map geometries. On a pseudo-plane geometry, a straight line does not always exist again. These pseudo-plane geometries also relate with differential equations and plane integral curves. Conditions for existing singular points of differential equations are gotten. In Section 5.4, we define a kind of even more general spaces called *metric pseudo-spaces* or *bounded metric pseudo-spaces.* Applying these bounded or unbounded metric pseudo-spaces, *bounded pseudo-plane geometries*, *pseudo-surface geometries*, *pseudo-space geometries* and *pseudo-manifold geometries* are defined.

By choice different smooth function ω, such as $\omega =$ a Finsler or a Riemann norm, we immediately get the Finsler geometry or Riemann geometry.

For the further research of interested readers, each of the final sections of Chapters $1-5$ contains a number of open problems and conjectures which can be seen as additional materials.

Part three only consists of Chapter 6 which is concentrated on applications of multi-spaces to theoretical physics. The view that *every observation of human beings for cosmos is only a pseudo-face of our space* is discussed in the first section by a mathematical manner. A brief introduction to Einstein's relative theory and M-theory is in Section 6.2 and 6.3.1. By a view of multi-spaces, models for p-branes and cosmos are constructed in Section 6.3. It is very interesting that the multi-space model of cosmos contains the *shelf structure* as a special case. The later is a fundamental structure in the *modern algebraic geometry* in recent years.

This book is began to write in the July, 2005 when I finished my post-doctor report: *On automorphisms of maps, surfaces and Smarandache geometries* for the *Chinese Academy of Sciences.* Many colleagues and friends of mine have given me enthusiastic support and endless helps in preparing this book. Without their help, this book will never appears today. Here I must mention some of them. On the first, I would like to give my sincerely thanks to Dr.Perze for his encourage and endless help. Without his encourage and suggestion, I would do some else works, can not investigate multi-spaces and finish this book. Second, I would like to thank Prof. Feng Tian, Yanpei Liu and Jiyi Yan for them interested in my post-doctor report: *On automorphisms of maps, surfaces and Smarandache geometries.* Their encourage and warmhearted support advance this book. Thanks are also given to Professor Mingyao Xu, Xiaodong Hu, Yanxun Chang, Han Ren, Rongxia Hao, Weili He and Erling Wei for their kindly helps and often discussing problems in mathematics altogether. Of course, I am responsible for the correctness all of these materials presented here. Any suggestions for improving this book and solutions for open problems in this book are welcome.

L.F.Mao
AMSS, Beijing
February, 2006

Contents

Chapter 1 Smarandache Multi-Spaces

The notion of multi-spaces was introduced by Smarandache in 1969, see his article uploaded to *arXiv* [86] under his idea of hybrid mathematics: *combining different fields into a unifying field*([85]), which is more closer to our real life world. Today, this idea is widely accepted by the world of sciences. For mathematics, a definite or an exact solution under a given condition is not the only object for mathematician. New creation power has emerged and new era for mathematics has come now. Applying the Smarandache's notion, this chapter concentrates on constructing various multi-spaces by algebraic structures, such as those of groups, rings, fields, vector spaces, $\cdots$,etc., also by metric spaces, which are more useful for constructing multi-voltage graphs, maps and map geometries in the following chapters.

§1.1 Sets

1.1.1. Sets

A *set* Ξ is a collection of objects with some common property P, denoted by

$$\Xi = \{x | x \textit{ has property } P\},$$

where, x is said an element of the set Ξ, denoted by $x \in \Xi$. For an element y not possessing the property P, i.e., not an element in the set Ξ, we denote it by $y \notin \Xi$.

The *cardinality* (or the number of elements if Ξ is finite) of a set Ξ is denoted by $|\Xi|$.

Some examples of sets are as follows.

$$A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\};$$

$$B = \{p|\ p\ is\ a\ prime\ number\};$$

$$C = \{(x, y)|x^2 + y^2 = 1\};$$

$$D = \{the\ cities\ in\ the\ world\}.$$

The sets A and D are finite with $|A| = 10$ and $|D| < +\infty$, but these sets B and C are infinite.

Two sets Ξ_1 and Ξ_2 are said to be *identical* if and only if for $\forall x \in \Xi_1$, we have $x \in \Xi_2$ and for $\forall x \in \Xi_2$, we also have $x \in \Xi_1$. For example, the following two sets

$$E = \{1, 2, -2\} \text{ and } F = \{\ x\ |x^3 - x^2 - 4x + 4 = 0\}$$

are identical since we can solve the equation $x^3 - x^2 - 4x + 4 = 0$ and get the solutions $x = 1, 2$ or -2. Similarly, for the cardinality of a set, we know the following result.

Theorem 1.1.1([6]) *For sets Ξ_1, Ξ_2, $|\Xi_1| = |\Xi_2|$ if and only if there is an $1 - 1$ mapping between Ξ_1 and Ξ_2.*

According to this theorem, we know that $|B| \neq |C|$ although they are infinite. Since B is countable, i.e., there is an $1 - 1$ mapping between B and the natural number set $N = \{1, 2, 3, \cdots, n, \cdots\}$, however C is not.

Let A_1, A_2 be two sets. If for $\forall a \in A_1 \Rightarrow a \in A_2$, then A_1 is said to be a *subset* of A_2, denoted by $A_1 \subseteq A_2$. If a set has no elements, we say it an empty set, denoted by $\emptyset$.

Definition 1.1.1 *For two sets Ξ_1, Ξ_2, two operations "$\bigcup$" and "$\bigcap$" on Ξ_1, Ξ_2 are defined as follows:*

$$\Xi_1 \bigcup \Xi_2 = \{x|x \in \Xi_1\ or\ x \in \Xi_2\},$$

$$\Xi_1 \bigcap \Xi_2 = \{x|x \in \Xi_1\ and\ x \in \Xi_2\}$$

and Ξ_1 *minus* Ξ_2 *is defined by*

$$\Xi_1 \setminus \Xi_2 = \{x|x \in \Xi_1 \text{ but } x \notin \Xi_2\}.$$

For the sets A and E, calculation shows that

$$A\bigcup E = \{1, 2, -2, 3, 4, 5, 6, 7, 8, 9, 10\},$$

$$A\bigcap E = \{1, 2\}$$

and

$$A \setminus E = \{3, 4, 5, 6, 7, 8, 9, 10\},$$

$$E \setminus A = \{-2\}.$$

For a set Ξ and $H \subseteq \Xi$, the set $\Xi \setminus H$ is said the *complement* of H in Ξ, denoted by $\overline{H}(\Xi)$. We also abbreviate it to $\overline{H}$ if each set considered in the situation is a subset of $\Xi = \Omega$, i.e., the *universal set*.

These operations defined in Definition 1.1.1 observe the following laws.

L1 Itempotent law. For $\forall S \subseteq \Omega$,

$$A\bigcup A = A\bigcap A = A.$$

L2 Commutative law. For $\forall U, V \subseteq \Omega$,

$$U\bigcup V = V\bigcup U;\ U\bigcap V = V\bigcap U.$$

L3 Associative law. For $\forall U, V, W \subseteq \Omega$,

$$U\bigcup(V\bigcup W) = (U\bigcup V)\bigcup W;\ U\bigcap(V\bigcap W) = (U\bigcap V)\bigcap W.$$

L4 Absorption law. For $\forall U, V \subseteq \Omega$,

$$U\bigcap(U\bigcup V) = U\bigcup(U\bigcap V) = U.$$

L5 Distributive law. For $\forall U, V, W \subseteq \Omega$,

$$U\bigcup(V\bigcap W) = (U\bigcup V)\bigcap(U\bigcup W);\ U\bigcap(V\bigcup W) = (U\bigcap V)\bigcup(U\bigcap W).$$

L6 Universal bound law. For $\forall U \subseteq \Omega$,

$$\emptyset\bigcap U = \emptyset, \emptyset\bigcup U = U;\ \Omega\bigcap U = U, \Omega\bigcup U = \Omega.$$

L7 Unary complement law. For $\forall U \subseteq \Omega$,

$$U\bigcap\overline{U} = \emptyset;\ U\bigcup\overline{U} = \Omega.$$

A set with two operations "$\bigcap$" and "$\bigcup$" satisfying the laws $L1 \sim L7$ is said to be a *Boolean algebra*. Whence, we get the following result.

Theorem 1.1.2 *For any set U, all its subsets form a Boolean algebra under the operations "$\bigcap$" and "$\bigcup$".*

1.1.2 **Partially order sets**

For a set Ξ, define its *Cartesian product* to be

$$\Xi \times \Xi = \{(x, y)|\forall x, y \in \Xi\}.$$

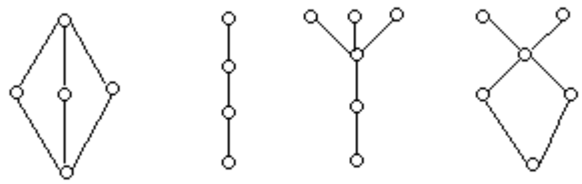
A subset $R \subseteq \Xi\times\Xi$ is called a *binary relation* on Ξ. If $(x, y) \in R$, we write xRy. A *partially order set* is a set Ξ with a binary relation "$\preceq$" such that the following laws hold.

O1 Reflective law. For $x \in \Xi$, xRx.

O2 Antisymmetry law. For $x, y \in \Xi$, xRy and $yRx \Rightarrow x = y$.

O3 Transitive law. For $x, y, z \in \Xi$, xRy and $yRz \Rightarrow xRz$.

A partially order set Ξ with a binary relation "$\preceq$" is denoted by $(\Xi, \preceq)$. Partially ordered sets with a finite number of elements can be conveniently represented by a diagram in such a way that each element in the set Ξ is represented by a point so placed on the plane that a point a is above another point b if and only if $b \prec a$. This kind of diagram is essentially a directed graph (see also Chapter 2 in this book). In fact, a directed graph is correspondent with a partially set and vice versa. Examples for the partially order sets are shown in Fig.1.1 where each diagram represents a finite partially order set.



(1) (2) (3) (4)

Fig.1.1

An element a in a partially order set $(\Xi, \preceq)$ is called *maximal* (or *minimal*) if for $\forall x \in \Xi,\ a \preceq x \Rightarrow x = a$ (or $x \preceq a \Rightarrow x = a$). The following result is obtained by the definition of partially order sets and the induction principle.

Theorem 1.1.3 *Any finite non-empty partially order set* $(\Xi, \preceq)$ *has maximal and minimal elements.*

A partially order set $(\Xi, \preceq)$ is an *order set* if for any $\forall x, y \in \Xi$, there must be $x \preceq y$ or $y \preceq x$. It is obvious that any partially order set contains an order subset, finding this fact in Fig.1.1.

An *equivalence relation* $R \subseteq \Xi \times \Xi$ on a set Ξ is defined by

R1 Reflective law. For $x \in \Xi,\ \ xRx$.

R2 Symmetry law. For $x, y \in \Xi$, $xRy \Rightarrow yRx$

R3 Transitive law. For $x, y, z \in \Xi$, xRy and $yRz \Rightarrow xRz$.

For a set Ξ with an equivalence relation R, we can classify elements in Ξ by R as follows:

$$R(x) = \{y|\ y \in \Xi \text{ and } yRx\ \}.$$

Then, we get the following useful result for the combinatorial enumeration.

Theorem 1.1.4 *For a finite set* Ξ *with an equivalence* R*,* $\forall x, y \in \Xi$*, if there is an bijection* ς *between* $R(x)$ *and* $R(y)$*, then the number of equivalence classes under* R

is

$$\frac{|\Xi|}{|R(x)|},$$

where x is a chosen element in Ξ.

Proof Notice that there is an bijection ς between $R(x)$ and $R(y)$ for $\forall x, y \in \Xi$. Whence, $|R(x)| = |R(y)|$. By definition, for $\forall x, y \in \Xi$, $R(x) \bigcap R(y) = \emptyset$ or $R(x) = R(y)$. Let T be a representation set of equivalence classes, i.e., choice one element in each class. Then we get that

$$\begin{aligned} |\Xi| &= \sum_{x \in T} |R(x)| \\ &= |T||R(x)|. \end{aligned}$$

Whence, we know that

$$|T| = \frac{|\Xi|}{|R(x)|}. \qquad \natural$$

1.1.3 Neutrosophic set

Let $[0,1]$ be a closed interval. For three subsets $T, I, F \subseteq [0,1]$ and $S \subseteq \Omega$, define a relation of an element $x \in \Omega$ with the subset S to be $x(T, I, F)$, i,e., the *confidence set* for $x \in S$ is T, the *indefinite set* is I and *fail set* is F. A set S with three subsets T, I, F is said to be a *neutrosophic set* ([85]). We clarify the conception of neutrosophic sets by abstract set theory as follows.

Let Ξ be a set and $A_1, A_2, \cdots, A_k \subseteq \Xi$. Define $3k$ functions $f_1^x, f_2^x, \cdots, f_k^x$ by $f_i^x : A_i \to [0,1],\ 1 \leq i \leq k$, where $x = T, I, F$. Denote by $(A_i; f_i^T, f_i^I, f_i^F)$ the subset A_i with three functions f_i^T, f_i^I, f_i^F, $1 \leq i \leq k$. Then

$$\bigcup_{i=1}^{k} (A_i; f_i^T, f_i^I, f_i^F)$$

is a union of neutrosophic sets. Some extremal cases for this union is in the following, which convince us that neutrosophic sets are a generalization of classical sets.

Case 1 $f_i^T = 1,\ f_i^I = f_i^F = 0$ for $i = 1, 2, \cdots, k$.

In this case,

$$\bigcup_{i=1}^{k}(A_i; f_i^T, f_i^I, f_i^F) = \bigcup_{i=1}^{k} A_i.$$

Case 2 $f_i^T = f_i^I = 0,\ f_i^F = 1$ for $i = 1, 2, \cdots, k$.

In this case,

$$\bigcup_{i=1}^{k}(A_i; f_i^T, f_i^I, f_i^F) = \overline{\bigcup_{i=1}^{k} A_i}.$$

Case 3 There is an integer s such that $f_i^T = 1\ f_i^I = f_i^F = 0,\ 1 \leq i \leq s$ but $f_j^T = f_j^I = 0, f_j^F = 1$ for $s+1 \leq j \leq k$.

In this case,

$$\bigcup_{i=1}^{k}(A_i, f_i) = \bigcup_{i=1}^{s} A_i \bigcup \overline{\bigcup_{i=s+1}^{k} A_i}.$$

Case 4 There is an integer l such that $f_l^T \neq 1$ or $f_l^F \neq 1$.

In this case, the union is a general neutrosophic set. It can not be represented by abstract sets.

If $A \bigcap B = \emptyset$, define the function value of a function f on the union set $A \bigcup B$ to be

$$f(A \bigcup B) = f(A) + f(B)$$

and

$$f(A \bigcap B) = f(A)f(B)$$

.
Then if $A \bigcap B \neq \emptyset$, we get that

$$f(A \bigcup B) = f(A) + f(B) - f(A)f(B).$$

Generally, by applying the Inclusion-Exclusion Principle to a union of sets, we get the following formulae.

$$f(\bigcap_{i=1}^{l} A_i) = \prod_{i=1}^{l} f(A_i),$$

$$f(\bigcup_{i=1}^{k} A_i) = \sum_{j=1}^{k} (-1)^{j-1} \prod_{s=1}^{j} f(A_s).$$

§1.2 Algebraic Structures

In this section, we recall some conceptions and results without proofs in algebra, such as, these groups, rings, fields, vectors $\cdots$, all of these can be viewed as a sole-space system.

1.2.1. Groups

A set G with a binary operation "$\circ$", denoted by $(G;\circ)$, is called a *group* if $x \circ y \in G$ for $\forall x, y \in G$ such that the following conditions hold.

(i) $(x \circ y) \circ z = x \circ (y \circ z)$ for $\forall x, y, z \in G$;

(ii) There is an element $1_G, 1_G \in G$ such that $x \circ 1_G = x$;

(iii) For $\forall x \in G$, there is an element $y, y \in G$, such that $x \circ y = 1_G$.

A group G is *abelian* if the following additional condition holds.

(iv) For $\forall x, y \in G$, $x \circ y = y \circ x$.

A set G with a binary operation "$\circ$" satisfying the condition (i) is called a *semigroup*. Similarly, if it satisfies the conditions (i) and (iv), then it is called a *abelian semigroup*.

Some examples of groups are as follows.

(1) $(R\ ;+)$ and $(R\ ;\cdot)$, where R is the set of real numbers.

(2) $(U_2;\cdot)$, where $U_2 = \{1, -1\}$ and generally, $(U_n;\cdot)$, where $U_n = \{e^{i\frac{2\pi k}{n}}, k = 1, 2, \cdots, n\}$.

(3) For a finite set X, the set $SymX$ of all permutations on X with respect to permutation composition.

The cases (1) and (2) are abelian group, but (3) is not in general.

A subset H of a group G is said to be *subgroup* if H is also a group under the same operation in G, denoted by $H \prec G$. The following results are well-known.

Theorem 1.2.1 *A non-empty subset H of a group $(G\ ;\circ)$ is a group if and only if for $\forall x,y\in H$, $x\circ y\in H$.*

Theorem 1.2.2(Lagrange theorem) *For any subgroup H of a finite group G, the order $|H|$ is a divisor of $|G|$.*

For $\forall x\in G$, denote the set $\{xh|\forall h\in H\}$ by xH and $\{hx|\forall h\in H\}$ by Hx. A subgroup H of a group $(G\ ;\circ)$ is *normal*, denoted by $H\triangleleft G$, if for $\forall x\in G$, $xH=Hx$.

For two subsets A,B of a group $(G\ ;\circ)$, define their product $A\circ B$ by

$$A\circ B=\{a\circ b|\ \forall a\in A,\ \forall b\in b\ \}.$$

For a subgroup $H, H\triangleleft G$, it can be shown that

$$(xH)\circ(yH)=(x\circ y)H \text{ and } (Hx)\circ(Hy)=H(x\circ y).$$

for $\forall x,y\in G$. Whence, the operation "$\circ$" is closed in the sets $\{xH|x\in G\}=\{Hx|x\in G\}$, denote this set by G/H. We know G/H is also a group by the facts

$$(xH\circ yH)\circ zH=xH\circ(yH\circ zH),\ \forall x,y,z\in G$$

and

$$(xH)\circ H=xH,\ (xH)\circ(x^{-1}H)=H.$$

For two groups G,G', let σ be a mapping from G to G'. If

$$\sigma(x\circ y)=\sigma(x)\circ\sigma(y),$$

for $\forall x,y\in G$, then call σ a *homomorphism* from G to G'. The *image* $Im\sigma$ and the *kernel* $Ker\sigma$ of a homomorphism $\sigma:G\to G'$ are defined as follows:

$$Im\sigma=G^{\sigma}=\{\sigma(x)|\ \forall x\in G\ \},$$

$$Ker\sigma=\{x|\ \forall x\in G,\ \sigma(x)=1_{G'}\ \}.$$

A one to one homomorphism is called a *monomorphism* and an onto homomorphism an *epimorphism*. A homomorphism is called a *bijection* if it is one to one and onto. Two groups G, G' are said to be *isomorphic* if there exists a bijective homomorphism σ between them, denoted by $G \cong G'$.

Theorem 1.2.3 *Let $\sigma : G \to G'$ be a homomorphism of group. Then*

$$(G, \circ)/Ker\sigma \cong Im\sigma.$$

1.2.2. **Rings**

A set R with two binary operations "$+$" and "$\circ$", denoted by $(R\,;+,\circ)$, is said to be a *ring* if $x + y \in R$, $x \circ y \in R$ for $\forall x, y \in R$ such that the following conditions hold.

(i) $(R\,;+)$ is an abelian group;
(ii) $(R\,;\circ)$ is a semigroup;
(iii) For $\forall x, y, z \in R$, $x \circ (y + z) = x \circ y + x \circ z$ and $(x + y) \circ z = x \circ z + y \circ z$.

Some examples of rings are as follows.

(1) $(Z\,;+,\cdot)$, where Z is the set of integers.
(2) $(pZ\,;+,\cdot)$, where p is a prime number and $pZ = \{pn | n \in Z\}$.
(3) $(\mathcal{M}_n(Z)\,;+,\cdot)$, where $\mathcal{M}_n(Z)$ is a set of $n \times n$ matrices with each entry being an integer, $n \geq 2$.

For a ring $(R\,;+,\circ)$, if $x \circ y = y \circ x$ for $\forall x, y \in R$, then it is called a *commutative ring*. The examples of (1) and (2) are commutative, but (3) is not.

If R contains an element 1_R such that for $\forall x \in R$, $x \circ 1_R = 1_R \circ x = x$, we call R a *ring with unit*. All of these examples of rings in the above are rings with unit. For (1), the unit is 1, (2) is Z and (3) is $I_{n\times n}$.

The unit of $(R\,;+)$ in a ring $(R\,;+,\circ)$ is called *zero*, denoted by 0. For $\forall a, b \in R$, if

$$a \circ b = 0,$$

then a and b are called *divisors of zero*. In some rings, such as the $(Z\,;+,\cdot)$ and $(pZ\,;+,\cdot)$, there must be a or b be 0. We call it only has a *trivial divisor of zero*. But in the ring $(pqZ\,;+,\cdot)$ with p, q both being prime, since

$$pZ \cdot qZ = 0$$

and $pZ \neq 0$, $qZ \neq 0$, we get non-zero divisors of zero, which is called to have *non-trivial divisors of zero*. The ring $(\mathcal{M}_n(Z);+,\cdot)$ also has non-trivial divisors of zero, since

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} = O_{n\times n}.$$

A *division ring* is a ring which has no non-trivial divisors of zero and an *integral domain* is a commutative ring having no non-trivial divisors of zero.

A *body* is a ring $(R\,;+,\circ)$ with a unit, $|R| \geq 2$ and $(R \setminus \{0\};\circ)$ is a group and a *field* is a commutative body. The examples (1) and (2) of rings are fields. The following result is well-known.

Theorem 1.2.4 *Any finite integral domain is a field.*

A non-empty subset R' of a ring $(R\,;+,\circ)$ is called a *subring* if $(R'\,;+,\circ)$ is also a ring. The following result for subrings can be obtained immediately by definition.

Theorem 1.2.5 *For a subset R' of a ring $(R\,;+,\circ)$, if*

(i) $(R'\,;+)$ is a subgroup of $(R\,;+)$,

(ii) R' is closed under the operation "$\circ$",

then $(R'\,;+,\circ)$ is a subring of $(R\,,+.\circ)$.

An *ideal* I of a ring $(R\,;+,\circ)$ is a non-void subset of R with properties:

(i) $(I\,;+)$ is a subgroup of $(R\,;+)$;

(ii) $a\circ x \in I$ and $x \circ a \in I$ for $\forall a \in I, \forall x \in R$.

Let $(R\,;+,\circ)$ be a ring. A chain

$$R \succ R_1 \succ \cdots \succ R_l = \{1_\circ\}$$

satisfying that R_{i+1} is an ideal of R_i for any integer $i, 1 \leq i \leq l$, is called an *ideal chain* of $(R\,,+,\circ)$. A ring whose every ideal chain only has finite terms is called an *Artin ring*. Similar to normal subgroups, consider the set $x+I$ in the group $(R\,;+)$.

Calculation shows that $R/I = \{x + I|\ x \in R\}$ is also a ring under these operations "+" and "$\circ$". Call it a *quotient ring of R to I.*

For two rings $(R\ ; +, \circ), (R'\ ; *, \bullet)$, let ι be a mapping from R to R'. If

$$\iota(x + y) = \iota(x) * \iota(y),$$

$$\iota(x \circ y) = \iota(x) \bullet \iota(y),$$

for $\forall x, y \in R$, then ι is called a *homomorphism* from $(R\ ; +, \circ)$ to $(R'\ ; *, \bullet)$. Similar to Theorem 2.3, we know that

Theorem 1.2.6 *Let $\iota : R \to R'$ be a homomorphism from $(R\ ; +, \circ)$ to $(R'\ ; *, \bullet)$. Then*

$$(R\ ; +, \circ)/Ker\iota \cong Im\iota.$$

1.2.3 **Vector spaces**

A *vector space* or *linear space* consists of the following:

(i) a field F of scalars;

(ii) a set V of objects, called vectors;

(iii) an operation, called vector addition, which associates with each pair of vectors $\mathbf{a}, \mathbf{b}$ in V a vector $\mathbf{a} + \mathbf{b}$ in V, called the sum of $\mathbf{a}$ and $\mathbf{b}$, in such a way that

(1) addition is commutative, $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$;

(2) addition is associative, $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$;

(3) there is a unique vector $\mathbf{0}$ in V, called the zero vector, such that $\mathbf{a} + \mathbf{0} = \mathbf{a}$ for all $\mathbf{a}$ in V;

(4) for each vector $\mathbf{a}$ in V there is a unique vector $-\mathbf{a}$ in V such that $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$;

(iv) an operation "$\cdot$", called scalar multiplication, which associates with each scalar k in F and a vector $\mathbf{a}$ in V a vector $k \cdot \mathbf{a}$ in V, called the product of k with $\mathbf{a}$, in such a way that

(1) $1 \cdot \mathbf{a} = \mathbf{a}$ for every $\mathbf{a}$ in V;

(2) $(k_1 k_2) \cdot \mathbf{a} = k_1(k_2 \cdot \mathbf{a})$;

(3) $k \cdot (\mathbf{a} + \mathbf{b}) = k \cdot \mathbf{a} + k \cdot \mathbf{b}$;

(4) $(k_1 + k_2) \cdot \mathbf{a} = k_1 \cdot \mathbf{a} + k_2 \cdot \mathbf{a}$.

We say that V is a *vector space over the field F, denoted by* $(V\ ;+,\cdot)$.

Some examples of vector spaces are as follows.

(1) *The n-tuple space R^n over the real number field R.* Let V be the set of all n-tuples $(x_1, x_2, \cdots, x_n)$ with $x_i \in R, 1 \leq i \leq n$. If $\forall \mathbf{a} = (x_1, x_2, \cdots, x_n)$, $\mathbf{b} = (y_1, y_2, \cdots, y_n) \in V$, then the sum of $\mathbf{a}$ and $\mathbf{b}$ is defined by

$$\mathbf{a} + \mathbf{b} = (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n).$$

The product of a real number k with $\mathbf{a}$ is defined by

$$k\mathbf{a} = (kx_1, kx_2, \cdots, kx_n).$$

(2) *The space $Q^{m\times n}$ of $m \times n$ matrices over the rational number field Q.* Let $Q^{m\times n}$ be the set of all $m \times n$ matrices over the natural number field Q. The sum of two vectors A and B in $Q^{m\times n}$ is defined by

$$(A + B)_{ij} = A_{ij} + B_{ij},$$

and the product of a rational number p with a matrix A is defined by

$$(pA)_{ij} = pA_{ij}.$$

A *subspace* W of a vector space V is a subset W of V which is itself a vector space over F with the operations of vector addition and scalar multiplication on V. The following result for subspaces is known in references [6] and [33].

Theorem 1.2.7 *A non-empty subset W of a vector space $(V\ ;+,\cdot)$ over the field F is a subspace of $(V\ ;+,\cdot)$ if and only if for each pair of vectors $\mathbf{a}, \mathbf{b}$ in W and each scalar k in F the vector $k \cdot \mathbf{a} + \mathbf{b}$ is also in W.*

Therefore, the intersection of two subspaces of a vector space V is still a subspace of V. Let U be a set of some vectors in a vector space V over F. The subspace spanned by U is defined by

$$\langle U\rangle\ = \{\ k_1 \cdot \mathbf{a_1} + k_2 \cdot \mathbf{a_2} + \cdots + k_l \cdot \mathbf{a}_l \mid l \geq 1, k_i \in F,\ \text{and } \mathbf{a_j} \in S, 1 \leq i \leq l\ \}.$$

A subset W of V is said to be linearly dependent if there exist distinct vectors $\mathbf{a_1}, \mathbf{a_2}, \cdots, \mathbf{a_n}$ in W and scalars $k_1, k_2, \cdots, k_n$ in F, not all of which are 0, such that

$$k_1 \cdot \mathbf{a_1} + k_2 \cdot \mathbf{a_2} + \cdots + k_n \cdot \mathbf{a_n} = \mathbf{0}.$$

For a vector space V, its *basis* is a linearly independent set of vectors in V which spans the space V. Call a space V *finite-dimensional* if it has a finite basis. Denoted by $dimV$ the number of elements in a basis of V.

For two subspaces U, W of a space V, the sum of subspaces U, W is defined by

$$U + W = \{\ \mathbf{u} + \mathbf{w} \mid \mathbf{u} \in U,\ \mathbf{w} \in W\ \}.$$

Then, we have results in the following ([6][33]).

Theorem 1.2.8 *Any finite-dimensional vector space V over a field F is isomorphic to one and only one space F^n, where $n = dimV$.*

Theorem 1.2.9 *If W_1 and W_2 are finite-dimensional subspaces of a vector space V, then $W_1 + W_2$ is finite-dimensional and*

$$dimW_1 + dimW_2 = dim(W_1 \bigcap W_2) + dim(W_1 + W_2).$$

§1.3 Algebraic Multi-Spaces

The notion of a multi-space was introduced by Smarandache in 1969 ([86]). Algebraic multi-spaces had be researched in references [58] – [61] and [103]. Vasantha Kandasamy researched various bispaces in [101], such as those of bigroups, bisemigroups, biquasigroups, biloops, bigroupoids, birings, bisemirings, bivectors, bisemivectors, bilnear-rings, $\cdots$, etc., considered two operation systems on two different sets.

1.3.1. Algebraic multi-spaces

Definition 1.3.1 *For any integers n, i, $n \geq 2$ and $1 \leq i \leq n$, let A_i be a set with ensemble of law L_i, and the intersection of k sets $A_{i_1}, A_{i_2}, \cdots, A_{i_k}$ of them constrains the law $I(A_{i_1}, A_{i_2}, \cdots, A_{i_k})$. Then the union $\widetilde{A}$*

$$\widetilde{A} = \bigcup_{i=1}^{n} A_i$$

is called a multi-space.

Notice that in this definition, each law may be contain more than one binary operation. For a binary operation "$\times$", if there exists an element $1_\times^l$ (or $1_\times^r$) such that

$$1_\times^l \times a = a \ \text{ or } \ a \times 1_\times^r = a$$

for $\forall a \in A_i, 1 \leq i \leq n$, then $1_\times^l$ $(1_\times^r)$ is called a *left (right) unit*. If $1_\times^l$ and $1_\times^r$ exist simultaneously, then there must be

$$1_\times^l = 1_\times^l \times 1_\times^r = 1_\times^r = 1_\times.$$

Call $1_\times$ a *unit* of A_i.

Remark 1.3.1 In Definition 1.3.1, the following three cases are permitted:

(i) $A_1 = A_2 = \cdots = A_n$, i.e., n laws on one set.

(ii) $L_1 = L_2 = \cdots = L_n$, i.e., n set with one law

(iii) there exist integers $s_1, s_2, \cdots, s_l$ such that $I(s_j) = \emptyset, 1 \leq j \leq l$, i.e., some laws on the intersections may be not existed.

We give some examples for Definition 1.3.1.

Example 1.3.1 Take n disjoint two by two cyclic groups $C_1, C_2, \cdots, C_n,\ n \geq 2$ with

$$C_1 = (\langle a\rangle\,; +_1), C_2 = (\langle b\rangle\,; +_2), \cdots, C_n = (\langle c\rangle\,; +_n).$$

Where "$+_1, +_2, \cdots, +_n$" are n binary operations. Then their union

$$\widetilde{C} = \bigcup_{i=1}^{n} C_i$$

is a multi-space with the empty intersection laws. In this multi-space, for $\forall x, y \in \widetilde{C}$, if $x, y \in C_k$ for some integer k, then we know $x +_k y \in C_k$. But if $x \in C_s$, $y \in C_t$ and $s \neq t$, then we do not know which binary operation between them and what is the resulting element corresponds to them.

A general multi-space of this kind is constructed by choosing n algebraic systems $A_1, A_2, \cdots, A_n$ satisfying that

$$A_i \bigcap A_j = \emptyset \ \text{ and } \ O(A_i) \bigcap O(A_j) = \emptyset,$$

for any integers $i, j, i \neq j, \ 1 \leq i, j \leq n$, where $O(A_i)$ denotes the binary operation set in A_i. Then

$$\widetilde{A} = \bigcup_{i=1}^{n} A_i$$

with $O(\widetilde{A}) = \bigcup\limits_{i=1}^{n} O(A_i)$ is a multi-space. This kind of multi-spaces can be seen as a model of spaces with a empty intersection.

Example 1.3.2 Let $(G\ ; \circ)$ be a group with a binary operation "$\circ$". Choose n different elements $h_1, h_2, \cdots, h_n$, $n \geq 2$ and make the extension of the group $(G\ ; \circ)$ by $h_1, h_2, \cdots, h_n$ respectively as follows:

$(G \bigcup \{h_1\}; \times_1)$, where the binary operation $\times_1 = \circ$ for elements in G, otherwise, new operation;

$(G \bigcup \{h_2\}; \times_2)$, where the binary operation $\times_2 = \circ$ for elements in G, otherwise, new operation;

$\cdots\cdots\cdots\cdots\cdots\cdots$;

$(G \bigcup \{h_n\}; \times_n)$, where the binary operation $\times_n = \circ$ for elements in G, otherwise, new operation.

Define

$$\widetilde{G} = \bigcup_{i=1}^{n} (G \bigcup \{h_i\}; \times_i).$$

Then $\widetilde{G}$ is a multi-space with binary operations "$\times_1, \times_2, \cdots, \times_n$". In this multi-space, for $\forall x, y \in \widetilde{G}$, unless the exception cases $x = h_i, y = h_j$ and $i \neq j$, we know the binary operation between x and y and the resulting element by them.

For $n = 3$, this multi-space can be shown as in Fig.1.2, in where the central circle represents the group G and each angle field the extension of G. Whence, we call this kind of multi-space a *fan multi-space*.

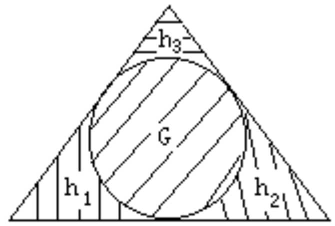


Fig.1.2

Similarly, we can also use a ring R to get fan multi-spaces. For example, let $(R\ ;+,\circ)$ be a ring and let $r_1, r_2, \cdots, r_s$ be two by two different elements. Make these extensions of $(R\ ;+,\circ)$ by $r_1, r_2, \cdots, r_s$ respectively as follows:

$(R\bigcup\{r_1\};+_1,\times_1)$, where binary operations $+_1 = +$, $\times_1 = \circ$ for elements in R, otherwise, new operation;

$(R\bigcup\{r_2\};+_2,\times_2)$, where binary operations $+_2 = +$, $\times_2 = \circ$ for elements in R, otherwise, new operation;

$\cdots\cdots\cdots\cdots\cdots\cdots;$

$(R\bigcup\{r_s\};+_s,\times_s)$, where binary operations $+_s = +$, $\times_s = \circ$ for elements in R, otherwise, new operation.

Define

$$\widetilde{R} = \bigcup_{j=1}^{s}(R\bigcup\{r_j\};+_j,\times_j).$$

Then $\widetilde{R}$ is a fan multi-space with ring-like structure. Also we can define a fan multi-space with field-like, vector-like, semigroup-like,$\cdots$, etc. structures.

These multi-spaces constructed in Examples 1.3.1 and 1.3.2 are not *completed*, i.e., there exist some elements in this space not have binary operation between them. In algebra, we wish to construct a *completed multi-space*, i.e., there is a binary operation between any two elements at least and their resulting is still in this space. The following example is a completed multi-space constructed by applying *Latin squares* in the combinatorial design.

Example 1.3.3 Let S be a finite set with $|S| = n \geq 2$. Constructing an $n\times n$ Latin square by elements in S, i.e., every element just appears one time on its each row and each column. Now choose k Latin squares $M_1, M_2, \cdots, M_k$, $k \leq \prod\limits_{s=1}^{n} s!$.

By a result in the reference [83], there are at least $\prod_{s=1}^{n} s!$ distinct $n \times n$ Latin squares. Whence, we can always choose $M_1, M_2, \cdots, M_k$ distinct two by two. For a Latin square $M_i, 1 \leq i \leq k$, define an operation "$\times_i$" as follows:

$$\times_i : (s, f) \in S \times S \rightarrow (M_i)_{sf}.$$

The case of $n = 3$ is explained in the following. Here $S = \{1, 2, 3\}$ and there are 2 Latin squares L_1, L_2 as follows:

$$L_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix} \qquad L_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}$$

.

Therefore, by the Latin square L_1, we get an operation "$\times_1$" as in table 1.3.1.

$\times_1$	1	2	3
1	1	2	3
2	2	3	1
3	3	1	2

table 1.3.1

and by the Latin square L_2, we also get an operation "$\times_2$" as in table 1.3.2.

$\times_2$	1	2	3
1	1	2	3
2	3	1	2
3	2	3	1

table 1.3.2

For $\forall x, y, z \in S$ and two operations "$\times_i$" and "$\times_j$", $1 \leq i, j \leq k$, define

$$x \times_i y \times_j z = (x \times_i y) \times_j z.$$

For example, in the case $n = 3$, we know that

$$1 \times_1 2 \times_2 3 = (1 \times_2) \times_2 3 = 2 \times_2 3 = 2;$$

and

$$2 \times_1 3 \times_2 2 = (2 \times_1 3) \times_2 2 = 1 \times_2 3 = 3.$$

Whence S is a completed multi-space with k operations.

The following example is also a completed multi-space constructed by an algebraic system.

Example 1.3.4 For constructing a completed multi-space, let $(S\,;\circ)$ be an algebraic system, i.e., $a \circ b \in S$ for $\forall a, b \in S$. Whence, we can take $C, C \subseteq S$ being a cyclic group. Now consider a partition of S

$$S \;=\; \bigcup_{k=1}^{m} G_k$$

with $m \geq 2$ such that $G_i \bigcap G_j = C$ for $\forall i, j, 1 \leq i, j \leq m$.

For an integer $k, 1 \leq k \leq m$, assume $G_k = \{g_{k1}, g_{k2}, \cdots, g_{kl}\}$. We define an operation "$\times_k$" on G_k as follows, which enables $(G_k; \times_k)$ to be a cyclic group.

$$g_{k1} \times_k g_{k1} = g_{k2},$$

$$g_{k2} \times_k g_{k1} = g_{k3},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$g_{k(l-1)} \times_k g_{k1} = g_{kl},$$

and

$$g_{kl)} \times_k g_{k1} = g_{k1}.$$

Then $S \;=\; \bigcup\limits_{k=1}^{m} G_k$ is a completed multi-space with $m + 1$ operations.

The approach used in Example 1.3.4 enables us to construct a complete multi-spaces $\widetilde{A} = \bigcup\limits_{i=1}^{n}$ with k operations for $k \geq n+1$, i.e., the intersection law $I(A_1, A_2, \cdots, A_n) \neq \emptyset$.

Definition 1.3.2 *A mapping f on a set X is called faithful if $f(x) = x$ for $\forall x \in X$, then $f = 1_X$, the unit mapping on X fixing each element in X.*

Notice that if f is faithful and $f_1(x) = f(x)$ for $\forall x \in X$, then $f_1^{-1}f = 1_X$, i.e., $f_1 = f$.

For each operation "$\times$" and a chosen element g in a subspace $A_i, A_i \subset \widetilde{A}, 1 \leq i \leq n$, there is a *left-mapping* $f_g^l : A_i \rightarrow A_i$ defined by

$$f_g^l : a \rightarrow g \times a, \quad a \in A_i.$$

Similarly, we can also define the *right-mapping* f_g^r.

We adopt the following convention for multi-spaces in this book.

Convention 1.3.1 *Each operation "$\times$" in a subset $A_i, A_i \subset \widetilde{A}, 1 \leq i \leq n$ is faithful, i.e., for $\forall g \in A_i$, $\varsigma : g \rightarrow f_g^l$ (or $\tau : g \rightarrow f_g^r$) is faithful.*

Define the kernel $Ker\varsigma$ of a mapping ς by

$$\mathrm{Ker}\varsigma = \{g | g \in A_i \text{ and } \varsigma(g) = 1_{A_i}\}.$$

Then Convention 1.3.1 is equivalent to the next convention.

Convention 1.3.2 *For each $\varsigma : g \rightarrow f_g^l$ (or $\varsigma : g \rightarrow f_g^r$) induced by an operation "$\times$" has kernel*

$$\mathrm{Ker}\varsigma = \{1_\times^l\}$$

if $1_\times^l$ exists. Otherwise, $\mathrm{Ker}\varsigma = \emptyset$.

We have the following results for multi-spaces $\widetilde{A}$.

Theorem 1.3.1 *For a multi-space $\widetilde{A}$ and an operation "$\times$", the left unit $1_\times^l$ and right unit $1_\times^r$ are unique if they exist.*

Proof If there are two left units $1_\times^l, I_\times^l$ in a subset A_i of a multi-space $\widetilde{A}$, then for $\forall x \in A_i$, their induced left-mappings $f_{1_\times^l}^l$ and $f_{I_\times^l}^l$ satisfy

$$f_{1_\times^l}^l(x) = 1_\times^l \times x = x$$

and

$$f_{I_\times^l}^l(x) = I_\times^l \times x = x.$$

Therefore, we get that $f^l_{1^l_\times} = f^l_{I^l_\times}$. Since the mappings $\varsigma_1 : 1^l_\times \to f^l_{1^l_\times}$ and $\varsigma_2 : I^l_\times \to f^l_{I^l_\times}$ are faithful, we know that

$$1^l_\times = I^l_\times.$$

Similarly, we can also prove that the right unit $1^r_\times$ is also unique. ♮

For two elements a, b of a multi-space $\widetilde{A}$, if $a \times b = 1^l_\times$, then b is called a *left-inverse* of a. If $a \times b = 1^r_\times$, then a is called a *right-inverse* of b. Certainly, if $a \times b = 1_\times$, then a is called an *inverse* of b and b an *inverse* of a.

Theorem 1.3.2 *For a multi-space $\widetilde{A}$, $a \in \widetilde{A}$, the left-inverse and right-inverse of a are unique if they exist.*

Proof Notice that $\kappa_a : x \to ax$ is faithful, i.e., $\mathrm{Ker}\kappa = \{1^l_\times\}$ for $1^l_\times$ existing now.

If there exist two left-inverses b_1, b_2 in $\widetilde{A}$ such that $a \times b_1 = 1^l_\times$ and $a \times b_2 = 1^l_\times$, then we know that

$$b_1 = b_2 = 1^l_\times.$$

Similarly, we can also prove that the right-inverse of a is also unique. ♮

Corollary 1.3.1 *If "$\times$" is an operation of a multi-space $\widetilde{A}$ with unit $1_\times$, then the equation*

$$a \times x = b$$

has at most one solution for the indeterminate x.

Proof According to Theorem 1.3.2, we know there is at most one left-inverse a_1 of a such that $a_1 \times a = 1_\times$. Whence, we know that

$$x = a_1 \times a \times x = a_1 \times b. \quad ♮$$

We also get a consequence for solutions of an equation in a multi-space by this result.

Corollary 1.3.2 *Let $\widetilde{A}$ be a multi-space with a operation set $O(\widetilde{A})$. Then the equation*

$$a \circ x = b$$

has at most $o(\widetilde{A})$ solutions, where "$\circ$" is any binary operation of $\widetilde{A}$.

Two multi-spaces $\widetilde{A_1}, \widetilde{A_2}$ are said to be *isomorphic* if there is a one to one mapping $\zeta : \widetilde{A_1} \rightarrow \widetilde{A_2}$ such that for $\forall x, y \in \widetilde{A}_1$ with binary operation "$\times$", $\zeta(x), \zeta(y)$ in $\widetilde{A_2}$ with binary operation "$\circ$" satisfying the following condition

$$\zeta(x \times y) = \zeta(x) \circ \zeta(y).$$

If $\widetilde{A_1} = \widetilde{A_2} = \widetilde{A}$, then an isomorphism between $\widetilde{A_1}$ and $\widetilde{A_2}$ is called an *automorphism* of $\widetilde{A}$. All automorphisms of $\widetilde{A}$ form a group under the composition operation between mappings, denoted by $\mathrm{Aut}\widetilde{A}$.

Notice that $\mathrm{Aut}Z_n \cong Z_n^*$, where Z_n^* is the group of reduced residue class mod$n$ under the multiply operation ([108]). It is known that $|\mathrm{Aut}Z_n| = \varphi(n)$, where $\varphi(n)$ is the Euler function. We know the automorphism group of the multi-space $\widetilde{C}$ in Example 1.3.1 is

$$\mathrm{Aut}\widetilde{C} = S_n[Z_n^*].$$

Whence, $|\mathrm{Aut}\widetilde{C}| = \varphi(n)^n n!$. For Example 1.3.3, determining its automorphism group is a more interesting problem for the combinatorial design (see also the final section in this chapter).

1.3.2 **Multi-Groups**

The conception of multi-groups is a generalization of classical algebraic structures, such as those of groups, fields, bodies, $\cdots$, etc., which is defined in the following definition.

Definition 1.3.3 *Let $\widetilde{G} = \bigcup\limits_{i=1}^{n} G_i$ be a complete multi-space with an operation set $O(\widetilde{G}) = \{\times_i, 1 \leq i \leq n\}$. If $(G_i; \times_i)$ is a group for any integer $i, 1 \leq i \leq n$ and for $\forall x, y, z \in \widetilde{G}$ and $\forall \times, \circ \in O(\widetilde{G})$, $\times \neq \circ$, there is one operation, for example the operation "$\times$" satisfying the distribution law to the operation "$\circ$" provided all of these operating results exist , i.e.,*

$$x \times (y \circ z) = (x \times y) \circ (x \times z),$$

$$(y \circ z) \times x = (y \times x) \circ (z \times x),$$

then $\widetilde{G}$ is called a multi-group.

Remark 1.3.2 The following special cases for $n = 2$ convince us that multi-groups are a generalization of groups, fields and bodies, $\cdots$, etc..

(i) If $G_1 = G_2 = \widetilde{G}$, then $\widetilde{G}$ is a body.

(ii) If $(G_1; \times_1)$ and $(G_2; \times_2)$ are commutative groups, then $\widetilde{G}$ is a field.

For a multi-group $\widetilde{G}$ and a subset $\widetilde{G_1} \subset \widetilde{G}$, if $\widetilde{G_1}$ is also a multi-group under a subset $O(\widetilde{G_1}), O(\widetilde{G_1}) \subset O(\widetilde{G})$, then $\widetilde{G}_1$ is called a *sub-multi-group* of $\widetilde{G}$, denoted by $\widetilde{G_1} \preceq \widetilde{G}$. We get a criterion for sub-multi-groups in the following.

Theorem 1.3.3 *For a multi-group $\widetilde{G} = \bigcup\limits_{i=1}^{n} G_i$ with an operation set $O(\widetilde{G}) = \{\times_i | 1 \leq i \leq n\}$, a subset $\widetilde{G_1} \subset \widetilde{G}$ is a sub-multi-group of $\widetilde{G}$ if and only if $(\widetilde{G_1} \bigcap G_k; \times_k)$ is a subgroup of $(G_k; \times_k)$ or $\widetilde{G_1} \bigcap G_k = \emptyset$ for any integer $k, 1 \leq k \leq n$.*

Proof If $\widetilde{G_1}$ is a multi-group with an operation set $O(\widetilde{G_1}) = \{\times_{i_j} | 1 \leq j \leq s\} \subset O(\widetilde{G})$, then

$$\widetilde{G_1} = \bigcup_{i=1}^{n} (\widetilde{G_1} \bigcap G_i) = \bigcup_{j=1}^{s} G'_{i_j}$$

where $G'_{i_j} \preceq G_{i_j}$ and $(G_{i_j}; \times_{i_j})$ is a group. Whence, if $\widetilde{G_1} \bigcap G_k \neq \emptyset$, then there exist an integer $l, k = i_l$ such that $\widetilde{G_1} \bigcap G_k = G'_{i_l}$, i.e., $(\widetilde{G_1} \bigcap G_k; \times_k)$ is a subgroup of $(G_k; \times_k)$.

Now if $(\widetilde{G_1} \bigcap G_k; \times_k)$ is a subgroup of $(G_k; \times_k)$ or $\widetilde{G_1} \bigcap G_k = \emptyset$ for any integer k, let N denote the index set k with $\widetilde{G_1} \bigcap G_k \neq \emptyset$, then

$$\widetilde{G_1} = \bigcup_{j \in N} (\widetilde{G_1} \bigcap G_j)$$

and $(\widetilde{G_1} \bigcap G_j, \times_j)$ is a group. Since $\widetilde{G_1} \subset \widetilde{G}$, $O(\widetilde{G_1}) \subset O(\widetilde{G})$, the associative law and distribute law are true for the $\widetilde{G_1}$. Therefore, $\widetilde{G_1}$ is a sub-multi-group of $\widetilde{G}$. $\natural$

For finite sub-multi-groups, we get a criterion as in the following.

Theorem 1.3.4 *Let $\widetilde{G}$ be a finite multi-group with an operation set $O(\widetilde{G}) = \{\times_i | 1 \leq i \leq n\}$. A subset $\widetilde{G_1}$ of $\widetilde{G}$ is a sub-multi-group under an operation subset $O(\widetilde{G_1}) \subset O(\widetilde{G})$ if and only if $(\widetilde{G_1}; \times)$ is complete for each operation "$\times$" in $O(\widetilde{G_1})$.*

Proof Notice that for a multi-group $\widetilde{G}$, its each sub-multi-group $\widetilde{G_1}$ is complete.

Now if $\widetilde{G_1}$ is a complete set under each operation "$\times_i$" in $O(\widetilde{G_1})$, we know that $(\widetilde{G_1} \bigcap G_i; \times_i)$ is a group or an empty set. Whence, we get that

$$\widetilde{G_1} = \bigcup_{i=1}^{n} (\widetilde{G_1} \bigcap G_i).$$

Therefore, $\widetilde{G_1}$ is a sub-multi-group of $\widetilde{G}$ under the operation set $O(\widetilde{G_1})$. ♮

For a sub-multi-group $\widetilde{H}$ of a multi-group $\widetilde{G}$, $g \in \widetilde{G}$, define

$$g\widetilde{H} = \{g \times h | h \in \widetilde{H}, \times \in O(\widetilde{H})\}.$$

Then for $\forall x, y \in \widetilde{G}$,

$$x\widetilde{H} \bigcap y\widetilde{H} = \emptyset \quad \text{or} \quad x\widetilde{H} = y\widetilde{H}.$$

In fact, if $x\widetilde{H} \bigcap y\widetilde{H} \neq \emptyset$, let $z \in x\widetilde{H} \bigcap y\widetilde{H}$, then there exist elements $h_1, h_2 \in \widetilde{H}$ and operations "$\times_i$" and "$\times_j$" such that

$$z = x \times_i h_1 = y \times_j h_2.$$

Since $\widetilde{H}$ is a sub-multi-group, $(\widetilde{H} \bigcap G_i; \times_i)$ is a subgroup. Whence, there exists an inverse element h_1^{-1} in $(\widetilde{H} \bigcap G_i; \times_i)$. We get that

$$x \times_i h_1 \times_i h_1^{-1} = y \times_j h_2 \times_i h_1^{-1}.$$

i.e.,

$$x = y \times_j h_2 \times_i h_1^{-1}.$$

Whence,

$$x\widetilde{H} \subseteq y\widetilde{H}.$$

Similarly, we can also get that

$$x\widetilde{H} \supseteq y\widetilde{H}.$$

Thereafter, we get that

$$x\widetilde{H} = y\widetilde{H}.$$

Denote the union of two set A and B by $A \bigoplus B$ if $A \bigcap B = \emptyset$. Then the following result is implied in the previous proof.

Theorem 1.3.5 *For any sub-multi-group $\widetilde{H}$ of a multi-group $\widetilde{G}$, there is a representation set T, $T \subset \widetilde{G}$, such that*

$$\widetilde{G} = \bigoplus_{x \in T} x\widetilde{H}.$$

For the case of finite groups, since there is only one binary operation "$\times$" and $|x\widetilde{H}| = |y\widetilde{H}|$ for any $x, y \in \widetilde{G}$, We get a consequence in the following, which is just the Lagrange theorem for finite groups.

Corollary 1.3.3(Lagrange theorem) *For any finite group G, if H is a subgroup of G, then $|H|$ is a divisor of $|G|$.*

For a multi-group $\widetilde{G}$ and $g \in \widetilde{G}$, denote all the binary operations associative with g by $\overrightarrow{O(g)}$ and the elements associative with the binary operation "$\times$" by $\widetilde{G}(\times)$. For a sub-multi-group $\widetilde{H}$ of $\widetilde{G}$, $\times \in O(\widetilde{H})$, if

$$g \times h \times g^{-1} \in \widetilde{H},$$

for $\forall h \in \widetilde{H}$ and $\forall g \in \widetilde{G}(\times)$, then we call $\widetilde{H}$ a *normal sub-multi-group* of $\widetilde{G}$, denoted by $\widetilde{H} \lhd \widetilde{G}$. If $\widetilde{H}$ is a normal sub-multi-group of $\widetilde{G}$, similar to the normal subgroups of groups, it can be shown that $g \times \widetilde{H} = \widetilde{H} \times g$, where $g \in \widetilde{G}(\times)$. Thereby we get a result as in the following.

Theorem 1.3.6 *Let $\widetilde{G} = \bigcup\limits_{i=1}^{n} G_i$ be a multi-group with an operation set $O(\widetilde{G}) = \{\times_i | 1 \leq i \leq n\}$. Then a sub-multi-group $\widetilde{H}$ of $\widetilde{G}$ is normal if and only if $(\widetilde{H} \bigcap G_i; \times_i)$ is a normal subgroup of $(G_i; \times_i)$ or $\widetilde{H} \bigcap G_i = \emptyset$ for any integer $i, 1 \leq i \leq n$.*

Proof We have known that

$$\widetilde{H}=\bigcup_{i=1}^{n}(\widetilde{H}\bigcap G_i).$$

If $(\widetilde{H}\bigcap G_i;\times_i)$ is a normal subgroup of $(G_i;\times_i)$ for any integer $i, 1\leq i\leq n$, then we know that

$$g\times_i(\widetilde{H}\bigcap G_i)\times_i g^{-1}=\widetilde{H}\bigcap G_i$$

for $\forall g\in G_i, 1\leq i\leq n$. Whence,

$$g\circ\widetilde{H}\circ g^{-1}=\widetilde{H}$$

for $\forall\circ\in O(\widetilde{H})$ and $\forall g\in\overrightarrow{\widetilde{G}(\circ)}$. That is, $\widetilde{H}$ is a normal sub-multi-group of $\widetilde{G}$.

Now if $\widetilde{H}$ is a normal sub-multi-group of $\widetilde{G}$, by definition we know that

$$g\circ\widetilde{H}\circ g^{-1}=\widetilde{H}$$

for $\forall\circ\in O(\widetilde{H})$ and $\forall g\in\widetilde{G}(\circ)$. Not loss of generality, we assume that $\circ=\times_k$, then we get

$$g\times_k(\widetilde{H}\bigcap G_k)\times_k g^{-1}=\widetilde{H}\bigcap G_k.$$

Therefore, $(\widetilde{H}\bigcap G_k;\times_k)$ is a normal subgroup of $(G_k,\times_k)$. Since the operation "$\circ$" is chosen arbitrarily, we know that $(\widetilde{H}\bigcap G_i;\times_i)$ is a normal subgroup of $(G_i;\times_i)$ or an empty set for any integer i, $1\leq i\leq n$. ♮

For a multi-group $\widetilde{G}$ with an operation set $O(\widetilde{G})=\{\times_i|\ 1\leq i\leq n\}$, an order of operations in $O(\widetilde{G})$ is said to be an *oriented operation sequence*, denoted by $\overrightarrow{O}(\widetilde{G})$. For example, if $O(\widetilde{G})=\{\times_1,\times_2\times_3\}$, then $\times_1\succ\times_2\succ\times_3$ is an oriented operation sequence and $\times_2\succ\times_1\succ\times_3$ is also an oriented operation sequence.

For a given oriented operation sequence $\overrightarrow{O}(\widetilde{G})$, we construct a series of normal sub-multi-group

$$\widetilde{G}\rhd\widetilde{G}_1\rhd\widetilde{G}_2\rhd\cdots\rhd\widetilde{G}_m=\{1_{\times_n}\}$$

by the following programming.

STEP 1: *Construct a series*

$$\widetilde{G} \rhd \widetilde{G}_{11} \rhd \widetilde{G}_{12} \rhd \cdots \rhd \widetilde{G}_{1l_1}$$

under the operation "$\times_1$".

STEP 2: *If a series*

$$\widetilde{G}_{(k-1)l_1} \rhd \widetilde{G}_{k1} \rhd \widetilde{G}_{k2} \rhd \cdots \rhd \widetilde{G}_{kl_k}$$

has be constructed under the operation "$\times_k$" and $\widetilde{G}_{kl_k} \neq \{1_{\times_n}\}$, then construct a series

$$\widetilde{G}_{kl_1} \rhd \widetilde{G}_{(k+1)1} \rhd \widetilde{G}_{(k+1)2} \rhd \cdots \rhd \widetilde{G}_{(k+1)l_{k+1}}$$

under the operation "$\times_{k+1}$".

This programming is terminated until the series

$$\widetilde{G}_{(n-1)l_1} \rhd \widetilde{G}_{n1} \rhd \widetilde{G}_{n2} \rhd \cdots \rhd \widetilde{G}_{nl_n} = \{1_{\times_n}\}$$

has be constructed under the operation "$\times_n$".

The number m is called the *length of the series of normal sub-multi-groups*. Call a series of normal sub-multi-group

$$\widetilde{G} \rhd \widetilde{G}_1 \rhd \widetilde{G}_2 \rhd \cdots \rhd \widetilde{G}_n = \{1_{\times_n}\}$$

maximal if there exists a normal sub-multi-group $\widetilde{H}$ for any integer $k, s, 1 \leq k \leq n, 1 \leq s \leq l_k$ such that

$$\widetilde{G}_{ks} \rhd \widetilde{H} \rhd \widetilde{G}_{k(s+1)},$$

then $\widetilde{H} = \widetilde{G}_{ks}$ or $\widetilde{H} = \widetilde{G}_{k(s+1)}$. For a maximal series of finite normal sub-multi-group, we get a result as in the following.

Theorem 1.3.7 *For a finite multi-group $\widetilde{G} = \bigcup\limits_{i=1}^{n} G_i$ and an oriented operation sequence $\overrightarrow{O}(\widetilde{G})$, the length of the maximal series of normal sub-multi-group in $\widetilde{G}$ is a constant, only dependent on $\widetilde{G}$ itself.*

Proof The proof is by the induction principle on the integer n.

For $n = 1$, the maximal series of normal sub-multi-groups of $\widetilde{G}$ is just a composition series of a finite group. By the Jordan-Hölder theorem (see [73] or [107]), we know the length of a composition series is a constant, only dependent on $\widetilde{G}$. Whence, the assertion is true in the case of $n = 1$.

Assume that the assertion is true for all cases of $n \leq k$. We prove it is also true in the case of $n = k + 1$. Not loss of generality, assume the order of those binary operations in $\overrightarrow{O}(\widetilde{G})$ being $\times_1 \succ \times_2 \succ \cdots \succ \times_n$ and the composition series of the group $(G_1, \times_1)$ being

$$G_1 \triangleright G_2 \triangleright \cdots \triangleright G_s = \{1_{\times_1}\}.$$

By the Jordan-Hölder theorem, we know the length of this composition series is a constant, dependent only on $(G_1; \times_1)$. According to Theorem 3.6, we know a maximal series of normal sub-multi-groups of $\widetilde{G}$ gotten by STEP 1 under the operation "$\times_1$" is

$$\widetilde{G} \triangleright \widetilde{G} \setminus (G_1 \setminus G_2) \triangleright \widetilde{G} \setminus (G_1 \setminus G_3) \triangleright \cdots \triangleright \widetilde{G} \setminus (G_1 \setminus \{1_{\times_1}\}).$$

Notice that $\widetilde{G} \setminus (G_1 \setminus \{1_{\times_1}\})$ is still a multi-group with less or equal to k operations. By the induction assumption, we know the length of the maximal series of normal sub-multi-groups in $\widetilde{G} \setminus (G_1 \setminus \{1_{\times_1}\})$ is a constant only dependent on $\widetilde{G} \setminus (G_1 \setminus \{1_{\times_1}\})$. Therefore, the length of a maximal series of normal sub-multi-groups is also a constant, only dependent on $\widetilde{G}$.

Applying the induction principle, we know that the length of a maximal series of normal sub-multi-groups of $\widetilde{G}$ is a constant under an oriented operations $\overrightarrow{O}(\widetilde{G})$, only dependent on $\widetilde{G}$ itself. $\natural$

As a special case of Theorem 1.3.7, we get a consequence in the following.

Corollary 1.3.4(Jordan-Hölder theorem) *For a finite group G, the length of its composition series is a constant, only dependent on G.*

Certainly, we can also find other characteristics for multi-groups similar to group theory, such as those to establish the decomposition theory for multi-groups similar to the decomposition theory of abelian groups, to characterize finite generated multi-groups, $\cdots$, etc.. More observations can be seen in the finial section of this chapter.

1.3.3 Multi-Rings

Definition 1.3.4 *Let* $\widetilde{R} = \bigcup\limits_{i=1}^{m} R_i$ *be a complete multi-space with a double operation set* $O(\widetilde{R}) = \{(+_i, \times_i), 1 \leq i \leq m\}$. *If for any integers* $i, j,\ i \neq j, 1 \leq i, j \leq m$, $(R_i; +_i, \times_i)$ *is a ring and*

$$(x +_i y) +_j z = x +_i (y +_j z), \quad (x \times_i y) \times_j z = x \times_i (y \times_j z)$$

for $\forall x, y, z \in \widetilde{R}$ *and*

$$x \times_i (y +_j z) = x \times_i y +_j x \times_i z, \quad (y +_j z) \times_i x = y \times_i x +_j z \times_i x$$

if all of these operating results exist, then $\widetilde{R}$ *is called a multi-ring. If* $(R; +_i, \times_i)$ *is a field for any integer* $1 \leq i \leq m$, *then* $\widetilde{R}$ *is called a multi-field.*

For a multi-ring $\widetilde{R} = \bigcup\limits_{i=1}^{m} R_i$, let $\widetilde{S} \subset \widetilde{R}$ and $O(\widetilde{S}) \subset O(\widetilde{R})$, if $\widetilde{S}$ is also a multi-ring with a double operation set $O(\widetilde{S})$, then we call $\widetilde{S}$ a *sub-multi-ring* of $\widetilde{R}$. We get a criterion for sub-multi-rings in the following.

Theorem 1.3.8 *For a multi-ring* $\widetilde{R} = \bigcup\limits_{i=1}^{m} R_i$, *a subset* $\widetilde{S} \subset \widetilde{R}$ *with* $O(\widetilde{S}) \subset O(\widetilde{R})$ *is a sub-multi-ring of* $\widetilde{R}$ *if and only if* $(\widetilde{S} \bigcap R_k; +_k, \times_k)$ *is a subring of* $(R_k; +_k, \times_k)$ *or* $\widetilde{S} \bigcap R_k = \emptyset$ *for any integer* $k, 1 \leq k \leq m$.

Proof For any integer $k, 1 \leq k \leq m$, if $(\widetilde{S} \bigcap R_k; +_k, \times_k)$ is a subring of $(R_k; +_k, \times_k)$ or $\widetilde{S} \bigcap R_k = \emptyset$, then since $\widetilde{S} = \bigcup\limits_{i=1}^{m} (\widetilde{S} \bigcap R_i)$, we know that $\widetilde{S}$ is a sub-multi-ring by the definition of a sub-multi-ring.

Now if $\widetilde{S} = \bigcup\limits_{j=1}^{s} S_{i_j}$ is a sub-multi-ring of $\widetilde{R}$ with a double operation set $O(\widetilde{S}) = \{(+_{i_j}, \times_{i_j}), 1 \leq j \leq s\}$, then $(S_{i_j}; +_{i_j}, \times_{i_j})$ is a subring of $(R_{i_j}; +_{i_j}, \times_{i_j})$. Therefore, $S_{i_j} = R_{i_j} \bigcap \widetilde{S}$ for any integer $j, 1 \leq j \leq s$. But $\widetilde{S} \bigcap S_l = \emptyset$ for other integer $l \in \{i; 1 \leq i \leq m\} \setminus \{i_j; 1 \leq j \leq s\}$. $\natural$

Applying these criterions for subrings of a ring, we get a result in the following.

Theorem 1.3.9 *For a multi-ring* $\widetilde{R} = \bigcup\limits_{i=1}^{m} R_i$, *a subset* $\widetilde{S} \subset \widetilde{R}$ *with* $O(\widetilde{S}) \subset O(\widetilde{R})$ *is a sub-multi-ring of* $\widetilde{R}$ *if and only if* $(\widetilde{S} \bigcap R_j; +_j) \prec (R_j; +_j)$ *and* $(\widetilde{S}; \times_j)$ *is complete for any double operation* $(+_j, \times_j) \in O(\widetilde{S})$.

Proof According to Theorem 1.3.8, we know that $\widetilde{S}$ is a sub-multi-ring if and only if $(\widetilde{S}\bigcap R_i;+_i,\times_i)$ is a subring of $(R_i;+_i,\times_i)$ or $\widetilde{S}\bigcap R_i=\emptyset$ for any integer $i, 1\leq i\leq m$. By a well known criterion for subrings of a ring (see also [73]), we know that $(\widetilde{S}\bigcap R_i;+_i,\times_i)$ is a subring of $(R_i;+_i,\times_i)$ if and only if $(\widetilde{S}\bigcap R_j;+_j)\prec(R_j;+_j)$ and $(\widetilde{S};\times_j)$ is a complete set for any double operation $(+_j,\times_j)\in O(\widetilde{S})$. This completes the proof. ♮

We use multi-ideal chains of a multi-ring to characteristic its structure properties. A *multi-ideal* $\widetilde{I}$ of a multi-ring $\widetilde{R}=\bigcup\limits_{i=1}^{m}R_i$ with a double operation set $O(\widetilde{R})$ is a sub-multi-ring of $\widetilde{R}$ satisfying the following conditions:

(i) $\widetilde{I}$ is a sub-multi-group with an operation set $\{+|\ (+,\times)\in O(\widetilde{I})\}$;

(ii) for any $r\in\widetilde{R}, a\in\widetilde{I}$ and $(+,\times)\in O(\widetilde{I})$, $r\times a\in\widetilde{I}$ and $a\times r\in\widetilde{I}$ if all of these operating results exist.

Theorem 1.3.10 *A subset $\widetilde{I}$ with $O(\widetilde{I}),O(\widetilde{I})\subset O(\widetilde{R})$ of a multi-ring $\widetilde{R}=\bigcup\limits_{i=1}^{m}R_i$ with a double operation set $O(\widetilde{R})=\{(+_i,\times_i)|\ 1\leq i\leq m\}$ is a multi-ideal if and only if $(\widetilde{I}\bigcap R_i,+_i,\times_i)$ is an ideal of the ring $(R_i,+_i,\times_i)$ or $\widetilde{I}\bigcap R_i=\emptyset$ for any integer $i,1\leq i\leq m$.*

Proof By the definition of a multi-ideal, the necessity of these conditions is obvious.

For the sufficiency, denote by $\widetilde{R}(+,\times)$ the set of elements in $\widetilde{R}$ with binary operations "$+$" and "$\times$". If there exists an integer i such that $\widetilde{I}\bigcap R_i\neq\emptyset$ and $(\widetilde{I}\bigcap R_i,+_i,\times_i)$ is an ideal of $(R_i,+_i,\times_i)$, then for $\forall a\in\widetilde{I}\bigcap R_i$, $\forall r_i\in R_i$, we know that

$$r_i\times_i a\in\widetilde{I}\bigcap R_i;\quad a\times_i r_i\in\widetilde{I}\bigcap R_i.$$

Notice that $\widetilde{R}(+_i,\times_i)=R_i$. Thereafter, we get that

$$r\times_i a\in\widetilde{I}\bigcap R_i\ \text{ and }\ a\times_i r\in\widetilde{I}\bigcap R_i,$$

for $\forall r\in\widetilde{R}$ if all of these operating results exist. Whence, $\widetilde{I}$ is a multi-ideal of $\widetilde{R}$. ♮

A multi-ideal $\widetilde{I}$ of a multi-ring $\widetilde{R}$ is said to be *maximal* if for any multi-ideal $\widetilde{I}'$, $\widetilde{R}\supseteq\widetilde{I}'\supseteq\widetilde{I}$ implies that $\widetilde{I}'=\widetilde{R}$ or $\widetilde{I}'=\widetilde{I}$. For an order of the double operations in the set $O(\widetilde{R})$ of a multi-ring $\widetilde{R} = \bigcup\limits_{i=1}^{m} R_i$, not loss of generality, let the order be $(+_1, \times_1) \succ (+_2, \times_2) \succ \cdots \succ (+_m, \times_m)$, we can define a *multi-ideal chain* of $\widetilde{R}$ by the following programming.

(i) Construct a multi-ideal chain

$$\widetilde{R} \supset \widetilde{R}_{11} \supset \widetilde{R}_{12} \supset \cdots \supset \widetilde{R}_{1s_1}$$

under the double operation $(+_1, \times_1)$, where $\widetilde{R}_{11}$ is a maximal multi-ideal of $\widetilde{R}$ and in general, $\widetilde{R}_{1(i+1)}$ is a maximal multi-ideal of $\widetilde{R}_{1i}$ for any integer $i,\ 1 \leq i \leq m-1$.

(ii) If a multi-ideal chain

$$\widetilde{R} \supset \widetilde{R}_{11} \supset \widetilde{R}_{12} \supset \cdots \supset \widetilde{R}_{1s_1} \supset \cdots \supset \widetilde{R}_{i1} \supset \cdots \supset \widetilde{R}_{is_i}$$

has been constructed for $(+_1, \times_1) \succ (+_2, \times_2) \succ \cdots \succ (+_i, \times_i)$, $1 \leq i \leq m-1$, then construct a multi-ideal chain of $\widetilde{R}_{is_i}$

$$\widetilde{R}_{is_i} \supset \widetilde{R}_{(i+1)1} \supset \widetilde{R}_{(i+1)2} \supset \cdots \supset \widetilde{R}_{(i+1)s_1}$$

under the double operation $(+_{i+1}, \times_{i+1})$, where $\widetilde{R}_{(i+1)1}$ is a maximal multi-ideal of $\widetilde{R}_{is_i}$ and in general, $\widetilde{R}_{(i+1)(i+1)}$ is a maximal multi-ideal of $\widetilde{R}_{(i+1)j}$ for any integer $j, 1 \leq j \leq s_i - 1$. Define a multi-ideal chain of $\widetilde{R}$ under $(+_1, \times_1) \succ (+_2, \times_2) \succ \cdots \succ (+_{i+1}, \times_{i+1})$ to be

$$\widetilde{R} \supset \widetilde{R}_{11} \supset \cdots \supset \widetilde{R}_{1s_1} \supset \cdots \supset \widetilde{R}_{i1} \supset \cdots \supset \widetilde{R}_{is_i} \supset \widetilde{R}_{(i+1)1} \supset \cdots \supset \widetilde{R}_{(i+1)s_{i+1}}.$$

Similar to multi-groups, we get a result for multi-ideal chains of a multi-ring in the following.

Theorem 1.3.11 *For a multi-ring $\widetilde{R} = \bigcup\limits_{i=1}^{m} R_i$, its multi-ideal chain only has finite terms if and only if the ideal chain of the ring $(R_i; +_i, \times_i)$ has finite terms, i.e., each ring $(R_i; +_i, \times_i)$ is an Artin ring for any integer $i, 1 \leq i \leq m$.*

Proof Let the order of these double operations in $\overrightarrow{O}(\widetilde{R})$ be

$$(+_1, \times_1) \succ (+_2, \times_2) \succ \cdots \succ (+_m, \times_m)$$

and let a maximal ideal chain in the ring $(R_1; +_1, \times_1)$ be

$$R_1 \succ R_{11} \succ \cdots \succ R_{1t_1}.$$

Calculate

$$\widetilde{R}_{11} = \widetilde{R} \setminus \{R_1 \setminus R_{11}\} = R_{11} \bigcup (\bigcup_{i=2}^{m} R_i),$$

$$\widetilde{R}_{12} = \widetilde{R}_{11} \setminus \{R_{11} \setminus R_{12}\} = R_{12} \bigcup (\bigcup_{i=2}^{m} R_i),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\widetilde{R}_{1t_1} = \widetilde{R}_{1t_1} \setminus \{R_{1(t_1-1)} \setminus R_{1t_1}\} = R_{1t_1} \bigcup (\bigcup_{i=2}^{m} R_i).$$

According to Theorem 1.3.10, we know that

$$\widetilde{R} \supset \widetilde{R}_{11} \supset \widetilde{R}_{12} \supset \cdots \supset \widetilde{R}_{1t_1}$$

is a maximal multi-ideal chain of $\widetilde{R}$ under the double operation $(+_1, \times_1)$. In general, for any integer $i, 1 \leq i \leq m-1$, assume

$$R_i \succ R_{i1} \succ \cdots \succ R_{it_i}$$

is a maximal ideal chain in the ring $(R_{(i-1)t_{i-1}}; +_i, \times_i)$. Calculate

$$\widetilde{R}_{ik} = R_{ik} \bigcup (\bigcup_{j=i+1}^{m} \widetilde{R}_{ik} \bigcap R_i).$$

Then we know that

$$\widetilde{R}_{(i-1)t_{i-1}} \supset \widetilde{R}_{i1} \supset \widetilde{R}_{i2} \supset \cdots \supset \widetilde{R}_{it_i}$$

is a maximal multi-ideal chain of $\widetilde{R}_{(i-1)t_{i-1}}$ under the double operation $(+_i, \times_i)$ by Theorem 3.10. Whence, if the ideal chain of the ring $(R_i; +_i, \times_i)$ has finite terms for any integer $i, 1 \leq i \leq m$, then the multi-ideal chain of the multi-ring $\widetilde{R}$ only has finite terms. Now if there exists an integer i_0 such that the ideal chain of the ring $(R_{i_0}, +_{i_0}, \times_{i_0})$ has infinite terms, then there must also be infinite terms in a multi-ideal chain of the multi-ring $\widetilde{R}$. $\natural$.

A multi-ring is called an *Artin multi-ring* if its each multi-ideal chain only has finite terms. We get a consequence by Theorem 1.3.11.

Corollary 1.3.5 *A multi-ring* $\widetilde{R} = \bigcup\limits_{i=1}^{m} R_i$ *with a double operation set* $O(\widetilde{R}) = \{(+_i, \times_i)|\ 1 \leq i \leq m\}$ *is an Artin multi-ring if and only if the ring* $(R_i; +_i, \times_i)$ *is an Artin ring for any integer* $i, 1 \leq i \leq m$.

For a multi-ring $\widetilde{R} = \bigcup\limits_{i=1}^{m} R_i$ with a double operation set $O(\widetilde{R}) = \{(+_i, \times_i)|\ 1 \leq i \leq m\}$, an element e is an *idempotent* element if $e_\times^2 = e \times e = e$ for a double binary operation $(+, \times) \in O(\widetilde{R})$. We define the *directed sum* $\widetilde{I}$ of two multi-ideals $\widetilde{I}_1$ and $\widetilde{I}_2$ as follows:

(i) $\widetilde{I} = \widetilde{I}_1 \bigcup \widetilde{I}_2$;

(ii) $\widetilde{I}_1 \bigcap \widetilde{I}_2 = \{0_+\}$, or $\widetilde{I}_1 \bigcap \widetilde{I}_2 = \emptyset$, where 0_+ denotes an unit element under the operation $+$.

Denote the directed sum of $\widetilde{I}_1$ and $\widetilde{I}_2$ by

$$\widetilde{I} = \widetilde{I}_1 \bigoplus \widetilde{I}_2.$$

If $\widetilde{I} = \widetilde{I}_1 \bigoplus \widetilde{I}_2$ for any $\widetilde{I}_1, \widetilde{I}_2$ implies that $\widetilde{I}_1 = \widetilde{I}$ or $\widetilde{I}_2 = \widetilde{I}$, then $\widetilde{I}$ is called *non-reducible*. We get the following result for Artin multi-rings similar to a well-known result for Artin rings (see [107] for details).

Theorem 1.3.12 *Any Artin multi-ring* $\widetilde{R} = \bigcup\limits_{i=1}^{m} R_i$ *with a double operation set* $O(\widetilde{R}) = \{(+_i, \times_i)|\ 1 \leq i \leq m\}$ *is a directed sum of finite non-reducible multi-ideals, and if* $(R_i; +_i, \times_i)$ *has unit* $1_{\times_i}$ *for any integer* $i, 1 \leq i \leq m$, *then*

$$\widetilde{R} = \bigoplus_{i=1}^{m} \Big(\bigoplus_{j=1}^{s_i} (R_i \times_i e_{ij}) \bigcup (e_{ij} \times_i R_i)\Big),$$

where $e_{ij}, 1 \leq j \leq s_i$ *are orthogonal idempotent elements of the ring* R_i.

Proof Denote by $\widetilde{M}$ the set of multi-ideals which can not be represented by a directed sum of finite multi-ideals in $\widetilde{R}$. According to Theorem 3.11, there is a minimal multi-ideal $\widetilde{I}_0$ in $\widetilde{M}$. It is obvious that $\widetilde{I}_0$ is reducible.

Assume that $\widetilde{I}_0 = \widetilde{I}_1 + \widetilde{I}_2$. Then $\widetilde{I}_1 \notin \widetilde{M}$ and $\widetilde{I}_2 \notin \widetilde{M}$. Therefore, $\widetilde{I}_1$ and $\widetilde{I}_2$ can be represented by a directed sum of finite multi-ideals. Thereby $\widetilde{I}_0$ can be also represented by a directed sum of finite multi-ideals. Contradicts that $\widetilde{I}_0 \in \widetilde{M}$.

Now let

$$\widetilde{R} = \bigoplus_{i=1}^{s} \widetilde{I}_i,$$

where each $\widetilde{I}_i, 1 \leq i \leq s$ is non-reducible. Notice that for a double operation $(+, \times)$, each non-reducible multi-ideal of $\widetilde{R}$ has the form

$$(e \times R(\times)) \bigcup (R(\times) \times e), \quad e \in R(\times).$$

Whence, we know that there is a set $T \subset \widetilde{R}$ such that

$$\widetilde{R} = \bigoplus_{e \in T,\ \times \in O(\widetilde{R})} (e \times R(\times)) \bigcup (R(\times) \times e).$$

For any operation $\times \in O(\widetilde{R})$ and the unit $1_\times$, assume that

$$1_\times = e_1 \oplus e_2 \oplus \cdots \oplus e_l,\ e_i \in T,\ 1 \leq i \leq s.$$

Then

$$e_i \times 1_\times = (e_i \times e_1) \oplus (e_i \times e_2) \oplus \cdots \oplus (e_i \times e_l).$$

Therefore, we get that

$$e_i = e_i \times e_i = e_i^2 \text{ and } e_i \times e_j = 0_i \text{ for } i \neq j.$$

That is, $e_i, 1 \leq i \leq l$ are orthogonal idempotent elements of $\widetilde{R}(\times)$. Notice that $\widetilde{R}(\times) = R_h$ for some integer h. We know that $e_i, 1 \leq i \leq l$ are orthogonal idempotent elements of the ring $(R_h, +_h, \times_h)$. Denote by e_{hi} for e_i, $1 \leq i \leq l$. Consider all units in $\widetilde{R}$, we get that

$$\widetilde{R} = \bigoplus_{i=1}^{m} (\bigoplus_{j=1}^{s_i} (R_i \times_i e_{ij}) \bigcup (e_{ij} \times_i R_i)).$$

This completes the proof. $\natural$

Corollary 1.3.6 *Any Artin ring $(R\ ;+,\times)$ is a directed sum of finite ideals, and if $(R\ ;+,\times)$ has unit $1_\times$, then*

$$R=\bigoplus_{i=1}^{s}R_ie_i,$$

where $e_i, 1\leq i\leq s$ are orthogonal idempotent elements of the ring $(R;+,\times)$.

Similarly, we can also define *Noether multi-rings, simple multi-rings, half-simple multi-rings*, $\cdots$, etc. and find their algebraic structures.

1.3.4 **Multi-Vector spaces**

Definition 1.3.5 *Let $\widetilde{V}=\bigcup\limits_{i=1}^{k}V_i$ be a complete multi-space with an operation set $O(\widetilde{V})=\{(\dot{+}_i,\cdot_i)\mid 1\leq i\leq m\}$ and let $\widetilde{F}=\bigcup\limits_{i=1}^{k}F_i$ be a multi-filed with a double operation set $O(\widetilde{F})=\{(+_i,\times_i)\mid 1\leq i\leq k\}$. If for any integers $i,j,\ 1\leq i,j\leq k$ and $\forall\mathbf{a},\mathbf{b},\mathbf{c}\in\widetilde{V},\ k_1,k_2\in\widetilde{F}$,*

(i) $(V_i;\dot{+}_i,\cdot_i)$ is a vector space on F_i with vector additive "$\dot{+}_i$" and scalar multiplication "$\cdot_i$";

(ii) $(\mathbf{a}\dot{+}_i\mathbf{b})\dot{+}_j\mathbf{c}=\mathbf{a}\dot{+}_i(\mathbf{b}\dot{+}_j\mathbf{c})$;

(iii) $(k_1+_i k_2)\cdot_j\mathbf{a}=k_1+_i(k_2\cdot_j\mathbf{a})$;

provided these operating results exist, then $\widetilde{V}$ is called a multi-vector space on the multi-filed space $\widetilde{F}$ with an double operation set $O(\widetilde{V})$, denoted by $(\widetilde{V};\widetilde{F})$.

For subsets $\widetilde{V}_1\subset\widetilde{V}$ and $\widetilde{F}_1\subset\widetilde{F}$, if $(\widetilde{V}_1;\widetilde{F}_1)$ is also a multi-vector space, then we call $(\widetilde{V}_1;\widetilde{F}_1)$ a *multi-vector subspace* of $(\widetilde{V};\widetilde{F})$. Similar to the linear space theory, we get the following criterion for multi-vector subspaces.

Theorem 1.3.13 *For a multi-vector space $(\widetilde{V};\widetilde{F})$, $\widetilde{V}_1\subset\widetilde{V}$ and $\widetilde{F}_1\subset\widetilde{F}$, $(\widetilde{V}_1;\widetilde{F}_1)$ is a multi-vector subspace of $(\widetilde{V};\widetilde{F})$ if and only if for any vector additive "$\dot{+}$", scalar multiplication "$\cdot$" in $(\widetilde{V}_1;\widetilde{F}_1)$ and $\forall\mathbf{a},\mathbf{b}\in\widetilde{V}$, $\forall\alpha\in\widetilde{F}$,*

$$\alpha\cdot\mathbf{a}\dot{+}\mathbf{b}\in\widetilde{V}_1$$

provided these operating results exist.

Proof Denote by $\widetilde{V}=\bigcup\limits_{i=1}^{k}V_i$, $\widetilde{F}=\bigcup\limits_{i=1}^{k}F_i$. Notice that $\widetilde{V}_1=\bigcup\limits_{i=1}^{k}(\widetilde{V}_1\bigcap V_i)$. By definition, we know that $(\widetilde{V}_1;\widetilde{F}_1)$ is a multi-vector subspace of $(\widetilde{V};\widetilde{F})$ if and only if for any integer $i, 1 \leq i \leq k$, $(\widetilde{V}_1 \bigcap V_i; \dot{+}_i, \cdot_i)$ is a vector subspace of $(V_i, \dot{+}_i, \cdot_i)$ and $\widetilde{F}_1$ is a multi-filed subspace of $\widetilde{F}$ or $\widetilde{V}_1 \bigcap V_i = \emptyset$.

According to a criterion for linear subspaces of a linear space ([33]), we know that $(\widetilde{V}_1 \bigcap V_i; \dot{+}_i, \cdot_i)$ is a vector subspace of $(V_i, \dot{+}_i, \cdot_i)$ for any integer $i, 1 \leq i \leq k$ if and only if for $\forall \mathbf{a}, \mathbf{b} \in \widetilde{V}_1 \bigcap V_i$, $\alpha \in F_i$,

$$\alpha \cdot_i \mathbf{a} \dot{+}_i \mathbf{b} \in \widetilde{V}_1 \bigcap V_i.$$

That is, for any vector additive "$\dot{+}$", scalar multiplication "$\cdot$" in $(\widetilde{V}_1;\widetilde{F}_1)$ and $\forall \mathbf{a}, \mathbf{b} \in \widetilde{V}$, $\forall \alpha \in \widetilde{F}$, if $\alpha \cdot \mathbf{a} \dot{+} \mathbf{b}$ exists, then $\alpha \cdot \mathbf{a} \dot{+} \mathbf{b} \in \widetilde{V}_1$. $\natural$

Corollary 1.3.7 *Let $(\widetilde{U};\widetilde{F}_1), (\widetilde{W};\widetilde{F}_2)$ be two multi-vector subspaces of a multi-vector space $(\widetilde{V};\widetilde{F})$. Then $(\widetilde{U} \bigcap \widetilde{W}; \widetilde{F}_1 \bigcap \widetilde{F}_2)$ is a multi-vector space.*

For a multi-vector space $(\widetilde{V};\widetilde{F})$, vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n \in \widetilde{V}$, if there are scalars $\alpha_1, \alpha_2, \cdots, \alpha_n \in \widetilde{F}$ such that

$$\alpha_1 \cdot_1 \mathbf{a}_1 \dot{+}_1 \alpha_2 \cdot_2 \mathbf{a}_2 \dot{+}_2 \cdots \dot{+}_{n-1} \alpha_n \cdot_n \mathbf{a}_n = \mathbf{0},$$

where $\mathbf{0} \in \widetilde{V}$ is a unit under an operation "$+$" in $\widetilde{V}$ and $\dot{+}_i, \cdot_i \in O(\widetilde{V})$, then these vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$ are said to be *linearly dependent*. Otherwise, $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$ are said to be *linearly independent*.

Notice that there are two cases for linearly independent vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$ in a multi-vector space:

(i) for scalars $\alpha_1, \alpha_2, \cdots, \alpha_n \in \widetilde{F}$, if

$$\alpha_1 \cdot_1 \mathbf{a}_1 \dot{+}_1 \alpha_2 \cdot_2 \mathbf{a}_2 \dot{+}_2 \cdots \dot{+}_{n-1} \alpha_n \cdot_n \mathbf{a}_n = \mathbf{0},$$

where $\mathbf{0}$ is a unit of $\widetilde{V}$ under an operation "$+$" in $O(\widetilde{V})$, then $\alpha_1 = 0_{+_1}, \alpha_2 = 0_{+_2}, \cdots, \alpha_n = 0_{+_n}$, where 0_{+_i} is the unit under the operation "$+_i$" in $\widetilde{F}$ for integer $i, 1 \leq i \leq n$.

(ii) the operating result of $\alpha_1 \cdot_1 \mathbf{a}_1 \dot{+}_1 \alpha_2 \cdot_2 \mathbf{a}_2 \dot{+}_2 \cdots \dot{+}_{n-1} \alpha_n \cdot_n \mathbf{a}_n$ does not exist.

Now for a subset $\widehat{S} \subset \widetilde{V}$, define its *linearly spanning set* $\left\langle \widehat{S} \right\rangle$ to be

$$\left\langle \widehat{S} \right\rangle = \{\ \mathbf{a} \mid \mathbf{a} = \alpha_1 \cdot_1 \mathbf{a}_1 \dot{+}_1 \alpha_2 \cdot_2 \mathbf{a}_2 \dot{+}_2 \cdots \in \widetilde{V}, \mathbf{a}_i \in \widehat{S}, \alpha_i \in \widetilde{F}, i \geq 1\}.$$

For a multi-vector space $(\widetilde{V};\widetilde{F})$, if there exists a subset $\widehat{S}, \widehat{S} \subset \widetilde{V}$ such that $\widetilde{V} = \left\langle \widehat{S} \right\rangle$, then we say $\widehat{S}$ is a *linearly spanning set* of the multi-vector space $\widetilde{V}$. If these vectors in a linearly spanning set $\widehat{S}$ of the multi-vector space $\widetilde{V}$ are linearly independent, then $\widehat{S}$ is said to be a *basis* of $\widetilde{V}$.

Theorem 1.3.14 Any multi-vector space $(\widetilde{V};\widetilde{F})$ has a basis.

Proof Assume $\widetilde{V} = \bigcup\limits_{i=1}^{k} V_i, \widetilde{F} = \bigcup\limits_{i=1}^{k} F_i$ and the basis of the vector space $(V_i; \dot{+}_i, \cdot_i)$ is $\Delta_i = \{\mathbf{a}_{i1}, \mathbf{a}_{i2}, \cdots, \mathbf{a}_{in_i}\}$, $1 \leq i \leq k$. Define

$$\widehat{\Delta} = \bigcup_{i=1}^{k} \Delta_i.$$

Then $\widehat{\Delta}$ is a linearly spanning set for $\widetilde{V}$ by definition.

If these vectors in $\widehat{\Delta}$ are linearly independent, then $\widehat{\Delta}$ is a basis of $\widetilde{V}$. Otherwise, choose a vector $\mathbf{b}_1 \in \widehat{\Delta}$ and define $\widehat{\Delta}_1 = \widehat{\Delta} \setminus \{\mathbf{b}_1\}$.

If we have obtained a set $\widehat{\Delta}_s, s \geq 1$ and it is not a basis, choose a vector $\mathbf{b}_{s+1} \in \widehat{\Delta}_s$ and define $\widehat{\Delta}_{s+1} = \widehat{\Delta}_s \setminus \{\mathbf{b}_{s+1}\}$.

If these vectors in $\widehat{\Delta}_{s+1}$ are linearly independent, then $\widehat{\Delta}_{s+1}$ is a basis of $\widetilde{V}$. Otherwise, we can define a set $\widehat{\Delta}_{s+2}$ again. Continue this process. Notice that all vectors in Δ_i are linearly independent for any integer $i, 1 \leq i \leq k$. Therefore, we can finally get a basis of $\widetilde{V}$. $\natural$

Now we consider finite-dimensional multi-vector spaces. A multi-vector space $\widetilde{V}$ is *finite-dimensional* if it has a finite basis. By Theorem 1.2.14, if the vector space $(V_i; +_i, \cdot_i)$ is finite-dimensional for any integer $i, 1 \leq i \leq k$, then $(\widetilde{V};\widetilde{F})$ is finite-dimensional. On the other hand, if there is an integer $i_0, 1 \leq i_0 \leq k$ such that the vector space $(V_{i_0}; +_{i_0}, \cdot_{i_0})$ is infinite-dimensional, then $(\widetilde{V};\widetilde{F})$ is also infinite-dimensional. This enables us to get a consequence in the following.

Corollary 1.3.8 *Let $(\widetilde{V};\widetilde{F})$ be a multi-vector space with $\widetilde{V} = \bigcup\limits_{i=1}^{k} V_i, \widetilde{F} = \bigcup\limits_{i=1}^{k} F_i$. Then $(\widetilde{V};\widetilde{F})$ is finite-dimensional if and only if $(V_i; +_i, \cdot_i)$ is finite-dimensional for any integer $i, 1 \leq i \leq k$.*

Theorem 1.3.15 *For a finite-dimensional multi-vector space $(\widetilde{V};\widetilde{F})$, any two bases have the same number of vectors.*

Proof Let $\widetilde{V} = \bigcup\limits_{i=1}^{k} V_i$ and $\widetilde{F} = \bigcup\limits_{i=1}^{k} F_i$. The proof is by the induction on k. For $k = 1$, the assertion is true by Theorem 4 of Chapter 2 in [33].

For the case of $k = 2$, notice that by a result in linearly vector spaces (see also [33]), for two subspaces W_1, W_2 of a finite-dimensional vector space, if the basis of $W_1 \bigcap W_2$ is $\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_t\}$, then the basis of $W_1 \bigcup W_2$ is

$$\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_t, \mathbf{b}_{t+1}, \mathbf{b}_{t+2}, \cdots, \mathbf{b}_{dimW_1}, \mathbf{c}_{t+1}, \mathbf{c}_{t+2}, \cdots, \mathbf{c}_{dimW_2}\},$$

where, $\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_t, \mathbf{b}_{t+1}, \mathbf{b}_{t+2}, \cdots, \mathbf{b}_{dimW_1}\}$ is a basis of W_1 and $\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_t, \mathbf{c}_{t+1}, \mathbf{c}_{t+2}, \cdots, \mathbf{c}_{dimW_2}\}$ a basis of W_2.

Whence, if $\widetilde{V} = W_1 \bigcup W_2$ and $\widetilde{F} = F_1 \bigcup F_2$, then the basis of $\widetilde{V}$ is also

$$\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_t, \mathbf{b}_{t+1}, \mathbf{b}_{t+2}, \cdots, \mathbf{b}_{dimW_1}, \mathbf{c}_{t+1}, \mathbf{c}_{t+2}, \cdots, \mathbf{c}_{dimW_2}\}.$$

Assume the assertion is true for $k = l, l \geq 2$. Now we consider the case of $k = l + 1$. In this case, since

$$\widetilde{V} = (\bigcup_{i=1}^{l} V_i) \bigcup V_{l+1}, \ \ \widetilde{F} = (\bigcup_{i=1}^{l} F_i) \bigcup F_{l+1},$$

by the induction assumption, we know that any two bases of the multi-vector space $(\bigcup\limits_{i=1}^{l} V_i; \bigcup\limits_{i=1}^{l} F_i)$ have the same number p of vectors. If the basis of $(\bigcup\limits_{i=1}^{l} V_i) \bigcap V_{l+1}$ is $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$, then the basis of $\widetilde{V}$ is

$$\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n, \mathbf{f}_{n+1}, \mathbf{f}_{n+2}, \cdots, \mathbf{f}_p, \mathbf{g}_{n+1}, \mathbf{g}_{n+2}, \cdots, \mathbf{g}_{dimV_{l+1}}\},$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n, \mathbf{f}_{n+1}, \mathbf{f}_{n+2}, \cdots, \mathbf{f}_p\}$ is a basis of $(\bigcup\limits_{i=1}^{l} V_i; \bigcup\limits_{i=1}^{l} F_i)$ and $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n, \mathbf{g}_{n+1}, \mathbf{g}_{n+2}, \cdots, \mathbf{g}_{dimV_{l+1}}\}$ is a basis of V_{l+1}. Whence, the number of vectors in a basis of $\widetilde{V}$ is $p + dimV_{l+1} - n$ for the case $n = l + 1$.

Therefore, we know the assertion is true for any integer k by the induction principle. ♮

The cardinal number of a basis of a finite dimensional multi-vector space $\widetilde{V}$ is called its *dimension*, denoted by $dim\widetilde{V}$.

Theorem 1.3.16(*dimensional formula*) *For a multi-vector space* $(\widetilde{V}; \widetilde{F})$ *with* $\widetilde{V} = \bigcup\limits_{i=1}^{k} V_i$ *and* $\widetilde{F} = \bigcup\limits_{i=1}^{k} F_i$, *the dimension* $dim\widetilde{V}$ *of* $\widetilde{V}$ *is*

$$dim\widetilde{V} = \sum_{i=1}^{k}(-1)^{i-1}\sum_{\{i1,i2,\cdots,ii\}\subset\{1,2,\cdots,k\}} dim(V_{i1}\bigcap V_{i2}\bigcap\cdots\bigcap V_{ii}).$$

Proof The proof is by the induction on k. For $k = 1$, the formula is turn to a trivial case of $dim\widetilde{V} = dimV_1$. for $k = 2$, the formula is

$$dim\widetilde{V} = dimV_1 + dimV_2 - dim(V_1\bigcap dimV_2),$$

which is true by the proof of Theorem 1.3.15.

Now we assume the formula is true for $k = n$. Consider the case of $k = n + 1$. According to the proof of Theorem 1.3.15, we know that

$$\begin{aligned} dim\widetilde{V} &= dim(\bigcup_{i=1}^{n} V_i) + dimV_{n+1} - dim((\bigcup_{i=1}^{n} V_i)\bigcap V_{n+1}) \\ &= dim(\bigcup_{i=1}^{n} V_i) + dimV_{n+1} - dim(\bigcup_{i=1}^{n}(V_i\bigcap V_{n+1})) \\ &= dimV_{n+1} + \sum_{i=1}^{n}(-1)^{i-1}\sum_{\{i1,i2,\cdots,ii\}\subset\{1,2,\cdots,n\}} dim(V_{i1}\bigcap V_{i2}\bigcap\cdots\bigcap V_{ii}) \\ &+ \sum_{i=1}^{n}(-1)^{i-1}\sum_{\{i1,i2,\cdots,ii\}\subset\{1,2,\cdots,n\}} dim(V_{i1}\bigcap V_{i2}\bigcap\cdots\bigcap V_{ii}\bigcap V_{n+1}) \\ &= \sum_{i=1}^{n}(-1)^{i-1}\sum_{\{i1,i2,\cdots,ii\}\subset\{1,2,\cdots,k\}} dim(V_{i1}\bigcap V_{i2}\bigcap\cdots\bigcap V_{ii}). \end{aligned}$$

By the induction principle, we know the formula is true for any integer k. ♮

As a consequence, we get the following formula.

Corollary 1.3.9(*additive formula*) *For any two multi-vector spaces* $\widetilde{V}_1, \widetilde{V}_2$,

$$dim(\widetilde{V}_1\bigcup\widetilde{V}_2) = dim\widetilde{V}_1 + dim\widetilde{V}_2 - dim(\widetilde{V}_1\bigcap\widetilde{V}_2).$$

§1.4 Multi-Metric Spaces

1.4.1. Metric spaces

A set M associated with a metric function $\rho : M \times M \to R^+ = \{x \mid x \in R, x \geq 0\}$ is called a *metric space* if for $\forall x, y, z \in M$, the following conditions for ρ hold:

(1)(*definiteness*) $\rho(x, y) = 0$ if and only if $x = y$;

(*ii*)(*symmetry*) $\rho(x, y) = \rho(y, x)$;

(*iii*)(*triangle inequality*) $\rho(x, y) + \rho(y, z) \geq \rho(x, z)$.

A metric space M with a metric function ρ is usually denoted by $(M\ ; \rho)$. Any $x, x \in M$ is called a point of $(M\ ; \rho)$. A sequence $\{x_n\}$ is said to be *convergent to* x if for any number $\epsilon > 0$ there is an integer N such that $n \geq N$ implies $\rho(x_n, x)\ <\ 0$, denoted by $\lim_n x_n = x$. We have known the following result in metric spaces.

Theorem 1.4.1 *Any sequence $\{x_n\}$ in a metric space has at most one limit point.*

For $x_0 \in M$ and $\epsilon > 0$, a ϵ-disk about x_0 is defined by

$$B(x_0, \epsilon) = \{\ x \mid x \in M, \rho(x, x_0)\ <\ \epsilon\}.$$

If $A \subset M$ and there is an ϵ-disk $B(x_0, \epsilon) \supset A$, we say A is a bounded point set of M.

Theorem 1.4.2 *Any convergent sequence $\{x_n\}$ in a metric space is a bounded point set.*

Now let (M, ρ) be a metric space and $\{x_n\}$ a sequence in M. If for any number $\epsilon > 0, \epsilon \in \mathbf{R}$, there is an integer N such that $n, m \geq N$ implies $\rho(x_n, x_m)\ <\ \epsilon$, we call $\{x_n\}$ a *Cauchy sequence*. A metric space (M, ρ) is called to be *completed* if its every Cauchy sequence converges.

Theorem 1.4.3 *For a completed metric space (M, ρ), if an ϵ-disk sequence $\{B_n\}$ satisfies*

(*i*) $B_1 \supset B_2 \supset \cdots \supset B_n \supset \cdots$;

(*ii*) $\lim_n \epsilon_n = 0$,

where $\epsilon_n > 0$ and $B_n = \{\ x \mid x \in M, \rho(x, x_n) \leq \epsilon_n\}$ for any integer $n, n = 1, 2, \cdots$, then $\bigcap_{n=1}^{\infty} B_n$ only has one point.

For a metric space (M, ρ) and $T : M \to M$ a mapping on (M, ρ), if there exists a point $x^* \in M$ such that

$$Tx^* = x^*,$$

then x^* is called a *fixed point* of T. If there exists a constant $\eta, 0 < \eta < 1$ such that

$$\rho(Tx, Ty) \leq \eta\rho(x, y)$$

for $\forall x, y \in M$, then T is called a *contraction*.

Theorem 1.4.4 (Banach) *Let (M, ρ) be a completed metric space and let $T : M \to M$ be a contraction. Then T has only one fixed point.*

1.4.2. **Multi-Metric spaces**

Definition 1.4.1 *A multi-metric space is a union $\widetilde{M} = \bigcup\limits_{i=1}^{m} M_i$ such that each M_i is a space with a metric ρ_i for $\forall i, 1 \leq i \leq m$.*

When we say a *multi-metric space* $\widetilde{M} = \bigcup\limits_{i=1}^{m} M_i$, it means that a multi-metric space with metrics $\rho_1, \rho_2, \cdots, \rho_m$ such that (M_i, ρ_i) is a metric space for any integer $i, 1 \leq i \leq m$. For a multi-metric space $\widetilde{M} = \bigcup\limits_{i=1}^{m} M_i$, $x \in \widetilde{M}$ and a positive number R, a *R-disk* $B(x, R)$ in $\widetilde{M}$ is defined by

$$B(x, R) = \{\ y \mid \text{there exists an integer } k, 1 \leq k \leq m \text{ such that } \rho_k(y, x)\ < R, y \in \widetilde{M}\}$$

Remark 1.4.1 The following two extremal cases are permitted in Definition 1.4.1:

(i) there are integers $i_1, i_2, \cdots, i_s$ such that $M_{i_1} = M_{i_2} = \cdots = M_{i_s}$, where $i_j \in \{1, 2, \cdots, m\}$, $1 \leq j \leq s$;

(ii) there are integers $l_1, l_2, \cdots, l_s$ such that $\rho_{l_1} = \rho_{l_2} = \cdots = \rho_{l_s}$, where $l_j \in \{1, 2, \cdots, m\}$, $1 \leq j \leq s$.

For metrics on a space, we have the following result.

Theorem 1.4.5 *Let $\rho_1, \rho_2, \cdots, \rho_m$ be m metrics on a space M and let F be a function on $\mathbf{R}^m$ such that the following conditions hold:*

(i) $F(x_1, x_2, \cdots, x_m) \geq F(y_1, y_2, \cdots, y_m)$ *for* $\forall i, 1 \leq i \leq m$, $x_i \geq y_i$;

(ii) $F(x_1, x_2, \cdots, x_m) = 0$ *only if* $x_1 = x_2 = \cdots = x_m = 0$;

(iii) *for two m-tuples* $(x_1, x_2, \cdots, x_m)$ *and* $(y_1, y_2, \cdots, y_m)$,

$$F(x_1, x_2, \cdots, x_m) + F(y_1, y_2, \cdots, y_m) \geq F(x_1 + y_1, x_2 + y_2, \cdots, x_m + y_m).$$

Then $F(\rho_1, \rho_2, \cdots, \rho_m)$ is also a metric on M.

Proof We only need to prove that $F(\rho_1, \rho_2, \cdots, \rho_m)$ satisfies those of metric conditions for $\forall x, y, z \in M$.

By (ii), $F(\rho_1(x,y), \rho_2(x,y), \cdots, \rho_m(x,y)) = 0$ only if $\rho_i(x,y) = 0$ for any integer i. Since ρ_i is a metric on M, we know that $x = y$.

For any integer $i, 1 \leq i \leq m$, since ρ_i is a metric on M, we know that $\rho_i(x,y) = \rho_i(y,x)$. Whence,

$$F(\rho_1(x,y), \rho_2(x,y), \cdots, \rho_m(x,y)) = F(\rho_1(y,x), \rho_2(y,x), \cdots, \rho_m(y,x)).$$

Now by (i) and (iii), we get that

$$\begin{aligned}
&F(\rho_1(x,y), \rho_2(x,y), \cdots, \rho_m(x,y)) + F(\rho_1(y,z), \rho_2(y,z), \cdots, \rho_m(y,z)) \\
&\geq F(\rho_1(x,y) + \rho_1(y,z), \rho_2(x,y) + \rho_2(y,z), \cdots, \rho_m(x,y) + \rho_m(y,z)) \\
&\geq F(\rho_1(x,z), \rho_2(x,z), \cdots, \rho_m(x,z)).
\end{aligned}$$

Therefore, $F(\rho_1, \rho_2, \cdots, \rho_m)$ is a metric on M. ♮

Corollary 1.4.1 If $\rho_1, \rho_2, \cdots, \rho_m$ are m metrics on a space M, then $\rho_1 + \rho_2 + \cdots + \rho_m$ and $\frac{\rho_1}{1+\rho_1} + \frac{\rho_2}{1+\rho_2} + \cdots + \frac{\rho_m}{1+\rho_m}$ are also metrics on M.

A sequence $\{x_n\}$ in a multi-metric space $\widetilde{M} = \bigcup\limits_{i=1}^{m} M_i$ is said to be *convergent to a point* $x, x \in \widetilde{M}$ if for any number $\epsilon > 0$, there exist numbers N and $i, 1 \leq i \leq m$ such that

$$\rho_i(x_n, x) \ < \ \epsilon$$

provided $n \geq N$. If $\{x_n\}$ is convergent to a point $x, x \in \widetilde{M}$, we denote it by $\lim\limits_{n} x_n = x$.

We get a characteristic for convergent sequences in a multi-metric space as in the following.

Theorem 1.4.6 *A sequence $\{x_n\}$ in a multi-metric space $\widetilde{M} = \bigcup\limits_{i=1}^{m} M_i$ is convergent if and only if there exist integers N and $k, 1 \leq k \leq m$ such that the subsequence $\{x_n | n \geq N\}$ is a convergent sequence in (M_k, ρ_k).*

Proof If there exist integers N and $k, 1 \leq k \leq m$ such that $\{x_n | n \geq N\}$ is a convergent sequence in (M_k, ρ_k), then for any number $\epsilon > 0$, by definition there exist an integer P and a point $x, x \in M_k$ such that

$$\rho_k(x_n, x) \ < \ \epsilon$$

if $n \geq max\{N, \ P\}$.

Now if $\{x_n\}$ is a convergent sequence in the multi-space $\widetilde{M}$, by definition for any positive number $\epsilon > 0$, there exist a point $x, x \in \widetilde{M}$, natural numbers $N(\epsilon)$ and integer $k, 1 \leq k \leq m$ such that if $n \geq N(\epsilon)$, then

$$\rho_k(x_n, x) \ < \ \epsilon.$$

That is, $\{x_n | n \geq N(\epsilon)\} \subset M_k$ and $\{x_n | n \geq N(\epsilon)\}$ is a convergent sequence in (M_k, ρ_k). $\natural$

Theorem 1.4.7 *Let $\widetilde{M} = \bigcup\limits_{i=1}^{m} M_i$ be a multi-metric space. For two sequences $\{x_n\}$, $\{y_n\}$ in $\widetilde{M}$, if $\lim\limits_{n} x_n = x_0$, $\lim\limits_{n} y_n = y_0$ and there is an integer p such that $x_0, y_0 \in M_p$, then $\lim\limits_{n} \rho_p(x_n, y_n) = \rho_p(x_0, y_0)$.*

Proof According to Theorem 1.4.6, there exist integers N_1 and N_2 such that if $n \geq max\{N_1, N_2\}$, then $x_n, y_n \in M_p$. Whence, we know that

$$\rho_p(x_n, y_n) \leq \rho_p(x_n, x_0) + \rho_p(x_0, y_0) + \rho_p(y_n, y_0)$$

and

$$\rho_p(x_0, y_0) \leq \rho_p(x_n, x_0) + \rho_p(x_n, y_n) + \rho_p(y_n, y_0).$$

Therefore,

$$|\rho_p(x_n, y_n) - \rho_p(x_0, y_0)| \ \leq \ \rho_p(x_n, x_0) + \rho_p(y_n, y_0).$$

Now for any number $\epsilon > 0$, since $\lim\limits_{n} x_n = x_0$ and $\lim\limits_{n} y_n = y_0$, there exist numbers $N_1(\epsilon), N_1(\epsilon) \geq N_1$ and $N_2(\epsilon), N_2(\epsilon) \geq N_2$ such that $\rho_p(x_n, x_0) \ \leq \ \frac{\epsilon}{2}$ if $n \geq N_1(\epsilon)$ and $\rho_p(y_n, y_0) \ \leq \ \frac{\epsilon}{2}$ if $n \geq N_2(\epsilon)$. Whence, if we choose $n \geq max\{N_1(\epsilon), N_2(\epsilon)\}$, then

$$|\rho_p(x_n, y_n) - \rho_p(x_0, y_0)| \ < \ \epsilon. \qquad \natural$$

Whether can a convergent sequence have more than one limiting points? The following result answers this question.

Theorem 1.4.8 *If $\{x_n\}$ is a convergent sequence in a multi-metric space $\widetilde{M} = \bigcup\limits_{i=1}^{m} M_i$, then $\{x_n\}$ has only one limit point.*

Proof According to Theorem 1.4.6, there exist integers N and $i, 1 \leq i \leq m$ such that $x_n \in M_i$ if $n \geq N$. Now if

$$\lim_n x_n = x_1 \text{ and } \lim_n x_n = x_2,$$

and $n \geq N$, by definition,

$$0 \leq \rho_i(x_1, x_2) \leq \rho_i(x_n, x_1) + \rho_i(x_n, x_2).$$

Whence, we get that $\rho_i(x_1, x_2) = 0$. Therefore, $x_1 = x_2$. $\natural$

Theorem 1.4.9 *Any convergent sequence in a multi-metric space is a bounded points set.*

Proof According to Theorem 1.4.8, we obtain this result immediately. $\natural$

A sequence $\{x_n\}$ in a multi-metric space $\widetilde{M} = \bigcup\limits_{i=1}^{m} M_i$ is called a *Cauchy sequence* if for any number $\epsilon > 0$, there exist integers $N(\epsilon)$ and $s, 1 \leq s \leq m$ such that for any integers $m, n \geq N(\epsilon)$, $\rho_s(x_m, x_n) \ < \epsilon$.

Theorem 1.4.10 *A Cauchy sequence $\{x_n\}$ in a multi-metric space $\widetilde{M} = \bigcup\limits_{i=1}^{m} M_i$ is convergent if and only if $|\{x_n\} \bigcap M_k|$ is finite or infinite but $\{x_n\} \bigcap M_k$ is convergent in (M_k, ρ_k) for $\forall k, 1 \leq k \leq m$.*

Proof The necessity of these conditions in this theorem is known by Theorem 1.4.6.

Now we prove the sufficiency. By definition, there exist integers $s, 1 \leq s \leq m$ and N_1 such that $x_n \in M_s$ if $n \geq N_1$. Whence, if $|\{x_n\} \bigcap M_k|$ is infinite and $\lim\limits_n \{x_n\} \bigcap M_k = x$, then there must be $k = s$. Denote by $\{x_n\} \bigcap M_k = \{x_{k1}, x_{k2}, \cdots, x_{kn}, \cdots\}$.

For any positive number $\epsilon > 0$, there exists an integer $N_2, N_2 \geq N_1$ such that $\rho_k(x_m, x_n) < \frac{\epsilon}{2}$ and $\rho_k(x_{kn}, x) < \frac{\epsilon}{2}$ if $m, n \geq N_2$. According to Theorem 1.4.7, we get that

$$\rho_k(x_n, x) \leq \rho_k(x_n, x_{kn}) + \rho_k(x_{kn}, x) < \epsilon$$

if $n \geq N_2$. Whence, $\lim_{n} x_n = x$. ♮

A multi-metric space $\widetilde{M}$ is said to be *completed* if its every Cauchy sequence is convergent. For a completed multi-metric space, we obtain two important results similar to Theorems 1.4.3 and 1.4.4 in metric spaces.

Theorem 1.4.11 *Let $\widetilde{M} = \bigcup\limits_{i=1}^{m} M_i$ be a completed multi-metric space. For an ϵ-disk sequence $\{B(\epsilon_n, x_n)\}$, where $\epsilon_n > 0$ for $n = 1, 2, 3, \cdots$, if the following conditions hold:*

(*i*) $B(\epsilon_1, x_1) \supset B(\epsilon_2, x_2) \supset B(\epsilon_3, x_3) \supset \cdots \supset B(\epsilon_n, x_n) \supset \cdots$;

(*ii*) $\lim\limits_{n\to+\infty} \epsilon_n = 0$,

then $\bigcap\limits_{n=1}^{+\infty} B(\epsilon_n, x_n)$ only has one point.

Proof First, we prove that the sequence $\{x_n\}$ is a Cauchy sequence in $\widetilde{M}$. By the condition (i), we know that if $m \geq n$, then $x_m \in B(\epsilon_m, x_m) \subset B(\epsilon_n, x_n)$. Whence $\rho_i(x_m, x_n) < \epsilon_n$ provided $x_m, x_n \in M_i$ for $\forall i, 1 \leq i \leq m$.

Now for any positive number ϵ, since $\lim\limits_{n\to+\infty} \epsilon_n = 0$, there exists an integer $N(\epsilon)$ such that if $n \geq N(\epsilon)$, then $\epsilon_n < \epsilon$. Therefore, if $x_n \in M_l$, then $\lim\limits_{m\to+\infty} x_m = x_n$. Thereby there exists an integer N such that if $m \geq N$, then $x_m \in M_l$ by Theorem 1.4.6. Choice integers $m, n \geq max\{N, N(\epsilon)\}$, we know that

$$\rho_l(x_m, x_n) < \epsilon_n < \epsilon.$$

So $\{x_n\}$ is a Cauchy sequence.

By the assumption that $\widetilde{M}$ is completed, we know that the sequence $\{x_n\}$ is convergent to a point $x_0, x_0 \in \widetilde{M}$. By conditions of (i) and (ii), we get that $\rho_l(x_0, x_n) < \epsilon_n$ if $m \to +\infty$. Whence, $x_0 \in \bigcap\limits_{n=1}^{+\infty} B(\epsilon_n, x_n)$.

Now if there is a point $y \in \bigcap\limits_{n=1}^{+\infty} B(\epsilon_n, x_n)$, then there must be $y \in M_l$. We get that

$$0 \leq \rho_l(y, x_0) = \lim_n \rho_l(y, x_n) \leq \lim_{n\to+\infty} \epsilon_n = 0$$

by Theorem 1.4.7. Therefore, $\rho_l(y, x_0) = 0$. By the definition of a metric function, we get that $y = x_0$. ♮

Let $\widetilde{M}_1$ and $\widetilde{M}_2$ be two multi-metric spaces and let $f : \widetilde{M}_1 \to \widetilde{M}_2$ be a mapping, $x_0 \in \widetilde{M}_1, f(x_0) = y_0$. For $\forall\epsilon > 0$, if there exists a number δ such that $f(x) = y \in B(\epsilon, y_0) \subset \widetilde{M}_2$ for $\forall x \in B(\delta, x_0)$, i.e.,

$$f(B(\delta, x_0)) \subset B(\epsilon, y_0),$$

then we say that f is *continuous at point* x_0. A mapping $f : \widetilde{M}_1 \to \widetilde{M}_2$ is called a *continuous mapping* from $\widetilde{M}_1$ to $\widetilde{M}_2$ if f is continuous at every point of $\widetilde{M}_1$.

For a continuous mapping f from $\widetilde{M}_1$ to $\widetilde{M}_2$ and a convergent sequence $\{x_n\}$ in $\widetilde{M}_1$, $\lim_n x_n = x_0$, we can prove that

$$\lim_n f(x_n) = f(x_0).$$

For a multi-metric space $\widetilde{M} = \bigcup_{i=1}^{m} M_i$ and a mapping $T : \widetilde{M} \to \widetilde{M}$, if there is a point $x^* \in \widetilde{M}$ such that $Tx^* = x^*$, then x^* is called a *fixed point* of T. Denote the number of fixed points of a mapping T in $\widetilde{M}$ by $^{\#}\Phi(T)$. A mapping T is called a *contraction* on a multi-metric space $\widetilde{M}$ if there are a constant $\alpha, 0 < \alpha < 1$ and integers $i, j, 1 \leq i, j \leq m$ such that for $\forall x, y \in M_i$, $Tx, Ty \in M_j$ and

$$\rho_j(Tx, Ty) \leq \alpha\rho_i(x, y).$$

Theorem 1.4.12 *Let* $\widetilde{M} = \bigcup_{i=1}^{m} M_i$ *be a completed multi-metric space and let* T *be a contraction on* $\widetilde{M}$. *Then*

$$1 \leq^{\#} \Phi(T) \leq m.$$

Proof Choose arbitrary points $x_0, y_0 \in M_1$ and define recursively

$$x_{n+1} = Tx_n, \quad y_{n+1} = Tx_n$$

for $n = 1, 2, 3, \cdots$. By definition, we know that for any integer $n, n \geq 1$, there exists an integer $i, 1 \leq i \leq m$ such that $x_n, y_n \in M_i$. Whence, we inductively get that

$$0 \leq \rho_i(x_n, y_n) \leq \alpha^n \rho_1(x_0, y_0).$$

Notice that $0 < \alpha < 1$, we know that $\lim\limits_{n \to +\infty} \alpha^n = 0$. Thereby there exists an integer i_0 such that

$$\rho_{i_0}(\lim_n x_n, \lim_n y_n) = 0.$$

Therefore, there exists an integer N_1 such that $x_n, y_n \in M_{i_0}$ if $n \geq N_1$. Now if $n \geq N_1$, we get that

$$\begin{aligned}
\rho_{i_0}(x_{n+1}, x_n) &= \rho_{i_0}(Tx_n, Tx_{n-1}) \\
&\leq \alpha\rho_{i_0}(x_n, x_{n-1}) = \alpha\rho_{i_0}(Tx_{n-1}, Tx_{n-2}) \\
&\leq \alpha^2\rho_{i_0}(x_{n-1}, x_{n-2}) \leq \cdots \leq \alpha^{n-N_1}\rho_{i_0}(x_{N_1+1}, x_{N_1}).
\end{aligned}$$

and generally, for $m \geq n \geq N_1$,

$$\begin{aligned}
\rho_{i_0}(x_m, x_n) &\leq \rho_{i_0}(x_n, x_{n+1}) + \rho_{i_0}(x_{n+1}, x_{n+2}) + \cdots + \rho_{i_0}(x_{n-1}, x_n) \\
&\leq (\alpha^{m-1} + \alpha^{m-2} + \cdots + \alpha^n)\rho_{i_0}(x_{N_1+1}, x_{N_1}) \\
&\leq \frac{\alpha^n}{1-\alpha}\rho_{i_0}(x_{N_1+1}, x_{N_1}) \to 0 (m, n \to +\infty).
\end{aligned}$$

Therefore, $\{x_n\}$ is a Cauchy sequence in $\widetilde{M}$. Similarly, we can also prove $\{y_n\}$ is a Cauchy sequence.

Because $\widetilde{M}$ is a completed multi-metric space, we know that

$$\lim_n x_n = \lim_n y_n = z^*.$$

Now we prove z^* is a fixed point of T in $\widetilde{M}$. In fact, by $\rho_{i_0}(\lim\limits_n x_n, \lim\limits_n y_n) = 0$, there exists an integer N such that

$$x_n, y_n, Tx_n, Ty_n \in M_{i_0}$$

if $n \geq N + 1$. Whence, we know that

$$\begin{aligned} 0 \le \rho_{i_0}(z^*, Tz^*) &\le \rho_{i_0}(z^*, x_n) + \rho_{i_0}(y_n, Tz^*) + \rho_{i_0}(x_n, y_n) \\ &\le \rho_{i_0}(z^*, x_n) + \alpha\rho_{i_0}(y_{n-1}, z^*) + \rho_{i_0}(x_n, y_n). \end{aligned}$$

Notice that

$$\lim_{n\to+\infty} \rho_{i_0}(z^*, x_n) = \lim_{n\to+\infty} \rho_{i_0}(y_{n-1}, z^*) = \lim_{n\to+\infty} \rho_{i_0}(x_n, y_n) = 0.$$

We get $\rho_{i_0}(z^*, Tz^*) = 0$, i.e., $Tz^* = z^*$.

For other chosen points $u_0, v_0 \in M_1$, we can also define recursively

$$u_{n+1} = Tu_n, \quad v_{n+1} = Tv_n$$

and get a limiting point $\lim\limits_n u_n = \lim\limits_n v_n = u^* \in M_{i_0}, Tu^* \in M_{i_0}$. Since

$$\rho_{i_0}(z^*, u^*) = \rho_{i_0}(Tz^*, Tu^*) \le \alpha\rho_{i_0}(z^*, u^*)$$

and $0 < \alpha < 1$, there must be $z^* = u^*$.

Similarly consider the points in $M_i, 2 \le i \le m$, we get that

$$1 \le^{\#} \Phi(T) \le m. \qquad \natural$$

As a consequence, we get the *Banach theorem* in metric spaces.

Corollary 1.4.2(Banach) *Let M be a metric space and let T be a contraction on M. Then T has just one fixed point.*

§1.5 Remarks and Open Problems

The central idea of Smarandache multi-spaces is to combine different fields (spaces, systems, objects, $\cdots$) into a unifying field and find its behaviors. Which is entirely new, also an application of combinatorial approaches to classical mathematics but more important than combinatorics itself. This idea arouses us to think why an assertion is true or not in classical mathematics. Then combine an assertion with its non-assertion and enlarge the filed of truths. A famous fable says that *each theorem in mathematics is an absolute truth*. But we do not think so. Our thinking is that *each theorem in mathematics is just a relative truth*. Thereby we can establish new theorems and present new problems boundless in mathematics. Results obtained in Section 1.3 and 1.4 are applications of this idea to these groups, rings, vector spaces or metric spaces. Certainly, more and more multi-spaces and their good behaviors can be found under this thinking. Here we present some remarks and open problems for multi-spaces.

1.5.1. **Algebraic Multi-Spaces** The algebraic multi-spaces are discrete representations for phenomena in the natural world. They maybe completed or not in cases. For a completed algebraic multi-space, it is a reflection of an equilibrium phenomenon. Otherwise, a reflection of a non-equilibrium phenomenon. Whence, more consideration should be done for algebraic multi-spaces, especially, by an analogous thinking as in classical algebra.

Problem 1.5.1 *Establish a decomposition theory for multi-groups.*

In group theory, we know the following decomposition result([107][82]) for groups.

Let G be a finite Ω-group. Then G can be uniquely decomposed as a direct product of finite non-decomposition Ω-subgroups.

Each finite abelian group is a direct product of its Sylow p-subgroups.

Then Problem 1.5.1 can be restated as follows.

Whether can we establish a decomposition theory for multi-groups similar to the above two results in group theory, especially, for finite multi-groups?

Problem 1.5.2 *Define the conception of simple multi-groups. For finite multi-groups, whether can we find all simple multi-groups?*

For finite groups, we know that there are four simple group classes ([108]):

Class 1: the cyclic groups of prime order;

Class 2: the alternating groups $A_n, n \geq 5$;

Class 3: the 16 groups of Lie types;

Class 4: the 26 sporadic simple groups.

Problem 1.5.3 *Determine the structure properties of multi-groups generated by finite elements.*

For a subset A of a multi-group $\widetilde{G}$, define its spanning set by

$$\langle A\rangle = \{a \circ b | a, b \in A \text{ and } \circ \in O(\widetilde{G})\}.$$

If there exists a subset $A \subset \widetilde{G}$ such that $\widetilde{G} = \langle A\rangle$, then call $\widetilde{G}$ is generated by A. Call $\widetilde{G}$ is *finitely generated* if there exist a finite set A such that $\widetilde{G} = \langle A\rangle$. Then Problem 5.3 can be restated by

Can we establish a finite generated multi-group theory similar to the finite generated group theory?

Problem 1.5.4 *Determine the structure of a Noether multi-ring.*

Let R be a ring. Call R a *Noether ring* if its every ideal chain only has finite terms. Similarly, for a multi-ring $\widetilde{R}$, if its every multi-ideal chain only has finite terms, it is called a *Noether multi-ring*. *Whether can we find its structures similar to Corollary* 1.3.5 *and Theorem* 1.3.12*?*

Problem 1.5.5 *Similar to ring theory, define a Jacobson or Brown-McCoy radical for multi-rings and determine their contribution to multi-rings.*

Notice that Theorem 1.3.14 has told us there is a similar linear theory for multi-vector spaces, but the situation is more complex.

Problem 1.5.6 *Similar to linear spaces, define linear transformations on multi-vector spaces. Can we establish a matrix theory for these linear transformations?*

Problem 1.5.7 *Whether a multi-vector space must be a linear space?*

Conjecture 1.5.1 *There are non-linear multi-vector spaces in multi-vector spaces.*

Based on Conjecture 1.5.1, there is a fundamental problem for multi-vector spaces.

Problem 1.5.8 *Can we apply multi-vector spaces to non-linear spaces?*

1.5.2. **Multi-Metric Spaces** On a tradition notion, only one metric maybe considered in a space to ensure the same on all the time and on all the situation.

Essentially, this notion is based on an assumption that all spaces are homogeneous. In fact, it is not true in general.

Multi-metric spaces can be used to simplify or beautify geometrical figures and algebraic equations. For an explanation, an example is shown in Fig.1.3, in where the left elliptic curve is transformed to the right circle by changing the metric along x, y-axes and an elliptic equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

to equation

$$x^2 + y^2 = r^2$$

of a circle of radius r.

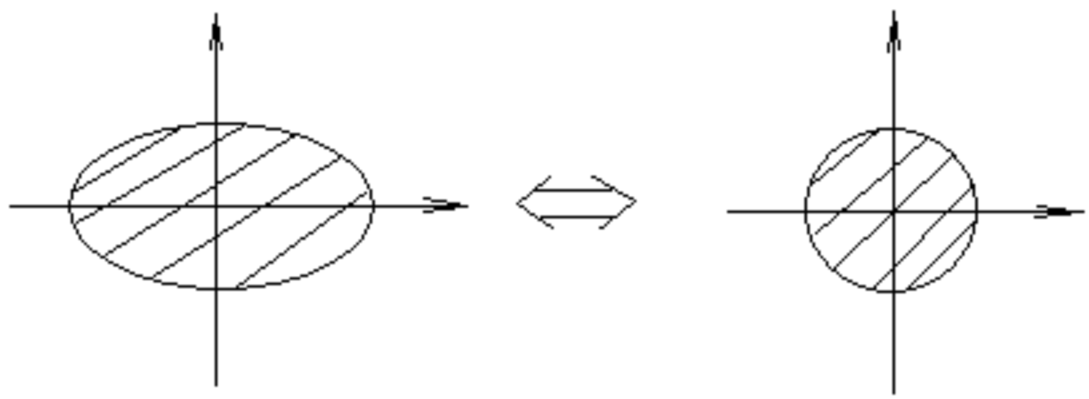


Fig.1.3

Generally, in a multi-metric space, we can simplify a polynomial similar to the approach used in the projective geometry. *Whether this approach can be contributed to mathematics with metrics?*

Problem 1.5.9 *Choose suitable metrics to simplify the equations of surfaces or curves in* $\mathbf{R}^3$.

Problem 1.5.10 *Choose suitable metrics to simplify the knot problem. Whether can it be used for classifying* 3*-dimensional manifolds?*

Problem 1.5.11 *Construct multi-metric spaces or non-linear spaces by Banach spaces. Simplify equations or problems to linear problems.*

1.5.3. **Multi-Operation Systems** By a complete Smarandache multi-space $\widetilde{A}$ with an operation set $O(\widetilde{A})$, we can get a *multi-operation system* $\widetilde{A}$. For example, if $\widetilde{A}$ is a multi-field $\widetilde{F} = \bigcup\limits_{i=1}^{n} F_i$ with an operation set $O(\widetilde{F}) = \{(+_i, \times_i) |\ 1 \leq i \leq n\}$, then $(\widetilde{F}; +_1, +_2, \cdots, +_n)$, $(\widetilde{F}; \times_1, \times_2, \cdots, \times_n)$ and $(\widetilde{F}; (+_1, \times_1), (+_2, \times_2), \cdots, (+_n, \times_n))$ are multi-operation systems. On this view, the classical operation system $(R\ ; +)$ and $(R\ ; \times)$ are only *sole operation systems*. For a multi-operation system $\widetilde{A}$, we can define these conceptions of equality and inequality, $\cdots$, etc.. For example, in the multi-operation system $(\widetilde{F}; +_1, +_2, \cdots, +_n)$, we define the equalities $=_1, =_2, \cdots, =_n$ such as those in sole operation systems $(\widetilde{F}; +_1), (\widetilde{F}; +_2), \cdots, (\widetilde{F}; +_n)$, for example, $2 =_1 2, 1.4 =_2 1.4, \cdots, \sqrt{3} =_n \sqrt{3}$ which is the same as the usual meaning and similarly, for the conceptions $\geq_1, \geq_2, \cdots, \geq_n$ and $\leq_1, \leq_2, \cdots, \leq_n$.

In a classical operation system $(R\ ; +)$, the equation system

$$\begin{aligned} x+2+4+6 &= 15 \\ x+1+3+6 &= 12 \\ x+1+4+7 &= 13 \end{aligned}$$

can not has a solution. But in the multi-operation system $(\widetilde{F}; +_1, +_2, \cdots, +_n)$, the equation system

$$\begin{aligned} x +_1 2 +_1 4 +_1 6 &=_1 15 \\ x +_2 1 +_2 3 +_2 6 &=_2 12 \\ x +_3 1 +_3 4 +_3 7 &=_3 13 \end{aligned}$$

may have a solution x if

$$\begin{aligned} 15 +_1 (-1) +_1 (-4) +_1 (-16) &= 12 +_2 (-1) +_2 (-3) +_2 (-6) \\ &= 13 +_3 (-1) +_3 (-4) +_3 (-7). \end{aligned}$$

in $(\widetilde{F}; +_1, +_2, \cdots, +_n)$. Whence, an element maybe have different disguises in a multi-operation system.

For the multi-operation systems, a number of open problems needs to research further.

Problem 1.5.12 *Find necessary and sufficient conditions for a multi-operation system with more than* 3 *operations to be the rational number field* Q*, the real number field* R *or the complex number field* C.

For a multi-operation system $(N\ ;(+_1,\times_1),(+_2,\times_2),\cdots,(+_n,\times_n))$ and integers $a,b,c\in N$, if $a=b\times_i c$ for an integer $i, 1\leq i\leq n$, then b and c are called *factors* of a. An integer p is called a *prime* if there exist integers n_1,n_2 and $i, 1\leq i\leq n$ such that $p=n_1\times_i n_2$, then $p=n_1$ or $p=n_2$. Two problems for primes of a multi-operation system $(N\ ;(+_1,\times_1),\ (+_2,\times_2),\cdots,(+_n,\times_n))$ are presented in the following.

Problem 1.5.13 *For a positive real number* x*, denote by* $\pi_m(x)$ *the number of primes* $\leq x$ *in* $(N\ ;(+_1,\times_1),(+_2,\times_2),\cdots,(+_n,\times_n))$*. Determine or estimate* $\pi_m(x)$.

Notice that for the positive integer system, by a well-known theorem, i.e., *Gauss prime theorem*, we have known that([15])

$$\pi(x)\sim\frac{x}{\log x}.$$

Problem 1.5.14 *Find the additive number properties for* $(N\ ;(+_1,\times_1),(+_2,\times_2),\cdots,$ $(+_n,\times_n))$*, for example, we have weakly forms for Goldbach's conjecture and Fermat's problem* ([34]) *as follows.*

Conjecture 1.5.2 *For any even integer* $n, n\geq 4$*, there exist odd primes* p_1,p_2 *and an integer* $i, 1\leq i\leq n$ *such that* $n=p_1+_i p_2$.

Conjecture 1.5.3 *For any positive integer* q*, the Diophantine equation* $x^q+y^q=z^q$ *has non-trivial integer solutions* (x,y,z) *at least for an operation "*$+_i$*" with* $1\leq i\leq n$.

A *Smarandache* n*-structure on a set* S means a weak structure $\{w(0)\}$ on S such that there exists a chain of proper subsets $P(n-1)\subset P(n-2)\subset\cdots\subset P(1)\subset S$ whose corresponding structures verify the inverse chain $\{w(n-1)\}\supset\{w(n-2)\}\supset\cdots\supset\{w(1)\}\supset\{w(0)\}$, i.e., structures satisfying more axioms.

Problem 1.5.15 *For Smarandache multi-structures, solves these Problems* 1.5.1 – 1.5.8.

1.5.4. **Multi-Manifolds** Manifolds are important objects in topology, Riemann geometry and modern mechanics. It can be seen as a local generalization of Euclid spaces. By the Smarandache's notion, we can also define multi-manifolds. To determine their behaviors or structure properties will useful for modern mathematics.

In an Euclid space $\mathbf{R}^n$, an *n-ball of radius r* is defined by

$$B^n(r) = \{(x_1, x_2, \cdots, x_n) | x_1^2 + x_2^2 + \cdots + x_n^2 \leq r\}.$$

Now we choose m n-balls $B_1^n(r_1), B_2^n(r_2), \cdots, B_m^n(r_m)$, where for any integers $i, j, 1 \leq i, j \leq m$, $B_i^n(r_i) \bigcap B_j^n(r_j) =$ or not and $r_i = r_j$ or not. An *n-multi-ball* is a union

$$\widetilde{B} = \bigcup_{k=1}^{m} B_k^n(r_k).$$

Then an *n-multi-manifold* is a Hausdorff space with each point in this space has a neighborhood homeomorphic to an n-multi-ball.

Problem 1.5.16 *For an integer $n, n \geq 2$, classifies n-multi-manifolds. Especially, classifies 2-multi-manifolds.*

For closed 2-manifolds, i.e., locally orientable surfaces, we have known a classification theorem for them.

Problem 1.5.17 *If we replace the word* **"***homeomorphic***"** *by* **"***points equivalent***"** *or* **"***isomorphic***"** *, what can we obtain for n-multi-manifolds? Can we classify them?*

Similarly, we can also define differential multi-manifolds and consider their contributions to modern differential geometry, Riemann geometry or modern mechanics, $\cdots$, etc..

Chapter 2 Multi-Spaces on Graphs

As a useful tool for dealing with relations of events, graph theory has rapidly grown in theoretical results as well as its applications to real-world problems, for example see [9], [11] and [80] for graph theory, [42] – [44] for topological graphs and combinatorial map theory, [7], [12] and [104] for its applications to probability, electrical network and real-life problems. By applying the Smarandache's notion, graphs are models of multi-spaces and matters in the natural world. For the later, graphs are a generalization of p-branes and seems to be useful for mechanics and quantum physics.

§2.1 Graphs

2.1.1. What is a graph?

A *graph* G is an ordered 3-tuple $(V, E; I)$, where V, E are finite sets, $V \neq \emptyset$ and $I : E \to V \times V$. Call V the *vertex set* and E the *edge set* of G, denoted by $V(G)$ and $E(G)$, respectively. Two elements $v \in V(G)$ and $e \in E(G)$ are said to be *incident* if $I(e) = (v, x)$ or (x, v), where $x \in V(G)$. If $(u, v) = (v, u)$ for $\forall u, v \in V$, the graph G is called a graph, otherwise, a directed graph with an orientation $u \to v$ on each edge (u, v). Unless Section 2.4, graphs considered in this chapter are non-directed.

The cardinal numbers of $|V(G)|$ and $|E(G)|$ are called the *order* and the *size* of a graph G, denoted by $|G|$ and $\varepsilon(G)$, respectively.

We can draw a graph G on a plane $\sum$ by representing each vertex u of G by a point $p(u)$, $p(u) \neq p(v)$ if $u \neq v$ and an edge (u, v) by a plane curve connecting points $p(u)$ and $p(v)$ on $\sum$, where $p : G \to P$ is a mapping from the graph G to P.

For example, a graph $G = (V, E; I)$ with $V = \{v_1, v_2, v_3, v_4\}$, $E = \{e_1, e_2, e_3, e_4, e_5,$

$e_6, e_7, e_8, e_9, e_{10}\}$ and $I(e_i) = (v_i, v_i), 1 \leq i \leq 4; I(e_5) = (v_1, v_2) = (v_2, v_1), I(e_8) = (v_3, v_4) = (v_4, v_3), I(e_6) = I(e_7) = (v_2, v_3) = (v_3, v_2), I(e_8) = I(e_9) = (v_4, v_1) = (v_1, v_4)$ can be drawn on a plane as shown in Fig.2.1

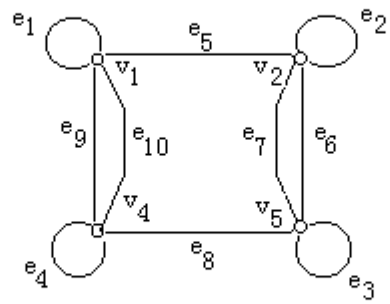


Fig 2.1

In a graph $G = (V, E; I)$, for $\forall e \in E$, if $I(e) = (u, u), u \in V$, then e is called a *loop*. For $\forall e_1, e_2 \in E$, if $I(e_1) = I(e_2)$ and they are not loops, then e_1 and e_2 are called *multiple edges* of G. A graph is *simple* if it is loopless and without multiple edges, i.e., $\forall e_1, e_2 \in E(\Gamma)$, $I(e_1) \neq I(e_2)$ if $e_1 \neq e_2$ and for $\forall e \in E$, if $I(e) = (u, v)$, then $u \neq v$. In a simple graph, an edge (u, v) can be abbreviated to uv.

An edge $e \in E(G)$ can be divided into two semi-arcs e_u, e_v if $I(e) = (u, v)$. Call u the *root vertex* of the semi-arc e_u. Two semi-arc e_u, f_v are said to be *v-incident* or *$e-$incident* if $u = v$ or $e = f$. The set of all semi-arcs of a graph G is denoted by $X_{\frac{1}{2}}(G)$.

A *walk* of a graph Γ is an alternating sequence of vertices and edges $u_1, e_1, u_2, e_2, \cdots, e_n, u_{n_1}$ with $e_i = (u_i, u_{i+1})$ for $1 \leq i \leq n$. The number n is the *length of the walk*. If $u_1 = u_{n+1}$, the walk is said to be *closed*, and *open* otherwise. For example, $v_1e_1v_1e_5v_2e_6v_3e_3v_3e_7v_2e_2v_2$ is a walk in Fig.2.1. A walk is called a *trail* if all its edges are distinct and a *path* if all the vertices are distinct. A closed path is said to be a *circuit*.

A graph $G = (V, E; I)$ is *connected* if there is a path connecting any two vertices in this graph. In a graph, a maximal connected subgraph is called a *component*. A graph G is *k-connected* if removing vertices less than k from G still remains a connected graph. Let G be a graph. For $\forall u \in V(G)$, the neighborhood $N_G(u)$ of the vertex u in G is defined by $N_G(u) = \{v|\forall(u,v) \in E(G)\}$. The cardinal number $|N_G(u)|$ is called the *valency of the vertex* u in the graph G and denoted by $\rho_G(u)$. A vertex v with $\rho_G(v) = 0$ is called an *isolated vertex* and $\rho_G(v) = 1$ a *pendent vertex*. Now we arrange all vertices valency of G as a sequence $\rho_G(u) \geq \rho_G(v) \geq \cdots \geq \rho_G(w)$. Call this sequence the *valency sequence* of G. By enumerating edges in $E(G)$, the following result holds.

$$\sum_{u \in V(G)} \rho_G(u) = 2|E(G)|.$$

Give a sequence $\rho_1, \rho_2, \cdots, \rho_p$ of non-negative integers. If there exists a graph whose valency sequence is $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_p$, then we say that $\rho_1, \rho_2, \cdots, \rho_p$ is a *graphical sequence*. We have known the following results (see [11] for details).

Theorem 2.1.1(Havel,1955 and Hakimi,1962) *A sequence $\rho_1, \rho_2, \cdots, \rho_p$ of non-negative integers with $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_p$, $p \geq 2, \rho_1 \geq 1$ is graphical if and only if the sequence $\rho_2 - 1, \rho_3 - 1, \cdots, \rho_{\rho_1+1} - 1, \rho_{\rho_1+2}, \cdots, \rho_p$ is graphical.*

Theorem 2.1.2(Erdös and Gallai,1960) *A sequence $\rho_1, \rho_2, \cdots, \rho_p$ of non-negative integers with $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_p$ is graphical if and only if $\sum\limits_{i=1}^{p} \rho_i$ is even and for each integer $n, 1 \leq n \leq p-1$,*

$$\sum_{i=1}^{n} \rho_i \leq n(n-1) + \sum_{i=n+1}^{p} min\{n, \rho_i\}.$$

A graph G with a vertex set $V(G) = \{v_1, v_2, \cdots, v_p\}$ and an edge set $E(G) = \{e_1, e_2, \cdots, e_q\}$ can be also described by means of matrix. One such matrix is a $p \times q$ *adjacency matrix* $A(G) = [a_{ij}]_{p \times q}$, where $a_{ij} = |I^{-1}(v_i, v_j)|$. Thus, the adjacency matrix of a graph G is symmetric and is a $0,1$-matrix having 0 entries on its main diagonal if G is simple. For example, the adjacency matrix $A(G)$ of the graph in Fig.2.1 is

$$A(G) = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 1 & 1 & 2 & 0 \\ 0 & 2 & 1 & 1 \\ 2 & 0 & 1 & 1 \end{bmatrix}$$

Let $G_1 = (V_1, E_1; I_1)$ and $G_2 = (V_2, E_2; I_2)$ be two graphs. They are *identical*, denoted by $G_1 = G_2$ if $V_1 = V_2, E_1 = E_2$ and $I_1 = I_2$. If there exists a $1-1$

mapping $\phi : E_1 \to E_2$ and $\phi : V_1 \to V_2$ such that $\phi I_1(e) = I_2\phi(e)$ for $\forall e \in E_1$ with the convention that $\phi(u, v) = (\phi(u), \phi(v))$, then we say that G_1 is *isomorphic* to G_2, denoted by $G_1 \cong G_2$ and ϕ an *isomorphism* between G_1 and G_2. For simple graphs H_1, H_2, this definition can be simplified by $(u, v) \in I_1(E_1)$ if and only if $(\phi(u), \phi(v)) \in I_2(E_2)$ for $\forall u, v \in V_1$.

For example, let $G_1 = (V_1, E_1; I_1)$ and $G_2 = (V_2, E_2; I_2)$ be two graphs with

$$V_1 = \{v_1, v_2, v_3\},$$

$$E_1 = \{e_1, e_2, e_3, e_4\},$$

$$I_1(e_1) = (v_1, v_2), I_1(e_2) = (v_2, v_3), I_1(e_3) = (v_3, v_1), I_1(e_4) = (v_1, v_1)$$

and

$$V_2 = \{u_1, u_2, u_3\},$$

$$E_2 = \{f_1, f_2, f_3, f_4\},$$

$$I_2(f_1) = (u_1, u_2), I_2(f_2) = (u_2, u_3), I_2(f_3) = (u_3, u_1), I_2(f_4) = (u_2, u_2),$$

i.e., the graphs shown in Fig.2.2.

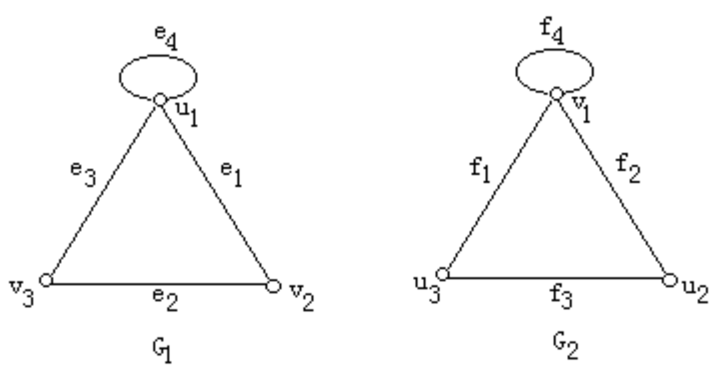


Fig 2.2

Then they are isomorphic since we can define a mapping $\phi : E_1 \bigcup V_1 \rightarrow E_2 \bigcup V_2$ by

$$\phi(e_1) = f_2, \phi(e_2) = f_3, \phi(e_3) = f_1, \phi(e_4) = f_4$$

and $\phi(v_i) = u_i$ for $1 \leq i \leq 3$. It can be verified immediately that $\phi I_1(e) = I_2\phi(e)$ for $\forall e \in E_1$. Therefore, ϕ is an isomorphism between G_1 and G_2.

If $G_1 = G_2 = G$, an isomorphism between G_1 and G_2 is said to be an *automorphism* of G. All automorphisms of a graph G form a group under the composition operation, i.e., $\phi\theta(x) = \phi(\theta(x))$, where $x \in E(G) \bigcup V(G)$. We denote the automorphism group of a graph G by $\mathrm{Aut}G$.

For a simple graph G of n vertices, it is easy to verify that $\mathrm{Aut}G \leq S_n$, the symmetry group action on these n vertices of G. But for non-simple graph, the situation is more complex. The automorphism groups of graphs $K_m, m = |V(K_m)|$ and $B_n, n = |E(B_n)|$ in Fig.2.3 are $\mathrm{Aut}K_m = S_m$ and $\mathrm{Aut}B_n = S_n$.

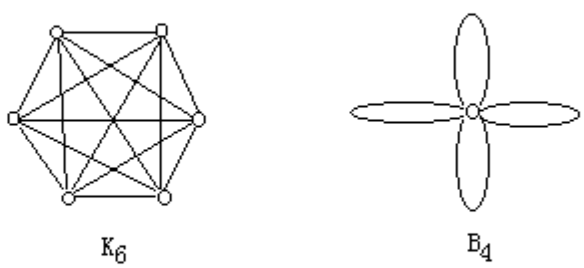


Fig 2.3

For generalizing the conception of automorphisms, the semi-arc automorphisms of a graph were introduced in [53], which is defined in the following definition.

Definition 2.1.1 *A one-to-one mapping ξ on $X_{\frac{1}{2}}(G)$ is called a semi-arc automorphism of a graph G if $\xi(e_u)$ and $\xi(f_v)$ are $v-$incident or $e-$incident if e_u and f_v are $v-$incident or $e-$incident for $\forall e_u, f_v \in X_{\frac{1}{2}}(G)$.*

All semi-arc automorphisms of a graph also form a group, denoted by $\mathrm{Aut}_{\frac{1}{2}}G$. For example, $\mathrm{Aut}_{\frac{1}{2}}B_n = S_n[S_2]$.

For $\forall g \in \mathrm{Aut}G$, there is an induced action $g|^{\frac{1}{2}} : X_{\frac{1}{2}}(G) \rightarrow X_{\frac{1}{2}}(G)$ on $X_{\frac{1}{2}}(G)$ defined by

$$\forall e_u \in X_{\frac{1}{2}}(G), g(e_u) = g(e)_{g(u)}.$$

All induced action of elements in Aut$G$ is denoted by Aut$G|^{\frac{1}{2}}$.

The graph B_n shows that $\mathrm{Aut}_{\frac{1}{2}}G$ may be not the same as Aut$G|^{\frac{1}{2}}$. However, we get a result in the following.

Theorem 2.1.3([56]) *For a graph* Γ *without loops,*

$$\mathrm{Aut}_{\frac{1}{2}}\Gamma = \mathrm{Aut}\Gamma|^{\frac{1}{2}}.$$

Various applications of this theorem to graphs, especially, to combinatorial maps can be found in references [55] − [56] and [66] − [67].

2.1.2. **Subgraphs in a graph**

A graph $H = (V_1, E_1; I_1)$ is a *subgraph* of a graph $G = (V, E; I)$ if $V_1 \subseteq V$, $E_1 \subseteq E$ and $I_1 : E_1 \rightarrow V_1 \times V_1$. We denote that H is a subgraph of G by $H \subset G$. For example, graphs G_1, G_2, G_3 are subgraphs of the graph G in Fig.2.4.

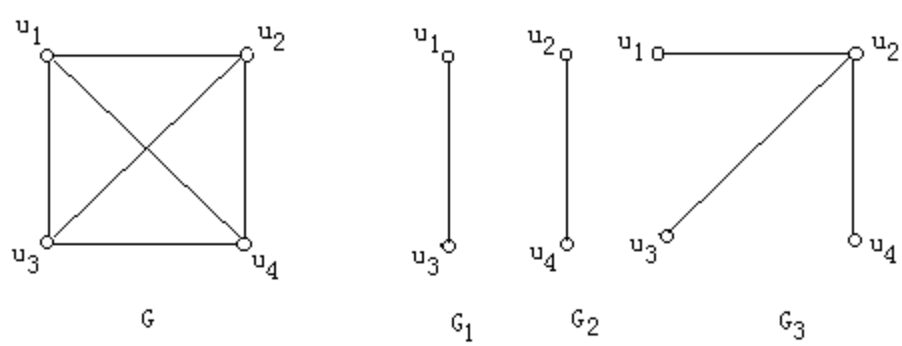


Fig 2.4

For a nonempty subset U of the vertex set $V(G)$ of a graph G, the subgraph $\langle U\rangle$ of G *induced* by U is a graph having vertex set U and whose edge set consists of these edges of G incident with elements of U. A subgraph H of G is called *vertex-induced* if $H \cong \langle U\rangle$ for some subset U of $V(G)$. Similarly, for a nonempty subset F of $E(G)$, the subgraph $\langle F\rangle$ induced by F in G is a graph having edge set F and whose vertex set consists of vertices of G incident with at least one edge of F. A subgraph H of G is *edge-induced* if $H \cong \langle F\rangle$ for some subset F of $E(G)$. In Fig.2.4, subgraphs G_1 and G_2 are both vertex-induced subgraphs $\langle\{u_1, u_4\}\rangle$, $\langle\{u_2, u_3\}\rangle$ and edge-induced subgraphs $\langle\{(u_1, u_4)\}\rangle$, $\langle\{(u_2, u_3)\}\rangle$.

For a subgraph H of G, if $|V(H)| = |V(G)|$, then H is called a *spanning subgraph* of G. In Fig.2.4, the subgraph G_3 is a spanning subgraph of the graph G. Spanning subgraphs are useful for constructing multi-spaces on graphs, see also Section 2.4.

A spanning subgraph without circuits is called a *spanning forest*. It is called a *spanning tree* if it is connected. The following characteristic for spanning trees of a connected graph is well-known.

Theorem 2.1.4 *A subgraph T of a connected graph G is a spanning tree if and only if T is connected and $E(T) = |V(G)| - 1$.*

Proof The necessity is obvious. For its sufficiency, since T is connected and $E(T) = |V(G)| - 1$, there are no circuits in T. Whence, T is a spanning tree. ♮

A path is also a tree in which each vertex has valency 2 unless the two pendent vertices valency 1. We denote a path with n vertices by P_n and define the *length* of P_n to be $n - 1$. For a connected graph G, $x, y \in V(G)$, the distance $d(x, y)$ of x to y in G is defined by

$$d_G(x, y) = min\{\ |V(P(x, y))| - 1 \mid P(x, y)\ is\ a\ path\ connecting\ x\ and\ y\ \}.$$

For $\forall u \in V(G)$, the *eccentricity* $e_G(u)$ of u is defined by

$$e_G(u) = max\{\ d_G(u, x)\ |x \in V(G)\}.$$

A vertex u^+ is called an *ultimate vertex* of a vertex u if $d(u, u^+) = e_G(u)$. Not loss of generality, we arrange these eccentricities of vertices in G in an order $e_G(v_1), e_G(v_2), \cdots$, $e_G(v_n)$ with $e_G(v_1) \leq e_G(v_2) \leq \cdots \leq e_G(v_n)$, where $\{v_1, v_2, \cdots, v_n\} = V(G)$. The sequence $\{e_G(v_i)\}_{1\leq i\leq s}$ is called an *eccentricity sequence* of G. If $\{e_1, e_2, \cdots$, $e_s\} = \{e_G(v_1), e_G(v_2), \cdots, e_G(v_n)\}$ and $e_1 < e_2 < \cdots < e_s$, the sequence $\{e_i\}_{1\leq i\leq s}$ is called an *eccentricity value sequence* of G. For convenience, we abbreviate an integer sequence $\{r - 1 + i\}_{1\leq i\leq s+1}$ to $[r, r + s]$.

The *radius* $r(G)$ and the *diameter* $D(G)$ of G are defined by

$$r(G) = min\{e_G(u)|u \in V(G)\}\ \ \text{and}\ \ D(G) = max\{e_G(u)|u \in V(G)\},$$

respectively. For a given graph G, if $r(G) = D(G)$, then G is called a *self-centered graph*, i.e., the eccentricity value sequence of G is $[r(G), r(G)]$. Some characteristics of self-centered graphs can be found in [47], [64] and [108].

For $\forall x \in V(G)$, we define a *distance decomposition* $\{V_i(x)\}_{1\leq i\leq e_G(x)}$ *of G with root x* by

$$G = V_1(x)\bigoplus V_2(x)\bigoplus\cdots\bigoplus V_{e_G(x)}(x)$$

where $V_i(x) = \{\ u\ |d(x,u) = i, u \in V(G)\}$ for any integer $i, 0 \leq i \leq e_G(x)$. We get a necessary and sufficient condition for the eccentricity value sequence of a simple graph in the following.

Theorem 2.1.5 *A non-decreasing integer sequence $\{r_i\}_{1\leq i\leq s}$ is a graphical eccentricity value sequence if and only if*

(i) $r_1 \leq r_s \leq 2r_1$;

(ii) $\triangle(r_{i+1}, r_i) = |r_{i+1} - r_i| = 1$ *for any integer* $i, 1 \leq i \leq s-1$.

Proof If there is a graph G whose eccentricity value sequence is $\{r_i\}_{1\leq i\leq s}$, then $r_1 \leq r_s$ is trivial. Now we choose three different vertices u_1, u_2, u_3 in G such that $e_G(u_1) = r_1$ and $d_G(u_2, u_3) = r_s$. By definition, we know that $d(u_1, u_2) \leq r_1$ and $d(u_1, u_3) \leq r_1$. According to the triangle inequality for distances, we get that $r_s = d(u_2, u_3) \leq d_G(u_2, u_1) + d_G(u_1, u_3) = d_G(u_1, u_2) + d_G(u_1, u_3) \leq 2r_1$. So $r_1 \leq r_s \leq 2r_1$.

Assume $\{e_i\}_{1\leq i\leq s}$ is the eccentricity value sequence of a graph G. Define $\triangle(i) = e_{i+1} - e_i$, $1 \leq i \leq n-1$. We assert that $0 \leq \triangle(i) \leq 1$. If this assertion is not true, then there must exists a positive integer $I, 1 \leq I \leq n-1$ such that $\triangle(I) = e_{I+1} - e_I \geq 2$. Choose a vertex $x \in V(G)$ such that $e_G(x) = e_I$ and consider the distance decomposition $\{V_i(x)\}_{0\leq i\leq e_G(x)}$ of G with root x.

Notice that it is obvious that $e_G(x) - 1 \leq e_G(u_1) \leq e_G(x) + 1$ for any vertex $u_1 \in V_1(G)$. Since $\triangle(I) \geq 2$, there does not exist a vertex with the eccentricity $e_G(x) + 1$. Whence, we get $e_G(u_1) \leq e_G(x)$ for $\forall u_1 \in V_1(x)$. If we have proved that $e_G(u_j) \leq e_G(x)$ for $\forall u_j \in V_j(x)$, $1 \leq j < e_G(x)$, we consider these eccentricity values of vertices in $V_{j+1}(x)$. Let $u_{j+1} \in V_{j+1}(x)$. According to the definition of $\{V_i(x)\}_{0\leq i\leq e_G(x)}$, there must exists a vertex $u_j \in V_j(x)$ such that $(u_j, u_{j+1}) \in E(G)$. Now consider the distance decomposition $\{V_i(u_j)\}_{0\leq j\leq e_G(u)}$ of G with root u_j. Notice that $u_{j+1} \in V_1(u_j)$. Thereby we get that

$$e_G(u_{j+1}) \le e_G(u_j) + 1 \le e_G(x) + 1.$$

Because we have assumed that there are no vertices with the eccentricity $e_G(x)+1$, so $e_G(u_{j+1}) \le e_G(x)$ for any vertex $u_{j+1} \in V_{j+1}(x)$. Continuing this process, we know that $e_G(y) \le e_G(x) = e_I$ for any vertex $y \in V(G)$. But then there are no vertices with the eccentricity e_I+1, which contradicts the assumption that $\triangle(I) \ge 2$. Therefore $0 \le \triangle(i) \le 1$ and $\triangle(r_{i+1}, r_i) = 1, 1 \le i \le s-1$.

For any integer sequence $\{r_i\}_{1\le i\le s}$ with conditions (i) and (ii) hold, it can be simply written as $\{r, r+1, \cdots, r+s-1\} = [r, r+s-1]$, where $s \le r$. We construct a graph with the eccentricity value sequence $[r, r+s-1]$ in the following.

Case 1 $s = 1$

In this case, $\{r_i\}_{1\le i\le s} = [r, r]$. We can choose any self-centered graph with $r(G) = r$, especially, the circuit C_{2r} of order $2r$. Then its eccentricity value sequence is $[r, r]$.

Case 2 $s \ge 2$

Choose a self-centered graph H with $r(H) = r, x \in V(H)$ and a path $P_s = u_0u_1\cdots u_{s-1}$. Define a new graph $G = P_s \bigodot H$ as follows:

$V(G) = V(P_s)\bigcup V(H) \setminus \{u_0\}$,

$E(G) = (E(P_s)\bigcup\{(x, u_1)\}\bigcup E(H) \setminus \{(u_1, u_0)\}$

such as the graph G shown in Fig.2.5.

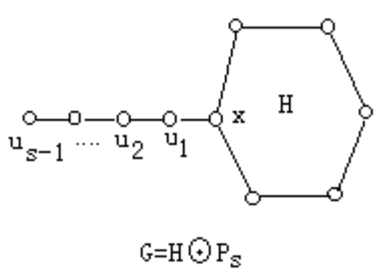


Fig 2.5

Then we know that $e_G(x) = r$, $e_G(u_{s-1}) = r+s-1$ and $r \le e_G(x) \le r+s-1$ for all other vertices $x \in V(G)$. Therefore, the eccentricity value sequence of G is $[r, r+s-1]$. This completes the proof. ♮

For a given eccentricity value l, the *multiplicity set* $N_G(l)$ is defined by $N_G(l) = \{\ x \mid x \in V(G), e(x) = l\ \}$. Jordan proved that the $\langle N_G(r(G))\rangle$ in a tree is a vertex or two adjacent vertices in 1869([11]). For a graph must not being a tree, we get the following result which generalizes Jordan's result for trees.

Theorem 2.1.6 *Let $\{r_i\}_{1\leq i\leq s}$ be a graphical eccentricity value sequence. If $|N_G(r_I)| = 1$, then there must be $I = 1$, i.e., $|N_G(r_i)| \geq 2$ for any integer $i, 2 \leq i \leq s$.*

Proof Let G be a graph with the eccentricity value sequence $\{r_i\}_{1\leq i\leq s}$ and $N_G(r_I) = \{x_0\}, e_G(x_0) = r_I$. We prove that $e_G(x) > e_G(x_0)$ for any vertex $x \in V(G)\setminus\{x_0\}$. Consider the distance decomposition $\{V_i(x_0)\}_{0\leq i\leq e_G(x_0)}$ of G with root x_0. First, we prove that $e_G(v_1) = e_G(x_0) + 1$ for any vertex $v_1 \in V_1(x_0)$. Since $e_G(x_0) - 1 \leq e_G(v_1) \leq e_G(x_0) + 1$ for any vertex $v_1 \in V_1(x_0)$, we only need to prove that $e_G(v_1) > e_G(x_0)$ for any vertex $v_1 \in V_1(x_0)$. In fact, since for any ultimate vertex x_0^+ of x_0, we have that $d_G(x_0, x_0^+) = e_G(x_0)$. So $e_G(x_0^+) \geq e_G(x_0)$. Since $N_G(e_G(x_0)) = \{x_0\}, x_0^+ \notin N_G(e_G(x_0))$. Therefore, $e_G(x_0^+) > e_G(x_0)$. Choose $v_1 \in V_1(x_0)$. Assume the shortest path from v_1 to x_0^+ is $P_1 = v_1v_2\cdots v_sx_0^+$ and $x_0 \notin V(P_1)$. Otherwise, we already have $e_G(v_1) > e_G(x_0)$. Now consider the distance decomposition $\{V_i(x_0^+)\}_{0\leq i\leq e_G(x_0^+)}$ of G with root x_0^+. We know that $v_s \in V_1(x_0^+)$. So we get that

$$e_G(x_0^+) - 1 \leq e_G(v_s) \leq e_G(x_0^+) + 1.$$

Thereafter we get that $e_G(v_s) \geq e_G(x_0^+) - 1 \geq e_G(x_0)$. Because $N_G(e_G(x_0)) = \{x_0\}$, so $v_s \notin N_G(e_G(x_0))$. We finally get that $e_G(v_s) > e_G(x_0)$.

Similarly, choose $v_s, v_{s-1}, \cdots, v_2$ to be root vertices respectively and consider these distance decompositions of G with roots $v_s, v_{s-1}, \cdots, v_2$, we get that

$e_G(v_s) > e_G(x_0)$,

$e_G(v_{s-1}) > e_G(x_0)$,

$\cdots\cdots\cdots\cdots\cdots$,

and

$e_G(v_1) > e_G(x_0)$.

Therefore, $e_G(v_1) = e_G(x_0) + 1$ for any vertex $v_1 \in V_1(x_0)$.

Now consider these vertices in $V_2(x_0)$. For $\forall v_2 \in V_2(x_0)$, assume that v_2 is adjacent to $u_1, u_1 \in V_1(x_0)$. We know that $e_G(v_2) \geq e_G(u_1) - 1 \geq e_G(x_0)$. Since

$|N_G(e_G(x_0))| = |N_G(r_I)| = 1$, we get $e_G(v_2) \geq e_G(x_0) + 1$.

Now assume that we have proved $e_G(v_k) \geq e_G(x_0)+1$ for any vertex $v_k \in V_1(x_0) \bigcup V_2(x_0) \bigcup \cdots \bigcup V_k(x_0)$ for $1 \leq k < e_G(x_0)$. Let $v_{k+1} \in V_{k+1}(x_0)$ and assume that v_{k+1} is adjacent to u_k in $V_k(x_0)$. Then we know that $e_G(v_{k+1}) \geq e_G(u_k) - 1 \geq e_G(x_0)$. Since $|N_G(e_G(x_0))| = 1$, we get that $e_G(v_{k+1}) \geq e_G(x_0) + 1$. Therefore, $e_G(x) > e_G(x_0)$ for any vertex $x, x \in V(G) \setminus \{x_0\}$. That is, if $|N_G(r_I)| = 1$, then there must be $I = 1$. ♮

Theorem 2.1.6 is the best possible in some cases of trees. For example, the eccentricity value sequence of a path P_{2r+1} is $[r, 2r]$ and we have that $|N_G(r)| = 1$ and $|N_G(k)| = 2$ for $r + 1 \leq k \leq 2r$. But for graphs not being trees, we only found some examples satisfying $|N_G(r_1)| = 1$ and $|N_G(r_i)| > 2$. A non-tree graph with the eccentricity value sequence $[2, 3]$ and $|NG(2)| = 1$ can be found in Fig.2 in the reference [64].

For a given graph G and $V_1, V_2 \in V(G)$, define an *edge cut* $E_G(V_1, V_2)$ by

$$E_G(V_1, V_2) = \{ \ (u, v) \in E(G) \mid u \in V_1, v \in V_2\}.$$

A graph G is *hamiltonian* if it has a circuit containing all vertices of G. This circuit is called a *hamiltonian circuit*. A path containing all vertices of a graph G is called a *hamiltonian path*. For hamiltonian circuits, we have the following characteristic.

Theorem 2.1.7 *A circuit C of a graph G without isolated vertices is a hamiltonian circuit if and only if for any edge cut $\mathcal{C}$, $|E(C) \bigcap E(\mathcal{C})| \equiv 0(mod2)$ and $|E(C) \bigcap E(\mathcal{C})| \geq 2$.*

Proof For any circuit C and an edge cut $\mathcal{C}$, the times crossing $\mathcal{C}$ as we travel along C must be even. Otherwise, we can not come back to the initial vertex. if C is a hamiltonian circuit, then $|E(C) \bigcap E(\mathcal{C})| \neq 0$. Whence, $|E(C) \bigcap E(\mathcal{C})| \geq 2$ and $|E(C) \bigcap E(\mathcal{C})| \equiv 0(mod2)$ for any edge cut $\mathcal{C}$.

Now if a circuit C satisfies $|E(C) \bigcap E(\mathcal{C})| \geq 2$ and $|E(C) \bigcap E(\mathcal{C})| \equiv 0(mod2)$ for any edge cut $\mathcal{C}$, we prove that C is a hamiltonian circuit of G. In fact, if $V(G) \setminus V(C) \neq \emptyset$, choose $x \in V(G) \setminus V(C)$. Consider an edge cut $E_G(\{x\}, V(G) \setminus \{x\})$. Since $\rho_G(x) \neq 0$, we know that $|E_G(\{x\}, V(G) \setminus \{x\})| \geq 1$. But since $V(C) \bigcap (V(G) \setminus V(C)) = \emptyset$, we know that $|E_G(\{x\}, V(G) \setminus \{x\}) \bigcap E(C)| = 0$. Contradicts the fact that $|E(C)\bigcap E(\mathcal{C})| \geq 2$ for any edge cut $\mathcal{C}$. Therefore $V(C) = V(G)$ and C is a hamiltonian circuit of G. $\natural$

Let G be a simple graph. The *closure* of G, denoted by $C(G)$, is a graph obtained from G by recursively joining pairs of non-adjacent vertices whose valency sum is at least $|G|$ until no such pair remains. In 1976, Bondy and Chvátal proved a very useful theorem for hamiltonian graphs.

Theorem 2.1.8([5][8]) *A simple graph is hamiltonian if and only if its closure is hamiltonian.*

This theorem generalizes Dirac's and Ore's theorems simultaneously stated as follows:

Dirac (1952): *Every connected simple graph G of order $n \geq 3$ with the minimum valency$\geq \frac{n}{2}$ is hamiltonian.*

Ore (1960): *If G is a simple graph of order $n \geq 3$ such that $\rho_G(u) + \rho_G(v) \geq n$ for all distinct non-adjacent vertices u and v, then G is hamiltonian.*

In 1984, Fan generalized Dirac's theorem to a localized form ([41]). He proved that

Let G be a 2-connected simple graph of order n. If Fan's condition:

$max\{\rho_G(u), \rho_G(v)\} \geq \frac{n}{2}$

holds for $\forall u, v \in V(G)$ provided $d_G(u, v) = 2$, then G is hamiltonian.

After Fan's paper [17], many researches concentrated on weakening Fan's condition and found new localized conditions for hamiltonian graphs. For example, those results in references $[4], [48] - [50], [52], [63]$ and $[65]$ are this type. The next result on hamiltonian graphs is obtained by Shi in 1992 ([84]).

Theorem 2.1.9(Shi, 1992) *Let G be a 2-connected simple graph of order n. Then G contains a circuit passing through all vertices of valency$\geq \frac{n}{2}$.*

Proof Assume the assertion is false. Let $C = v_1v_2\cdots v_kv_1$ be a circuit containing as many vertices of valency$\geq \frac{n}{2}$ as possible and with an orientation on it. For $\forall v \in V(C)$, v^+ denotes the successor and v^- the predecessor of v on C. Set $R = V(G) \setminus V(C)$. Since G is 2-connected, there exists a path length than 2 connecting two vertices of C that is internally disjoint from C and containing one internal vertex x of valency$\geq \frac{n}{2}$ at least. Assume C and P are chosen in such a way that the length of P as small as possible. Let $N_R(x) = N_G(x)\bigcap R$, $N_C(x) = N_G(x)\bigcap C$, $N_C^+(x) = \{v|v^- \in N_C(x)\}$ and $N_C^-(x) = \{v|v^+ \in N_C(x)\}$.

Not loss of generality, we may assume $v_1 \in V(P)\bigcap V(C)$. Let v_t be the other vertex in $V(P)\bigcap V(C)$. By the way C was chosen, there exists a vertex v_s with $1 < s < t$ such that $\rho_G(v_s) \geq \frac{n}{2}$ and $\rho(v_i) < \frac{n}{2}$ for $1 < i < s$.

If $s \geq 3$, by the choice of C and P the sets

$$N_C^-(v_s) \setminus \{v_1\}, N_C(x), N_R(v_s), N_R(x), \{x, v_{s-1}\}$$

are pairwise disjoint, implying that

$$\begin{aligned} n &\geq |N_C^-(v_s) \setminus \{v_1\}| + |N_C(x)| + |N_R(v_s)| + |N_R(x)| + |\{x, v_{s-1}\}| \\ &= \rho_G(v_s) + \rho_G(x) + 1 \geq n + 1, \end{aligned}$$

a contradiction. If $s = 2$, then the sets

$$N_C^-(v_s), N_C(x), N_R(v_s), N_R(x), \{x\}$$

are pairwise disjoint, which yields a similar contradiction. ♮

Three induced subgraphs used in the next result for hamiltonian graphs are shown in Fig.2.6.

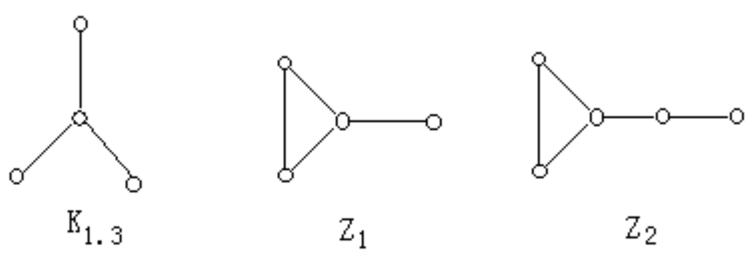


Fig 2.6

For an induced subgraph L of a simple graph G, a condition is called a *localized condition* $D_L(l)$ if $D_L(x, y) = l$ implies that $max\{\rho_G(x), \rho_G(y)\} \geq \frac{|G|}{2}$ for $\forall x, y \in V(L)$. Then we get the following result.

Theorem 2.1.10 *Let G be a 2-connected simple graph. If the localized condition $D_L(2)$ holds for induced subgraphs $L \cong K_{1.3}$ or Z_2 in G, then G is hamiltonian.*

Proof By Theorem 2.1.9, we denote by $c_{\frac{n}{2}}(G)$ the maximum length of circuits passing through all vertices$\geq \frac{n}{2}$. Similar to Theorem 2.1.8, we know that for $x, y \in V(G)$, if $\rho_G(x) \geq \frac{n}{2}$, $\rho_G(y) \geq \frac{n}{2}$ and $xy \notin E(G)$, then $c_{\frac{n}{2}}(G\bigcup\{xy\}) = c_{\frac{n}{2}}(G)$. Otherwise, if $c_{\frac{n}{2}}(G\bigcup\{xy\}) > c_{\frac{n}{2}}(G)$, there exists a circuit of length $c_{\frac{n}{2}}(G\bigcup\{xy\})$ and passing through all vertices$\geq \frac{n}{2}$. Let $C_{\frac{n}{2}}$ be such a circuit and $C_{\frac{n}{2}} = xx_1x_2\cdots x_syx$ with $s = c_{\frac{n}{2}}(G\bigcup\{xy\}) - 2$. Notice that

$$N_G(x)\bigcap(V(G)\setminus V(C_{\frac{n}{2}}(G\bigcup\{xy\}))) = \emptyset$$

and

$$N_G(y)\bigcap(V(G)\setminus V(C_{\frac{n}{2}}(G\bigcup\{xy\}))) = \emptyset.$$

If there exists an integer $i, 1 \leq i \leq s$, $xx_i \in E(G)$, then $x_{i-1}y \notin E(G)$. Otherwise, there is a circuit $C' = xx_ix_{i+1}\cdots x_syx_{i-1}x_{i-2}\cdots x$ in G passing through all vertices$\geq \frac{n}{2}$ with length $c_{\frac{n}{2}}(G\bigcup\{xy\})$. Contradicts the assumption that $c_{\frac{n}{2}}(G\bigcup\{xy\}) > c_{\frac{n}{2}}(G)$. Whence,

$$\rho_G(x) + \rho_G(y) \leq |V(G)\setminus V(C(C_{\frac{n}{2}}))| + |V(C(C_{\frac{n}{2}}))| - 1 = n - 1,$$

also contradicts that $\rho_G(x) \geq \frac{n}{2}$ and $\rho_G(y) \geq \frac{n}{2}$. Therefore, $c_{\frac{n}{2}}(G\bigcup\{xy\}) = c_{\frac{n}{2}}(G)$ and generally, $c_{\frac{n}{2}}(C(G)) = c_{\frac{n}{2}}(G)$.

Now let C be a maximal circuit passing through all vertices$\geq \frac{n}{2}$ in the closure $C(G)$ of G with an orientation $\overrightarrow{C}$. According to Theorem 2.1.8, if $C(G)$ is non-hamiltonian, we can choose H be a component in $C(G)\setminus C$. Define $N_C(H) = (\bigcup\limits_{x\in H} N_{C(G)}(x))\bigcap V(C)$. Since $C(G)$ is 2-connected, we get that $|N_C(H)| \geq 2$. This enables us choose vertices $x_1, x_2 \in N_C(H)$, $x_1 \neq x_2$ and x_1 can arrive at x_2 along $\overrightarrow{C}$. Denote by $x_1\overrightarrow{C}x_2$ the path from x_1 to x_2 on $\overrightarrow{C}$ and $x_2\overleftarrow{C}x_1$ the reverse. Let P be a shortest path connecting x_1, x_2 in $C(G)$ and

$$u_1 \in N_{C(G)}(x_1)\bigcap V(H)\bigcap V(P), \quad u_2 \in N_{C(G)}(x_2)\bigcap V(H)\bigcap V(P).$$

Then

$$E(C(G))\bigcap(\{x_1^-x_2^-,x_1^+x_2^+\}\bigcup E_{C(G)}(\{u_1,u_2\},\{x_1^-,x_1^+,x_2^-,x_2^+\}))=\emptyset$$

and

$$\left\langle\{x_1^-,x_1,x_1^+,u_1\}\right\rangle\not\cong K_{1.3}\text{ or }\left\langle\{x_2^-,x_2,x_2^+,u_2\}\right\rangle\not\cong K_{1.3}.$$

Otherwise, there exists a circuit longer than C, a contradiction. To prove this theorem, we consider two cases.

Case 1 $\left\langle\{x_1^-,x_1,x_1^+,u_1\}\right\rangle\not\cong K_{1.3}$ and $\left\langle\{x_2^-,x_2,x_2^+,u_2\}\right\rangle\not\cong K_{1.3}$

In this case, $x_1^-x_1^+\in E(C(G))$ and $x_2^-x_2^+\in E(C(G))$. By the maximality of C in $C(G)$, we have two claims.

Claim 1.1 $u_1=u_2=u$

Otherwise, let $P=x_1u_1y_1\cdots y_lu_2$. By the choice of P, there must be

$$\left\langle\{x_1^-,x_1,x_1^+,u_1,y_1\}\right\rangle\cong Z_2\text{ and }\left\langle\{x_2^-,x_2,x_2^+,u_2,y_l\}\right\rangle\cong Z_2$$

Since $C(G)$ also has the $D_L(2)$ property, we get that

$$max\{\rho_{C(G)}(x_1^-),\rho_{C(G)}(u_1)\}\geq\frac{n}{2},\quad max\{\rho_{C(G)}(x_12^-),\rho_{C(G)}(u_2)\}\geq\frac{n}{2}.$$

Whence, $\rho_{C(G)}(x_1^-)\geq\frac{n}{2}$, $\rho_{C(G)}(x_2^-)\geq\frac{n}{2}$ and $x_1^-x_2^-\in E(C(G))$, a contradiction.

Claim 1.2 $x_1x_2\in E(C(G))$

If $x_1x_2\notin E(C(G))$, then $\left\langle\{x_1^-,x_1,x_1^+,u,x_2\}\right\rangle\cong Z_2$. Otherwise, we get $x_2x_1^-\in E(C(G))$ or $x_2x_1^+\in E(C(G))$. But then there is a circuit

$$C_1=x_2^+\overrightarrow{C}x_1^-x_2ux_1\overrightarrow{C}x_2^-x_2^+\text{ or }C_2=x_2^+\overrightarrow{C}x_1ux_2x_1^+\overrightarrow{C}x_2^-x_2^+.$$

Contradicts the maximality of C. Therefore, we know that

$$\left\langle\{x_1^-,x_1,x_1^+,u,x_2\}\right\rangle\cong Z_2.$$

By the property $D_L(2)$, we get that $\rho_{C(G)}(x_1^-)\geq\frac{n}{2}$

Similarly, consider the induced subgraph $\left\langle\{x_2^-, x_2, x_2^+, u, x_2\}\right\rangle$, we get that $\rho_{C(G)}(x_2^-) \geq \frac{n}{2}$. Whence, $x_1^- x_2^- \in E(C(G))$, also a contradiction. Thereby we know the structure of G as shown in Fig.2.7.

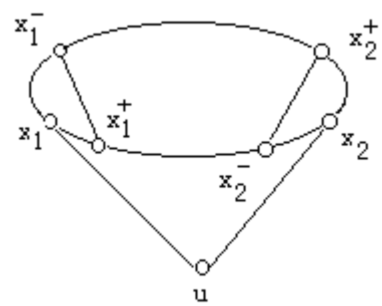


Fig 2.7

By the maximality of C in $C(G)$, it is obvious that $x_1^{--} \neq x_2^+$. We construct an induced subgraph sequence $\{G_i\}_{1\leq i\leq m}$, $m = |V(x_1^- \overleftarrow{C} x_2^+)| - 2$ and prove there exists an integer $r, 1 \leq r \leq m$ such that $G_r \cong Z_2$.

First, we consider the induced subgraph $G_1 = \left\langle\{x_1, u, x_2, x_1^-, x_1^{--}\}\right\rangle$. If $G_1 \cong Z_2$, take $r = 1$. Otherwise, there must be

$$\{x_1^- x_2, x_1^{--} x_2, x_1^{--} u, x_1^{--} x_1\} \bigcap E(C(G)) \neq \emptyset.$$

If $x_1^- x_2 \in E(C(G))$, or $x_1^{--} x_2 \in E(C(G))$, or $x_1^{--} u \in E(C(G))$, there is a circuit $C_3 = x_1^- \overleftarrow{C} x_2^+ x_2^- \overleftarrow{C} x_1 u x_2 x_1^-$, or $C_4 = x_1^{--} \overleftarrow{C} x_2^+ x_2^- \overleftarrow{C} x_1^+ x_1^- x_1 u x_2 x_1^{--}$, or $C_5 = x_1^{--} \overleftarrow{C} x_1^+ x_1^- x_1 u x_1^{--}$. Each of these circuits contradicts the maximality of C. Therefore, $x_1^{--} x_1 \in E(C(G))$.

Now let $x_1^- \overleftarrow{C} x_2^+ = x_1^- y_1 y_2 \cdots y_m x_2^+$, where $y_0 = x_1^-, y_1 = x_1^{--}$ and $y_m = x_2^{++}$. If we have defined an induced subgraph G_k for any integer k and have gotten $y_i x_1 \in E(C(G))$ for any integer $i, 1 \leq i \leq k$ and $y_{k+1} \neq x_2^{++}$, then we define

$$G_{k+1} = \langle\{y_{k+1}, y_k, x_1, x_2, u\}\rangle .$$

If $G_{k+1} \cong Z_2$, then $r = k + 1$. Otherwise, there must be

$$\{y_k u, y_k x_2, y_{k+1} u, y_{k+1} x_2, y_{k+1} x_1\} \bigcap E(C(G)) \neq \emptyset.$$

If $y_k u \in E(C(G))$, or $y_k x_2 \in E(C(G))$, or $y_{k+1} u \in E(C(G))$, or $y_{k+1} x_2 \in$

$E(C(G))$, there is a circuit $C_6 = y_k\overleftarrow{C}x_1^+x_1^-\overleftarrow{C}y_{k-1}x_1uy_k$, or $C_7 = y_k\overleftarrow{C}x_2^+x_2^-\overleftarrow{C}x_1^+x_1^-\overleftarrow{C}$ $y_{k-1}x_1ux_2y_k$, or $C_8 = y_{k+1}\overleftarrow{C}x_1^+x_1^-\overleftarrow{C}y_kx_1uy_{k+1}$, or $C_9 = y_{k+1}\overleftarrow{C}x_2^+x_2^-\overleftarrow{C}x_1^+x_1^-\overleftarrow{C}y_kx_1u$ x_2y_{k+1}. Each of these circuits contradicts the maximality of C. Thereby, $y_{k+1}x_1 \in E(C(G))$.

Continue this process. If there are no subgraphs in $\{G_i\}_{1\leq i\leq m}$ isomorphic to Z_2, we finally get $x_1x_2^{++} \in E(C(G))$. But then there is a circuit $C_{10} = x_1^-\overleftarrow{C}x_2^{++}x_1ux_2x_2^+$ $\overleftarrow{C}x_1^+x_1^-$ in $C(G)$. Also contradicts the maximality of C in $C(G)$. Therefore, there must be an integer $r, 1 \leq r \leq m$ such that $G_r \cong Z_2$.

Similarly, let $x_2^-\overleftarrow{C}x_1^+ = x_2^-z_1z_2\cdots z_tx_1^-$, where $t = |V(x_2^-\overleftarrow{C}x_1^+)| - 2$, $z_0 = x_2^-, z_1^{++} = x_2, z_t = x_1^{++}$. We can also construct an induced subgraph sequence $\{G^i\}_{1\leq i\leq t}$ and know that there exists an integer $h, 1 \leq h \leq t$ such that $G^h \cong Z_2$ and $x_2z_i \in E(C(G))$ for $0 \leq i \leq h-1$.

Since the localized condition $D_L(2)$ holds for an induced subgraph Z_2 in $C(G)$, we get that $max\{\rho_{C(G)}(u), \rho_{C(G)}(y_{r-1})\} \geq \frac{n}{2}$ and $max\{\rho_{C(G)}(u), \rho_{C(G)}(z_{h-1})\} \geq \frac{n}{2}$. Whence $\rho_{C(G)}(y_{r-1}) \geq \frac{n}{2}$, $\rho_{C(G)}(z_{h-1}) \geq \frac{n}{2}$ and $y_{r-1}z_{h-1} \in E(C(G))$. But then there is a circuit

$$C_{11} = y_{r-1}\overleftarrow{C}x_2^+x_2^-\overleftarrow{C}z_{h-2}x_2ux_1y_{r-2}\overrightarrow{C}x_1^-x_1^+\overrightarrow{C}z_{h-1}y_{r-1}$$

in $C(G)$, where if $h = 1$, or $r = 1$, $x_2^-\overleftarrow{C}z_{h-2} = \emptyset$, or $y_{r-2}\overrightarrow{C}x_1^- = \emptyset$. Also contradicts the maximality of C in $C(G)$.

Case 2 $\left\langle\{x_1^-, x_1, x_1^+, u_1\}\right\rangle \not\cong K_{1.3}$, $\left\langle\{x_2^-, x_2, x_2^+, u_2\}\right\rangle \cong K_{1.3}$ or $\left\langle\{x_1^-, x_1, x_1^+, u_1\}\right\rangle \cong K_{1.3}$, $\left\langle\{x_2^-, x_2, x_2^+, u_2\}\right\rangle \not\cong K_{1.3}$

Not loss of generality, we assume that $\left\langle\{x_1^-, x_1, x_1^+, u_1\}\right\rangle \not\cong K_{1.3}$, $\left\langle\{x_2^-, x_2, x_2^+, u_2\}\right\rangle$ $\cong K_{1.3}$. Since each induced subgraph $K_{1.3}$ in $C(G)$ possesses $D_L(2)$, we get that $max\{\rho_{C(G)}(u), \rho_{C(G)}(x_2^-)\} \geq \frac{n}{2}$ and $max\{\rho_{C(G)}(u), \rho_{C(G)}(x_2^+)\} \geq \frac{n}{2}$. Whence $\rho_{C(G)}(x_2^-)$ $\geq \frac{n}{2}$, $\rho_{C(G)}(x_2^+) \geq \frac{n}{2}$ and $x_2^-x_2^+ \in E(C(G))$. Therefore, the discussion of Case 1 also holds in this case and yields similar contradictions.

Combining Case 1 with Case 2, the proof is complete. ♮

Let G, $F_1, F_2, \cdots, F_k$ be $k+1$ graphs. If there are no induced subgraphs of G isomorphic to $F_i, 1 \leq i \leq k$, then G is called $\{F_1, F_2, \cdots, F_k\}$*-free*. we get a immediately consequence by Theorem 2.1.10.

Corollary 2.1.1 *Every* 2*-connected* $\{K_{1.3}, Z_2\}$*-free graph is hamiltonian.*

Let G be a graph. For $\forall u \in V(G)$, $\rho_G(u) = d$, let H be a graph with d pendent vertices $v_1, v_2, \cdots, v_d$. Define a splitting operator $\vartheta : G \to G^{\vartheta(u)}$ on u by

$$V(G^{\vartheta(u)}) = (V(G) \setminus \{u\})\bigcup(V(H) \setminus \{v_1, v_2, \cdots, v_d\}),$$

$$\begin{aligned} E(G^{\vartheta(u)}) &= (E(G) \setminus \{ux_i \in E(G), 1 \le i \le d\}) \\ &\bigcup (E(H) \setminus \{v_iy_i \in E(H), 1 \le i \le d\})\bigcup\{x_iy_i, 1 \le i \le d\}. \end{aligned}$$

We call d the *degree of the splitting operator* ϑ and $N(\vartheta(u)) = H \setminus \{x_iy_i, 1 \le i \le d\}$ the *nucleus of* ϑ . A splitting operator is shown in Fig.2.8.

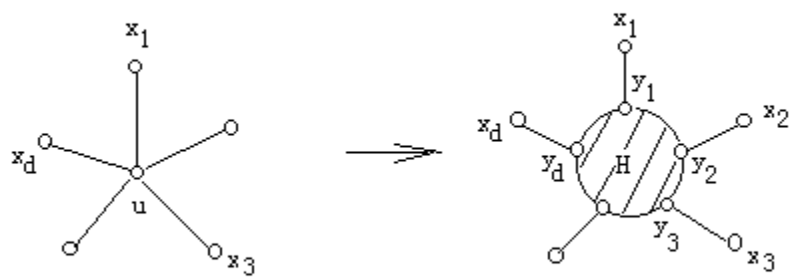


Fig 2.9

Erdös and Rényi raised a question in 1961 ([7]): *in what model of random graphs is it true that almost every graph is hamiltonian?* Pósa and Korshuuov proved independently that for some constant c almost every labelled graph with n vertices and at least $n \log n$ edges is hamiltonian in 1974. Contrasting this probabilistic result, there is another property for hamiltonian graphs, i.e., there is a splitting operator ϑ such that $G^{\vartheta(u)}$ is non-hamiltonian for $\forall u \in V(G)$ of a graph G.

Theorem 2.1.11 *Let G be a graph. For $\forall u \in V(G)$, $\rho_G(u) = d$, there exists a splitting operator ϑ of degree d on u such that $G^{\vartheta(u)}$ is non-hamiltonian.*

Proof For any positive integer i, define a simple graph Θ_i by $V(\Theta_i) = \{x_i, y_i, z_i, u_i\}$ and $E(\Theta_i) = \{x_iy_i, x_iz_i, y_iz_i, y_iu_i, z_iu_i\}$. For integers $\forall i, j \ge 1$, the point product $\Theta_i \odot \Theta_j$ of Θ_i and Θ_j is defined by

$$V(\Theta_i \odot \Theta_j) = V(\Theta_i)\bigcup V(\Theta_j) \setminus \{u_j\},$$

$$E(\Theta_i \odot \Theta_j) = E(\Theta_i)\bigcup E(\Theta_j)\bigcup\{x_iy_j, x_iz_j\} \setminus \{x_jy_j, x_jz_j\}.$$

Now let H_d be a simple graph with

$$V(H_d) = V(\Theta_1 \odot \Theta_2 \odot \cdots \Theta_{d+1})\bigcup\{v_1, v_2, \cdots, v_d\},$$

$$E(H_d) = E(\Theta_1 \odot \Theta_2 \odot \cdots \Theta_{d+1})\bigcup\{v_1u_1, v_2u_2, \cdots, v_du_d\}.$$

Ther

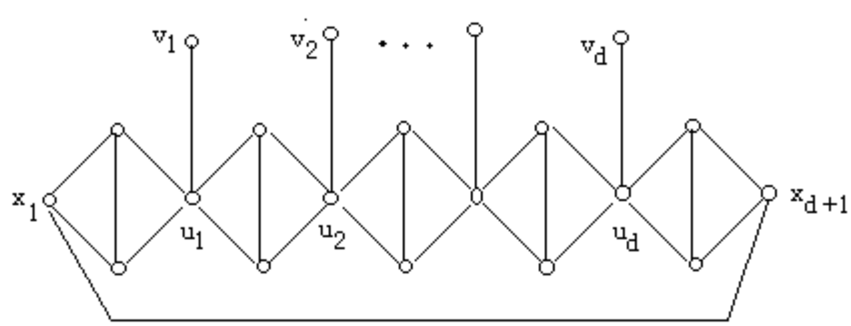


Fig 2.10

For any graph G and $w \in V(G), \rho_G(w) = d$, we prove that $G^{\vartheta(w)}$ is non-hamiltonian. In fact, If $G^{\vartheta(w)}$ is a hamiltonian graph, then there must be a hamiltonian path $P(u_i, u_j)$ connecting two vertices u_i, u_j for some integers $i, j, 1 \leq i, j \leq d$ in the graph $H_d \setminus \{v_1, v_2, \cdots, v_d\}$. However, there are no hamiltonian path connecting vertices u_i, u_j in the graph $H_d \setminus \{v_1, v_2, \cdots, v_d\}$ for any integer $i, j, 1 \leq i, j \leq d$. Therefore, $G^{\vartheta(w)}$ is non-hamiltonian. ♮

2.1.3. **Classes of graphs with decomposition**

(1) **Typical classes of graphs**

C1. Bouquets and Dipoles

In graphs, two simple cases is these graphs with one or two vertices, which are just bouquets or dipoles. A graph $B_n = (V_b, E_b; I_b)$ with $V_b = \{\ O\ \}$, $E_b = \{e_1, e_2, \cdots, e_n\}$ and $I_b(e_i) = (O, O)$ for any integer $i, 1 \leq i \leq n$ is called a *bouquet* of n edges. Similarly, a graph $D_{s.l.t} = (V_d, E_d; I_d)$ is called a *dipole* if $V_d = \{O_1, O_2\}$, $E_d = \{e_1, e_2, \cdots, e_s, e_{s+1}, \cdots, e_{s+l}, e_{s+l+1}, \cdots, e_{s+l+t}\}$ and

$$I_d(e_i) = \begin{cases} (O_1, O_1), & \text{if } 1 \le i \le s, \\ (O_1, O_2), & \text{if } s+1 \le i \le s+l, \\ (O_2, O_2), & \text{if } s+l+1 \le i \le s+l+t. \end{cases}$$

For example, B_3 and $D_{2,3,2}$ are shown in Fig.2.11.

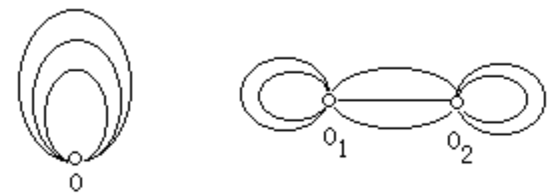


Fig 2.11

In the past two decades, the behavior of bouquets on surfaces fascinated many mathematicians. A typical example for its application to mathematics is the classification theorem of surfaces. By a combinatorial view, these connected sums of tori, or these connected sums of projective planes used in this theorem are just bouquets on surfaces. In Section 2.4, we will use them to construct completed multi-spaces.

C2. Complete graphs

A *complete graph* $K_n = (V_c, E_c; I_c)$ is a simple graph with $V_c = \{v_1, v_2, \cdots, v_n\}$, $E_c = \{e_{ij}, 1 \le i, j \le n, i \ne j\}$ and $I_c(e_{ij}) = (v_i, v_j)$. Since K_n is simple, it can be also defined by a pair (V, E) with $V = \{v_1, v_2, \cdots, v_n\}$ and $E = \{v_i v_j, 1 \le i, j \le n, i \ne j\}$. The one edge graph K_2 and the triangle graph K_3 are both complete graphs.

A complete subgraph in a graph is called a *clique*. Obviously, every graph is a union of its cliques.

C3. r-Partite graphs

A simple graph $G = (V, E; I)$ is *r-partite* for an integer $r \ge 1$ if it is possible to partition V into r subsets $V_1, V_2, \cdots, V_r$ such that for $\forall e \in E$, $I(e) = (v_i, v_j)$ for $v_i \in V_i$, $v_j \in V_j$ and $i \ne j, 1 \le i, j \le r$. Notice that by definition, there are no edges between vertices of V_i, $1 \le i \le r$. A vertex subset of this kind in a graph is called an *independent vertex subset*.

For $n = 2$, a 2-partite graph is also called a *bipartite graph*. It can be shown that *a graph is bipartite if and only if there are no odd circuits in this graph*. As a consequence, a tree or a forest is a bipartite graph since they are circuit-free.

Let $G = (V, E; I)$ be an r-partite graph and let $V_1, V_2, \cdots, V_r$ be its r-partite vertex subsets. If there is an edge $e_{ij} \in E$ for $\forall v_i \in V_i$ and $\forall v_j \in V_j$, where $1 \leq i, j \leq r, i \neq j$ such that $I(e) = (v_i, v_j)$, then we call G a *complete r-partite graph*, denoted by $G = K(|V_1|, |V_2|, \cdots, |V_r|)$. Whence, a complete graph is just a complete 1-partite graph. For an integer n, the complete bipartite graph $K(n, 1)$ is called a *star*. For a graph G, we have an obvious formula shown in the following, which corresponds to the neighborhood decomposition in topology.

$$E(G) = \bigcup_{x \in V(G)} E_G(x, N_G(x)).$$

C4. Regular graphs

A graph G is *regular of valency k* if $\rho_G(u) = k$ for $\forall u \in V(G)$. These graphs are also called *k-regular*. There 3-regular graphs are referred to as *cubic graphs*. A k-regular vertex-spanning subgraph of a graph G is also called a k-factor of G.

For a k-regular graph G, since $k|V(G)| = 2|E(G)|$, thereby one of k and $|V(G)|$ must be an even number, i.e., there are no k-regular graphs of odd order with $k \equiv 1(mod 2)$. A complete graph K_n is $(n-1)$-regular and a complete s-partite graph $K(p_1, p_2, \cdots, p_s)$ of order n with $p_1 = p_2 = \cdots = p_s = p$ is $(n-p)$-regular.

In regular graphs, those of simple graphs with high symmetry are particularly important to mathematics. They are related combinatorics with group theory and crystal geometry. We briefly introduce them in the following.

Let G be a simple graph and H a subgroup of Aut$G$. G is said to be *H-vertex transitive*, *H-edge transitive* or *H-symmetric* if H acts transitively on the vertex set $V(G)$, the edge set $E(G)$ or the set of ordered adjacent pairs of vertex of G. If $H = \mathrm{Aut}G$, an H-vertex transitive, an H-edge transitive or an H-symmetric graph is abbreviated to a *vertex-transitive*, an *edge-transitive* or a *symmetric* graph.

Now let Γ be a finite generated group and $S \subseteq \Gamma$ such that $1_\Gamma \notin S$ and $S^{-1} = \{x^{-1} | x \in S\} = S$. A *Cayley graph Cay*$(\Gamma : S)$ is a simple graph with vertex set $V(G) = \Gamma$ and edge set $E(G) = \{(g, h) | g^{-1}h \in S\}$. By the definition of Cayley graphs, we know that *a Cayley graph Cay*$(\Gamma : S)$ *is complete if and only if*

$S = \Gamma \setminus \{1_\Gamma\}$ *and connected if and only if* $\Gamma = \langle S\rangle$.

Theorem 2.1.12 *A Cayley graph* $Cay(\Gamma : S)$ *is vertex-transitive.*

Proof For $\forall g \in \Gamma$, define a permutation ζ_g on $V(\mathrm{Cay}(\Gamma : S)) = \Gamma$ by $\zeta_g(h) = gh, h \in \Gamma$. Then ζ_g is an automorphism of $\mathrm{Cay}(\Gamma : S)$ for $(h,k) \in E(\mathrm{Cay}(\Gamma : S)) \Rightarrow h^{-1}k \in S \Rightarrow (gh)^{-1}(gk) \in S \Rightarrow (\zeta_g(h), \zeta_g(k)) \in E(\mathrm{Cay}(\Gamma : S))$.

Now we know that $\zeta_{kh^{-1}}(h) = (kh^{-1})h = k$ for $\forall h, k \in \Gamma$. Whence, $\mathrm{Cay}(\Gamma : S)$ is vertex-transitive. ♮

Not every vertex-transitive graph is a Cayley graph of a finite group. For example, the Petersen graph is vertex-transitive but not a Cayley graph(see [10], [21]] and [110] for details). However, every vertex-transitive graph can be constructed almost like a Cayley graph. This result is due to Sabidussi in 1964. The readers can see [110] for a complete proof of this result.

Theorem 2.1.13 *Let* G *be a vertex-transitive graph whose automorphism group is* A*. Let* $H = A_b$ *be the stabilizer of* $b \in V(G)$*. Then* G *is isomorphic with the group-coset graph* $C(A/H, S)$*, where* S *is the set of all automorphisms* x *of* G *such that* $(b, x(b)) \in E(G)$*,* $V(C(A/H,S)) = A/H$ *and* $E(C(A/H,S)) = \{(xH, yH) | x^{-1}y \in HSH\}$.

C5. Planar graphs

Every graph is drawn on the plane. A graph is *planar* if it can be drawn on the plane in such a way that edges are disjoint expect possibly for endpoints. When we remove vertices and edges of a planar graph G from the plane, each remained connected region is called a *face* of G. The length of the boundary of a face is called its *valency*. Two planar graphs are shown in Fig.2.12.

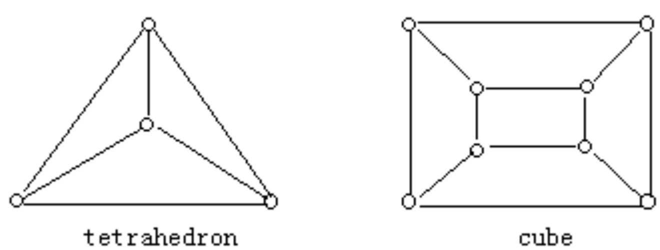


Fig 2.12

For a planar graph G, its order, size and number of faces are related by a well-known formula discovered by Euler.

Theorem 2.1.14 *let G be a planar graph with $\phi(G)$ faces. Then*

$$|G| - \varepsilon(G) + \phi(G) = 2.$$

Proof We can prove this result by employing induction on $\varepsilon(G)$. See [42] or [23], [69] for a complete proof. $\natural$

For an integer $s, s \geq 3$, an s-regular planar graph with the same length r for all faces is often called an (s,r)*-polyhedron*, which are completely classified by the ancient Greeks.

Theorem 2.1.15 There are exactly five polyhedrons, two of them are shown in Fig.2.12, the others are shown in Fig.2.13.

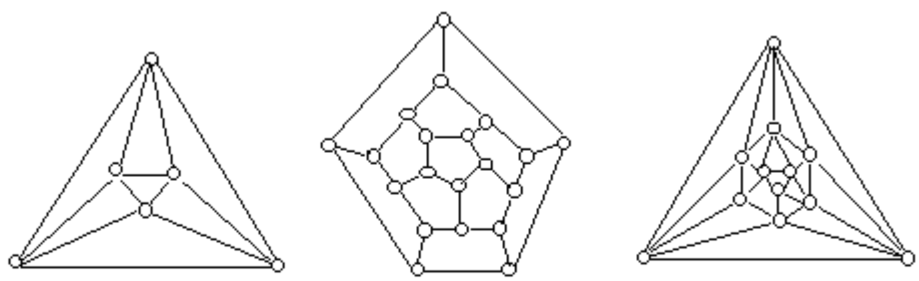


Fig 2.13

Proof Let G be a k-regular planar graph with l faces. By definition, we know that $|G|k = \phi(G)l = 2\varepsilon(G)$. Whence, we get that $|G| = \frac{2\varepsilon(G)}{k}$ and $\phi(G) = \frac{2\varepsilon(G)}{l}$. According to Theorem 2.1.14, we get that

$$\frac{2\varepsilon(G)}{k} - \varepsilon(G) + \frac{2\varepsilon(G)}{l} = 2.$$

i.e.,

$$\varepsilon(G) = \frac{2}{\frac{2}{k} - 1 + \frac{2}{l}}.$$

Whence, $\frac{2}{k}+\frac{2}{l}-1>0$. Since k,l are both integers and $k\geq 3, l\geq 3$, if $k\geq 6$, we get

$$\frac{2}{k}+\frac{2}{l}-1\leq\frac{2}{3}+\frac{2}{6}-1=0.$$

Contradicts that $\frac{2}{k}+\frac{2}{l}-1>0$. Therefore, $k\leq 5$. Similarly, $l\leq 5$. So we have $3\leq k\leq 5$ and $3\leq l\leq 5$. Calculation shows that all possibilities for (k,l) are $(k,l)=(3,3),(3,4),(3,5),(4,3)$ and $(5,3)$. The $(3,3)$ and $(3,4)$ polyhedrons have be shown in Fig.2.12 and the remainder $(3,5),(4,3)$ and $(5,3)$ polyhedrons are shown in Fig.2.13. $\natural$

An *elementary subdivision* on a graph G is a graph obtained from G replacing an edge $e=uv$ by a path uwv, where, $w\not\in V(G)$. A *subdivision* of G is a graph obtained from G by a succession of elementary subdivision. A graph H is defined to be a *homeomorphism of* G if either $H\cong G$ or H is isomorphic to a subdivision of G. Kuratowski found the following characterization for planar graphs in 1930. For its a complete proof, see [9], [11] for details.

Theorem 2.1.16 *A graph is planar if and only if it contains no subgraph homeomorphic with K_5 or $K(3,3)$.*

(2) **Decomposition of graphs**

A complete graph K_6 with vertex set $\{1,2,3,4,5,6\}$ has two families of subgraphs $\{C_6, C_3^1, C_3^2, P_2^1, P_2^2, P_2^3\}$ and $\{S_{1.5}, S_{1.4}, S_{1.3}, S_{1.2}, S_{1.1}\}$, such as those shown in Fig.2.14 and Fig.2.15.

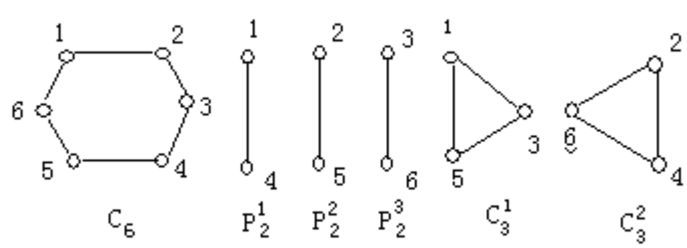


Fig 2.14

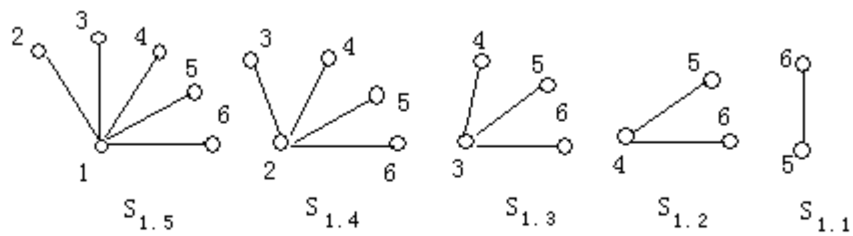


Fig 2.15

We know that

$$E(K_6) = E(C_6)\bigcup E(C_3^1)\bigcup E(C_3^2)\bigcup E(P_2^1)\bigcup E(P_2^2)\bigcup E(P_2^3);$$

$$E(K_6) = E(S_{1.5})\bigcup E(S_{1.4})\bigcup E(S_{1.3})\bigcup E(S_{1.2})\bigcup E(S_{1.1}).$$

These formulae imply the conception of decomposition of graphs. For a graph G, a *decomposition* of G is a collection $\{H_i\}_{1\leq i\leq s}$ of subgraphs of G such that for any integer $i, 1 \leq i \leq s$, $H_i = \langle E_i\rangle$ for some subsets E_i of $E(G)$ and $\{E_i\}_{1\leq i\leq s}$ is a partition of $E(G)$, denoted by $G = H_1 \bigoplus H_2 \bigoplus \cdots \bigoplus H_s$. The following result is obvious.

Theorem 2.1.17 *Any graph G can be decomposed to bouquets and dipoles, in where K_2 is seen as a dipole $D_{0.1.0}$.*

Theorem 2.1.18 *For every positive integer n, the complete graph K_{2n+1} can be decomposed to n hamiltonian circuits.*

Proof For $n = 1$, K_3 is just a hamiltonian circuit. Now let $n \geq 2$ and $V(K_{2n+1}) = \{v_0, v_1, v_2, \cdots, v_{2n}\}$. Arrange these vertices $v_1, v_2, \cdots, v_{2n}$ on vertices of a regular $2n$-gon and place v_0 in a convenient position not in the $2n$-gon. For $i = 1, 2, \cdots, n$, we define the edge set of H_i to be consisted of v_0v_i, v_0v_{n+i} and edges parallel to v_iv_{i+1} or edges parallel to $v_{i-1}v_{i+1}$, where the subscripts are expressed modulo $2n$. Then we get that

$$K_{2n+1} = H_1 \bigoplus H_2 \bigoplus \cdots \bigoplus H_n$$

with each $H_i, 1 \leq i \leq n$ being a hamiltonian circuit

$$v_0 v_i v_{i+1} v_{i-1} v_{i+1} v_{i-2} \cdots v_{n+i-1} v_{n+i+1} v_{n+i} v_0. \quad \natural$$

Every Cayley graph of a finite group Γ can be decomposed into 1-factors or 2-factors in a natural way as stated in the following theorems.

Theorem 2.1.19 *Let G be a vertex-transitive graph and let H be a regular subgroup of* Aut$G$*. Then for any chosen vertex $x, x \in V(G)$, there is a factorization*

$$G = (\bigoplus_{y \in N_G(x), |H_{(x,y)}|=1} (x,y)^H) \bigoplus (\bigoplus_{y \in N_G(x), |H_{(x,y)}|=2} (x,y)^H),$$

for G such that $(x,y)^H$ is a 2-factor if $|H_{(x,y)}| = 1$ and a 1-factor if $|H_{(x,y)}| = 2$.

Proof First, We prove the following claims.

Claim 1 $\forall x \in V(G), x^H = V(G)$ *and* $H_x = 1_H$.

Claim 2 *For* $\forall (x,y), (u,w) \in E(G)$, $(x,y)^H \bigcap (u,w)^H = \emptyset$ *or* $(x,y)^H = (u,w)^H$.

Claims 1 and 2 are holden by definition.

Claim 3 *For* $\forall (x,y) \in E(G), |H_{(x,y)}| = 1 \quad or \quad 2$.

Assume that $|H_{(x,y)}| \neq 1$. Since we know that $(x,y)^h = (x,y)$, i.e., $(x^h, y^h) = (x,y)$ for any element $h \in H_{(x,y)}$. Thereby we get that $x^h = x$ and $y^h = y$ or $x^h = y$ and $y^h = x$. For the first case we know $h = 1_H$ by Claim 1. For the second, we get that $x^{h^2} = x$. Therefore, $h^2 = 1_H$.

Now if there exists an element $g \in H_{(x,y)} \setminus \{1_H, h\}$, then we get $x^g = y = x^h$ and $y^g = x = y^h$. Thereby we get $g = h$ by Claim 1, a contradiction. So we get that $|H_{(x,y)}| = 2$.

Claim 4 *For any* $(x,y) \in E(G)$, *if* $|H_{(x,y)}| = 1$, *then* $(x,y)^H$ *is a 2-factor.*

Because $x^H = V(G) \subset V(\left\langle (x,y)^H \right\rangle) \subset V(G)$, so $V(\left\langle (x,y)^H \right\rangle) = V(G)$. Therefore, $(x,y)^H$ is a spanning subgraph of G.

Since H acting on $V(G)$ is transitive, there exists an element $h \in H$ such that $x^h = y$. It is obvious that $o(h)$ is finite and $o(h) \neq 2$. Otherwise, we have $|H_{(x,y)}| \geq 2$, a contradiction. Now $(x,y)^{\langle h \rangle} = x x^h x^{h^2} \cdots x^{h^{o(h)-1}} x$ is a circuit in the graph

G. Consider the right coset decomposition of H on $\langle h\rangle$. Suppose $H = \bigcup\limits_{i=1}^{s} \langle h\rangle a_i$, $\langle h\rangle a_i \bigcap \langle h\rangle a_j = \emptyset$, if $i \neq j$, and $a_1 = 1_H$.

Now let $X = \{a_1, a_2, ..., a_s\}$. We know that for any $a, b \in X$, $(\langle h\rangle a)\bigcap(\langle h\rangle b) = \emptyset$ if $a \neq b$. Since $(x,y)^{\langle h\rangle a} = ((x,y)^{\langle h\rangle})^a$ and $(x,y)^{\langle h\rangle b} = ((x,y)^{\langle h\rangle})^b$ are also circuits, if $V(\left\langle (x,y)^{\langle h\rangle a}\right\rangle)\bigcap V(\left\langle (x,y)^{\langle h\rangle b}\right\rangle) \neq \emptyset$ for some $a, b \in X, a \neq b$, then there must be two elements $f, g \in \langle h\rangle$ such that $x^{fa} = x^{gb}$. According to Claim 1, we get that $fa = gb$, that is $ab^{-1} \in \langle h\rangle$. So $\langle h\rangle a = \langle h\rangle b$ and $a = b$, contradicts to the assumption that $a \neq b$.

Thereafter we know that $(x,y)^H = \bigcup\limits_{a\in X} (x,y)^{\langle h\rangle a}$ is a disjoint union of circuits. So $(x,y)^H$ is a 2-factor of the graph G.

Claim 5 *For any $(x,y) \in E(G)$, $(x,y)^H$ is an 1-factor if $|H_{(x,y)}| = 2$.*

Similar to the proof of Claim 4, we know that $V(\left\langle (x,y)^H\right\rangle) = V(G)$ and $(x,y)^H$ is a spanning subgraph of the graph G.

Let $H_{(x,y)} = \{1_H, h\}$, where $x^h = y$ and $y^h = x$. Notice that $(x,y)^a = (x,y)$ for $\forall a \in H_{(x,y)}$. Consider the coset decomposition of H on $H_{(x,y)}$, we know that $H = \bigcup\limits_{i=1}^{t} H_{(x,y)}b_i$, where $H_{(x,y)}b_i \bigcap H_{(x,y)}b_j = \emptyset$ if $i \neq j, 1 \leq i, j \leq t$. Now let $L = \{H_{(x,y)}b_i, 1 \leq i \leq t\}$. We get a decomposition

$$(x,y)^H = \bigcup_{b\in L} (x,y)^b$$

for $(x,y)^H$. Notice that if $b = H_{(x,y)}b_i \in L$, $(x,y)^b$ is an edge of G. Now if there exist two elements $c, d \in L, c = H_{(x,y)}f$ and $d = H_{(x,y)}g$, $f \neq g$ such that $V(\langle (x,y)^c\rangle)\bigcap V(\left\langle (x,y)^d\right\rangle) \neq \emptyset$, there must be $x^f = x^g$ or $x^f = y^g$. If $x^f = x^g$, we get $f = g$ by Claim 1, contradicts to the assumption that $f \neq g$. If $x^f = y^g = x^{hg}$, where $h \in H_{(x,y)}$, we get $f = hg$ and $fg^{-1} \in H_{(x,y)}$, so $H_{(x,y)}f = H_{(x,y)}g$. According to the definition of L, we get $f = g$, also contradicts to the assumption that $f \neq g$. Therefore, $(x,y)^H$ is an 1-factor of the graph G.

Now we can prove the assertion in this theorem. According to Claim 1- Claim 4, we get that

$$G = \Big(\bigoplus_{y\in N_G(x),|H_{(x,y)}|=1} (x,y)^H\Big)\bigoplus\Big(\bigoplus_{y\in N_G(x),|H_{(x,y)}|=2} (x,y)^H\Big).$$

for any chosen vertex $x, x \in V(G)$. By Claims 5 and 6, we know that $(x,y)^H$ is a 2-factor if $|H_{(x,y)}| = 1$ and is a 1-factor if $|H_{(x,y)}| = 2$. Whence, the desired factorization for G is obtained. $\natural$

Now for a Cayley graph $\mathrm{Cay}(\Gamma : S)$, by Theorem 2.1.13, we can always choose the vertex $x = 1_\Gamma$ and H the right regular transformation group on Γ. After then, Theorem 2.1.19 can be restated as follows.

Theorem 2.1.20 *Let Γ be a finite group with a subset $S, S^{-1} = S$, $1_\Gamma \notin S$ and H is the right transformation group on Γ. Then there is a factorization*

$$G = (\bigoplus_{s \in S, s^2 \neq 1_\Gamma} (1_\Gamma, s)^H) \bigoplus (\bigoplus_{s \in S, s^2 = 1_\Gamma} (1_\Gamma, s)^H)$$

for the Cayley graph $\mathrm{Cay}(\Gamma : S)$ *such that $(1_\Gamma, s)^H$ is a 2-factor if $s^2 \neq 1_\Gamma$ and* 1*-factor if $s^2 = 1_\Gamma$.*

Proof For any $h \in H_{(1_\Gamma, s)}$, if $h \neq 1_\Gamma$, then we get that $1_\Gamma h = s$ and $sh = 1_\Gamma$, that is $s^2 = 1_\Gamma$. According to Theorem 2.1.19, we get the factorization for the Cayley graph $\mathrm{Cay}(\Gamma : S)$. $\natural$

More factorial properties for Cayley graphs of a finite group can be found in the reference [51].

2.1.4. **Operations on graphs**

For two given graphs $G_1 = (V_1.E_1; I_1)$ and $G_2 = (V_2, E_2; I_2)$, there are a number of ways to produce new graphs from G_1 and G_2. Some of them are described in the following.

Operation 1. **Union**

The *union* $G_1 \bigcup G_2$ of graphs G_1 and G_2 is defined by

$$V(G_1 \bigcup G_2) = V_1 \bigcup V_2,\ E(G_1 \bigcup G_2) = E_1 \bigcup E_2 \text{ and } I(E_1 \bigcup E_2) = I_1(E_1) \bigcup I_2(E_2).$$

If a graph consists of k disjoint copies of a graph H, $k \geq 1$, then we write $G = kH$. Therefore, we get that $K_6 = C_6 \bigcup 3K_2 \bigcup 2K_3 = \bigcup\limits_{i=1}^{5} S_{1.i}$ for graphs in Fig.2.14 and

Fig.2.15 and generally, $K_n = \bigcup\limits_{i=1}^{n-1} S_{1.i}$. For an integer $k, k \geq 2$ and a simple graph G, kG is a multigraph with edge multiple k by definition.

By the definition of a union of two graphs, we get decompositions for some well-known graphs such as

$$B_n = \bigcup_{i=1}^{n} B_1(O), \quad D_{k,m,n} = (\bigcup_{i=1}^{k} B_1(O_1))\bigcup(\bigcup_{i=1}^{m} K_2)\bigcup(\bigcup_{i=1}^{n} B_1(O_2)),$$

where $V(B_1)(O_1) = \{O_1\}, V(B_1)(O_2) = \{O_2\}$ and $V(K_2) = \{O_1, O_2\}$. By Theorem 1.18, we get that

$$K_{2n+1} = \bigcup_{i=1}^{n} H_i$$

with $H_i = v_0 v_i v_{i+1} v_{i-1} v_{i+1} v_{i-2} \cdots v_{n+i-1} v_{n+i+1} v_{n+i} v_0$.

In Fig.2.16, we show two graphs C_6 and K_4 with a nonempty intersection and their union $C_6 \bigcup K_4$.

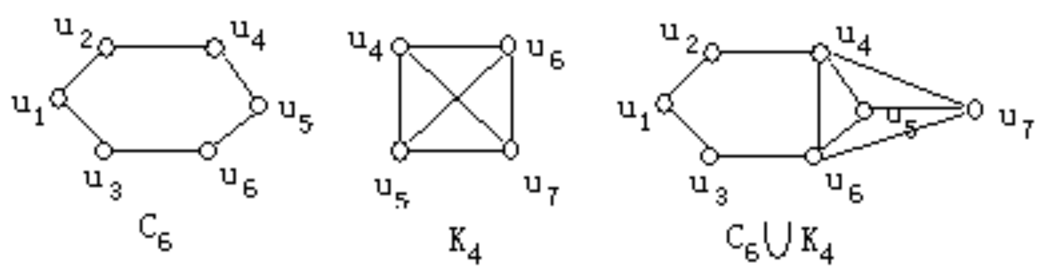


Fig 2.16

Operation 2. **Join**

The *complement* $\overline{G}$ of a graph G is a graph with the vertex set $V(G)$ such that two vertices are adjacent in $\overline{G}$ if and only if these vertices are not adjacent in G. The *join* $G_1 + G_2$ of G_1 and G_2 is defined by

$$V(G_1 + G_2) = V(G_1)\bigcup V(G_2),$$

$$E(G_1 + G_2) = E(G_1)\bigcup E(G_2)\bigcup\{(u, v)|u \in V(G_1), v \in V(G_2)\}$$

and

$$I(G_1+G_2)=I(G_1)\bigcup I(G_2)\bigcup\{I(u,v)=(u,v)|u\in V(G_1),v\in V(G_2)\}.$$

Using this operation, we can represent $K(m,n)\cong\overline{K_m}+\overline{K_n}$. The join graph of circuits C_3 and C_4 is given in Fig.2.17.

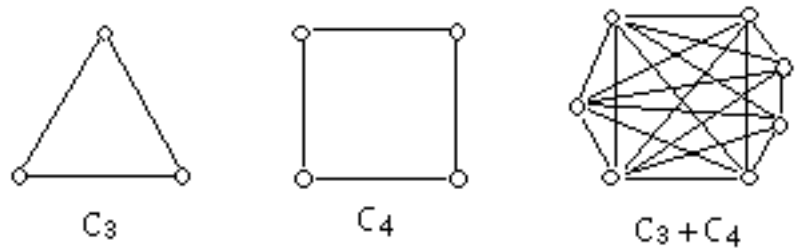


Fig 2.17

Operation 3. **Cartesian product**

The *cartesian product* $G_1\times G_2$ of graphs G_1 and G_2 is defined by $V(G_1\times G_2)=V(G_1)\times V(G_2)$ and two vertices (u_1,u_2) and (v_1,v_2) of $G_1\times G_2$ are adjacent if and only if either $u_1=v_1$ and $(u_2,v_2)\in E(G_2)$ or $u_2=v_2$ and $(u_1,v_1)\in E(G_1)$.

For example, the cartesian product $C_3\times C_3$ of circuits C_3 and C_3 is shown in Fig.2.18.

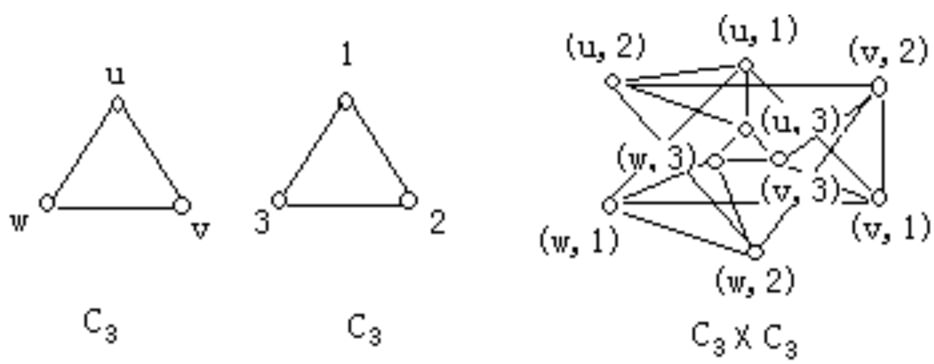


Fig 2.18

§2.2 Multi-Voltage Graphs

There is a convenient way for constructing a covering space of a graph G in topological graph theory, i.e., by a voltage graph (G,α) of G which was firstly introduced by

Gustin in 1963 and then generalized by Gross in 1974. Youngs extensively used voltage graphs in proving Heawood map coloring theorem([23]). Today, it has become a convenient way for finding regular maps on surface. In this section, we generalize voltage graphs to two types of multi-voltage graphs by using finite multi-groups.

2.2.1. **Type** 1

Definition 2.2.1 *Let* $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma_i$ *be a finite multi-group with an operation set* $O(\widetilde{\Gamma}) = \{\circ_i | 1 \leq i \leq n\}$ *and* G *a graph. If there is a mapping* $\psi : X_{\frac{1}{2}}(G) \to \widetilde{\Gamma}$ *such that* $\psi(e^{-1}) = (\psi(e^+))^{-1}$ *for* $\forall e^+ \in X_{\frac{1}{2}}(G)$, *then* (G, ψ) *is called a multi-voltage graph of type* 1.

Geometrically, a multi-voltage graph is nothing but a weighted graph with weights in a multi-group. Similar to voltage graphs, the importance of a multi-voltage graph is in its *lifting* defined in the next definition.

Definition 2.2.2 *For a multi-voltage graph* (G, ψ) *of type* 1, *the lifting graph* $G^\psi = (V(G^\psi),\ E(G^\psi); I(G^\psi))$ *of* (G, ψ) *is defined by*

$$V(G^\psi) = V(G) \times \widetilde{\Gamma},$$

$$E(G^\psi) = \{(u_a, v_{a\circ b}) | e^+ = (u, v) \in X_{\frac{1}{2}}(G), \psi(e^+) = b, a \circ b \in \widetilde{\Gamma}\}$$

and

$$I(G^\psi) = \{(u_a, v_{a\circ b}) | I(e) = (u_a, v_{a\circ b})\ if\ e = (u_a, v_{a\circ b}) \in E(G^\psi)\}.$$

For abbreviation, a vertex (x, g) in G^ψ is denoted by x_g. Now for $\forall v \in V(G)$, $v \times \widetilde{\Gamma} = \{v_g | g \in \widetilde{\Gamma}\}$ is called a *fiber over* v, denoted by F_v. Similarly, for $\forall e^+ = (u, v) \in X_{\frac{1}{2}}(G)$ with $\psi(e^+) = b$, all edges $\{(u_g, v_{g\circ b}) | g, g \circ b \in \widetilde{\Gamma}\}$ is called the *fiber over* e, denoted by F_e.

For a multi-voltage graph (G, ψ) and its lifting G^ψ, there is a *natural projection* $p : G^\psi \to G$ defined by $p(F_v) = v$ for $\forall v \in V(G)$. It can be verfied that $p(F_e) = e$ for $\forall e \in E(G)$.

Choose $\widetilde{\Gamma} = \Gamma_1 \bigcup \Gamma_2$ with $\Gamma_1 = \{1, a, a^2\}$, $\Gamma_2 = \{1, b, b^2\}$ and $a \neq b$. A multi-voltage graph and its lifting are shown in Fig.2.19.

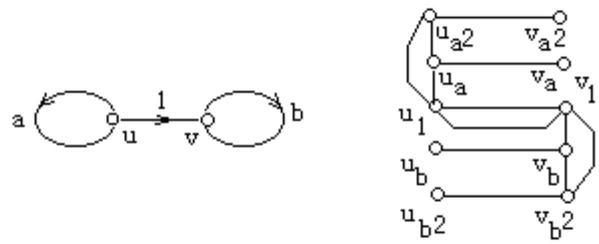


Fig 2.19

Let $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma_i$ be a finite multi-group with groups $(\Gamma_i; \circ_i), 1 \leq i \leq n$. Similar to the unique walk lifting theorem for voltage graphs, we know the following *walk multi-lifting theorem* for multi-voltage graphs of type 1.

Theorem 2.2.1 *Let $W = e^1 e^2 \cdots e^k$ be a walk in a multi-voltage graph (G, ψ) with initial vertex u. Then there exists a lifting W^ψ start at u_a in G^ψ if and only if there are integers $i_1, i_2, \cdots, i_k$ such that*

$$a \circ_{i_1} \psi(e_1^+) \circ_{i_2} \cdots \circ_{i_{j-1}} \psi(e_j^+) \in \Gamma_{i_{j+1}} \text{ and } \psi(e_{j+1}^+) \in \Gamma_{i_{j+1}}$$

for any integer $j, 1 \leq j \leq k$

Proof Consider the first semi-arc in the walk W, i.e., e_1^+. Each lifting of e_1 must be $(u_a, u_{a \circ \psi(e_1^+)})$. Whence, there is a lifting of e_1 in G^ψ if and only if there exists an integer i_1 such that $\circ = \circ_{i_1}$ and $a, a \circ_{i_1} \psi(e_1^+) \in \Gamma_{i_1}$.

Now if we have proved there is a lifting of a sub-walk $W_l = e_1 e_2 \cdots e_l$ in G^ψ if and only if there are integers $i_1, i_2, \cdots, i_l$, $1 \leq l < k$ such that

$$a \circ_{i_1} \psi(e_1^+) \circ_{i_2} \cdots \circ_{i_{j-1}} \psi(e_j^+) \in \Gamma_{i_{j+1}}, \quad \psi(e_{j+1}^+) \in \Gamma_{i_{j+1}}$$

for any integer $j, 1 \leq j \leq l$, we consider the semi-arc e_{l+1}^+. By definition, there is a lifting of e_{l+1}^+ in G^ψ with initial vertex $u_{a \circ_{i_1} \psi(e_1^+) \circ_{i_2} \cdots \circ_{i_{j-1}} \psi(e_l^+)}$ if and only if there exists an integer i_{l+1} such that

$$a \circ_{i_1} \psi(e_1^+) \circ_{i_2} \cdots \circ_{i_{j-1}} \psi(e_l^+) \in \Gamma_{l+1} \text{ and } \psi(e_{l+1}^+) \in \Gamma_{l+1}.$$

According to the induction principle, we know that there exists a lifting W^ψ start at u_a in G^ψ if and only if there are integers $i_1, i_2, \cdots, i_k$ such that

$$a \circ_{i_1} \psi(e_1^+) \circ_{i_2} \cdots \circ_{i_{j-1}} \psi(e_j^+) \in \Gamma_{i_{j+1}}, \text{ and } \psi(e_{j+1}^+) \in \Gamma_{i_{j+1}}$$

for any integer $j, 1 \leq j \leq k$. $\natural$

For two elements $g, h \in \widetilde{\Gamma}$, if there exist integers $i_1, i_2, \cdots, i_k$ such that $g, h \in \bigcap\limits_{j=1}^{k} \Gamma_{i_j}$ but for $\forall i_{k+1} \in \{1, 2, \cdots, n\} \setminus \{i_1, i_2, \cdots, i_k\}$, $g, h \notin \bigcap\limits_{j=1}^{k+1} \Gamma_{i_j}$, we call $k = \Pi[g, h]$ the *joint number of g and h*. Denote $O(g, h) = \{\circ_{i_j}; 1 \leq j \leq k\}$. Define $\widetilde{\Pi}[g, h] = \sum\limits_{\circ \in O(\widetilde{\Gamma})} \Pi[g, g \circ h]$, where $\Pi[g, g \circ h] = \Pi[g \circ h, h] = 0$ if $g \circ h$ does not exist in $\widetilde{\Gamma}$. According to Theorem 2.2.1, we get an upper bound for the number of liftings in G^ψ for a walk W in (G, ψ).

Corollary 2.2.1 *If those conditions in Theorem* 2.2.1 *hold, the number of liftings of W with initial vertex u_a in G^ψ is not in excess of*

$$\widetilde{\Pi}[a, \psi(e_1^+)] \times$$
$$\prod_{i=1}^{k} \sum_{\circ_1 \in O(a, \psi(e_1^+))} \cdots \sum_{\circ_i \in O(a; \circ_j, \psi(e_j^+), 1 \leq j \leq i-1)} \widetilde{\Pi}[a \circ_1 \psi(e_1^+) \circ_2 \cdots \circ_i \psi(e_i^+), \psi(e_{i+1}^+)],$$

where $O(a; \circ_j, \psi(e_j^+), 1 \leq j \leq i-1) = O(a \circ_1 \psi(e_1^+) \circ_2 \cdots \circ_{i-1} \psi(e_{i-1}^+), \psi(e_i^+))$.

The natural projection of a multi-voltage graph is not regular in general. For finding a regular covering of a graph, a typical class of multi-voltage graphs is the case of $\Gamma_i = \Gamma$ for any integer $i, 1 \leq i \leq n$ in these multi-groups $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma_i$. In this case, we can find the exact number of liftings in G^ψ for a walk in (G, ψ).

Theorem 2.2.2 *Let $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma$ be a finite multi-group with groups $(\Gamma; \circ_i), 1 \leq i \leq n$ and let $W = e^1 e^2 \cdots e^k$ be a walk in a multi-voltage graph (G, ψ) , $\psi : X_{\frac{1}{2}}(G) \rightarrow \widetilde{\Gamma}$ of type* 1 *with initial vertex u. Then there are n^k liftings of W in G^ψ with initial vertex u_a for $\forall a \in \widetilde{\Gamma}$.*

Proof The existence of lifting of W in G^ψ is obvious by Theorem 2.2.1. Consider the semi-arc e_1^+. Since $\Gamma_i = \Gamma$ for $1 \leq i \leq n$, we know that there are n liftings of e_1 in G^ψ with initial vertex u_a for any $a \in \widetilde{\Gamma}$, each with a form $(u_a, u_{a \circ \psi(e_1^+)}), \circ \in O(\widetilde{\Gamma})$.

Now if we have gotten $n^s, 1 \le s \le k-1$ liftings in G^ψ for a sub-walk $W_s = e^1e^2\cdots e^s$. Consider the semi-arc e^+_{s+1}. By definition we know that there are also n liftings of e_{s+1} in G^ψ with initial vertex $u_{a\circ_{i_1}\psi(e_1^+)\circ_{i_2}\cdots\circ_s\psi(e_s^+)}$, where $\circ_i \in O(\widetilde{\Gamma}), 1 \le i \le s$. Whence, there are n^{s+1} liftings in G^ψ for a sub-walk $W_s = e^1e^2\cdots e^{s+1}$ in $(G;\psi)$.

By the induction principle, we know the assertion is true. ♮

Corollary 2.2.2([23]) *Let W be a walk in a voltage graph $(G,\psi), \psi : X_{\frac{1}{2}}(G) \to \Gamma$ with initial vertex u. Then there is an unique lifting of W in G^ψ with initial vertex u_a for $\forall a \in \Gamma$.*

If a lifting W^ψ of a multi-voltage graph (G,ψ) is the same as the lifting of a voltage graph $(G,\alpha), \alpha : X_{\frac{1}{2}}(G) \to \Gamma_i$, then this lifting is called a *homogeneous lifting of* Γ_i. For lifting a circuit in a multi-voltage graph, we get the following result.

Theorem 2.2.3 *Let $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma$ be a finite multi-group with groups $(\Gamma;\circ_i), 1 \le i \le n$, $C = u_1u_2\cdots u_mu_1$ a circuit in a multi-voltage graph (G,ψ) and $\psi : X_{\frac{1}{2}}(G) \to \widetilde{\Gamma}$. Then there are $\frac{|\Gamma|}{o(\psi(C,\circ_i))}$ homogenous liftings of length $o(\psi(C,\circ_i))m$ in G^ψ of C for any integer $i, 1 \le i \le n$, where $\psi(C,\circ_i) = \psi(u_1,u_2)\circ_i\psi(u_2,u_3)\circ_i\cdots\circ_i\psi(u_{m-1},u_m)\circ_i \psi(u_m,u_1)$ and there are*

$$\sum_{i=1}^{n}\frac{|\Gamma|}{o(\psi(C,\circ_i))}$$

homogenous liftings of C in G^ψ altogether.

Proof According to Theorem 2.2.2, there are liftings with initial vertex $(u_1)_a$ of C in G^ψ for $\forall a \in \widetilde{\Gamma}$. Whence, for any integer $i, 1 \le i \le n$, walks

$$W_a = (u_1)_a(u_2)_{a\circ_i\psi(u_1,u_2)}\cdots(u_m)_{a\circ_i\psi(u_1,u_2)\circ_i\cdots\circ_i\psi(u_{m-1},u_m)}(u_1)_{a\circ_i\psi(C,\circ_i)},$$

$$\begin{aligned} W_{a\circ_i\psi(C,\circ_i)} &= (u_1)_{a\circ_i\psi(C,\circ_i)}(u_2)_{a\circ_i\psi(C,\circ_i)\circ_i\psi(u_1,u_2)} \\ &\cdots (u_m)_{a\circ_i\psi(C,\circ_i)\circ_i\psi(u_1,u_2)\circ_i\cdots\circ_i\psi(u_{m-1},u_m)}(u_1)_{a\circ_i\psi^2(C,\circ_i)}, \end{aligned}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

and

$$\begin{aligned} W_{a\circ_i\psi^{o(\psi(C,\circ_i))-1}(C,\circ_i)} &= (u_1)_{a\circ_i\psi^{o(\psi(C,\circ_i))-1}(C,\circ_i)}(u_2)_{a\circ_i\psi^{o(\psi(C,\circ_i))-1}(C,\circ_i)\circ_i\psi(u_1,u_2)} \\ &\cdots (u_m)_{a\circ_i\psi^{o(\psi(C,\circ_i))-1}(C,\circ_i)\circ_i\psi(u_1,u_2)\circ_i\cdots\circ_i\psi(u_{m-1},u_m)}(u_1)_a \end{aligned}$$

are attached end-to-end to form a circuit of length $o(\psi(C,\circ_i))m$. Notice that there are $\frac{|\Gamma|}{o(\psi(C,\circ_i))}$ left cosets of the cyclic group generated by $\psi(C,\circ_i)$ in the group $(\Gamma,\circ_i)$ and each is correspondent with a homogenous lifting of C in G^ψ. Therefore, we get

$$\sum_{i=1}^{n}\frac{|\Gamma|}{o(\psi(C,\circ_i))}$$

homogenous liftings of C in G^ψ. $\natural$

Corollary 2.2.3([23]) *Let C be a k-circuit in a voltage graph (G,ψ) such that the order of $\psi(C,\circ)$ is m in the voltage group $(\Gamma;\circ)$. Then each component of the preimage $p^{-1}(C)$ is a km-circuit, and there are $\frac{|\Gamma|}{m}$ such components.*

The lifting G^ζ of a multi-voltage graph (G,ζ) of type 1 has a natural decomposition described in the next result.

Theorem 2.2.4 *Let $(G,\zeta),\zeta: X_{\frac{1}{2}}(G)\to\widetilde{\Gamma}=\bigcup\limits_{i=1}^{n}\Gamma_i$, be a multi-voltage graph of type 1. Then*

$$G^\zeta=\bigoplus_{i=1}^{n}H_i,$$

where H_i is an induced subgraph $\langle E_i\rangle$ of G^ζ for an integer $i, 1\leq i\leq n$ with

$$E_i=\{(u_a,v_{a\circ_i b})|a,b\in\Gamma_i \text{ and } (u,v)\in E(G),\zeta(u,v)=b\}.$$

For a finite multi-group $\widetilde{\Gamma}=\bigcup\limits_{i=1}^{n}\Gamma_i$ with an operation set $O(\widetilde{\Gamma})=\{\circ_i, 1\leq i\leq n\}$ and a graph G, if there exists a decomposition $G=\bigoplus\limits_{j=1}^{n}H_i$ and we can associate each element $g_i\in\Gamma_i$ a homeomorphism φ_{g_i} on the vertex set $V(H_i)$ for any integer $i, 1\leq i\leq n$ such that

(i) $\varphi_{g_i\circ_i h_i}=\varphi_{g_i}\times\varphi_{h_i}$ for all $g_i,h_i\in\Gamma_i$, where "$\times$" is an operation between homeomorphisms;

(ii) φ_{g_i} is the identity homeomorphism if and only if g_i is the identity element of the group $(\Gamma_i;\circ_i)$,

then we say this association to be a *subaction of a multi-group* $\widetilde{\Gamma}$ *on the graph* G. If there exists a subaction of $\widetilde{\Gamma}$ on G such that $\varphi_{g_i}(u)=u$ only if $g_i=\mathbf{1}_{\Gamma_i}$ for any integer $i, 1\leq i\leq n$, $g_i\in\Gamma_i$ and $u\in V_i$, then we call it a *fixed-free subaction.*

A *left subaction* lA of $\widetilde{\Gamma}$ on G^{ψ} is defined as follows:

For any integer $i, 1\leq i\leq n$, *let* $V_i=\{u_a|u\in V(G), a\in\widetilde{\Gamma}\}$ *and* $g_i\in\Gamma_i$. *Define* $lA(g_i)(u_a)=u_{g_i\circ_i a}$ *if* $a\in V_i$. *Otherwise,* $g_i(u_a)=u_a$.

Then the following result holds.

Theorem 2.2.5 *Let* (G,ψ) *be a multi-voltage graph with* $\psi: X_{\frac{1}{2}}(G)\rightarrow\widetilde{\Gamma}=\bigcup\limits_{i=1}^{n}\Gamma_i$ *and* $G=\bigoplus\limits_{j=1}^{n}H_i$ *with* $H_i=\langle E_i\rangle$, $1\leq i\leq n$, *where* $E_i=\{(u_a,v_{a\circ_i b})|a,b\in\Gamma_i$ *and* $(u,v)\in E(G),\zeta(u,v)=b\}$. *Then for any integer* $i, 1\leq i\leq n$,

(i) *for* $\forall g_i\in\Gamma_i$, *the left subaction* $lA(g_i)$ *is a fixed-free subaction of an automorphism of* H_i*;*

(ii) Γ_i *is an automorphism group of* H_i.

Proof Notice that $lA(g_i)$ is a one-to-one mapping on $V(H_i)$ for any integer $i, 1\leq i\leq n$, $\forall g_i\in\Gamma_i$. By the definition of a lifting, an edge in H_i has the form $(u_a,v_{a\circ_i b})$ if $a,b\in\Gamma_i$. Whence,

$$(lA(g_i)(u_a), lA(g_i)(v_{a\circ_i b}))=(u_{g_i\circ_i a}, v_{g_i\circ_i a\circ_i b})\in E(H_i).$$

As a result, $lA(g_i)$ is an automorphism of the graph H_i.

Notice that $lA:\Gamma_i\rightarrow\mathrm{Aut}H_i$ is an injection from Γ_i to $\mathrm{Aut}G^{\psi}$. Since $lA(g_i)\neq lA(h_i)$ for $\forall g_i,h_i\in\Gamma_i, g_i\neq h_i, 1\leq i\leq n$. Otherwise, if $lA(g_i)=lA(h_i)$ for $\forall a\in\Gamma_i$, then $g_i\circ_i a=h_i\circ_i a$. Whence, $g_i=h_i$, a contradiction. Therefore, Γ_i is an automorphism group of H_i.

For any integer $i, 1\leq i\leq n$, $g_i\in\Gamma_i$, it is implied by definition that $lA(g_i)$ is a fixed-free subaction on G^{ψ}. This completes the proof. $\natural$

Corollary 2.2.4([23]) *Let* (G,α) *be a voltage graph with* $\alpha: X_{\frac{1}{2}}(G)\rightarrow\Gamma$. *Then* Γ *is an automorphism group of* G^{α}.

For a finite multi-group $\widetilde{\Gamma}=\bigcup\limits_{i=1}^{n}\Gamma_i$ action on a graph $\widetilde{G}$, the vertex orbit $orb(v)$

of a vertex $v \in V(\widetilde{G})$ and the edge orbit $orb(e)$ of an edge $e \in E(\widetilde{G})$ are defined as follows:

$$orb(v) = \{g(v)|g \in \widetilde{\Gamma}\} \text{ and } orb(e) = \{g(e)|g \in \widetilde{\Gamma}\}.$$

The *quotient graph* $\widetilde{G}/\widetilde{\Gamma}$ of $\widetilde{G}$ under the action of $\widetilde{\Gamma}$ is defined by

$$V(\widetilde{G}/\widetilde{\Gamma}) = \{\ orb(v)\ |\ v \in V(\widetilde{G})\}, \quad E(\widetilde{G}/\widetilde{\Gamma}) = \{orb(e)|e \in E(\widetilde{G})\}$$

and

$$I(orb(e)) = (orb(u), orb(v))\ if\ there\ exists\ (u,v) \in E(\widetilde{G})$$

.

For example, a quotient graph is shown in Fig.2.20, where, $\widetilde{\Gamma} = Z_5$.

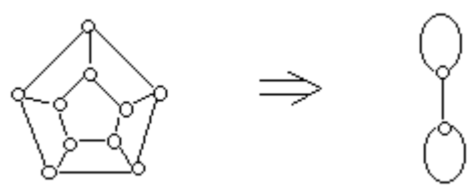


Fig 2.20

Then we get a necessary and sufficient condition for the lifting of a multi-voltage graph in next result.

Theorem 2.2.6 *If the subaction $\mathcal{A}$ of a finite multi-group $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma_i$ on a graph $\widetilde{G} = \bigoplus\limits_{i=1}^{n} H_i$ is fixed-free, then there is a multi-voltage graph $(\widetilde{G}/\widetilde{\Gamma}, \zeta)$, $\zeta : X_{\frac{1}{2}}(\widetilde{G}/\widetilde{\Gamma}) \to \widetilde{\Gamma}$ of type 1 such that*

$$\widetilde{G} \ \cong\ (\widetilde{G}/\widetilde{\Gamma})^{\zeta}.$$

Proof First, we choose positive directions for edges of $\widetilde{G}/\widetilde{\Gamma}$ and $\widetilde{G}$ so that the quotient map $q_{\widetilde{\Gamma}} : \widetilde{G} \to \widetilde{G}/\widetilde{\Gamma}$ is direction-preserving and that the action $\mathcal{A}$ of $\widetilde{\Gamma}$ on $\widetilde{G}$ preserves directions. Next, for any integer $i, 1 \leq i \leq n$ and $\forall v \in V(\widetilde{G}/\widetilde{\Gamma})$, label one vertex of the orbit $q_{\widetilde{\Gamma}}^{-1}(v)$ in $\widetilde{G}$ as $v_{1_{\Gamma_i}}$ and for every group element $g_i \in \Gamma_i, g_i \neq 1_{\Gamma_i}$, label the vertex $\mathcal{A}(g_i)(v_{1_{\Gamma_i}})$ as v_{g_i}. Now if the edge e of $\widetilde{G}/\widetilde{\Gamma}$ runs from u to w, we assigns the label e_{g_i} to the edge of the orbit $q_{\Gamma_i}^{-1}(e)$ that originates at the vertex u_{g_i}. Since Γ_i acts freely on H_i, there are just $|\Gamma_i|$ edges in the orbit $q_{\Gamma_i}^{-1}(e)$ for each integer $i, 1 \leq i \leq n$, one originating at each of the vertices in the vertex orbit $q_{\Gamma_i}^{-1}(v)$. Thus the choice of an edge to be labelled e_{g_i} is unique for any integer $i, 1 \leq i \leq n$. Finally, if the terminal vertex of the edge $e_{1_{\Gamma_i}}$ is w_{h_i}, one assigns a voltage h_i to the edge e in the quotient $\widetilde{G}/\widetilde{\Gamma}$, which enables us to get a multi-voltage graph $(\widetilde{G}/\widetilde{\Gamma}, \zeta)$. To show that this labelling of edges in $q_{\Gamma_i}^{-1}(e)$ and the choice of voltages $h_i, 1 \leq i \leq n$ for the edge e really yields an isomorphism $\vartheta : \widetilde{G} \rightarrow (\widetilde{G}/\widetilde{\Gamma})^{\zeta}$, one needs to show that for $\forall g_i \in \Gamma_i, 1 \leq i \leq n$ that the edge e_{g_i} terminates at the vertex $w_{g_i \circ_i h_i}$. However, since $e_{g_i} = \mathcal{A}(g_i)(e_{1_{\Gamma_i}})$, the terminal vertex of the edge e_{g_i} must be the terminal vertex of the edge $\mathcal{A}(g_i)(e_{1_{\Gamma_i}})$, which is

$$\mathcal{A}(g_i)(w_{h_i}) = \mathcal{A}(g_i)\mathcal{A}(h_i)(w_{1_{\Gamma_i}}) = \mathcal{A}(g_i \circ_i h_i)(w_{1_{\Gamma_i}}) = w_{g_i \circ_i h_i}.$$

Under this labelling process, the isomorphism $\vartheta : \widetilde{G} \rightarrow (\widetilde{G}/\widetilde{\Gamma})^{\zeta}$ identifies orbits in $\widetilde{G}$ with fibers of G^{ζ}. Moreover, it is defined precisely so that the action of $\widetilde{\Gamma}$ on $\widetilde{G}$ is consistent with the left subaction lA on the lifting graph G^{ζ}. This completes the proof. ♮

Corollary 2.2.5([23]) *Let Γ be a group acting freely on a graph $\widetilde{G}$ and let G be the resulting quotient graph. Then there is an assignment α of voltages in Γ to the quotient graph G and a labelling of the vertices $\widetilde{G}$ by the elements of $V(G) \times \Gamma$ such that $\widetilde{G} = G^{\alpha}$ and that the given action of Γ on $\widetilde{G}$ is the natural left action of Γ on G^{α}.*

2.2.2. **Type** 2

Definition 2.2.3 *Let $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma_i$ be a finite multi-group and let G be a graph with vertices partition $V(G) = \bigcup\limits_{i=1}^{n} V_i$. For any integers $i, j, 1 \leq i, j \leq n$, if there is a mapping $\tau : X_{\frac{1}{2}}(\langle E_G(V_i, V_j)\rangle) \rightarrow \Gamma_i \bigcap \Gamma_j$ and $\varsigma : V_i \rightarrow \Gamma_i$ such that $\tau(e^{-1}) = (\tau(e^+))^{-1}$ for $\forall e^+ \in X_{\frac{1}{2}}(G)$ and the vertex subset V_i is associated with the group $(\Gamma_i, \circ_i)$ for any integer $i, 1 \leq i \leq n$, then (G, τ, ς) is called a multi-voltage graph of type 2.*

Similar to multi-voltage graphs of type 1, we construct a lifting from a multi-voltage graph of type 2.

Definition 2.2.4 *For a multi-voltage graph (G,τ,ς) of type 2, the lifting graph $G^{(\tau,\varsigma)} = (V(G^{(\tau,\varsigma)}),\ E(G^{(\tau,\varsigma)}); I(G^{(\tau,\varsigma)}))$ of (G,τ,ς) is defined by*

$$V(G^{(\tau,\varsigma)}) = \bigcup_{i=1}^{n}\{V_i \times \Gamma_i\},$$

$$E(G^{(\tau,\varsigma)}) = \{(u_a, v_{a\circ b}) | e^+ = (u,v) \in X_{\frac{1}{2}}(G), \psi(e^+) = b, a\circ b \in \widetilde{\Gamma}\}$$

and

$$I(G^{(\tau,\varsigma)}) = \{(u_a, v_{a\circ b}) | I(e) = (u_a, v_{a\circ b})\ if\ e = (u_a, v_{a\circ b}) \in E(G^{(\tau,\varsigma)})\}.$$

Two multi-voltage graphs of type 2 are shown on the left and their lifting on the right in (a) and (b) of Fig.21. In where, $\widetilde{\Gamma} = Z_2 \bigcup Z_3$, $V_1 = \{u\}$, $V_2 = \{v\}$ and $\varsigma : V_1 \to Z_2$, $\varsigma : V_2 \to Z_3$.

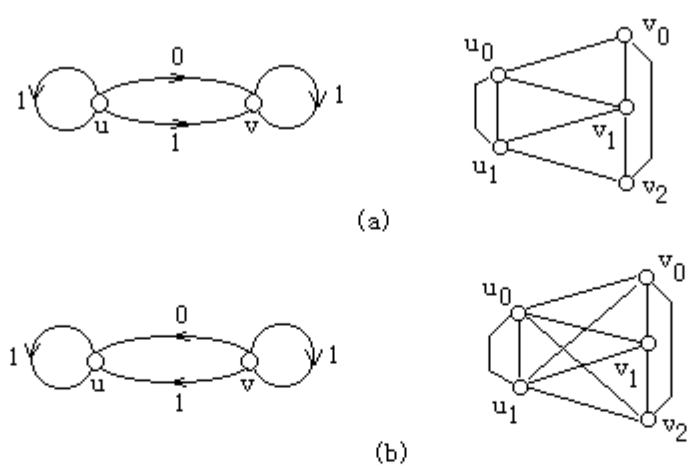


Fig 2.21

Theorem 2.2.7 *Let (G,τ,ς) be a multi-voltage graph of type 2 and let $W_k = u_1u_2\cdots u_k$ be a walk in G. Then there exists a lifting of $W^{(\tau,\varsigma)}$ with an initial vertex $(u_1)_a, a \in \varsigma^{-1}(u_1)$ in $G^{(\tau,\varsigma)}$ if and only if $a \in \varsigma^{-1}(u_1)\bigcap\varsigma^{-1}(u_2)$ and for any integer $s, 1 \le s < k$, $a \circ_{i_1} \tau(u_1u_2) \circ_{i_2} \tau(u_2u_3) \circ_{i_3} \cdots \circ_{i_{s-1}} \tau(u_{s-2}u_{s-1}) \in \varsigma^{-1}(u_{s-1})\bigcap\varsigma^{-1}(u_s)$, where "$\circ_{i_j}$" is an operation in the group $\varsigma^{-1}(u_{j+1})$ for any integer $j, 1 \le j \le s$.*

Proof By the definition of the lifting of a multi-voltage graph of type 2, there exists a lifting of the edge u_1u_2 in $G^{(\tau,\varsigma)}$ if and only if $a \circ_{i_1} \tau(u_1u_2) \in \varsigma^{-1}(u_2)$, where "$\circ_{i_j}$" is an operation in the group $\varsigma^{-1}(u_2)$. Since $\tau(u_1u_2) \in \varsigma^{-1}(u_1) \bigcap \varsigma^{-1}(u_2)$, we get that $a \in \varsigma^{-1}(u_1) \bigcap \varsigma^{-1}(u_2)$. Similarly, there exists a lifting of the subwalk $W_2 = u_1u_2u_3$ in $G^{(\tau,\varsigma)}$ if and only if $a \in \varsigma^{-1}(u_1) \bigcap \varsigma^{-1}(u_2)$ and $a \circ_{i_1} \tau(u_1u_2) \in \varsigma^{-1}(u_2) \bigcap \varsigma^{-1}(u_3)$.

Now assume there exists a lifting of the subwalk $W_l = u_1u_2u_3 \cdots u_l$ in $G^{(\tau,\varsigma)}$ if and only if $a \circ_{i_1} \tau(u_1u_2) \circ_{i_2} \cdots \circ_{i_{t-2}} \tau(u_{t-2}u_{t-1}) \in \varsigma^{-1}(u_{t-1}) \bigcap \varsigma^{-1}(u_t)$ for any integer $t, 1 \leq t \leq l$, where "$\circ_{i_j}$" is an operation in the group $\varsigma^{-1}(u_{j+1})$ for any integer $j, 1 \leq j \leq l$. We consider the lifting of the subwalk $W_{l+1} = u_1u_2u_3 \cdots u_{l+1}$. Notice that if there exists a lifting of the subwalk W_l in $G^{(\tau,\varsigma)}$, then the terminal vertex of W_l in $G^{(\tau,\varsigma)}$ is $(u_l)_{a \circ_{i_1} \tau(u_1u_2) \circ_{i_2} \cdots \circ_{i_{l-1}} \tau(u_{l-1}u_l)}$. We only need to find a necessary and sufficient condition for existing a lifting of u_lu_{l+1} with an initial vertex $(u_l)_{a \circ_{i_1} \tau(u_1u_2) \circ_{i_2} \cdots \circ_{i_{l-1}}} \tau(u_{l-1}u_l)$. By definition, there exists such a lifting of the edge u_lu_{l+1} if and only if $(a \circ_{i_1} \tau(u_1u_2) \circ_{i_2} \cdots \circ_{i_{l-1}}) \tau(u_{l-1}u_l)) \circ_l \tau(u_lu_{l+1}) \in \varsigma^{-1}(u_{l+1})$. Since $\tau(u_lu_{l+1}) \in \varsigma^{-1}(u_{l+1})$ by the definition of multi-voltage graphs of type 2, we know that $a \circ_{i_1} \tau(u_1u_2) \circ_{i_2} \cdots \circ_{i_{l-1}} \tau(u_{l-1}u_l) \in \varsigma^{-1}(u_{l+1})$.

Continuing this process, we get the assertion of this theorem by the induction principle. $\natural$

Corollary 2.2.7 *Let G a graph with vertices partition $V(G) = \bigcup\limits_{i=1}^{n} V_i$ and let $(\Gamma; \circ)$ be a finite group, $\Gamma_i \prec \Gamma$ for any integer $i, 1 \leq i \leq n$. If (G, τ, ς) is a multi-voltage graph with $\tau : X_{\frac{1}{2}}(G) \rightarrow \Gamma$ and $\varsigma : V_i \rightarrow \Gamma_i$ for any integer $i, 1 \leq i \leq n$, then for a walk W in G with an initial vertex u, there exists a lifting $W^{(\tau,\varsigma)}$ in $G^{(\tau,\varsigma)}$ with the initial vertex $u_a, a \in \varsigma^{-1}(u)$ if and only if $a \in \bigcap_{v \in V(W)} \varsigma^{-1}(v)$.*

Similar to multi-voltage graphs of type 1, we can get the exact number of liftings of a walk in the case of $\Gamma_i = \Gamma$ and $V_i = V(G)$ for any integer $i, 1 \leq i \leq n$.

Theorem 2.2.8 *Let $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma$ be a finite multi-group with groups $(\Gamma; \circ_i), 1 \leq i \leq n$ and let $W = e^1e^2 \cdots e^k$ be a walk with an initial vertex u in a multi-voltage graph (G, τ, ς) , $\tau : X_{\frac{1}{2}}(G) \rightarrow \bigcap\limits_{i=1}^{n} \Gamma$ and $\varsigma : V(G) \rightarrow \Gamma$, of type 2. Then there are n^k liftings of W in $G^{(\tau,\varsigma)}$ with an initial vertex u_a for $\forall a \in \widetilde{\Gamma}$.*

Proof The proof is similar to the proof of Theorem 2.2.2. $\natural$

Theorem 2.2.9 *Let $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma$ be a finite multi-group with groups $(\Gamma; \circ_i), 1 \leq i \leq n$, $C = u_1 u_2 \cdots u_m u_1$ a circuit in a multi-voltage graph (G, τ, ς) of type 2 where $\tau : X_{\frac{1}{2}}(G) \to \bigcap\limits_{i=1}^{n} \Gamma$ and $\varsigma : V(G) \to \Gamma$. Then there are $\frac{|\Gamma|}{o(\tau(C,\circ_i))}$ liftings of length $o(tau(C, \circ_i))m$ in $G^{(\tau,\varsigma)}$ of C for any integer $i, 1 \leq i \leq n$, where $\tau(C, \circ_i) = \tau(u_1, u_2) \circ_i \tau(u_2, u_3) \circ_i \cdots \circ_i \tau(u_{m-1}, u_m) \circ_i \tau(u_m, u_1)$ and there are*

$$\sum_{i=1}^{n} \frac{|\Gamma|}{o(\tau(C, \circ_i))}$$

liftings of C in $G^{(\tau,\varsigma)}$ altogether.

Proof The proof is similar to the proof of Theorem 2.2.3. $\natural$

Definition 2.2.5 *Let G_1, G_2 be two graphs and H a subgraph of G_1 and G_2. A one-to-one mapping ξ between G_1 and G_2 is called an H-isomorphism if for any subgraph J isomorphic to H in G_1, $\xi(J)$ is also a subgraph isomorphic to H in G_2.*

If $G_1 = G_2 = G$, then an H-isomorphism between G_1 and G_2 is called an H-automorphism of G. Certainly, all H-automorphisms form a group under the composition operation, denoted by $\mathrm{Aut}_H G$ and $\mathrm{Aut}_H G = \mathrm{Aut} G$ if we take $H = K_2$.

For example, let $H = \langle E(x, N_G(x)) \rangle$ for $\forall x \in V(G)$. Then the H-automorphism group of a complete bipartite graph $K(n, m)$ is $\mathrm{Aut}_H K(n, m) = S_n[S_m] = \mathrm{Aut} K(n, m)$. There H-automorphisms are called *star-automorphisms*.

Theorem 2.2.10 *Let G be a graph. If there is a decomposition $G = \bigoplus\limits_{i=1}^{n} H_i$ with $H_i \cong H$ for $1 \leq i \leq n$ and $H = \bigoplus\limits_{j=1}^{m} J_j$ with $J_j \cong J$ for $1 \leq j \leq m$, then*

(i) $\langle \iota_i, \iota_i : H_1 \to H_i$, an isomorphism, $1 \leq i \leq n \rangle = S_n \preceq \mathrm{Aut}_H G$, and particularly, $S_n \preceq \mathrm{Aut}_H K_{2n+1}$ if $H = C$, a hamiltonian circuit in K_{2n+1}.

(ii) $\mathrm{Aut}_J G \preceq \mathrm{Aut}_H G$, and particularly, $\mathrm{Aut} G \preceq \mathrm{Aut}_H G$ for a simple graph G.

Proof (i) For any integer $i, 1 \leq i \leq n$, we prove there is a such H-automorphism ι on G that $\iota_i : H_1 \to H_i$. In fact, since $H_i \cong H$, $1 \leq i \leq n$, there is an isomorphism $\theta : H_1 \to H_i$. We define ι_i as follows:

$$\iota_i(e) = \begin{cases} \theta(e), & \text{if } e \in V(H_1) \bigcup E(H_1), \\ e, & \text{if} e \in (V(G) \setminus V(H_1)) \bigcup (E(G) \setminus E(H_1)). \end{cases}$$

Then ι_i is a one-to-one mapping on the graph G and is also an H-isomorphism by definition. Whence,

$$\langle \iota_i, \iota_i : H_1 \to H_i, \text{ an isomorphism}, \ 1 \le i \le n\rangle \preceq \mathrm{Aut}_H G.$$

Since $\langle \iota_i, 1 \le i \le n\rangle \cong \langle (1,i), 1 \le i \le n\rangle = S_n$, thereby we get that $S_n \preceq \mathrm{Aut}_H G$.

For a complete graph K_{2n+1}, we know a decomposition $K_{2n+1} = \bigoplus\limits_{i=1}^{n} C_i$ with

$$C_i = v_0 v_i v_{i+1} v_{i-1} v_{i-2} \cdots v_{n+i-1} v_{n+i+1} v_{n+i} v_0$$

for any integer $i, 1 \le i \le n$ by Theorem 2.1.18. Therefore, we get that

$$S_n \preceq \mathrm{Aut}_H K_{2n+1}$$

if we choose a hamiltonian circuit H in K_{2n+1}.

(ii) Choose $\sigma \in \mathrm{Aut}_J G$. By definition, for any subgraph A of G, if $A \cong J$, then $\sigma(A) \cong J$. Notice that $H = \bigoplus\limits_{j=1}^{m} J_j$ with $J_j \cong J$ for $1 \le j \le m$. Therefore, for any subgraph $B, B \cong H$ of G, $\sigma(B) \cong \bigoplus\limits_{j=1}^{m} \sigma(J_j) \cong H$. This fact implies that $\sigma \in \mathrm{Aut}_H G$.

Notice that for a simple graph G, we have a decomposition $G = \bigoplus\limits_{i=1}^{\varepsilon(G)} K_2$ and $\mathrm{Aut}_{K_2} G = \mathrm{Aut} G$. Whence, $\mathrm{Aut} G \preceq \mathrm{Aut}_H G$. $\natural$

The equality in Theorem 2.2.10(ii) does not always hold. For example, a one-to-one mapping σ on the lifting graph of Fig.2.21(a): $\sigma(u_0) = u_1$, $\sigma(u_1) = u_0$, $\sigma(v_0) = v_1$, $\sigma(v_1) = v_2$ and $\sigma(v_2) = v_0$ is not an automorphism, but it is an H-automorphism with H being a star $S_{1.2}$.

For automorphisms of the lifting $G^{(\tau,\varsigma)}$ of a multi-voltage graph (G, τ, ς) of type 2, we get a result in the following.

Theorem 2.2.11 *Let (G, τ, ς) be a multi-voltage graph of type 2 with $\tau : X_{\frac{1}{2}}(G) \to \bigcap\limits_{i=1}^{n} \Gamma_i$ and $\varsigma : V_i \to \Gamma_i$. Then for any integers $i, j, 1 \le i, j \le n$,*

(i) for $\forall g_i \in \Gamma_i$, the left action $lA(g_i)$ on $\langle V_i\rangle^{(\tau,\varsigma)}$ is a fixed-free action of an automorphism of $\langle V_i\rangle^{(\tau,\varsigma)}$;

(ii) for $\forall g_{ij} \in \Gamma_i \bigcap \Gamma_j$, the left action $lA(g_{ij})$ on $\langle E_G(V_i, V_j)\rangle^{(\tau,\varsigma)}$ is a star-automorphism of $\langle E_G(V_i, V_j)\rangle^{(\tau,\varsigma)}$.

Proof The proof of (i) is similar to the proof of Theorem 2.2.4. We prove the assertion (ii). A star with a central vertex u_a, $u \in V_i, a \in \Gamma_i \bigcap \Gamma_j$ is the graph $S_{star} = \left\langle \{(u_a, v_{a\circ_j b}) \text{ if } (u,v) \in E_G(V_i, V_j), \tau(u,v) = b\} \right\rangle$. By definition, the left action $lA(g_{ij})$ is a one-to-one mapping on $\langle E_G(V_i, V_j)\rangle^{(\tau,\varsigma)}$. Now for any element $g_{ij}, g_{ij} \in \Gamma_i \bigcap \Gamma_j$, the left action $lA(g_{ij})$ of g_{ij} on a star S_{star} is

$$lA(g_{ij})(S_{star}) = \left\langle \{(u_{g_{ij}\circ_i a}, v_{(g_{ij}\circ_i a)\circ_j b}) \text{ if } (u,v) \in E_G(V_i, V_j), \tau(u,v) = b\} \right\rangle = S_{star}.$$

Whence, $lA(g_{ij})$ is a star-automorphism of $\langle E_G(V_i, V_j)\rangle^{(\tau,\varsigma)}$. ♮

Let $\widetilde{G}$ be a graph and let $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma_i$ be a finite multi-group. If there is a partition for the vertex set $V(\widetilde{G}) = \bigcup\limits_{i=1}^{n} V_i$ such that the action of $\widetilde{\Gamma}$ on $\widetilde{G}$ consists of Γ_i action on $\langle V_i\rangle$ and $\Gamma_i \bigcap \Gamma_j$ on $\langle E_G(V_i, v_j)\rangle$ for $1 \le i, j \le n$, then we say this action to be a *partially-action*. A partially-action is called *fixed-free* if Γ_i is fixed-free on $\langle V_i\rangle$ and the action of each element in $\Gamma_i \bigcap \Gamma_j$ is a star-automorphism and fixed-free on $\langle E_G(V_i, V_j)\rangle$ for any integers $i, j, 1 \le i, j \le n$. These orbits of a partially-action are defined to be

$$orb_i(v) = \{g(v) | g \in \Gamma_i, v \in V_i\}$$

for any integer $i, 1 \le i \le n$ and

$$orb(e) = \{g(e) | e \in E(\widetilde{G}), g \in \widetilde{\Gamma}\}.$$

A *partially-quotient graph* $\widetilde{G}/_p\widetilde{\Gamma}$ is defined by

$$V(\widetilde{G}/_p\widetilde{\Gamma}) = \bigcup_{i=1}^{n} \{\ orb_i(v) \mid v \in V_i\}, \quad E(\widetilde{G}/_p\widetilde{\Gamma}) = \{orb(e) | e \in E(\widetilde{G})\}$$

and $I(\widetilde{G}/_p\widetilde{\Gamma}) = \{I(e) = (orb_i(u), orb_j(v)) \text{ if } u \in V_i, v \in V_j \text{ and } (u,v) \in E(\widetilde{G}), 1 \le i, j \le n\}$. An example for partially-quotient graph is shown in Fig.2.22, where $V_1 = \{u_0, u_1, u_2, u_3\}$, $V_2 = \{v_0, v_1, v_2\}$ and $\Gamma_1 = Z_4$, $\Gamma_2 = Z_3$.

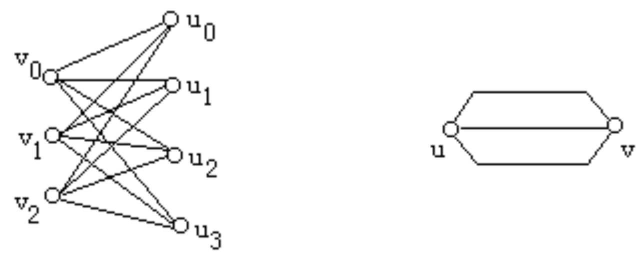


Fig 2.22

Then we have a necessary and sufficient condition for the lifting of a multi-voltage graph of type 2.

Theorem 2.2.12 *If the partially-action $\mathcal{P}_a$ of a finite multi-group $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma_i$ on a graph $\widetilde{G}$, $V(\widetilde{G}) = \bigcup\limits_{i=1}^{n} V_i$ is fixed-free, then there is a multi-voltage graph $(\widetilde{G}/_p\widetilde{\Gamma}, \tau, \varsigma)$, $\tau : X_{\frac{1}{2}}(\widetilde{G}/\widetilde{\Gamma}) \to \widetilde{\Gamma}$, $\varsigma : V_i \to \Gamma_i$ of type 2 such that*

$$\widetilde{G} \cong (\widetilde{G}/_p\widetilde{\Gamma})^{(\tau,\varsigma)}.$$

Proof Similar to the proof of Theorem 2.2.6, we also choose positive directions on these edges of $\widetilde{G}/_p\widetilde{\Gamma}$ and $\widetilde{G}$ so that the partially-quotient map $p_{\widetilde{\Gamma}} : \widetilde{G} \to \widetilde{G}/_p\widetilde{\Gamma}$ is direction-preserving and the partially-action of $\widetilde{\Gamma}$ on $\widetilde{G}$ preserves directions.

For any integer $i, 1 \leq i \leq n$ and $\forall v^i \in V_i$, we can label v^i as $v^i_{1_{\Gamma_i}}$ and for every group element $g_i \in \Gamma_i, g_i \neq 1_{\Gamma_i}$, label the vertex $\mathcal{P}_a(g_i)((v_i)_{1_{\Gamma_i}})$ as $v^i_{g_i}$. Now if the edge e of $\widetilde{G}/_p\widetilde{\Gamma}$ runs from u to w, we assign the label e_{g_i} to the edge of the orbit $p^{-1}(e)$ that originates at the vertex $u^i_{g_i}$ and terminates at $w^j_{h_j}$.

Since Γ_i acts freely on $\langle V_i \rangle$, there are just $|\Gamma_i|$ edges in the orbit $p^{-1}_{\Gamma_i}(e)$ for each integer $i, 1 \leq i \leq n$, one originating at each of the vertices in the vertex orbit $p^{-1}_{\Gamma_i}(v)$. Thus for any integer $i, 1 \leq i \leq n$, the choice of an edge in $p^{-1}(e)$ to be labelled e_{g_i} is unique. Finally, if the terminal vertex of the edge e_{g_i} is $w^j_{h_j}$, one assigns voltage $g_i^{-1} \circ_j h_j$ to the edge e in the partially-quotient graph $\widetilde{G}/_p\widetilde{\Gamma}$ if $g_i, h_j \in \Gamma_i \bigcap \Gamma_j$ for $1 \leq i, j \leq n$.

Under this labelling process, the isomorphism $\vartheta : \widetilde{G} \to (\widetilde{G}/_p\widetilde{\Gamma})^{(\tau,\varsigma)}$ identifies orbits in $\widetilde{G}$ with fibers of $G^{(\tau,\varsigma)}$. $\natural$

The multi-voltage graphs defined in this section enables us to enlarge the application field of voltage graphs. For example, a complete bipartite graph $K(n,m)$ is a lifting of a multi-voltage graph, but it is not a lifting of a voltage graph in general if $n \neq m$.

§2.3 Graphs in a Space

For two topological spaces $\mathcal{E}_1$ and $\mathcal{E}_2$, an embedding of $\mathcal{E}_1$ in $\mathcal{E}_2$ is a one-to-one continuous mapping $f : \mathcal{E}_1 \to \mathcal{E}_2$ (see [92] for details). Certainly, the same problem can be also considered for $\mathcal{E}_2$ being a metric space. By a topological view, a graph is nothing but a 1-complex, we consider the embedding problem for graphs in spaces or on surfaces in this section. The same problem had been considered by Grümbaum in [25]-[26] for graphs in spaces and in these references [6], [23],[42] – [44],[56], [69] and [106] for graphs on surfaces.

2.3.1. Graphs in an n-manifold

For a positive integer n, an n-manifold $\mathbf{M}^n$ is a Hausdorff space such that each point has an open neighborhood homeomorphic to an open n-dimensional ball $B^n = \{(x_1, x_2, \cdots, x_n)|x_1^2+x_2^2+\cdots+x_n^2 < 1\}$. For a given graph G and an n-manifold $\mathbf{M}^n$ with $n \geq 3$, the embeddability of G in $\mathbf{M}^n$ is trivial. We characterize an embedding of a graph in an n-dimensional manifold $\mathbf{M}^n$ for $n \geq 3$ similar to the rotation embedding scheme of a graph on a surface (see [23], [42] – [44], [69] for details) in this section.

For $\forall v \in V(G)$, a *space permutation* $P(v)$ of v is a permutation on $N_G(v) = \{u_1, u_2, \cdots, u_{\rho_G(v)}\}$ and all space permutation of a vertex v is denoted by $\mathcal{P}_s(v)$. We define a *space permutation* $P_s(G)$ *of a graph* G to be

$$P_s(G) = \{P(v)|\forall v \in V(G), P(v) \in \mathcal{P}_s(v)\}$$

and a *permutation system* $\mathcal{P}_s(G)$ *of* G to be all space permutation of G. Then we have the following characteristic for an embedded graph in an n-manifold $\mathbf{M^n}$ with $n \geq 3$.

Theorem 2.3.1 *For an integer $n \geq 3$, every space permutation $P_s(G)$ of a graph G defines a unique embedding of $G \to \mathbf{M}^n$. Conversely, every embedding of a graph $G \to \mathbf{M}^n$ defines a space permutation of G.*

Proof Assume G is embedded in an n-manifold $\mathbf{M}^n$. For $\forall v \in V(G)$, define an $(n-1)$-ball $B^{n-1}(v)$ to be $x_1^2 + x_2^2 + \cdots + x_n^2 = r^2$ with center at v and radius r as small as needed. Notice that all autohomeomorphisms $\mathrm{Aut}B^{n-1}(v)$ of $B^{n-1}(v)$ is a group under the composition operation and two points $A = (x_1, x_2, \cdots, x_n)$ and $B = (y_1, y_2, \cdots, y_n)$ in $B^{n-1}(v)$ are said to be combinatorially equivalent if there exists an autohomeomorphism $\varsigma \in \mathrm{Aut}B^{n-1}(v)$ such that $\varsigma(A) = B$. Consider intersection points of edges in $E_G(v, N_G(v))$ with $B^{n-1}(v)$. We get a permutation $P(v)$ on these points, or equivalently on $N_G(v)$ by $(A, B, \cdots, C, D)$ being a cycle of $P(v)$ if and only if there exists $\varsigma \in \mathrm{Aut}B^{n-1}(v)$ such that $\varsigma^i(A) = B$, $\cdots$, $\varsigma^j(C) = D$ and $\varsigma^l(D) = A$, where $i, \cdots, j, l$ are integers. Thereby we get a space permutation $P_s(G)$ of G.

Conversely, for a space permutation $P_s(G)$, we can embed G in $\mathbf{M}^n$ by embedding each vertex $v \in V(G)$ to a point X of $\mathbf{M}^n$ and arranging vertices in one cycle of $P_s(G)$ of $N_G(v)$ as the same orbit of $\langle\sigma\rangle$ action on points of $N_G(v)$ for $\sigma \in \mathrm{Aut}B^{n-1}(X)$. Whence we get an embedding of G in the manifold $\mathbf{M}^n$. ♮

Theorem 2.3.1 establishes a relation for an embedded graph in an n-dimensional manifold with a permutation, which enables us to give a combinatorial definition for graphs embedded in n-dimensional manifolds, see Definition 2.3.6 in the finial part of this section.

Corollary 2.3.1 *For a graph G, the number of embeddings of G in $\mathbf{M}^n, n \geq 3$ is*

$$\prod_{v \in V(G)} \rho_G(v)!.$$

For applying graphs in spaces to theoretical physics, we consider an embedding of a graph in an manifold with some additional conditions which enables us to find good behavior of a graph in spaces. On the first, we consider rectilinear embeddings of a graph in an Euclid space.

Definition 2.3.1 *For a given graph G and an Euclid space $\mathbf{E}$, a rectilinear embedding of G in $\mathbf{E}$ is a one-to-one continuous mapping $\pi : G \to \mathbf{E}$ such that*

(i) for $\forall e \in E(G)$, $\pi(e)$ is a segment of a straight line in $\mathbf{E}$;

(ii) for any two edges $e_1 = (u, v), e_2 = (x, y)$ in $E(G)$, $(\pi(e_1) \setminus \{\pi(u), \pi(v)\}) \bigcap (\pi(e_2) \setminus \{\pi(x), \pi(y)\}) = \emptyset$.

In $\mathbf{R}^3$, a rectilinear embedding of K_4 and a cube Q_3 are shown in Fig.2.23.

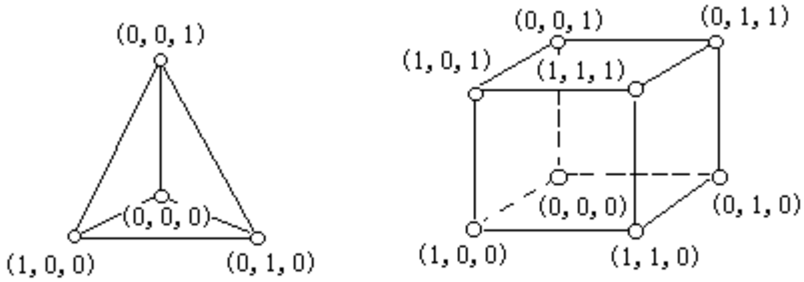


Fig 2.23

In general, we know the following result for rectilinear embedding of G in an Euclid space $\mathbf{R}^n, n \geq 3$.

Theorem 2.3.2 *For any simple graph G of order n, there is a rectilinear embedding of G in $\mathbf{R}^n$ with $n \geq 3$.*

Proof We only need to prove this assertion for $n = 3$. In $\mathbf{R}^3$, choose n points $(t_1, t_1^2, t_1^3), (t_2, t_2^2, t_2^3), \cdots, (t_n, t_n^2, t_n^3)$, where $t_1, t_2, \cdots, t_n$ are n different real numbers. For integers $i, j, k, l, 1 \leq i, j, k, l \leq n$, if a straight line passing through vertices (t_i, t_i^2, t_i^3) and (t_j, t_j^2, t_j^3) intersects with a straight line passing through vertices (t_k, t_k^2, t_k^3) and (t_l, t_l^2, t_l^3), then there must be

$$\begin{vmatrix} t_k - t_i & t_j - t_i & t_l - t_k \\ t_k^2 - t_i^2 & t_j^2 - t_i^2 & t_l^2 - t_k^2 \\ t_k^3 - t_i^3 & t_j^3 - t_i^3 & t_l^3 - t_k^3 \end{vmatrix} = 0,$$

which implies that there exist integers $s, f \in \{k, l, i, j\}$, $s \neq f$ such that $t_s = t_f$, a contradiction.

Let $V(G) = \{v_1, v_2, \cdots, v_n\}$. We embed the graph G in $\mathbf{R}^3$ by a mapping $\pi : G \to \mathbf{R}^3$ with $\pi(v_i) = (t_i, t_i^2, t_i^3)$ for $1 \leq i \leq n$ and if $v_iv_j \in E(G)$, define $\pi(v_iv_j)$ being the segment between points (t_i, t_i^2, t_i^3) and (t_j, t_j^2, t_j^3) of a straight line passing through points (t_i, t_i^2, t_i^3) and (t_j, t_j^2, t_j^3). Then π is a rectilinear embedding of the graph G in $\mathbf{R}^3$. $\natural$

For a graph G and a surface S, an *immersion* ι of G on S is a one-to-one continuous mapping $\iota : G \to S$ such that for $\forall e \in E(G)$, if $e = (u, v)$, then $\iota(e)$ is a curve connecting $\iota(u)$ and $\iota(v)$ on S. The following two definitions are generalization of embedding of a graph on a surface.

Definition 2.3.2 *Let G be a graph and S a surface in a metric space $\mathcal{E}$. A pseudo-embedding of G on S is a one-to-one continuous mapping $\pi : G \rightarrow \mathcal{E}$ such that there exists vertices $V_1 \subset V(G)$, $\pi|_{\langle V_1 \rangle}$ is an immersion on S with each component of $S \setminus \pi(\langle V_1 \rangle)$ isomorphic to an open 2-disk.*

Definition 2.3.3 *Let G be a graph with a vertex set partition $V(G) = \bigcup\limits_{j=1}^{k} V_i$, $V_i \bigcap V_j = \emptyset$ for $1 \leq i, j \leq k$ and let $S_1, S_2, \cdots, S_k$ be surfaces in a metric space $\mathcal{E}$ with $k \geq 1$. A multi-embedding of G on $S_1, S_2, \cdots, S_k$ is a one-to-one continuous mapping $\pi : G \rightarrow \mathcal{E}$ such that for any integer $i, 1 \leq i \leq k$, $\pi|_{\langle V_i \rangle}$ is an immersion with each component of $S_i \setminus \pi(\langle V_i \rangle)$ isomorphic to an open 2-disk.*

Notice that if $\pi(G) \bigcap (S_1 \bigcup S_2 \cdots \bigcup S_k) = \pi(V(G))$, then every $\pi : G \rightarrow \mathbf{R}^3$ is a multi-embedding of G. We say it to be a *trivial multi-embedding* of G on $S_1, S_2, \cdots, S_k$. If $k = 1$, then every trivial multi-embedding is a trivial pseudo-embedding of G on S_1. The main object of this section is to find nontrivial multi-embedding of G on $S_1, S_2, \cdots, S_k$ with $k \geq 1$. The existence pseudo-embedding of a graph G is obvious by definition. We concentrate our attention on characteristics of multi-embeddings of a graph.

For a graph G, let $G_1, G_2, \cdots, G_k$ be k vertex-induced subgraphs of G. If $V(G_i) \bigcap V(G_j) = \emptyset$ for any integers $i, j, 1 \leq i, j \leq k$, it is called a *block decomposition* of G and denoted by

$$G = \biguplus_{i=1}^{k} G_i.$$

The *planar block number* $n_p(G)$ of G is defined by

$$n_p(G) = min\{k | G = \biguplus_{i=1}^{k} G_i, \text{For any integer } i, 1 \leq i \leq k, G_i \text{ is planar}\}.$$

Then we get a result for the planar black number of a graph G in the following.

Theorem 2.3.3 *A graph G has a nontrivial multi-embedding on s spheres $P_1, P_2, \cdots, P_s$ with empty overlapping if and only if $n_p(G) \leq s \leq |G|$.*

Proof Assume G has a nontrivial multi-embedding on spheres $P_1, P_2, \cdots, P_s$. Since $|V(G)\bigcap P_i| \geq 1$ for any integer $i, 1 \leq i \leq s$, we know that

$$|G| = \sum_{i=1}^{s} |V(G)\bigcap P_i| \geq s.$$

By definition, if $\pi : G \rightarrow \mathbf{R}^3$ is a nontrivial multi-embedding of G on $P_1, P_2, \cdots, P_s$, then for any integer $i, 1 \leq i \leq s$, $\pi^{-1}(P_i)$ is a planar induced graph. Therefore,

$$G = \biguplus_{i=1}^{s} \pi^{-1}(P_i),$$

and we get that $s \geq n_p(G)$.

Now if $n_p(G) \leq s \leq |G|$, there is a block decomposition $G = \biguplus\limits_{i=1}^{s} G_s$ of G such that G_i is a planar graph for any integer $i, 1 \leq i \leq s$. Whence we can take s spheres $P_1, P_2, \cdots, P_s$ and define an embedding $\pi_i : G_i \rightarrow P_i$ of G_i on sphere P_i for any integer $i, 1 \leq i \leq s$.

Now define an immersion $\pi : G \rightarrow \mathbf{R}^3$ of G on $\mathbf{R}^3$ by

$$\pi(G) = (\bigcup_{i=1}^{s} \pi(G_i)) \bigcup \{(v_i, v_j) | v_i \in V(G_i), v_j \in V(G_j), (v_i, v_j) \in E(G), 1 \leq i, j \leq s\}.$$

Then $\pi : G \rightarrow \mathbf{R}^3$ is a multi-embedding of G on spheres $P_1, P_2, \cdots, P_s$. ♮

For example, a multi-embedding of K_6 on two spheres is shown in Fig.2.24, in where, $\langle\{x, y, z\}\rangle$ is on one sphere and $\langle\{u, v, w\}\rangle$ on another.

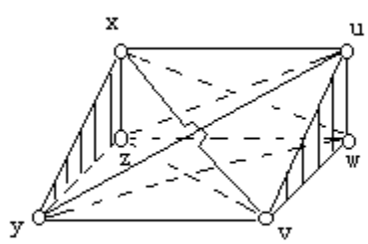


Fig 2.24

For a complete or a complete bipartite graph, we get the number $n_p(G)$ as follows.

Theorem 2.3.4 *For any integers $n, m, n, m \geq 1$, the numbers $n_p(K_n)$ and $n_p(K(m, n))$ are*

$$n_p(K_n) = \lceil \frac{n}{4} \rceil \ \ and \ \ n_p(K(m,n)) = 2,$$

if $m \geq 3, n \geq 3$, *otherwise* 1, *respectively.*

Proof Notice that every vertex-induced subgraph of a complete graph K_n is also a complete graph. By Theorem 2.1.16, we know that K_5 is non-planar. Thereby we get that

$$n_p(K_n) = \lceil \frac{n}{4} \rceil$$

by definition of $n_p(K_n)$. Now for a complete bipartite graph K(m,n), any vertex-induced subgraph by choosing s and l vertices from its two partite vertex sets is still a complete bipartite graph. According to Theorem 2.1.16, $K(3,3)$ is non-planar and $K(2,k)$ is planar. If $m \leq 2$ or $n \leq 2$, we get that $n_p(K(m,n)) = 1$. Otherwise, $K(m,n)$ is non-planar. Thereby we know that $n_p(K(m,n)) \geq 2$.

Let $V(K(m,n)) = V_1 \bigcup V_2$, where V_1, V_2 are its partite vertex sets. If $m \geq 3$ and $n \geq 3$, we choose vertices $u, v \in V_1$ and $x, y \in V_2$. Then the vertex-induced subgraphs $\langle \{u,v\} \bigcup V_2 \setminus \{x,y\} \rangle$ and $\langle \{x,y\} \bigcup V_2 \setminus \{u,v\} \rangle$ in $K(m,n)$ are planar graphs. Whence, $n_p(K(m,n)) = 2$ by definition. $\natural$

The position of surfaces $S_1, S_2, \cdots, S_k$ in a metric space $\mathcal{E}$ also influences the existence of multi-embeddings of a graph. Among these cases an interesting case is there exists an arrangement $S_{i_1}, S_{i_2}, \cdots, S_{i_k}$ for $S_1, S_2, \cdots, S_k$ such that in $\mathcal{E}$, S_{i_j} is a subspace of $S_{i_{j+1}}$ for any integer $j, 1 \leq j \leq k$. In this case, the multi-embedding is called an *including multi-embedding* of G on surfaces $S_1, S_2, \cdots, S_k$.

Theorem 2.3.5 *A graph G has a nontrivial including multi-embedding on spheres $P_1 \supset P_2 \supset \cdots \supset P_s$ if and only if there is a block decomposition $G = \biguplus_{i=1}^{s} G_i$ of G such that for any integer $i, 1 < i < s$,*

(i) G_i is planar;

(ii) for $\forall v \in V(G_i)$, $N_G(x) \subseteq (\bigcup_{j=i-1}^{i+1} V(G_j))$.

Proof Notice that in the case of spheres, if the radius of a sphere is tending to infinite, an embedding of a graph on this sphere is tending to a planar embedding. From this observation, we get the necessity of these conditions.

Now if there is a block decomposition $G = \biguplus_{i=1}^{s} G_i$ of G such that G_i is planar for any integer $i, 1 < i < s$ and $N_G(x) \subseteq (\bigcup_{j=i-1}^{i+1} V(G_j))$ for $\forall v \in V(G_i)$, we can so place s spheres $P_1, P_2, \cdots, P_s$ in $\mathbf{R}^3$ that $P_1 \supset P_2 \supset \cdots \supset P_s$. For any integer $i, 1 \le i \le s$, we define an embedding $\pi_i : G_i \to P_i$ of G_i on sphere P_i.

Since $N_G(x) \subseteq (\bigcup_{j=i-1}^{i+1} V(G_j))$ for $\forall v \in V(G_i)$, define an immersion $\pi : G \to \mathbf{R}^3$ of G on $\mathbf{R}^3$ by

$$\pi(G) = (\bigcup_{i=1}^{s} \pi(G_i)) \bigcup \{(v_i, v_j) | j = i-1, i, i+1 \ for \ 1 < i < s \ and \ (v_i, v_j) \in E(G)\}.$$

Then $\pi : G \to \mathbf{R}^3$ is a multi-embedding of G on spheres $P_1, P_2, \cdots, P_s$. $\natural$

Corollary 2.3.2 *If a graph G has a nontrivial including multi-embedding on spheres $P_1 \supset P_2 \supset \cdots \supset P_s$, then the diameter $D(G) \ge s - 1$.*

2.3.2. **Graphs on a surface**

In recent years, many books concern the embedding problem of graphs on surfaces, such as Biggs and White's [6], Gross and Tucker's [23], Mohar and Thomassen's [69] and White's [106] on embeddings of graphs on surfaces and Liu's [42]-[44], Mao's [56] and Tutte's [100] for combinatorial maps. Two disguises of graphs on surfaces, i.e., *graph embedding* and *combinatorial map* consist of two main streams in the development of topological graph theory in the past decades. For relations of these disguises with Klein surfaces, differential geometry and Riemman geometry, one can see in Mao's [55]-[56] for details.

(1) **The embedding of a graph**

For a graph $G = (V(G), E(G), I(G))$ and a surface S, an embedding of G on S is the case of $k = 1$ in Definition 2.3.3, which is also an embedding of a graph in a 2-manifold. It can be shown immediately that if there exists an embedding of G on S, then G is connected. Otherwise, we can get a component in $S \setminus \pi(G)$ not isomorphic to an open 2-disk. Thereafter all graphs considered in this subsection are connected.

Let G be a graph. For $v \in V(G)$, denote all of edges incident with the vertex v by $N_G^e(v) = \{e_1, e_2, \cdots, e_{\rho_G(v)}\}$. A permutation $C(v)$ on $e_1, e_2, \cdots, e_{\rho_G(v)}$ is said a *pure rotation* of v. All pure rotations incident with a vertex v is denoted by $\varrho(v)$. A *pure rotation system* of G is defined by

$$\rho(G) = \{C(v)|C(v) \in \varrho(v) \text{ for } \forall v \in V(G)\}$$

and all pure rotation systems of G is denoted by $\varrho(G)$.

Notice that in the case of embedded graphs on surfaces, a 1-dimensional ball is just a circle. By Theorem 2.3.1, we get a useful characteristic for embedding of graphs on orientable surfaces first found by Heffter in 1891 and then formulated by Edmonds in 1962. It can be restated as follows.

Theorem 2.3.6([23]) *Every pure rotation system for a graph G induces a unique embedding of G into an orientable surface. Conversely, every embedding of a graph G into an orientable surface induces a unique pure rotation system of G.*

According to this theorem, we know that the number of all embeddings of a graph G on orientable surfaces is $\prod_{v\in V(G)}(\rho_G(v)-1)!$.

By a topological view, an embedded vertex or face can be viewed as a disk, and an embedded edge can be viewed as a 1-band which is defined as a topological space B together with a homeomorphism $h: I \times I \to B$, where $I = [0,1]$, the unit interval. Whence, an edge in an embedded graph has two sides. One side is $h((0,x)), x \in I$. Another is $h((1,x)), x \in I$.

For an embedded graph G on a surface, the two sides of an edge $e \in E(G)$ may lie in two different faces f_1 and f_2, or in one face f without a twist ,or in one face f with a twist such as those cases (a), or (b), or (c) shown in Fig.25.

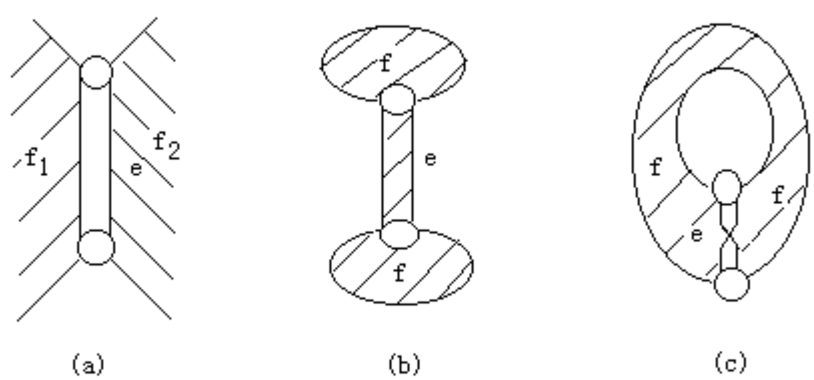


Fig 2.25

Now we define a rotation system $\rho^L(G)$ to be a pair $(\mathcal{J}, \lambda)$, where $\mathcal{J}$ is a pure rotation system of G, and $\lambda : E(G) \rightarrow Z_2$. The edge with $\lambda(e) = 0$ or $\lambda(e) = 1$ is called *type* 0 or *type* 1 edge, respectively. The *rotation system* $\varrho^L(G)$ of a graph G are defined by

$$\varrho^L(G) = \{(\mathcal{J}, \lambda) | \mathcal{J} \in \varrho(G), \lambda : E(G) \rightarrow Z_2\}.$$

By Theorem 2.3.1 we know the following characteristic for embedding graphs on locally orientable surfaces.

Theorem 2.3.7([23],[91]) *Every rotation system on a graph G defines a unique locally orientable embedding of $G \rightarrow S$. Conversely, every embedding of a graph $G \rightarrow S$ defines a rotation system for G.*

Notice that in any embedding of a graph G, there exists a spanning tree T such that every edge on this tree is type 0 (see also [23],[91] for details). Whence, the number of all embeddings of a graph G on locally orientable surfaces is

$$2^{\beta(G)} \prod_{v \in V(G)} (\rho_G(v) - 1)!$$

and the number of all embedding of G on non-orientable surfaces is

$$(2^{\beta(G)} - 1) \prod_{v \in V(G)} (\rho(v) - 1)!.$$

The following result is the famous *Euler-Poincaré* formula for embedding a graph on a surface.

Theorem 2.3.8 *If a graph G can be embedded into a surface S, then*

$$\nu(G) - \varepsilon(G) + \phi(G) = \chi(S),$$

where $\nu(G), \varepsilon(G)$ and $\phi(G)$ are the order, size and the number of faces of G on S, and $\chi(S)$ is the Euler characteristic of S, i.e.,

$$\chi(S) = \begin{cases} 2 - 2p, & if\ S\ is\ orientable, \\ 2 - q, & if\ S\ is\ non-orientable. \end{cases}$$

For a given graph G and a surface S, whether G embeddable on S is uncertain. We use the notation $G \to S$ denoting that G can be embeddable on S. Define the *orientable genus range* $GR^O(G)$ and the *non-orientable genus range* $GR^N(G)$ of a graph G by

$$GR^O(G) = \{\frac{2-\chi(S)}{2} | G \to S, S \ is \ an \ orientable \ surface\},$$

$$GR^N(G) = \{2-\chi(S) | G \to S, S \ is \ a \ non-orientable \ surface\},$$

respectively and the orientable or non-orientable genus $\gamma(G)$, $\overline{\gamma}(G)$ by

$$\gamma(G) = min\{p|p \in GR^O(G)\}, \quad \gamma_M(G) = max\{p|p \in GR^O(G)\},$$

$$\widetilde{\gamma}(G) = min\{q|q \in GR^N(G)\}, \quad \widetilde{\gamma}_M(G) = max\{q|q \in GR^O(G)\}.$$

Theorem 2.3.9(Duke 1966) *Let G be a connected graph. Then*

$$GR^O(G) = [\gamma(G), \gamma_M(G)].$$

Proof Notice that if we delete an edge e and its adjacent faces from an embedded graph G on a surface S, we get two holes at most, see Fig.25 also. This implies that $|\phi(G) - \phi(G-e)| \leq 1$.

Now assume G has been embedded on a surface of genus $\gamma(G)$ and $V(G) = \{u, v, \cdots, w\}$. Consider those of edges adjacent with u. Not loss of generality, we assume the rotation of G at vertex v is $(e_1, e_2, \cdots, e_{\rho_G(u)})$. Construct an embedded graph sequence $G_1, G_2, \cdots, G_{\rho_G(u)!}$ by

$\varrho(G_1) = \varrho(G)$;
$\varrho(G_2) = (\varrho(G) \setminus \{\varrho(u)\}) \bigcup \{(e_2, e_1, e_3, \cdots, e_{\rho_G(u)})\}$;
$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$;
$\varrho(G_{\rho_G(u)-1}) = (\varrho(G) \setminus \{\varrho(u)\}) \bigcup \{(e_2, e_3, \cdots, e_{\rho_G(u)}, e_1)\}$;
$\varrho(G_{\rho_G(u)}) = (\varrho(G) \setminus \{\varrho(u)\}) \bigcup \{(e_3, e_2, \cdots, e_{\rho_G(u)}, e_1)\}$;
$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$;

$\varrho(G_{\rho_G(u)!}) = (\varrho(G) \setminus \{\varrho(u)\}) \bigcup \{(e_{\rho_G(u)}, \cdots, e_2, e_1,)\}$.

For any integer $i, 1 \leq i \leq \rho_G(u)!$, since $|\phi(G) - \phi(G-e)| \leq 1$ for $\forall e \in E(G)$, we know that $|\phi(G_{i+1}) - \phi(G_i)| \leq 1$. Whence, $|\chi(G_{i+1}) - \chi(G_i)| \leq 1$.

Continuing the above process for every vertex in G we finally get an embedding of G with the maximum genus $\gamma_M(G)$. Since in this sequence of embeddings of G, the genus of two successive surfaces differs by at most one, we get that

$$GR^O(G) = [\gamma(G), \gamma_M(G)]. \qquad \natural$$

The genus problem, i.e., *to determine the minimum orientable or non-orientable genus of a graph* is NP-complete (see [23] for details). Ringel and Youngs got the genus of K_n completely by *current graphs* (a dual form of voltage graphs) as follows.

Theorem 2.3.10 *For a complete graph K_n and a complete bipartite graph $K(m,n)$, $m, n \geq 3$,*

$$\gamma(K_n) = \lceil \frac{(n-3)(n-4)}{12} \rceil \text{ and } \gamma(K(m,n)) = \lceil \frac{(m-2)(n-2)}{4} \rceil.$$

Outline proofs for $\gamma(K_n)$ in Theorem 2.3.10 can be found in [42], [23],[69] and a complete proof is contained in [81]. For a proof of $\gamma(K(m,n))$ in Theorem 2.3.10 can be also found in [42], [23],[69].

For the maximum genus $\gamma_M(G)$ of a graph, the time needed for computation is bounded by a polynomial function on the number of $\nu(G)$ ([23]). In 1979, Xuong got the following result.

Theorem 2.3.11(Xuong 1979) *Let G be a connected graph with n vertices and q edges. Then*

$$\gamma_M(G) = \frac{1}{2}(q-n+1) - \frac{1}{2}\min_T c_{odd}(G \setminus E(T)),$$

where the minimum is taken over all spanning trees T of G and $c_{odd}(G \setminus E(T))$ denotes the number of components of $G \setminus E(T)$ with an odd number of edges.

In 1981, Nebeský derived another important formula for the maximum genus of a graph. For a connected graph G and $A \subset E(G)$, let $c(A)$ be the number of connected component of $G \setminus A$ and let $b(A)$ be the number of connected components

X of $G \setminus A$ such that $|E(X)| \equiv |V(X)|(mod2)$. With these notations, his formula can be restated as in the next theorem.

Theorem 2.3.12(Nebeský 1981) *Let G be a connected graph with n vertices and q edges. Then*

$$\gamma_M(G) = \frac{1}{2}(q - n + 2) - \max_{A \subseteq E(G)} \{c(A) + b(A) - |A|\}.$$

Corollary 2.3.3 *The maximum genus of K_n and $K(m,n)$ are given by*

$$\gamma_M(K_n) = \lfloor \frac{(n-1)(n-2)}{4} \rfloor \text{ and } \gamma_M(K(m,n)) = \lfloor \frac{(m-1)(n-1)}{2} \rfloor,$$

respectively.

Now we turn to non-orientable embedding of a graph G. For $\forall e \in E(G)$, we define an *edge-twisting surgery* $\otimes(e)$ to be given the band of e an extra twist such as that shown in Fig.26.

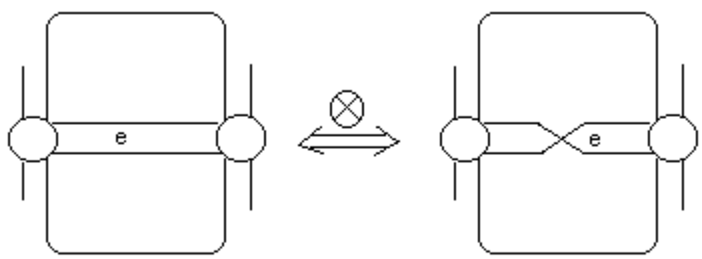


Fig 2.26

Notice that for an embedded graph G on a surface S, $e \in E(G)$, if two sides of e are in two different faces, then $\otimes(e)$ will make these faces into one and if two sides of e are in one face, $\otimes(e)$ will divide the one face into two. This property of $\otimes(e)$ enables us to get the following result for the crosscap range of a graph.

Theorem 2.3.13(Edmonds 1965, Stahl 1978) *Let G be a connected graph. Then*

$$GR^N(G) = [\tilde{\gamma}(G), \beta(G)],$$

where $\beta(G) = \varepsilon(G) - \nu(G) + 1$ is called the Betti number of G.

Proof It can be checked immediately that $\widetilde{\gamma}(G) = \widetilde{\gamma}_M(G) = 0$ for a tree G. If G is not a tree, we have known there exists a spanning tree T such that every edge on this tree is type 0 for any embedding of G.

Let $E(G) \setminus E(T) = \{e_1, e_2, \cdots, e_{\beta(G)}\}$. Adding the edge e_1 to T, we get a two faces embedding of $T + e_1$. Now make edge-twisting surgery on e_1. Then we get a one face embedding of $T + e_1$ on a surface. If we have get a one face embedding of $T + (e_1 + e_2 + \cdots + e_i)$, $1 \leq i < \beta(G)$, adding the edge e_{i+1} to $T + (e_1 + e_2 + \cdots + e_i)$ and make $\otimes(e_{i+1})$ on the edge e_{i+1}. We also get a one face embedding of $T + (e_1 + e_2 + \cdots + e_{i+1})$ on a surface again.

Continuing this process until all edges in $E(G) \setminus E(T)$ have a twist, we finally get a one face embedding of $T + (E(G) \setminus E(T)) = G$ on a surface. Since the number of twists in each circuit of this embedding of G is $1(mod2)$, this embedding is non-orientable with only one face. By the Euler-Poincaré formula, we know its genus $\widetilde{g}(G)$

$$\widetilde{g}(G) = 2 - (\nu(G) - \varepsilon(G) + 1) = \beta(G).$$

For a minimum non-orientable embedding $\mathcal{E}_G$ of G, i.e., $\widetilde{\gamma}(\mathcal{E}_G) = \widetilde{\gamma}(G)$, one can selects an edge e that lies in two faces of the embedding $\mathcal{E}_G$ and makes $\otimes(e)$. Thus in at most $\widetilde{\gamma}_M(G) - \widetilde{\gamma}(G)$ steps, one has obtained all of embeddings of G on every non-orientable surface N_s with $s \in [\widetilde{\gamma}(G), \widetilde{\gamma}_M(G)]$. Therefore,

$$GR^N(G) = [\widetilde{\gamma}(G), \beta(G)] \qquad \natural$$

Corollary 2.3.4 *Let G be a connected graph with p vertices and q edges. Then*

$$\widetilde{\gamma}_M(G) = q - p + 1.$$

Theorem 2.3.14 For a complete graph K_n and a complete bipartite graph $K(m, n)$, $m, n \geq 3$,

$$\widetilde{\gamma}(K_n) = \lceil \frac{(n-3)(n-4)}{6} \rceil$$

with an exception value $\widetilde{\gamma}(K_7) = 3$ and

$$\widetilde{\gamma}(K(m,n)) = \lceil\frac{(m-2)(n-2)}{2}\rceil.$$

A complete proof of this theorem is contained in [81], Outline proofs of Theorem 2.3.14 can be found in [42].

(2) **Combinatorial maps**

Geometrically, an embedded graph of G on a surface is called a combinatorial map M and say G underlying M. Tutte found an algebraic representation for an embedded graph on a locally orientable surface in 1973 ([98], which transfers a geometrical partition of a surface to a permutation in algebra.

According to the summaries in Liu's [43] − [44], a *combinatorial map* $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ is defined to be a permutation $\mathcal{P}$ acting on $\mathcal{X}_{\alpha,\beta}$ of a disjoint union of quadricells Kx of $x \in X$, where X is a finite set and $K = \{1, \alpha, \beta, \alpha\beta\}$ is Klein 4-group with the following conditions hold.

(i) $\forall x \in \mathcal{X}_{\alpha,\beta}$, there does not exist an integer k such that $\mathcal{P}^k x = \alpha x$;

(ii) $\alpha\mathcal{P} = \mathcal{P}^{-1}\alpha$;

(iii) The group $\Psi_J = \langle\alpha, \beta, \mathcal{P}\rangle$ is transitive on $\mathcal{X}_{\alpha,\beta}$.

The *vertices* of a combinatorial map are defined to be pairs of conjugate orbits of $\mathcal{P}$ action on $\mathcal{X}_{\alpha,\beta}$, *edges* to be orbits of K on $\mathcal{X}_{\alpha,\beta}$ and *faces* to be pairs of conjugate orbits of $\mathcal{P}\alpha\beta$ action on $\mathcal{X}_{\alpha,\beta}$.

For determining a map $(\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ is orientable or not, the following condition is needed.

(iv) If the group $\Psi_I = \langle\alpha\beta, \mathcal{P}\rangle$ is transitive on $\mathcal{X}_{\alpha,\beta}$, then M is non-orientable. Otherwise, orientable.

For example, the graph $D_{0.4.0}$ (a dipole with 4 multiple edges) on Klein bottle shown in Fig.27,

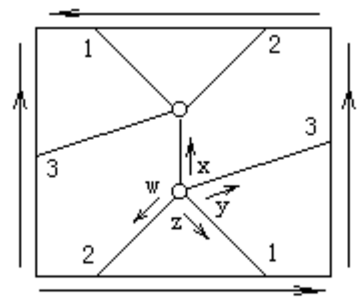


Fig 2.27

can be algebraic represented by a combinatorial map $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ with

$$\mathcal{X}_{\alpha,\beta} = \bigcup_{e \in \{x,y,z,w\}} \{e, \alpha e, \beta e, \alpha\beta e\},$$

$$\begin{aligned}\mathcal{P} &= (x, y, z, w)(\alpha\beta x, \alpha\beta y, \beta z, \beta w) \\ &\times (\alpha x, \alpha w, \alpha z, \alpha y)(\beta x, \alpha\beta w, \alpha\beta z, \beta y).\end{aligned}$$

This map has 2 vertices $v_1 = \{(x, y, z, w), (\alpha x, \alpha w, \alpha z, \alpha y)\}$, $v_2 = \{(\alpha\beta x, \alpha\beta y, \beta z, \beta w), (\beta x, \alpha\beta w, \alpha\beta z, \beta y)\}$, 4 edges $e_1 = \{x, \alpha x, \beta x, \alpha\beta x\}$, $e_2 = \{y, \alpha y, \beta y, \alpha\beta y\}$, $e_3 = \{z, \alpha z, \beta z, \alpha\beta z\}$, $e_4 = \{w, \alpha w, \beta w, \alpha\beta w\}$ and 2 faces $f_2 = \{(x, \alpha\beta y, z, \beta y, \alpha x, \alpha\beta w), (\beta x, \alpha w, \alpha\beta x, y, \beta z, \alpha y)\}$, $f_2 = \{(\beta w, \alpha z), (w, \alpha\beta z)\}$. The Euler characteristic of this map is

$$\chi(M) = 2 - 4 + 2 = 0$$

and $\Psi_I = \langle \alpha\beta, \mathcal{P} \rangle$ is transitive on $\mathcal{X}_{\alpha,\beta}$. Thereby it is a map of $D_{0.4.0}$ on a Klein bottle with 2 faces accordance with its geometrical figure.

The following result was gotten by Tutte in [98], which establishes a relation for an embedded graph with a combinatorial map.

Theorem 2.3.15 *For an embedded graph G on a locally orientable surface S, there exists one combinatorial map $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ with an underlying graph G and for a combinatorial map $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$, there is an embedded graph G underlying M on S.*

Similar to the definition of a multi-voltage graph (see [56] for details), we can define a multi-voltage map and its lifting by applying a multi-group $\widetilde{\Gamma} = \bigcup_{i=1}^{n} \Gamma_i$ with $\Gamma_i = \Gamma_j$ for any integers $i, j, 1 \leq i, j \leq n$.

Definition 2.3.4 *Let $\widetilde{\Gamma} = \bigcup_{i=1}^{n} \Gamma$ be a finite multi-group with $\Gamma = \{g_1, g_2, \cdots, g_m\}$ and an operation set $O(\widetilde{\Gamma}) = \{\circ_i | 1 \leq i \leq n\}$ and let $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ be a combinatorial map. If there is a mapping $\psi : \mathcal{X}_{\alpha,\beta} \rightarrow \widetilde{\Gamma}$ such that*

(i) for $\forall x \in \mathcal{X}_{\alpha,\beta}, \forall \sigma \in K = \{1, \alpha, \beta, \alpha\beta\}$, $\psi(\alpha x) = \psi(x)$, $\psi(\beta x) = \psi(\alpha\beta x) = \psi(x)^{-1}$;

(ii) for any face $f = (x, y, \cdots, z)(\beta z, \cdots, \beta y, \beta x)$, $\psi(f, i) = \psi(x) \circ_i \psi(y) \circ_i \cdots \circ_i \psi(z)$, where $\circ_i \in O(\widetilde{\Gamma})$, $1 \leq i \leq n$ and $\langle \psi(f, i) | f \in F(v) \rangle = G$ for $\forall v \in V(G)$, where $F(v)$ denotes all faces incident with v,

then (M, ψ) is called a multi-voltage map.

The *lifting of a multi-voltage map* is defined in the next definition.

Definition 2.3.5 *For a multi-voltage map (M, ψ), the lifting map $M^\psi = (\mathcal{X}^\psi_{\alpha^\psi,\beta^\psi}, \mathcal{P}^\psi)$ is defined by*

$$\mathcal{X}^\psi_{\alpha^\psi,\beta^\psi} = \{x_g | x \in \mathcal{X}_{\alpha,\beta}, g \in \widetilde{\Gamma}\},$$

$$\mathcal{P}^\psi = \prod_{g \in \widetilde{\Gamma}} \prod_{(x,y,\cdots,z)(\alpha z,\cdots,\alpha y,\alpha x) \in V(M)} (x_g, y_g, \cdots, z_g)(\alpha z_g, \cdots, \alpha y_g, \alpha x_g),$$

$$\alpha^\psi = \prod_{x \in \mathcal{X}_{\alpha,\beta}, g \in \widetilde{\Gamma}} (x_g, \alpha x_g),$$

$$\beta^\psi = \prod_{i=1}^{m} \prod_{x \in \mathcal{X}_{\alpha,\beta}} (x_{g_i}, (\beta x)_{g_i \circ_i \psi(x)})$$

with a convention that $(\beta x)_{g_i \circ_i \psi(x)} = y_{g_i}$ for some quadricells $y \in \mathcal{X}_{\alpha,\beta}$.

Notice that the lifting M^ψ is connected and $\Psi^\psi_I = \left\langle \alpha^\psi \beta^\psi, \mathcal{P}^\psi \right\rangle$ is transitive on $\mathcal{X}^\psi_{\alpha^\psi,\beta^\psi}$ if and only if $\Psi_I = \langle \alpha\beta, \mathcal{P} \rangle$ is transitive on $\mathcal{X}_{\alpha,\beta}$. We get a result in the following.

Theorem 2.3.16 *The Euler characteristic $\chi(M^\psi)$ of the lifting map M^ψ of a multi-voltage map $(M, \widetilde{\Gamma})$ is*

$$\chi(M^{\psi}) = |\Gamma|(\chi(M) + \sum_{i=1}^{n}\sum_{f\in F(M)}(\frac{1}{o(\psi(f,\circ_i))} - \frac{1}{n})),$$

where $F(M)$ and $o(\psi(f,\circ_i))$ denote the set of faces in M and the order of $\psi(f,\circ_i)$ in $(\Gamma;\circ_i)$, respectively.

Proof By definition the lifting map M^{ϑ} has $|\Gamma|\nu(M)$ vertices, $|\Gamma|\varepsilon(M)$ edges. Notice that each lifting of the boundary walk of a face is a homogenous lifting by definition of β^{ψ}. Similar to the proof of Theorem 2.2.3, we know that M^{ϑ} has $\sum\limits_{i=1}^{n}\sum\limits_{f\in F(M)}\frac{|\Gamma|}{o(\psi(f,\circ_i))}$ faces. By the Eular-Poincaré formula we get that

$$\begin{aligned}
\chi(M^{\psi}) &= \nu(M^{\psi}) - \varepsilon(M^{\psi}) + \phi(M^{\psi})\\
&= |\Gamma|\nu(M) - |\Gamma|\varepsilon(M) + \sum_{i=1}^{n}\sum_{f\in F(M)}\frac{|\Gamma|}{o(\psi(f,\circ_i))}\\
&= |\Gamma|(\chi(M) - \phi(M) + \sum_{i=1}^{n}\sum_{f\in F(M)}\frac{1}{o(\psi(f,\circ_i))}\\
&= |G|(\chi(M) + \sum_{i=1}^{n}\sum_{f\in F(M)}\frac{1}{o(\psi(f,\circ_i))} - \frac{1}{n}). \qquad \natural
\end{aligned}$$

Recently, more and more papers concentrated on finding *regular maps* on surface, which are related with *discrete groups, discrete geometry* and *crystal physics.* For this object, an important way is by the voltage assignment on a map. In this field, general results for automorphisms of the lifting map are known, see [45] − [46] and [71] − [72] for details. It is also an interesting problem for applying these multi-voltage maps to finding non-regular or other maps with some constraint conditions.

Motivated by the Four Color Conjecture, Tait conjectured that *every simple 3-polytope is hamiltonian* in 1880. By Steinitz's a famous result (see [24]), this conjecture is equivalent to that *every 3-connected cubic planar graph is hamiltonian.* Tutte disproved this conjecture by giving a 3-connected non-hamiltonian cubic planar graph with 46 vertices in 1946 and proved that *every* 4-*connected planar graph is hamiltonian* in 1956([97],[99]). In [56], Grünbaum conjectured that *each* 4-*connected graph embeddable in the torus or in the projective plane is hamiltonian.* This conjecture had been solved for the projective plane case by Thomas and Yu in 1994 ([93]). Notice that the splitting operator ϑ constructed in the proof of Theorem 2.1.11 is a planar operator. Applying Theorem 2.1.11 on surfaces we know that *for every map*

M on a surface, M^{ϑ} is non-hamiltonian. In fact, we can further get an interesting result related with Tait's conjecture.

Theorem 2.3.17 *There exist infinite $3-$connected non-hamiltonian cubic maps on each locally orientable surface.*

Proof Notice that there exist 3-connected triangulations on every locally orientable surface S. Each dual of them is a 3-connected cubic map on S. Now we define a splitting operator σ as shown in Fig.2.28.

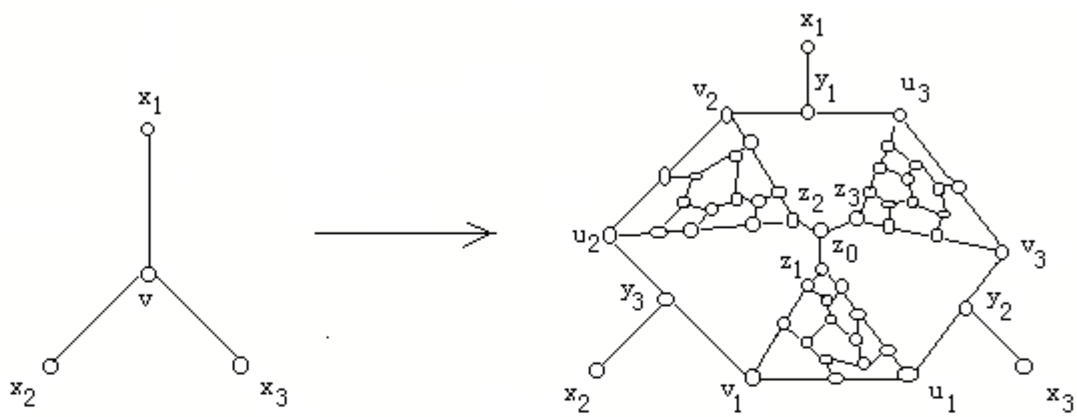


Fig.2.28

For a 3-connected cubic map M, we prove that $M^{\sigma(v)}$ is non-hamiltonian for $\forall v \in V(M)$. According to Theorem 2.1.7, we only need to prove that there are no $y_1 - y_2$, or $y_1 - y_3$, or $y_2 - y_3$ hamiltonian path in the nucleus $N(\sigma(v))$ of operator σ.

Let $H(z_i)$ be a component of $N(\sigma(v))\backslash\{z_0z_i, y_{i-1}u_{i+1}, y_{i+1}v_{i-1}\}$ which contains the vertex $z_i, 1 \leq i \leq 3$(all these indices mod 3). If there exists a $y_1 - y_2$ hamiltonian path P in $N(\sigma(v))$, we prove that there must be a $u_i - v_i$ hamiltonian path in the subgraph $H(z_i)$ for an integer $i, 1 \leq i \leq 3$.

Since P is a hamiltonian path in $N(\sigma(v))$, there must be that $v_1y_3u_2$ or $u_2y_3v_1$ is a subpath of P. Now let $E_1 = \{y_1u_3, z_0z_3, y_2v_3\}$, we know that $|E(P) \bigcap E_1| = 2$. Since P is a $y_1 - y_2$ hamiltonian path in the graph $N(\sigma(v))$, we must have $y_1u_3 \notin E(P)$ or $y_2v_3 \notin E(P)$. Otherwise, by $|E(P) \bigcap S_1| = 2$ we get that $z_0z_3 \notin E(P)$. But in this case, P can not be a $y_1 - y_2$ hamiltonian path in $N(\sigma(v))$, a contradiction.

Assume $y_2v_3 \notin E(P)$. Then $y_2u_1 \in E(P)$. Let $E_2 = \{u_1y_2, z_1z_0, v_1y_3\}$. We also know that $|E(P)\bigcap E_2| = 2$ by the assumption that P is a hamiltonian path in $N(\sigma(v))$. Hence $z_0z_1 \notin E(P)$ and the $v_1 - u_1$ subpath in P is a $v_1 - u_1$ hamiltonian path in the subgraph $H(z_1)$.

Similarly, if $y_1u_3 \notin E(P)$, then $y_1v_2 \in E(P)$. Let $E_3 = \{y_1v_2, z_0z_2, y_3u_2\}$. We can also get that $|E(P)\bigcap E_3| = 2$ and a $v_2 - u_2$ hamiltonian path in the subgraph $H(z_2)$.

Now if there is a $v_1 - u_1$ hamiltonian path in the subgraph $H(z_1)$, then the graph $H(z_1) + u_1v_1$ must be hamiltonian. According to the Grinberg's criterion for planar hamiltonian graphs, we know that

$$\phi'_3 - \phi"_3 + 2(\phi'_4 - \phi"_4) + 3(\phi'_5 - \phi"_5) + 6(\phi'_8 - \phi"_8) = 0, \qquad (*)$$

where ϕ'_i or $\phi"_i$ is the number of i-gons in the interior or exterior of a chosen hamiltonian circuit C passing through u_1v_1 in the graph $H(z_1) + u_1v_1$. Since it is obvious that

$$\phi'_3 = \phi"_8 = 1, \quad \phi"_3 = \phi'_8 = 0,$$

we get that

$$2(\phi'_4 - \phi"_4) + 3(\phi'_5 - \phi"_5) = 5, \qquad (**)$$

by (*).

Because $\phi'_4 + \phi"_4 = 2$, so $\phi'_4 - \phi"_4 = 0, 2$ or -2. Now the valency of z_1 in $H(z_1)$ is 2, so the 4-gon containing the vertex z_1 must be in the interior of C, that is $\phi'_4 - \phi"_4 \neq -2$. If $\phi'_4 - \phi"_4 = 0$ or $\phi'_4 - \phi"_4 = 2$, we get $3(\phi'_5 - \phi"_5) = 5$ or $3(\phi'_5 - \phi"_5) = 1$, a contradiction.

Notice that $H(z_1) \cong H(z_2) \cong H(z_3)$. If there exists a $v_2 - u_2$ hamiltonian path in $H(z_2)$, a contradiction can be also gotten. So there does not exist a $y_1 - y_2$ hamiltonian path in the graph $N(\sigma(v))$. Similarly , there are no $y_1 - y_3$ or $y_2 - y_3$ hamiltonian paths in the graph $N(\sigma(v))$. Whence, $M^{\sigma(v)}$ is non-hamiltonian.

Now let n be an integer, $n \geq 1$. We get that

$$\begin{array}{rcl} M_1 & = & (M)^{\sigma(u)}, \quad u \in V(M); \\ M_2 & = & (M_1)^{N(\sigma(v))(v)}, \quad v \in V(M_1); \\ \cdots & \cdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots; \\ M_n & = & (M_{n-1})^{N(\sigma(v))(w)}, \quad w \in V(M_{n-1}); \\ \cdots & \cdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots. \end{array}$$

All of these maps are 3-connected non-hamiltonian cubic maps on the surface S. This completes the proof. $\natural$

Corollary 2.3.5 *There is not a locally orientable surface on which every 3-connected cubic map is hamiltonian.*

2.3.3. **Multi-Embeddings in an n-manifold**

We come back to determine multi-embeddings of graphs in this subsection. Let $S_1, S_2, \cdots, S_k$ be k locally orientable surfaces and G a connected graph. Define numbers

$$\gamma(G; S_1, S_2, \cdots, S_k) = \min\{\sum_{i=1}^{k} \gamma(G_i) | G = \biguplus_{i=1}^{k} G_i, G_i \to S_i, 1 \le i \le k\},$$

$$\gamma_M(G; S_1, S_2, \cdots, S_k) = \max\{\sum_{i=1}^{k} \gamma(G_i) | G = \biguplus_{i=1}^{k} G_i, G_i \to S_i, 1 \le i \le k\}.$$

and the *multi-genus range* $GR(G; S_1, S_2, \cdots, S_k)$ by

$$GR(G; S_1, S_2, \cdots, S_k) = \{\sum_{i=1}^{k} g(G_i) | G = \biguplus_{i=1}^{k} G_i, G_i \to S_i, 1 \le i \le k\},$$

where G_i is embeddable on a surface of genus $g(G_i)$. Then we get the following result.

Theorem 2.3.18 *Let G be a connected graph and let $S_1, S_2, \cdots, S_k$ be locally orientable surfaces with empty overlapping. Then*

$$GR(G;S_1,S_2,\cdots,S_k)=[\gamma(G;S_1,S_2,\cdots,S_k),\gamma_M(G;S_1,S_2,\cdots,S_k)].$$

Proof Let $G=\biguplus_{i=1}^{k}G_i, G_i\rightarrow S_i, 1\leq i\leq k$. We prove that there are no gap in the multi-genus range from $\gamma(G_1)+\gamma(G_2)+\cdots+\gamma(G_k)$ to $\gamma_M(G_1)+\gamma_M(G_2)+\cdots+\gamma_M(G_k)$. According to Theorems 2.3.8 and 2.3.12, we know that the genus range $GR^O(G_i)$ or $GR^N(G)$ is $[\gamma(G_i),\gamma_M(G_i)]$ or $[\tilde{\gamma}(G_i),\tilde{\gamma}_M(G_i)]$ for any integer $i, 1\leq i\leq k$. Whence, there exists a multi-embedding of G on k locally orientable surfaces $P_1,P_2,\cdots,P_k$ with $g(P_1)=\gamma(G_1)$, $g(P_2)=\gamma(G_2)$,$\cdots$, $g(P_k)=\gamma(G_k)$. Consider the graph G_1, then G_2, and then G_3, $\cdots$ to get multi-embedding of G on k locally orientable surfaces step by step. We get a multi-embedding of G on k surfaces with genus sum at least being an unbroken interval $[\gamma(G_1)+\gamma(G_2)+\cdots+\gamma(G_k),\gamma_M(G_1)+\gamma_M(G_2)+\cdots+\gamma_M(G_k)]$ of integers.

By definitions of $\gamma(G;S_1,S_2,\cdots,S_k)$ and $\gamma_M(G;S_1,S_2,\cdots,S_k)$, we assume that $G=\biguplus_{i=1}^{k}G_i', G_i'\rightarrow S_i, 1\leq i\leq k$ and $G=\biguplus_{i=1}^{k}G_i'', G_i''\rightarrow S_i, 1\leq i\leq k$ attain the extremal values $\gamma(G;S_1,S_2,\cdots,S_k)$ and $\gamma_M(G;S_1,S_2,\cdots,S_k)$, respectively. Then we know that the multi-embedding of G on k surfaces with genus sum is at least an unbroken intervals $[\sum_{i=1}^{k}\gamma(G_i'),\sum_{i=1}^{k}\gamma_M(G_i')]$ and $[\sum_{i=1}^{k}\gamma(G_i''),\sum_{i=1}^{k}\gamma_M(G_i'')]$ of integers.

Since

$$\sum_{i=1}^{k}g(S_i)\in[\sum_{i=1}^{k}\gamma(G_i'),\sum_{i=1}^{k}\gamma_M(G_i')]\bigcap[\sum_{i=1}^{k}\gamma(G_i''),\sum_{i=1}^{k}\gamma_M(G_i'')],$$

we get that

$$GR(G;S_1,S_2,\cdots,S_k)=[\gamma(G;S_1,S_2,\cdots,S_k),\gamma_M(G;S_1,S_2,\cdots,S_k)].$$

This completes the proof. $\natural$

For multi-embeddings of a complete graph, we get the following result.

Theorem 2.3.19 *Let $P_1,P_2,\cdots,P_k$ and $Q_1,Q_2,\cdots,Q_k$ be respective k orientable and non-orientable surfaces of genus≥ 1. A complete graph K_n is multi-embeddable in $P_1,P_2,\cdots,P_k$ with empty overlapping if and only if*

$$\sum_{i=1}^{k}\lceil\frac{3+\sqrt{16g(P_i)+1}}{2}\rceil \leq n \leq \sum_{i=1}^{k}\lfloor\frac{7+\sqrt{48g(P_i)+1}}{2}\rfloor$$

and is multi-embeddable in $Q_1, Q_2, \cdots, Q_k$ with empty overlapping if and only if

$$\sum_{i=1}^{k}\lceil 1+\sqrt{2g(Q_i)}\rceil \leq n \leq \sum_{i=1}^{k}\lfloor\frac{7+\sqrt{24g(Q_i)+1}}{2}\rfloor.$$

Proof According to Theorem 2.3.9 and Corollary 2.3.2, we know that the genus $g(P)$ of an orientable surface P on which a complete graph K_n is embeddable satisfies

$$\lceil\frac{(n-3)(n-4)}{12}\rceil \leq g(P) \leq \lfloor\frac{(n-1)(n-2)}{4}\rfloor,$$

i.e.,

$$\frac{(n-3)(n-4)}{12} \leq g(P) \leq \frac{(n-1)(n-2)}{4}.$$

If $g(P) \geq 1$, we get that

$$\lceil\frac{3+\sqrt{16g(P)+1}}{2}\rceil \leq n \leq \lfloor\frac{7+\sqrt{48g(P)+1}}{2}\rfloor.$$

Similarly, if K_n is embeddable on a non-orientable surface Q, then

$$\lceil\frac{(n-3)(n-4)}{6}\rceil \leq g(Q) \leq \lfloor\frac{(n-1)^2}{2}\rfloor,$$

i.e.,

$$\lceil 1+\sqrt{2g(Q)}\rceil \leq n \leq \lfloor\frac{7+\sqrt{24g(Q)+1}}{2}\rfloor.$$

Now if K_n is multi-embeddable in $P_1, P_2, \cdots, P_k$ with empty overlapping, then there must exists a partition $n = n_1 + n_2 + \cdots + n_k$, $n_i \geq 1, 1 \leq i \leq k$. Since each vertex-induced subgraph of a complete graph is still a complete graph, we know that for any integer $i, 1 \leq i \leq k$,

$$\lceil\frac{3+\sqrt{16g(P_i)+1}}{2}\rceil \leq n_i \leq \lfloor\frac{7+\sqrt{48g(P_i)+1}}{2}\rfloor.$$

Whence, we know that

$$\sum_{i=1}^{k}\lceil\frac{3+\sqrt{16g(P_i)+1}}{2}\rceil \leq n \leq \sum_{i=1}^{k}\lfloor\frac{7+\sqrt{48g(P_i)+1}}{2}\rfloor. \qquad (*)$$

On the other hand, if the inequality (*) holds, we can find positive integers $n_1, n_2, \cdots, n_k$ with $n = n_1 + n_2 + \cdots + n_k$ and

$$\lceil\frac{3+\sqrt{16g(P_i)+1}}{2}\rceil \leq n_i \leq \lfloor\frac{7+\sqrt{48g(P_i)+1}}{2}\rfloor.$$

for any integer $i, 1 \leq i \leq k$. This enables us to establish a partition $K_n = \biguplus_{i=1}^{k} K_{n_i}$ for K_n and embed each K_{n_i} on P_i for $1 \leq i \leq k$. Therefore, we get a multi-embedding of K_n in $P_1, P_2, \cdots, P_k$ with empty overlapping.

Similarly, if K_n is multi-embeddable in $Q_1, Q_2, \cdots Q_k$ with empty overlapping, there must exists a partition $n = m_1 + m_2 + \cdots + m_k$, $m_i \geq 1, 1 \leq i \leq k$ and

$$\lceil 1+\sqrt{2g(Q_i)}\rceil \leq m_i \leq \lfloor\frac{7+\sqrt{24g(Q_i)+1}}{2}\rfloor.$$

for any integer $i, 1 \leq i \leq k$. Whence, we get that

$$\sum_{i=1}^{k}\lceil 1+\sqrt{2g(Q_i)}\rceil \leq n \leq \sum_{i=1}^{k}\lfloor\frac{7+\sqrt{24g(Q_i)+1}}{2}\rfloor. \qquad (**)$$

Now if the inequality (**) holds, we can also find positive integers $m_1, m_2, \cdots, m_k$ with $n = m_1 + m_2 + \cdots + m_k$ and

$$\lceil 1+\sqrt{2g(Q_i)}\rceil \leq m_i \leq \lfloor\frac{7+\sqrt{24g(Q_i)+1}}{2}\rfloor.$$

for any integer $i, 1 \leq i \leq k$. Similar to those of orientable cases, we get a multi-embedding of K_n in $Q_1, Q_2, \cdots, Q_k$ with empty overlapping. $\natural$

Corollary 2.3.6 *A complete graph K_n is multi-embeddable in $k, k \geq 1$ orientable surfaces of genus $p, p \geq 1$ with empty overlapping if and only if*

$$\lceil\frac{3+\sqrt{16p+1}}{2} \leq \frac{n}{k} \leq \lfloor\frac{7+\sqrt{48p+1}}{2}\rfloor$$

and is multi-embeddable in $l, l \geq 1$ non-orientable surfaces of genus $q, q \geq 1$ with empty overlapping if and only if

$$\lceil 1+\sqrt{2q}\rceil \le \frac{n}{k} \le \lfloor \frac{7+\sqrt{24q+1}}{2} \rfloor.$$

Corollary 2.3.7 *A complete graph K_n is multi-embeddable in $s, s \ge 1$ tori with empty overlapping if and only if*

$$4s \le n \le 7s$$

and is multi-embeddable in $t, t \ge 1$ projective planes with empty overlapping if and only if

$$3t \le n \le 6t.$$

Similarly, the following result holds for a complete bipartite graph $K(n,n)$.

Theorem 2.3.20 *Let $P_1, P_2, \cdots, P_k$ and $Q_1, Q_2, \cdots, Q_k$ be respective k orientable and k non-orientable surfaces of genus≥ 1. A complete bipartite graph $K(n,n)$ is multi-embeddable in $P_1, P_2, \cdots, P_k$ with empty overlapping if and only if*

$$\sum_{i=1}^{k} \lceil 1+\sqrt{2g(P_i)}\rceil \le n \le \sum_{i=1}^{k} \lfloor 2+2\sqrt{g(P_i)}\rfloor$$

and is multi-embeddable in $Q_1, Q_2, \cdots, Q_k$ with empty overlapping if and only if

$$\sum_{i=1}^{k} \lceil 1+\sqrt{g(Q_i)}\rceil \le n \le \sum_{i=1}^{k} \lfloor 2+\sqrt{2g(Q_i)}\rfloor.$$

Proof Similar to the proof of Theorem 2.3.18, we get this result. ♮

2.3.4. **Classification of graphs in an n-manifold**

By Theorem 2.3.1 we can give a combinatorial definition for a graph embedded in an n-manifold, i.e., a *manifold graph* similar to the Tutte's definition for a map.

Definition 2.3.6 *For any integer $n, n \ge 2$, an n-dimensional manifold graph ${}^n\mathcal{G}$ is a pair ${}^n\mathcal{G} = (\mathcal{E}_\Gamma, \mathcal{L})$ in where a permutation $\mathcal{L}$ acting on $\mathcal{E}_\Gamma$ of a disjoint union $\Gamma x = \{\sigma x | \sigma \in \Gamma\}$ for $\forall x \in E$, where E is a finite set and $\Gamma = \{\mu, o | \mu^2 = o^n = 1, \mu o = o\mu\}$ is a commutative group of order $2n$ with the following conditions hold.*

(i) $\forall x \in \mathcal{E}_K$, *there does not exist an integer* k *such that* $\mathcal{L}^k x = o^i x$ *for* $\forall i, 1 \leq i \leq n-1$;

(ii) $\mu\mathcal{L} = \mathcal{L}^{-1}\mu$;

(iii) *The group* $\Psi_J = \langle \mu, o, \mathcal{L}\rangle$ *is transitive on* $\mathcal{E}_\Gamma$.

According to (i) and (ii), a *vertex* v *of an* n*-dimensional manifold graph* is defined to be an n-tuple $\{(o^i x_1, o^i x_2, \cdots, o^i x_{s_l(v)})(o^i y_1, o^i y_2, \cdots, o^i y_{s_2(v)}) \cdots (o^i z_1, o^i z_2, \cdots, o^i z_{s_{l(v)}(v)}); 1 \leq i \leq n\}$ of permutations of $\mathcal{L}$ action on $\mathcal{E}_\Gamma$, edges to be these orbits of Γ action on $\mathcal{E}_\Gamma$. The number $s_1(v) + s_2(v) + \cdots + s_{l(v)}(v)$ is called the *valency of* v, denoted by $\rho_G^{s_1, s_2, \cdots, s_{l(v)}}(v)$. The condition (iii) is used to ensure that an n-dimensional manifold graph is connected. Comparing definitions of a map with an n-dimensional manifold graph, the following result holds.

Theorem 2.3.21 *For any integer* $n, n \geq 2$, *every* n*-dimensional manifold graph* ${}^n\mathcal{G} = (\mathcal{E}_\Gamma, \mathcal{L})$ *is correspondent to a unique map* $M = (\mathcal{E}_{\alpha,\beta}, \mathcal{P})$ *in which each vertex* v *in* ${}^n\mathcal{G}$ *is converted to* $l(v)$ *vertices* $v_1, v_2, \cdots, v_{l(v)}$ *of* M. *Conversely, a map* $M = (\mathcal{E}_{\alpha,\beta}, \mathcal{P})$ *is also correspondent to an* n*-dimensional manifold graph* ${}^n\mathcal{G} = (\mathcal{E}_\Gamma, \mathcal{L})$ *in which* $l(v)$ *vertices* $u_1, u_2, \cdots, u_{l(v)}$ *of* M *are converted to one vertex* u *of* ${}^n\mathcal{G}$.

Two n-dimensional manifold graphs ${}^n\mathcal{G}_1 = (\mathcal{E}^1_{\Gamma_1}, \mathcal{L}_1)$ and ${}^n\mathcal{G}_2 = (\mathcal{E}^2_{\Gamma_2}, \mathcal{L}_2)$ are said to be *isomorphic* if there exists a one-to-one mapping $\kappa : \mathcal{E}^1_{\Gamma_1} \to \mathcal{E}^2_{\Gamma_2}$ such that $\kappa\mu = \mu\kappa, \kappa o = o\kappa$ and $\kappa\mathcal{L}_1 = \mathcal{L}_2\kappa$. If $\mathcal{E}^1_{\Gamma_1} = \mathcal{E}^2_{\Gamma_2} = \mathcal{E}_\Gamma$ and $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{L}$, an isomorphism between ${}^n\mathcal{G}_1$ and ${}^n\mathcal{G}_2$ is called an automorphism of ${}^n\mathcal{G} = (\mathcal{E}_\Gamma, \mathcal{L})$. It is immediately that all automorphisms of ${}^n\mathcal{G}$ form a group under the composition operation. We denote this group by $\mathrm{Aut}^n\mathcal{G}$.

It is obvious that for two isomorphic n-dimensional manifold graphs ${}^n\mathcal{G}_1$ and ${}^n\mathcal{G}_2$, their underlying graphs G_1 and G_2 are isomorphic. For an embedding ${}^n\mathcal{G} = (\mathcal{E}_\Gamma, \mathcal{L})$ in an n-dimensional manifold and $\forall\zeta \in \mathrm{Aut}_{\frac{1}{2}}G$, an induced action of ζ on $\mathcal{E}_\Gamma$ is defined by

$$\zeta(gx) = g\zeta(x)$$

for $\forall x \in \mathcal{E}_\Gamma$ and $\forall g \in \Gamma$. Then the following result holds.

Theorem 2.3.22 $\mathrm{Aut}^n\mathcal{G} \preceq \mathrm{Aut}_{\frac{1}{2}}G \times \langle\mu\rangle$.

Proof First we prove that two n-dimensional manifold graphs ${}^n\mathcal{G}_1 = (\mathcal{E}^1_{\Gamma_1}, \mathcal{L}_1)$ and ${}^n\mathcal{G}_2 = (\mathcal{E}^2_{\Gamma_2}, \mathcal{L}_2)$ are isomorphic if and only if there is an element $\zeta \in \mathrm{Aut}_{\frac{1}{2}}\Gamma$ such that $\mathcal{L}_1^\zeta = \mathcal{L}_2$ or $\mathcal{L}_2^{-1}$.

If there is an element $\zeta \in \mathrm{Aut}_{\frac{1}{2}}\Gamma$ such that $\mathcal{L}_1^\zeta = \mathcal{L}_2$, then the n-dimensional manifold graph ${}^n\mathcal{G}_1$ is isomorphic to ${}^n\mathcal{G}_2$ by definition. If $\mathcal{L}_1^\zeta = \mathcal{L}_2^{-1}$, then $\mathcal{L}_1^{\zeta\mu} = \mathcal{L}_2$. The n-dimensional manifold graph ${}^n\mathcal{G}_1$ is also isomorphic to ${}^n\mathcal{G}_2$.

By the definition of an isomorphism ξ between n-dimensional manifold graphs ${}^n\mathcal{G}_1$ and ${}^n\mathcal{G}_2$, we know that

$$\mu\xi(x) = \xi\mu(x), \ o\xi(x) = \xi o(x) \text{ and } \mathcal{L}_1^\xi(x) = \mathcal{L}_2(x).$$

$\forall x \in \mathcal{E}_\Gamma$. By definition these conditions

$$o\xi(x) = \xi o(x) \text{ and } \mathcal{L}_1^\xi(x) = \mathcal{L}_2(x).$$

are just the condition of an automorphism ξ or $\alpha\xi$ on $X_{\frac{1}{2}}(\Gamma)$. Whence, the assertion is true.

Now let $\mathcal{E}^1_{\Gamma_1} = \mathcal{E}^2_{\Gamma_2} = \mathcal{E}_\Gamma$ and $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{L}$. We know that

$$\mathrm{Aut}^n\mathcal{G} \preceq \mathrm{Aut}_{\frac{1}{2}}G \times \langle\mu\rangle . \qquad \natural$$

Similar to combinatorial maps, the action of an automorphism of a manifold graph on $\mathcal{E}_\Gamma$ is fixed-free.

Theorem 2.3.23 *Let ${}^n\mathcal{G} = (\mathcal{E}_\Gamma, \mathcal{L})$ be an n-dimensional manifold graph. Then $(\mathrm{Aut}^n\mathcal{G})_x$ is trivial for $\forall x \in \mathcal{E}_\Gamma$.*

Proof For $\forall g \in (\mathrm{Aut}^n\mathcal{G})_x$, we prove that $g(y) = y$ for $\forall y \in \mathcal{E}_\Gamma$. In fact, since the group $\Psi_J = \langle\mu, o, \mathcal{L}\rangle$ is transitive on $\mathcal{E}_\Gamma$, there exists an element $\tau \in \Psi_J$ such that $y = \tau(x)$. By definition we know that every element in Ψ_J is commutative with automorphisms of ${}^n\mathcal{G}$. Whence, we get that

$$g(y) = g(\tau(x)) = \tau(g(x)) = \tau(x) = y.$$

i.e., $(\mathrm{Aut}^n\mathcal{G})_x$ is trivial. $\natural$

Corollary 2.3.8 *Let $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ be a map. Then for $\forall x \in \mathcal{X}_{\alpha,\beta}$, $(\mathrm{Aut}M)_x$ is trivial.*

For an n-dimensional manifold graph ${}^n\mathcal{G} = (\mathcal{E}_\Gamma, \mathcal{L})$, an $x \in \mathcal{E}_\Gamma$ is said a *root* of ${}^n\mathcal{G}$. If we have chosen a root r on an n-dimensional manifold graph ${}^n\mathcal{G}$, then ${}^n\mathcal{G}$ is called a *rooted n-dimensional manifold graph*, denoted by ${}^n\mathcal{G}^r$. Two rooted n-dimensional manifold graphs ${}^n\mathcal{G}^{r_1}$ and ${}^n\mathcal{G}^{r_2}$ are said to be *isomorphic* if there is an isomorphism ς between them such that $\varsigma(r_1) = r_2$. Applying Theorem 2.3.23 and Corollary 2.3.1, we get an enumeration result for n-dimensional manifold graphs underlying a graph G in the following.

Theorem 2.3.24 *For any integer $n, n \geq 3$, the number $r_n^S(G)$ of rooted n-dimensional manifold graphs underlying a graph G is*

$$r_n^S(G) = \frac{n\varepsilon(G) \prod\limits_{v \in V(G)} \rho_G(v)!}{|Aut_{\frac{1}{2}} G|}.$$

Proof Denote the set of all non-isomorphic n-dimensional manifold graphs underlying a graph G by $\mathcal{G}^S(G)$. For an n-dimensional graph ${}^n\mathcal{G} = (\mathcal{E}_\Gamma, \mathcal{L}) \in \mathcal{G}^S(G)$, denote the number of non-isomorphic rooted n-dimensional manifold graphs underlying ${}^n\mathcal{G}$ by $r({}^n\mathcal{G})$. By a result in permutation groups theory, for $\forall x \in \mathcal{E}_\Gamma$ we know that

$$|\mathrm{Aut}^n\mathcal{G}| = |(\mathrm{Aut}^n\mathcal{G})_x||x^{\mathrm{Aut}^n\mathcal{G}}|.$$

According to Theorem 2.3.23, $|(\mathrm{Aut}^n\mathcal{G})_x| = 1$. Whence, $|x^{\mathrm{Aut}^n\mathcal{G}}| = |\mathrm{Aut}^n\mathcal{G}|$. However there are $|\mathcal{E}_\Gamma| = 2n\varepsilon(G)$ roots in ${}^n\mathcal{G}$ by definition. Therefore, the number of non-isomorphic rooted n-dimensional manifold graphs underlying an n-dimensional graph ${}^n\mathcal{G}$ is

$$r({}^n\mathcal{G}) = \frac{|\mathcal{E}_\Gamma|}{|\mathrm{Aut}^n\mathcal{G}|} = \frac{2n\varepsilon(G)}{|\mathrm{Aut}^n\mathcal{G}|}.$$

Whence, the number of non-isomorphic rooted n-dimensional manifold graphs underlying a graph G is

$$r_n^S(G) = \sum_{{}^n\mathcal{G} \in \mathcal{G}^S(G)} \frac{2n\varepsilon(G)}{|\mathrm{Aut}^n\mathcal{G}|}.$$

According to Theorem 2.3.22, $\mathrm{Aut}^n\mathcal{G} \preceq \mathrm{Aut}_{\frac{1}{2}} G \times \langle\mu\rangle$. Whence $\tau \in \mathrm{Aut}^n\mathcal{G}$ for ${}^n\mathcal{G} \in \mathcal{G}^S(G)$ if and only if $\tau \in (\mathrm{Aut}_{\frac{1}{2}} G \times \langle\mu\rangle)_{{}^n\mathcal{G}}$. Therefore, we know that $\mathrm{Aut}^n\mathcal{G} =$

$(\mathrm{Aut}_{\frac{1}{2}}G\times\langle\mu\rangle)_{^n\mathcal{G}}$. Because of $|\mathrm{Aut}_{\frac{1}{2}}G\times\langle\mu\rangle\,| = |(\mathrm{Aut}_{\frac{1}{2}}G\times\langle\mu\rangle)_{^n\mathcal{G}}||{}^n\mathcal{G}^{\mathrm{Aut}_{\frac{1}{2}}G\times\langle\mu\rangle}|$, we get that

$$|{}^n\mathcal{G}^{\mathrm{Aut}_{\frac{1}{2}}G\times\langle\mu\rangle}| = \frac{2|\mathrm{Aut}_{\frac{1}{2}}G|}{|\mathrm{Aut}^n\mathcal{G}|}.$$

Therefore,

$$\begin{aligned} r_n^S(G) &= \sum_{^n\mathcal{G}\in\mathcal{G}^S(G)} \frac{2n\varepsilon(G)}{|\mathrm{Aut}^n\mathcal{G}|} \\ &= \frac{2n\varepsilon(G)}{|\mathrm{Aut}_{\frac{1}{2}}G\times\langle\mu\rangle\,|}\sum_{^n\mathcal{G}\in\mathcal{G}^S(G)}\frac{|\mathrm{Aut}_{\frac{1}{2}}G\times\langle\mu\rangle\,|}{|\mathrm{Aut}^n\mathcal{G}|} \\ &= \frac{2n\varepsilon(G)}{|\mathrm{Aut}_{\frac{1}{2}}G\times\langle\mu\rangle\,|}\sum_{^n\mathcal{G}\in\mathcal{G}^S(G)}|{}^n\mathcal{G}^{\mathrm{Aut}_{\frac{1}{2}}G\times\langle\mu\rangle|} \\ &= \frac{n\varepsilon(G)\prod\limits_{v\in V(G)}\rho_G(v)!}{|\mathrm{Aut}_{\frac{1}{2}}G|} \end{aligned}$$

by applying Corollary 2.3.1. ♮

Notice the fact that an embedded graph in a 2-dimensional manifolds is just a map. Then Definition 3.6 is converted to Tutte's definition for combinatorial maps in this case. We can also get an enumeration result for rooted maps on surfaces underlying a graph G by applying Theorems 2.3.7 and 2.3.23 in the following.

Theorem 2.3.25([66],[67]) *The number $r^L(\Gamma)$ of rooted maps on locally orientable surfaces underlying a connected graph G is*

$$r^L(G) = \frac{2^{\beta(G)+1}\varepsilon(G)\prod\limits_{v\in V(G)}(\rho(v)-1)!}{|Aut_{\frac{1}{2}}G|},$$

where $\beta(G)=\varepsilon(G)-\nu(G)+1$ is the Betti number of G.

Similarly, for a graph $G=\bigoplus\limits_{i=1}^{l}G_i$ and a multi-manifold $\widetilde{M}=\bigcup\limits_{i=1}^{l}\mathbf{M}^{l_i}$, choose l commutative groups $\Gamma_1,\Gamma_2,\cdots,\Gamma_l$, where $\Gamma_i=\left\langle\mu_i,o_i|\mu_i^2=o^{h_i}=1\right\rangle$ for any integer $i, 1\leq i\leq l$. Consider permutations acting on $\bigcup\limits_{i=1}^{l}\mathcal{E}_{\Gamma_i}$, where for any integer $i, 1\leq i\leq l$, $\mathcal{E}_{\Gamma_i}$ is a disjoint union $\Gamma_i x=\{\sigma_i x|\sigma_i\in\Gamma\}$ for $\forall x\in E(G_i)$. Similar to Definition 2.3.6, we can also get a multi-embedding of G in $\widetilde{M}=\bigcup\limits_{i=1}^{l}\mathbf{M}^{h_i}$.

§2.4 Multi-Spaces on Graphs

A Smarandache multi-space is a union of k spaces $A_1, A_2, \cdots, A_k$ for an integer $k, k \geq 2$ with some additional constraint conditions. For describing a finite algebraic multi-space, graphs are a useful way. All graphs considered in this section are directed graphs.

2.4.1. **A graph model for an operation system**

A graph is called a *directed graph* if there is an orientation on its every edge. A directed graph $\overrightarrow{G}$ is called an *Euler graph* if we can travel all edges of $\overrightarrow{G}$ alone orientations on its edges with no repeat starting at any vertex $u \in V(\overrightarrow{G})$ and come back to u. For a directed graph $\overrightarrow{G}$, we use the convention that the orientation on the edge e is $u \to v$ for $\forall e = (u, v) \in E(\overrightarrow{G})$ and say that e is *incident from* u and *incident to* v. For $u \in V(\overrightarrow{G})$, the *outdegree* $\rho^+_{\overrightarrow{G}}(u)$ of u is the number of edges in $\overrightarrow{G}$ incident from u and the *indegree* $\rho^-_{\overrightarrow{G}}(u)$ of u is the number of edges in $\overrightarrow{G}$ incident to u. Whence, we know that

$$\rho^+_{\overrightarrow{G}}(u) + \rho^-_{\overrightarrow{G}}(u) = \rho_{\overrightarrow{G}}(u).$$

It is well-known that a graph $\overrightarrow{G}$ is Eulerian if and only if $\rho^+_{\overrightarrow{G}}(u) = \rho^-_{\overrightarrow{G}}(u)$ for $\forall u \in V(\overrightarrow{G})$, seeing examples in [11] for details. For a multiple 2-edge (a, b), if two orientations on edges are both to a or both to b, then we say it to be a *parallel multiple* 2*-edge*. If one orientation is to a and another is to b, then we say it to be an *opposite multiple* 2*-edge*.

Now let $(A; \circ)$ be an algebraic system with operation "$\circ$". We associate a *weighted graph* $G[A]$ for $(A; \circ)$ defined as in the next definition.

Definition 2.4.1 *Let $(A; \circ)$ be an algebraic system. Define a weighted graph $G[A]$ associated with $(A; \circ)$ by*

$$V(G[A]) = A$$

and

$$E(G[A]) = \{(a, c) \ with \ weight \ \circ b \mid if \ a \circ b = c \ for \ \forall a, b, c \in A\}$$

as shown in Fig.2.29.

Fig.2.29

For example, the associated graph $G[Z_4]$ for the commutative group Z_4 is shown in Fig.2.30.

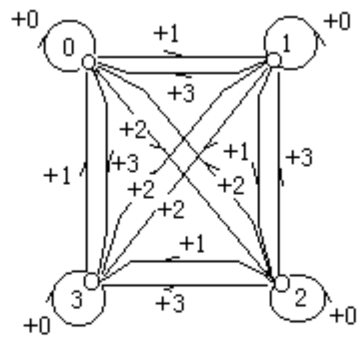


Fig.2.30

The advantage of Definition 2.4.1 is that for any edge in $G[A]$, if its vertices are a,c with a weight $\circ b$, then a$\circ b = c$ and vice versa, if a$\circ b = c$, then there is one and only one edge in $G[A]$ with vertices a, c and weight $\circ b$. This property enables us to find some structure properties of $G[A]$ for an algebraic system $(A; \circ)$.

*P*1. *$G[A]$ is connected if and only if there are no partition $A = A_1 \bigcup A_2$ such that for $\forall a_1 \in A_1$, $\forall a_2 \in A_2$, there are no definition for $a_1 \circ a_2$ in $(A; \circ)$.*

If $G[A]$ is disconnected, we choose one component C and let $A_1 = V(C)$. Define $A_2 = V(G[A]) \setminus V(C)$. Then we get a partition $A = A_1 \bigcup A_2$ and for $\forall a_1 \in A_1$, $\forall a_2 \in A_2$, there are no definition for $a_1 \circ a_2$ in $(A; \circ)$, a contradiction and vice versa.

*P*2. *If there is a unit $\mathbf{1}_A$ in $(A; \circ)$, then there exists a vertex $\mathbf{1}_A$ in $G[A]$ such that the weight on the edge $(\mathbf{1}_A, x)$ is $\circ x$ if $\mathbf{1}_A \circ x$ is defined in $(A; \circ)$ and vice versa.*

*P*3. *For $\forall a \in A$, if a^{-1} exists, then there is an opposite multiple 2-edge $(\mathbf{1}_A, a)$ in $G[A]$ with weights $\circ a$ and $\circ a^{-1}$, respectively and vice versa.*

*P*4. *For $\forall a, b \in A$ if $a \circ b = b \circ a$, then there are edges (a, x) and (b, x), $x \in A$ in $(A; \circ)$ with weights $w(a, x) = \circ b$ and $w(b, x) = \circ a$, respectively and vice versa.*

*P*5. *If the cancellation law holds in $(A; \circ)$, i.e., for $\forall a, b, c \in A$, if $a \circ b = a \circ c$ then $b = c$, then there are no parallel multiple 2-edges in $G[A]$ and vice versa.*

The property $P2, P3, P4$ and $P5$ are gotten by definition. Each of these cases is shown in Fig.2.31(1), (2), (3) and (4), respectively.

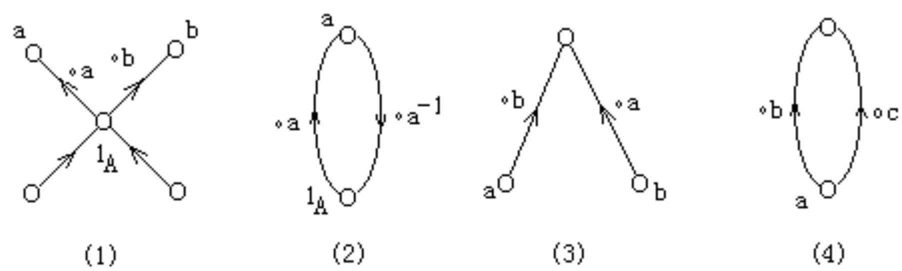


Fig.2.31

Definition 2.4.2 *An algebraic system $(A; \circ)$ is called to be a one-way system if there exists a mapping $\varpi : A \rightarrow A$ such that if $a \circ b \in A$, then there exists a unique $c \in A$, $c \circ \varpi(b) \in A$. ϖ is called a one-way function on $(A; \circ)$.*

We have the following results for an algebraic system $(A; \circ)$ with its associated weighted graph $G[A]$.

Theorem 2.4.1 *Let $(A; \circ)$ be an algebraic system with a associated weighted graph $G[A]$. Then*

(i) if there is a one-way function ϖ on $(A; \circ)$, then $G[A]$ is an Euler graph, and vice versa, if $G[A]$ is an Euler graph, then there exist a one-way function ϖ on $(A; \circ)$.

(ii) if $(A; \circ)$ is a complete algebraic system, then the outdegree of every vertex in $G[A]$ is $|A|$; in addition, if the cancellation law holds in $(A; \circ)$, then $G[A]$ is a complete multiple 2-graph with a loop attaching at each of its vertices such that each edge between two vertices in $G[A]$ is an opposite multiple 2-edge, and vice versa.

Proof (i) Assume ϖ is a one-way function ϖ on $(A; \circ)$. By definition there exists $c \in A$, $c \circ \varpi(b) \in A$ for $\forall a \in A$, $a \circ b \in A$. Thereby there is a one-to-one correspondence between edges from a with edges to a. That is, $\rho^{+}_{G[A]}(a) = \rho^{-}_{G[A]}(a)$ for $\forall a \in V(G[A])$. Therefore, $G[A]$ is an Euler graph.

Now if $G[A]$ is an Euler graph, then there is a one-to-one correspondence between edges in $E^{-} = \{e_i^{-}; 1 \leq i \leq k\}$ from a vertex a with edges $E^{+} = \{e_i^{+}; 1 \leq i \leq k\}$ to the vertex a. For any integer $i, 1 \leq i \leq k$, define $\varpi : w(e_i^{-}) \rightarrow w(e_i^{+})$. Therefore, ϖ is a well-defined one-way function on $(A; \circ)$.

(ii) If $(A; \circ)$ is complete, then for $\forall a \in A$ and $\forall b \in A$, $a \circ b \in A$. Therefore, $\rho^{+}_{\overrightarrow{G}}(a) = |A|$ for any vertex $a \in V(G[A])$.

If the cancellation law holds in $(A; \circ)$, by $P5$ there are no parallel multiple 2-edges in $G[A]$. Whence, each edge between two vertices is an opposite 2-edge and weights on loops are $\circ \mathbf{1}_A$.

By definition, if $G[A]$ is a complete multiple 2-graph with a loop attaching at each of its vertices such that each edge between two vertices in $G[A]$ is an opposite multiple 2-edge, we know that $(A; \circ)$ is a complete algebraic system with the cancellation law holding by the definition of $G[A]$. ♮

Corollary 2.4.1 *Let Γ be a semigroup. Then $G[\Gamma]$ is a complete multiple 2-graph with a loop attaching at each of its vertices such that each edge between two vertices in $G[A]$ is an opposite multiple 2-edge.*

Notice that in a group Γ, $\forall g \in \Gamma$, if $g^2 \neq \mathbf{1}_\Gamma$, then $g^{-1} \neq g$. Whence, all elements of order> 2 in Γ can be classified into pairs. This fact enables us to know the following result.

Corollary 2.4.2 *Let Γ be a group of even order. Then there are opposite multiple 2-edges in $G[\Gamma]$ such that weights on its 2 directed edges are the same.*

2.4.2. **Multi-Spaces on graphs**

Let $\widetilde{\Gamma}$ be a Smarandache multi-space. Its associated weighted graph is defined in the following.

Definition 2.4.3 *Let $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma_i$ be an algebraic multi-space with $(\Gamma_i; \circ_i)$ being an algebraic system for any integer $i, 1 \leq i \leq n$. Define a weighted graph $G(\widetilde{\Gamma})$ associated with $\widetilde{\Gamma}$ by*

$$G(\widetilde{\Gamma}) = \bigcup_{i=1}^{n} G[\Gamma_i],$$

where $G[\Gamma_i]$ *is the associated weighted graph of* $(\Gamma_i; \circ_i)$ *for* $1 \leq i \leq n$.

For example, the weighted graph shown in Fig.2.32 is correspondent with a multi-space $\widetilde{\Gamma} = \Gamma_1 \bigcup \Gamma_2 \bigcup \Gamma_3$, where $(\Gamma_1; +) = (Z_3, +)$, $\Gamma_2 = \{e, a, b\}$, $\Gamma_3 = \{1, 2, a, b\}$ and these operations "$\cdot$" on Γ_2 and "$\circ$" on Γ_3 are shown in tables 2.4.1 and 2.4.2.

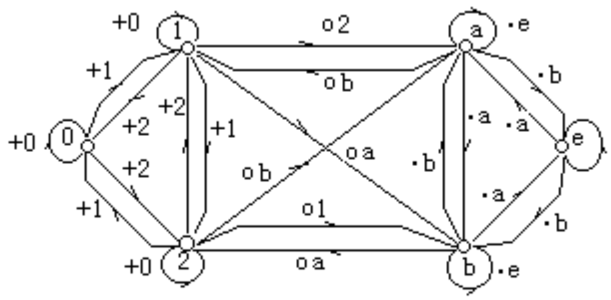


Fig.2.32

$\cdot$	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

table 2.4.1

$\circ$	1	2	a	b
1	$*$	a	b	$*$
2	b	$*$	$*$	a
a	$*$	$*$	$*$	1
b	$*$	$*$	2	$*$

table 2.4.2

Notice that the correspondence between the multi-space $\widetilde{\Gamma}$ and the weighted graph $G[\widetilde{\Gamma}]$ is one-to-one. We immediately get the following result.

Theorem 2.4.2 *The mappings* $\pi : \widetilde{\Gamma} \rightarrow G[\widetilde{\Gamma}]$ *and* $\pi^{-1} : G[\widetilde{\Gamma}] \rightarrow \widetilde{\Gamma}$ *are all one-to-one.*

According to Theorems 2.4.1 and 2.4.2, we get some consequences in the following.

Corollary 2.4.3 *Let $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma_i$ be a multi-space with an algebraic system $(\Gamma_i; \circ_i)$ for any integer $i, 1 \leq i \leq n$. If for any integer $i, 1 \leq i \leq n$, $G[\Gamma_i]$ is a complete multiple 2-graph with a loop attaching at each of its vertices such that each edge between two vertices in $G[\Gamma_i]$ is an opposite multiple 2-edge, then $\widetilde{\Gamma}$ is a complete multi-space.*

Corollary 2.4.4 *Let $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma_i$ be a multi-group with an operation set $O(\widetilde{\Gamma}) = \{\circ_i; 1 \leq i \leq n\}$. Then there is a partition $G[\widetilde{\Gamma}] = \bigcup\limits_{i=1}^{n} G_i$ such that each G_i being a complete multiple 2-graph attaching with a loop at each of its vertices such that each edge between two vertices in $V(G_i)$ is an opposite multiple 2-edge for any integer $i, 1 \leq i \leq n$.*

Corollary 2.4.5 *Let F be a body. Then $G[F]$ is a union of two graphs $K^2(F)$ and $K^2(F^*)$, where $K^2(F)$ or $K^2(F^*)$ is a complete multiple 2-graph with vertex set F or $F^* = F \setminus \{0\}$ and with a loop attaching at each of its vertices such that each edge between two different vertices is an opposite multiple 2-edge.*

2.4.3. **Cayley graphs of a multi-group**

Similar to the definition of Cayley graphs of a finite generated group, we can also define *Cayley graphs of a finite generated multi-group*, where a multi-group $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma_i$ is said to be *finite generated* if the group Γ_i is finite generated for any integer $i, 1 \leq i \leq n$, i.e., $\Gamma_i = \langle x_i, y_i, \cdots, z_{s_i} \rangle$. We denote by $\widetilde{\Gamma} = \langle x_i, y_i, \cdots, z_{s_i}; 1 \leq i \leq n \rangle$ if $\widetilde{\Gamma}$ is finite generated by $\{x_i, y_i, \cdots, z_{s_i}; 1 \leq i \leq n\}$.

Definition 2.4.4 *Let $\widetilde{\Gamma} = \langle x_i, y_i, \cdots, z_{s_i}; 1 \leq i \leq n \rangle$ be a finite generated multi-group, $\widetilde{S} = \bigcup\limits_{i=1}^{n} S_i$, where $1_{\Gamma_i} \notin S_i$, $\widetilde{S}^{-1} = \{a^{-1} | a \in \widetilde{S}\} = \widetilde{S}$ and $\langle S_i \rangle = \Gamma_i$ for any integer $i, 1 \leq i \leq n$. A Cayley graph $Cay(\widetilde{\Gamma} : \widetilde{S})$ is defined by*

$$V(Cay(\widetilde{\Gamma} : \widetilde{S})) = \widetilde{\Gamma}$$

and

$$E(Cay(\widetilde{\Gamma} : \widetilde{S})) = \{(g, h) | \ if \ there \ exists \ an \ integer \ i, g^{-1} \circ_i h \in S_i, 1 \leq i \leq n\}.$$

By Definition 2.4.4, we immediately get the following result for Cayley graphs of a finite generated multi-group.

Theorem 2.4.3 *For a Cayley graph $Cay(\widetilde{\Gamma} : \widetilde{S})$ with $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma_i$ and $\widetilde{S} = \bigcup\limits_{i=1}^{n} S_i$,*

$$Cay(\widetilde{\Gamma} : \widetilde{S}) = \bigcup_{i=1}^{n} Cay(\Gamma_i : S_i).$$

It is well-known that *every Cayley graph of order≥ 3 is 2-connected*. But in general, a Cayley graph of a multi-group is not connected. For the connectedness of Cayley graphs of multi-groups, we get the following result.

Theorem 2.4.4 *A Cayley graph $Cay(\widetilde{\Gamma} : \widetilde{S})$ with $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma_i$ and $\widetilde{S} = \bigcup\limits_{i=1}^{n} S_i$ is connected if and only if for any integer $i, 1 \leq i \leq n$, there exists an integer $j, 1 \leq j \leq n$ and $j \neq i$ such that $\Gamma_i \bigcap \Gamma_j \neq \emptyset$.*

Proof According to Theorem 2.4.3, if there is an integer $i, 1 \leq i \leq n$ such that $\Gamma_i \bigcap \Gamma_j = \emptyset$ for any integer $j, 1 \leq j \leq n$, $j \neq i$, then there are no edges with the form (g_i, h), $g_i \in \Gamma_i$, $h \in \widetilde{\Gamma} \setminus \Gamma_i$. Thereby $Cay(\widetilde{\Gamma} : \widetilde{S})$ is not connected.

Notice that $Cay(\widetilde{\Gamma} : \widetilde{S}) = \bigcup\limits_{i=1}^{n} Cay(\Gamma_i : S_i)$. Not loss of generality, we assume that $g \in \Gamma_k$ and $h \in \Gamma_l$, where $1 \leq k, l \leq n$ for any two elements $g, h \in \widetilde{\Gamma}$. If $k = l$, then there must exists a path connecting g and h in $Cay(\widetilde{\Gamma} : \widetilde{S})$.

Now if $k \neq l$ and for any integer $i, 1 \leq i \leq n$, there is an integer $j, 1 \leq j \leq n$ and $j \neq i$ such that $\Gamma_i \bigcap \Gamma_j \neq \emptyset$, then we can find integers $i_1, i_2, \cdots, i_s$, $1 \leq i_1, i_2, \cdots, i_s \leq n$ such that

$$\Gamma_k \bigcap \Gamma_{i_1} \neq \emptyset,$$

$$\Gamma_{i_1} \bigcap \Gamma_{i_2} \neq \emptyset,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$\Gamma_{i_s} \bigcap \Gamma_l \neq \emptyset.$$

Thereby we can find a path connecting g and h in $Cay(\widetilde{\Gamma} : \widetilde{S})$ passing through these vertices in $Cay(\Gamma_{i_1} : S_{i_1})$, $Cay(\Gamma_{i_2} : S_{i_2})$, $\cdots$, and $Cay(\Gamma_{i_s} : S_{i_s})$. Therefore, $Cay(\widetilde{\Gamma} : \widetilde{S})$ is connected. ♮

The following theorem is gotten by the definition of a Cayley graph and Theorem 2.4.4.

Theorem 2.4.5 *If* $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma$ *with* $|\Gamma| \geq 3$, *then a Cayley graph* $Cay(\widetilde{\Gamma} : \widetilde{S})$

(*i*) *is an* $|\widetilde{S}|$*-regular graph;*

(*ii*) *the edge connectivity* $\kappa(Cay(\widetilde{\Gamma} : \widetilde{S})) \geq 2n$.

Proof The assertion (i) is gotten by the definition of $Cay(\widetilde{\Gamma} : \widetilde{S})$. For (ii) since every Cayley graph of order≥ 3 is 2-connected, for any two vertices g, h in $Cay(\widetilde{\Gamma} : \widetilde{S})$, there are at least $2n$ edge disjoint paths connecting g and h. Whence, the edge connectivity $\kappa(Cay(\widetilde{\Gamma} : \widetilde{S})) \geq 2n$. ♮

Applying multi-voltage graphs, we get a structure result for Cayley graphs of a finite multi-group similar to that of Cayley graphs of a finite group.

Theorem 2.4.6 *For a Cayley graph* $Cay(\widetilde{\Gamma} : \widetilde{S})$ *of a finite multi-group* $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma_i$ *with* $\widetilde{S} = \bigcup\limits_{i=1}^{n} S_i$, *there is a multi-voltage bouquet* $\varsigma : B_{|\widetilde{S}|} \rightarrow \widetilde{S}$ *such that* $Cay(\widetilde{\Gamma} : \widetilde{S}) \cong (B_{|\widetilde{S}|})^{\varsigma}$.

Proof Let $\widetilde{S} = \{s_i; 1 \leq i \leq |\widetilde{S}|\}$ and $E(B_{|\widetilde{S}|}) = \{L_i; 1 \leq i \leq |\widetilde{S}|\}$. Define a multi-voltage graph on a bouquet $B_{|\widetilde{S}|}$ by

$$\varsigma : L_i \rightarrow s_i, \quad 1 \leq i \leq |\widetilde{S}|.$$

Then we know that there is an isomorphism τ between $(B_{|\widetilde{S}|})^{\varsigma}$ and $Cay(\widetilde{\Gamma} : \widetilde{S})$ by defining $\tau(O_g) = g$ for $\forall g \in \widetilde{\Gamma}$, where $V(B_{|\widetilde{S}|}) = \{O\}$. ♮

Corollary 2.4.6 *For a Cayley graph* $Cay(\Gamma : S)$ *of a finite group* Γ, *there exists a voltage bouquet* $\alpha : B_{|S|} \rightarrow S$ *such that* $Cay(\Gamma : S) \cong (B_{|S|})^{\alpha}$.

§2.5 Graph Phase Spaces

The behavior of a graph in an m-manifold is related with theoretical physics since it can be viewed as a model of p-branes in M-theory both for a microcosmic and macrocosmic world. For more details one can see in Chapter 6. This section concentrates on surveying some useful fundamental elements for graphs in n-manifolds.

2.5.1. **Graph phase in a multi-space**

For convenience, we introduce some notations used in this section in the following.

$\widetilde{\mathbf{M}}$ – a multi-manifold $\widetilde{\mathbf{M}} = \bigcup\limits_{i=1}^{n} \mathbf{M}^{n_i}$, where $\mathbf{M}^{n_i}$ is an n_i-manifold, $n_i \geq 2$. For multi-manifolds, see also those materials in Subsection 1.5.4.

$\overline{u} \in \widetilde{\mathbf{M}}$ – a point $\overline{u}$ of $\widetilde{\mathbf{M}}$.

$\mathcal{G}$ – a graph G embedded in $\widetilde{\mathbf{M}}$.

$C(\widetilde{\mathbf{M}})$ – the set of smooth mappings $\omega : \widetilde{\mathbf{M}} \to \widetilde{\mathbf{M}}$, differentiable at each point $\overline{u}$ in $\widetilde{\mathbf{M}}$.

Now we define the phase of a graph in a multi-space.

Definition 2.5.1 *Let $\mathcal{G}$ be a graph embedded in a multi-manifold $\widetilde{\mathbf{M}}$. A phase of $\mathcal{G}$ in $\widetilde{\mathbf{M}}$ is a triple $(\mathcal{G};\omega,\Lambda)$ with an operation $\circ$ on $C(\widetilde{M})$, where $\omega : V(G) \to C(\widetilde{\mathbf{M}})$ and $\Lambda : E(\mathcal{G}) \to C(\widetilde{\mathbf{M}})$ such that $\Lambda(\overline{u},\overline{v}) = \frac{\omega(\overline{u})\circ\omega(\overline{v})}{\|\overline{u}-\overline{v}\|}$ for $\forall(\overline{u},\overline{v}) \in E(\mathcal{G})$, where $\| \overline{u} \|$ denotes the norm of $\overline{u}$.*

For examples, the complete graph K_4 embedded in $\mathbf{R}^3$ has a phase as shown in Fig.2.33, where $g \in C(\mathbf{R}^3)$ and $h \in C(\mathbf{R}^3)$.

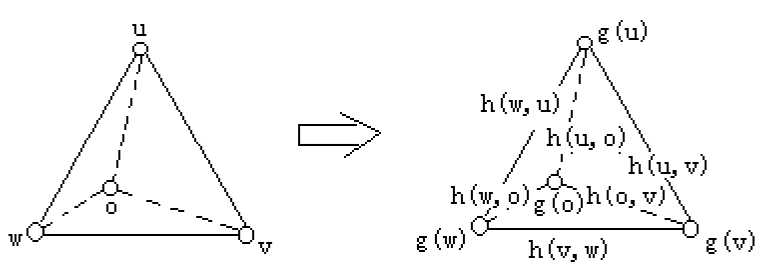


Fig.2.33

Similar to the definition of a adjacent matrix on a graph, we can also define matrixes on graph phases .

Definition 2.5.2 *Let $(\mathcal{G};\omega,\Lambda)$ be a phase and $A[G] = [a_{ij}]_{p\times p}$ the adjacent matrix of a graph G with $V(G) = \{v_1, v_2, \cdots, v_p\}$. Define matrixes $V[\mathcal{G}] = [V_{ij}]_{p\times p}$ and*

$\Lambda[\mathcal{G}] = [\Lambda_{ij}]_{p\times p}$ *by*

$$V_{ij} = \frac{\omega(\overline{v}_i)}{\parallel \overline{v}_i - \overline{v}_j \parallel} \ if \ a_{ij} \neq 0; \ otherwise, V_{ij} = 0$$

and

$$\Lambda_{ij} = \frac{\omega(\overline{v}_i) \circ \omega(\overline{v}_j)}{\parallel \overline{v}_i - \overline{v}_j \parallel^2} \ if \ a_{ij} \neq 0; \ otherwise, \Lambda_{ij} = 0,$$

where "$\circ$" *is an operation on* $C(\widetilde{M})$.

For example, for the phase of K_4 in Fig.2.33, if choice $g(u) = (x_1, x_2, x_3)$, $g(v) = (y_1, y_2, y_3)$, $g(w) = (z_1, z_2, z_3)$, $g(o) = (t_1, t_2, t_3)$ and $\circ = \times$, the multiplication of vectors in $\mathbf{R}^3$, then we get that

$$V(\mathcal{G}) = \begin{bmatrix} 0 & \frac{g(u)}{\rho(u,v)} & \frac{g(u)}{\rho(u,w)} & \frac{g(u)}{\rho(u,o)} \\ \frac{g(v)}{\rho(v,u)} & 0 & \frac{g(v)}{\rho(v,w)} & \frac{g(v)}{\rho(v,t)} \\ \frac{g(w)}{\rho(w,u)} & \frac{g(w)}{\rho(w,v)} & 0 & \frac{g(w)}{\rho(w,o)} \\ \frac{g(o)}{\rho(o,u)} & \frac{g(o)}{\rho(o,v)} & \frac{g(o)}{\rho(o,w)} & 0 \end{bmatrix}$$

where

$$\rho(u,v) = \rho(v,u) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2},$$

$$\rho(u,w) = \rho(w,u) = \sqrt{(x_1 - z_1)^2 + (x_2 - z_2)^2 + (x_3 - z_3)^2},$$

$$\rho(u,o) = \rho(o,u) = \sqrt{(x_1 - t_1)^2 + (x_2 - t_2)^2 + (x_3 - t_3)^2},$$

$$\rho(v,w) = \rho(w,v) = \sqrt{(y_1 - z_1)^2 + (y_2 - z_2)^2 + (y_3 - z_3)^2},$$

$$\rho(v,o) = \rho(o,v) = \sqrt{(y_1 - t_1)^2 + (y_2 - t_2)^2 + (y_3 - t_3)^2},$$

$$\rho(w,o) = \rho(o,w) = \sqrt{(z_1 - t_1)^2 + (z_2 - t_2)^2 + (z_3 - t_3)^2}$$

and

$$\Lambda(\mathcal{G}) = \begin{bmatrix} 0 & \frac{g(u)\times g(v)}{\rho^2(u,v)} & \frac{g(u)\times g(w)}{\rho^2(u,w)} & \frac{g(u)\times g(o)}{\rho^2(u,o)} \\ \frac{g(v)\times g(u)}{\rho^2(v,u)} & 0 & \frac{g(v)\times g(w)}{\rho^2(v,w)} & \frac{g(v\times g(o)}{\rho^2(v,o)} \\ \frac{g(w)\times g(u)}{\rho^2(w,u)} & \frac{g(w)\times g(v)}{\rho^2(w,v)} & 0 & \frac{g(w)\times g(o)}{\rho^2(w,o)} \\ \frac{g(o)\times g(u)}{\rho^2(o,u)} & \frac{g(o)\times g(v)}{\rho^2(o,v)} & \frac{g(o)\times g(w)}{\rho^2(o,w)} & 0 \end{bmatrix}.$$

where

$$g(u)\times g(v) = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1),$$

$$g(u)\times g(w) = (x_2z_3 - x_3z_2, x_3z_1 - x_1z_3, x_1z_2 - x_2z_1),$$

$$g(u)\times g(o) = (x_2t_3 - x_3t_2, x_3t_1 - x_1t_3, x_1t_2 - x_2t_1),$$

$$g(v)\times g(u) = (y_2x_3 - y_3x_2, y_3x_1 - y_1x_3, y_1x_2 - y_2x_1),$$

$$g(v)\times g(w) = (y_2z_3 - y_3z_2, y_3z_1 - y_1z_3, y_1z_2 - y_2z_1),$$

$$g(v)\times g(o) = (y_2t_3 - y_3t_2, y_3t_1 - y_1t_3, y_1t_2 - y_2t_1),$$

$$g(w)\times g(u) = (z_2x_3 - z_3x_2, z_3x_1 - z_1x_3, z_1x_2 - z_2x_1),$$

$$g(w)\times g(v) = (z_2y_3 - z_3y_2, z_3y_1 - z_1y_3, z_1y_2 - z_2y_1),$$

$$g(w)\times g(o) = (z_2t_3 - z_3t_2, z_3t_1 - z_1t_3, z_1t_2 - z_2t_1),$$

$$g(o)\times g(u) = (t_2x_3 - t_3x_2, t_3x_1 - t_1x_3, t_1x_2 - t_2x_1),$$

$$g(o)\times g(v) = (t_2y_3 - t_3y_2, t_3y_1 - t_1y_3, t_1y_2 - t_2y_1),$$

$$g(o) \times g(w) = (t_2z_3 - t_3z_2, t_3z_1 - t_1z_3, t_1z_2 - t_2z_1).$$

For two given matrixes $A = [a_{ij}]_{p\times p}$ and $B = [b_{ij}]_{p\times p}$, the *star product* "$*$" on an operation "$\circ$" is defined by $A * B = [a_{ij} \circ b_{ij}]_{p\times p}$. We get the following result for matrixes $V[\mathcal{G}]$ and $\Lambda[\mathcal{G}]$.

Theorem 2.5.1 $V[\mathcal{G}] * V^t[\mathcal{G}] = \Lambda[\mathcal{G}]$.

Proof Calculation shows that each (i,j) entry in $V[\mathcal{G}] * V^t[\mathcal{G}]$ is

$$\frac{\omega(\overline{v}_i)}{\| \overline{v}_i - \overline{v}_j \|} \circ \frac{\omega(\overline{v}_j)}{\| \overline{v}_j - \overline{v}_i \|} = \frac{\omega(\overline{v}_i) \circ \omega(\overline{v}_j)}{\| \overline{v}_i - \overline{v}_j \|^2} = \Lambda_{ij},$$

where $1 \leq i,j \leq p$. Therefore, we get that

$$V[\mathcal{G}] * V^t[\mathcal{G}] = \Lambda[\mathcal{G}]. \qquad \natural$$

An operation called *addition on graph phases* is defined in the next.

Definition 2.5.3 *For two phase spaces $(\mathcal{G}_1;\omega_1,\Lambda_1)$, $(\mathcal{G}_2;\omega_2,\Lambda_2)$ of graphs G_1, G_2 in $\widetilde{M}$ and two operations "$\bullet$" and "$\circ$" on $C(\widetilde{M})$, their addition is defined by*

$$(\mathcal{G}_1;\omega_1,\Lambda_1)\bigoplus(\mathcal{G}_2;\omega_2,\Lambda_2) = (\mathcal{G}_1\bigoplus\mathcal{G}_2;\omega_1\bullet\omega_2,\Lambda_1\bullet\Lambda_2),$$

where $\omega_1\bullet\omega_2 : V(\mathcal{G}_1\bigcup\mathcal{G}_2) \to C(\widetilde{M})$ satisfying

$$\omega_1\bullet\omega_2(\overline{u}) = \begin{cases} \omega_1(\overline{u})\bullet\omega_2(\overline{u}), & if\ \overline{u} \in V(\mathcal{G}_1)\bigcap V(\mathcal{G}_2),\\ \omega_1(\overline{u}), & if\ \overline{u} \in V(\mathcal{G}_1)\setminus V(\mathcal{G}_2),\\ \omega_2(\overline{u}), & if\ \overline{u} \in V(\mathcal{G}_2)\setminus V(\mathcal{G}_1).\end{cases}$$

and

$$\Lambda_1\bullet\Lambda_2(\overline{u},\overline{v}) = \frac{\omega_1\bullet\omega_2(\overline{u})\circ\omega_1\bullet\omega_2(\overline{v})}{\| \overline{u}-\overline{v} \|^2}$$

for $(\overline{u},\overline{v}) \in E(\mathcal{G}_1)\bigcup E(\mathcal{G}_2)$

The following result is immediately gotten by Definition 2.5.3.

Theorem 2.5.2 *For two given operations "$\bullet$" and "$\circ$" on $C(\widetilde{M})$, all graph phases in $\widetilde{M}$ form a linear space on the field Z_2 with a phase $\bigoplus$ for any graph phases $(\mathcal{G}_1;\omega_1,\Lambda_1)$ and $(\mathcal{G}_2;\omega_2,\Lambda_2)$ in $\widetilde{M}$.*

2.5.2. **Transformation of a graph phase**

Definition 2.5.4 *Let $(\mathcal{G}_1;\omega_1,\Lambda_1)$ and $(\mathcal{G}_2;\omega_2,\Lambda_2)$ be graph phases of graphs G_1 and G_2 in a multi-space $\widetilde{M}$ with operations "$\circ_1,\circ_2$", respectively. If there exists a smooth mapping $\tau\in C(\widetilde{M})$ such that*

$$\tau:(\mathcal{G}_1;\omega_1,\Lambda_1)\rightarrow(\mathcal{G}_2;\omega_2,\Lambda_2),$$

i.e., for $\forall\overline{u}\in V(\mathcal{G}_1)$, $\forall(\overline{u},\overline{v})\in E(\mathcal{G}_1)$, $\tau(\mathcal{G}_1)=\mathcal{G}_2$, $\tau(\omega_1(\overline{u}))=\omega_2(\tau(\overline{u}))$ and $\tau(\Lambda_1(\overline{u},\overline{v}))=\Lambda_2(\tau(\overline{u},\overline{v}))$, then we say $(\mathcal{G}_1;\omega_1,\Lambda_1)$ and $(\mathcal{G}_2;\omega_2,\Lambda_2)$ are transformable and τ a transform mapping.

For examples, a transform mapping t for embeddings of K_4 in $\mathbf{R}^3$ and on the plane is shown in Fig.2.34

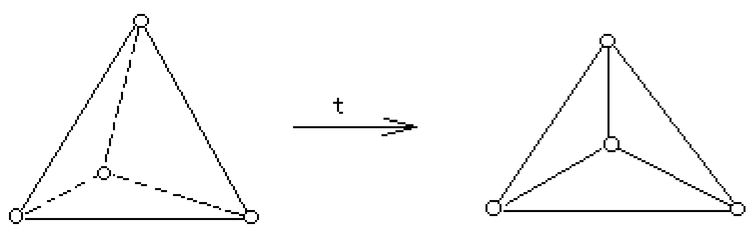


Fig.2.34

Theorem 2.5.3 *Let $(\mathcal{G}_1;\omega_1,\Lambda_1)$ and $(\mathcal{G}_2;\omega_2,\Lambda_2)$ be transformable graph phases with transform mapping τ. If τ is one-to-one on $\mathcal{G}_1$ and $\mathcal{G}_2$, then $\mathcal{G}_1$ is isomorphic to $\mathcal{G}_2$.*

Proof By definitions, if τ is one-to-one on $\mathcal{G}_1$ and $\mathcal{G}_2$, then τ is an isomorphism between $\mathcal{G}_1$ and $\mathcal{G}_2$. $\natural$

A very useful case among transformable graph phases is that one can find parameters $t_1,t_2,\cdots,t_q$, $q\geq 1$, such that each vertex of a graph phase is a smooth mapping of $t_1,t_2,\cdots,t_q$, i.e., for $\forall\overline{u}\in\widetilde{M}$, we consider it as $\overline{u}(t_1,t_2,\cdots,t_q)$. In this case, we introduce two conceptions on graph phases.

Definition 2.5.5 *For a graph phase $(\mathcal{G};\omega,\Lambda)$, define its capacity $Ca(\mathcal{G};\omega,\Lambda)$ and entropy $En(\mathcal{G};\omega,\Lambda)$ by*

$$Ca(\mathcal{G};\omega,\Lambda) = \sum_{\overline{u}\in V(\mathcal{G})} \omega(\overline{u})$$

and

$$En(\mathcal{G};\omega,\Lambda) = \log(\prod_{\overline{u}\in V(\mathcal{G})} \parallel \omega(\overline{u}) \parallel).$$

Then we know the following result.

Theorem 2.5.4 *For a graph phase $(\mathcal{G};\omega,\Lambda)$, its capacity $Ca(\mathcal{G};\omega,\Lambda)$ and entropy $En(\mathcal{G};\omega,\Lambda)$ satisfy the following differential equations*

$$\mathrm{d}Ca(\mathcal{G};\omega,\Lambda) = \frac{\partial Ca(\mathcal{G};\omega,\Lambda)}{\partial u_i}\mathrm{d}u_i \ \ \textit{and} \ \ \mathrm{d}En(\mathcal{G};\omega,\Lambda) = \frac{\partial En(\mathcal{G};\omega,\Lambda)}{\partial u_i}\mathrm{d}u_i,$$

where we use the Einstein summation convention, i.e., a sum is over i if it is appearing both in upper and lower indices.

Proof Not loss of generality, we assume $\overline{u} = (u_1, u_2, \cdots, u_p)$ for $\forall \overline{u} \in \widetilde{M}$. According to the invariance of differential form, we know that

$$\mathrm{d}\omega = \frac{\partial \omega}{\partial u_i}\mathrm{d}u_i.$$

By the definition of the capacity $Ca(\mathcal{G};\omega,\Lambda)$ and entropy $En(\mathcal{G};\omega,\Lambda)$ of a graph phase, we get that

$$\begin{aligned}
\mathrm{d}Ca(\mathcal{G};\omega,\Lambda) &= \sum_{\overline{u}\in V(\mathcal{G})} \mathrm{d}(\omega(\overline{u})) \\
&= \sum_{\overline{u}\in V(\mathcal{G})} \frac{\partial\omega(\overline{u})}{\partial u_i}\mathrm{d}u_i = \frac{\partial(\sum\limits_{\overline{u}\in V(\mathcal{G})} \omega(\overline{u}))}{\partial u_i}\mathrm{d}u_i \\
&= \frac{\partial Ca(\mathcal{G};\omega,\Lambda)}{\partial u_i}\mathrm{d}u_i.
\end{aligned}$$

Similarly, we also obtain that

$$\mathrm{d}En(\mathcal{G};\omega,\Lambda) = \sum_{\overline{u}\in V(\mathcal{G})} \mathrm{d}(\log \parallel \omega(\overline{u}) \parallel)$$

$$= \sum_{\overline{u}\in V(\mathcal{G})} \frac{\partial \log|\omega(\overline{u})|}{\partial u_i} \mathrm{d}u_i = \frac{\partial(\sum\limits_{\overline{u}\in V(\mathcal{G})} \log \parallel \omega(\overline{u}) \parallel)}{\partial u_i} \mathrm{d}u_i$$
$$= \frac{\partial En(\mathcal{G};\omega,\Lambda)}{\partial u_i} \mathrm{d}u_i.$$

This completes the proof. $\natural$

In a 3-dimensional Euclid space we can get more concrete results for graph phases $(\mathcal{G};\omega,\Lambda)$. In this case, we get some formulae in the following by choice $\overline{u} = (x_1, x_2, x_3)$ and $\overline{v} = (y_1, y_2, y_3)$.

$$\omega(\overline{u}) = (x_1, x_2, x_3) \text{ for } \forall \overline{u} \in V(\mathcal{G}),$$

$$\Lambda(\overline{u},\overline{v}) = \frac{x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1}{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2} \text{ for } \forall(\overline{u},\overline{v}) \in E(\mathcal{G}),$$

$$Ca(\mathcal{G};\omega,\Lambda) = (\sum_{\overline{u}\in V(\mathcal{G})} x_1(\overline{u}), \sum_{\overline{u}\in V(\mathcal{G})} x_2(\overline{u}), \sum_{\overline{u}\in V(\mathcal{G})} x_3(\overline{u}))$$

and

$$En(\mathcal{G};\omega,\Lambda) = \sum_{\overline{u}\in V(\mathcal{G})} \log(x_1^2(\overline{u}) + x_2^2(\overline{u}) + x_3^2(\overline{u}).$$

§2.6 Remarks and Open Problems

2.6.1 A graphical property $P(G)$ is called to be *subgraph hereditary* if for any subgraph $H \subseteq G$, H posses $P(G)$ whenever G posses the property $P(G)$. For example, the properties: G *is complete* and *the vertex coloring number* $\chi(G) \leq k$ both are subgraph hereditary. The hereditary property of a graph can be generalized by the following way.

Let G and H be two graphs in a space $\widetilde{M}$. If there is a smooth mapping ς in $C(\widetilde{M})$ such that $\varsigma(G) = H$, then we say G and H are *equivalent in* $\widetilde{M}$. Many conceptions in graph theory can be included in this definition, such as *graph homomorphism, graph equivalent*, $\cdots$, etc.

Problem 2.6.1 *Applying different smooth mappings in a space such as smooth mappings in* $\mathbf{R}^3$ *or* $\mathbf{R}^4$ *to classify graphs and to find their invariants.*

Problem 2.6.2 *Find which parameters already known in graph theory for a graph is invariant or to find the smooth mapping in a space on which this parameter is invariant.*

2.6.2 As an efficient way for finding regular covering spaces of a graph, voltage graphs have been gotten more attentions in the past half-century by mathematicians. Works for regular covering spaces of a graph can seen in [23], [45] – [46] and [71] – [72]. But few works are found in publication for irregular covering spaces of a graph. The multi-voltage graph of type 1 or type 2 with multi-groups defined in Section 2.2 are candidate for further research on irregular covering spaces of graphs.

Problem 2.6.3 *Applying multi-voltage graphs to get the genus of a graph with less symmetries.*

Problem 2.6.4 *Find new actions of a multi-group on a graph, such as the left subaction and its contribution to topological graph theory. What can we say for automorphisms of the lifting of a multi-voltage graph?*

There is a famous conjecture for Cayley graphs of a finite group in algebraic graph theory, i.e., *every connected Cayley graph of order* ≥ 3 *is hamiltonian.* Similarly, we can also present a conjecture for Cayley graphs of a multi-group.

Conjecture 2.6.1 *Every Cayley graph of a finite multi-group* $\widetilde{\Gamma} = \bigcup\limits_{i=1}^{n} \Gamma_i$ *with order*≥ 3 *and* $|\bigcap\limits_{i=1}^{n} \Gamma_i| \geq 2$ *is hamiltonian.*

2.6.3 As pointed out in [56], for applying combinatorics to other sciences, a good idea is pullback measures on combinatorial objects, initially ignored by the classical combinatorics and reconstructed or make a combinatorial generalization for the classical mathematics, such as, the algebra, the differential geometry, the Riemann geometry, $\cdots$ and the mechanics, the theoretical physics, $\cdots$. For this object, a more natural way is to put a graph in a metric space and find its good behaviors. The problem discussed in Sections 2.3 is just an elementary step for this target. More works should be done and more techniques should be designed. The following open problems are valuable to research for a researcher on combinatorics.

Problem 2.6.5 *Find which parameters for a graph can be used to a graph in a* *space. Determine combinatorial properties of a graph in a space.*

Consider a graph in an Euclid space of dimension 3. All of its edges are seen as a structural member, such as steel bars or rods and its vertices are hinged points. Then we raise the following problem.

Problem 2.6.6 *Applying structural mechanics to classify what kind of graph structures are stable or unstable. Whether can we discover structural mechanics of dimension≥ 4 by this idea?*

We have known the orbit of a point under an action of a group, for example, a torus is an orbit of $Z \times Z$ action on a point in $\mathbf{R}^3$. Similarly, we can also define an *orbit of a graph in a space* under an action on this space.

Let $\mathcal{G}$ be a graph in a multi-space $\widetilde{M}$ and Π a family of actions on $\widetilde{M}$. Define an orbit $Or(\mathcal{G})$ by

$$Or(\mathcal{G}) = \{\pi(\mathcal{G}) | \ \forall \pi \in \Pi\}.$$

Problem 2.6.7 *Given an action π, continuous or discontinuous on a space $\widetilde{M}$, for example $\mathbf{R}^3$ and a graph $\mathcal{G}$ in $\widetilde{M}$, find the orbit of $\mathcal{G}$ under the action of π. When can we get a closed geometrical object by this action?*

Problem 2.6.8 *Given a family $\mathcal{A}$ of actions, continuous or discontinuous on a space $\widetilde{M}$ and a graph $\mathcal{G}$ in $\widetilde{M}$, find the orbit of $\mathcal{G}$ under these actions in $\mathcal{A}$. Find the orbit of a vertex or an edge of $\mathcal{G}$ under the action of $\mathcal{G}$, and when are they closed?*

2.6.4 The central idea in Section 2.4 is that a graph is equivalent to Smarandache multi-spaces. This fact enables us to investigate Smarandache multi-spaces possible by a combinatorial approach. Applying infinite graph theory (see [94] for details), we can also define an infinite graph for an infinite Smarandache multi-space similar to Definition 2.4.3.

Problem 2.6.9 *Find its structural properties of an infinite graph of an infinite Smarandache multi-space.*

2.6.5 There is an alternative way for defining transformable graph phases, i.e., by homotopy groups in a topological space, which is stated as follows.

Let $(\mathcal{G}_1;\omega_1,\Lambda_1)$ and $(\mathcal{G}_2;\omega_2,\Lambda_2)$ be two graph phases. If there is a continuous mapping $H : C(\widetilde{M})\times I \to C(\widetilde{M})\times I$, $I=[0,1]$ such that $H(C(\widetilde{M}),0)=(\mathcal{G}_1;\omega_1,\Lambda_1)$ and $H(C(\widetilde{M}),1)=(\mathcal{G}_2;\omega_2,\Lambda_2)$, then $(\mathcal{G}_1;\omega_1,\Lambda_1)$ and $(\mathcal{G}_2;\omega_2,\Lambda_2)$ are said two *transformable graph phases*.

Similar to topology, we can also introduce product on homotopy equivalence classes and prove that all homotopy equivalence classes form a group. This group is called a *fundamental group* and denote it by $\pi(\mathcal{G};\omega,\Lambda)$. In topology there is a famous theorem, called the *Seifert and Van Kampen theorem* for characterizing fundamental groups $\pi_1(\mathcal{A})$ of topological spaces $\mathcal{A}$ restated as follows (see [92] for details).

Suppose $\mathcal{E}$ is a space which can be expressed as the union of path-connected open sets $\mathcal{A}$, $\mathcal{B}$ such that $\mathcal{A}\bigcap\mathcal{B}$ is path-connected and $\pi_1(\mathcal{A})$ and $\pi_1(\mathcal{B})$ have respective presentations

$$\langle a_1,\cdots,a_m;r_1,\cdots,r_n\rangle\,,$$

$$\langle b_1,\cdots,b_m;s_1,\cdots,s_n\rangle$$

while $\pi_1(\mathcal{A}\bigcap\mathcal{B})$ is finitely generated. Then $\pi_1(\mathcal{E})$ has a presentation

$$\langle a_1,\cdots,a_m,b_1,\cdots,b_m;r_1,\cdots,r_n,s_1,\cdots,s_n,u_1=v_1,\cdots,u_t=v_t\rangle\,,$$

where $u_i,v_i,i=1,\cdots,t$ are expressions for the generators of $\pi_1(\mathcal{A}\bigcap\mathcal{B})$ in terms of the generators of $\pi_1(\mathcal{A})$ and $\pi_1(\mathcal{B})$ respectively.

Then there is a problem for the fundamental group $\pi(\mathcal{G};\omega,\Lambda)$ of a graph phase $(\mathcal{G};\omega,\Lambda)$.

Problem 2.6.10 *Find a result similar to the Seifert and Van Kampen theorem for the fundamental group of a graph phase.*

Chapter 3 Map Geometries

As a kind of multi-metric spaces, Smarandache geometries were introduced by Smarandache in [86] and investigated by many mathematicians. These geometries are related with the *Euclid geometry*, the *Lobachevshy-Bolyai-Gauss geometry* and the *Riemann geometry*, also related with *relativity theory* and *parallel universes* (see [56], [35] – [36], [38] and [77] – [78] for details). As a generalization of Smarandache manifolds of dimension 2, *Map geometries* were introduced in [55], [57] and [62], which can be also seen as a realization of Smarandache geometries on surfaces or Smarandache geometries on maps.

§3.1 Smarandache Geometries

3.1.1. What are lost in classical mathematics?

As we known, mathematics is a powerful tool of sciences for its unity and neatness, without any shade of mankind. On the other hand, it is also a kind of aesthetics deep down in one's mind. There is a famous proverb says that *only the beautiful things can be handed down to today*, which is also true for the mathematics.

Here, the terms *unity* and *neatness* are relative and local, maybe also have various conditions. For obtaining a good result, many unimportant matters are abandoned in the research process. Whether are those matters still unimportant in another time? It is not true. That is why we need to think a queer question: *what are lost in the classical mathematics?*

For example, a compact surface is topological equivalent to a polygon with even number of edges by identifying each pairs of edges along its a given direction

([68], [92]). If label each pair of edges by a letter $e, e \in \mathcal{E}$, a surface S is also identified to a cyclic permutation such that each edge $e, e \in \mathcal{E}$ just appears two times in S, one is e and another is e^{-1} (orientable) or e (non-orientable). Let $a, b, c, \cdots$ denote letters in $\mathcal{E}$ and $A, B, C, \cdots$ the sections of successive letters in a linear order on a surface S (or a string of letters on S). Then, an orientable surface can be represented by

$$S = (\cdots, A, a, B, a^{-1}, C, \cdots),$$

where, $a \in \mathcal{E}$ and A, B, C denote strings of letter. Three elementary transformations are defined as follows:

(O_1) $\quad (A, a, a^{-1}, B) \Leftrightarrow (A, B);$

(O_2) $\quad (i)\quad (A, a, b, B, b^{-1}, a^{-1}) \Leftrightarrow (A, c, B, c^{-1});$

$(ii)\quad (A, a, b, B, a, b) \Leftrightarrow (A, c, B, c);$

(O_3) $\quad (i)\quad (A, a, B, C, a^{-1}, D) \Leftrightarrow (B, a, A, D, a^{-1}, C);$

$(ii)\quad (A, a, B, C, a, D) \Leftrightarrow (B, a, A, C^{-1}, a, D^{-1}).$

If a surface S_0 can be obtained by these elementary transformations O_1-O_3 from a surface S, it is said that S is *elementary equivalent* with S_0, denoted by $S \sim_{El} S_0$.

We have known the following formulae from [43]:

$(i)\quad (A, a, B, b, C, a^{-1}, D, b^{-1}, E) \sim_{El} (A, D, C, B, E, a, b, a^{-1}, b^{-1});$

$(ii)\quad (A, c, B, c) \sim_{El} (A, B^{-1}, C, c, c);$

$(iii)\quad (A, c, c, a, b, a^{-1}, b^{-1}) \sim_{El} (A, c, c, a, a, b, b).$

Then we can get the classification theorem of compact surfaces as follows([68]):

Any compact surface is homeomorphic to one of the following standard surfaces:

(P_0) The sphere: aa^{-1};

(P_n) The connected sum of $n, n \geq 1$, tori:

$$a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}\cdots a_nb_na_n^{-1}b_n^{-1};$$

(Q_n) The connected sum of $n, n \geq 1$, projective planes:

$$a_1a_1a_2a_2\cdots a_na_n.$$

As we have discussed in Chapter 2, a combinatorial map is just a kind of decomposition of a surface. Notice that all the standard surfaces are one face map underlying an one vertex graph, i.e., a bouquet B_n with $n \geq 1$. By a combinatorial view, *a combinatorial map is nothing but a surface*. This assertion is needed clarifying. For example, let us see the left graph Π_4 in Fig. 3.1, which is a tetrahedron.

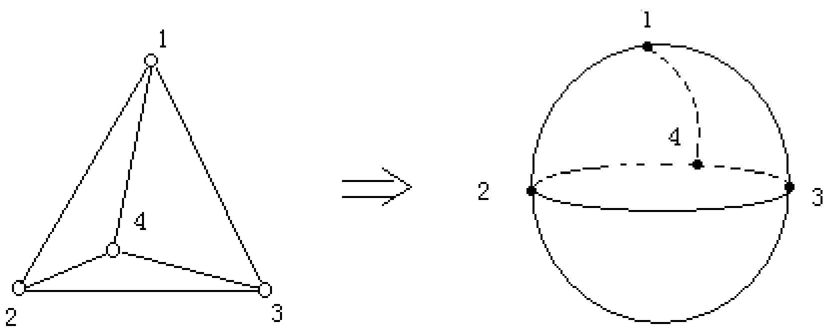


Fig.3.1

Whether can we say $\prod_4$ is a sphere? Certainly NOT. Since any point u on a sphere has a neighborhood $N(u)$ homeomorphic to an open disc, thereby all angles incident with the point 1 must be $120°$ degree on a sphere. But in Π_4, those are only $60°$ degree. For making them same in a topological sense, i.e., homeomorphism, we must blow up the Π_4 and make it become a sphere. This physical processing is shown in the Fig.3.1. Whence, for getting the classification theorem of compact surfaces, we lose the *angle,area, volume,distance,curvature,*$\cdots$, etc, which are also lost in combinatorial maps.

By a geometrical view, *Klein Erlanger Program* says that *any geometry is nothing but find invariants under a transformation group of this geometry.* This is essentially the group action idea and widely used in mathematics today. Surveying topics appearing in publications for combinatorial maps, we know the following problems are applications of *Klein Erlanger Program*:

(i) *to determine isomorphism maps or rooted maps;*

(ii) *to determine equivalent embeddings of a graph;*

(iii) *to determine an embedding whether exists or not;*

(iv) *to enumerate maps or rooted maps on a surface;*

(v) *to enumerate embeddings of a graph on a surface;*

(vi) $\cdots$, etc.

All the problems are extensively investigated by researches in the last century and papers related those problems are still frequently appearing in journals today. Then,

what are their importance to classical mathematics?

and

what are their contributions to sciences?

Today, we have found that combinatorial maps can contribute an underlying frame for applying mathematics to sciences, i.e., through by map geometries or by graphs in spaces.

3.1.2. **Smarandache geometries**

Smarandache geometries were proposed by Smarandache in [86] which are generalization of classical geometries, i.e., these *Euclid, Lobachevshy-Bolyai-Gauss* and *Riemann geometries* may be united altogether in a same space, by some Smarandache geometries. These last geometries can be either partially Euclidean and partially Non-Euclidean, or Non-Euclidean. Smarandache geometries are also connected with the *Relativity Theory* because they include Riemann geometry in a subspace and with the *Parallel Universes* because they combine separate spaces into one space too. For a detail illustration, we need to consider classical geometries first.

As we known, the axiom system of an *Euclid geometry* is in the following:

(A1) *there is a straight line between any two points.*

(A2) *a finite straight line can produce a infinite straight line continuously.*

(A3) *any point and a distance can describe a circle.*

(A4) *all right angles are equal to one another.*

(A5) *if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, then the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.*

The axiom (A5) can be also replaced by:

(A5') *given a line and a point exterior this line, there is one line parallel to this line.*

The *Lobachevshy-Bolyai-Gauss geometry*, also called *hyperbolic geometry*, is a geometry with axioms $(A1) - (A4)$ and the following axiom $(L5)$:

(L5) *there are infinitely many lines parallel to a given line passing through an exterior point.*

The *Riemann geometry*, also called *elliptic geometry*, is a geometry with axioms $(A1) - (A4)$ and the following axiom $(R5)$:

there is no parallel to a given line passing through an exterior point.

By a thought of anti-mathematics: *not in a nihilistic way, but in a positive one, i.e., banish the old concepts by some new ones: their opposites*, Smarandache introduced thse *paradoxist geometry, non-geometry, counter-projective geometry* and *anti-geometry* in [86] by contradicts axioms $(A1) - (A5)$ in an Euclid geometry.

Paradoxist geometry

In this geometry, its axioms consist of $(A1) - (A4)$ and one of the following as the axiom $(P5)$:

(i) *there are at least a straight line and a point exterior to it in this space for which any line that passes through the point intersect the initial line.*

(ii) *there are at least a straight line and a point exterior to it in this space for which only one line passes through the point and does not intersect the initial line.*

(iii) *there are at least a straight line and a point exterior to it in this space for which only a finite number of lines* $l_1, l_2, \cdots, l_k, k \geq 2$ *pass through the point and do not intersect the initial line.*

(iv) *there are at least a straight line and a point exterior to it in this space for which an infinite number of lines pass through the point (but not all of them) and do not intersect the initial line.*

(v) *there are at least a straight line and a point exterior to it in this space for which any line that passes through the point and does not intersect the initial line.*

Non-Geometry

The non-geometry is a geometry by denial some axioms of $(A1) - (A5)$, such as:

$(A1^-)$ *It is not always possible to draw a line from an arbitrary point to another arbitrary point.*

$(A2^-)$ *It is not always possible to extend by continuity a finite line to an infinite line.*

$(A3^-)$ *It is not always possible to draw a circle from an arbitrary point and of an arbitrary interval.*

$(A4^-)$ *not all the right angles are congruent.*

$(A5^-)$ *if a line, cutting two other lines, forms the interior angles of the same side of it strictly less than two right angle, then not always the two lines extended towards infinite cut each other in the side where the angles are strictly less than two right angle.*

Counter-Projective geometry

Denoted by P the point set, L the line set and R a relation included in $P \times L$. A counter-projective geometry is a geometry with these counter-axioms $(C_1) - (C_3)$:

$(C1)$ *there exist: either at least two lines, or no line, that contains two given distinct points.*

$(C2)$ *let p_1, p_2, p_3 be three non-collinear points, and q_1, q_2 two distinct points. Suppose that $\{p_1.q_1, p_3\}$ and $\{p_2, q_2, p_3\}$ are collinear triples. Then the line containing p_1, p_2 and the line containing q_1, q_2 do not intersect.*

$(C3)$ *every line contains at most two distinct points.*

Anti-Geometry

A geometry by denial some axioms of the Hilbert's 21 axioms of Euclidean geometry. As shown in [38], there are at least $2^{21} - 1$ anti-geometries.

In general, Smarandache geometries are defined as follows.

Definition 3.1.1 *An axiom is said to be Smarandachely denied if the axiom behaves in at least two different ways within the same space, i.e., validated and invalided, or only invalided but in multiple distinct ways.*

*A Smarandache geometry is a geometry which has at least one Smarandachely denied axiom(*1969*).*

In a Smarandache geometries, points, lines, planes, spaces, triangles, $\cdots$, etc are called s-points, s-lines, s-planes, s-spaces, s-triangles, $\cdots$, respectively in order to distinguish them from classical geometries. An example of Smarandache geometries in the classical geometrical sense is in the following.

Example 3.1.1 Let us consider an Euclidean plane $\mathbf{R}^2$ and three non-collinear points A, B and C. Define s-points as all usual Euclidean points on $\mathbf{R}^2$ and s-lines any Euclidean line that passes through one and only one of points A, B and C. Then this geometry is a Smarandache geometry because two axioms are Smarandachely denied comparing with an Euclid geometry:

(i) The axiom (A5) that through a point exterior to a given line there is only one parallel passing through it is now replaced by two statements: *one parallel*, and *no parallel*. Let L be an s-line passes through C and is parallel in the euclidean sense to AB. Notice that through any s-point not lying on AB there is one s-line parallel to L and through any other s-point lying on AB there is no s-lines parallel to L such as those shown in Fig.3.2(a).

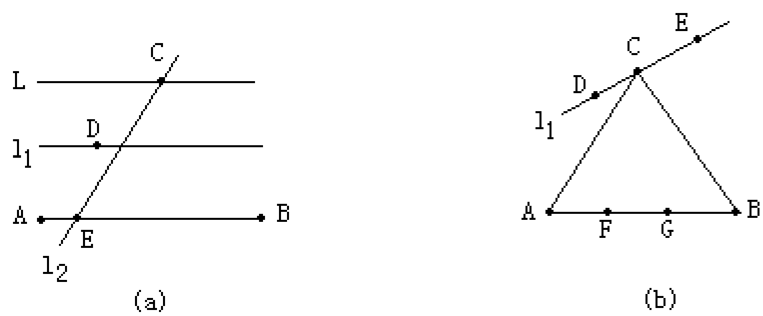


Fig.3.2

(ii) The axiom that through any two distinct points there exist one line passing through them is now replaced by; *one s-line*, and *no s-line*. Notice that through any two distinct s-points D, E collinear with one of A, B and C, there is one s-line passing through them and through any two distinct s-points F, G lying on AB or non-collinear with one of A, B and C, there is no s-line passing through them such as those shown in Fig.3.2(b).

3.1.3. **Smarandache manifolds**

Generally, a *Smarandache manifold* is an n-dimensional manifold that support a

Smarandache geometry. For $n = 2$, a nice model for Smarandache geometries called *s-manifolds* was found by Iseri in [35][36], which is defined as follows:

An s-manifold is any collection $\mathcal{C}(T, n)$ of these equilateral triangular disks $T_i, 1 \leq i \leq n$ satisfying the following conditions:

(i) each edge e is the identification of at most two edges e_i, e_j in two distinct triangular disks $T_i, T_j, 1 \leq i, j \leq n$ and $i \neq j$;

(ii) each vertex v is the identification of one vertex in each of five, six or seven distinct triangular disks.

The vertices are classified by the number of the disks around them. A vertex around five, six or seven triangular disks is called an *elliptic vertex*, an *euclidean vertex* or a *hyperbolic vertex*, respectively.

In a plane, an elliptic vertex O, an euclidean vertex P and a hyperbolic vertex Q and an s-line L_1, L_2 or L_3 passes through points O, P or Q are shown in Fig.3.3$(a), (b), (c)$, respectively.

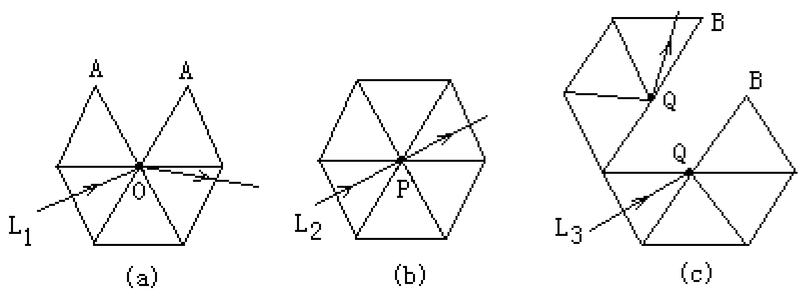


Fig.3.3

Smarandache paradoxist geometries and non-geometries can be realized by s-manifolds, but other Smarandache geometries can be only partly realized by this kind of manifolds. Readers are inferred to Iseri's book [35] for those geometries.

An s-manifold is called closed if each edge is shared exactly by two triangular disks. An elementary classification for closed s-manifolds by triangulation were introduced in [56]. They are classified into 7 classes. Each of those classes is defined in the following.

Classical Type:

(1) $\Delta_1 = \{5 - regular\ triangular\ maps\}$ $(elliptic)$;

(2) $\Delta_2 = \{6 - regular\ triangular\ maps\}(euclidean)$;

(3) $\Delta_3 = \{7 - \textit{regular triangular maps}\}(\textit{hyperbolic})$.

Smarandache Type:

(4) $\Delta_4 = \{\textit{triangular maps with vertex valency } 5 \textit{ and } 6\}$ (*euclid-elliptic*);

(5) $\Delta_5 = \{\textit{triangular maps with vertex valency } 5 \textit{ and } 7\}$ (*elliptic-hyperbolic*);

(6) $\Delta_6 = \{\textit{triangular maps with vertex valency } 6 \textit{ and } 7\}$ (*euclid-hyperbolic*);

(7) $\Delta_7 = \{\textit{triangular maps with vertex valency } 5, 6 \textit{ and } 7\}$ (*mixed*).

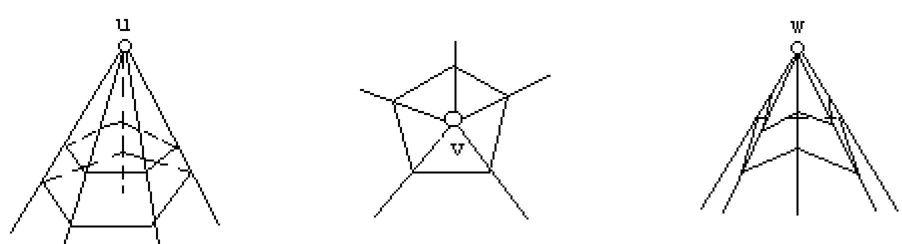
It is proved in [56] that $|\Delta_1| = 2$, $|\Delta_5| \geq 2$ and $|\Delta_i|, i = 2, 3, 4, 6, 7$ are infinite. Iseri proposed a question in [35]: *Do the other closed 2-manifolds correspond to s-manifolds with only hyperbolic vertices?* Since there are infinite Hurwitz maps, i.e., $|\Delta_3|$ is infinite, the answer is affirmative.

§3.2 Map Geometries without Boundary

A combinatorial map can be also used to construct new models for Smarandache geometries. By a geometrical view, these models are generalizations of Isier's model for Smarandache geometries. For a given map on a locally orientable surface, map geometries without boundary are defined in the following definition.

Definition 3.2.1 *For a combinatorial map M with each vertex valency≥ 3, associates a real number $\mu(u), 0 < \mu(u) < \frac{4\pi}{\rho_M(u)}$, to each vertex $u, u \in V(M)$. Call (M, μ) a map geometry without boundary, $\mu(u)$ an angle factor of the vertex u and orientablle or non-orientable if M is orientable or not.*

The realization for vertices $u, v, w \in V(M)$ in a space $\mathbf{R}^3$ is shown in Fig.3.4, where $\rho_M(u)\mu(u) < 2\pi$ for the vertex u, $\rho_M(v)\mu(v) = 2\pi$ for the vertex v and $\rho_M(w)\mu(w) > 2\pi$ for the vertex w, respectively.



$\rho_M(u)\mu(u) < 2\pi$ $\rho_M(u)\mu(u) = 2\pi$ $\rho_M(u)\mu(u) > 2\pi$

Fig.3.4

As we have pointed out in Section 3.1, this kind of realization is not a surface, but it is homeomorphic to a locally orientable surface by a view of topological equivalence. Similar to s-manifolds, we also classify points in a map geometry (M,μ) without boundary into *elliptic points*, *euclidean points* and *hyperbolic points*, defined in the next definition.

Definition 3.2.2 *A point u in a map geometry (M,μ) is said to be elliptic, euclidean or hyperbolic if $\rho_M(u)\mu(u) < 2\pi$, $\rho_M(u)\mu(u) = 2\pi$ or $\rho_M(u)\mu(u) > 2\pi$.*

Then we get the following results.

Theorem 3.2.1 *Let M be a map with $\forall u \in V(M), \rho_M(u) \geq 3$. Then for $\forall u \in V(M)$, there is a map geometry (M,μ) without boundary such that u is elliptic, euclidean or hyperbolic.*

Proof Since $\rho_M(u) \geq 3$, we can choose an angle factor $\mu(u)$ such that $\mu(u)\rho_M(u) < 2\pi$, $\mu(u)\rho_M(u) = 2\pi$ or $\mu(u)\rho_M(u) > 2\pi$. Notice that

$$0 < \frac{2\pi}{\rho_M(u)} < \frac{4\pi}{\rho_M(u)}.$$

Thereby we can always choose $\mu(u)$ satisfying that $0 < \mu(u) < \frac{4\pi}{\rho_M(u)}$. ♮

Theorem 3.2.2 *Let M be a map of order ≥ 3 and $\forall u \in V(M), \rho_M(u) \geq 3$. Then there exists a map geometry (M,μ) without boundary in which elliptic, euclidean and hyperbolic points appear simultaneously.*

Proof According to Theorem 3.2.1, we can always choose an angle factor μ such that a vertex $u, u \in V(M)$ to be elliptic, or euclidean, or hyperbolic. Since $|V(M)| \geq 3$, we can even choose the angle factor μ such that any two different vertices $v, w \in V(M)\setminus\{u\}$ to be elliptic, or euclidean, or hyperbolic as we wish. Then the map geometry (M,μ) makes the assertion hold. ♮

A *geodesic* in a manifold is a curve as straight as possible. Applying conceptions such as angles and straight lines in an Euclid geometry, we define m-lines and m-points in a map geometry in the next definition.

Definition 3.2.3 *Let (M,μ) be a map geometry without boundary and let $S(M)$ be the locally orientable surface represented by a plane polygon on which M is embedded.*

A point P on $S(M)$ is called an m-point. A line L on $S(M)$ is called an m-line if it is straight in each face of M and each angle on L has measure $\frac{\rho_M(v)\mu(v)}{2}$ when it passes through a vertex v on M.

Two examples for m-lines on the torus are shown in the Fig.3.5(a) and (b), where $M = M(B_2)$, $\mu(u) = \frac{\pi}{2}$ for the vertex u in (a) and

$$\mu(u) = \frac{135 - \arctan(2)}{360}\pi$$

for the vertex u in (b), i.e., u is euclidean in (a) but elliptic in (b). Notice that in (b), the m-line L_2 is self-intersected.

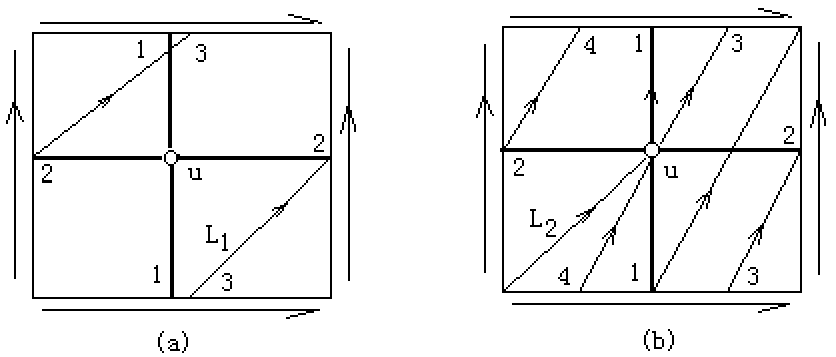
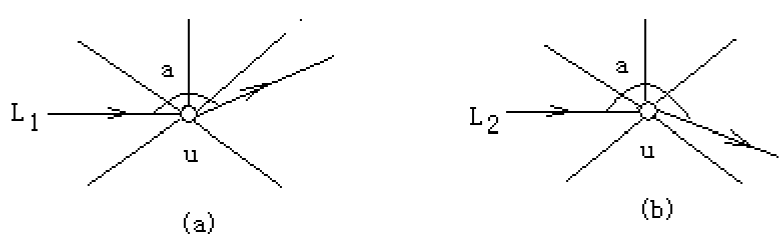


Fig.3.5

If an m-line passes through an elliptic point or a hyperbolic point u, it must has an angle $\frac{\mu(u)\rho_M(u)}{2}$ with the entering line, not $180°$ which are explained in Fig.3.6.



a $= \frac{\mu(u)\rho_M(u)}{2} < \pi$ a $= \frac{\mu(u)\rho_M(u)}{2} > \pi$

Fig.3.6

In an Euclid geometry, a right angle is an angle with measure $\frac{\pi}{2}$, half of a straight angle and parallel lines are straight lines never intersecting. They are very important research objects. Many theorems characterize properties of them in classical Euclid geometry.

In a map geometry, we can also define a straight angle, a right angle and parallel m-lines by Definition 3.2.2. Now a *straight angle* is an angle with measure π for points not being vertices of M and $\frac{\rho_M(u)\mu(u)}{2}$ for $\forall u \in V(M)$. A *right angle* is an angle with a half measure of a straight angle. Two m-lines are said *parallel* if they are never intersecting. The following result asserts that map geometries without boundary are paradoxist geometries.

Theorem 3.2.3 *For a map M on a locally orientable surface with $|M| \geq 3$ and $\rho_M(u) \geq 3$ for $\forall u \in V(M)$, there exists an angle factor μ such that (M, μ) is a Smarandache geometry by denial the axiom (A5) with these axioms (A5),(L5) and (R5).*

Proof According to Theorem 3.2.1, we know that there exists an angle factor μ such that there are elliptic vertices, euclidean vertices and hyperbolic vertices in (M, μ) simultaneously. The proof is divided into three cases according to M is planar, orientable or non-orientable. Not loss of generality, we assume that an angle is measured along a clockwise direction, i.e., as these cases in Fig.3.6 for an m-line passing through an elliptic point or a hyperbolic point.

Case 1. M **is a planar map**

Notice that for a given line L not intersection with the map M and a point u in (M, μ), if u is an euclidean point, then there is one and only one line passing through u not intersecting with L, and if u is an elliptic point, then there are infinite lines passing through u not intersecting with L, but if u is a hyperbolic point, then each line passing through u will intersect with L. See also in Fig.3.7, where the planar graph is a complete graph K_4 and points $1, 2$ are elliptic, the point 3 is euclidean but the point 4 is hyperbolic. Then all m-lines in the field A do not intersect with L and each m-line passing through the point 4 will intersect with the line L. Therefore, (M, μ) is a Smarandache geometry by denial the axiom (A5) with these axioms (A5), (L5) and (R5).

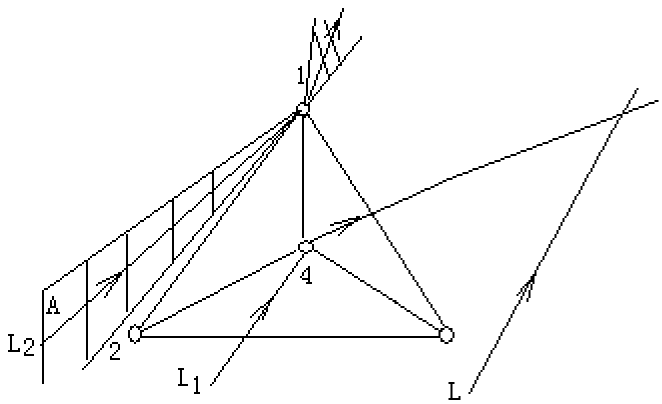


Fig.3.7

Case 2. M **is an orientable map**

According to the classification theorem of compact surfaces, We only need to prove this result for a torus. Notice that m-lines on a torus has the following property (see [82] for details):

If the slope ς of an m-line L is a rational number, then L is a closed line on the torus. Otherwise, L is infinite, and moreover L passes arbitrarily close to every point on the torus.

Whence, if L_1 is an m-line on a torus with an irrational slope not passing through an elliptic or a hyperbolic point, then for any point u exterior to L_1, if u is an euclidean point, then there is only one m-line passing through u not intersecting with L_1, and if u is elliptic or hyperbolic, any m-line passing through u will intersect with L_1.

Now let L_2 be an m-line on the torus with an rational slope not passing through an elliptic or a hyperbolic point, such as the m-line L_2 in Fig.3.8, v is an euclidean point. If u is an euclidean point, then each m-line L passing through u with rational slope in the area A will not intersect with L_2 but each m-line passing through u with irrational slope in the area A will intersect with L_2.

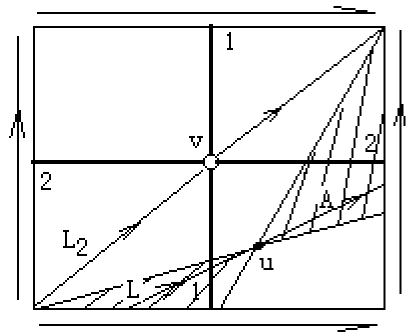


Fig.3.8

Therefore, (M, μ) is a Smarandache geometry by denial the axiom (A5) with axioms (A5),(L5) and (R5) in this case.

Case 3. M **is a non-orientable map**

Similar to the Case 2, we only need to prove this result for the projective plane. An m-line in a projective plane is shown in Fig.3.9(a), (b) or (c), in where case (a) is an m-line passing through an euclidean point, (b) passing through an elliptic point and (c) passing through an hyperbolic point.

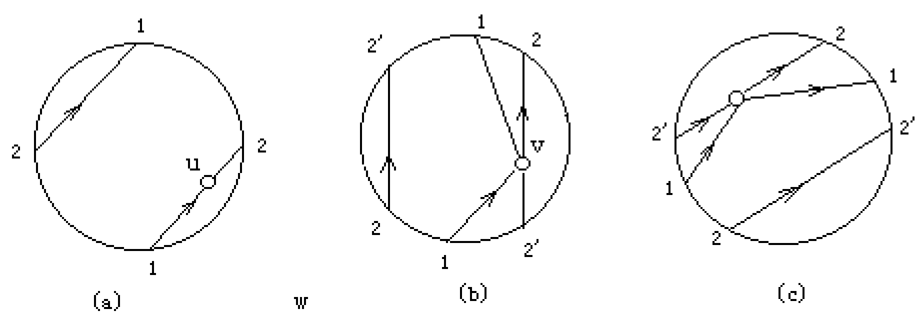


Fig.3.9

Now let L be an m-line passing through the center in the circle. Then if u is an euclidean point, there is only one m-line passing through u such as the case (a) in Fig.3.10. If v is an elliptic point then there is an m-line passing through it and intersecting with L such as the case (b) in Fig.3.10. We assume the point 1 is a point such that there exists an m-line passing through 1 and 0, then any m-line in the shade of Fig.3.10(b) passing through v will intersect with L.

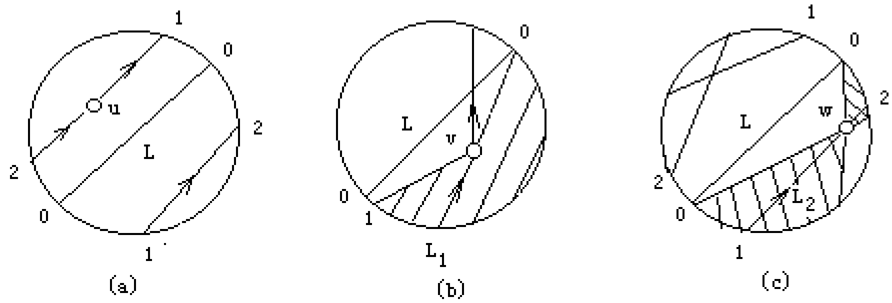


Fig.3.10

If w is an euclidean point and there is an m-line passing through it not intersecting with L such as the case (c) in Fig.3.10, then any m-line in the shade of Fig.3.10(c) passing through w will not intersect with L. Since the position of the vertices of a map M on a projective plane can be choose as our wish, we know (M,μ) is a Smarandache geometry by denial the axiom (A5) with axioms (A5),(L5) and (R5).

Combining these discussions of Cases 1, 2 and 3, the proof is complete. ♮.

Similar to Iseri's s-manifolds, among map geometries without boundary there are non-geometries, anti-geometries and counter-projective geometries, $\cdots$, etc..

Theorem 3.2.4 *There are non-geometries in map geometries without boundary.*

Proof We prove there are map geometries without boundary satisfying axioms $(A_1^-)-(A_5^-)$. Let (M,μ) be such a map geometry with elliptic or hyperbolic points.

(i) Assume u is an eulicdean point and v is an elliptic or hyperbolic point on (M,μ). Let L be an m-line passing through points u and v in an Euclid plane. Choose a point w in L after but nearly enough to v when we travel on L from u to v. Then there does not exist a line from u to w in the map geometry (M,μ) since v is an elliptic or hyperbolic point. So the axiom (A_1^-) is true in (M,μ).

(ii) In a map geometry (M,μ), an m-line maybe closed such as we have illustrated in the proof of Theorem 3.2.3. Choose any two points A,B on a closed m-line L in a map geometry. Then the m-line between A and B can not continuously extend to indefinite in (M,μ). Whence the axiom (A_2^-) is true in (M,μ).

(iii) An m-circle in a map geometry is defined to be a set of continuous points in which all points have a given distance to a given point. Let C be a m-circle in an Euclid plane. Choose an elliptic or a hyperbolic point A on C which enables us to get a map geometry (M, μ). Then C has a gap in A by definition of an elliptic or hyperbolic point. So the axiom (A_3^-) is true in a map geometry without boundary.

(iv) By the definition of a right angle, we know that a right angle on an elliptic point can not equal to a right angle on a hyperbolic point. So the axiom (A_4^-) is held in a map geometry with elliptic or hyperbolic points.

(v) The axiom (A_5^-) is true by Theorem 3.2.3.

Combining these discussions of (i)-(v), we know that there are non-geometries in map geometries. This completes the proof. $\natural$

The *Hilbert's axiom system* for an Euclid plane geometry consists five group axioms stated in the following, where we denote each group by a capital *Roman* numeral.

I. **Incidence**

$I-1$. *For every two points A and B, there exists a line L that contains each of the points A and B.*

$I-2$. *For every two points A and B, there exists no more than one line that contains each of the points A and B.*

$I-3$. *There are at least two points on a line. There are at least three points not on a line.*

II. **Betweenness**

$II-1$. *If a point B lies between points A and C, then the points A, B and C are distinct points of a line, and B also lies between C and A.*

$II-2$. *For two points A and C, there always exists at least one point B on the line AC such that C lies between A and B.*

$II-3$. *Of any three points on a line, there exists no more than one that lies between the other two.*

$II-4$. *Let A, B and C be three points that do not lie on a line, and let L be a line which does not meet any of the points A, B and C. If the line L passes through a point of the segment AB, it also passes through a point of the segment AC, or through a point of the segment BC.*

III. **Congruence**

$III-1$. *If A_1 and B_1 are two points on a line L_1, and A_2 is a point on a line L_2 then it is always possible to find a point B_2 on a given side of the line L_2 through A_2 such that the segment A_1B_1 is congruent to the segment A_2B_2.*
$III-2$. *If a segment A_1B_1 and a segment A_2B_2 are congruent to the segment AB, then the segment A_1B_1 is also congruent to the segment A_2B_2.*
$III-3$. *On the line L, let AB and BC be two segments which except for B have no point in common. Furthermore, on the same or on another line L_1, let A_1B_1 and B_1C_1 be two segments, which except for B_1 also have no point in common. In that case, if AB is congruent to A_1B_1 and BC is congruent to B_1C_1, then AC is congruent to A_1C_1.*
$III-4$. *Every angle can be copied on a given side of a given ray in a uniquely determined way.*
$III-5$ *If for two triangles ABC and $A_1B_1C_1$, AB is congruent to A_1B_1, AC is congruent to A_1C_1 and $\angle BAC$ is congruent to $\angle B_1A_1C_1$, then $\angle ABC$ is congruent to $\angle A_1B_1C_1$.*

IV. **Parallels**

$IV-1$. *There is at most one line passes through a point P exterior a line L that is parallel to L.*

V. **Continuity**

$V-1$(Archimedes) *Let AB and CD be two line segments with $|AB| \geq |CD|$. Then there is an integer m such that*

$$m|CD| \leq |AB| \leq (m+1)|CD|.$$

$V-2$(Cantor) Let $A_1B_1, A_2B_2, \cdots, A_nB_n, \cdots$ be a segment sequence on a line L. If

$$A_1B_1 \supseteq A_2B_2 \supseteq \cdots \supseteq A_nB_n \supseteq \cdots,$$

then there exists a common point X on each line segment A_nB_n for any integer $n, n \geq 1$.

Smarandache defined an anti-geometries by denial some axioms of Hilbert axiom system for an Euclid geometry. Similar to the discussion in the reference [35], We obtain the following result for anti-geometries in map geometries without boundary.

Theorem 3.2.5 *Unless axioms $I-3$, $II-3$, $III-2$, $V-1$ and $V-2$, an anti-geometry can be gotten from map geometries without boundary by denial other axioms in Hilbert axiom system.*

Proof The axiom $I-1$ has been denied in the proof of Theorem 3.2.4. Since there maybe exists more than one line passing through two points A and B in a map geometry with elliptic or hyperbolic points u such as those shown in Fig.3.11. So the axiom $II-2$ can be Smarandachely denied.

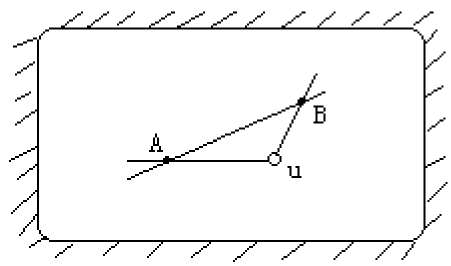


Fig.3.11

Notice that an m-line maybe has self-intersection points in a map geometry without boundary. So the axiom $II-1$ can be denied. By the proof of Theorem 3.2.4, we know that for two points A and B, an m-line passing through A and B may not exist. Whence, the axiom $II-2$ can be denied. For the axiom $II-4$, see Fig.3.12, in where v is a non-euclidean point such that $\rho_M(v)\mu(v) \geq 2(\pi + \angle ACB)$ in a map geometry.

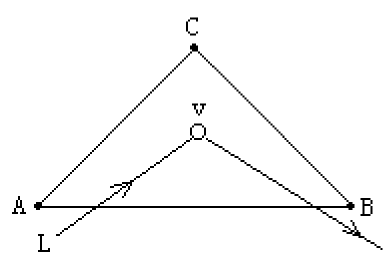


Fig.3.12

So $II-4$ can be also denied. Notice that an m-line maybe has self-intersection points. There are maybe more than one m-lines passing through two given points A, B. Therefore, the axioms $III-1$ and $III-3$ are deniable. For denial the axiom $III-4$, since an elliptic point u can be measured at most by a number $\frac{\rho_M(u)\mu(u)}{2} < \pi$, i.e., there is a limitation for an elliptic point u. Whence, an angle with measure bigger than $\frac{\rho_M(u)\mu(u)}{2}$ can not be copied on an elliptic point on a given ray.

Because there are maybe more than one m-lines passing through two given points A and B in a map geometry without boundary, the axiom $III-5$ can be Smarandachely denied in general such as those shown in Fig.3.13(a) and (b) where u is an elliptic point.

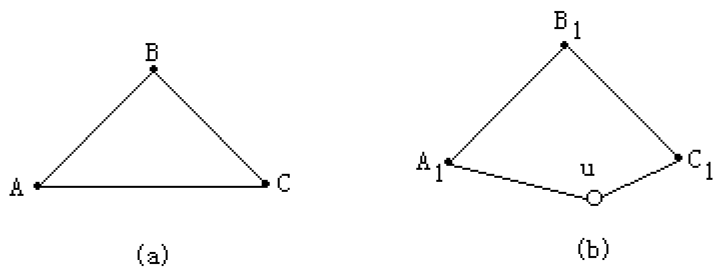


Fig.3.13

For the parallel axiom $IV-1$, it has been denied by the proofs of Theorems 3.2.3 and 3.2.4.

Notice that axioms $I-3$, $II-3$ $III-2$, $V-1$ and $V-2$ can not be denied in a map geometry without boundary. This completes the proof. ♮

For counter-projective geometries, we have a result as in the following.

Theorem 3.2.6 *Unless the axiom* $(C3)$, *a counter-projective geometry can be gotten from map geometries without boundary by denial axioms* $(C1)$ *and* $(C2)$.

Proof Notice that axioms $(C1)$ and $(C2)$ have been denied in the proof of Theorem 3.2.5. Since a map is embedded on a locally orientable surface, every m-line in a map geometry without boundary may contains infinite points. Therefore the axiom $(C3)$ can not be Smarandachely denied. ♮

§3.3 Map Geometries with Boundary

A *Poincaré's model* for a hyperbolic geometry is an upper half-plane in which lines are upper half-circles with center on the x-axis or upper straight lines perpendicular to the x-axis such as those shown in Fig.3.14.

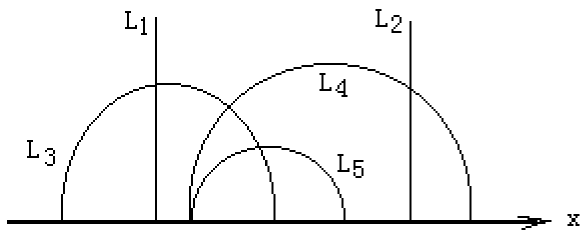


Fig.3.14

If we think that all infinite points are the same, then a Poincaré's model for a hyperbolic geometry is turned to a *Klein model* for a hyperbolic geometry which uses a boundary circle and lines are straight line segment in this circle, such as those shown in Fig.3.15.

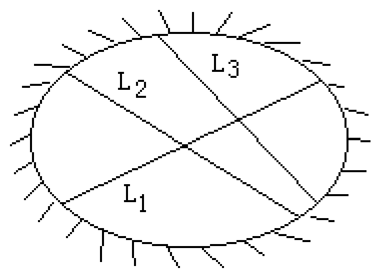


Fig.3.15

By a combinatorial map view, a Klein's model is nothing but a one face map geometry. This fact hints us to introduce map geometries with boundary, which is defined in the next definition.

Definition 3.3.1 *For a map geometry (M,μ) without boundary and faces $f_1, f_2, \cdots, f_l \in F(M), 1 \leq l \leq \phi(M) - 1$, if $S(M) \setminus \{f_1, f_2, \cdots, f_l\}$ is connected, then call $(M,\mu)^{-l} = (S(M) \setminus \{f_1, f_2, \cdots, f_l\}, \mu)$ a map geometry with boundary $f_1, f_2, \cdots, f_l$ and orientable or not if (M,μ) is orientable or not, where $S(M)$ denotes the locally orientable surface on which M is embedded.*

The m-points and m-lines in a map geometry $(M,\mu)^{-l}$ are defined as same as Definition 3.2.3 by adding an m-line terminated at the boundary of this map geometry. Two m^{-}-lines on the torus and projective plane are shown in these Fig.3.16 and Fig.3.17, where the shade field denotes the boundary.

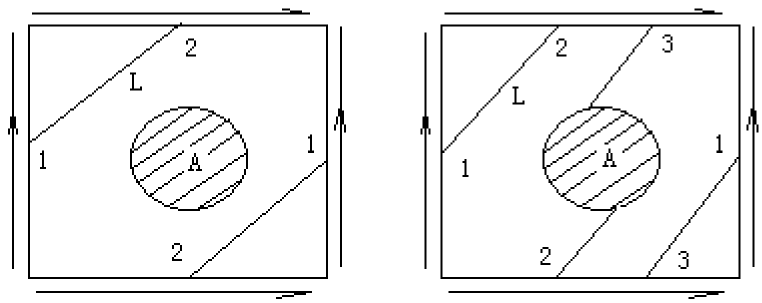


Fig.3.16

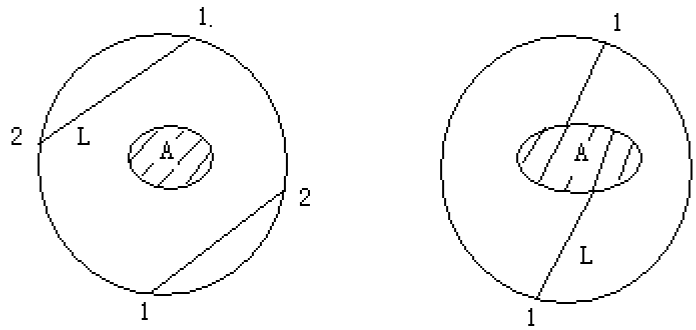


Fig.3.17

All map geometries with boundary are also Smarandache geometries which is convince by a result in the following.

Theorem 3.3.1 *For a map M on a locally orientable surface with order≥ 3, vertex valency≥ 3 and a face $f \in F(M)$, there is an angle factor μ such that $(M, \mu)^{-1}$ is a Smarandache geometry by denial the axiom (A5) with these axioms (A5),(L5) and (R5).*

Proof Similar to the proof of Theorem 3.2.3, we consider a map M being a planar map, an orientable map on a torus or a non-orientable map on a projective plane, respectively. We can get the assertion. In fact, by applying the property that m-lines in a map geometry with boundary are terminated at the boundary, we can get an more simpler proof for this theorem. $\natural$

Notice that in a one face map geometry $(M, \mu)^{-1}$ with boundary is just a Klein's model for hyperbolic geometry if we choose all points being euclidean.

Similar to map geometries without boundary, we can also get non-geometries, anti-geometries and counter-projective geometries from map geometries with boundary.

Theorem 3.3.2 *There are non-geometries in map geometries with boundary.*

Proof The proof is similar to the proof of Theorem 3.2.4 for map geometries without boundary. Each of axioms $(A_1^-)-(A_5^-)$ is hold, for example, cases $(a)-(e)$ in Fig.3.18,

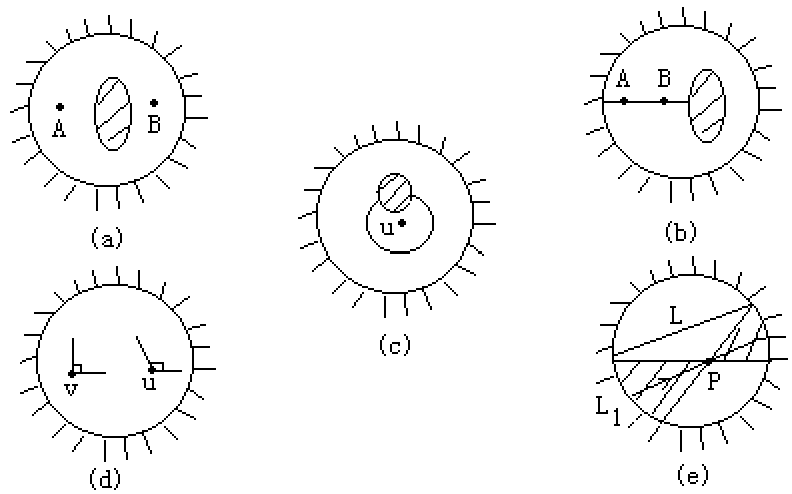


Fig.3.18

in where there are no an m-line from points A to B in (a), the line AB can not be continuously extended to indefinite in (b), the circle has gap in (c), a right angle at an euclidean point v is not equal to a right angle at an elliptic point u in (d) and there are infinite m-lines passing through a point P not intersecting with the m-line L in (e). Whence, there are non-geometries in map geometries with boundary. ♮

Theorem 3.3.3 *Unless axioms $I-3$, $II-3$ $III-2$, $V-1$ and $V-2$ in the Hilbert's axiom system for an Euclid geometry, an anti-geometry can be gotten from map geometries with boundary by denial other axioms in this axiom system.*

Theorem 3.3.4 *Unless the axiom $(C3)$, a counter-projective geometry can be gotten from map geometries with boundary by denial axioms $(C1)$ and $(C2)$.*

Proof The proofs of Theorems 3.3.3 and 3.3.4 are similar to the proofs of Theorems 3.2.5 and 3.2.6. The reader is required to complete their proof. ♮

§3.4 The Enumeration of Map Geometries

For classifying map geometries, the following definition is needed.

Definition 3.4.1 *Two map geometries (M_1, μ_1) and (M_2, μ_2) or $(M_1, \mu_1)^{-l}$ and $(M_2, \mu_2)^{-l}$ are said to be equivalent each other if there is a bijection $\theta : M_1 \to M_2$ such that for $\forall u \in V(M)$, $\theta(u)$ is euclidean, elliptic or hyperbolic if and only if u is euclidean, elliptic or hyperbolic.*

A relation for the numbers of unrooted maps with map geometries is in the following result.

Theorem 3.4.1 *Let $\mathcal{M}$ be a set of non-isomorphic maps of order n and with m faces. Then the number of map geometries without boundary is $3^n|\mathcal{M}|$ and the number of map geometries with one face being its boundary is $3^n m|\mathcal{M}|$.*

Proof By the definition of equivalent map geometries, for a given map $M \in \mathcal{M}$, there are 3^n map geometries without boundary and $3^n m$ map geometries with one face being its boundary by Theorem 3.3.1. Whence, we get $3^n|\mathcal{M}|$ map geometries without boundary and $3^n m|\mathcal{M}|$ map geometries with one face being its boundary from $\mathcal{M}$. $\natural$.

We get an enumeration result for non-equivalent map geometries without boundary as follows.

Theorem 3.4.2 *The numbers $n^O(\Gamma, g)$ and $n^N(\Gamma, g)$ of non-equivalent orientable and non-orientable map geometries without boundary underlying a simple graph Γ by denial the axiom (A5) by (A5), (L5) or (R5) are*

$$n^O(\Gamma, g) = \frac{3^{|\Gamma|} \prod\limits_{v \in V(\Gamma)} (\rho(v) - 1)!}{2|\mathrm{Aut}\Gamma|},$$

and

$$n^N(\Gamma, g) = \frac{(2^{\beta(\Gamma)} - 1)3^{|\Gamma|} \prod\limits_{v \in V(\Gamma)} (\rho(v) - 1)!}{2|\mathrm{Aut}\Gamma|},$$

where $\beta(\Gamma) = \varepsilon(\Gamma) - \nu(\Gamma) + 1$ is the Betti number of the graph Γ.

Proof Denote the set of non-isomorphic maps underlying the graph Γ on locally orientable surfaces by $\mathcal{M}(\Gamma)$ and the set of embeddings of the graph Γ on locally orientable surfaces by $\mathcal{E}(\Gamma)$. For a map $M, M \in \mathcal{M}(\Gamma)$, there are $\frac{3^{|M|}}{|\mathrm{Aut}M|}$ different map geometries without boundary by choice the angle factor μ on a vertex u such that u is euclidean, elliptic or hyperbolic. From permutation groups, we know that

$$|\mathrm{Aut}\Gamma \times \langle\alpha\rangle| = |(\mathrm{Aut}\Gamma)_M||M^{\mathrm{Aut}\Gamma\times\langle\alpha\rangle}| = |\mathrm{Aut}M||M^{\mathrm{Aut}\Gamma\times\langle\alpha\rangle}|.$$

Therefore, we get that

$$\begin{aligned} n^O(\Gamma, g) &= \sum_{M\in\mathcal{M}(\Gamma)} \frac{3^{|M|}}{|\mathrm{Aut}M|} \\ &= \frac{3^{|\Gamma|}}{|\mathrm{Aut}\Gamma \times \langle\alpha\rangle|} \sum_{M\in\mathcal{M}(\Gamma)} \frac{|\mathrm{Aut}\Gamma \times \langle\alpha\rangle|}{|\mathrm{Aut}M|} \\ &= \frac{3^{|\Gamma|}}{|\mathrm{Aut}\Gamma \times \langle\alpha\rangle|} \sum_{M\in\mathcal{M}(\Gamma)} |M^{\mathrm{Aut}\Gamma\times\langle\alpha\rangle}| \\ &= \frac{3^{|\Gamma|}}{|\mathrm{Aut}\Gamma \times \langle\alpha\rangle|} |\mathcal{E}^O(\Gamma)| \\ &= \frac{3^{|\Gamma|} \prod\limits_{v\in V(\Gamma)} (\rho(v)-1)!}{2|\mathrm{Aut}\Gamma|}. \end{aligned}$$

Similarly, we can also get that

$$\begin{aligned} n^N(\Gamma, g) &= \frac{3^{|\Gamma|}}{|\mathrm{Aut}\Gamma \times \langle\alpha\rangle|} |\mathcal{E}^N(\Gamma)| \\ &= \frac{(2^{\beta(\Gamma)}-1)3^{|\Gamma|} \prod\limits_{v\in V(\Gamma)} (\rho(v)-1)!}{2|\mathrm{Aut}\Gamma|}. \end{aligned}$$

This completes the proof. ♮

For classifying map geometries with boundary, we get a result as in the following.

Theorem 3.4.3 *The numbers $n^O(\Gamma, -g)$, $n^N(\Gamma, -g)$ of non-equivalent orientable, non-orientable map geometries with one face being its boundary underlying a simple graph Γ by denial the axiom (A5) by (A5), (L5) or (R5) are respective*

$$n^O(\Gamma,-g)=\frac{3^{|\Gamma|}}{2|\mathrm{Aut}\Gamma|}[(\beta(\Gamma)+1)\prod_{v\in V(\Gamma)}(\rho(v)-1)!-\frac{2d(g[\Gamma](x))}{dx}|_{x=1}]$$

and

$$n^N(\Gamma,-g)=\frac{(2^{\beta(\Gamma)}-1)3^{|\Gamma|}}{2|\mathrm{Aut}\Gamma|}[(\beta(\Gamma)+1)\prod_{v\in V(\Gamma)}(\rho(v)-1)!-\frac{2d(g[\Gamma](x))}{dx}|_{x=1}],$$

where $g[\Gamma](x)$ *is the genus polynomial of the graph* Γ*, i.e.,* $g[\Gamma](x)=\sum\limits_{k=\gamma(\Gamma)}^{\gamma_m(\Gamma)}g_k[\Gamma]x^k$ *with* $g_k[\Gamma]$ *being the number of embeddings of* Γ *on the orientable surface of genus* k*.*

Proof Notice that $\nu(M)-\varepsilon(M)+\phi(M)=2-2g(M)$ for an orientable map M by the Euler-Poincar*é* formula. Similar to the proof of Theorem 3.4.2 with the same meaning for $\mathcal{M}(\Gamma)$, we know that

$$\begin{aligned}
n^O(\Gamma,-g) &= \sum_{M\in\mathcal{M}(\Gamma)}\frac{\phi(M)3^{|M|}}{|\mathrm{Aut}M|}\\
&= \sum_{M\in\mathcal{M}(\Gamma)}\frac{(2+\varepsilon(\Gamma)-\nu(\Gamma)-2g(M))3^{|M|}}{|\mathrm{Aut}M|}\\
&= \sum_{M\in\mathcal{M}(\Gamma)}\frac{(2+\varepsilon(\Gamma)-\nu(\Gamma))3^{|M|}}{|\mathrm{Aut}M|}-\sum_{M\in\mathcal{M}(\Gamma)}\frac{2g(M)3^{|M|}}{|\mathrm{Aut}M|}\\
&= \frac{(2+\varepsilon(\Gamma)-\nu(\Gamma))3^{|M|}}{|\mathrm{Aut}\Gamma\times\langle\alpha\rangle|}\sum_{M\in\mathcal{M}(\Gamma)}\frac{|\mathrm{Aut}\Gamma\times\langle\alpha\rangle|}{|\mathrm{Aut}M|}\\
&- \frac{2\times 3^{|\Gamma|}}{|\mathrm{Aut}\Gamma\times\langle\alpha\rangle|}\sum_{M\in\mathcal{M}(\Gamma)}\frac{g(M)|\mathrm{Aut}\Gamma\times\langle\alpha\rangle|}{|\mathrm{Aut}M|}\\
&= \frac{(\beta(\Gamma)+1)3^{|M|}}{|\mathrm{Aut}\Gamma\times\langle\alpha\rangle|}\sum_{M\in\mathcal{M}}(\Gamma)|M^{\mathrm{Aut}\Gamma\times\langle\alpha\rangle}|\\
&- \frac{3^{|\Gamma|}}{|\mathrm{Aut}\Gamma|}\sum_{M\in\mathcal{M}(\Gamma)}g(M)|M^{\mathrm{Aut}\Gamma\times\langle\alpha\rangle}|\\
&= \frac{(\beta(\Gamma)+1)3^{|\Gamma|}}{2|\mathrm{Aut}\Gamma|}\prod_{v\in V(\Gamma)}(\rho(v)-1)!-\frac{3^{|\Gamma|}}{|\mathrm{Aut}\Gamma|}\sum_{k=\gamma(\Gamma)}^{\gamma_m(\Gamma)}kg_k[\Gamma]\\
&= \frac{3^{|\Gamma|}}{2|\mathrm{Aut}\Gamma|}[(\beta(\Gamma)+1)\prod_{v\in V(\Gamma)}(\rho(v)-1)!-\frac{2d(g[\Gamma](x))}{dx}|_{x=1}].
\end{aligned}$$

by Theorem 3.4.1.

Notice that $n^L(\Gamma,-g)=n^O(\Gamma,-g)+n^N(\Gamma,-g)$ and the number of re-embeddings an orientable map M on surfaces is $2^{\beta(M)}$ (see also [56] for details). We know that

$$\begin{aligned} n^L(\Gamma,-g) &= \sum_{M\in\mathcal{M}(\Gamma)} \frac{2^{\beta(M)}\times 3^{|M|}\phi(M)}{|\mathrm{Aut}M|} \\ &= 2^{\beta(M)}n^O(\Gamma,-g). \end{aligned}$$

Whence, we get that

$$\begin{aligned} n^N(\Gamma,-g) &= (2^{\beta(M)}-1)n^O(\Gamma,-g) \\ &= \frac{(2^{\beta(M)}-1)3^{|\Gamma|}}{2|\mathrm{Aut}\Gamma|}[(\beta(\Gamma)+1)\prod_{v\in V(\Gamma)}(\rho(v)-1)!-\frac{2d(g[\Gamma](x))}{dx}|_{x=1}]. \end{aligned}$$

This completes the proof. $\natural$

§3.5 Remarks and Open Problems

3.5.1 A complete Hilbert axiom system for an Euclid geometry contains axioms $I-i, 1\leq i\leq 8$; $II-j, 1\leq j\leq 4$; $III-k, 1\leq k\leq 5$; $IV-1$ and $V-l, 1\leq l\leq 2$, which can be also applied to the geometry of space. Unless $I-i, 4\leq i\leq 8$, other axioms are presented in Section 3.2. Each of axioms $I-i, 4\leq i\leq 8$ is described in the following.

$I-4$ *For three non-collinear points A, B and C, there is one and only one plane passing through them.*

$I-5$ *Each plane has at least one point.*

$I-6$ *If two points A and B of a line L are in a plane $\sum$, then every point of L is in the plane $\sum$.*

$I-7$ *If two planes $\sum_1$ and $\sum_2$ have a common point A, then they have another common point B.*

$I-8$ *There are at least four points not in one plane.*

By the Hilbert's axiom system, the following result for parallel planes can be obtained.

(**T**) *Passing through a given point A exterior to a given plane $\sum$ there is one and only one plane parallel to $\sum$.*

This result seems like the Euclid's fifth axiom. Similar to the Smarandache's notion, we present problems by denial this theorem for the geometry of space as follows.

Problem 3.5.1 *Construct a geometry of space by denial the parallel theorem of planes with*

(T_1^-) *there are at least a plane $\sum$ and a point A exterior to the plane $\sum$ such that no parallel plane to $\sum$ passing through the point A.*

(T_2^-) *there are at least a plane $\sum$ and a point A exterior to the plane $\sum$ such that there are finite parallel planes to $\sum$ passing through the point A.*

(T_3^-) *there are at least a plane $\sum$ and a point A exterior to the plane $\sum$ such that there are infinite parallel planes to $\sum$ passing through the point A.*

Problem 3.5.2 *Similar to the Iseri's idea define an elliptic, euclidean, or hyperbolic point or plane in $\mathbf{R}^3$ and apply these Plato polyhedrons to construct Smarandache geometries of a space $\mathbf{R}^3$.*

Problem 3.5.3 *Similar to map geometries define graph in a space geometries and apply graphs in $\mathbf{R}^3$ to construct Smarandache geometries of a space $\mathbf{R}^3$.*

Problem 3.5.4 *For an integer $n, n \geq 4$, define Smarandache geometries in $\mathbf{R}^n$ by denial some axioms for an Euclid geometry in $\mathbf{R}^n$ and construct them.*

3.5.2 The terminology *map geometry* was first appeared in [55] which enables us to find non-homogenous spaces from already known homogenous spaces and is also a typical example for application combinatorial maps to metric geometries. Among them there are many problems not solved yet until today. Here we would like to describe some of them.

Problem 3.5.5 *For a given graph G, determine non-equivalent map geometries with an underlying graph G, particularly, for graphs K_n, $K(m,n), m,n \geq 4$ and enumerate them.*

Problem 3.5.6 *For a given locally orientable surface S, determine non-equivalent map geometries on S, such as a sphere, a torus or a projective plane, $\cdots$ and enumerate them.*

Problem 3.5.7 *Find characteristics for equivalent map geometries or establish new ways for classifying map geometries.*

Problem 3.5.8 *Whether can we rebuilt an intrinsic geometry on surfaces, such as a sphere, a torus or a projective plane, $\cdots$, by map geometries?*

Chapter 4 Planar Map Geometries

Fundamental elements in an Euclid geometry are those of points, lines, polygons and circles. For a map geometry, the situation is more complex since a point maybe an elliptic, euclidean or a hyperbolic point, a polygon maybe a line, $\cdots$, etc.. This chapter concentrates on discussing fundamental elements and measures such as angle, area, curvature, $\cdots$, etc., also parallel bundles in planar map geometries, which can be seen as a first step for comprehending map geometries on surfaces. All materials of this chapter will be used in Chapters 5-6 for establishing relations of an integral curve with a differential equation system in a pseudo-plane geometry and continuous phenomena with discrete phenomena

§4.1 Points in a Planar Map Geometry

Points in a map geometry are classified into three classes: *elliptic, euclidean* and *hyperbolic*. There are only finite non-euclidean points considered in Chapter 3 because we had only defined an elliptic, euclidean or a hyperbolic point on vertices of a map. In a planar map geometry, we can present an even more delicate consideration for euclidean or non-euclidean points and find infinite non-euclidean points in a plane.

Let (M,μ) be a planar map geometry on a plane $\sum$. Choose vertices $u,v \in V(M)$. A mapping is called an *angle function between u and v* if there is a smooth monotone mapping $f: \sum \to \sum$ such that $f(u) = \frac{\rho_M(u)\mu(u)}{2}$ and $f(v) = \frac{\rho_M(v)\mu(v)}{2}$. Not loss of generality, we can assume that each edge in a planar map geometry is an angle function. Then we know a result as in the following.

Theorem 4.1.1 *A planar map geometry (M,μ) has infinite non-euclidean points if and only if there is an edge $e=(u,v)\in E(M)$ such that $\rho_M(u)\mu(u)\neq\rho_M(v)\mu(v)$, or $\rho_M(u)\mu(u)$ is a constant but $\neq 2\pi$ for $\forall u\in V(M)$, or a loop $(u,u)\in E(M)$ attaching a non-euclidean point u.*

Proof If there is an edge $e=(u,v)\in E(M)$ such that $\rho_M(u)\mu(u)\neq\rho_M(v)\mu(v)$, then at least one of vertices u and v in (M,μ) is non-euclidean. Not loss of generality, we assume the vertex u is non-euclidean.

If u and v are elliptic or u is elliptic but v is euclidean, then by the definition of angle functions, the edge (u,v) is correspondent with an angle function $f:\sum\to\sum$ such that $f(u)=\frac{\rho_M(u)\mu(u)}{2}$ and $f(v)=\frac{\rho_M(v)\mu(v)}{2}$, each points is non-euclidean in $(u,v)\setminus\{v\}$. If u is elliptic but v is hyperbolic, i.e., $\rho_M(u)\mu(u)<2\pi$ and $\rho_M(v)\mu(v)>2\pi$, since f is smooth and monotone on (u,v), there is one and only one point x^* in (u,v) such that $f(x^*)=\pi$. Thereby there are infinite non-euclidean points on (u,v).

Similar discussion can be gotten for the cases that u and v are both hyperbolic, or u is hyperbolic but v is euclidean, or u is hyperbolic but v is elliptic.

If $\rho_M(u)\mu(u)$ is a constant but $\neq 2\pi$ for $\forall u\in V(M)$, then each point on an edges is a non-euclidean point. Thereby there are infinite non-euclidean points in (M,μ).

Now if there is a loop $(u,u)\in E(M)$ and u is non-eucliean, then by definition, each point v on the loop (u,u) satisfying that $f(v)>$ or $<\pi$ according to $\rho_M(u)\mu(u)>\pi$ or $<\pi$. Therefore there are also infinite non-euclidean points on the loop (u,u).

On the other hand, if there are no an edge $e=(u,v)\in E(M)$ such that $\rho_M(u)\mu(u)\neq\rho_M(v)\mu(v)$, i.e., $\rho_M(u)\mu(u)=\rho_M(v)\mu(v)$ for $\forall(u,v)\in E(M)$, or there are no vertices $u\in V(M)$ such that $\rho_M(u)\mu(u)$ is a constant but $\neq 2\pi$ for $\forall$, or there are no loops $(u,u)\in E(M)$ with a non-eucliean point u, then all angle functions on these edges of M are an constant π. Therefore there are no non-euclidean points in the map geometry (M,μ). This completes the proof. $\natural$

For euclidean points in a planar map geometry (M,μ), we get the following result.

Theorem 4.1.2 *For a planar map geometry (M,μ) on a plane $\sum$,* *(i) every point in* $\sum \setminus E(M)$ *is an euclidean point;*

(ii) there are infinite euclidean points on M *if and only if there exists an edge* $(u, v) \in E(M)$ *(*$u = v$ *or* $u \neq v$*) such that* u *and* v *are both euclidean.*

Proof By the definition of angle functions, we know that every point is euclidean if it is not on M. So the assertion (i) is true.

According to the proof of Theorem 4.1.1, there are only finite euclidean points unless there is an edge $(u, v) \in E(M)$ with $\rho_M(u)\mu(u) = \rho_M(v)\mu(v) = 2\pi$. In this case, there are infinite euclidean points on the edge (u, v). Thereby the assertion (ii) is also holds. ♮

According to Theorems 4.1.1 and 4.1.2, we classify edges in a planar map geometry (M, μ) into six classes as follows.

C_E^1 **(euclidean-elliptic edges)**: *edges* $(u, v) \in E(M)$ *with* $\rho_M(u)\mu(u) = 2\pi$ *but* $\rho_M(v)\mu(v) < 2\pi$.

C_E^2 **(euclidean-euclidean edges)**: *edges* $(u, v) \in E(M)$ *with* $\rho_M(u)\mu(u) = 2\pi$ *and* $\rho_M(v)\mu(v) = 2\pi$.

C_E^3 **(euclidean-hyperbolic edges)**: *edges* $(u, v) \in E(M)$ *with* $\rho_M(u)\mu(u) = 2\pi$ *but* $\rho_M(v)\mu(v) > 2\pi$.

C_E^4 **(elliptic-elliptic edges)**: *edges* $(u, v) \in E(M)$ *with* $\rho_M(u)\mu(u) < 2\pi$ *and* $\rho_M(v)\mu(v) < 2\pi$.

C_E^5 **(elliptic-hyperbolic edges)**: *edges* $(u, v) \in E(M)$ *with* $\rho_M(u)\mu(u) < 2\pi$ *but* $\rho_M(v)\mu(v) > 2\pi$.

C_E^6 **(hyperbolic-hyperbolic edges)**: *edges* $(u, v) \in E(M)$ *with* $\rho_M(u)\mu(u) > 2\pi$ *and* $\rho_M(v)\mu(v) > 2\pi$.

In Fig.4.1$(a) - (f)$, these m-lines passing through an edge in one of classes of C_E^1-C_E^6 are shown, where u is elliptic and v is eucildean in (a), u and v are both euclidean in (b), u is eucildean but v is hyperbolic in (c), u and v are both elliptic in (d), u is elliptic but v is hyperbolic in (e) and u and v are both hyperbolic in (f), respectively.

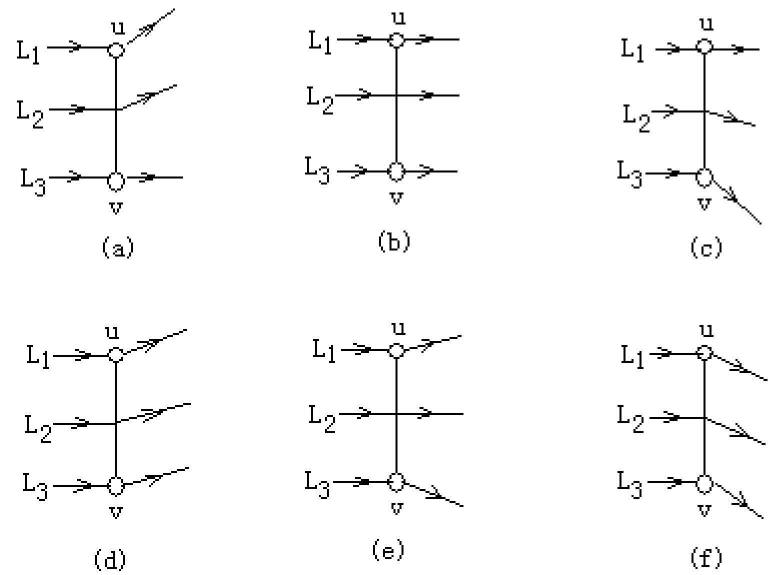


Fig.4.1

Denote by $V_{el}(M), V_{eu}(M)$ and $V_{hy}(M)$ the respective sets of elliptic, euclidean and hyperbolic points in $V(M)$ in a planar map geometry (M,μ). Then we get a result as in the following.

Theorem 4.1.3 *Let (M,μ) be a planar map geometry. Then*

$$\sum_{u\in V_{el}(M)} \rho_M(u) + \sum_{v\in V_{eu}(M)} \rho_M(v) + \sum_{w\in V_{hy}(M)} \rho_M(w) = 2\sum_{i=1}^{6} |C_E^i|$$

and

$$|V_{el}(M)| + |V_{eu}(M)| + |V_{hy}(M)| + \phi(M) = \sum_{i=1}^{6} |C_E^i| + 2.$$

where $\phi(M)$ denotes the number of faces of a map M.

Proof Notice that

$$|V(M)| = |V_{el}(M)| + |V_{eu}(M)| + |V_{hy}(M)| \text{ and } |E(M)| = \sum_{i=1}^{6} |C_E^i|$$

for a planar map geometry (M,μ). By two well-known results

$$\sum_{v\in V(M)} \rho_M(v) = 2|E(M)| \quad \text{and} \quad |V(M)| - |E(M)| + \phi(M) = 2$$

for a planar map M, we know that

$$\sum_{u \in V_{el}(M)} \rho_M(u) + \sum_{v \in V_{eu}(M)} \rho_M(v) + \sum_{w \in V_{hy}(M)} \rho_M(w) = 2 \sum_{i=1}^{6} |C_E^i|$$

and

$$|V_{el}(M)| + |V_{eu}(M)| + |V_{hy}(M)| + \phi(M) = \sum_{i=1}^{6} |C_E^i| + 2. \quad \natural$$

§4.2 Lines in a Planar Map Geometry

The situation of m-lines in a planar map geometry (M, μ) is more complex. Here an m-line maybe open or closed, with or without self-intersections in a plane. We discuss all of these m-lines and their behaviors in this section, .

4.2.1. Lines in a planar map geometry

As we have seen in Chapter 3, m-lines in a planar map geometry (M, μ) can be classified into three classes.

C_L^1**(opened lines without self-intersections)**: *m-lines in (M, μ) have an infinite number of continuous m-points without self-intersections and endpoints and may be extended indefinitely in both directions.*

C_L^2**(opened lines with self-intersections)**: *m-lines in (M, μ) have an infinite number of continuous m-points and self-intersections but without endpoints and may be extended indefinitely in both directions.*

C_L^3**(closed lines)**: *m-lines in (M, μ) have an infinite number of continuous m-points and will come back to the initial point as we travel along any one of these m-lines starting at an initial point.*

By this classification, a straight line in an Euclid plane is nothing but an opened m-line without non-euclidean points. Certainly, m-lines in a planar map geometry (M, μ) maybe contain non-euclidean points. In Fig.4.2, these m-lines shown in $(a), (b)$ and (c) are opened m-line without self-intersections, opened m-line with a self-intersection and closed m-line with A, B, C, D and E non-euclidean points, respectively.

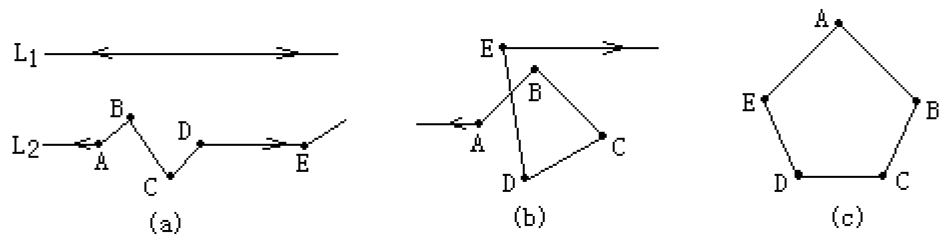


Fig.4.2

Notice that a closed m-line in a planar map geometry maybe also has self-intersections. A closed m-line is said to be *simply closed* if it has no self-intersections, such as the m-line in Fig.4.2(c). For simply closed m-lines, we know the following result.

Theorem 4.2.1 *Let (M,μ) be a planar map geometry. An m-line L in (M,μ) passing through n non-euclidean points $x_1, x_2, \cdots, x_n$ is simply closed if and only if*

$$\sum_{i=1}^{n} f(x_i) = (n-2)\pi,$$

where $f(x_i)$ denotes the angle function value at an m-point $x_i, 1 \le i \le n$.

Proof By results in an Euclid geometry of plane, we know that the angle sum of an n-polygon is $(n-2)\pi$. In a planar map geometry (M,μ), a simply closed m-line L passing through n non-euclidean points $x_1, x_2, \cdots, x_n$ is nothing but an n-polygon with vertices $x_1, x_2, \cdots, x_n$. Whence, we get that

$$\sum_{i=1}^{n} f(x_i) = (n-2)\pi.$$

Now if a simply m-line L passing through n non-euclidean points $x_1, x_2, \cdots, x_n$ with

$$\sum_{i=1}^{n} f(x_i) = (n-2)\pi$$

held, then L is nothing but an n-polygon with vertices $x_1, x_2, \cdots, x_n$. Therefore, L is simply closed. $\natural$

By applying Theorem 4.2.1, we can also find conditions for an opened m-line with or without self-intersections.

Theorem 4.2.2 *Let (M, μ) be a planar map geometry. An m-line L in (M, μ) passing through n non-euclidean points $x_1, x_2, \cdots, x_n$ is opened without self-intersections if and only if m-line segments $x_ix_{i+1}, 1 \leq i \leq n-1$ are not intersect two by two and*

$$\sum_{i=1}^{n} f(x_i) \geq (n-1)\pi.$$

Proof By the Euclid's fifth postulate for a plane geometry, two straight lines will meet on the side on which the angles less than two right angles if we extend them to indefinitely. Now for an m-line L in a planar map geometry (M, μ), if it is opened without self-intersections, then for any integer $i, 1 \leq i \leq n-1$, m-line segments x_ix_{i+1} will not intersect two by two and the m-line L will also not intersect before it enters x_1 or leaves x_n.

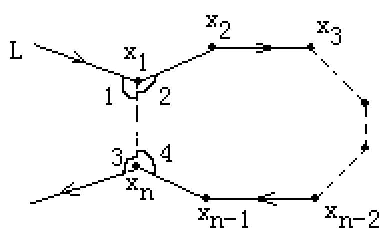


Fig.4.3

Now look at Fig.4.3, in where line segment x_1x_n is an added auxiliary m-line segment. We know that

$$\angle 1 + \angle 2 = f(x_1) \text{ and } \angle 3 + \angle 4 = f(x_n).$$

According to Theorem 4.2.1 and the Euclid's fifth postulate, we know that

$$\angle 2 + \angle 4 + \sum_{i=2}^{n-1} f(x_i) = (n-2)\pi$$

and

$$\angle 1 + \angle 3 \geq \pi$$

Therefore, we get that

$$\sum_{i=1}^{n} f(x_i) = (n-2)\pi + \angle 1 + \angle 3 \geq (n-1)\pi. \qquad \natural$$

For opened m-lines with self-intersections, we know a result as in the following.

Theorem 4.2.3 Let (M,μ) be a planar map geometry. An m-line L in (M,μ) passing through n non-euclidean points $x_1, x_2, \cdots, x_n$ is opened only with l self-intersections if and only if there exist integers i_j and $s_{i_j}, 1 \leq j \leq l$ with $1 \leq i_j, s_{i,j} \leq n$ and $i_j \neq i_t$ if $t \neq j$ such that

$$(s_{i_j} - 2)\pi \ < \ \sum_{h=1}^{s_{i_j}} f(x_{i_j+h}) \ < \ (s_{i_j} - 1)\pi.$$

Proof If an m-line L passing through m-points $x_{t+1}, x_{t+2}, \cdots, x_{t+s_t}$ only has one self-intersection point, let us look at Fig.4.4 in where $x_{t+1}x_{t+s_t}$ is an added auxiliary m-line segment.

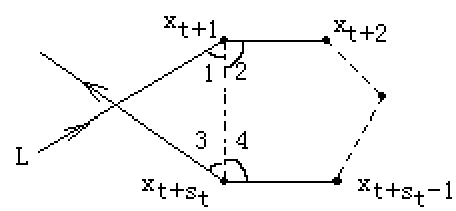


Fig.4.4

We know that

$$\angle 1 + \angle 2 = f(x_{t+1}) \text{ and } \angle 3 + \angle 4 = f(x_{t+s_t}).$$

Similar to the proof of Theorem 4.2.2, by Theorem 4.2.1 and the Euclid's fifth postulate, we know that

$$\angle 2 + \angle 4 + \sum_{j=2}^{s_t - 1} f(x_{t+j}) = (s_t - 2)\pi$$

and

$$\angle 1 + \angle 3 \ < \ \pi.$$

Whence, we get that

$$(s_t - 2)\pi \ < \ \sum_{j=1}^{s_t} f(x_{t+j}) \ < \ (s_t - 1)\pi.$$

Therefore, if L is opened only with l self-intersection points, we can find integers i_j and $s_{i_j}, 1 \leq j \leq l$ with $1 \leq i_j, s_{i,j} \leq n$ and $i_j \neq i_t$ if $t \neq j$ such that L passing through $x_{i_j+1}, x_{i_j+2}, \cdots, x_{i_j+s_j}$ only has one self-intersection point. By the previous discussion, we know that

$$(s_{i_j} - 2)\pi \ < \ \sum_{h=1}^{s_{i_j}} f(x_{i_j+h}) \ < \ (s_{i_j} - 1)\pi.$$

This completes the proof. ♮

Notice that all m-lines considered in this section are consisted by line segments or rays in an Euclid plane geometry. If the length of each line segment tends to zero, then we get a curve at the limitation in the usually sense. Whence, an m-line in a planar map geometry can be also seen as a discretization for plane curves and also has relation with differential equations. Readers interested in those materials can see in Chapter 5 for more details.

4.2.2. **Curvature of an m-line**

The curvature at a point of a curve C is a measure of how quickly the tangent vector changes direction with respect to the length of arc, such as those of the Gauss curvature, the Riemann curvature, $\cdots$, etc.. In Fig.4.5 we present a smooth curve and the changing of tangent vectors.

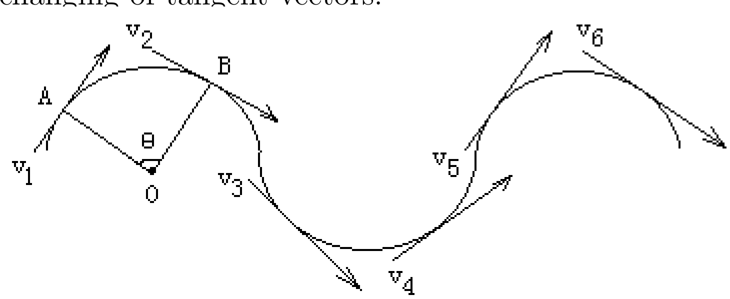


Fig.4.5

To measure the changing of vector v_1 to v_2, a simpler way is by the changing of the angle between vectors v_1 and v_2. If a curve $C = f(s)$ is smooth, then the changing rate of the angle between two tangent vector with respect to the length of arc, i.e., $\frac{df}{ds}$ is continuous. For example, as we known in the differential geometry, the Gauss curvature at every point of a circle $x^2 + y^2 = r^2$ of radius r is $\frac{1}{r}$. Whence, the changing of the angle from vectors v_1 to v_2 is

$$\int_A^B \frac{1}{r}ds = \frac{1}{r}|\widehat{AB}| = \frac{1}{r}r\theta = \theta.$$

By results in an Euclid plane geometry, we know that θ is also the angle between vectors v_1 and v_2. As we illustrated in Subsection 4.2.1, an m-line in a planar map geometry is consisted by line segments or rays. Therefore, the changing rate of the angle between two tangent vector with respect to the length of arc is not continuous. Similar to the definition of the set curvature in the reference [1], we present a discrete definition for the curvature of m-lines as follows.

Definition 4.2.1 *Let L be an m-line in a planar map geometry (M, μ) with the set W of non-euclidean points. The curvature $\omega(L)$ of L is defined by*

$$\omega(L) = \sum_{p \in W} (\pi - \varpi(p)),$$

where $\varpi(p) = f(p)$ if p is on an edge (u, v) in map M on a plane $\sum$ with an angle function $f: \sum \to \sum$.

In the classical differential geometry, the *Gauss mapping* and the *Gauss curvature* on surfaces are defined as follows:

Let $\mathcal{S} \subset R^3$ be a surface with an orientation $\mathbf{N}$. *The mapping $N : \mathcal{S} \to S^2$ takes its value in the unit sphere*

$$S^2 = \{(x, y, z) \in R^3 | x^2 + y^2 + z^2 = 1\}$$

along the orientation $\mathbf{N}$. *The map $N : \mathcal{S} \to S^2$, thus defined, is called a Gauss mapping and the determinant of $K(p) = d\mathbf{N}_p$ a Gauss curvature.*

We know that for a point $p \in \mathcal{S}$ such that the Gaussian curvature $K(p) \neq 0$ and a connected neighborhood V of p with K does not change sign,

$$K(p) = \lim_{A \to 0} \frac{N(A)}{A},$$

where A is the area of a region $B \subset V$ and $N(A)$ is the area of the image of B by the Gauss mapping $N : \mathcal{S} \to S^2$.

The well-known *Gauss-Bonnet theorem* for a compact surface says that

$$\int\int_{S} K d\sigma = 2\pi\chi(S),$$

for any orientable compact surface S.

For a simply closed m-line, we also have a result similar to the Gauss-Bonnet theorem, which can be also seen as a discrete Gauss-Bonnet theorem on a plane.

Theorem 4.2.4 *Let L be a simply closed m-line passing through n non-euclidean points $x_1, x_2, \cdots, x_n$ in a planar map geometry (M, μ). Then*

$$\omega(L) = 2\pi.$$

Proof According to Theorem 4.2.1, we know that

$$\sum_{i=1}^{n} f(x_i) = (n-2)\pi,$$

where $f(x_i)$ denotes the angle function value at an m-point $x_i, 1 \leq i \leq n$. Whence, by Definition 4.2.1 we know that

$$\begin{aligned} \omega(L) &= \sum_{i=1}^{n}(\pi - f(x_i)) \\ &= \pi n - \sum_{i=1}^{n} f(x_i) \\ &= \pi n - (n-2)\pi = 2\pi. \qquad \natural \end{aligned}$$

Similarly, we get a result for the sum of curvatures on the planar map M in a planar geometry (M, μ).

Theorem 4.2.6 *Let (M, μ) be a planar map geometry. Then the sum $\omega(M)$ of curvatures on edges in a map M is*

$$\omega(M) = 2\pi s(M),$$

where $s(M)$ denotes the sum of length of edges in M.

Proof Notice that the sum $\omega(u, v)$ of curvatures on an edge (u, v) of M is

$$\omega(u,v) = \int\limits_{v}^{u} (\pi - f(s))ds = \pi|\widehat{(u,v)}| - \int\limits_{v}^{u} f(s)ds.$$

Since M is a planar map, each of its edges appears just two times with an opposite direction. Whence, we get that

$$\begin{aligned}
\omega(M) &= \sum_{(u,v)\in E(M)} \omega(u,v) + \sum_{(v,u)\in E(M)} \omega(v,u) \\
&= \pi \sum_{(u,v)\in E(M)} (|\widehat{(u,v)}| + |\widehat{(v,u)}|) - (\int\limits_{v}^{u} f(s)ds + \int\limits_{u}^{v} f(s)ds) \\
&= 2\pi s(M) \qquad \natural
\end{aligned}$$

Notice that if we assume $s(M) = 1$, then Theorem 4.2.6 turns to the Gauss-bonnet theorem for a sphere. Similarly, if we consider general map geometry on an orientable surface, similar results can be also obtained such as those materials in Problem 4.7.8 and Conjecture 4.7.1 in the final section of this chapter.

§4.3 Polygons in a Planar Map Geometry

4.3.1. Existence

In an Euclid plane geometry, we have encountered triangles, quadrilaterals, $\cdots$, and generally, n-polygons, i.e., these graphs on a plane with n straight line segments not on the same line connected with one after another. There are no 1 and 2-polygons in an Euclid plane geometry since every point is euclidean. The definition of n-polygons in a planar map geometry (M, μ) is similar to that of an Euclid plane geometry.

Definition 4.3.1 *An n-polygon in a planar map geometry (M, μ) is defined to be a graph on (M, μ) with n m-line segments two by two without self-intersections and connected with one after another.*

Although their definition is similar, the situation is more complex in a planar map geometry (M, μ). We have found a necessary and sufficient condition for 1-polygon in Theorem 4.2.1, i.e., 1-polygons maybe exist in a planar map geometry. In general, we can find n-polygons in a planar map geometry for any integer $n, n \geq 1$.

Examples of polygon in a planar map geometry (M,μ) are shown in Fig.4.6, in where (a) is a 1-polygon with u,v,w and t being non-euclidean points, (b) is a 2-polygon with vertices A,B and non-euclidean points u,v, (c) is a triangle with vertices A,B,C and a non-euclidean point u and (d) is a quadrilateral with vertices A,B,C and D.

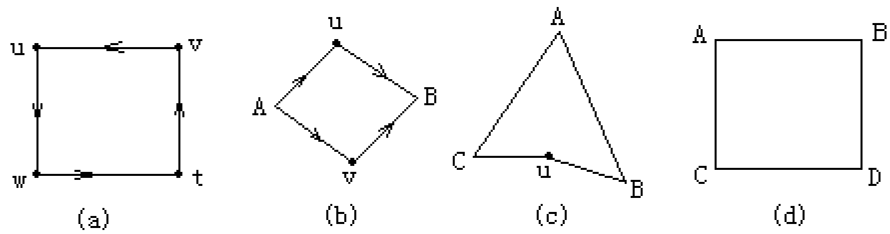


Fig.4.6

Theorem 4.3.1 *There exists a* 1*-polygon in a planar map geometry* (M,μ) *if and only if there are non-euclidean points* $u_1,u_2,\cdots,u_l$ *with* $l \geq 3$ *such that*

$$\sum_{i=1}^{l} f(u_i) = (l-2)\pi,$$

where $f(u_i)$ *denotes the angle function value at the point* u_i*,* $1 \leq i \leq l$*.*

Proof According to Theorem 4.2.1, an m-line passing through l non-euclidean points $u_1,u_2,\cdots,u_l$ is simply closed if and only if

$$\sum_{i=1}^{l} f(u_i) = (l-2)\pi,$$

i.e., 1-polygon exists in (M,μ) if and only if there are non-euclidean points $u_1,u_2,\cdots,u_l$ with the above formula hold.

Whence, we only need to prove $l \geq 3$. Since there are no 1-polygons or 2-polygons in an Euclid plane geometry, we must have $l \geq 3$ by the Hilbert's axiom $I-2$. In fact, for $l=3$ we can really find a planar map geometry (M,μ) with a 1-polygon passing through three non-euclidean points u,v and w. Look at Fig.4.7,

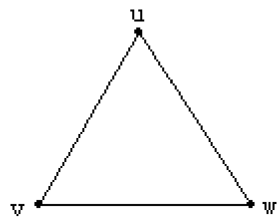


Fig.4.7

in where the angle function values are $f(u) = f(v) = f(w) = \frac{2}{3}\pi$ at u, v and w. ♮

Similarly, for 2-polygons we get the following result.

Theorem 4.3.2 *There are* 2*-polygons in a planar map geometry* (M, μ) *only if there are at least one non-euclidean point in* (M, μ)*.*

Proof In fact, if there is a non-euclidean point u in (M, μ), then each straight line enter u will turn an angle $\theta = \pi - \frac{f(u)}{2}$ or $\frac{f(u)}{2} - \pi$ from the initial straight line dependent on that u is elliptic or hyperbolic. Therefore, we can get a 2-polygon in (M, μ) by choice a straight line AB passing through euclidean points in (M, μ), such as the graph shown in Fig.4.8.

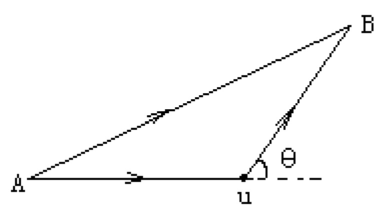


Fig.4.8

This completes the proof. ♮

For the existence of n-polygons with $n \geq 3$, we have a general result as in the following.

Theorem 4.3.3 *For any integer* $n, n \geq 3$*, there are* n*-polygons in a planar map geometry* (M, μ)*.*

Proof Since in an Euclid plane geometry, there are n-polygons for any integer $n, n \geq 3$. Therefore, there are also n-polygons in a planar map geometry (M, μ) for any integer $n, n \geq 3$. ♮

4.3.2. **Sum of internal angles**

For the sum of the internal angles in an n-polygon, we have the following result.

Theorem 4.3.4 *Let $\prod$ be an n-polygon in a map geometry with its edges passing through non-euclidean points $x_1, x_2, \cdots, x_l$. Then the sum of internal angles in $\prod$ is*

$$(n+l-2)\pi - \sum_{i=1}^{l} f(x_i),$$

where $f(x_i)$ denotes the value of the angle function f at the point $x_i, 1 \leq i \leq l$.

Proof Denote by U, V the sets of elliptic points and hyperbolic points in $x_1, x_2, \cdots, x_l$ and $|U| = p, |V| = q$, respectively. If an m-line segment passes through an elliptic point u, add an auxiliary line segment AB in the plane as shown in Fig.4.9(1). Then we get that

$$\angle \mathrm{a} = \angle 1 + \angle 2 = \pi - f(u).$$

If an m-line passes through a hyperbolic point v, also add an auxiliary line segment AB in the plane as that shown in Fig.4.9(2). Then we get that

$$\text{angle } b = \text{angle} 3 + \text{angle} 4 = f(v) - \pi.$$

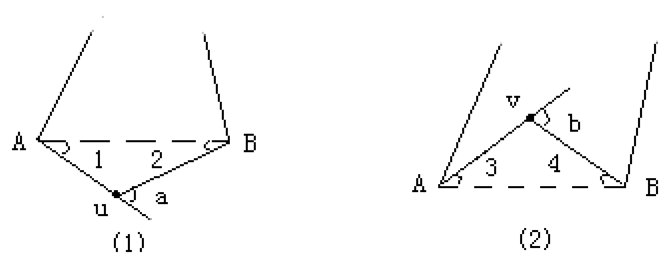


Fig.4.9

Since the sum of internal angles of an n-polygon in a plane is $(n-2)\pi$ whenever it is a convex or concave polygon, we know that the sum of the internal angles in $\prod$ is

$$(n-2)\pi + \sum_{x \in U} (\pi - f(x)) - \sum_{y \in V} (f(y) - \pi)$$

$$= (n+p+q-2)\pi - \sum_{i=1}^{l} f(x_i)$$
$$= (n+l-2)\pi - \sum_{i=1}^{l} f(x_i).$$

This completes the proof. ♮

A triangle is called *euclidean, elliptic* or *hyperbolic* if its edges only pass through one kind of euclidean, elliptic or hyperbolic points. As a consequence of Theorem 4.3.4, we get the sum of the internal angles of a triangle in a map geometry which is consistent with these already known results .

Corollary 4.3.1 *Let $\triangle$ be a triangle in a planar map geometry (M,μ). Then*

(i) the sum of its internal angles is equal to π if $\triangle$ is euclidean;

(ii) the sum of its internal angles is less than π if $\triangle$ is elliptic;

(iii) the sum of its internal angles is more than π if $\triangle$ is hyperbolic.

Proof Notice that the sum of internal angles of a triangle is

$$\pi + \sum_{i=1}^{l}(\pi - f(x_i))$$

if it passes through non-euclidean points $x_1, x_2, \cdots, x_l$. By definition, if these $x_i, 1 \leq i \leq l$ are one kind of euclidean, elliptic, or hyperbolic, then we have that $f(x_i) = \pi$, or $f(x_i) < \pi$, or $f(x_i) > \pi$ for any integer $i, 1 \leq i \leq l$. Whence, the sum of internal angles of an euclidean, elliptic or hyperbolic triangle is equal to, or lees than, or more than π. ♮

4.3.3. **Area of a polygon**

As it is well-known, calculation for the area $A(\triangle)$ of a triangle $\triangle$ with two sides a, b and the value of their include angle θ or three sides a, b and c in an Euclid plane is simple. Formulae for its area are

$$A(\triangle) = \frac{1}{2}ab\sin\theta \text{ or } A(\triangle) = \sqrt{s(s-a)(s-b)(s-c)},$$

where $s = \frac{1}{2}(a+b+c)$. But in a planar map geometry, calculation for the area of a triangle is complex since each of its edge maybe contains non-euclidean points.

Where, we only present a programming for calculation the area of a triangle in a planar map geometry.

STEP 1 *Divide a triangle into triangles in an Euclid plane such that no edges contain non-euclidean points unless their endpoints;*

STEP 2 *Calculate the area of each triangle;*

STEP 3 *Sum up all of areas of these triangles to get the area of the given triangle in a planar map geometry.*

The simplest cases for triangle is the cases with only one non-euclidean point such as those shown in Fig.4.10(1) and (2) with an elliptic point u or with a hyperbolic point v.

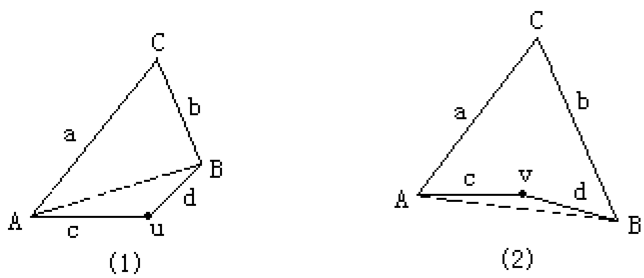


Fig.4.10

Add an auxiliary line segment AB in Fig.4.10. Then by formulae in the plane trigonometry, we know that

$$A(\triangle ABC) = \sqrt{s_1(s_1-a)(s_1-b)(s_1-t)} + \sqrt{s_2(s_2-c)(s_2-d)(s_2-t)}$$

for case (1) in Fig.4.10 and

$$A(\triangle ABC) = \sqrt{s_1(s_1-a)(s_1-b)(s_1-t)} - \sqrt{s_2(s_2-c)(s_2-d)(s_2-t)}$$

for case (2) in Fig.4.10, where

$$t = \sqrt{c^2 + d^2 - 2cd\cos\frac{f(x)}{2}}$$

with $x = u$ or v and

$$s_1 = \frac{1}{2}(a+b+t), \quad s_2 = \frac{1}{2}(c+d+t).$$

Generally, let $\triangle ABC$ be a triangle with its edge AB passing through p elliptic or p hyperbolic points $x_1, x_2, \cdots, x_p$ simultaneously, as those shown in Fig.4.11(1) and (2).

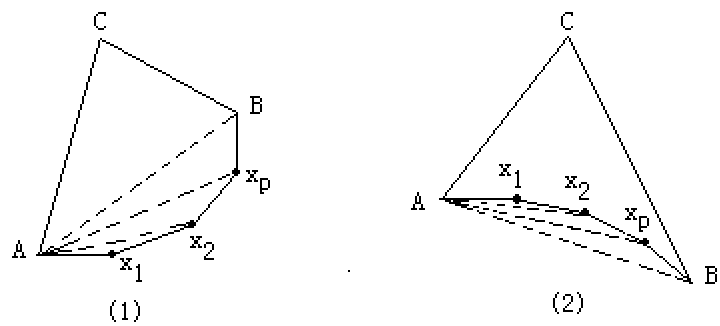


Fig.4.11

Where $|AC| = a$, $|BC| = b$ and $|Ax_1| = c_1$, $|x_1x_2| = c_2, \cdots, |x_{p-1}x_p| = c_p$ and $|x_pB| = c_{p+1}$. Adding auxiliary line segments $Ax_2, Ax_3, \cdots, Ax_p, AB$ in Fig.4.11, then we can find its area by the programming STEP 1 to STEP 3. By formulae in the plane trigonometry, we get that

$$|Ax_2| = \sqrt{c_1^2 + c_2^2 - 2c_1c_2 \cos\frac{f(x_1)}{2}},$$

$$\angle Ax_2x_1 = \cos^{-1}\frac{c_1^2 - c_2^2 - |Ax_1|^2}{2c_2|Ax_2|},$$

$$\angle Ax_2x_3 = \frac{f(x_2)}{2} - \angle Ax_2x_1 \quad \text{or} \quad 2\pi - \frac{f(x_2)}{2} - \angle Ax_2x_1,$$

$$|Ax_3| = \sqrt{|Ax_2|^2 + c_3^2 - 2|Ax_2|c_3\cos(\frac{f(x_2)}{2} - \angle Ax_2x_3)},$$

$$\angle Ax_3x_2 = \cos^{-1}\frac{|Ax_2|^2 - c_3^2 - |Ax_3|^2}{2c_3|Ax_3|},$$

$$\angle Ax_2x_3 = \frac{f(x_3)}{2} - \angle Ax_3x_2 \quad \text{or} \quad 2\pi - \frac{f(x_3)}{2} - \angle Ax_3x_2,$$

. . . .

and generally, we get that

$$|AB| = \sqrt{|Ax_p|^2 + c_{p+1}^2 - 2|Ax_p|c_{p+1}\cos\angle Ax_pB}.$$

Then the area of the triangle $\triangle ABC$ is

$$\begin{aligned} A(\triangle ABC) &= \sqrt{s_p(s_p-a)(s_p-b)(s_p-|AB|)} \\ &+ \sum_{i=1}^{p}\sqrt{s_i(s_i-|Ax_i|)(s_i-c_{i+1})(s_i-|Ax_{i+1}|)} \end{aligned}$$

for case (1) in Fig.4.11 and

$$\begin{aligned} A(\triangle ABC) &= \sqrt{s_p(s_p-a)(s_p-b)(s_p-|AB|)} \\ &- \sum_{i=1}^{p}\sqrt{s_i(s_i-|Ax_i|)(s_i-c_{i+1})(s_i-|Ax_{i+1}|)} \end{aligned}$$

for case (2) in Fig.4.11, where for any integer $i, 1 \leq i \leq p-1$,

$$s_i = \frac{1}{2}(|Ax_i| + c_{i+1} + |Ax_{i+1}|)$$

and

$$s_p = \frac{1}{2}(a + b + |AB|).$$

Certainly, this programming can be also applied to calculate the area of an n-polygon in a planar map geometry in general.

§4.4 Circles in a Planar Map Geometry

The length of an m-line segment in a planar map geometry is defined in the following definition.

Definition 4.4.1 *The length $|AB|$ of an m-line segment AB consisted by k straight line segments AC_1, C_1C_2, $C_2C_3, \cdots, C_{k-1}B$ in a planar map geometry (M,μ) is defined by*

$$|AB| = |AC_1| + |C_1C_2| + |C_2C_3| + \cdots + |C_{k-1}B|.$$

As that shown in Chapter 3, there are not always exist a circle with any center and a given radius in a planar map geometry in the sense of the Euclid's definition. Since we have introduced angle function on a planar map geometry, we can likewise the Euclid's definition to define an m-circle in a planar map geometry in the next definition.

Definition 4.4.2 *A closed curve C without self-intersection in a planar map geometry (M, μ) is called an m-circle if there exists an m-point O in (M, μ) and a real number r such that $|OP| = r$ for each m-point P on C.*

Two Examples for m-circles in a planar map geometry (M, μ) are shown in Fig.4.12(1) and (2). The m-circle in Fig.4.12(1) is a circle in the Euclid's sense, but (2) is not. Notice that in Fig.4.12(2), m-points u and v are elliptic and the length $|OQ| = |Ou| + |uQ| = r$ for an m-point Q on the m-circle C, which seems likely an ellipse but it is not. The m-circle C in Fig.4.12(2) also implied that m-circles are more complex than those in an Euclid plane geometry.

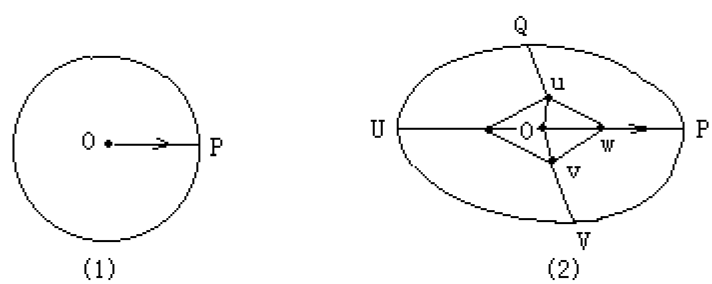


Fig.4.12

We have a necessary and sufficient condition for the existence of an m-circle in a planar map geometry.

Theorem 4.4.1 *Let (M, μ) be a planar map geometry on a plane $\sum$ and O an m-point on (M, μ). For a real number r, there is an m-circle of radius r with center O if and only if O is in a non-outer face of M or O is in the outer face of M but for any $\epsilon, r > \epsilon > 0$, the initial and final intersection points of a circle of radius ϵ with M in an Euclid plane $\sum$ are euclidean points.*

Proof If there is a solitary non-euclidean point A with $|OA| < r$, then by those materials in Chapter 3, there are no m-circles in (M, μ) of radius r with center O.

Now if O is in the outer face of M but there exists a number $\epsilon, r > \epsilon > 0$ such that one of the initial and final intersection points of a circle of radius ϵ with M on $\sum$ is non-euclidean point, then points with distance r to O in (M, μ) at least has a gap in a circle with an Euclid sense. See Fig.4.13 for details, in where u is a non-euclidean point and the shade field denotes the map M. Therefore there are no m-circles in (M, μ) of radius r with center O.

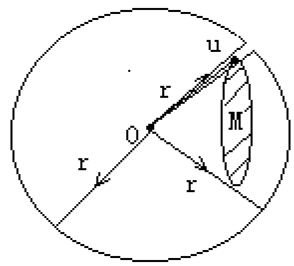


Fig.4.13

Now if O in the outer face of M but for any $\epsilon, r > \epsilon > 0$, the initial and final intersection points of a circle of radius ϵ with M on $\sum$ are euclidean points or O is in a non-outer face of M, then by the definition of angle functions, we know that all points with distance r to O is a closed smooth curve on $\sum$, for example, see Fig.4.14(1) and (2).

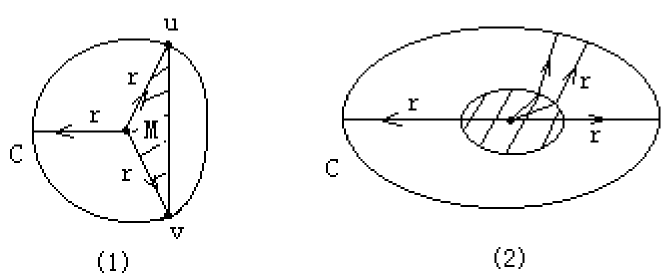


Fig.4.14

Whence it is an m-circle. ♮

We construct a polar axis OX with center O in a planar map geometry as that in an Euclid geometry. Then each m-point A has a coordinate (ρ, θ), where ρ is the length of the m-line segment OA and θ is the angle between OX and the straight line segment of OA containing the point A. We get an equation for an m-circle of radius r which has the same form as that in the analytic geometry of plane.

Theorem 4.4.2 *In a planar geometry (M,μ) with a polar axis OX of center O, the equation of each m-circle of radius r with center O is*

$$\rho = r.$$

Proof By the definition of an m-circle C of radius r, every m-point on C has a distance r to its center O. Whence, its equation is $\rho = r$ in a planar map geometry with a polar axis OX of center O. $\natural$

§4.5 Line Bundles in a Planar Map Geometry

The behaviors of m-line bundles is need to clarify from a geometrical sense. Among those m-line bundles the most important is parallel bundles defined in the next definition, which is also motivated by the Euclid's fifth postulate discussed in the reference [54] first.

Definition 4.5.1 *A family $\mathcal{L}$ of infinite m-lines not intersecting each other in a planar geometry (M,μ) is called a parallel bundle.*

In Fig.4.15, we present all cases of parallel bundles passing through an edge in planar geometries, where, (a) is the case with the same type points u, v and $\rho_M(u)\mu(u) = \rho_M(v)\mu(v) = 2\pi$, (b) and (c) are the same type cases with $\rho_M(u)\mu(u) > \rho_M(v)\mu(v)$ or $\rho_M(u)\mu(u) = \rho_M(v)\mu(v) > 2\pi$ or $< 2\pi$ and (d) is the case with an elliptic point u but a hyperbolic point v.

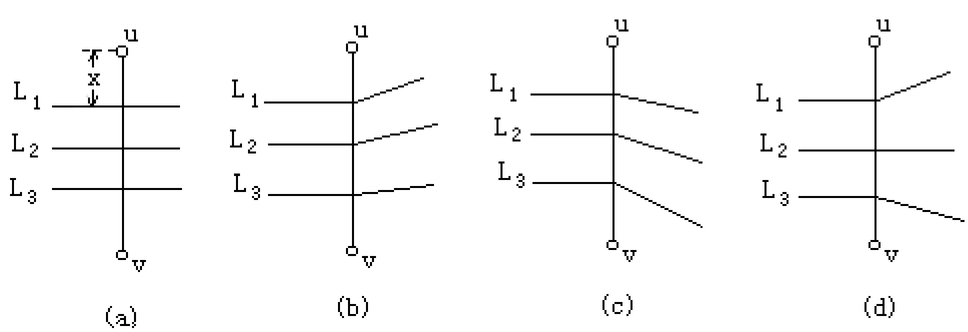


Fig.4.15

Here, we assume the angle at the intersection point is in clockwise, that is, a line passing through an elliptic point will bend up and passing through a hyperbolic point will bend down, such as those cases (b),(c) in the Fig.4.15. Generally, we define a *sign function $sign(f)$ of an angle function f* as follows.

Definition 4.5.2 *For a vector $\overrightarrow{O}$ on the Euclid plane called an orientation, a sign function $sign(f)$ of an angle function f at an m-point u is defined by*

$$sign(f)(u) = \begin{cases} 1, & \text{if } u \text{ is elliptic}, \\ 0, & \text{if } u \text{ is euclidean}, \\ -1, & \text{if } u \text{ is hyperbolic}. \end{cases}$$

We classify parallel bundles in planar map geometries along an orientation $\overrightarrow{O}$ in this section.

4.5.1. **A condition for parallel bundles**

We investigate the behaviors of parallel bundles in a planar map geometry (M, μ). Denote by $f(x)$ the angle function value at an intersection m-point of an m-line L with an edge (u, v) of M and a distance x to u on (u, v) as shown in Fig.4.15(a). Then we get an elementary result as in the following.

Theorem 4.5.1 *A family $\mathcal{L}$ of parallel m-lines passing through an edge (u, v) is a parallel bundle if and only if*

$$\left.\frac{df}{dx}\right|_{+} \geq 0.$$

Proof If $\mathcal{L}$ is a parallel bundle, then any two m-lines L_1, L_2 will not intersect after them passing through the edge uv. Therefore, if θ_1, θ_2 are the angles of L_1, L_2 at the intersection m-points of L_1, L_2 with (u, v) and L_2 is far from u than L_1, then we know $\theta_2 \geq \theta_1$. Thereby we know that

$$f(x + \Delta x) - f(x) \geq 0$$

for any point with distance x from u and $\Delta x > 0$. Therefore, we get that

$$\left.\frac{df}{dx}\right|_+ = \lim_{\Delta x \to +0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \geq 0.$$

As that shown in the Fig.4.15.

Now if $\left.\frac{df}{dx}\right|_+ \geq 0$, then $f(y) \geq f(x)$ if $y \geq x$. Since $\mathcal{L}$ is a family of parallel m-lines before meeting uv, any two m-lines in $\mathcal{L}$ will not intersect each other after them passing through (u, v). Therefore, $\mathcal{L}$ is a parallel bundle. ♮

A general condition for a family of parallel m-lines passing through a cut of a planar map being a parallel bundle is the following.

Theorem 4.5.2 *Let (M, μ) be a planar map geometry, $C = \{(u_1, v_1), (u_2, v_2), \cdots, (u_l, v_l)\}$ a cut of the map M with order $(u_1, v_1), (u_2, v_2), \cdots, (u_l, v_l)$ from the left to the right, $l \geq 1$ and the angle functions on them are $f_1, f_2, \cdots, f_l$ (also seeing Fig.*4.16*), respectively.*

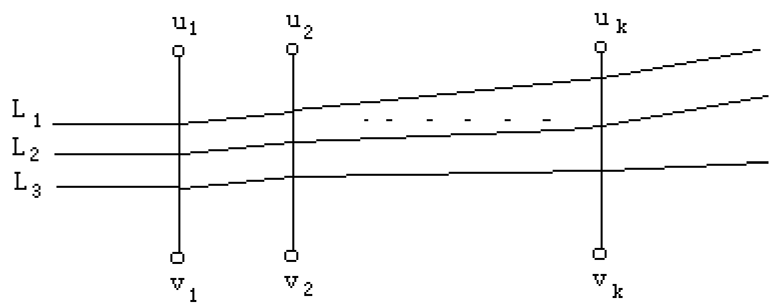


Fig.4.16

Then a family $\mathcal{L}$ of parallel m-lines passing through C is a parallel bundle if and only if for any $x, x \geq 0$,

$$\begin{aligned}
sign(f_1)(x)f'_{1+}(x) &\geq 0 \\
sign(f_1)(x)f'_{1+}(x) + sign(f_2)(x)f'_{2+}(x) &\geq 0 \\
sign(f_1)(x)f'_{1+}(x) + sign(f_2)(x)f'_{2+}(x) + sign(f_3)(x)f'_{3+}(x) &\geq 0 \\
\cdots\cdots\cdots\cdots \\
sign(f_1)(x)f'_{1+}(x) + sign(f_2)(x)f'_{2+}(x) + \cdots + sign(f_l)(x)f'_{l+}(x) &\geq 0.
\end{aligned}$$

Proof According to Theorem 4.5.1, we know that m-lines will not intersect after them passing through (u_1, v_1) and (u_2, v_2) if and only if for $\forall \Delta x > 0$ and $x \geq 0$,

$$sign(f_2)(x)f_2(x+\Delta x) + sign(f_1)(x)f'_{1+}(x)\Delta x \geq sign(f_2)(x)f_2(x),$$

seeing Fig.4.17 for an explanation.

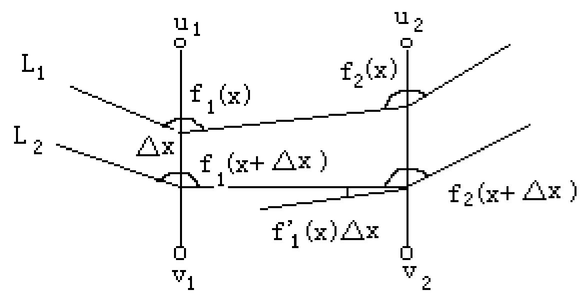


Fig.4.17

That is,

$$sign(f_1)(x)f'_{1+}(x) + sign(f_2)(x)f'_{2+}(x) \geq 0.$$

Similarly, m-lines will not intersect after them passing through $(u_1, v_1), (u_2, v_2)$ and (u_3, v_3) if and only if for $\forall \Delta x > 0$ and $x \geq 0$,

$$sign(f_3)(x)f_3(x+\Delta x) + sign(f_2)(x)f'_{2+}(x)\Delta x$$
$$+sign(f_1)(x)f'_{1+}(x)\Delta x \geq sign(f_3)(x)f_3(x).$$

That is,

$$sign(f_1)(x)f'_{1+}(x) + sign(f_2)(x)f'_{2+}(x) + sign(f_3)(x)f'_{3+}(x) \geq 0.$$

Generally, m-lines will not intersect after them passing through $(u_1, v_1), (u_2, v_2), \cdots,$ (u_{l-1}, v_{l-1}) and (u_l, v_l) if and only if for $\forall \Delta x > 0$ and $x \geq 0$,

$$sign(f_l)(x)f_l(x+\Delta x) + sign(f_{l-1})(x)f'_{l-1+}(x)\Delta x +$$
$$\cdots + sign(f_1)(x)f'_{1+}(x)\Delta x \geq sign(f_l)(x)f_l(x).$$

Whence, we get that

$$sign(f_1)(x)f'_{1+}(x) + sign(f_2)(x)f'_{2+}(x) + \cdots + sign(f_l)(x)f'_{l+}(x) \geq 0.$$

Therefore, a family $\mathcal{L}$ of parallel m-lines passing through C is a parallel bundle if and only if for any $x, x \geq 0$, we have that

$$\begin{aligned}
sign(f_1)(x)f'_{1+}(x) &\geq 0 \\
sign(f_1)(x)f'_{1+}(x) + sign(f_2)(x)f'_{2+}(x) &\geq 0 \\
sign(f_1)(x)f'_{1+}(x) + sign(f_2)(x)f'_{2+}(x) + sign(f_1)(x)f'_{3+}(x) &\geq 0 \\
\cdots\cdots\cdots\cdots \\
sign(f_1)(x)f'_{1+}(x) + sign(f_2)(x)f'_{2+}(x) + \cdots + sign(f_1)(x)f'_{l+}(x) &\geq 0.
\end{aligned}$$

This completes the proof. $\natural$.

Corollary 4.5.1 *Let (M, μ) be a planar map geometry, $C = \{(u_1, v_1), (u_2, v_2), \cdots, (u_l, v_l)\}$ a cut of the map M with order $(u_1, v_1), (u_2, v_2), \cdots, (u_l, v_l)$ from the left to the right, $l \geq 1$ and the angle functions on them are $f_1, f_2, \cdots, f_l$, respectively. Then a family $\mathcal{L}$ of parallel lines passing through C is still parallel lines after them leaving C if and only if for any $x, x \geq 0$,*

$$\begin{aligned}
sign(f_1)(x)f'_{1+}(x) &\geq 0 \\
sign(f_1)(x)f'_{1+}(x) + sign(f_2)(x)f'_{2+}(x) &\geq 0 \\
sign(f_1)(x)f'_{1+}(x) + sign(f_2)(x)f'_{2+}(x) + sign(f_1)(x)f'_{3+}(x) &\geq 0 \\
\cdots\cdots\cdots\cdots \\
sign(f_1)(x)f'_{1+}(x) + sign(f_2)(x)f'_{2+}(x) + \cdots + sign(f_1)(x)f'_{l-1+}(x) &\geq 0.
\end{aligned}$$

and

$$sign(f_1)(x)f'_{1+}(x) + sign(f_2)(x)f'_{2+}(x) + \cdots + sign(f_1)(x)f'_{l+}(x) = 0.$$

Proof According to Theorem 4.5.2, we know the condition is a necessary and sufficient condition for $\mathcal{L}$ being a parallel bundle. Now since lines in $\mathcal{L}$ are parallel lines after them leaving C if and only if for any $x \geq 0$ and $\Delta x \geq 0$, there must be that

$$sign(f_l)f_l(x+\Delta x)+sign(f_{l-1})f'_{l-1+}(x)\Delta x+\cdots+sign(f_1)f'_{1+}(x)\Delta x = sign(f_l)f_l(x).$$

Therefore, we get that

$$sign(f_1)(x)f'_{1+}(x) + sign(f_2)(x)f'_{2+}(x) + \cdots + sign(f_1)(x)f'_{l+}(x) = 0. \qquad \natural$$

When do some parallel m-lines parallel the initial parallel lines after them passing through a cut C in a planar map geometry? The answer is in the next result.

Theorem 4.5.3 *Let (M, μ) be a planar map geometry, $C = \{(u_1, v_1), (u_2, v_2), \cdots, (u_l, v_l)\}$ a cut of the map M with order $(u_1, v_1), (u_2, v_2), \cdots, (u_l, v_l)$ from the left to the right, $l \geq 1$ and the angle functions on them are $f_1, f_2, \cdots, f_l$, respectively. Then the parallel m-lines parallel the initial parallel lines after them passing through C if and only if for $\forall x \geq 0$,*

$$\begin{aligned}
sign(f_1)(x)f'_{1+}(x) &\geq 0 \\
sign(f_1)(x)f'_{1+}(x) + sign(f_2)(x)f'_{2+}(x) &\geq 0 \\
sign(f_1)(x)f'_{1+}(x) + sign(f_2)(x)f'_{2+}(x) + sign(f_1)(x)f'_{3+}(x) &\geq 0 \\
\cdots\cdots\cdots\cdots \\
sign(f_1)(x)f'_{1+}(x) + sign(f_2)(x)f'_{2+}(x) + \cdots + sign(f_1)(x)f'_{l-1+}(x) &\geq 0.
\end{aligned}$$

and

$$sign(f_1)f_1(x) + sign(f_2)f_2(x) + \cdots + sign(f_1)(x)f_l(x) = l\pi.$$

Proof According to Theorem 4.5.2 and Corollary 4.5.1, we know that these parallel m-lines satisfying conditions of this theorem is a parallel bundle.

We calculate the angle $\alpha(i, x)$ of an m-line L passing through an edge $u_iv_i, 1 \leq i \leq l$ with the line before it meeting C at the intersection of L with the edge

(u_i, v_i), where x is the distance of the intersection point to u_1 on (u_1, v_1), see also Fig.4.18. By definition, we know the angle $\alpha(1, x) = sign(f_1)f(x)$ and $\alpha(2, x) = sign(f_2)f_2(x) - (\pi - sign(f_1)f_1(x)) = sign(f_1)f_1(x) + sign(f_2)f_2(x) - \pi$.

Now if $\alpha(i, x) = sign(f_1)f_1(x) + sign(f_2)f_2(x) + \cdots + sign(f_i)f_i(x) - (i-1)\pi$, then we know that $\alpha(i+1, x) = sign(f_{i+1})f_{i+1}(x) - (\pi - \alpha(i, x)) = sign(f_{i+1})f_{i+1}(x) + \alpha(i, x) - \pi$ similar to the case $i = 2$. Thereby we get that

$$\alpha(i+1, x) = sign(f_1)f_1(x) + sign(f_2)f_2(x) + \cdots + sign(f_{i+1})f_{i+1}(x) - i\pi.$$

Notice that an m-line L parallel the initial parallel line after it passing through C if and only if $\alpha(l, x) = \pi$, i.e.,

$$sign(f_1)f_1(x) + sign(f_2)f_2(x) + \cdots + sign(f_l)f_l(x) = l\pi.$$

This completes the proof. ♮

4.5.2. **Linear conditions and combinatorial realization for parallel bundles**

For the simplicity, we can assume even that the function $f(x)$ is linear and denoted it by $f_l(x)$. We calculate $f_l(x)$ in the first.

Theorem 4.5.4 *The angle function $f_l(x)$ of an m-line L passing through an edge (u, v) at a point with distance x to u is*

$$f_l(x) = (1 - \frac{x}{d(u,v)})\frac{\rho(u)\mu(v)}{2} + \frac{x}{d(u,v)}\frac{\rho(v)\mu(v)}{2},$$

where, $d(u, v)$ is the length of the edge (u, v).

Proof Since $f_l(x)$ is linear, we know that $f_l(x)$ satisfies the following equation.

$$\frac{f_l(x) - \frac{\rho(u)\mu(u)}{2}}{\frac{\rho(v)\mu(v)}{2} - \frac{\rho(u)\mu(u)}{2}} = \frac{x}{d(u,v)},$$

Calculation shows that

$$f_l(x) = (1 - \frac{x}{d(u,v)})\frac{\rho(u)\mu(v)}{2} + \frac{x}{d(u,v)}\frac{\rho(v)\mu(v)}{2}. \quad ♮$$

Corollary 4.5.2 *Under the linear assumption, a family $\mathcal{L}$ of parallel m-lines passing through an edge (u,v) is a parallel bundle if and only if*

$$\frac{\rho(u)}{\rho(v)} \leq \frac{\mu(v)}{\mu(u)}.$$

Proof According to Theorem 4.5.1, a family of parallel m-lines passing through an edge (u,v) is a parallel bundle if and only if $f'(x) \geq 0$ for $\forall x, x \geq 0$, i.e.,

$$\frac{\rho(v)\mu(v)}{2d(u,v)} - \frac{\rho(u)\mu(u)}{2d(u,v)} \geq 0.$$

Therefore, a family $\mathcal{L}$ of parallel m-lines passing through an edge (u,v) is a parallel bundle if and only if

$$\rho(v)\mu(v) \geq \rho(u)\mu(u).$$

Whence,

$$\frac{\rho(u)}{\rho(v)} \leq \frac{\mu(v)}{\mu(u)}. \qquad \natural$$

For a family of parallel m-lines passing through a cut, we get the following condition.

Theorem 4.5.5 *Let (M,μ) be a planar map geometry, $C = \{(u_1,v_1),(u_2,v_2),\cdots,(u_l,v_l)\}$ a cut of the map M with order $(u_1,v_1),(u_2,v_2),\cdots,(u_l,v_l)$ from the left to the right, $l \geq 1$. Then under the linear assumption, a family L of parallel m-lines passing through C is a parallel bundle if and only if the angle factor μ satisfies the following linear inequality system*

$$\rho(v_1)\mu(v_1) \geq \rho(u_1)\mu(u_1)$$

$$\frac{\rho(v_1)\mu(v_1)}{d(u_1,v_1)} + \frac{\rho(v_2)\mu(v_2)}{d(u_2,v_2)} \geq \frac{\rho(u_1)\mu(u_1)}{d(u_1,v_1)} + \frac{\rho(u_2)\mu(u_2)}{d(u_2,v_2)}$$

$$\cdots\cdots\cdots\cdots$$

$$\begin{aligned}\frac{\rho(v_1)\mu(v_1)}{d(u_1,v_1)} &+ \frac{\rho(v_2)\mu(v_2)}{d(u_2,v_2)}+\cdots+\frac{\rho(v_l)\mu(v_l)}{d(u_l,v_l)}\\ &\geq \frac{\rho(u_1)\mu(u_1)}{d(u_1,v_1)}+\frac{\rho(u_2)\mu(u_2)}{d(u_2,v_2)}+\cdots+\frac{\rho(u_l)\mu(u_l)}{d(u_l,v_l)}.\end{aligned}$$

Proof Under the linear assumption, for any integer $i, i \geq 1$ we know that

$$f'_{i+}(x) = \frac{\rho(v_i)\mu(v_i) - \rho(u_i)\mu(u_i)}{2d(u_i,v_i)}$$

by Theorem 4.5.4. Thereby according to Theorem 4.5.2, we get that a family L of parallel m-lines passing through C is a parallel bundle if and only if the angle factor μ satisfies the following linear inequality system

$$\rho(v_1)\mu(v_1) \geq \rho(u_1)\mu(u_1)$$

$$\frac{\rho(v_1)\mu(v_1)}{d(u_1,v_1)}+\frac{\rho(v_2)\mu(v_2)}{d(u_2,v_2)} \geq \frac{\rho(u_1)\mu(u_1)}{d(u_1,v_1)}+\frac{\rho(u_2)\mu(u_2)}{d(u_2,v_2)}$$

$$\cdots\cdots\cdots\cdots$$

$$\begin{aligned}\frac{\rho(v_1)\mu(v_1)}{d(u_1,v_1)} &+ \frac{\rho(v_2)\mu(v_2)}{d(u_2,v_2)}+\cdots+\frac{\rho(v_l)\mu(v_l)}{d(u_l,v_l)}\\ &\geq \frac{\rho(u_1)\mu(u_1)}{d(u_1,v_1)}+\frac{\rho(u_2)\mu(u_2)}{d(u_2,v_2)}+\cdots+\frac{\rho(u_l)\mu(u_l)}{d(u_l,v_l)}.\end{aligned}$$

This completes the proof. $\natural$

For planar maps underlying a regular graph, we have an interesting consequence for parallel bundles in the following.

Corollary 4.5.3 *Let (M,μ) be a planar map geometry with M underlying a regular graph, $C = \{(u_1,v_1),(u_2,v_2),\cdots,(u_l,v_l)\}$ a cut of the map M with order $(u_1,v_1),(u_2,v_2),\cdots,(u_l,v_l)$ from the left to the right, $l \geq 1$. Then under the linear assumption, a family L of parallel lines passing through C is a parallel bundle if and only if the angle factor μ satisfies the following linear inequality system.*

$$\mu(v_1) \geq \mu(u_1)$$

$$\frac{\mu(v_1)}{d(u_1,v_1)} + \frac{\mu(v_2)}{d(u_2,v_2)} \geq \frac{\mu(u_1)}{d(u_1,v_1)} + \frac{\mu(u_2)}{d(u_2,v_2)}$$

$$\cdots\cdots\cdots\cdots$$

$$\frac{\mu(v_1)}{d(u_1,v_1)} + \frac{\mu(v_2)}{d(u_2,v_2)} + \cdots + \frac{\mu(v_l)}{d(u_l,v_l)} \geq \frac{\mu(u_1)}{d(u_1,v_1)} + \frac{\mu(u_2)}{d(u_2,v_2)} + \cdots + \frac{\mu(u_l)}{d(u_l,v_l)}$$

and particularly, if assume that all the lengths of edges in C are the same, then

$$\begin{aligned} \mu(v_1) &\geq \mu(u_1) \\ \mu(v_1)+\mu(v_2) &\geq \mu(u_1)+\mu(u_2) \\ \cdots\cdots &\cdots \cdots\cdots \\ \mu(v_1)+\mu(v_2)+\cdots+\mu(v_l) &\geq \mu(u_1)+\mu(u_2)+\cdots+\mu(u_l). \end{aligned}$$

Certainly, by choice different angle factors, we can also get combinatorial conditions for the existence of parallel bundles under the linear assumption.

Theorem 4.5.6 *Let (M,μ) be a planar map geometry, $C = \{(u_1,v_1),(u_2,v_2),\cdots,(u_l,v_l)\}$ a cut of the map M with order $(u_1,v_1),(u_2,v_2),\cdots,(u_l,v_l)$ from the left to the right, $l \geq 1$. If*

$$\frac{\rho(u_i)}{\rho(v_i)} \leq \frac{\mu(v_i)}{\mu(u_i)}$$

for any integer $i, i \geq 1$, then a family L of parallel m-lines passing through C is a parallel bundle under the linear assumption.

Proof Under the linear assumption we know that

$$f'_{i+}(x) = \frac{\rho(v_i)\mu(v_i) - \rho(u_i)\mu(u_i)}{2d(u_i,v_i)}$$

for any integer $i, i \geq 1$ by Theorem 4.5.4. Thereby $f'_{i+}(x) \geq 0$ for $i = 1,2,\cdots,l$. We get that

$$f_1'(x) \geq 0$$
$$f_{1+}'(x) + f_{2+}'(x) \geq 0$$
$$f_{1+}'(x) + f_{2+}'(x) + f_{3+}'(x) \geq 0$$
$$\cdots\cdots\cdots\cdots$$
$$f_{1+}'(x) + f_{2+}'(x) + \cdots + f_{l+}'(x) \geq 0.$$

By Theorem 4.5.2 we know that a family L of parallel m-lines passing through C is still a parallel bundle. $\natural$

§4.6 Examples of Planar Map Geometries

By choice different planar maps and define angle factors on their vertices, we can get various planar map geometries. In this section, we present some concrete examples for planar map geometries.

Example 4.6.1 *A complete planar map* K_4

We take a complete map K_4 embedded on a plane $\sum$ with vertices u, v, w and t and angle factors

$$\mu(u) = \frac{\pi}{2}, \quad \mu(v) = \mu(w) = \pi \text{ and } \mu(t) = \frac{2\pi}{3},$$

such as shown in Fig.4.18 where each number on the side of a vertex denotes $\rho_M(x)\mu(x)$ for $x = u, v, w$ and t.

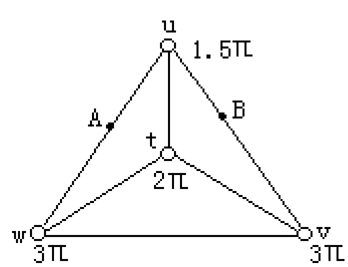


Fig.4.18

We assume the linear assumption is holds in this planar map geometry (M, μ). Then we get a classifications for m-points in (M, μ) as follows.

$$V_{el} = \{\text{points in } (\text{uA} \setminus \{\text{A}\}) \bigcup (\text{uB} \setminus \{\text{B}\}) \bigcup (\text{ut} \setminus \{\text{t}\})\},$$

where A and B are euclidean points on (u,w) and (u,v), respectively.

$$V_{eu} = \{A, B, t\} \bigcup (P \setminus E(K_4))$$

and

$$V_{hy} = \{\text{points in } (\text{wA} \setminus \{\text{A}\}) \bigcup (\text{wt} \setminus \{\text{t}\}) \bigcup \text{wv} \bigcup (\text{tv} \setminus \{\text{t}\}) \bigcup (\text{vB} \setminus \{\text{B}\})\}.$$

Edges in K_4 are classified into $(u,t) \in C_E^1$, $(t,w),(t,v) \in C_E^3$, $(u,w),(u,v) \in C_E^5$ and $(w,u) \in C_E^6$.

Various m-lines in this planar map geometry are shown in Fig.4.19.

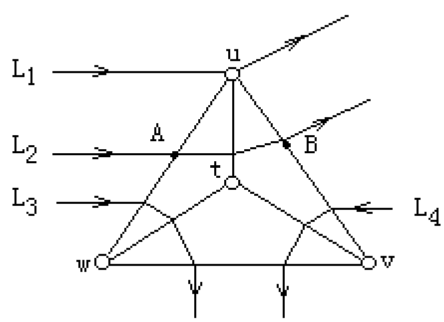


Fig.4.19

There are no 1-polygons in this planar map geometry. One 2-polygon and various triangles are shown in Fig.4.20.

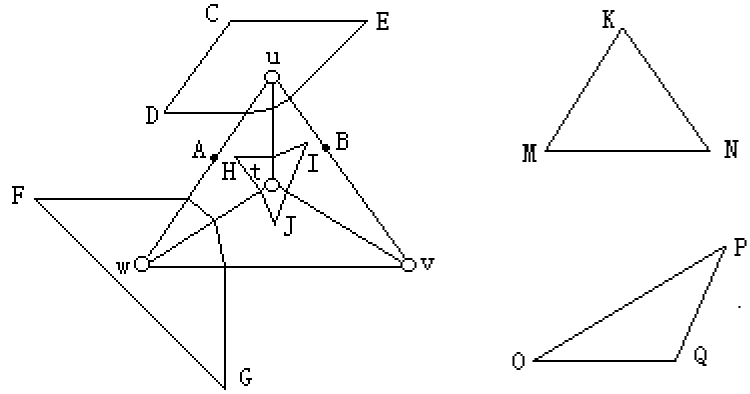


Fig.4.20

Example 4.6.2 *A wheel planar map $W_{1.4}$*

We take a wheel $W_{1.4}$ embedded on a plane $\sum$ with vertices O and u, v, w, t and angle factors

$$\mu(O) = \frac{\pi}{2}, \text{ and } \mu(u) = \mu(v) = \mu(w) = \mu(t) = \frac{4\pi}{3},$$

such as shown in Fig.4.21.

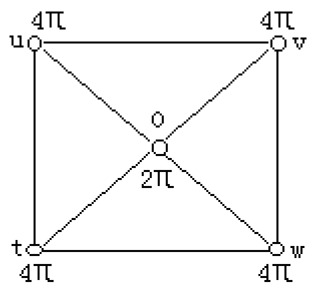


Fig.4.21

There are no elliptic points in this planar map geometries. Euclidean and hyperbolic points V_{eu}, V_{hy} are as follows.

$$V_{eu} = P\bigcup\backslash(E(W_{1.4}) \setminus \{O\})$$

and

$$V_{hy} = E(W_{1.4}) \setminus \{O\}.$$

Edges are classified into $(O, u), (O, v), (O, w), (O, t) \in C_E^3$ and $(u, v), (v, w), (w, t), (t, u) \in C_E^6$. Various m-lines and one 1-polygon are shown in Fig.4.22 where each m-line will turn to its opposite direction after it meeting $W_{1.4}$ such as those m-lines L_1, L_2 and L_4, L_5 in Fig.4.22.

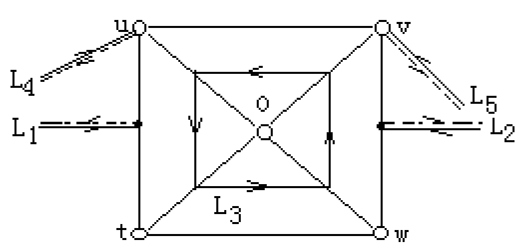


Fig.4.22

Example 4.6.3 *A parallel bundle in a planar map geometry*

We choose a planar ladder and define its angle factor as shown in Fig.4.23 where each number on the side of a vertex u denotes the number $\rho_M(u)\mu(u)$. Then we find a parallel bundle $\{L_i; 1 \leq i \leq 6\}$ as those shown in Fig.4.23.

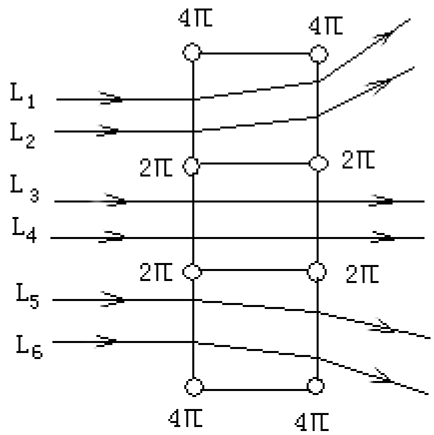


Fig.4.23

§4.7 Remarks and Open Problems

4.7.1. Unless the Einstein's relativity theory, nearly all other branches of physics use an Euclid space as their spacetime model. This has their own reason, also due to one's sight because the moving of an object is more likely as it is described by an Euclid geometry. As a generalization of an Euclid geometry of plane by the Smarandache's notion, planar map geometries were introduced in the references [54] and [62]. The same research can be also done for an Euclid geometry of a space $\mathbf{R}^3$ and open problems are selected in the following.

Problem 4.7.1 *Establish Smarandache geometries of a space $\mathbf{R}^3$ and classify their fundamental elements, such as points, lines, polyhedrons, $\cdots$, etc..*

Problem 4.7.2 *Determine various surfaces in a Smarandache geometry of a space $\mathbf{R}^3$, such as a sphere, a surface of cylinder, circular cone, a torus, a double torus and a projective plane, a Klein bottle, $\cdots$, also determine various convex polyhedrons such as a tetrahedron, a pentahedron, a hexahedron, $\cdots$, etc..*

Problem 4.7.3 *Define the conception of volume and find formulae for volumes of* *convex polyhedrons in a Smarandache geometry of a space* $\mathbf{R}^3$*, such as a tetrahedron, a pentahedron or a hexahedron,* $\cdots$*, etc..*

Problem 4.7.4 *Apply Smarandache geometries of a space* $\mathbf{R}^3$ *to find knots and characterize them.*

4.7.2. As those proved in Chapter 3, we can also research these map geometries on a locally orientable surfaces and find its fundamental elements in a surface, such as a sphere, a torus, a double torus, $\cdots$ and a projective plane, a Klein bottle, $\cdots$, i.e., to establish an intrinsic geometry on a surface. For this target, open problems for surfaces with small genus should be solved in the first.

Problem 4.7.5 *Establish an intrinsic geometry by map geometries on a sphere or a torus and find its fundamental elements.*

Problem 4.7.6 *Establish an intrinsic geometry on a projective or a Klein bottle and find its fundamental elements.*

Problem 4.7.7 *Define various measures of map geometries on a locally orientable surface* S *and apply them to characterize the surface* S*.*

Problem 4.7.8 *Define the conception of curvature for a map geometry* (M,μ) *on a locally orientable surface and calculate the sum* $\omega(M)$ *of curvatures on all edges in* M*.*

Conjecture 4.7.1 $\omega(M) = 2\pi\chi(M)s(M)$*, where* $s(M)$ *denotes the sum of length of edges in* M*.*

Chapter 5. Pseudo-Plane Geometries

The essential idea in planar map geometries is to associate each point in a planar map with an angle factor which turns flatness of a plane to tortuous as we have seen in Chapter 4. When the order of a planar map tends to infinite and its diameter of each face tends to zero (such planar maps exist, for example, triangulations of a plane), we get a tortuous plane at the limiting point, i.e., a plane equipped with a vector and straight lines maybe not exist. We concentrate on discussing these pseudo-planes in this chapter. A relation for integral curves with differential equations is established, which enables us to find good behaviors of plane curves.

§5.1 Pseudo-Planes

In the classical analytic geometry of plane, each point is correspondent with the Descartes coordinate (x, y), where x and y are real numbers which ensures the flatness of a plane. Motivated by the ideas in Chapters 3 and 4, we find a new kind of planes, called *pseudo-planes* which distort the flatness of a plane and can be applied to classical mathematics.

Definition 5.1.1 *Let $\sum$ be an Euclid plane. For $\forall u \in \sum$, if there is a continuous mapping $\omega : u \to \omega(u)$ where $\omega(u) \in \mathbf{R}^n$ for an integer $n, n \geq 1$ such that for any chosen number $\epsilon > 0$, there exists a number $\delta > 0$ and a point $v \in \sum$, $\|u - v\| < \delta$ such that $\|\omega(u) - \omega(v)\| < \epsilon$, then $\sum$ is called a pseudo-plane, denoted by $(\sum, \omega)$, where $\|u - v\|$ denotes the norm between points u and v in $\sum$.*

An explanation for Definition 5.1.1 is shown in Fig.5.1, in where $n = 1$ and

$\omega(u)$ is an angle function $\forall u \in \sum$.

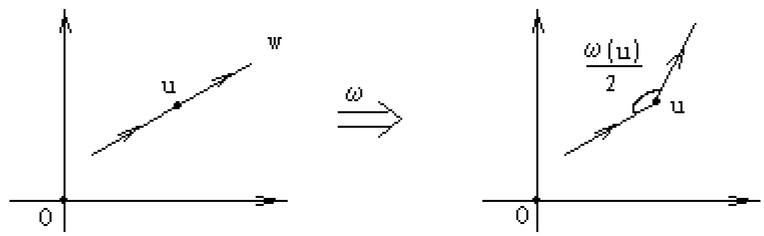


Fig.5.1

We can also explain $\omega(u)$, $u \in \mathcal{P}$ to be the coordinate z in $u = (x, y, z) \in \mathbf{R}^3$ by taking also $n = 1$. Thereby a pseudo-plane can be also seen as a projection of an Euclid space $\mathbf{R}^{n+2}$ on an Euclid plane. This fact implies that some characteristic of the geometry of space may reflected by a pseudo-plane.

We only discuss the case of $n = 1$ and explain $\omega(u), u \in \sum$ being a periodic function in this chapter, i.e., for any integer k, $4k\pi + \omega(u) \equiv \omega(u)(mod\ 4\pi)$. Not loss of generality, we assume that $0 < \omega(u) \leq 4\pi$ for $\forall u \in \sum$. Similar to map geometries, points in a pseudo-plane are classified into three classes, i.e., *elliptic points* V_{el}, *euclidean points* V_{eu} and *hyperbolic points* V_{hy}, defined by

$$V_{el} = \{u \in \sum |\omega(u) \ < 2\pi\},$$

$$V_{eu} = \{v \in \sum |\omega(v) \ = 2\pi\}$$

and

$$V_{hy} = \{w \in \sum |\omega(w) \ > 2\pi\}.$$

We define a sign function $sign(v)$ on a point of a pseudo-plane $(\sum, \omega)$

$$sign(v) = \begin{cases} 1, & \text{if } v \text{ is elliptic}, \\ 0, & \text{if } v \text{ is euclidean}, \\ -1, & \text{if } v \text{ is hyperbolic}. \end{cases}$$

Then we get a result as in the following.

Theorem 5.1.1 *There is a straight line segment AB in a pseudo-plane $(\sum, \omega)$ if and only if for $\forall u \in AB$, $\omega(u) = 2\pi$, i.e., every point on AB is euclidean.*

Proof Since $\omega(u)$ is an angle function for $\forall u \in \sum$, we know that AB is a straight line segment if and only if for $\forall u \in AB$,

$$\frac{\omega(u)}{2} = \pi,$$

i.e., $\omega(u) = 2\pi$, u is an euclidean point. $\natural$

Theorem 5.1.1 implies that not every pseudo-plane has straight line segments.

Corollary 5.1.1 *If there are only finite euclidean points in a pseudo-plane* $(\sum, \omega)$*, then there are no straight line segments in* $(\sum, \omega)$*.*

Corollary 5.1.2 *There are not always exist a straight line between two given points* u *and* v *in a pseudo-plane* $(\mathcal{P}, \omega)$*.*

By the intermediate value theorem in calculus, we know the following result for points in a pseudo-plane.

Theorem 5.1.2 *In a pseudo-plane* $(\sum, \omega)$*, if* $V_{el} \neq \emptyset$ *and* $V_{hy} \neq \emptyset$*, then*

$$V_{eu} \neq \emptyset.$$

Proof By these assumptions, we can choose points $u \in V_{el}$ and $v \in V_{hy}$. Consider points on line segment uv in an Euclid plane $\sum$. Since $\omega(u) < 2\pi$ and $\omega(v) > 2\pi$, there exists at least a point $w, w \in uv$ such that $\omega(w) = 2\pi$, i.e., $w \in V_{eu}$ by the intermediate value theorem in calculus. Whence, $V_{eu} \neq \emptyset$. $\natural$

Corollary 5.1.3 *In a pseudo-plane* $(\sum, \omega)$*, if* $V_{eu} = \emptyset$*, then every point of* $(\sum, \omega)$ *is elliptic or every point of* $\sum$ *is hyperbolic.*

According to Corollary 5.1.3, pseudo-planes can be classified into four classes as follows.

C_P^1(euclidean): *pseudo-planes whose each point is euclidean.*

C_P^2(elliptic): *pseudo-planes whose each point is elliptic.*

C_P^3(hyperbolic): *pseudo-planes whose each point is hyperbolic.*

C_P^4(Smarandache's): *pseudo-planes in which there are euclidean, elliptic and hyperbolic points simultaneously.*

For the existence of an algebraic curve C in a pseudo-plane $(\sum, \omega)$, we get a criteria as in the following.

Theorem 5.1.3 *There is an algebraic curve $F(x,y)=0$ passing through (x_0,y_0) in a domain D of a pseudo-plane $(\sum,\omega)$ with Descartes coordinate system if and only if $F(x_0,y_0)=0$ and for $\forall(x,y)\in D$,*

$$(\pi - \frac{\omega(x,y)}{2})(1+(\frac{dy}{dx})^2) = sign(x,y).$$

Proof By the definition of pseudo-planes in the case of that ω being an angle function and the geometrical meaning of the differential value of a function at a point, we know that an algebraic curve $F(x,y)=0$ exists in a domain D of $(\sum,\omega)$ if and only if

$$(\pi - \frac{\omega(x,y)}{2}) = sign(x,y)\frac{d(\arctan(\frac{dy}{dx}))}{dx},$$

for $\forall(x,y)\in D$, i.e.,

$$(\pi - \frac{\omega(x,y)}{2}) = \frac{sign(x,y)}{1+(\frac{dy}{dx})^2},$$

such as shown in Fig.5.2, where $\theta = \pi - \angle 2 + \angle 1$, $\lim\limits_{\triangle x\to 0}\theta = \omega(x,y)$ and (x,y) is an elliptic point.

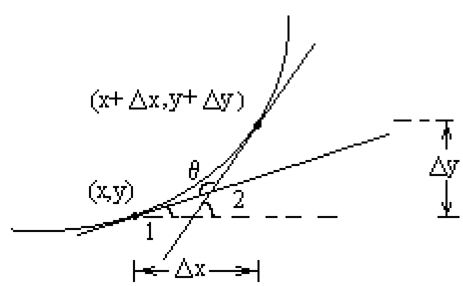


Fig.5.2

Therefore we get that

$$(\pi - \frac{\omega(x,y)}{2})(1+(\frac{dy}{dx})^2) = sign(x,y). \quad \natural$$

A plane curve C is called *elliptic* or *hyperbolic* if $sign(x,y)=1$ or -1 for each point (x,y) on C. We know a result for the existence of an elliptic or a hyperbolic curve in a pseudo-plane.

Corollary 5.1.4 *An elliptic curve $F(x,y)=0$ exists in a pseudo-plane $(\sum,\omega)$ with the Descartes coordinate system passing through (x_0,y_0) if and only if there is a domain $D\subset\sum$ such that $F(x_0,y_0)=0$ and for $\forall(x,y)\in D$,*

$$(\pi-\frac{\omega(x,y)}{2})(1+(\frac{dy}{dx})^2)=1$$

and there exists a hyperbolic curve $H(x,y)=0$ in a pseudo-plane $(\sum,\omega)$ with the Descartes coordinate system passing through (x_0,y_0) if and only if there is a domain $U\subset\sum$ such that for $H(x_0,y_0)=0$ and $\forall(x,y)\in U$,

$$(\pi-\frac{\omega(x,y)}{2})(1+(\frac{dy}{dx})^2)=-1.$$

Now construct a polar axis (ρ,θ) in a pseudo-plane $(\sum,\omega)$. Then we get a result as in the following.

Theorem 5.1.4 *There is an algebraic curve $f(\rho,\theta)=0$ passing through (ρ_0,θ_0) in a domain F of a pseudo-plane $(\sum,\omega)$ with a polar coordinate system if and only if $f(\rho_0,\theta_0)=0$ and for $\forall(\rho,\theta)\in F$,*

$$\pi-\frac{\omega(\rho,\theta)}{2}=sign(\rho,\theta)\frac{d\theta}{d\rho}.$$

Proof Similar to the proof of Theorem 5.1.3, we know that $\lim\limits_{\triangle x\to 0}\theta=\omega(x,y)$ and $\theta=\pi-\angle 2+\angle 1$ if (ρ,θ) is elliptic, or $\theta=\pi-\angle 1+\angle 2$ if (ρ,θ) is hyperbolic in Fig.5.2. Whence, we get that

$$\pi-\frac{\omega(\rho,\theta)}{2}=sign(\rho,\theta)\frac{d\theta}{d\rho}.$$

Corollary 5.1.5 *An elliptic curve $F(\rho,\theta)=0$ exists in a pseudo-plane $(\sum,\omega)$ with a polar coordinate system passing through (ρ_0,θ_0) if and only if there is a domain $F\subset\sum$ such that $F(\rho_0,\theta_0)=0$ and for $\forall(\rho,\theta)\in F$,*

$$\pi-\frac{\omega(\rho,\theta)}{2}=\frac{d\theta}{d\rho}$$

and there exists a hyperbolic curve $h(x,y)=0$ *in a pseudo-plane* $(\sum,\omega)$ *with a polar coordinate system passing through* (ρ_0,θ_0) *if and only if there is a domain* $U\subset\sum$ *such that* $h(\rho_0,\theta_0)=0$ *and for* $\forall(\rho,\theta)\in U$,

$$\pi-\frac{\omega(\rho,\theta)}{2}=-\frac{d\theta}{d\rho}.$$

Now we discuss a kind of expressions in an Euclid plane $\mathbf{R}^2$ for points in $\mathbf{R}^3$ and its characteristics.

Definition 5.1.2 *For a point* $P=(x,y,z)\in\mathbf{R}^3$ *with center* O, *let* ϑ *be the angle of vector* $\overrightarrow{OP}$ *with the plane* XOY. *Then define an angle function* $\omega:(x,y)\to 2(\pi-\vartheta)$, *i.e., the presentation of a point* (x,y,z) *in* $\mathbf{R}^3$ *is a point* (x,y) *with* $\omega(x,y)=2(\pi-\angle(\overrightarrow{OP},XOY))$ *in a pseudo-plane* $(\sum,\omega)$.

An explanation for Definition 5.2.1 is shown in Fig.5.3 where θ is an angle between the vector $\overrightarrow{OP}$ and plane XOY.

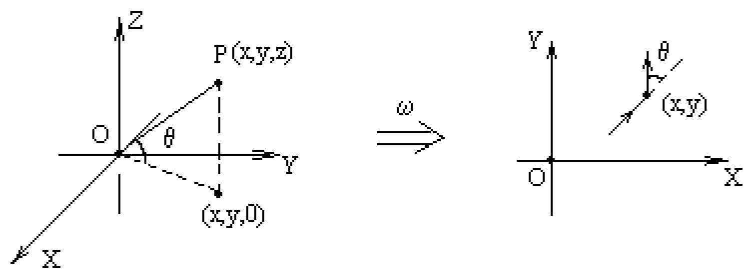


Fig.5.3

Theorem 5.1.5 *Let* $(\sum,\omega)$ *be a pseudo-plane and* $P=(x,y,z)$ *a point in* $\mathbf{R}^3$. *Then the point* (x,y) *is elliptic, euclidean or hyperbolic if and only if* $z>0$, $z=0$ *or* $z<0$.

Proof By Definition 5.1.2, we know that $\omega(x,y)>2\pi$, $=2\pi$ or $<2\pi$ if and only if $\theta>0,=0$ or <0 since $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$. Those conditions are equivalent to $z>0$, $=0$ or <0. ♮

The following result reveals the shape of points with a constant angle function value in a pseudo-plane $(\sum,\omega)$.

Theorem 5.1.6 *For a constant* $\eta, 0<\eta\le 4\pi$, *all points* (x,y,z) *in* $\mathbf{R}^3$ *with* $\omega(x,y)=\eta$ *consist an infinite circular cone with vertex* O *and an angle* $\pi-\frac{\eta}{2}$

between its generatrix and the plane XOY.

Proof Notice that $\omega(x_1, y_1) = \omega(x_2, y_2)$ for two points A, B in $\mathbf{R}^3$ with $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$ if and only if

$$\angle(\overrightarrow{OA}, XOY) = \angle(\overrightarrow{OB}, XOY) = \pi - \frac{\eta}{2},$$

that is, points A and B is on a circular cone with vertex O and an angle $\pi - \frac{\eta}{2}$ between $\overrightarrow{OA}$ or $\overrightarrow{OB}$ and the plane XOY. Since $z \to +\infty$, we get an infinite circular cone in $\mathbf{R}^3$ with vertex O and an angle $\pi - \frac{\eta}{2}$ between its generatrix and the plane XOY. $\natural$

§5.2 Integral Curves

An integral curve in an Euclid plane is defined by the next definition.

Definition 5.2.1 If the solution of a differential equation

$$\frac{dy}{dx} = f(x, y)$$

with an initial condition $y(x_0) = y_0$ exists, then all points (x, y) consisted by their solutions of this initial problem on an Euclid plane $\sum$ is called an integral curve.

By the ordinary differential equation theory, a well-known result for the unique solution of an ordinary differential equation is stated in the following. See also the reference [3] for details.

If the following conditions hold:

(i) $f(x, y)$ is continuous in a field F:

$$F : x_0 - a \leq x \leq x_0 + a, \quad y_0 - b \leq y \leq y_0 + b$$

(ii) there exist a constant ς such that for $\forall (x, y), (x, \overline{y}) \in F$,

$$|f(x, y) - f(x, \overline{y})| \leq \varsigma |y - \overline{y}|,$$

then there is an unique solution

$$y = \varphi(x), \quad \varphi(x_0) = y_0$$

for the differential equation

$$\frac{dy}{dx} = f(x, y)$$

with an initial condition $y(x_0) = y_0$ *in the interval* $[x_0 - h_0, x_0 + h_0]$, *where* $h_0 = \min(a, \frac{b}{M})$, $M = \max\limits_{(x,y)\in R} |f(x, y)|$.

The conditions in this theorem are complex and can not be applied conveniently. As we have seen in Section 5.1 of this chapter, a pseudo-plane $(\sum, \omega)$ is related with differential equations in an Euclid plane $\sum$. Whence, by a geometrical view, to find an integral curve in a pseudo-plane $(\sum, \omega)$ is equivalent to solve an initial problem for an ordinary differential equation. Thereby we concentrate on to find integral curves in a pseudo-plane in this section.

According to Theorem 5.1.3, we get the following result.

Theorem 5.2.1 *A curve* C,

$$C = \{(x, y(x))|\frac{dy}{dx} = f(x, y), y(x_0) = y_0\}$$

exists in a pseudo-plane $(\sum, \omega)$ *if and only if there is an interval* $I = [x_0 - h, x_0 + h]$ *and an angle function* $\omega : \sum \to \mathbf{R}$ *such that*

$$\omega(x, y(x)) = 2(\pi - \frac{sign(x, y(x))}{1 + f^2(x, y)})$$

for $\forall x \in I$ *with*

$$\omega(x_0, y(x_0)) = 2(\pi - \frac{sign(x, y(x))}{1 + f^2(x_0, y(x_0))}).$$

Proof According to Theorem 5.1.3, a curve passing through the point $(x_0, y(x_0))$ in a pseudo-plane $(\sum, \omega)$ if and only if $y(x_0) = y_0$ and for $\forall x \in I$,

$$(\pi - \frac{\omega(x, y(x))}{2})(1 + (\frac{dy}{dx})^2) = sign(x, y(x)).$$

Solving $\omega(x, y(x))$ from this equation, we get that

$$\omega(x,y(x)) = 2(\pi - \frac{sign(x,y(x))}{1+(\frac{dy}{dx})^2}) = 2(\pi - \frac{sign(x,y(x))}{1+f^2(x,y)}). \qquad \natural$$

Now we consider curves with an constant angle function value at each of its point.

Theorem 5.2.2 *Let $(\sum,\omega)$ be a pseudo-plane. Then for a constant $0<\theta\leq 4\pi$,*

(i) a curve C passing through a point (x_0,y_0) and $\omega(x,y)=\eta$ for $\forall(x,y)\in C$ is closed without self-intersections on $(\sum,\omega)$ if and only if there exists a real number s such that

$$s\eta = 2(s-2)\pi.$$

(ii) a curve C passing through a point (x_0,y_0) with $\omega(x,y)=\theta$ for $\forall(x,y)\in C$ is a circle on $(\sum,\omega)$ if and only if

$$\eta = 2\pi - \frac{2}{r},$$

where $r=\sqrt{x_0^2+y_0^2}$, i.e., C is a projection of a section circle passing through a point (x_0,y_0) on the plane XOY.

Proof Similar to Theorem 4.3.1, we know that a curve C passing through a point (x_0,y_0) in a pseudo-plane $(\sum,\omega)$ is closed if and only if

$$\int_0^s (\pi - \frac{\omega(s)}{2})ds = 2\pi.$$

Now since $\omega(x,y)=\eta$ is constant for $\forall(x,y)\in C$, we get that

$$\int_0^s (\pi - \frac{\omega(s)}{2})ds = s(\pi - \frac{\eta}{2}).$$

Whence, we get that

$$s(\pi - \frac{\eta}{2}) = 2\pi,$$

i.e.,

$$s\eta = 2(s-2)\pi.$$

Now if C is a circle passing through a point (x_0, y_0) with $\omega(x,y) = \theta$ for $\forall(x,y) \in C$, then by the Euclid plane geometry we know that $s = 2\pi r$, where $r = \sqrt{x_0^2 + y_0^2}$. Therefore, there must be that

$$\eta = 2\pi - \frac{2}{r}.$$

This completes the proof. ♮

Two spiral curves without self-intersections are shown in Fig.5.4, in where (a) is an input but (b) an output curve.

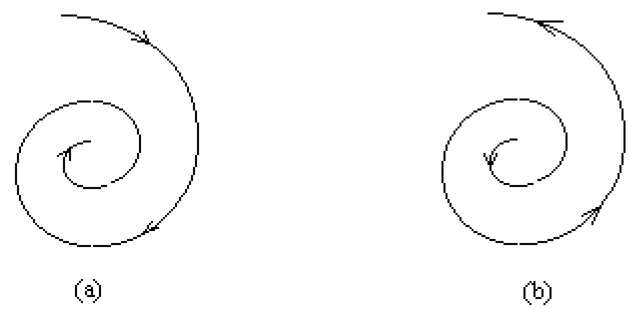


Fig.5.4

We call the curve in Fig.5.4(a) an *elliptic in-spiral* and Fig.5.4(b) an *elliptic out-spiral*, correspondent to the right hand rule. In a polar coordinate system (ρ, θ), a spiral curve has equation

$$\rho = ce^{\theta t},$$

where c, t are real numbers and $c > 0$. If $t < 0$, then the curve is an in-spiral as the curve shown in Fig.5.4(a). If $t > 0$, then the curve is an out-spiral as shown in Fig.5.4(b).

For the case $t = 0$, we get a circle $\rho = c$ (or $x^2 + y^2 = c^2$ in the Descartes coordinate system).

Now in a pseudo-plane, we can easily find conditions for in-spiral or out-spiral curves. That is the following theorem.

Theorem 5.2.3 *Let $(\sum, \omega)$ be a pseudo-plane and let η, ζ be constants. Then an elliptic in-spiral curve C with $\omega(x,y) = \eta$ for $\forall(x,y) \in C$ exists in $(\sum, \omega)$ if and only if there exist numbers $s_1 > s_2 > \cdots > s_l > \cdots$, $s_i > 0$ for $i \geq 1$ such that*

$$s_i\eta < 2(s_i - 2i)\pi$$

for any integer $i, i \geq 1$ *and an elliptic out-spiral curve* C *with* $\omega(x,y) = \zeta$ *for* $\forall(x,y) \in C$ *exists in* $(\sum,\omega)$ *if and only if there exist numbers* $s_1 > s_2 > \cdots > s_l > \cdots$, $s_i > 0$ *for* $i \geq 1$ *such that*

$$s_i\zeta > 2(s_i - 2i)\pi$$

for any integer $i, i \geq 1$.

Proof Let L be an m-line like an elliptic in-spiral shown in Fig.5.5, in where x_1, $x_2,\cdots$, x_n are non-euclidean points and x_1x_6 is an auxiliary line segment.

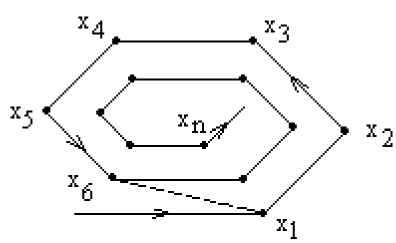


Fig.5.5

Then we know that

$$\sum_{i=1}^{6}(\pi - f(x_1)) < 2\pi,$$

$$\sum_{i=1}^{12}(\pi - f(x_1)) < 4\pi,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots.$$

Similarly from any initial point O to a point P far s to O on C, the sum of lost angles at P is

$$\int_0^s(\pi - \frac{\eta}{2})ds = (\pi - \frac{\eta}{2})s.$$

Whence, the curve C is an elliptic in-spiral if and only if there exist numbers $s_1 > s_2 > \cdots > s_l > \cdots$, $s_i > 0$ for $i \geq 1$ such that

$$(\pi - \frac{\eta}{2})s_1 < 2\pi,$$

$$(\pi - \frac{\eta}{2})s_2 < 4\pi,$$

$$(\pi - \frac{\eta}{2})s_3 < 6\pi,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$(\pi - \frac{\eta}{2})s_l < 2l\pi.$$

Therefore, we get that

$$s_i\eta < 2(s_i - 2i)\pi$$

for any integer $i, i \geq 1$.

Similarly, consider an m-line like an elliptic out-spiral with x_1, x_2,$\cdots$, x_n non-euclidean points. We can also find that C is an elliptic out-spiral if and only if there exist numbers $s_1 > s_2 > \cdots > s_l > \cdots$, $s_i > 0$ for $i \geq 1$ such that

$$(\pi - \frac{\zeta}{2})s_1 > 2\pi,$$

$$(\pi - \frac{\zeta}{2})s_2 > 4\pi,$$

$$(\pi - \frac{\zeta}{2})s_3 > 6\pi,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$(\pi - \frac{\zeta}{2})s_l > 2l\pi.$$

Whence, we get that

$$s_i\eta < 2(s_i - 2i)\pi.$$

for any integer $i, i \geq 1$. $\natural$

Similar to elliptic in or out-spirals, we can also define a *hyperbolic in-spiral* or *hyperbolic out-spiral* correspondent to the left hand rule, which are mirrors of curves in Fig.5.4. We get the following result for a hyperbolic in or out-spiral in a pseudo-plane.

Theorem 5.2.4 *Let $(\sum, \omega)$ be a pseudo-plane and let η, ζ be constants. Then a hyperbolic in-spiral curve C with $\omega(x, y) = \eta$ for $\forall(x, y) \in C$ exists in $(\sum, \omega)$ if and only if there exist numbers $s_1 > s_2 > \cdots > s_l > \cdots$, $s_i > 0$ for $i \geq 1$ such that*

$$s_i\eta > 2(s_i - 2i)\pi$$

for any integer $i, i \geq 1$ and a hyperbolic out-spiral curve C with $\omega(x, y) = \zeta$ for $\forall(x, y) \in C$ exists in $(\sum, \omega)$ if and only if there exist numbers $s_1 > s_2 > \cdots > s_l > \cdots$, $s_i > 0$ for $i \geq 1$ such that

$$s_i\zeta < 2(s_i - 2i)\pi$$

for any integer $i, i \geq 1$.

Proof The proof for (i) and (ii) is similar to the proof of Theorem 5.2.3. $\natural$

§5.3 Stability of a Differential Equation

For an ordinary differential equation system

$$\frac{dx}{dt} = P(x, y),$$
$$\frac{dy}{dt} = Q(x, y), \quad (A^*)$$

where t is a time parameter, the Euclid plane XOY with the Descartes coordinate system is called its a *phase plane* and the orbit $(x(t), y(t))$ of its a solution $x = x(t), y = y(t)$ is called an *orbit curve.* If there exists a point (x_0, y_0) on XOY such that

$$P(x_0, y_0) = Q(x_0, y_0) = 0,$$

then there is an obit curve which is only a point (x_0, y_0) on XOY. The point (x_0, y_0) is called a *singular point of* (A^*). Singular points of an ordinary differential equation are classified into four classes: *knot, saddle, focal* and *central points*. Each of these classes are introduced in the following.

Class 1. **Knots**

A *knot* O of a differential equation is shown in Fig.5.6 where (a) denotes that O is stable but (b) is unstable.

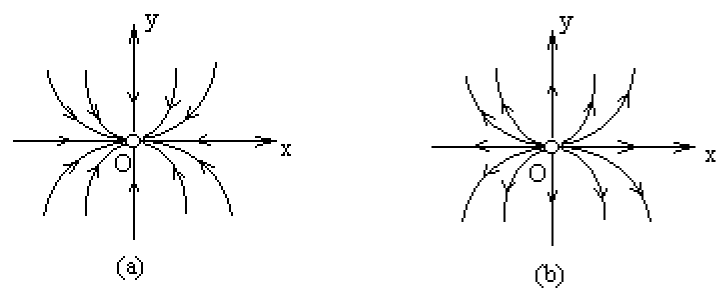


Fig.5.6

A *critical knot* O of a differential equation is shown in Fig.5.7 where (a) denotes that O is stable but (b) is unstable.

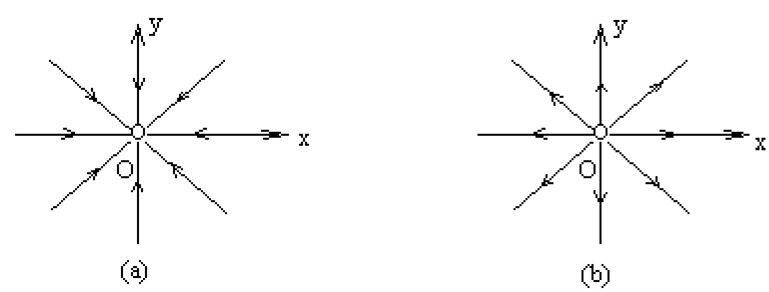


Fig.5.7

A *degenerate knot* O of a differential equation is shown in Fig.5.8 where (a) denotes that O is stable but (b) is unstable.

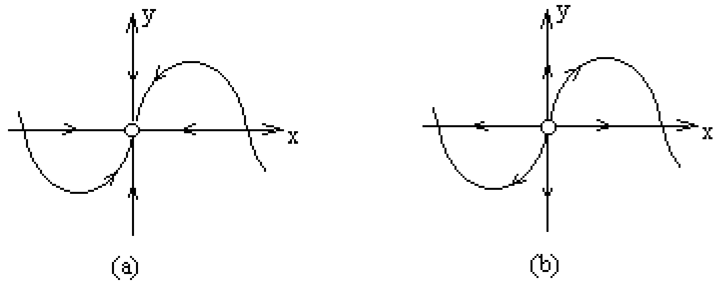


Fig.5.8

Class 2. **Saddle points**

A *saddle point* O of a differential equation is shown in Fig.5.9.

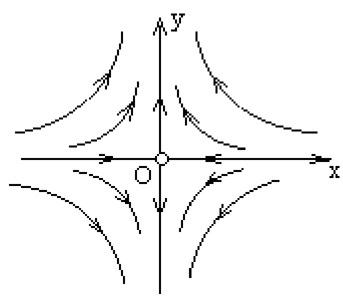


Fig.5.9

Class 3. **Focal points**

A *focal point* O of a differential equation is shown in Fig.5.10 where (a) denotes that O is stable but (b) is unstable.

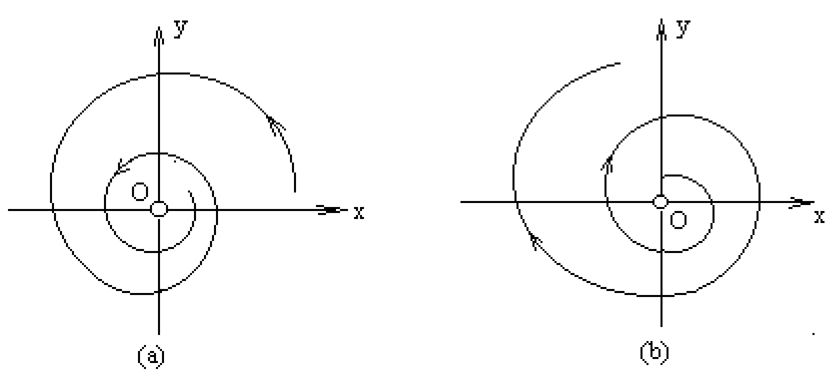


Fig.5.10

Class 4. **Central points**

A *central point* O of a differential equation is shown in Fig.5.11, which is just the center of a circle.

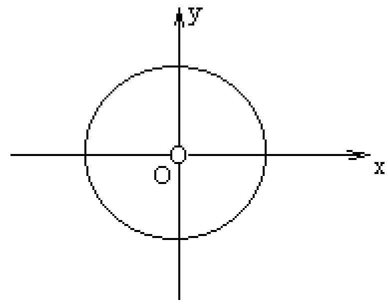


Fig.5.11

In a pseudo-plane $(\sum,\omega)$, not all kinds of singular points exist. We get a result for singular points in a pseudo-plane as in the following.

Theorem 5.3.1 *There are no saddle points and stable knots in a pseudo-plane plane* $(\sum,\omega)$.

Proof On a saddle point or a stable knot O, there are two rays to O, seeing Fig.5.6(a) and Fig.5.10 for details. Notice that if this kind of orbit curves in Fig.5.6(a) or Fig.5.10 appears, then there must be that

$$\omega(O) = 4\pi.$$

Now according to Theorem 5.1.1, every point u on those two rays should be euclidean, i.e., $\omega(u) = 2\pi$, unless the point O. But then ω is not continuous at the point O, which contradicts Definition 5.1.1. ♮

If an ordinary differential equation system (A^*) has a closed orbit curve C but all other orbit curves are not closed in a neighborhood of C nearly enough to C and those orbits curve tend to C when $t \to +\infty$ or $t \to -\infty$, then C is called a *limiting ring of* (A^*) and *stable* or *unstable* if $t \to +\infty$ or $t \to -\infty$.

Theorem 5.3.2 *For two constants* ρ_0, θ_0*,* $\rho_0 > 0$ *and* $\theta_0 \neq 0$*, there is a pseudo-plane* $(\sum,\omega)$ *with*

$$\omega(\rho,\theta) = 2(\pi - \frac{\rho_0}{\theta_0\rho})$$

or

$$\omega(\rho,\theta) = 2(\pi + \frac{\rho_0}{\theta_0\rho})$$

such that

$$\rho = \rho_0$$

is a limiting ring in $(\sum,\omega)$.

Proof Notice that for two given constants ρ_0, θ_0, $\rho_0 > 0$ and $\theta_0 \neq 0$, the equation

$$\rho(t) = \rho_0 e^{\theta_0\theta(t)}$$

has a stable or unstable limiting ring

$$\rho = \rho_0$$

if $\theta(t) \to 0$ when $t \to +\infty$ or $t \to -\infty$. Whence, we know that

$$\theta(t) = \frac{1}{\theta_0}\ln\frac{\rho_0}{\rho(t)}.$$

Therefore,

$$\frac{d\theta}{d\rho} = \frac{\rho_0}{\theta_0\rho(t)}.$$

According to Theorem 5.1.4, we get that

$$\omega(\rho,\theta) = 2(\pi - sign(\rho,\theta)\frac{d\theta}{d\rho}),$$

for any point $(\rho,\theta) \in \sum$, i.e.,

$$\omega(\rho,\theta) = 2(\pi - \frac{\rho_0}{\theta_0\rho})$$

or

$$\omega(\rho,\theta) = 2(\pi + \frac{\rho_0}{\theta_0\rho}). \quad \natural$$

A general pseudo-space is discussed in the next section which enables us to know the Finsler geometry is a particular case of Smnarandache geometries.

§5.4 Remarks and Open Problems

Definition 5.1.1 can be generalized as follows, which enables us to enlarge our fields of mathematics for further research.

Definition 5.4.1 *Let U and W be two metric spaces with metric ρ, $W \subseteq U$. For $\forall u \in U$, if there is a continuous mapping $\omega : u \rightarrow \omega(u)$, where $\omega(u) \in \mathbf{R}^n$ for an integer $n, n \geq 1$ such that for any number $\epsilon > 0$, there exists a number $\delta > 0$ and a point $v \in W$, $\rho(u-v) < \delta$ such that $\rho(\omega(u) - \omega(v)) < \epsilon$, then U is called a metric pseudo-space if $U = W$ or a bounded metric pseudo-space if there is a number $N > 0$ such that $\forall w \in W$, $\rho(w) \leq N$, denoted by (U, ω) or (U^-, ω), respectively.*

By choice different metric spaces U and W in this definition, we can get various metric pseudo-spaces. For the case $n = 1$, we can also explain $\omega(u)$ being an angle function with $0 < \omega(u) \leq 4\pi$, i.e.,

$$\omega(u) = \begin{cases} \omega(u)(mod 4\pi), & \text{if u} \in \text{W}, \\ 2\pi, & \text{if u} \in \text{U} \setminus \text{W} \end{cases} \qquad (*)$$

and get some interesting metric pseudo-spaces.

5.4.1. **Bounded pseudo-plane geometries** Let C be a closed curve in an Euclid plane $\sum$ without self-intersections. Then C divides $\sum$ into two domains. One of them is finite. Denote by D_{fin} the finite one. Call C a boundary of D_{fin}. Now let $U = \sum$ and $W = D_{fin}$ in Definition 5.4.1 for the case of $n = 1$. For example, choose C be a 6-polygon such as shown in Fig.5.12.

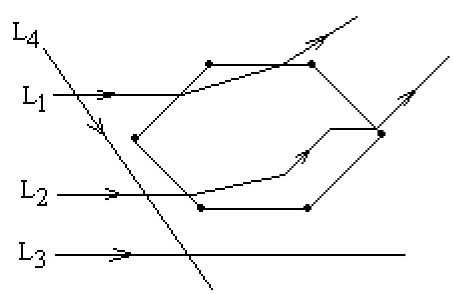


Fig.5.12

Then we get a geometry $(\sum^-, \omega)$ partially euclidean and partially non-euclidean.

Problem 5.4.1 *Similar to Theorem 4.5.2, find conditions for parallel bundles on*

$(\sum^{-}, \omega)$.

Problem 5.4.2 *Find conditions for existing an algebraic curve* $F(x, y) = 0$ *on* $(\sum^{-}, \omega)$.

Problem 5.4.3 *Find conditions for existing an integer curve* C *on* $(\sum^{-}, \omega)$.

5.4.2. **Pseudo-Space geometries** For any integer $m, m \geq 3$ and a point $\overline{u} \in \mathbf{R}^m$. Choose $U = W = \mathbf{R}^m$ in Definition 5.4.1 for the case of $n = 1$ and $\omega(\overline{u})$ an angle function. Then we get a pseudo-space geometry $(\mathbf{R}^m, \omega)$ on $\mathbf{R}^m$.

Problem 5.4.4 *Find conditions for existing an algebraic surface* $F(x_1, x_2, \cdots, x_m) = 0$ *in* $(\mathbf{R}^m, \omega)$*, particularly, for an algebraic surface* $F(x_1, x_2, x_3) = 0$ *existing in* $(\mathbf{R}^3, \omega)$.

Problem 5.4.5 *Find conditions for existing an integer surface in* $(\mathbf{R}^m, \omega)$.

If we take $U = \mathbf{R}^m$ and W a bounded convex point set of $\mathbf{R}^m$ in Definition 5.4.1. Then we get a bounded pseudo-space $(\mathbf{R}^{m-}, \omega)$, which is partially euclidean and partially non-euclidean.

Problem 5.4.6 *For a bounded pseudo-space* $(\mathbf{R}^{m-}, \omega)$*, solve Problems* 5.4.4 *and* 5.4.5 *again.*

5.4.3. **Pseudo-Surface geometries** For a locally orientable surface S and $\forall u \in S$, we choose $U = W = S$ in Definition 5.4.1 for $n = 1$ and $\omega(u)$ an angle function. Then we get a pseudo-surface geometry (S, ω) on the surface S.

Problem 5.4.7 *Characterize curves on a surface* S *by choice angle function* ω*. Whether can we classify automorphisms on* S *by applying pseudo-surface geometries* (S, ω)*?*

Notice that Thurston had classified automorphisms of a surface $S, \chi(S) \leq 0$ into three classes in [86]: *reducible, periodic* or *pseudo-Anosov.*

If we take $U = S$ and W a bounded simply connected domain of S in Definition 5.4.1. Then we get a bounded pseudo-surface (S^{-}, ω).

Problem 5.4.8 *For a bounded pseudo-surface* (S^{-}, ω)*, solve Problem* 5.4.7.

5.4.4. **Pseudo-Manifold geometries** For an m-manifold M^m and $\forall u \in M^m$, choose $U = W = M^m$ in Definition 5.4.1 for $n = 1$ and $\omega(u)$ a smooth function. Then we get a pseudo-manifold geometry (M^m, ω) on the m-manifold M^m. This geometry includes the *Finsler geometry*, i.e., equipped each m-manifold with a Minkowski norm defined in the following ([13], [39]).

A *Minkowski norm* on M^m is a function $F : M^m \to [0, +\infty)$ such that

(i) F is smooth on $M^m \setminus \{0\}$;

(ii) F is 1-homogeneous, i.e., $F(\lambda\overline{u}) = \lambda F(\overline{u})$ for $\overline{u} \in M^m$ and $\lambda > 0$;

(iii) for $\forall y \in M^m \setminus \{0\}$, the symmetric bilinear form $g_y : M^m \times M^m \to R$ with

$$g_y(\overline{u}, \overline{v}) = \frac{1}{2}\frac{\partial^2 F^2(y + s\overline{u} + t\overline{v})}{\partial s \partial t}|_{t=s=0}$$

.

is positive definite.

Then a *Finsler manifold* is a manifold M^m and a function $F : TM^m \to [0, +\infty)$ such that

(i) F is smooth on $TM^m \setminus \{0\} = \bigcup\{T_{\overline{x}}M^m \setminus \{0\} : \overline{x} \in M^m\}$;

(ii) $F|_{T_{\overline{x}}M^m} \to [0, +\infty)$ is a Minkowski norm for $\forall \overline{x} \in M^m$.

As a special case of pseudo-manifold geometries, we choose $\omega(\overline{x}) = F(\overline{x})$ for $\overline{x} \in M^m$, then (M^m, ω) is a *Finsler manifold*, particularly, if $\omega(\overline{x}) = g_{\overline{x}}(y, y) = F^2(x, y)$, then (M^m, ω) is a *Riemann manifold*. Thereby, *Smarandache geometries, particularly pseudo-manifold geometries include the Finsler geometry.*

Open problems for pseudo-manifold geometries are presented in the following.

Problem 5.4.9 *Characterize these pseudo-manifold geometries (M^m, ω) without boundary and apply them to classical mathematics and to classical mechanics.*

Similarly, if we take $U = M^m$ and W a bounded submanifold of M^m in Definition 5.4.1. Then we get a bounded pseudo-manifold (M^{m-}, ω).

Problem 5.4.10 *Characterize these pseudo-manifold geometries (M^{m-}, ω) with boundary and apply them to classical mathematics and to classical mechanics, particularly, to hamiltonian mechanics.*

Chapter 6. Applications to Theoretical Physics

Whether are there finite, or infinite cosmoses? Is there just one? What is the dimension of our cosmos? Those simpler but more puzzling problems have confused the eyes of human beings thousands years and one does not know the answer even until today. The dimension of the cosmos in the eyes of the ancient Greeks is 3, but Einstein's is 4. In recent decades, 10 or 11 is the dimension of our cosmos in superstring theory or M-theory. All these assumptions acknowledge that there is just one cosmos. Which one is the correct and whether can human beings realize the cosmos or cosmoses? By applying results gotten in Chapters 3-5, we tentatively answer those problems and explain the Einstein's or Hawking's model for cosmos in this chapter.

§6.1 Pseudo-Faces of Spaces

Throughout this chapter, $\mathbf{R}^n$ denotes an Euclid space of dimensional n. In this section, we consider a problem related to how to represent an Euclid space in another. First, we introduce the conception of pseudo-faces of Euclid spaces in the following.

Definition 6.1.1 *Let $\mathbf{R}^m$ and $(\mathbf{R}^n, \omega)$ be an Euclid space and a pseudo-metric space. If there is a continuous mapping $p : \mathbf{R}^m \to (\mathbf{R}^n, \omega)$, then the pseudo-metric space $(\mathbf{R}^n, \omega(p(\mathbf{R}^m)))$ is called a pseudo-face of $\mathbf{R}^m$ in $(\mathbf{R}^n, \omega)$.*

Notice that these pseudo-faces of $\mathbf{R}^3$ in $\mathbf{R}^2$ have been considered in Chapter 5. For the existence of a pseudo-face of an Euclid space $\mathbf{R}^m$ in $\mathbf{R}^n$, we have a result as in the following.

Theorem 6.1.1 *Let $\mathbf{R}^m$ and $(\mathbf{R}^n, \omega)$ be an Euclid space and a pseudo-metric space. Then there exists a pseudo-face of $\mathbf{R}^m$ in $(\mathbf{R}^n, \omega)$ if and only if for any number $\epsilon > 0$, there exists a number $\delta > 0$ such that for $\forall \overline{u}, \overline{v} \in \mathbf{R}^m$ with $\|\overline{u} - \overline{v}\| < \delta$,*

$$\|\omega(p(\overline{u})) - \omega(p(\overline{v}))\| < \epsilon,$$

where $\|\overline{u}\|$ denotes the norm of a vector $\overline{u}$ in the Euclid space.

Proof We only need to prove that there exists a continuous mapping $p : \mathbf{R}^m \to (\mathbf{R}^n, \omega)$ if and only if all of these conditions in this theorem hold. By the definition of a pseudo-space $(\mathbf{R}^n, \omega)$, since ω is continuous, we know that for any number $\epsilon > 0$, $\|\omega(\overline{x}) - \omega(\overline{y})\| < \epsilon$ for $\forall \overline{x}, \overline{y} \in \mathbf{R}^n$ if and only if there exists a number $\delta_1 > 0$ such that $\|\overline{x} - \overline{y})\| < \delta_1$.

By definition, a mapping $q : \mathbf{R}^m \to \mathbf{R}^n$ is continuous between Euclid spaces if and only if for any number $\delta_1 > 0$, there exists a number $\delta_2 > 0$ such that $\|q(\overline{x}) - q(\overline{y})\| < \delta_1$ for $\forall \overline{u}, \overline{v} \in \mathbf{R}^m$ with $\|\overline{u} - \overline{v})\| < \delta_2$.

Combining these assertions, we know that $p : \mathbf{R}^m \to (\mathbf{R}^n, \omega)$ is continuous if and only if for any number $\epsilon > 0$, there is number $\delta = \min\{\delta_1, \delta_2\}$ such that

$$\|\omega(p(\overline{u})) - \omega(p(\overline{v}))\| < \epsilon$$

for $\forall \overline{u}, \overline{v} \in \mathbf{R}^m$ with $\|\overline{u} - \overline{v})\| < \delta$. $\natural$

Corollary 6.1.1 *If $m \geq n+1$, let $\omega : \mathbf{R}^n \to \mathbf{R}^{m-n}$ be a continuous mapping, then $(\mathbf{R}^n, \omega(p(\mathbf{R}^m)))$ is a pseudo-face of $\mathbf{R}^m$ in $(\mathbf{R}^n, \omega)$ with*

$$p(x_1, x_2, \cdots, x_n, x_{n+1}, \cdots, x_m) = \omega(x_1, x_2, \cdots, x_n).$$

Particularly, if $m = 3, n = 2$ and ω is an angle function, then $(\mathbf{R}^n, \omega(p(\mathbf{R}^m)))$ is a pseudo-face with $p(x_1, x_2, x_3) = \omega(x_1, x_2)$.

There is a simple relation for a continuous mapping between Euclid spaces and that of between pseudo-faces established in the next result.

Theorem 6.1.2 *Let $g : \mathbf{R}^m \to \mathbf{R}^m$ and $p : \mathbf{R}^m \to (\mathbf{R}^n, \omega)$ be continuous mappings. Then $pgp^{-1} : (\mathbf{R}^n, \omega) \to (\mathbf{R}^n, \omega)$ is also a continuous mapping.*

Proof Because a composition of continuous mappings is a continuous mapping, we know that pgp^{-1} is continuous.

Now for $\forall\omega(x_1, x_2, \cdots, x_n) \in (\mathbf{R}^n, \omega)$, assume that $p(y_1, y_2, \cdots, y_m) = \omega(x_1, x_2, \cdots, x_n)$, $g(y_1, y_2, \cdots, y_m) = (z_1, z_2, \cdots, z_m)$ and $p(z_1, z_2, \cdots, z_m) = \omega(t_1, t_2, \cdots, t_n)$. Then calculation shows that

$$\begin{aligned} pgp^{-}(\omega(x_1, x_2, \cdots, x_n)) &= pg(y_1, y_2, \cdots, y_m) \\ &= p(z_1, z_2, \cdots, z_m) = \omega(t_1, t_2, \cdots, t_n) \in (\mathbf{R}^n, \omega). \end{aligned}$$

Whence, pgp^{-1} is a continuous mapping and $pgp^{-1} : (\mathbf{R}^n, \omega) \to (\mathbf{R}^n, \omega)$. ♮

Corollary 6.1.2 *Let $C(\mathbf{R}^m)$ and $C(\mathbf{R}^n, \omega)$ be sets of continuous mapping on an Euclid space $\mathbf{R}^m$ and an pseudo-metric space $(\mathbf{R}^n, \omega)$. If there is a pseudo-space for $\mathbf{R}^m$ in $(\mathbf{R}^n, \omega)$. Then there is a bijection between $C(\mathbf{R}^m)$ and $C(\mathbf{R}^n, \omega)$.*

For a body $\mathcal{B}$ in an Euclid space $\mathbf{R}^m$, its shape in a pseudo-face $(\mathbf{R}^n, \omega(p(\mathbf{R}^m)))$ of $\mathbf{R}^m$ in $(\mathbf{R}^n, \omega)$ is called a *pseudo-shape* of $\mathcal{B}$. We get results for pseudo-shapes of a ball in the following.

Theorem 6.1.3 *Let $\mathcal{B}$ be an $(n+1)$-ball of radius R in a space $\mathbf{R}^{n+1}$, i.e.,*

$$x_1^2 + x_2^2 + \cdots + x_n^2 + t^2 \leq R^2.$$

Define a continuous mapping $\omega : \mathbf{R}^n \to \mathbf{R}^n$ by

$$\omega(x_1, x_2, \cdots, x_n) = \varsigma t(x_1, x_2, \cdots, x_n)$$

for a real number ς and a continuous mapping $p : \mathbf{R}^{n+1} \to \mathbf{R}^n$ by

$$p(x_1, x_2, \cdots, x_n, t) = \omega(x_1, x_2, \cdots, x_n).$$

*Then the pseudo-shape of $\mathcal{B}$ in $(\mathbf{R}^n, \omega)$ is a ball of radius $\frac{\sqrt{R^2-t^2}}{\varsigma t}$ for any parameter $t, -R \leq t \leq R$. Particularly, for the case of $n = 2$ and $\varsigma = \frac{1}{2}$, it is a circle of radius $\sqrt{R^2 - t^2}$ for parameter t and an elliptic ball in $\mathbf{R}^3$ as shown in Fig.*6.1.

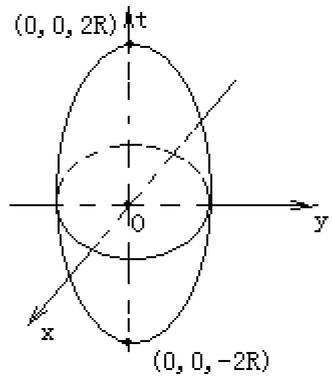


Fig.6.1

Proof For any parameter t, an $(n+1)$-ball

$$x_1^2 + x_2^2 + \cdots + x_n^2 + t^2 \leq R^2$$

can be transferred to an n-ball

$$x_1^2 + x_2^2 + \cdots + x_n^2 \leq R^2 - t^2$$

of radius $\sqrt{R^2 - t^2}$. Whence, if we define a continuous mapping on $\mathbf{R}^n$ by

$$\omega(x_1, x_2, \cdots, x_n) = \varsigma t(x_1, x_2, \cdots, x_n)$$

and

$$p(x_1, x_2, \cdots, x_n, t) = \omega(x_1, x_2, \cdots, x_n),$$

then we get an n-ball

$$x_1^2 + x_2^2 + \cdots + x_n^2 \leq \frac{R^2 - t^2}{\varsigma^2 t^2},$$

of $\mathcal{B}$ under p for any parameter t, which is the pseudo-face of $\mathcal{B}$ for a parameter t by definition.

For the case of $n = 2$ and $\varsigma = \frac{1}{2}$, since its pseudo-face is a circle in an Euclid plane and $-R \leq t \leq R$, we get an elliptic ball as shown in Fig.6.1. $\natural$

Similarly, if we define $\omega(x_1, x_2, \cdots, x_n) = 2\angle(\overrightarrow{OP}, Ot)$ for a point $P = (x_1, x_2, \cdots, x_n, t)$, i.e., an angle function, then we can also get a result like Theorem 6.1.2 for these pseudo-shapes of an $(n+1)$-ball.

Theorem 6.1.4 *Let $\mathcal{B}$ be an $(n+1)$-ball of radius R in a space $\mathbf{R}^{n+1}$, i.e.,*

$$x_1^2 + x_2^2 + \cdots + x_n^2 + t^2 \leq R^2.$$

Define a continuous mapping $\omega : \mathbf{R}^n \to \mathbf{R}^n$ by

$$\omega(x_1, x_2, \cdots, x_n) = 2\angle(\overrightarrow{OP}, Ot)$$

for a point P on $\mathcal{B}$ and a continuous mapping $p : \mathbf{R}^{n+1} \to \mathbf{R}^n$ by

$$p(x_1, x_2, \cdots, x_n, t) = \omega(x_1, x_2, \cdots, x_n).$$

Then the pseudo-shape of $\mathcal{B}$ in $(\mathbf{R}^n, \omega)$ is a ball of radius $\sqrt{R^2 - t^2}$ for any parameter $t, -R \leq t \leq R$. Particularly, for the case of $n = 2$, it is a circle of radius $\sqrt{R^2 - t^2}$ for parameter t and a body in $\mathbf{R}^3$ with equations

$$\oint \arctan(\frac{t}{x}) = 2\pi \quad \textit{and} \quad \oint \arctan(\frac{t}{y}) = 2\pi$$

for curves of its intersection with planes XOT and YOT.

Proof The proof is similar to the proof of Theorem 6.1.3. For these equations

$$\oint \arctan(\frac{t}{x}) = 2\pi \quad \text{or} \quad \oint \arctan(\frac{t}{y}) = 2\pi$$

of curves on planes XOT or YOT in the case of $n = 2$, they are implied by the geometrical meaning of an angle function. ♮

For an Euclid space $\mathbf{R}^n$, we can get a subspace sequence

$$\mathbf{R}_0 \supset \mathbf{R}_1 \supset \cdots \supset \mathbf{R}_{n-1} \supset \mathbf{R}_n,$$

where the dimensional of $\mathbf{R}_i$ is $n - i$ for $1 \leq i \leq n$ and $\mathbf{R}_n$ is just a point. But we can not get a sequence reversing the order, i.e., a sequence

$$\mathbf{R}_0 \subset \mathbf{R}_1 \subset \cdots \subset \mathbf{R}_{n-1} \subset \mathbf{R}_n$$

in classical space theory. By applying Smarandache multi-spaces, we can really find this kind of sequence by the next result, which can be used to explain a well-known model for our cosmos in M-theory.

Theorem 6.1.5 *Let $P = (x_1, x_2, \cdots, x_n)$ be a point of $\mathbf{R}^n$. Then there are subspaces of dimensional s in P for any integer $s, 1 \leq s \leq n$.*

Proof Notice that in an Euclid space $\mathbf{R}^n$, there is a basis $e_1 = (1, 0, 0, \cdots, 0)$, $e_2 = (0, 1, 0, \cdots, 0)$, $\cdots$, $e_i = (0, \cdots, 0, 1, 0, \cdots, 0)$ (every entry is 0 unless the i-th entry is 1), $\cdots$, $e_n = (0, 0, \cdots, 0, 1)$ such that

$$(x_1, x_2, \cdots, x_n) = x_1e_1 + x_2e_2 + \cdots + x_ne_n$$

for any point $(x_1, x_2, \cdots, x_n)$ of $\mathbf{R}^n$. Now we consider a linear space $\mathbf{R}^- = (V, +_{new}, \circ_{new})$ on a field $F = \{a_i, b_i, c_i, \cdots, d_i; i \geq 1\}$, where

$$V = \{x_1, x_2, \cdots, x_n\}.$$

Not loss of generality, we assume that $x_1, x_2, \cdots, x_s$ are independent, i.e., if there exist scalars $a_1, a_2, \cdots, a_s$ such that

$$a_1 \circ_{new} x_1 +_{new} a_2 \circ_{new} x_2 +_{new} \cdots +_{new} a_s \circ_{new} x_s = 0,$$

then $a_1 = a_2 = \cdots = 0_{new}$ and there are scalars $b_i, c_i, \cdots, d_i$ with $1 \leq i \leq s$ in $\mathbf{R}^-$ such that

$$x_{s+1} = b_1 \circ_{new} x_1 +_{new} b_2 \circ_{new} x_2 +_{new} \cdots +_{new} b_s \circ_{new} x_s;$$

$$x_{s+2} = c_1 \circ_{new} x_1 +_{new} c_2 \circ_{new} x_2 +_{new} \cdots +_{new} c_s \circ_{new} x_s;$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots;$$

$$x_n = d_1 \circ_{new} x_1 +_{new} d_2 \circ_{new} x_2 +_{new} \cdots +_{new} d_s \circ_{new} x_s.$$

Therefore, we get a subspace of dimensional s in a point P of $\mathbf{R}^n$. $\natural$

Corollary 6.1.3 *Let P be a point of an Euclid space $\mathbf{R}^n$. Then there is a subspace sequence*

$$\mathbf{R}_0^- \subset \mathbf{R}_1^- \subset \cdots \subset \mathbf{R}_{n-1}^- \subset \mathbf{R}_n^-$$

such that $\mathbf{R}_n^- = \{P\}$ and the dimensional of the subspace $\mathbf{R}_i^-$ is $n - i$, where $1 \leq i \leq n$.

Proof Applying Theorem 6.1.5 repeatedly, we get the desired sequence. ♮

§6.2. Relativity Theory

In theoretical physics, these spacetimes are used to describe various states of particles dependent on the time parameter in an Euclid space $\mathbf{R}^3$. There are two kinds of spacetimes. An *absolute spacetime* is an Euclid space $\mathbf{R}^3$ with an independent time, denoted by $(x_1, x_2, x_3|t)$ and a *relative spacetime* is an Euclid space $\mathbf{R}^4$, where time is the t-axis, seeing also in [30] – [31] for details.

A point in a spacetime is called an *event*, i.e., represented by

$$(x_1, x_2, x_3) \in \mathbf{R}^3 \ \text{ and } \ t \in \mathbf{R}^+$$

in an absolute spacetime in the Newton's mechanics and

$$(x_1, x_2, x_3, t) \in \mathbf{R}^4$$

with time parameter t in a relative space-time used in the Einstein's relativity theory.

For two events $A_1 = (x_1, x_2, x_3|t_1)$ and $A_2 = (y_1, y_2, y_3|t_2)$, the *time interval* $\triangle t$ is defined by $\triangle t = t_1 - t_2$ and the *space interval* $\triangle(A_1, A_2)$ by

$$\triangle(A_1, A_2) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}.$$

Similarly, for two events $B_1 = (x_1, x_2, x_3, t_1)$ and $B_2 = (y_1, y_2, y_3, t_2)$, the *space-time interval* $\triangle s$ is defined by

$$\triangle^2 s = -c^2 \triangle t^2 + \triangle^2(B_1, B_2),$$

where c is the speed of the light in vacuum. In Fig.6.2, a spacetime only with two parameters x, y and the time parameter t is shown.

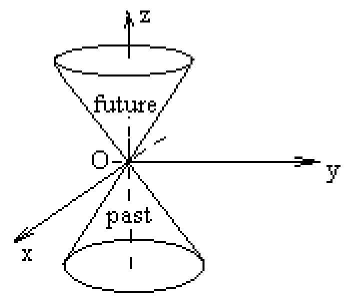


Fig.6.2

The Einstein's spacetime is an uniform linear space. By the assumption of linearity of a spacetime and invariance of the light speed, it can be shown that the invariance of the space-time intervals, i.e.,

For two reference systems S_1 *and* S_2 *with a homogenous relative velocity, there must be*

$$\triangle s^2 = \triangle s'^2.$$

We can also get the Lorentz transformation of spacetimes or velocities by this assumption. For two parallel reference systems S_1 and S_2, if the velocity of S_2 relative to S_1 is v along x-axis such as shown in Fig.6.3,

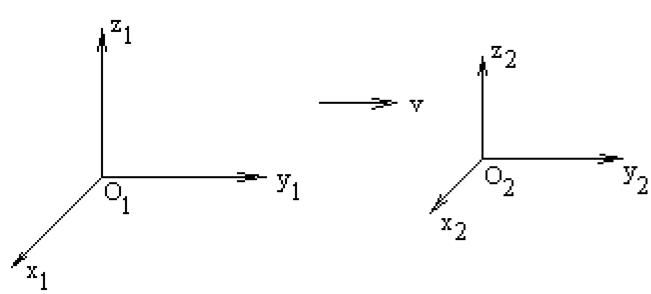


Fig.6.3

then we know the *Lorentz transformation of spacetimes*

$$\begin{cases} x_2 = \frac{x_1 - vt_1}{\sqrt{1-(\frac{v}{c})^2}} \\ y_2 = y_1 \\ z_2 = z_1 \\ t_2 = \frac{t_1 - \frac{v}{c}x_1}{\sqrt{1-(\frac{v}{c})^2}} \end{cases}$$

and the *transformation of velocities*

$$\begin{cases} v_{x_2} = \frac{v_{x_1}-v}{1-\frac{vv_{x_1}}{c^2}} \\ v_{y_2} = \frac{v_{y_1}\sqrt{1-(\frac{v}{c})^2}}{1-\frac{vv_{x_1}}{c^2}} \\ v_{z_2} = \frac{v_{z_1}\sqrt{1-(\frac{v}{c})^2}}{1-\frac{vv_{x_1}}{c^2}}. \end{cases}$$

In the relative spacetime, the *general interval* is defined by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu,$$

where $g_{\mu\nu} = g_{\mu\nu}(x^\sigma, t)$ is a metric dependent on the space and time. We can also introduce the invariance of general intervals, i.e.,

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = g'_{\mu\nu}dx'^\mu dx'^\nu.$$

Then the *Einstein's equivalence principle* says that

There are no difference for physical effects of the inertial force and the gravitation in a field small enough.

An immediately consequence of the Einstein's equivalence principle is the idea of the *geometrization of gravitation*, i.e., considering the curvature at each point in a spacetime to be all effect of gravitation([18]), which is called a *gravitational factor* at this point.

Combining these discussions in Section 6.1 with the Einstein's idea of the geometrization of gravitation, we get a result for spacetimes in the theoretical physics.

Theorem 6.2.1 *Every spacetime is a pseudo-face in an Euclid pseudo-space, especially, the Einstein's space-time is $\mathbf{R}^n$ in $(\mathbf{R}^4, \omega)$ for an integer $n, n \geq 4$.*

By the uniformity of a spacetime, we get an equation by equilibrium of vectors in a cosmos.

Theorem 6.2.2 *By the assumption of uniformity for a spacetime in $(\mathbf{R}^4, \omega)$, there exists an anti-vector ω_O^- of ω_O along any orientation $\overrightarrow{O}$ in $\mathbf{R}^4$ such that*

$$\omega_O + \omega_O^- = 0.$$

Proof Since $\mathbf{R}^4$ is uniformity, By the principle of equilibrium in a uniform space, along any orientation $\overrightarrow{O}$ in $\mathbf{R}^4$, there must exists an anti-vector ω_O^- of ω_O such that

$$\omega_O + \omega_O^- = 0. \natural$$

Theorem 6.2.2 has many useful applications. For example, let

$$\omega_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \lambda g_{\mu\nu},$$

then we know that

$$\omega_{\mu\nu}^- = -8\pi G T_{\mu\nu}.$$

in a gravitational field. Whence, we get the *Einstein's equation of gravitational field*

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \lambda g_{\mu\nu} = -8\pi G T_{\mu\nu}$$

by the equation in Theorem 6.2.2 which is widely used for our cosmos by physicists. In fact, there are two assumptions for our cosmos in the following. One is partially adopted from the Einstein's, another is just suggested by ours.

Postulate 6.2.1 *At the beginning our cosmos is homogenous.*

Postulate 6.2.2 *Human beings can only survey pseudo-faces of our cosmos by observations and experiments.*

Applying these postulates, the Einstein's equation of gravitational field and the *cosmological principle*, i.e., *there are no difference at different points and different orientations at a point of a cosmos on the metric* $10^4 l.y.$, we can get a standard model of cosmos, also called the *Friedmann cosmos*, seeing [18],[26], [28],[30] – [31],[79] and [95] for details. In this model, its line element ds is

$$ds^2 = -c^2dt^2 + a^2(t)[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)]$$

and cosmoses are classified into three types:

Static Cosmos: $da/dt = 0$;

Contracting Cosmos: $da/dt < 0$;

Expanding Cosmos: $da/dt > 0$.

By the Einstein's view, our living cosmos is the static cosmos. That is why he added a cosmological constant λ in his equation of gravitational field. But unfortunately, our cosmos is an expanding cosmos found by Hubble in 1929. As a by-product, the shape of our cosmos described by S.Hawking in [30] – [32] is coincide with these results gotten in Section 6.1.

§6.3 A Multi-Space Model for Cosmoses

6.3.1. What is M-theory

Today, we have know that all matter are made of atoms and sub-atomic particles, held together by four fundamental forces: *gravity, electro-magnetism, strong nuclear force* and *weak force*. Their features are partially explained by the *quantum theory* and the *relativity theory*. The former is a theory for the microcosm but the later is for the macrocosm. However, these two theories do not resemble each other in any way. The quantum theory reduces forces to the exchange of discrete packet of quanta, while the relativity theory explains the cosmic forces by postulating the smooth deformation of the fabric spacetime.

As we known, there are two string theories : the $E_8 \times E_8$ heterotic string, the *SO(32) heterotic string* and three superstring theories: the *SO(32) Type I string*, the *Type IIA* and *Type IIB* in superstring theories. Two physical theories are *dual* to each other if they have identical physics after a certain mathematical transformation. There are *T-duality* and *S-duality* in superstring theories defined in the following table 6.1([15]).

	fundamental string	dual string
$T-duality$	$Radius \leftrightarrow 1/(radius)$ $Kaluza-Klein \leftrightarrow Winding$	$charge \leftrightarrow 1/(charge)$ $Electric \leftrightarrow Magnet$
$S-duality$	$charge \leftrightarrow 1/(charge)$ $Electric \leftrightarrow Magnetic$	$Radius \leftrightarrow 1/(Radius)$ $Kaluza-Klein \leftrightarrow Winding$

table 6.1

We already know some profound properties for these spring or superspring theories, such as:

(i) *Type IIA and IIB are related by T-duality, as are the two heterotic theories.*

(ii) *Type I and heterotic $SO(32)$ are related by S-duality and Type IIB is also S-dual with itself.*

(iii) *Type II theories have two supersymmetries in the 10-dimensional sense, but the rest just one.*

(iv) *Type I theory is special in that it is based on unoriented open and closed strings, but the other four are based on oriented closed strings.*

(v) *The IIA theory is special because it is non-chiral(parity conserving), but the other four are chiral(parity violating).*

(vi) *In each of these cases there is an 11th dimension that becomes large at strong coupling. For substance, in the IIA case the 11th dimension is a circle and in IIB case it is a line interval, which makes 11-dimensional spacetime display two 10-dimensional boundaries.*

(vii) *The strong coupling limit of either theory produces an 11-dimensional spacetime.*

$(viii)$ $\cdots$, etc..

The M-theory was established by Witten in 1995 for the unity of those two string theories and three superstring theories, which postulates that all matter and energy can be reduced to *branes* of energy vibrating in an 11 dimensional space. This theory gives us a compelling explanation of the origin of our cosmos and combines all of existed string theories by showing those are just special cases of M-theory such as shown in table 6.2.

$$M-theory\left\{\begin{array}{l} E_8\times E_8\ heterotic\ string\\ SO(32)\ heterotic\ string\\ SO(32)\ Type\ I\ string\\ Type\ IIA\\ Type\ IIB.\end{array}\right.$$

Table 6.2

See Fig.6.4 for the M-theory planet in which we can find a relation of M-theory with these two strings or three superstring theories.

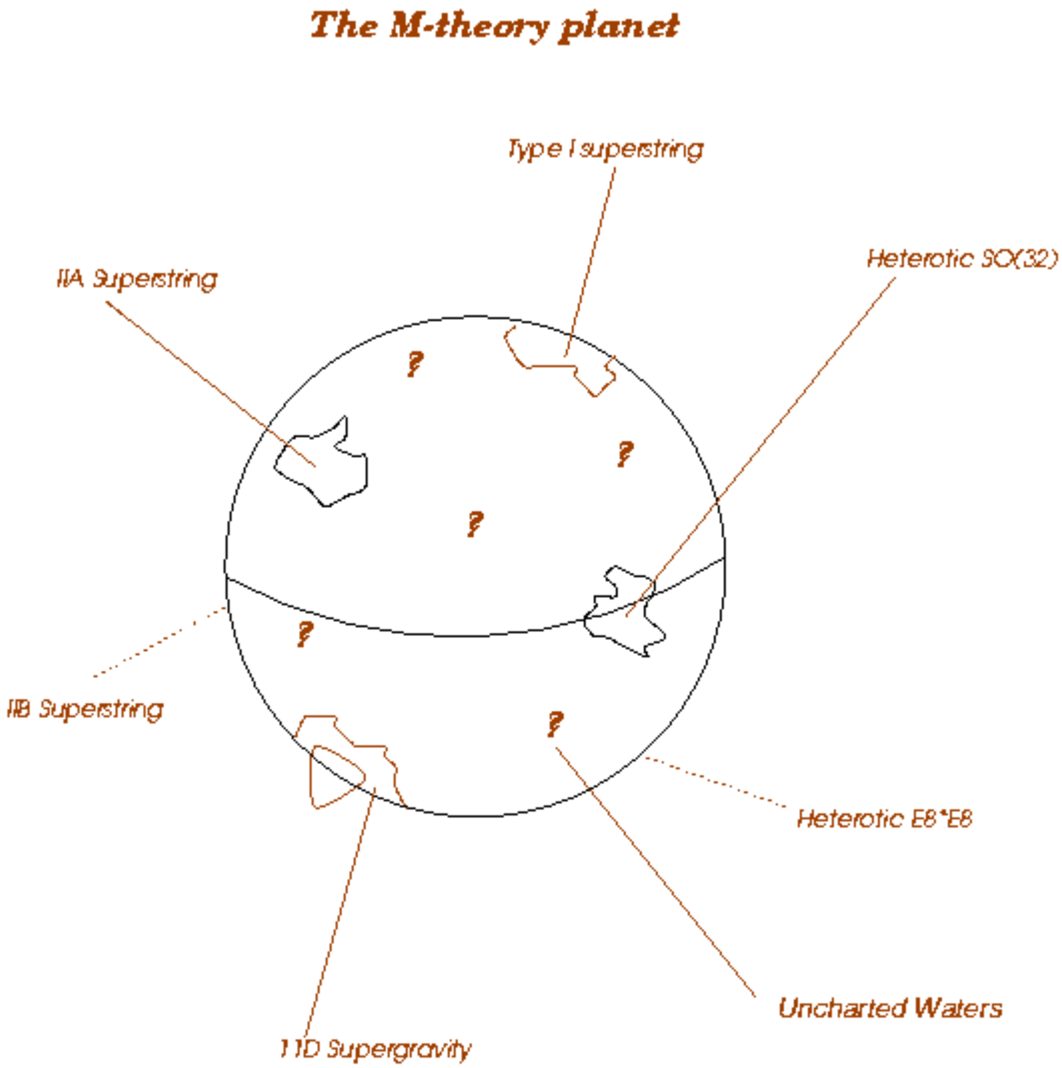


Fig.6.4

As it is widely accepted that our cosmos is in accelerating expansion, i.e., our cosmos is most possible an accelerating cosmos of expansion, it should satisfies the following condition

$$\frac{d^2a}{dt^2} > 0.$$

The *Kasner* type metric

$$ds^2 = -dt^2 + a(t)^2 d^2_{\mathbf{R}^3} + b(t)^2 ds^2(T^m)$$

solves the $4+m$ dimensional vacuum Einstein equations if

$$a(t) = t^{\mu} \ \text{ and } \ b(t) = t\nu$$

with

$$\mu = \frac{3 \pm \sqrt{3m(m+2)}}{3(m+3)}, \nu = \frac{3 \mp \sqrt{3m(m+2)}}{3(m+3)}.$$

These solutions in general do not give an accelerating expansion of spacetime of dimension 4. However, by using the time-shift symmetry

$$t \to t_{+\infty} - t, \quad a(t) = (t_{+\infty} - t)^{\mu},$$

we see that yields a really accelerating expansion since

$$\frac{da(t)}{dt} > 0 \text{ and } \frac{d^2 a(t)}{dt^2} > 0.$$

According to M-theory, our cosmos started as a perfect 11 dimensional space with nothing in it. However, this 11 dimensional space was unstable. The original 11 dimensional spacetime finally cracked into two pieces, a 4 and a 7 dimensional cosmos. The cosmos made the 7 of the 11 dimensions curled into a tiny ball, allowing the remaining 4 dimensional cosmos to inflate at enormous rates. This origin of our cosmos implies a multi-space result for our cosmos verified by Theorem 6.1.5.

Theorem 6.3.1 *The spacetime of M-theory is a multi-space with a warping* $\mathbf{R}^7$ *at each point of* $\mathbf{R}^4$.

Applying Theorem 6.3.1, an example for an accelerating expansion cosmos of 4-dimensional cosmos from supergravity compactification on hyperbolic spaces is the *Townsend-Wohlfarth type* in which the solution is

$$ds^2 = e^{-m\phi(t)}(-S^6 dt^2 + S^2 dx_3^2) + r_C^2 e^{2\phi(t)} ds_{H_m}^2,$$

where

$$\phi(t) = \frac{1}{m-1}(\ln K(t) - 3\lambda_0 t), \quad S^2 = K^{\frac{m}{m-1}} e^{-\frac{m+2}{m-1}\lambda_0 t}$$

and

$$K(t) = \frac{\lambda_0 \zeta r_c}{(m-1)\sin[\lambda_0 \zeta |t + t_1|]}$$

with $\zeta = \sqrt{3 + 6/m}$. This solution is obtainable from space-like brane solution and if the proper time ς is defined by $d\varsigma = S^3(t)dt$, then the conditions for expansion and acceleration are $\frac{dS}{d\varsigma} > 0$ and $\frac{d^2S}{d\varsigma^2} > 0$. For example, the expansion factor is 3.04 if $m = 7$, i.e., a really expanding cosmos.

6.3.2. **A pseudo-face model for p-branes**

In fact, M-theory contains much more than just strings, which is also implied in Fig.6.4. It contains both higher and lower dimensional objects, called *branes*. A *brane* is an object or subspace which can have various spatial dimensions. For any integer $p \geq 0$, a p-*brane* has length in p dimensions, for example, a 0-*brane* is just a point; a 1-*brane* is a string and a 2-*brane* is a surface or membrane $\cdots$.

Two branes and their motion have been shown in Fig.6.5 where (a) is a 1-brane and (b) is a 2-brane.

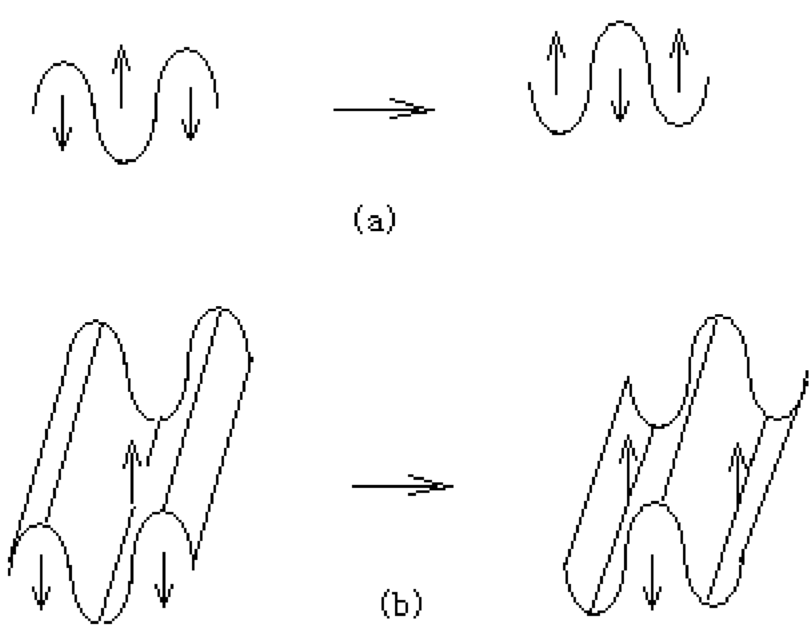


Fig.6.5

Combining these ideas in the pseudo-spaces theory and in M-theory, a model for $\mathbf{R}^m$ is constructed in the below.

Model 6.3.1 *For each m-brane $\mathbf{B}$ of a space $\mathbf{R}^m$, let $(n_1(\mathbf{B}), n_2(\mathbf{B}), \cdots, n_p(\mathbf{B}))$ be its unit vibrating normal vector along these p directions and $q : \mathbf{R}^m \to \mathbf{R}^4$ a continuous mapping. Now for $\forall P \in \mathbf{B}$, define*

$$\omega(q(P)) = (n_1(P), n_2(P), \cdots, n_p(P)).$$

Then $(\mathbf{R}^4, \omega)$ is a pseudo-face of $\mathbf{R}^m$, particularly, if $m = 11$, it is a pseudo-face for the M-theory.

For the case of $p = 4$, interesting results are obtained by applying results in Chapters 5.

Theorem 6.3.2 *For a sphere-like cosmos $\mathbf{B}^2$, there is a continuous mapping $q : \mathbf{B}^2 \to \mathbf{R}^2$ such that its spacetime is a pseudo-plane.*

Proof According to the classical geometry, we know that there is a projection $q : \mathbf{B}^2 \to \mathbf{R}^2$ from a 2-ball $\mathbf{B}^2$ to an Euclid plane $\mathbf{R}^2$, as shown in Fig.6.6.

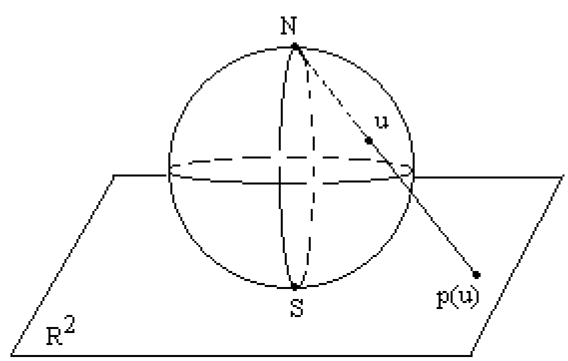


Fig.6.6

Now for any point $u \in \mathbf{B}^2$ with an unit vibrating normal vector $(x(u), y(u), z(u))$, define

$$\omega(q(u)) = (z(u), t),$$

where t is the time parameter. Then $(\mathbf{R}^2, \omega)$ is a pseudo-face of $(\mathbf{B}^2, t)$. $\natural$

Generally, we can also find pseudo-surfaces as a pseudo-face of sphere-like cosmoses.

Theorem 6.3.3 *For a sphere-like cosmos $\mathbf{B}^2$ and a surface S, there is a continuous mapping $q : \mathbf{B}^2 \to S$ such that its spacetime is a pseudo-surface on S.*

Proof According to the classification theorem of surfaces, an surface S can be combinatorially represented by a $2n$-polygon for an integer $n, n \geq 1$. If we assume that each edge of this polygon is at an infinite place, then the projection in Fig.6.6 also enables us to get a continuous mapping $q : \mathbf{B}^2 \to S$. Thereby we get a pseudo-face on S for the cosmos $\mathbf{B}^2$. ♮

Furthermore, we can construct a combinatorial model for our cosmos by applying materials in Section 2.5.

Model 6.3.2 *For each m-brane $\mathbf{B}$ of a space $\mathbf{R}^m$, let $(n_1(\mathbf{B}), n_2(\mathbf{B}), \cdots, n_p(\mathbf{B}))$ be its unit vibrating normal vector along these p directions and $q : \mathbf{R}^m \to \mathbf{R}^4$ a continuous mapping. Now construct a graph phase $(\mathcal{G}, \omega, \Lambda)$ by*

$$V(\mathcal{G}) = \{p - branes\ q(\mathbf{B})\},$$

$$E(\mathcal{G}) = \{(q(\mathbf{B}_1), q(\mathbf{B}_2)) | there\ is\ an\ action\ between\ \mathbf{B}_1\ and\ \mathbf{B}_2\},$$

$$\omega(q(\mathbf{B})) = (n_1(\mathbf{B}), n_2(\mathbf{B}), \cdots, n_p(\mathbf{B})),$$

and

$$\Lambda(q(\mathbf{B}_1), q(\mathbf{B}_2)) = \ forces\ between\ \mathbf{B}_1\ and\ \mathbf{B}_2.$$

Then we get a graph phase $(\mathcal{G}, \omega, \Lambda)$ in $\mathbf{R}^4$. Similarly, if $m = 11$, it is a graph phase for the M-theory.

If there are only finite p-branes in our cosmos, then Theorems 6.3.2 and 6.3.3 can be restated as follows.

Theorem 6.3.4 *For a sphere-like cosmos $\mathbf{B}^2$ with finite p-branes and a surface S, its spacetime is a map geometry on S.*

Now we consider the transport of a graph phase $(\mathcal{G}, \omega, \Lambda)$ in $\mathbf{R}^m$ by applying results in Sections 2.3 and 2.5.

Theorem 6.3.5 *A graph phase $(\mathcal{G}_1, \omega_1, \Lambda_1)$ of space $\mathbf{R}^m$ is transformable to a graph phase $(\mathcal{G}_2, \omega_2, \Lambda_2)$ of space $\mathbf{R}^n$ if and only if $\mathcal{G}_1$ is embeddable in $\mathbf{R}^n$ and there is a continuous mapping τ such that $\omega_2 = \tau(\omega_1)$ and $\Lambda_2 = \tau(\Lambda_1)$.*

Proof By the definition of transformations, if $(\mathcal{G}_1, \omega_1, \Lambda_1)$ is transformable to $(\mathcal{G}_2, \omega_2, \Lambda_2)$, then there must be $\mathcal{G}_1$ is embeddable in $\mathbf{R}^n$ and there is a continuous mapping τ such that $\omega_2 = \tau(\omega_1)$ and $\Lambda_2 = \tau(\Lambda_1)$.

Now if $\mathcal{G}_1$ is embeddable in $\mathbf{R}^n$ and there is a continuous mapping τ such that $\omega_2 = \tau(\omega_1)$, $\Lambda_2 = \tau(\Lambda_1)$, let $\varsigma : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ be a continuous mapping from $\mathcal{G}_1$ to $\mathcal{G}_2$, then (ς, τ) is continuous and

$$(\varsigma, \tau) : (\mathcal{G}_1, \omega_1, \Lambda_1) \rightarrow (\mathcal{G}_2, \omega_2, \Lambda_2).$$

Therefore $(\mathcal{G}_1, \omega_1, \Lambda_1)$ is transformable to $(\mathcal{G}_2, \omega_2, \Lambda_2)$. ♮

Theorem 6.3.5 has many interesting consequences as by-products.

Corollary 6.3.1 *A graph phase $(\mathcal{G}_1, \omega_1, \Lambda_1)$ in $\mathbf{R}^m$ is transformable to a planar graph phase $(\mathcal{G}_2, \omega_2, \Lambda_2)$ if and only if $\mathcal{G}_2$ is a planar embedding of $\mathcal{G}_1$ and there is a continuous mapping τ such that $\omega_2 = \tau(\omega_1)$, $\Lambda_2 = \tau(\Lambda_1)$ and vice via, a planar graph phase $(\mathcal{G}_2, \omega_2, \Lambda_2)$ is transformable to a graph phase $(\mathcal{G}_1, \omega_1, \Lambda_1)$ in $\mathbf{R}^m$ if and only if $\mathcal{G}_1$ is an embedding of $\mathcal{G}_2$ in $\mathbf{R}^m$ and there is a continuous mapping τ^{-1} such that $\omega_1 = \tau^{-1}(\omega_2)$, $\Lambda_1 = \tau^{-1}(\Lambda_2)$.*

Corollary 6.3.2 *For a continuous mapping τ, a graph phase $(\mathcal{G}_1, \omega_1, \Lambda_1)$ in $\mathbf{R}^m$ is transformable to a graph phase $(\mathcal{G}_2, \tau(\omega_1), \tau(\Lambda_1))$ in $\mathbf{R}^n$ with $m, n \geq 3$.*

Proof This result follows immediately from Theorems 2.3.2 and 6.3.5. ♮

This theorem can be also used to explain the problems of *travelling between cosmoses* or *getting into the heaven or hell* for a person. For example, water will go from a liquid phase to a steam phase by heating and then will go to a liquid phase by cooling because its phase is transformable between the steam phase and the liquid phase. For a person on the earth, he can only get into the heaven or hell after death because the dimension of the heaven is more than 4 and that of the hell is less than 4 and there does not exist a transformation for an alive person from our cosmos to the heaven or hell by the biological structure of his body. Whence, if black holes are really these tunnels between different cosmoses, the destiny for a cosmonaut unfortunately fell into a black hole is only the death ([30][32]). Perhaps, there are really other kind of beings in cosmoses or mankind in the further who can freely change from one phase in a space $\mathbf{R}^m$ to another in $\mathbf{R}^n$ with $m \neq n$, then the travelling between cosmoses is possible for those beings or mankind in that time.

6.3.3. **A multi-space model of cosmos**

Until today, many problems in cosmology are puzzling one's eyes. Comparing with these vast cosmoses, human beings are very tiny. In spite of this depressed fact, we can still investigate cosmoses by our deeply thinking. Motivated by this belief, a multi-space model for cosmoses is constructed in the following.

Model 6.3.3 *A mathematical cosmos is constructed by a triple* (Ω, Δ, T), *where*

$$\Omega = \bigcup_{i\geq 0} \Omega_i, \quad \Delta = \bigcup_{i\geq 0} O_i$$

and $T = \{t_i; i \geq 0\}$ *are respectively called the cosmos, the operation or the time set with the following conditions hold.*

(1) (Ω, Δ) *is a Smarandache multi-space dependent on* T, *i.e., the cosmos* (Ω_i, O_i) *is dependent on the time parameter* t_i *for any integer* $i, i \geq 0$.

(2) *For any integer* $i, i \geq 0$, *there is a sub-cosmos sequence*

$$(S): \ \Omega_i \supset \cdots \supset \Omega_{i1} \supset \Omega_{i0}$$

in the cosmos (Ω_i, O_i) *and for two sub-cosmoses* (Ω_{ij}, O_i) *and* (Ω_{il}, O_i), *if* $\Omega_{ij} \supset \Omega_{il}$, *then there is a homomorphism* $\rho_{\Omega_{ij},\Omega_{il}} : (\Omega_{ij}, O_i) \to (\Omega_{il}, O_i)$ *such that*

(*i*) *for* $\forall(\Omega_{i1}, O_i), (\Omega_{i2}, O_i)(\Omega_{i3}, O_i) \in (S)$, *if* $\Omega_{i1} \supset \Omega_{i2} \supset \Omega_{i3}$, *then*

$$\rho_{\Omega_{i1},\Omega_{i3}} = \rho_{\Omega_{i1},\Omega_{i2}} \circ \rho_{\Omega_{i2},\Omega_{i3}},$$

where "$\circ$" denotes the composition operation on homomorphisms.

(*ii*) *for* $\forall g, h \in \Omega_i$, *if for any integer* i, $\rho_{\Omega,\Omega_i}(g) = \rho_{\Omega,\Omega_i}(h)$, *then* $g = h$.

(*iii*) *for* $\forall i$, *if there is an* $f_i \in \Omega_i$ *with*

$$\rho_{\Omega_i,\Omega_i \bigcap \Omega_j}(f_i) = \rho_{\Omega_j,\Omega_i \bigcap \Omega_j}(f_j)$$

for integers $i, j, \Omega_i \bigcap \Omega_j \neq \emptyset$, *then there exists an* $f \in \Omega$ *such that* $\rho_{\Omega,\Omega_i}(f) = f_i$ *for any integer* i.

Notice that this model is a multi-cosmos model. In the Newton's mechanics, the Einstein's relativity theory or the M-theory, there is just one cosmos Ω and these sub-cosmos sequences are

$$\mathbf{R}^3 \supset \mathbf{R}^2 \supset \mathbf{R}^1 \supset \mathbf{R}^0 = \{P\},$$

$$\mathbf{R}^4 \supset \mathbf{R}^3 \supset \mathbf{R}^2 \supset \mathbf{R}^1 \supset \mathbf{R}^0 = \{P\}$$

and

$$\mathbf{R}^4 \supset \mathbf{R}^3 \supset \mathbf{R}^2 \supset \mathbf{R}^1 \supset \mathbf{R}^0 = \{P\} \supset \mathbf{R}_7^- \supset \cdots \supset \mathbf{R}_1^- \supset \mathbf{R}_0^- = \{Q\}.$$

These conditions in (2) are used to ensure that a mathematical cosmos posses a *general structure sheaf* of a topological space, for instance if we equip each multi-space (Ω_i, O_i) with an abelian group G_i for an integer $i, i \geq 0$, then we get a structure sheaf on a mathematical cosmos. For *general sheaf theory*, one can see in the reference [29] for details. This structure enables that a being in a cosmos of higher dimension can supervises those in lower dimension.

Motivated by this multi-space model of cosmos, we present a number of conjectures on cosmoses in the following. The first is on the number of cosmoses and their dimension.

Conjecture 6.3.1 *There are infinite many cosmoses and all dimensions of cosmoses make up an integer interval* $[1, +\infty]$.

A famous proverbs in Chinese says that *seeing is believing but hearing is unbelieving*, which is also a dogma in the pragmatism. Today, this view should be abandoned by a mathematician if he wish to investigate the 21st mathematics. On the first, we present a conjecture on the problem of travelling between cosmoses.

Conjecture 6.3.2 *There must exists a kind of beings who can get from one cosmos into another. There must exists a kind of being who can goes from a space of higher dimension into its subspace of lower dimension, especially, on the earth.*

Although nearly every physicist acknowledges the existence of black holes, those holes are really found by mathematical calculation. On the opposite, we present the next conjecture.

Conjecture 6.3.3 *Contrary to black holes, there are also white holes at where no matters can arrive including the light in our cosmos.*

Conjecture 6.3.4 *Every black hole is also a white hole in a cosmos.*

Our cosmonauts is good luck if Conjecture 6.3.4 is true since they do not need to worry about attracted by these black holes in our cosmos. Today, a very important task in theoretical and experimental physics is looking for dark matters. However, we do not think this would be success by the multi-model of cosmoses. This is included in the following conjecture.

Conjecture 6.3.5 *One can not find dark matters by experiments since they are in spatial can not be found by human beings.*

Few consideration is on the relation of the dark energy with dark matters. But we believe there exists a relation between the dark energy and dark matters such as stated in the next conjecture.

Conjecture 6.3.6 *Dark energy is just the effect of dark matters.*

References

[1] A.D.Aleksandrov and V.A.Zalgaller, *Intrinsic Geometry of Surfaces*, American Mathematical Society, Providence, Rhode Island, 1976.

[2] I.Antoniadis, Physics with large extra dimensions: String theory under experimental test, *Current Science*, Vol.81, No.12, 25(2001),1609-1613

[3] V.L.Arnold, *Ordinary differential Equations*, Massachusetts Institute of Technology, 1973.

[4] P.Bedrossian, G.Chen, R.Schelp, A generalization of Fan's condition for homiltonicity, pancyclicity and hamiltonian connectedness, *Discrete Math.*, Vol.115, 1993,39-50.

[5] J.C.Bermond, Hamiltonian graphs, *Selected Topics in Graph Theory* , Academic Press, London (1978).

[5] N.L.Biggs and A.T.White, *Permutation Groups and Combinatoric Structure*, Cambridge University Press, 1979.

[6] G.Birkhoff and S.Mac Lane, *A Survey of Modern Algebra*, Macmillan Publishing Co., Inc, 1977.

[7] B.Bollobás, *Random Graphs*, Academic Press Inc. (London) Ltd, 1985.

[8] J.A.Bondy and V.Chvátal, A method in graph theory, *Discrete Math.* 15(1976), 111-136.

[9] J.A.Bondy and U.S.R.Murty, *Graph Theory with Applications*, The Macmillan Press Ltd, 1976.

[10] M.Capobianco and J.C.Molluzzo, *Examples and Counterexamples in Graph Theory*, Elsevier North-Holland Inc, 1978.

[11] G.Chartrand and L.Lesniak, *Graphs & Digraphs*, Wadsworth, Inc., California, 1986.

[12] W.K.Chen, *Applied Graph Theory–Graphs and Electrical Theory*, North-Holland

Publishing Company, New York, 1976.

[13] S.S.Chern and W.H.Chern, *Lectures in Differential Geometry* (in Chinese), Peking University Press, 2001.

[14] D.Deleanu, *A Dictionary of Smarandache Mathematics*, Buxton University Press, London & New York,2

[15] M.J.Duff, A layman's guide to M-theory, *arXiv*: hep-th/9805177, v3, 2 July(1998).

[16] J.Edmonds, A combinatorial representation for polyhedral surfaces, *Notices Amer. Math. Soc.*, 7(1960).

[17] G.H.Fan, New sufficient conditions for cycle in graphs, *J.Combinat.Theory*, Ser.B(37),1984,221-227.

[18] B.J.Fei, *relativity theory and Non-Euclid Geometries*, Science Publisher Press, Beijing, 2005.

[19] C.Frenk, Where is the missing matter?
http://hom.flash.net/ csmith0/missing.htm.

[20] M.Gleiser, Where does matter come from?
http://hom.flash.net/ csmith0/matter.htm.

[21] C.Godsil and G.Royle, *Algebraic Graph Theory*, Springer-Verlag New York,Inc., 2001.

[22] J.E.Graver and M.E.Watkins, *Combinatorics with Emphasis on the Theory of Graphs*, Springer-Verlag, New York Inc,1977.

[23] J.L.Gross and T.W.Tucker, *Topological Graph Theory*, John Wiley & Sons, 1987.

[24] B.Grümbaum, *Convex Polytopes*, London Interscience Publishers, 1967.

[25] B.Grümbaum, Polytopes, graphs and complexes,*Bull.Amer.Math.Soc*, 76(1970), 1131-1201.

[26] B.Grümbaum, Polytopal graphs, *Studies in Mathematics (Mathematical Association Of America)* ,vol. 12(1975), 201-224.

[27] A.Guth, An eternity of bubbles,
http://hom.flash.net/ csmith0/bubbles.htm.

[28]] J.Hartle, Theories of everything and Hawking's wave function of the universe, in Gibbones, Shellard and Rankin eds:*The Future of Theoretical Physics and Cosmology*, Cambridge University Press, 2003.

[29] R.Hartshorne, *Algebraic Geometry*, Springer-Verlag New York, Inc., 1977.

[30] S.Hawking, *A Brief History of Times*, A Bantam Books/ November, 1996.

[31] S.Hawking, *The Universe in a Nutshell*, A Bantam Books/ November, 2001.

[32] S.Hawking, Sixty years in a nutshell, in Gibbones, Shellard and Rankin eds: *The Future of Theoretical Physics and Cosmology*, Cambridge University Press, 2003.

[33] K.Hoffman and R.Kunze, *Linear Algebra* (Second Edition), Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1971.

[34] L.G.Hua, *Introduction to Number Theory* (in Chinese), Science Publish Press, Beijing ,1979.

[35] H.Iseri, *Smarandache Manifolds*, American Research Press, Rehoboth, NM,2002.

[36] H.Iseri, *Partially Paradoxist Smarandache Geometries*, http://www.gallup.unm.edu/s̃marandache/Howard-Iseri-paper.htm.

[77] M.Kaku, *Hyperspace: A Scientific Odyssey through Parallel Universe, Time Warps and 10th Dimension*, Oxford Univ. Press.

[38] L.Kuciuk and M.Antholy, An Introduction to Smarandache Geometries, *Mathematics Magazine, Aurora, Canada*, Vol.12(2003), and online:
http://www.mathematicsmagazine.com/1-2004/Sm_Geom_1_2004.htm;
also at New Zealand Mathematics Colloquium, Massey University, Palmerston North, New Zealand, December 3-6,2001
http://atlas-conferences.com/c/a/h/f/09.htm;
also at the International Congress of Mathematicians (ICM2002), Beijing, China, 20-28, August, 2002,
http://www.icm2002.org.cn/B/Schedule.Section04.htm.

[39] J.M.Lee, *Riemannian Manifolds*, Springer-Verlag New York, Inc., 1997.

[40] E.V.Linder, Seeing darkness: the new cosmology, *arXiv*: astro-ph/0511197 v1, (7) Nov. 2005.

[41] S.D.Liu et.al, *Chaos and Fractals in Natural Sciences* (in Chinese), Peking University Press, 2003.

[42] Y.P.Liu, *Embeddability in Graphs*, Kluwer Academic Publishers, Dordrecht/ Boston/ London, 1995.

[43] Y.P.Liu, *Enumerative Theory of Maps*, Kluwer Academic Publishers, Dordrecht/ Boston/ London, 1999.

[44] Y.P.Liu, *Advances in Combinatorial Maps* (in Chinese), Northern Jiaotong Uni-

versity Publisher, Beijing, 2003.

[45] A.Malnic, Group action,coverings and lifts of automorphisms, *Discrete Math*, 182(1998),203-218.

[46] A.Malinic,R.Nedela and M.Škoviera, Lifting graph automorphisms by voltage assignment, *Europ. J.Combinatorics*, 21(2000),927-947.

[47] L.F.Mao, The maximum size of self-centered graphs with a given radius (in Chinese), J. Xidan University, Vol. 23(supp), 6-10, 1996.

[48] L.F.Mao, Hamiltonian graphs with constraints on the vertices degree in a subgraphs pair, *J. Institute of Taiyuan Machinery*, Vol.15(Supp.), 1994,79-90.

[49] L.F.Mao, A localization of Dirac's theorem for hamiltonian graphs, *J.Math.Res. & Exp.*, Vol.18, 2(1998), 188-190.

[50] L.F.Mao, An extension of localized Fan's condition (in Chinese), *J.Qufu Normal University*, Vol.26, 3(2000), 25-28.

[51] L.F.Mao, On the panfactorial property of Cayley graphs, *J.Math.Res.* & *Exp.*, Vol 22,No.3(2002),383-390.

[52] L.F.Mao, Localized neighborhood unions condition for hamiltonian graphs, *J.Henan Normal University*(Natural Science), Vol.30, 1(2002), 16-22.

[53] L.F.Mao, *A census of maps on surface with given underlying graphs*, A doctor thesis in Northern Jiaotong University, Beijing, 2002.

[54] L.F.Mao, Parallel bundles in planar map geometries, Scientia Magna, Vol.1(2005), No.2, 120-133.

[55] L.F.Mao, *On Automorphisms groups of Maps, Surfaces and Smarandache geometries*, A post-doctoral report at the Chinese Academy of Sciences(2005), also appearing in *Sientia Magna*, Vol.1(2005), No.2, 55-73.

[56] L.F.Mao, *Automorphism Groups of Maps, Surfaces and Smarandache Geometries*, American Research Press, 2005.

[57] L.F.Mao, A new view of combinatorial maps by Smarandache's notion, *arXiv*: math.GM/0506232.

[58] L.F.Mao, On algebraic multi-group spaces, *arXiv*:math.GM/05010427.

[59] L.F.Mao, On algebraic multi-ring spaces, *arXiv*:math.GM/05010478.

[60] L.F.Mao, On algebraic multi-vector spaces, *arXiv*:math.GM/05010479.

[61] L.F.Mao, On multi-metric spaces, *arXiv*:math.GM/05010480.

[62] L.F.Mao, An introduction to Smarandache geometries on maps, Reported at

the *2005 International Conference on Graphs and Combinatorics*, Zhejiang, P.R.China, *Scientia Magna*(accepted).

[63] L.F.Mao and F.Liu, New localized condition with $d(x,y) \geq 2$ for hamiltonian graphs (in Chinese), *J.Henan Normal University*(Natural Science), Vol.31, 1(2003), 17-21.

[64] L.F.Mao and Y.P.Liu, On the eccentricity value sequence of a simple graph, *J.Henan Normal University*(Natural Science), Vol.29, 4(2001),13-18.

[65] L.F.Mao and Y.P.Liu, New localized condition for hamiltonian graphs (in Chinese), *J.Qufu Normal University*, Vol. 27, 2(2001), 18-22.

[66] L.F.Mao and Y.P.Liu, Group action approach for enumerating maps on surfaces,*J.Applied Math. & Computing*, vol.13, No.1-2,201-215.

[67] L.F.Mao, Y.P.Liu, E.L.Wei, The semi-arc automorphism group of a graph with application to map enumeration, *Graphs and Combinatorics*(in pressing).

[68] W.S.Massey, *Algebraic topology: an introduction*, Springer-Verlag,New York, etc., 1977.

[69] B.Mohar and C.Thomassen, *Graphs on Surfaces*, The Johns Hopkins University Press, London, 2001.

[70] G.W.Moore, What is a brane? *Notices of the AMS*, Vol.52, No.2(2005), 214-215.

[71] R.Nedela and M Škoviera, Regular embeddings of canonical double coverings of graphs, *J.combinatorial Theory*,Ser.B 67, 249-277(1996).

[72] R.Nedela and M.Škoviera, Regular maps from voltage assignments and exponent groups, *Europ.J.Combinatorics*, 18(1997),807-823.

[73] L.Z.Nie and S.S.Ding, *Introduction to Algebra* (in Chinese), Higher Education Publishing Press, 1994.

[74] J.A.Nieto, Matroids and p-branes, *Adv. Theor. Math. Phys.*, 8(2004), 177-188.

[75] V.V.Nikulin and I.R.Shafarevlch, *Geometries and Groups*, Springer-Verlag Berlin Heidelberg, 1987.

[76] R.Penrose, The problem of spacetime singularities :implication for quantum gravity, in Gibbones, Shellard and Rankin eds:*The Future of Theoretical Physics and Cosmology*, Cambridge University Press, 2003.

[77] PlanetMath, *Smarandache Geometries*, http://planetmath.org/encyclopedia/SmarandacheGeometries.htm1.

[78] K.Polthier and M.Schmies, Straightest geodesics on polyhedral surfaces, in *Mathematical Visualization* (ed. by H.C.Hege and K.Polthier), Springer-Verlag, Berlin, 1998.

[79] M.Rees, Our complex cosmos and its future, in Gibbones, Shellard and Rankin eds:*The Future of Theoretical Physics and Cosmology*, Cambridge University Press, 2003.

[80] D.Reinhard, *Graph Theory*, Springer-Verlag New York,Inc., 2000.

[81] G.Ringel, *Map Color Theorem*, Springer-Verlag, Berlin, 1974.

[82] D.J.S.Robinson, *A Course in the Theory of Groups*, Springer-Verlag, New York Inc., 1982.

[83] H.J.Ryser, *Combinatorial Mathematics*, The Mathematical Association of America, 1963.

[84] R.H.Shi, 2-neighborhood and hamiltonian conditions, *J.Graph Theory*, Vol.16, 1992,267-271.

[85] F.Smarandache, *A Unifying Field in Logics. Neutrosopy: Neturosophic Probability, Set, and Logic*, American research Press, Rehoboth, 1999.

[86] F.Smarandache, Mixed noneuclidean geometries, *eprint arXiv: math/0010119*, 10/2000.

[87] F.Smarandache, Neutrosophy, a new Branch of Philosophy, *Multi-Valued Logic*, Vol.8, No.3(2002)(special issue on Neutrosophy and Neutrosophic Logic), 297-384.

[88] F.Smarandache, A Unifying Field in Logic: Neutrosophic Field, *Multi-Valued Logic*, Vol.8, No.3(2002)(special issue on Neutrosophy and Neutrosophic Logic), 385-438.

[89] F.Smarandache, There is no speed barrier for a wave phase nor for entangle particles, *Progress in Physics*, Vol.1, April(2005), 85-86.

[90] L.Smolin, What is the future of cosmology?
http://hom.flash.net/ csmith0/future.htm.

[91] S.Stahl, Generalized embedding schemes, *J.Graph Theory*, No.2,41-52, 1978.

[92] J.Stillwell, *Classical topology and combinatorial group theory*, Springer-Verlag New York Inc., 1980.

[93] R.Thomas and X.X.Yu, 4-connected projective graphs are hamiltonian, *J.Combin. Theory Ser.B*, 62(1994),114-132.

[94] C.Thomassen, infinite graphs, in *Selected Topics in Graph Theory*(edited by Lowell W.Beineke and Robin J. Wilson), Academic Press Inc. (London) LTD, 1983.

[95] K.Thone, Warping spacetime, in Gibbones, Shellard and Rankin eds:*The Future of Theoretical Physics and Cosmology*, Cambridge University Press, 2003.

[96] W.Thurston, On the geometry and dynamic of diffeomorphisms of surfaces, *Bull. Amer. Math. Soc.*, 19(1988), 417-431.

[97] W.T.Tutte, A theorem on planar graphs, *Trans, Amer. Math. Soc.*, 82(1956), 99-116.

[98] W.T.Tutte, What is a maps? in *New Directions in the Theory of Graphs* (ed.by F.Harary), Academic Press, 1973, 309-325.

[99] W.T.Tutte, Bridges and hamiltonian circuits in planar graphs. *Aequationes Mathematicae*, 15(1977),1-33.

[100] W.T.Tutte, *Graph Theory*, Combridge University Press, 2001.

[101]] W.B.Vasantha Kandasamy, *Bialgebraic structures and Smarandache bialgebraic structures*, American Research Press, 2003.

[102] W.B.Vasantha Kandasamy and F.Smarandache, *Basic Neutrosophic Algebraic Structures and Their Applications to Fuzzy and Neutrosophic Models*, Hexis, Church Rock, 2004.

[103] W.B.Vasantha Kandasamy and F.Smarandache, *N-Algebraic Structures and S-N-Algebraic Structures*, HEXIS, Phoenix, Arizona, 2005.

[104] H.Walther, *Ten Applications of Graph Theory*, D.Reidel Publishing Company, Dordrecht/Boston/Lancaster, 1984.

[105] C.V.Westenholz, *Differential Forms in Mathematical Physics*, North-Holland Publishing Company. New York, 1981.

[106] A.T.White, *Graphs of Group on Surfaces- interactions and models*, Elsevier Science B.V. 2001.

[107] Q.Y.Xong, *Ring Theory* (in Chinese), Wuhan University Press, 1993.

[108] M.Y.Xu, *Introduction to Group Theory* (in Chinese)(I)(II), Science Publish Press, Beijing ,1999.

[109] Y.C.Yang and L.F.Mao, On the cyclic structure of self-centered graphs with $r(G) = 3$ (in Chinese), *Pure and Applied Mathematics*, Vol.10(Special Issue), 88-98, 1994.

[110] H.P.Yap, *Some Topics in Graph Theory*, Cambridge University Press, 1986.

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About the Author

Linfan Mao is a researcher in the *Chinese Academy of Mathematics and Systems*, also a consultor in the Guoxin Tendering Co., Ltd, in Beijing. His main interesting focus on these *Smarandache geometries with applications to other sciences*, *combinatorial map theory* and *differential geometry*. Now he has published more than 30 papers on Smarandache geometries, combinatorial maps, graphs and operation research.

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Abstract

A Smarandache multi-space is a union of n different spaces equipped with some different structures for an integer $n \geq 2$, which can be both used for discrete or connected spaces, particularly for geometries and spacetimes in theoretical physics. This monograph concentrates on characterizing various multi-spaces including three parts altogether. The first part is on *algebraic multi-spaces with structures*, such as those of multi-groups, multi-rings, multi-vector spaces, multi-metric spaces, multi-operation systems and multi-manifolds, also multi-voltage graphs, multi-embedding of a graph in an n-manifold,$\cdots$, etc.. The second discusses *Smarandache geometries*, including those of map geometries, planar map geometries and pseudo-plane geometries, in which the *Finsler geometry*, particularly the *Riemann geometry* appears as a special case of these Smarandache geometries. The third part of this book considers the *applications of multi-spaces to theoretical physics*, including the relativity theory, the M-theory and the cosmology. Multi-space models for p-branes and cosmos are constructed and some questions in cosmology are clarified by multi-spaces. The first two parts are relative independence for reading and in each part open problems are included for further research of interested readers.

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