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On Special Curves According to Darboux Frame in the Three Dimensional Lorentz Space

H. S. Abdel-Aziz[1] and M. Khalifa Saad[1, 2, *]

Abstract: In the light of great importance of curves and their frames in many different branches of science, especially differential geometry as well as geometric properties and the uses in various fields, we are interested here to study a special kind of curves called Smarandache curves in Lorentz 3-space. Then, we present some characterizations for these curves and calculate their Darboux invariants. Moreover, we classify **TP**, **TU**, **PU** and **TPU**-Smarandache curves of a spacelike curve according to the causal character of the vector, curve and surface used in the study. Besides, we give some of differential geometric properties and important relations between that curves. Finally, to demonstrate our theoretical results a computational example is given with graph.

Keywords: Smarandache curves, spacelike curve, timelike surface, Darboux frame.

1 Introduction

The curves and their frames play an important role in differential geometry and in many branches of science such as mechanics and physics, so we are interested here in studying one of these curves which have many applications in Computer Aided Design (CAD), Computer Aided Geometric Design (CAGD) and mathematical modeling. Also, these curves can be used in the discrete model and equivalent model which are usually adopted for the design and mechanical analysis of grid structures [Dincel and Akbarov (2017)]. Smarandache Geometry is a geometry which has at least one Smarandachely denied axiom. It was developed by Smarandache [Smarandache (1969)]. We say that an axiom is Smarandachely denied if the axiom behaves in at least two different ways within the same space (i.e. validated and invalided, or only invalidated but in multiple distinct ways).

As a particular case, Euclidean, Lobachevsky-Bolyai-Gauss, and Riemannian geometries may be united altogether, in the same space, by some Smarandache geometries. Florentin Smarandache proposed a number of ways in which we could explore "new math concepts and theories, especially if they run counter to the classical ones".

In a manner consistent with his unique point of view, he defined several types of geometry that are purpose fully not Euclidean and that focus on structures that the rest of us can use to enhance our understanding of geometry in general.

To most of us, Euclidean geometry seems self-evident and natural. This feeling is so strong

[1] Math. Dept., Faculty of Science, Sohag University, 82524 Sohag, Egypt.

[2] Math. Dept., Faculty of Science, Islamic University of Madinah, 170 El-Madinah, KSA.

[*] Corresponding author: M. Khalifa Saad. Email: hsabdelaziz2005@yahoo.com.

that it took thousands of years for anyone to even consider an alternative to Euclid's teachings. These non-Euclidean ideas started, for the most part, with Gauss, Bolyai, and Lobachevski, and continued with Riemann, when they found counter examples to the notion that geometry is precisely Euclidean geometry. This opened a whole universe of possibilities for what geometry could be, and many years later, Smarandache's imagination has wandered off into this universe [Howard (2002)]. Curves are usually studied as subsets of an ambient space with a notion of equivalence. For example, one may study curves in the plane, the usual three dimensional space, the Minkowski space, curves on a sphere, etc. In three-dimensional curve theory, for a differentiable curve, at each point a triad of mutually orthogonal unit vectors (Frenet frame vectors) called tangent, normal and binormal can be constructed. In the light of the existing studies about the curves and their properties, authors introduced new curves. One of the important of among curves called Smarandache curve which using the Frenet frame vectors of a given curve. Among all space curves, Smarandache curves have special emplacement regarding their properties, this is the reason that they deserve special attention in Euclidean geometry as well as in other geometries. It is known that Smarandache geometry is a geometry which has at least one Smarandache denied axiom [Ashbacher (1997)]. An axiom is said to be Smarandache denied, if it behaves in at least two different ways within the same space.

Smarandache geometries are connected with the theory of relativity and the parallel universes and they are the objects of Smarandache geometry.

By definition, if the position vector of a curve δ is composed by Frenet frame's vectors of another curve β, then the curve δ is called a Smarandache curve [Turgut and Yilmaz (2008)]. The study of such curves is very important and many interesting results on these curves have been obtained by some geometers [Abdel-Aziz and Khalifa Saad (2015, 2017); Ali (2010); Bektas and Yunce (2013); Çetin and Kocayiğit (2013); Çetin, Tunçer and Karacan (2014)); Khalifa Saad (2016)]. Turgut et al. [Turgut and Yilmaz (2008)] introduced a particular circumstance of such curves. They entitled it Smarandache $\mathbf{TB}_2$ curves in the space E_4^1. Special Smarandache curves in such a manner that Smarandache curves $\mathbf{TN}_1$, $\mathbf{TN}_2$, $\mathbf{N}_1\mathbf{N}_2$ and $\mathbf{TN}_1\mathbf{N}_2$ with respect to Bishop frame in Euclidean 3-space have been seeked for by Çetin et al. [Çetin, Tunçer and Karacan (2014)]. Furthermore, they worked differential geometric properties of these special curves and they checked out first and second curvatures of these curves. Also, they get the centers of the curvature spheres and osculating spheres of Smarandache curves.

Recently, Abdel-Aziz et al. [Abdel-Aziz and Khalifa Saad (2015, 2016)] have studied special Smarandache curves of an arbitrary curve such as $\mathbf{TN}$, $\mathbf{TB}$ and $\mathbf{TNB}$ with respect to Frenet frame in the three-dimensional Galilean and pseudo-Galilean spaces. Also in Abdel-Aziz et al. [Abdel-Aziz and Khalifa Saad (2017)], authors have studied Smarandache curves of a timelike curve lying fully on a timelike surface according to Darboux frame in Minkowski 3-space.

In this work, for a given timelike surface and a spacelike curve lying fully on it, we study some special Smarandache curves with reference to Darboux frame in the three-dimensional Minkowski space E_1^3. We are looking forward to see that our results will be helpful to researchers who are specialized on mathematical modeling.

2 Basic concepts

Let Minkowski 3-space E_1^3 be the vector space E^3 provide with the Lorentzian inner product $\langle\ ,\ \rangle$ given by

$$\langle X, X\rangle = x_1^2 + x_2^2 - x_3^2$$

where $X = (x_1, x_2, x_3) \in E_1^3$. An arbitrary vector u in E_1^3 can have one of three Lorentzian causal characters; it can be spacelike, timelike and lightlike (null) if $\langle u,u\rangle > 0$ or $u = 0$, $\langle u,u\rangle < 0$ and $\langle u,u\rangle = 0$ and $u \neq 0$, respectively. Similarly, a curve r, locally parameterized by $r = r(s):\ I \subset R \to E_1^3$, where s is pseudo arc length parameter, is called a spacelike curve if $\langle r'(s), r'(s)\rangle > 0$, timelike if $\langle r'(s), r'(s)\rangle < 0$ and lightlike if $\langle r'(s), r'(s)\rangle = 0$ and $r'(s) \neq 0$ for all $s \in I$. The vectors $X = (x_1, x_2, x_3)$, $Y = (y_1, y_2, y_3) \in E_1^3$ are orthogonal if and only if $\langle X, Y\rangle = 0$ [O'Neil (1983)]. Also, the Lorentzian cross product of X and Y is given by

$$X \times Y = \begin{vmatrix} e_1 & e_2 & -e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}.$$

The norm of a vector $v \in E_1^3$ is given by $\|v\| = \sqrt{|\langle v, v\rangle|}$. We denote by $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ the moving Frenet frame along the curve $r(s)$ in the Minkowski space E_1^3, where the vectors **T**, **N** and **B** are called the tangent, principal normal and the binormal vectors of r, respectively. The following definition needs throughout this study.

Definition 2.1 A surface Ψ in the Minkowski 3-space E_1^3 is said to be spacelike, timelike surface if, respectively the induced metric on the surface is a positive definite Riemannian metric, Lorentz metric. In other words, the normal vector on the spacelike (timelike) surface is a timelike (spacelike) vector [O''Neil (1983)].

3 Smarandache curves of a spacelike curve

Let Ψ be an oriented timelike surface in Minkowski 3-space E_1^3 and $r = r(s)$ be a spacelike curve with timelike normal vector lying fully on it. Then, the Frenet equations of $r(s)$ are given by

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & \tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}, \tag{1}$$

where a prime denotes differentiation with respect to s . For this frame the following are satisfying

$$\langle \mathbf{T},\mathbf{T} \rangle = \langle \mathbf{B},\mathbf{B} \rangle = 1, \quad \langle \mathbf{N},\mathbf{N} \rangle = -1,$$
$$\langle \mathbf{T},\mathbf{N} \rangle = \langle \mathbf{T},\mathbf{B} \rangle = \langle \mathbf{N},\mathbf{B} \rangle = 0.$$

Let $\{\mathbf{T},\mathbf{P},\mathbf{U}\}$ be the Darboux frame of $r(s)$, then the relation between Frenet and Darboux frames takes the form [Do Carmo (1976); O'Neil (1983)]:

$$\begin{pmatrix} \mathbf{T} \\ \mathbf{P} \\ \mathbf{U} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & \sinh\theta \\ 0 & \sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}, \tag{2}$$

where $\mathbf{T}$ is the tangent vector of r and $\mathbf{U}$ is the unit normal to the surface Ψ and $\mathbf{P} = \mathbf{U} \times \mathbf{T}$. Therefore, the derivative formula of the Darboux frame of $r(s)$ is in the following form:

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{P}' \\ \mathbf{U}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa_g & -\kappa_n \\ \kappa_g & 0 & \tau_g \\ \kappa_n & \tau_g & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{P} \\ \mathbf{U} \end{pmatrix}. \tag{3}$$

The vectors $\mathbf{T},\mathbf{P}$ and $\mathbf{U}$ satisfy the following conditions:

$$\langle \mathbf{T},\mathbf{T} \rangle = \langle \mathbf{U},\mathbf{U} \rangle = 1, \ \langle \mathbf{P},\mathbf{P} \rangle = -1,$$
$$\langle \mathbf{T},\mathbf{U} \rangle = \langle \mathbf{T},\mathbf{P} \rangle = \langle \mathbf{P},\mathbf{U} \rangle = 0,$$
$$\mathbf{T} \times \mathbf{P} = -\mathbf{U}, \ \mathbf{P} \times \mathbf{U} = -\mathbf{T}, \ \mathbf{U} \times \mathbf{T} = \mathbf{P}.$$

In the differential geometry of surfaces, for a curve $r = r(s)$ lying on a surface M, the following are well-known [Do Carmo (1976)]

1) $r(s)$ is a geodesic curve if and only if $\kappa_g = 0$,
2) $r(s)$ is an asymptotic line if and only if $\kappa_n = 0$,
3) $r(s)$ is a principal line if and only if $\tau_g = 0$.

Definition 3.1 A regular curve in Minkowski space-time, whose position vector is composed by Frenet frame vectors on another regular curve, is called a Smarandache curve [Turgut and Yilmaz (2008)].

In the following, we investigate Smarandache curves $\mathbf{TP}$, $\mathbf{TU}$, $\mathbf{PU}$ and $\mathbf{TPU}$, and study some of their properties for a curve lies on a surface as follows:

3.1 TP-Smarandache curves

Definition 3.2 Let Ψ be an oriented timelike surface in E_1^3 and the unit speed spacelike curve $r = r(s)$ lying fully on Ψ with Darboux frame $\{\mathbf{T},\mathbf{P},\mathbf{U}\}$. Then the **TP-**Smarandache curves of r are defined by

$$\alpha(\bar{s}) = \frac{1}{\sqrt{2}}\left(\mathbf{T} + \mathbf{P}\right). \tag{4}$$

Theorem 3.1 Let $r = r(s)$ be a spacelike curve lying fully on a timelike surface Ψ in E_1^3 with Darboux frame $\{\mathbf{T}, \mathbf{P}, \mathbf{U}\}$, and non-zero curvatures; κ_n, κ_g *and* τ_g. Then the curvature functions of the **TP**-Smarandache curves of r satisfy the following equations:

$$\bar{\kappa}_\alpha = \left\| \frac{d\bar{\mathbf{T}}_\alpha}{d\bar{s}} \right\| = \frac{1}{\left(\kappa_n - \tau_g\right)^3} \sqrt{2\left(\omega_1^2 - \omega_2^2 - \omega_3^2\right)}, \tag{5}$$

$$\bar{\tau}_\alpha = -\frac{\left(\kappa_n - \tau_g\right)^4 \left[\begin{array}{c} 3\tau_g'^2\kappa_g + 3\kappa_n'^2\kappa_g + \tau_g'\left(-6\kappa_n'\kappa_g + \left(-3\kappa_g' - 5\kappa_g^{\,2} + \kappa_n\left(\kappa_n - \tau_g\right)\right)\left(\kappa_n - \tau_g\right)\right) \\ + \kappa_n'\left(\kappa_n - \tau_g\right)\left(3\kappa_g' + 5\kappa_g^{\,2} + \tau_g\left(-\kappa_n + \tau_g\right)\right) \\ + \left(\kappa_n - \tau_g\right)\left(\begin{array}{c} 5\kappa_g'\kappa_g\left(-\kappa_n + \tau_g\right) + 2\kappa_g^{\,3}\left(-\kappa_n + \tau_g\right) + \left(-\kappa_n + \tau_g\right)\kappa_g'' \\ + \kappa_g\left(2\left(\kappa_n - \tau_g\right)^2\left(\kappa_n + \tau_g\right) + \tau_g'' - \kappa_n''\right) \end{array} \right) \end{array} \right]}{\left(4\sqrt{2}\left(2\tau_g'\kappa_g - 2\kappa_n'\kappa_g + \left(\kappa_n - \tau_g\right)\left(2\kappa_g' + 3\kappa_g^{\,2} - \kappa_n^{\,2} + \tau_g^{\,2}\right)\right)\right)}. \tag{6}$$

Proof. Let $\alpha = \alpha(\bar{s})$ be a **TP-** Smarandache curve reference to a spacelike curve r. From Eq. (4), we get

$$\alpha' = \frac{d\alpha}{d\bar{s}}\frac{d\bar{s}}{ds} = \frac{1}{\sqrt{2}}\left(\kappa_g \mathbf{T} + \kappa_g \mathbf{P} + (\tau_g - \kappa_n)\mathbf{U}\right).$$

So, we have

$$\frac{d\bar{s}}{ds} = \frac{1}{\sqrt{2}}\left(\tau_g - \kappa_n\right), \tag{7}$$

this leads to

$$\bar{\mathbf{T}}_\alpha = \frac{1}{(\tau_g - \kappa_n)}\left(\kappa_g \mathbf{T} + \kappa_g \mathbf{P} + (\tau_g - \kappa_n)\mathbf{U}\right). \tag{8}$$

Differentiating Eq. (8) with respect to s and using Eq. (7), we obtain

$$\frac{d\bar{\mathbf{T}}_\alpha}{d\bar{s}} = \frac{\sqrt{2}}{\left(\kappa_n - \tau_g\right)^3}\left(\omega_1 \mathbf{T} + \omega_2 \mathbf{P} + \omega_3 \mathbf{U}\right),$$

where

$$\omega_1 = \kappa_g\left(\tau_g' - \kappa_n'\right) + \left(\kappa_n - \tau_g\right)\left(\kappa_g' + \kappa_g^{\,2} + \kappa_n\tau_g - \kappa_n^2\right),$$

$$\omega_2 = \kappa_g\left(\tau_g' - \kappa_n'\right) + \left(\kappa_n - \tau_g\right)\left(\kappa_g' + \kappa_g^{\,2} - \kappa_n\tau_g + \tau_g^{\,2}\right),$$

$\omega_3 = -\kappa_g(\kappa_n - \tau_g)^2.$

Then, the curvature is given by

$$\overline{\kappa}_\alpha = \left\| \frac{d\overline{\mathbf{T}}_\alpha}{d\overline{s}} \right\| = \frac{1}{(\kappa_n - \tau_g)^3}\sqrt{2(\omega_1^2 - \omega_2^2 - \omega_3^2)},$$

as denoted by Eq. (5).

And the principal normal vector field of the curve α is

$$\overline{\mathbf{N}}_\alpha = \frac{\omega_1\mathbf{T} + \omega_2\mathbf{P} + \omega_3\mathbf{U}}{\sqrt{\omega_1^2 - \omega_2^2 - \omega_3^2}}.$$

On the other hand, we express

$$\overline{\mathbf{B}}_\alpha = -(\overline{\mathbf{T}} \times \overline{\mathbf{N}}) = \frac{-1}{(\tau_g - \kappa_n)\sqrt{\omega_1^2 - \omega_2^2 - \omega_3^2}}\begin{vmatrix} \mathbf{T} & \mathbf{P} & -\mathbf{U} \\ \kappa_g & \kappa_g & (\tau_g - \kappa_n) \\ \omega_1 & \omega_2 & \omega_3 \end{vmatrix}.$$

So, the binormal vector of α is given by

$$\overline{\mathbf{B}}_\alpha = \frac{-1}{(\tau_g - \kappa_n)\sqrt{\omega_1^2 - \omega_2^2 - \omega_3^2}}(\overline{\omega}_1\mathbf{T} + \overline{\omega}_2\mathbf{P} + \overline{\omega}_3\mathbf{U}),$$

where

$$\overline{\omega}_1 = (\kappa_n - \tau_g)\omega_2 + \kappa_g\omega_3,$$
$$\overline{\omega}_2 = (\tau_g - \kappa_n)\omega_1 - \kappa_g\omega_3,$$
$$\overline{\omega}_3 = \kappa_g\omega_1 - \kappa_g\omega_2.$$

Now, in order to calculate the torsion of α, we consider the derivatives α'', α''' with respect to s as follows

$$\alpha'' = \frac{1}{\sqrt{2}}\{(\kappa_g' + \tau_g\kappa_n - \kappa_n^2 + \kappa_g^2)\mathbf{T} + (\kappa_g' + \tau_g^2 - \tau_g\kappa_n + \kappa_g^2)\mathbf{P} + (\tau_g' - \kappa_n' + \kappa_g\tau_g - \kappa_g\kappa_n)\mathbf{U}\},$$

$$\alpha''' = \frac{1}{\sqrt{2}}(\lambda_1\mathbf{T} + \lambda_2\mathbf{P} + \lambda_3\mathbf{U}),$$

where

$$\lambda_1 = (2\tau_g' - 3\kappa_n')\kappa_n + \kappa_n'\tau_g + \kappa_g(3\kappa_g' + \kappa_g{}^2 - \kappa_n{}^2 + \tau_g{}^2) + \kappa_g'',$$
$$\lambda_2 = -\tau_g'\kappa_n + (3\tau_g' - 2\kappa_n')\tau_g + \kappa_g(3\kappa_g' + \kappa_g{}^2 - \kappa_n{}^2 + \tau_g{}^2) + \kappa_g'',$$
$$\lambda_3 = \tau_g'\kappa_g - \kappa_n'\kappa_g - (\kappa_n - \tau_g)(2\kappa_g' + \kappa_g{}^2 - \kappa_n{}^2 + \tau_g{}^2) + \tau_g'' - \kappa_n''.$$

In the light of the above calculations, the torsion of α is calculated as Eq. (6).

Lemma 3.1 Let $\alpha(\bar{s})$ be a spacelike curve lies on a timelike surface Ψ in Minkowski 3-space E_1^3, then

1) If α is a geodesic curve, the following hold

$$\bar{\kappa}_\alpha = \sqrt{\frac{2(\kappa_n + \tau_g)}{(\kappa_n - \tau_g)}},$$

$$\bar{\tau}_\alpha = \frac{(\kappa_n - \tau_g)^3(\tau_g' \kappa_n - \kappa_n' \tau_g)}{4\sqrt{2}}.$$

2) If α is an asymptotic line, the following hold

$$\bar{\kappa}_\alpha = \sqrt{\frac{4\tau_g' \kappa_g + 2\tau_g\left(-2\kappa_g' - 3\kappa_g{}^2 - \tau_g{}^2\right)}{\tau_g{}^3}},$$

$$\bar{\tau}_\alpha = -\frac{\tau_g{}^4\left(-3\tau_g'^2\kappa_g - \tau_g'\left(3\kappa_g' + 5\kappa_g{}^2\right)\tau_g + \tau_g\left(\tau_g\left(\kappa_g\left(5\kappa_g' + 2\left(\kappa_g{}^2 + \tau_g{}^2\right)\right) + \kappa_g''\right) + \kappa_g\tau_g''\right)\right)}{4\sqrt{2}\left(-2\tau_g'\kappa_g + \tau_g\left(2\kappa_g' + 3\kappa_g{}^2 + \tau_g{}^2\right)\right)}.$$

3) If α is a principal line, the following hold

$$\bar{\kappa}_\alpha = \sqrt{\frac{4\kappa_n' \kappa_g + 2\kappa_n\left(-2\kappa_g' - 3\kappa_g{}^2 + \kappa_n{}^2\right)}{\kappa_n{}^3}},$$

$$\bar{\tau}_\alpha = \frac{\kappa_n{}^4\left(3\kappa_n'^2\kappa_g + \kappa_n'\left(3\kappa_g' + 5\kappa_g{}^2\right)\kappa_n + \kappa_n\left(-\kappa_n\left(\kappa_g\left(5\kappa_g' + 2\kappa_g{}^2 - 2\kappa_n{}^2\right) + \kappa_g''\right) - \kappa_g\kappa_n''\right)\right)}{4\sqrt{2}\left(2\kappa_n'\kappa_g + \kappa_n\left(-2\kappa_g' - 3\kappa_g{}^2 + \kappa_n{}^2\right)\right)}.$$

3.2 TU-Smarandache curves

Definition 3.3 Let Ψ be an oriented timelike surface in E_1^3 and the unit speed spacelike curve $r = r(s)$ lying fully on Ψ with Darboux frame $\{\mathbf{T}, \mathbf{P}, \mathbf{U}\}$. Then the **TU-**Smarandache curves of r are defined by

$$\beta(\bar{s}) = \frac{1}{\sqrt{2}}(\mathbf{T} + \mathbf{U}). \tag{9}$$

Theorem 3.2 Let $r = r(s)$ be a spacelike curve lying fully on a timelike surface Ψ in E_1^3 with Darboux frame $\{\mathbf{T}, \mathbf{P}, \mathbf{U}\}$, and non-zero curvatures; κ_n, κ_g *and* τ_g. Then the curvature functions of the **TU-** Smarandache curves of r satisfy the following equations:

$$\bar{\kappa}_\beta = \left\| \frac{d\bar{\mathbf{T}}_\beta}{d\bar{s}} \right\| = \frac{1}{(\kappa_g + \tau_g)^3}\sqrt{2\left(\varepsilon_1^2 - \varepsilon_2^2 - \varepsilon_3^2\right)}, \tag{10}$$

$$\bar{\tau}_\beta = \frac{\left[\begin{array}{c} \kappa_n\left(2\kappa_n' + \kappa_g{}^2 - \tau_g{}^2\right)\left((\kappa_g' - \tau_g')\kappa_n + 2\kappa_n'\left(\kappa_g - \tau_g\right) + \left(\kappa_g + \tau_g\right)\left(\kappa_g{}^2 - \kappa_n{}^2 + \tau_g{}^2\right) + \kappa_g'' + \tau_g''\right) \\ \left((\kappa_g' + \tau_g')\kappa_n + \kappa_g\left(\kappa_n' + 2\kappa_n{}^2\right) + \left(\kappa_n' - \kappa_g{}^2\right)\tau_g - 2\kappa_g\tau_g{}^2 - \tau_g{}^3\right) \\ \left(2\tau_g'\kappa_g + \kappa_g'\left(3\kappa_g + \tau_g\right) + \kappa_n\left(-3\kappa_n' + \kappa_g{}^2 - \kappa_n{}^2 + \tau_g{}^2\right) + \kappa_n''\right) \\ \left(-(\kappa_g' + \tau_g')\kappa_n + \kappa_n'\left(\kappa_g + \tau_g\right) + \kappa_g\left(-2\kappa_n{}^2 + \left(\kappa_g + \tau_g\right)^2\right)\right) \\ \left(-2\kappa_g'\tau_g - \tau_g'\left(\kappa_g + 3\tau_g\right) + \kappa_n[s]\left(3\kappa_n' + \kappa_g{}^2 - \kappa_n{}^2 + \tau_g{}^2\right) + \kappa_n''\right) \end{array}\right]}{4\sqrt{2}\left[-1 + \frac{1}{\left(\kappa_g + \tau_g\right)^5}\left(\begin{array}{c} 4(\kappa_g' + \tau_g')\kappa_n{}^3 - 4\left(\kappa_n' + \kappa_g{}^2\right)\kappa_n{}^2\left(\kappa_g + \tau_g\right) - \\ 2(\kappa_g' + \tau_g')\kappa_n\left(\kappa_g + \tau_g\right)^2 + \left(2\kappa_n' + \kappa_n{}^2\right)\left(\kappa_g + \tau_g\right)^3 + 2\kappa_g\left(\kappa_g + \tau_g\right)^4 \end{array}\right)\right]}. \tag{11}$$

Proof. Let $\beta = \beta(\bar{s})$ be a **TU-** Smarandache curve reference to a spacelike curve r. From Eq. (9), we get

$$\bar{\mathbf{T}}_\beta = \frac{1}{(\tau_g + \kappa_n)}\left(\kappa_n \mathbf{T} + \left(\kappa_g + \tau_g\right)\mathbf{P} - \kappa_n \mathbf{U}\right). \tag{12}$$

Differentiating Eq. (12) with respect to s, we get

$$\frac{d\bar{\mathbf{T}}_\beta}{d\bar{s}} = \frac{\sqrt{2}}{\left(\kappa_g + \tau_g\right)^3}\left(\varepsilon_1 \mathbf{T} + \varepsilon_2 \mathbf{P} + \varepsilon_3 \mathbf{U}\right),$$

where

$$\varepsilon_1 = \left((\kappa_g' + \tau_g')\kappa_n - \left(\kappa_g + \tau_g\right)\left(\kappa_n' - \kappa_n^2 + \kappa_g\left(\kappa_g + \tau_g\right)\right)\right),$$

$$\varepsilon_2 = \kappa_n\left(\tau_g^2 - \kappa_g^2\right),$$

$$\varepsilon_3 = \left(-(\kappa_g' + \tau_g')\kappa_n + \left(\kappa_g + \tau_g\right)\left(\kappa_n' + \kappa_n{}^2 - \tau_g\left(\kappa_g + \tau_g\right)\right)\right).$$

Therefore, the curvature $\bar{\kappa}$ is given by

$$\bar{\kappa}_\beta = \left\|\frac{d\bar{\mathbf{T}}_\beta}{d\bar{s}}\right\| = \frac{1}{\left(\kappa_g + \tau_g\right)^3}\sqrt{2\left(\varepsilon_1^2 - \varepsilon_2^2 - \varepsilon_3^2\right)},$$

as denoted by Eq. (10).

And principal normal vector field $\bar{\mathbf{N}}$ of β is

$$\bar{\mathbf{N}}_\beta = \frac{\varepsilon_1 \mathbf{T} + \varepsilon_2 \mathbf{P} + \varepsilon_3 \mathbf{U}}{\sqrt{\varepsilon_1^2 - \varepsilon_2^2 - \varepsilon_3^2}}.$$

Besides, the binormal vector of β is

$$\bar{\mathbf{B}}_\beta = \frac{-1}{(\tau_g + \kappa_n)\sqrt{\varepsilon_1^2 - \varepsilon_2^2 - \varepsilon_3^2}}\left(\bar{\varepsilon}_1 \mathbf{T} + \bar{\varepsilon}_2 \mathbf{P} + \bar{\varepsilon}_3 \mathbf{U}\right),$$

where

$$\bar{\varepsilon}_1 = \kappa_n \varepsilon_2 + \left(\kappa_g + \tau_g\right)\varepsilon_3,$$
$$\bar{\varepsilon}_2 = -\kappa_n \varepsilon_1 - \kappa_n \varepsilon_3,$$
$$\bar{\varepsilon}_3 = \left(\kappa_g + \tau_g\right)\varepsilon_1 - \kappa_n \varepsilon_2.$$

Differentiating β with respect to s, we get

$$\beta'' = \frac{1}{\sqrt{2}}\left\{(\kappa_n' + \tau_g\kappa_g - \kappa_n^2 + \kappa_g^2)\mathbf{T} + (\kappa_g' + \tau_g' - \tau_g\kappa_n + \kappa_n\kappa_g)\mathbf{P} + (-\kappa_n' + \kappa_g\tau_g + \tau_g^2 - \kappa_n^2)\mathbf{U}\right\},$$

similarly,

$$\beta''' = \frac{1}{\sqrt{2}}\left(\mu_1\mathbf{T} + \mu_2\mathbf{P} + \mu_3\mathbf{U}\right),$$

where

$$\mu_1 = 2\tau_g'\kappa_g + \kappa_g'\left(3\kappa_g + \tau_g\right) + \kappa_n\left(-3\kappa_n' + \kappa_g^{\,2} - \kappa_n^{\,2} + \tau_g^{\,2}\right) + \kappa_n'',$$
$$\mu_2 = (\kappa_g' - \tau_g')\kappa_n + 2\kappa_n'\left(\kappa_g - \tau_g\right) + \left(\kappa_g + \tau_g\right)\left(\kappa_g^{\,2} - \kappa_n^{\,2} + \tau_g^{\,2}\right) + \kappa_g'' + \tau_g'',$$
$$\mu_3 = 2\kappa_g'\tau_g + \tau_g'\left(\kappa_g + 3\tau_g\right) - \kappa_n\left(3\kappa_n' + \kappa_g^{\,2} - \kappa_n^{\,2} + \tau_g^{\,2}\right) - \kappa_n''.$$

It follows that, the torsion of β is expressed as in Eq. (11).

Lemma 3.2 Let $\beta(\bar{s})$ be a spacelike curve lies on Ψ in Minkowski 3-space E_1^3, then

1) If β is a geodesic curve, the following are satisfied

$$\bar{\kappa}_\beta = \sqrt{\frac{2\left(4\tau_g'\kappa_n^{\,3} - 4\kappa_n'\kappa_n^{\,2}\tau_g - 2\tau_g'\kappa_n\tau_g^{\,2} + \left(2\kappa_n' + \kappa_n^{\,2}\right)\tau_g^{\,3}\right) - 2\tau_g^{\,5}}{\tau_g^{\,5}}},$$

$$\bar{\tau}_\beta = \frac{\left\{\begin{array}{c} \kappa_n\left(2\kappa_n' - \tau_g^{\,2}\right)\left(-\tau_g'\kappa_n - 2\kappa_n'\tau_g + \tau_g\left(-\kappa_n^{\,2} + \tau_g^{\,2}\right) + \tau_g''\right) \\ -\left(\tau_g'\kappa_n + \kappa_n'\tau_g - \tau_g^{\,3}\right)\left(\kappa_n\left(-3\kappa_n' - \kappa_n^{\,2} + \tau_g^{\,2}\right) + \kappa_n''\right) \\ +\left(-\tau_g'\kappa_n + \kappa_n'\tau_g\right)\left(-3\tau_g'\tau_g + \kappa_n\left(3\kappa_n' - \kappa_n^{\,2} + \tau_g^{\,2}\right) + \kappa_n''\right) \end{array}\right\}}{\left(\frac{4\sqrt{2}\left(4\tau_g'\kappa_n^{\,3} - 4\kappa_n'\kappa_n^{\,2}\tau_g - 2\tau_g'\kappa_n\tau_g^{\,2} + \left(2\kappa_n' + \kappa_n^{\,2}\right)\tau_g^{\,3}\right)}{\tau_g^{\,5}} - 4\sqrt{2}\right)}.$$

2) If β is an asymptotic line, then

$$\bar{\kappa}_\beta = \sqrt{\frac{2\left(\kappa_g - \tau_g\right)}{\kappa_g + \tau_g}},$$

$$\bar{\tau}_\beta = \frac{\left(\kappa_g + \tau_g\right)^4\left(\kappa_g'\tau_g - \tau_g'\kappa_g\right)}{4\sqrt{2}\left(\kappa_g - \tau_g\right)}.$$

3) If β is a principal line, the following are satisfied

$$\bar{\kappa}_\beta=\sqrt{\frac{2\left(2\kappa_g{}^5-2\kappa_g'\kappa_g{}^2\kappa_n-4\kappa_g\left(\kappa_n'+\kappa_g{}^2\right)\kappa_n{}^2+4\kappa_g'\kappa_n{}^3+\kappa_g{}^3\left(2\kappa_n'+\kappa_n{}^2\right)\right)}{\kappa_g{}^5}-2},$$

$$\bar{\tau}_\beta=\frac{\kappa_g{}^5\begin{bmatrix}-3\kappa_g'^2\kappa_g\kappa_n+\kappa_n\left\{\kappa_g\begin{pmatrix}10\kappa_n'^2+7\kappa_n'\kappa_g{}^2+2\kappa_g{}^4\\-2\left(\kappa_n'+3\kappa_g{}^2\right)\kappa_n{}^2+4\kappa_n{}^4\end{pmatrix}+\left(2\kappa_n'+\kappa_g{}^2\right)\kappa_g''\right\}\\+\kappa_g'\left(\kappa_n'\left(-3\kappa_g{}^2+2\kappa_n{}^2\right)+\kappa_n\left(-7\kappa_g{}^2\kappa_n+2\kappa_n{}^3-2\kappa_n''\right)\right)\\+\kappa_g\left(\kappa_g{}^2-4\kappa_n{}^2\right)\kappa_n''\end{bmatrix}}{\left(4\sqrt{2}\left(2\kappa_n'\kappa_g{}^3+\kappa_g{}^5-2\kappa_g'\kappa_g{}^2\kappa_n-\kappa_g\left(4\kappa_n'+3\kappa_g{}^2\right)\kappa_n{}^2+4\kappa_g'\kappa_n{}^3\right)\right)}.$$

3.3 PU-Smarandache curves

Definition 3.4 Let Ψ be an oriented timelike surface in E_1^3 and the unit speed spacelike curve $r=r(s)$ lying fully on Ψ with Darboux frame $\{\mathbf{T},\mathbf{P},\mathbf{U}\}$. Then the **PU-**Smarandache curves of r are defined by

$$\gamma(\bar{s})=\frac{1}{\sqrt{2}}(\mathbf{P}+\mathbf{U}). \tag{13}$$

Theorem 3.3 Let $r=r(s)$ be a spacelike curve lying fully on a timelike surface Ψ in E_1^3 with Darboux frame $\{\mathbf{T},\mathbf{P},\mathbf{U}\}$, and non-zero curvatures; κ_n,κ_g *and* τ_g . Then the curvature functions of the **PU-**Smarandache curves of r satisfy the following equations:

$$\bar{\kappa}_\gamma=\left\|\frac{d\bar{\mathbf{T}}_\gamma}{d\bar{s}}\right\|=\frac{1}{\left(\left(\kappa_g+\kappa_n\right)^2-2\tau_g{}^2\right)^2}\sqrt{2\left(\zeta_1^2-\zeta_2^2-\zeta_3^2\right)}, \tag{14}$$

$$\bar{\tau}_\gamma=\frac{\begin{bmatrix}-\left((\kappa_g'+\kappa_n')\tau_g+\left(\kappa_g+\kappa_n\right)\left(\tau_g'-\kappa_n\left(\kappa_g+\kappa_n\right)+2\tau_g{}^2\right)\right)\\\left(2\kappa_n'\kappa_g+\kappa_g'\left(3\kappa_g+\kappa_n\right)+\tau_g\left(3\tau_g'+\kappa_g{}^2-\kappa_n{}^2+\tau_g{}^2\right)+\tau_g''\right)\\+\left(\left(\kappa_g+\kappa_n\right)\left(\tau_g'+\kappa_g\left(\kappa_g+\kappa_n\right)\right)-(\kappa_g'+\kappa_n')\tau_g\right)\\\left(-2\kappa_g'\kappa_n-\kappa_n'\left(\kappa_g+3\kappa_n\right)+\tau_g\left(3\tau_g'+\kappa_g{}^2-\kappa_n{}^2+\tau_g{}^2\right)+\tau_g''\right)\\+\tau_g\left(2\tau_g'+\kappa_g{}^2-\kappa_n{}^2+2\tau_g{}^2\right)\\\left((\kappa_g'+\kappa_n')\tau_g+\left(\kappa_g+\kappa_n\right)\left(2\tau_g'+\kappa_g{}^2-\kappa_n{}^2+\tau_g{}^2\right)+\kappa_g''+\kappa_n''\right)\end{bmatrix}}{2\sqrt{2}\left(\frac{1}{\left(\left(\kappa_g+\kappa_n\right)^2-2\tau_g{}^2\right)^2}\sqrt{2\left(\zeta_1^2-\zeta_2^2-\zeta_3^2\right)}\right)^2}. \tag{15}$$

Proof. Let $\gamma=\gamma(\bar{s})$ be a **PU-**Smarandache curve reference to a spacelike curve r . From Eq. (13), we obtain

$$\overline{\mathbf{T}}_\gamma = \frac{\left(\left(\kappa_g + \kappa_n\right)\mathbf{T} + \tau_g \mathbf{P} + \tau_g \mathbf{U}\right)}{\sqrt{\left(\kappa_g + \kappa_n\right)^2 - 2\tau_g^{\ 2}}}. \tag{16}$$

Differentiating Eq. (16) with respect to s, we have

$$\frac{d\overline{\mathbf{T}}_\gamma}{d\bar{s}} = \frac{\sqrt{2}}{\left(\left(\kappa_g + \kappa_n\right)^2 - 2\tau_g^{\ 2}\right)^2}\left(\zeta_1 \mathbf{T} + \zeta_2 \mathbf{P} + \zeta_3 \mathbf{U}\right),$$

where

$$\zeta_1 = \tau_g\left(-2(\kappa_g^{'} + \kappa_n^{'})\tau_g + \left(\kappa_g + \kappa_n\right)\left(2\tau_g^{'} + \left(\kappa_g + \kappa_n\right)^2 - 2\tau_g^{\ 2}\right)\right),$$

$$\zeta_2 = \begin{bmatrix} \left(\kappa_g + \kappa_n\right)^2\left(\tau_g^{'} + \kappa_g\left(\kappa_g + \kappa_n\right)\right) - (\kappa_g^{'} + \kappa_n^{'})\left(\kappa_g + \kappa_n\right)\tau_g \\ + \left(-\kappa_g^{\ 2} + \kappa_n^{\ 2}\right)\tau_g^{\ 2} - 2\tau_g^{\ 4} \end{bmatrix},$$

$$\zeta_3 = \begin{bmatrix} \tau_g^{'}\left(\kappa_g + \kappa_n\right)^2 - \kappa_n\left(\kappa_g + \kappa_n\right)^3 - (\kappa_g^{'} + \kappa_n^{'})\left(\kappa_g + \kappa_n\right)\tau_g \\ + \left(\kappa_g + \kappa_n\right)\left(\kappa_g + 3\kappa_n\right)\tau_g^{\ 2} - 2\tau_g^{\ 4} \end{bmatrix},$$

and then, the curvature of γ is given by

$$\bar{\kappa}_\gamma = \left\| \frac{d\overline{\mathbf{T}}_\gamma}{d\bar{s}} \right\| = \frac{1}{\left(\left(\kappa_g + \kappa_n\right)^2 - 2\tau_g^{\ 2}\right)^2}\sqrt{2\left(\zeta_1^2 - \zeta_2^2 - \zeta_3^2\right)},$$

which is denoted by Eq. (14).

Based on the above calculations, we can express the principal normal vector of γ as follows

$$\overline{\mathbf{N}}_\gamma = \frac{\zeta_1 \mathbf{T} + \zeta_2 \mathbf{P} + \zeta_3 \mathbf{U}}{\sqrt{\zeta_1^2 - \zeta_2^2 - \zeta_3^2}}.$$

Also,

$$\overline{\mathbf{B}}_\gamma = \frac{-1}{\sqrt{\left(\kappa_g + \kappa_n\right)^2 - 2\tau_g^{\ 2}}\sqrt{\zeta_1^2 - \zeta_2^2 - \zeta_3^2}}\left(\bar{\zeta}_1 \mathbf{T} + \bar{\zeta}_2 \mathbf{P} + \bar{\zeta}_3 \mathbf{U}\right),$$

where

$$\bar{\zeta}_1 = \tau_g \zeta_2 + \tau_g \zeta_3,$$
$$\bar{\zeta}_2 = \tau_g \zeta_1 - \left(\kappa_g + \kappa_n\right)\zeta_3,$$
$$\bar{\zeta}_3 = \tau_g \zeta_1 - \left(\kappa_g + \kappa_n\right)\zeta_2.$$

The derivatives γ'' and γ''' as follows

$$\gamma'' = \frac{1}{\sqrt{2}}\left\{(\kappa_g^{'} + \kappa_n^{'} + \tau_g \kappa_g + \tau_g \kappa_n)\mathbf{T} + (\tau_g^{'} + \kappa_g^2 + \kappa_n \kappa_g + \tau_g^{\ 2})\mathbf{P} + (\tau_g^{'} - \kappa_n \kappa_g - \kappa_n^2 + \tau_g^{\ 2})\mathbf{U}\right\},$$

$$\gamma''' = \frac{1}{\sqrt{2}}\left(\nu_1 \mathbf{T} + \nu_2 \mathbf{P} + \nu_3 \mathbf{U}\right),$$

where

$$\nu_1 = (\kappa_g' + \kappa_n')\tau_g + \left(\kappa_g + \kappa_n\right)\left(2\tau_g' + \kappa_g^{\ 2} - \kappa_n^{\ 2} + \tau_g^{\ 2}\right) + \kappa_g'' + \kappa_n'',$$

$$\nu_2 = 2\kappa_n'\kappa_g + \kappa_g'\left(3\kappa_g + \kappa_n\right) + \tau_g\left(3\tau_g' + \kappa_g^{\ 2} - \kappa_n^{\ 2} + \tau_g^{\ 2}\right) + \tau_g'',$$

$$\nu_3 = -2\kappa_g'\kappa_n - \kappa_n'\left(\kappa_g + 3\kappa_n\right) + \tau_g\left(3\tau_g' + \kappa_g^{\ 2} - \kappa_n^{\ 2} + \tau_g^{\ 2}\right) + \tau_g'',$$

According to the above calculations, we obtain the torsion of γ as in Eq. (15).

Lemma 3.3 Let $\gamma(\bar{s})$ be a spacelike curve lies on Ψ in Minkowski 3-space E_1^3, then

1) If γ is a geodesic curve, the curvature and torsion of γ are, respectively

$$\bar{\kappa}_\gamma = \sqrt{\left[\frac{1}{\left(\kappa_n^{\ 2} - 2\tau_g^{\ 2}\right)^3}\left\{\begin{array}{c} -2\kappa_n^{\ 2}\left(2\tau_g'^2 - 2\tau_g'\kappa_n^{\ 2} + \kappa_n^{\ 4}\right) \\ +4\kappa_n'\kappa_n\left(2\tau_g' - \kappa_n^{\ 2}\right)\tau_g \\ +2\left(-2\kappa_n'^2 + 5\kappa_n^{\ 4}\right)\tau_g^{\ 2} - 16\kappa_n^{\ 2}\tau_g^{\ 4} + 8\tau_g^{\ 6} \end{array}\right\}\right]},$$

$$\bar{\tau}_\gamma = \frac{1}{2\sqrt{2}}\left[\begin{array}{c} 4\tau_g'^2\kappa_n\tau_g + 3\kappa_n'^2\kappa_n\tau_g + \kappa_n'\tau_g\left(\kappa_n^{\ 2}\tau_g - 2\tau_g''\right) \\ -\tau_g'\left(\kappa_n'\left(3\kappa_n^{\ 2} + 4\tau_g^{\ 2}\right) + \tau_g\left(\kappa_n^{\ 3} - 2\kappa_n''\right)\right) \\ +\left(\kappa_n^{\ 2} - 2\tau_g^{\ 2}\right)\left(\kappa_n\tau_g'' - \tau_g\kappa_n''\right) \end{array}\right].$$

2) If γ is an asymptotic line, we get

$$\bar{\kappa}_\gamma = \sqrt{\frac{\left\{\begin{array}{c} -2\kappa_g^{\ 2}\left(2\tau_g'^2 + 2\tau_g'\kappa_g^{\ 2} + \kappa_g^{\ 4}\right) \\ +4\kappa_g'\kappa_g\left(2\tau_g' + \kappa_g^{\ 2}\right)\tau_g + 2\left(-2\kappa_g'^2 + \kappa_g^{\ 4}\right)\tau_g^{\ 2} + 8\tau_g^{\ 6} \end{array}\right\}}{\left(\kappa_g^{\ 2} - 2\tau_g^{\ 2}\right)^3}},$$

$$\bar{\tau}_\gamma = \frac{1}{2\sqrt{2}}\left[\begin{array}{c} -3\kappa_g'^2\kappa_g\tau_g + \tau_g\left(4\tau_g'^2\kappa_g + 7\tau_g'\kappa_g^{\ 3} + 2\kappa_g^{\ 3}\left(\kappa_g^{\ 2} + \tau_g^{\ 2}\right) + \left(2\tau_g' + \kappa_g^{\ 2} + 2\tau_g^{\ 2}\right)\kappa_g''\right) \\ +\kappa_g\left(\kappa_g^{\ 2} - 2\tau_g^{\ 2}\right)\tau_g'' - \kappa_g'\left(\tau_g'\left(3\kappa_g^{\ 2} + 4\tau_g^{\ 2}\right) + \tau_g\left(7\kappa_g^{\ 2}\tau_g + 2\tau_g''\right)\right) \end{array}\right].$$

3) If γ is a principal line, the following hold

$$\bar{\kappa}_\gamma = \sqrt{2\left(\frac{-\kappa_g^{\ 2} - \kappa_n^{\ 2}}{\left(\kappa_g + \kappa_n\right)^2}\right)},$$

$$\bar{\tau}_\gamma = \frac{\left(\kappa_g + \kappa_n\right)^3\left(\kappa_g'\kappa_n - \kappa_n'\kappa_g\right)}{2\sqrt{2}}. \qquad (17)$$

3.4 TPU-Smarandache curves

Definition 3.5 Let Ψ be an oriented timelike surface in E_1^3 and the unit speed spacelike curve $r = r(s)$ lying fully on Ψ with Darboux frame $\{\mathbf{T}, \mathbf{P}, \mathbf{U}\}$. Then the **TPU**-Smarandache curves of r are defined by

$$\delta(\bar{s}) = \frac{1}{\sqrt{3}}(\mathbf{T} + \mathbf{P} + \mathbf{U}). \tag{18}$$

Theorem 3.4 Let $r = r(s)$ be a spacelike curve lying fully on a timelike surface Ψ in E_1^3 with Darboux frame $\{\mathbf{T}, \mathbf{P}, \mathbf{U}\}$, and non-zero curvatures; κ_n, κ_g *and* τ_g. Then the curvature functions of the **TPU**-Smarandache curves of r satisfy the following equations:

$$\bar{\kappa}_\delta = \left\| \frac{d\bar{\mathbf{T}}_\delta}{d\bar{s}} \right\| = \frac{1}{4(\kappa_n - \tau_g)^2(\kappa_g + \tau_g)^2}\sqrt{3(\xi_1^2 - \xi_2^2 - \xi_3^2)}, \tag{19}$$

$$\bar{\tau}_\delta = \frac{(D_1 + (\kappa_n - \tau_g)D_2)}{3\sqrt{3}\left(\frac{1}{4(\kappa_n - \tau_g)^2(\kappa_g + \tau_g)^2}\sqrt{3(\xi_1^2 - \xi_2^2 - \xi_3^2)}\right)^2}.$$

(20)

Proof. Let $\delta = \delta(\bar{s})$ be a **TPU**-Smarandache curve reference to a spacelike curve r.

$$\bar{\mathbf{T}}_\delta = \frac{((\kappa_g + \kappa_n)\mathbf{T} + (\kappa_g + \tau_g)\mathbf{P} + (\tau_g - \kappa_n)\mathbf{U})}{\sqrt{2(\kappa_n - \tau_g)(\kappa_g + \tau_g)}}. \tag{21}$$

Differentiating (21) with respect to s, we get

$$\frac{d\bar{\mathbf{T}}_\delta}{d\bar{s}} = \frac{\sqrt{3}}{4(\kappa_n - \tau_g)^2(\kappa_g + \tau_g)^2}(\xi_1\mathbf{T} + \xi_2\mathbf{P} + \xi_3\mathbf{U}),$$

where

$$\xi_1 = \begin{array}{l}(\kappa_g + \tau_g)^2(\tau_g' - \kappa_n') - (\kappa_n - \tau_g)^2(\kappa_g' + \tau_g') - 2\kappa_n(\kappa_n - \tau_g)^2(\kappa_g + \tau_g) \\ \qquad + (\kappa_n - \tau_g)(\kappa_g + \tau_g)(\kappa_g' + \kappa_n' + 2\kappa_g(\kappa_g + \tau_g)),\end{array}$$

$$\xi_2 = (\kappa_g + \tau_g)(\tau_g'(\kappa_g + \kappa_n) - \kappa_n'(\kappa_g + \tau_g) + (\kappa_n - \tau_g)(\kappa_g' + 2(\kappa_g(\kappa_g + \kappa_n) - \kappa_n\tau_g + \tau_g^{\ 2}))),$$

$$\xi_3 = (\kappa_n - \tau_g)(\tau_g'(\kappa_g + \kappa_n) + \kappa_g'(\kappa_n - \tau_g) - (\kappa_g + \tau_g)(\kappa_n' + 2(\kappa_n - \tau_g)(\kappa_g + \kappa_n + \tau_g))).$$

Then, the curvature is given by

$$\bar{\kappa}_\delta = \left\| \frac{d\bar{\mathbf{T}}_\delta}{d\bar{s}} \right\| = \frac{1}{4(\kappa_n - \tau_g)^2(\kappa_g + \tau_g)^2}\sqrt{3(\xi_1^2 - \xi_2^2 - \xi_3^2)},$$

as denoted in Eq. (19).

And the principal normal vector field of δ is

$$\overline{\mathbf{N}}_\delta = \frac{\xi_1\mathbf{T}+\xi_2\mathbf{P}+\xi_3\mathbf{U}}{\sqrt{\xi_1^2-\xi_2^2-\xi_3^2}}.$$

So, the binormal vector of δ is

$$\overline{\mathbf{B}}_\delta = \frac{-1}{\sqrt{2(\kappa_n-\tau_g)(\kappa_g+\tau_g)}\sqrt{\xi_1^2-\xi_2^2-\xi_3^2}}\left(\bar{\xi}_1\mathbf{T}+\bar{\xi}_2\mathbf{P}+\bar{\xi}_3\mathbf{U}\right),$$

where

$$\bar{\xi}_1 = (\kappa_n-\tau_g)\xi_2+(\kappa_g+\tau_g)\xi_3,$$
$$\bar{\xi}_2 = (\tau_g-\kappa_n)\xi_1-(\kappa_g+\kappa_n)\xi_3,$$
$$\bar{\xi}_3 = (\kappa_g+\tau_g)\xi_1-(\kappa_g+\kappa_n)\xi_2.$$

For computing the torsion of δ , we are going to differentiate δ' with respect to s as follows

$$\delta'' = \frac{1}{\sqrt{3}}\left\{\begin{array}{c}(\kappa_g'+\kappa_n'-\kappa_n^2+\kappa_g^2+\tau_g\kappa_g+\tau_g\kappa_n)\mathbf{T}+(\kappa_g'+\tau_g'+\kappa_g^2+\kappa_n\kappa_g-\kappa_n\tau_g+\tau_g^{\ 2})\mathbf{P}\\ +(\tau_g'-\kappa_n'-\kappa_n\kappa_g+\tau_g\kappa_g-\kappa_n^2+\tau_g^{\ 2})\mathbf{U}\end{array}\right\},$$

and similarly

$$\delta''' = \frac{1}{\sqrt{3}}\left(\eta_1\mathbf{T}+\eta_2\mathbf{P}+\eta_3\mathbf{U}\right),$$

where

$$\eta_1 = 2\tau_g'(\kappa_g+\kappa_n)+\kappa_g'(3\kappa_g+\tau_g)+\kappa_n'(-3\kappa_n+\tau_g)+(\kappa_g+\kappa_n)(\kappa_g^{\ 2}-\kappa_n^{\ 2}+\tau_g^{\ 2})+\kappa_g''+\kappa_n'',$$
$$\eta_2 = -\tau_g'\kappa_n+\kappa_g'(3\kappa_g+\kappa_n)+2\kappa_n'(\kappa_g-\tau_g)+3\tau_g'\tau_g+(\kappa_g+\tau_g)(\kappa_g^{\ 2}-\kappa_n^{\ 2}+\tau_g^{\ 2})+\kappa_g''+\tau_g'',$$
$$\eta_3 = -\kappa_n'(\kappa_g+3\kappa_n)+\tau_g'(\kappa_g+3\tau_g)-(\kappa_n-\tau_g)(2\kappa_g'+\kappa_g^{\ 2}-\kappa_n^{\ 2}+\tau_g^{\ 2})+\tau_g''-\kappa_n''.$$

In the light of the above derivatives, the torsion of δ is computed as in Eq. (20), where

$$D_1 = \left[\begin{array}{c}\tau_g'^2(\kappa_g+\kappa_n)(3\kappa_g-\kappa_n+4\tau_g)\\ +\kappa_n'^2\left(3\kappa_g^{\ 2}+2\kappa_g(5\kappa_n-2\tau_g)+\tau_g(2\kappa_n+\tau_g)\right)\\ +2\kappa_n'(\kappa_n-\tau_g)\left(\kappa_g'(5\kappa_g+\kappa_n)+(2\kappa_g-\kappa_n)(\kappa_g+\kappa_n-\tau_g)(\kappa_g+\tau_g)+\kappa_g''+\tau_g''\right)\\ -2\tau_g'\left\{\begin{array}{c}\kappa_n'\left(3\kappa_g^{\ 2}+6\kappa_g\kappa_n+\kappa_n^{\ 2}+2\tau_g^{\ 2}\right)(\kappa_n-\tau_g)\\ \left(\kappa_g'(5\kappa_g+3\kappa_n-2\tau_g)+(\kappa_g+\kappa_n)\left(2\kappa_g^{\ 2}+\kappa_g(3\kappa_n-4\tau_g)+\kappa_n(-\kappa_n+\tau_g)\right)\right)\\ +\kappa_g''+\kappa_n''\end{array}\right\}\end{array}\right],$$

$$D_2=\left[\begin{array}{c}\kappa_g'^2\left(-\kappa_n+\tau_g\right)-2\kappa_g'\left(-\left(\kappa_n-\tau_g\right)\left(-2\kappa_g^{\ 2}-\kappa_g\left(\kappa_n-4\tau_g\right)+\kappa_n\left(\kappa_n+\tau_g\right)\right)-\tau_g''+\kappa_n''\right)\\ +2\left(\begin{array}{c}2\kappa_g^{\ 4}\left(-\kappa_n+\tau_g\right)+2\kappa_g^{\ 3}\kappa_n\left(-\kappa_n+\tau_g\right)\\ +\kappa_g\left(2\kappa_n^{\ 4}-2\kappa_n^{\ 3}\tau_g-2\kappa_n^{\ 2}\tau_g^{\ 2}+\tau_g\left(\tau_g''-\kappa_n''\right)+2\kappa_n\left(\tau_g^{\ 3}+\tau_g''-\kappa_n''\right)\right)\\ +2\kappa_g^{\ 2}\left(\left(\kappa_n-\tau_g\right)^2\left(\kappa_n+\tau_g\right)+\tau_g''-\kappa_n''\right)+\tau_g\left(\kappa_n\left(\kappa_g''+\tau_g''\right)-\tau_g\left(\kappa_g''+\kappa_n''\right)\right)\end{array}\right)\end{array}\right].$$

Thus the proof is completed.

Lemma 3.4 Let $\delta(\bar{s})$ be a spacelike curve lies on Ψ in Minkowski 3-space E_1^3, then

1) If δ is a geodesic curve, the curvature and torsion can be expressed as follows

$$\bar{\kappa}_\delta=\sqrt{\frac{3}{8}}\sqrt{\left[\frac{1}{\left(\tau_g-\kappa_n\right)^3\tau_g^{\ 3}}\left\{\begin{array}{c}\tau_g'^2\kappa_n^{\ 2}+\kappa_n'^2\tau_g^{\ 2}-4\kappa_n'\kappa_n\left(\kappa_n-\tau_g\right)\tau_g\left(-\kappa_n+\tau_g\right)\\ +4\left(\kappa_n-\tau_g\right)^2\tau_g\left(\kappa_n^{\ 2}\tau_g-\tau_g^{\ 3}\right)\\ +2\tau_g'\kappa_n\left(-\kappa_n'\tau_g+2\kappa_n\left(\kappa_n-\tau_g\right)\left(-\kappa_n+\tau_g\right)\right)\end{array}\right\}\right]},$$

$$\bar{\tau}_\delta=\frac{1}{3\sqrt{3}}\left[\begin{array}{c}-\tau_g'^2\kappa_n\left(\kappa_n-4\tau_g\right)+\kappa_n'^2\tau_g\left(2\kappa_n+\tau_g\right)\\ +2\kappa_n'\left(\kappa_n-\tau_g\right)\left(\kappa_n\tau_g\left(-\kappa_n+\tau_g\right)+\tau_g''\right)+2\left(\kappa_n-\tau_g\right)\tau_g\left(\kappa_n\tau_g''-\tau_g\kappa_n''\right)\\ -2\tau_g'\left(\kappa_n'\left(\kappa_n^{\ 2}+2\tau_g^{\ 2}\right)+\left(\kappa_n-\tau_g\right)\left(\kappa_n^{\ 2}\left(-\kappa_n+\tau_g\right)+\kappa_n''\right)\right)\end{array}\right].$$

2) If δ is an asymptotic line, we have

$$\bar{\kappa}_\delta=\sqrt{\frac{3}{8}}\sqrt{\left[\frac{1}{\tau_g^{\ 3}\left(\kappa_g+\tau_g\right)^3}\left\{\begin{array}{c}\tau_g'^2\kappa_g^{\ 2}-2\tau_g'\kappa_g\tau_g\left(\kappa_g'+2\kappa_g\left(\kappa_g+\tau_g\right)\right)\\ +\tau_g^{\ 2}\left(\begin{array}{c}\kappa_g'^2+4\kappa_g'\kappa_g\left(\kappa_g+\tau_g\right)\\ +4\left(\kappa_g+\tau_g\right)\left(\kappa_g^{\ 3}-\kappa_g^{\ 2}\tau_g-\kappa_g\tau_g^{\ 2}-\tau_g^{\ 3}\right)\end{array}\right)\end{array}\right\}\right]},$$

$$\bar{\tau}_\delta=\frac{1}{3\sqrt{3}}\left[\begin{array}{c}\tau_g'^2\kappa_g\left(3\kappa_g+4\tau_g\right)+2\tau_g'\tau_g\left(5\kappa_g'\kappa_g+2\kappa_g^{\ 3}-2\left(\kappa_g'+2\kappa_g^{\ 2}\right)\tau_g+\kappa_g''\right)\\ +\tau_g\left(\begin{array}{c}-\tau_g\left(\kappa_g'^2+4\kappa_g'\kappa_g\left(\kappa_g-2\tau_g\right)+4\kappa_g^{\ 2}\left(\kappa_g^{\ 2}+\tau_g^{\ 2}\right)-2\tau_g\kappa_g''\right)\\ -2\left(\kappa_g'+\kappa_g\left(2\kappa_g+\tau_g\right)\right)\tau_g''\end{array}\right)\end{array}\right].$$

3) If δ is a principal line, we obtain

$$\bar{\kappa}_\delta=\sqrt{\frac{3}{8}}\sqrt{\left[\frac{1}{\kappa_g^{\ 3}\kappa_n^{\ 3}}\left(\begin{array}{c}2\kappa_n'\kappa_g\kappa_n\left(\kappa_g'+2\left(\kappa_g-\kappa_n\right)\left(\kappa_g+\kappa_n\right)\right)-\kappa_n'^2\kappa_g^{\ 2}\\ +\kappa_n^{\ 2}\left(\kappa_g'^2+4\kappa_g'\left(\kappa_g-\kappa_n\right)\left(\kappa_g+\kappa_n\right)+4\kappa_g^{\ 2}\left(\kappa_g+\kappa_n\right)^2\right)\end{array}\right)\right]},$$

$$\bar{\tau}_\delta=\frac{1}{3\sqrt{3}}\left[\begin{array}{c}\kappa_n'^2\kappa_g\left(3\kappa_g+10\kappa_n\right)\\ +2\kappa_n'\kappa_n\left(\kappa_g\left(2\kappa_g-\kappa_n\right)\left(\kappa_g+\kappa_n\right)+\kappa_g'\left(5\kappa_g+\kappa_n\right)+\kappa_g''\right)\\ -\kappa_n\left(\kappa_g'+2\kappa_g\left(\kappa_g+\kappa_n\right)\right)\left(\kappa_n\left(\kappa_g'+2\kappa_g^{\ 2}-2\kappa_n^{\ 2}\right)+2\kappa_n''\right)\end{array}\right].$$

4 Computational example

In this section, we consider an example for a spacelike curve lying fully on an oriented timelike ruled surface in E_1^3 (see Fig. 1(b)), and compute its Smarandache curves. Suppose we are given a timelike ruled surface represented as

$$\Psi(s,v)=\left(\sqrt{2}\sin s,\sqrt{2}\cos s,s\right)+v\left(1,\sqrt{2}\cos s,-\sqrt{2}\sin s\right),$$

where the spacelike base curve is given by (see Fig. 1(a))

$$r(s)=\left(\sqrt{2}\sin s,\sqrt{2}\cos s,s\right),$$

and $Q=\left(1,\sqrt{2}\cos s,-\sqrt{2}\sin s\right)$ is the ruling vector of Ψ .

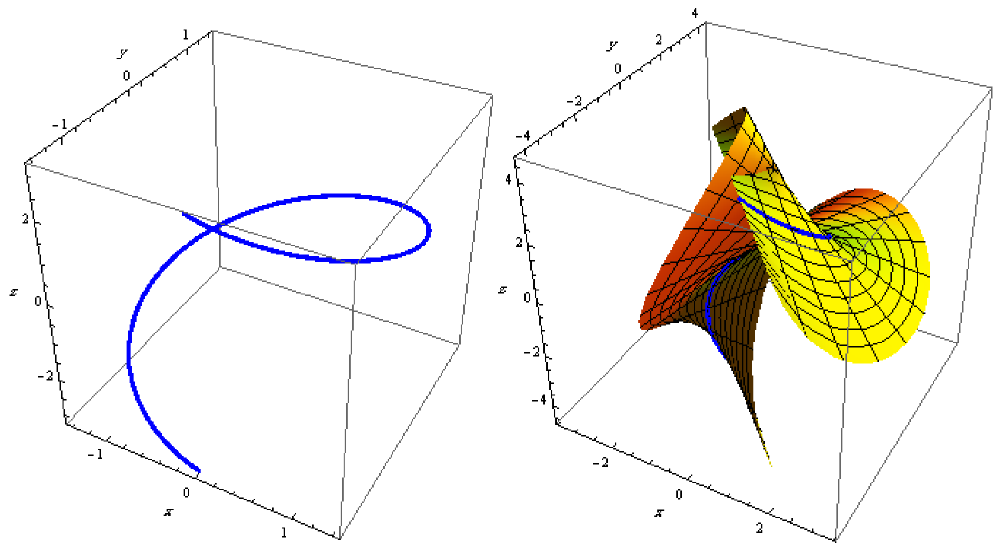


Figure 1: The spacelike curve r(s) on the timelike ruled surface Ψ

So, we can compute the Darboux frame of Ψ as follows

$$\mathbf{T}=\left(\sqrt{2}\cos(s),-\sqrt{2}\sin(s),1\right),$$

$$\mathbf{P}=\frac{(2A1,A2,2A3)}{2F},$$

where

$$F=\sqrt{\begin{array}{c}\left(-1-2v+\sqrt{2}\cos(s)+\cos(2s)\right)^2-\left(1+\cos(2s)+\sqrt{2}(1+v)\sin(s)\right)^2\\+\left(1-\sqrt{2}v\cos(s)+\sin(2s)\right)^2\end{array}},$$

$$A1=-v-\sqrt{2}v\cos(s)+(1+v)\cos(2s)-\frac{\sin(s)}{\sqrt{2}}+\sin(2s)-\frac{\sin(3s)}{\sqrt{2}},$$

$$A2=-\sqrt{2}\cos(s)+2\cos(2s)-\sqrt{2}\cos(3s)-2(1+2v+(1+v)\sin(2s)),$$

$$A3=-v+\sqrt{2}\cos(s)-v\cos(2s)+2\sqrt{2}(1+v)\sin(s)-\sin(2s),$$

and

$$\mathbf{U}=\frac{(B1,B2,B3)}{F},$$

where

$$B1=1+2v-\sqrt{2}\cos(s)-\cos(2s),$$
$$B2=1-\sqrt{2}v\cos(s)+\sin(2s),$$
$$B3=-1-\cos(2s)-\sqrt{2}(1+v)\sin(s).$$

According to Eq. (3), the geodesic curvature κ_g, the normal curvature κ_n and the geodesic torsion τ_g of the curve r are computed as follows

$$\kappa_g=\langle\mathbf{T}',\mathbf{P}\rangle=\frac{\left(4v\cos(s)+\sqrt{2}\cos(2s)+(2+4v)\sin(s)+\sqrt{2}(1+v\sin(2s))\right)}{\sqrt{\left\{\begin{array}{c}3+4v+8v^2-2\sqrt{2}(1+6v)\cos(s)+4(-1+(-1+v)v)\cos(2s)+\\ 2\sqrt{2}\cos(3s)-\cos(4s)-2\sqrt{2}(1+2v)\sin(s)+4\sin(2s)-2\sqrt{2}(1+2v)\sin(3s)\end{array}\right\}}}.$$

$$\kappa_n=\langle\mathbf{U}',\mathbf{T}\rangle=\frac{\left\{\begin{array}{c}-4(4+v(2+v)(7+2v))+2\sqrt{2}(7+v(4+v(13+12v)))\cos(s)\\ +v(11+4v(9+2v))\cos(2s)-\sqrt{2}(15+2v(12+v))\cos(3s)\\ +4(4+v(7+6v))\cos(4s)-\sqrt{2}(3+4v)\cos(5s)+v\cos(6s)+\\ 2\sqrt{2}(14+v(19+2v(5+2v)))\sin(s)-\left(31+84v+64v^2\right)\sin(2s)\\ +\sqrt{2}v\left(-5+6v+8v^2\right)\sin(3s)+2(4+5v)\sin(4s)\\ +\sqrt{2}(-4+v(-3+2v))\sin(5s)+\sin(6s)\end{array}\right\}}{4F^3}.$$

$$\tau_g=\langle\mathbf{P}',\mathbf{U}\rangle=\frac{\left(\begin{array}{c}-7-2v+\sqrt{2}(7+4v(2+v))\cos(s)+(2+6v)\cos(2s)\\ -\sqrt{2}(3+4v)\cos(3s)+\cos(4s)+\sqrt{2}(4+v(9+4v))\sin(s)\\ -2(2+v(3+2v))\sin(2s)+\sqrt{2}(2+v)\sin(3s)-v\sin(4s)\end{array}\right)}{\left(\begin{array}{c}3+4v+8v^2-2\sqrt{2}(1+6v)\cos(s)+4(-1+(-1+v)v)\cos(2s)\\ +2\sqrt{2}\cos(3s)-\cos(4s)-2\sqrt{2}(1+2v)\sin(s)\\ +4\sin(2s)-2\sqrt{2}(1+2v)\sin(3s)\end{array}\right)}.$$

In the case of ($s=0$ and $v=0$), we have

$$\kappa_g=2,\kappa_n=-\sqrt{2},\tau_g=2(1-\sqrt{2}).$$

TP − Smarandache curve

For this curve $\alpha=\alpha(\bar{s})$, (see Fig. 2(a)), we have

$$\alpha=(\alpha_1,\alpha_2,\alpha_3),$$

where

$$\alpha_1 = \cos(s) + \frac{-v - \sqrt{2}v\cos(s) + (1+v)\cos(2s) - \frac{\sin(s)}{\sqrt{2}} + \sin(2s) - \frac{\sin(3s)}{\sqrt{2}}}{\sqrt{2}F},$$

$$\alpha_2 = -\sin(s) + \frac{-\sqrt{2}\cos(s) + 2\cos(2s) - \sqrt{2}\cos(3s) - 2(1+2v+(1+v)\sin(2s))}{2\sqrt{2}F},$$

$$\alpha_3 = \frac{1}{\sqrt{2}} + \frac{-v + \sqrt{2}\cos(s) - v\cos(2s) + 2\sqrt{2}(1+v)\sin(s) - \sin(2s)}{\sqrt{2}F},$$

If we choose $(s=0)$ and $(v=0)$, the curvature and torsion of α are

$$\bar{\kappa}_\alpha = 7.89204, \quad \bar{\tau}_\alpha = -0.0122605.$$

As the above, we can calculate the other Smarandache curves as follows:

TU − Smarandache curve

For this curve (see Fig. 2(b)), we have

$$\beta = (\beta_1, \beta_2, \beta_3),$$

where

$$\beta_1 = \cos(s) + \frac{1}{\sqrt{2}}\left\{\frac{1 + 2v - \sqrt{2}\cos(s) - \cos(2s)}{F}\right\},$$

$$\beta_2 = -\sin(s) + \frac{1}{\sqrt{2}}\left\{\frac{1 - \sqrt{2}v\cos(s) + \sin(2s)}{F}\right\},$$

$$\beta_3 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left\{\frac{-1 - \cos(2s) - \sqrt{2}(1+v)\sin(s)}{F}\right\},$$

and therefore ($s=0$ and $v=0$)

$$\bar{\kappa}_\beta = 5.12132, \quad \bar{\tau}_\beta = -0.538206.$$

PU − Smarandache curve

For this curve (see Fig. 3(a)), we have

$$\gamma = (\gamma_1, \gamma_2, \gamma_3),$$

$$\gamma_1 = \frac{\sqrt{2} - 2(1+v)\cos(s) + 2\sqrt{2}v\cos(s)^2 - \sin(s) + \sqrt{2}\sin(2s) - \sin(3s)}{2F},$$

$$\gamma_2 = \frac{-\sqrt{2}(1+2v)\cos(s)+2\cos(2s)-\sqrt{2}\cos(3s)-2v(2+\sin(2s))}{2\sqrt{2}F},$$

$$\gamma_3 = \frac{-1-v+\sqrt{2}\cos(s)-(1+v)\cos(2s)+\sqrt{2}(1+v)\sin(s)-\sin(2s)}{\sqrt{2}F},$$

and then ($s=0$ and $v=0$)

$$\bar{\kappa}_\gamma = 3.22487, \quad \bar{\tau}_\gamma = -1.07994.$$

TPU − Smarandache curve

For this curve (see Fig. 3(b)), we have

$$\delta = (\delta_1, \delta_2, \delta_3),$$

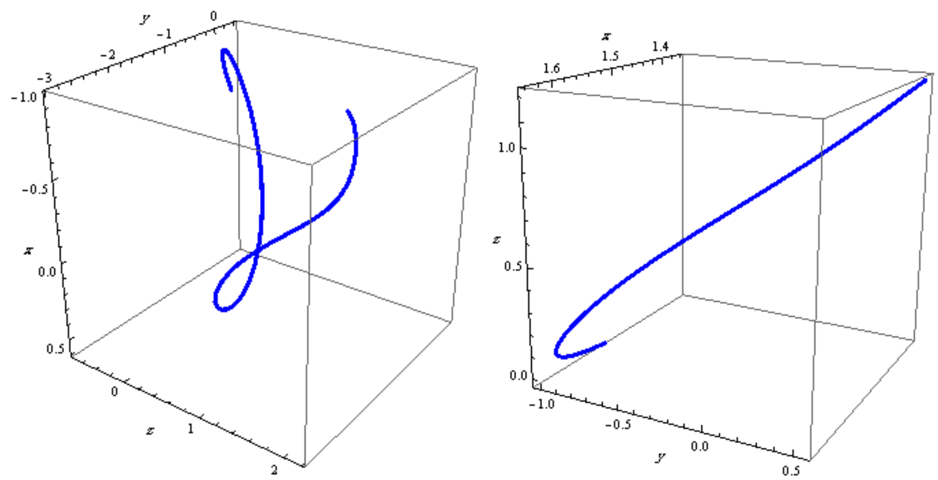
$$\delta_1 = \frac{1}{2\sqrt{3}F}\left\{\begin{array}{c} 2(1+v)-2\sqrt{2}(1+v)\cos(s)+2v\cos(2s)-\sqrt{2}\sin(s)+2\sin(2s) \\ +2\sqrt{2}\cos(s)\sqrt{\begin{array}{c}\left(-1-2v+\sqrt{2}\cos(s)+\cos(2s)\right)^2, \\ -\left(1+\cos(2s)+\sqrt{2}(1+v)\sin(s)\right)^2 \\ +\left(1-\sqrt{2}v\cos(s)+\sin(2s)\right)^2\end{array}} -\sqrt{2}\sin(3s)\end{array}\right\},$$

$$\delta_2 = \frac{-1}{2\sqrt{3}F}\left\{\begin{array}{c} 4v+\sqrt{2}(1+2v)\cos(s)-2\cos(2s)+\sqrt{2}\cos(3s)+2v\sin(2s) \\ +2\sqrt{2}\sin(s)\sqrt{\begin{array}{c}\left(-1-2v+\sqrt{2}\cos(s)+\cos(2s)\right)^2, \\ -\left(1+\cos(2s)+\sqrt{2}(1+v)\sin(s)\right)^2 \\ +\left(1-\sqrt{2}v\cos(s)+\sin(2s)\right)^2\end{array}}\end{array}\right\},$$

$$\delta_3 = \frac{1}{\sqrt{3}F}\left\{\begin{array}{c} -1-v+\sqrt{2}\cos(s)-(1+v)\cos(2s)+\sqrt{2}(1+v)\sin(s)-\sin(2s) \\ +\sqrt{\begin{array}{c}\left(-1-2v+\sqrt{2}\cos(s)+\cos(2s)\right)^2-\left(1+\cos(2s)+\sqrt{2}(1+v)\sin(s)\right)^2 \\ +\left(1-\sqrt{2}v\cos(s)+\sin(2s)\right)^2\end{array}}\end{array}\right\},$$

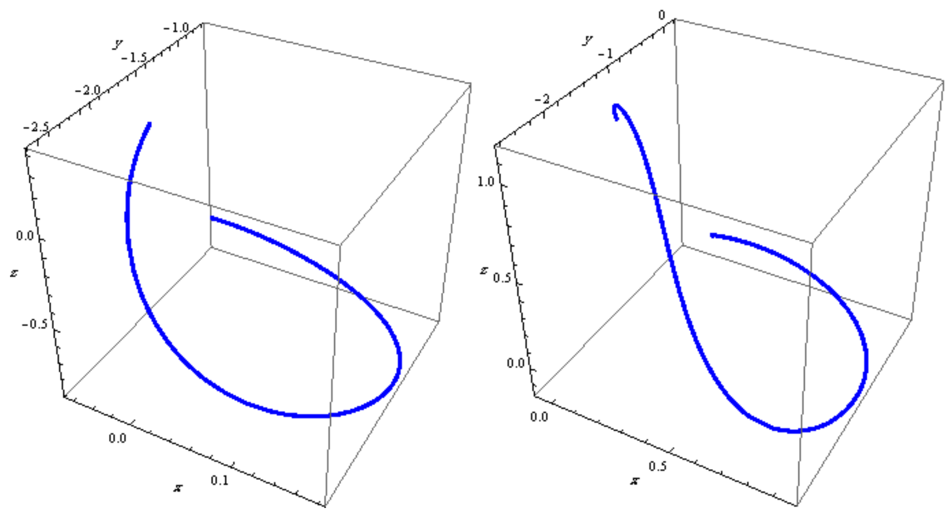
it follows that ($s=0$ and $v=0$)

$$\bar{\kappa}_\delta = 1.6945, \quad \bar{\tau}_\delta = -0.831341.$$



(a) **(b)**

Figure 2: The **TP** and **TU**-Smarandache curves α and β of the spacelike curve *r*



(a) **(b)**

Figure 3: The **PU** and **TPU**-Smarandache curves γ and δ of the spacelike curve *r*

5 Conclusion

In this study, Smarandache curves of a given spacelike curve with timelike normal lying on a timelike surface in the three-dimensional Minkowski space are investigated. According to the Lorentzian Darboux frame the curvatures and some characterizations for these curves are obtained. Finally, for confirming our main results, an example is given and plotted.

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