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Mannheim Partner Curve a Different View

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Abstract: In this study, we investigated special Smarandache curves belonging to Sabban frame drawn on the surface of the sphere by Darboux vector of Mannheim partner curve . We created Sabban frame belonging to this curve. It were explained Smarandache curves position vector is consisted by Sabban vectors belonging to this curve. Then, we calculated geodesic curvatures of this Smarandache curves. Found results were expressed depending on the Mannheim curve.

Key Words: Mannheim curve pair, Darboux vector, Smarandache curves, Sabban frame, geodesic curvature.

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§1. Introduction and Preliminaries

Let $\alpha : I \to E^3$ be a unit speed curve denote by $\{T, N, B, p, q\}$ the moving Frenet apparatus. The Frenet formulae is given by ([5])

$$\begin{cases} T'(s) = p(s)N(s) \\ N'(s) = -p(s)T(s) + q(s)B(s) \\ B'(s) = -q(s)N(s). \end{cases} \tag{1.1}$$

The Darboux vector defined by

$$W = qT + pB.$$

By the unit Darboux vector, we have

$$\sin\varphi = \frac{q}{\sqrt{p^2+q^2}}, \quad \cos\varphi = \frac{p}{\sqrt{p^2+q^2}}$$

including

$$C = \sin\varphi T + \cos\varphi B$$

where $\angle(W, B) = \varphi$, ([4]). Let α and α_1 be the C^2-class differentiable two curves and $T_1(s)$, $N_1(s)$, $B_1(s)$ be the Frenet vectors of α_1. If the binormal vector of the curve α_1 is linearly dependent on the principal normal vector of the curve α, then (α) is defined a Mannheim curve

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and (α_1) a Mannheim partner curve of (α). The relations between the Frenet vectors we can write [3, 6, 7]

$$\begin{cases} T_1 = \cos\sigma T - \sin\sigma B \\ N_1 = \sin\sigma T + \cos\sigma B \\ B_1 = N \end{cases} \tag{1.2}$$

and for the curvatures we get

$$\begin{cases} \overline{p} = \dfrac{p\sigma'}{\lambda q\sqrt{p^2+q^2}} \\ \overline{q} = \dfrac{p}{\lambda q}. \end{cases} \tag{1.3}$$

Let (α, α_1) be a curve pair in $\mathbb{E}^3$. For the vector C_1 is the direction of the Mannheim partner curve α_1 we have [3,8]

$$C_1 = \frac{\sqrt{p^2+q^2}}{\sqrt{p^2+q^2+\sigma'^2}}C + \frac{\sigma'}{\sqrt{p^2+q^2+\sigma'^2}}N. \tag{1.4}$$

Let $\omega : I \to S^2$ be a unit speed spherical curve. We can write ([10])

$$\omega(s) = \omega(s),\ \ t(s) = \omega'(s),\ \ d(s) = \omega(s) \wedge t(s), \tag{1.5}$$

where, $\{\omega(s), t(s), d(s)\}$ frame is expressed the Sabban frame of ω on S^2. Then we have equations ([10])

$$\omega'(s) = t(s),\ \ t'(s) = -\omega(s) + \kappa_g(s)d(s),\ \ d'(s) = -\kappa_g(s)t(s), \tag{1.6}$$

where κ_g is expressed the geodesic curvature of the curve ω on S^2 which is ([10])

$$\kappa_g(s) = \langle t'(s), d(s)\rangle. \tag{1.7}$$

§2. Mannheim Partner Curve a Different View

Let (C_1) be a unit speed spherical curve on S^2. Then we can write

$$\begin{cases} C_1 = \sin\overline{\varphi}T_1 + \cos\overline{\varphi}B_1, \\ T_{C_1} = \cos\overline{\varphi}T_1 - \sin\overline{\varphi}B_1, \\ C_1 \wedge T_{C_1} = N_1, \end{cases} \tag{2.1}$$

where $\angle(C_1, B_1) = \overline{\varphi}$. Then from the equation (1.6) we have the following equations of (C_1) are

$$\begin{cases} {C_1}' = T_{C_1} \\ T'_{C_1} = -C_1 + \dfrac{\sqrt{\overline{p}^2+\overline{q}^2}}{\overline{\varphi}'}C_1 \wedge T_{C_1} \\ (C_1 \wedge T_{C_1})' = -\dfrac{\sqrt{\overline{p}^2+\overline{q}^2}}{\overline{\varphi}'}T_{C_1}. \end{cases} \tag{2.2}$$

From the equation (1.7), we have the following geodesic curvature of (C_1) is

$$\kappa_g = \langle T'_{C_1}, C_1 \wedge T_{C_1} \rangle \Longrightarrow \kappa_g = \frac{\sqrt{\overline{p}^2+\overline{q}^2}}{\overline{\varphi}'}. \tag{2.3}$$

2.1 The β_1-*Smarandache curve* can be defined by

$$\beta_1(s) = \frac{1}{\sqrt{2}}(C_1 + T_{C_1}) \tag{2.4}$$

or substituting the equation (2.1) into equation (2.4) we obtain

$$\beta_1(s) = \frac{1}{\sqrt{2}}\Big((\sin\overline{\varphi} + \cos\overline{\varphi})T_1 + (\cos\overline{\varphi} - \sin\overline{\varphi})B_1\Big). \tag{2.5}$$

Differentiating (2.4) we can write

$$T_{\beta_1} = \frac{\overline{\varphi}'(\cos\overline{\varphi} - \sin\overline{\varphi})}{\sqrt{2\overline{\varphi}'^2 + \overline{p}^2 + \overline{q}^2}}T_1 + \frac{\sqrt{\overline{p}^2+\overline{q}^2}}{\sqrt{2\overline{\varphi}'^2 + \overline{p}^2 + \overline{q}^2}}N_1 - \frac{\overline{\varphi}'(\cos\overline{\varphi} + \sin\overline{\varphi})}{\sqrt{2\overline{\varphi}'^2 + \overline{p}^2 + \overline{q}^2}}B_1. \tag{2.6}$$

Considering the equations (2.5) and (2.6) we get

$$\beta_1 \wedge T_{\beta_1} = \frac{\sqrt{\overline{p}^2+\overline{q}^2}(\cos\overline{\varphi} + \sin\overline{\varphi})}{\sqrt{2\overline{p}^2 + 2\overline{q}^2 + 4\overline{\varphi}'^2}}T_1 - \frac{\overline{\varphi}'}{\sqrt{2\overline{p}^2 + 2\overline{q}^2 + 4\overline{\varphi}'^2}}N_1 + \frac{\sqrt{\overline{p}^2+\overline{q}^2}(\cos\overline{\varphi} + \sin\overline{\varphi})}{\sqrt{2\overline{p}^2 + 2\overline{q}^2 + 4\overline{\varphi}'^2}}B_1. \tag{2.7}$$

Differentiating (2.6) where coefficients

$$\begin{aligned}
\chi_1 &= -2 - \Big(\frac{\sqrt{\overline{p}^2+\overline{q}^2}}{\overline{\varphi}'}\Big)^2 + \Big(\frac{\sqrt{\overline{p}^2+\overline{q}^2}}{\overline{\varphi}'}\Big)'\Big(\frac{\sqrt{\overline{p}^2+\overline{q}^2}}{\overline{\varphi}'}\Big) \\
\chi_2 &= -2 - 3\Big(\frac{\sqrt{\overline{p}^2+\overline{q}^2}}{\overline{\varphi}'}\Big)^2 - \Big(\frac{\sqrt{\overline{p}^2+\overline{q}^2}}{\overline{\varphi}'}\Big)^4 - \Big(\frac{\sqrt{\overline{p}^2+\overline{q}^2}}{\overline{\varphi}'}\Big)'\Big(\frac{\sqrt{\overline{p}^2+\overline{q}^2}}{\overline{\varphi}'}\Big) \\
\chi_3 &= 2\Big(\frac{\sqrt{\overline{p}^2+\overline{q}^2}}{\overline{\varphi}'}\Big) + \Big(\frac{\sqrt{\overline{p}^2+\overline{q}^2}}{\overline{\varphi}'}\Big)^3 + \Big(\frac{\sqrt{\overline{p}^2+\overline{q}^2}}{\overline{\varphi}'}\Big)'
\end{aligned} \tag{2.8}$$

including we can reach,

$$T'_{\beta_1} = \frac{\overline{\varphi}'^4\sqrt{2}(\chi_1\sin\overline{\varphi} + \chi_2\cos\overline{\varphi})}{\Big(\overline{p}^2 + \overline{q}^2 + \overline{\varphi}'^2\Big)^2}T_1 + \frac{\chi_3\overline{\varphi}'^4\sqrt{2}}{\Big(\overline{p}^2 + \overline{q}^2 + \overline{\varphi}'^2\Big)^2}N_1 + \frac{\overline{\varphi}'^4\sqrt{2}(\chi_1\cos\overline{\varphi} - \chi_2\sin\overline{\varphi})}{\Big(\overline{p}^2 + \overline{q}^2 + \overline{\varphi}'^2\Big)^2}B_1. \tag{2.9}$$

From the equation (2.7) and (2.9) $\kappa_g^{\beta_1}$ geodesic curvature for Mannheim partner curve β_1 is

$$\begin{aligned}
\kappa_g^{\beta_1} &= \langle T'_{\beta_1}, (C_1 \wedge T_{C_1})_{\beta_1} \rangle \\
&= \frac{1}{\Big(2 + \big(\frac{\sqrt{\overline{p}^2+\overline{q}^2}}{\overline{\varphi}'}\big)^2\Big)^{\frac{5}{2}}}\Big(\frac{\sqrt{\overline{p}^2+\overline{q}^2}}{\overline{\varphi}'}\chi_1 - \frac{\sqrt{\overline{p}^2+\overline{q}^2}}{\overline{\varphi}'}\chi_2 + 2\chi_3\Big).
\end{aligned} \tag{2.10}$$

From the equation (1.2) and (1.3) Sabban apparatus of the β_1-Smarandache curve for

Mannheim curve are

$$\beta_1(s) = \frac{(\sigma' + \sqrt{p^2+q^2})\cos\sigma}{\sqrt{2\sigma'^2+2p^2+q^2}}T - \frac{\sigma' - \sqrt{p^2+q^2}}{\sqrt{2\sigma'^2+2p^2+q^2}}N - \frac{(\sigma' + \sqrt{p^2+q^2})\sin\sigma}{\sqrt{2\sigma'^2+2p^2+q^2}}B,$$

$$T_{\beta_1} = \frac{(\sigma' - \sqrt{p^2+q^2})\eta\cos\sigma - \sqrt{p^2+q^2+\sigma'^2}\cos\sigma}{\sqrt{p^2+q^2+\sigma'^2}\sqrt{1+2\eta^2}}T + \frac{\eta(\sigma' + \sqrt{p^2+q^2})}{\sqrt{p^2+q^2+\sigma'^2}\sqrt{1+2\eta^2}}N + \frac{(\sqrt{p^2+q^2} - \sigma')\eta\sin\sigma + \sqrt{p^2+q^2+\sigma'^2}\cos\sigma}{\sqrt{p^2+q^2+\sigma'^2}\sqrt{1+2\eta^2}}B,$$

$$\beta_1 \wedge T_{\beta_1} = \frac{(\sqrt{p^2+q^2} - \sigma')\cos\sigma + 2\eta\sqrt{p^2+q^2+\sigma'^2}\sin\sigma}{\sqrt{2+4\eta^2}\sqrt{p^2+q^2+\sigma'^2}}T + \frac{\sigma' + \eta\sqrt{p^2+q^2}}{\sqrt{2+4\eta^2}\sqrt{p^2+q^2+\sigma'^2}}N - \frac{(\sqrt{p^2+q^2} + \sigma')\sin\sigma - 2\eta\sqrt{p^2+q^2+\sigma'^2}\cos\sigma}{\sqrt{2+4\eta^2}\sqrt{p^2+q^2+\sigma'^2}}B,$$

$$T'_{\beta_1} = \frac{(\overline{\chi}_1\sqrt{p^2+q^2} + \overline{\chi}_2\sigma')\eta^4\sqrt{2}\cos\sigma - \overline{\chi}_3\eta^4\sqrt{2p^2+q^2+2\sigma'^2}\sin\sigma}{(1+2\eta^2)^2\sqrt{p^2+q^2+\sigma'^2}}T + \frac{(\overline{\chi}_1\sigma' - \overline{\chi}_2\sqrt{p^2+q^2})\eta^4\sqrt{2}}{(1+2\eta^2)^2\sqrt{p^2+q^2+\sigma'^2}}N + \frac{(\overline{\chi}_2\sigma' - \overline{\chi}_1\sqrt{p^2+q^2})\eta^4\sqrt{2}\sin\sigma + \overline{\chi}_3\eta^4\sqrt{2p^2+q^2+2\sigma'^2}\cos\sigma}{(1+2\eta^2)^2\sqrt{p^2+q^2+\sigma'^2}}B$$

and

$$\kappa_g^{\beta_1} = \frac{1}{\left(2+\frac{1}{\eta^2}\right)^{\frac{5}{2}}}\left(\frac{1}{\eta}\overline{\chi}_1 - \frac{1}{\eta}\overline{\chi}_2 + 2\overline{\chi}_3\right), \tag{2.11}$$

where

$$\frac{1}{\eta} = \frac{(\overline{\varphi})'}{\sqrt{p^2+q^2}} = \left(\frac{\sqrt{p^2+q^2}}{\sqrt{\sigma'^2+p^2+q^2}}\right)'\frac{\lambda q\sqrt{p^2+q^2}}{\sigma'} \tag{2.12}$$

and

$$\begin{cases} \overline{\chi}_1 = -2 - \frac{1}{\eta^2} + \frac{1}{\eta'}\frac{1}{\eta} \\ \overline{\chi}_2 = -2 - 3\frac{1}{\eta^2} - \frac{1}{\eta^4} - \frac{1}{\eta'}\frac{1}{\eta} \\ \overline{\chi}_3 = 2\frac{1}{\eta} + \frac{1}{\eta^3} + 2\frac{1}{\eta'} \end{cases} \tag{2.13}$$

2.2 The β_2-*Smarandache curve* can be defined by

$$\beta_2(s) = \frac{1}{\sqrt{2}}(C_1 + C_1 \wedge T_{C_1}) \tag{2.14}$$

or from the equation (1.2), (1.4) and (2.1) we can write

$$\beta_2(s) = \frac{\sqrt{p^2+q^2}\cos\sigma+\sqrt{\sigma'^2+p^2+q^2}\sin\sigma}{\sqrt{2\sigma'^2+2p^2+q^2}}T+\frac{\sigma'}{\sqrt{2\sigma'^2+2p^2+q^2}}N$$
$$+\frac{\sqrt{\sigma'^2+p^2+q^2}\cos\sigma-\sqrt{p^2+q^2}\sin\sigma}{\sqrt{2\sigma'^2+2p^2+q^2}}B. \quad (2.15)$$

Differentiating (2.15) we can write

$$T_{\beta_2}=\frac{\sigma'\cos\sigma}{\sqrt{p^2+q^2+\sigma'^2}}T-\frac{\sqrt{p^2+q^2}}{\sqrt{p^2+q^2+\sigma'^2}}N-\frac{\sigma'\sin\sigma}{\sqrt{p^2+q^2+\sigma'^2}}B. \quad (2.16)$$

Considering the equations (2.15) and (2.16) it is easily seen

$$\beta_2\wedge T_{\beta_2} = \frac{\sqrt{p^2+q^2+\sigma'^2}\sin\sigma-\sqrt{p^2+q^2}\cos\sigma}{\sqrt{2p^2+q^2+2\sigma'^2}}T-\frac{\sigma'}{\sqrt{2p^2+q^2+2\sigma'^2}}N$$
$$+\frac{\sqrt{p^2+q^2}\sin\sigma+\sqrt{p^2+q^2+\sigma'^2}\cos\sigma}{\sqrt{2p^2+q^2+2\sigma'^2}}B. \quad (2.17)$$

Differentiating (2.16) we can write

$$T'_{\beta_2} = \frac{\sqrt{2p^2+q^2+2\sigma'^2}\sin\sigma-\eta\sqrt{2}\sqrt{p^2+q^2}\cos\sigma}{(\eta-1)\sqrt{p^2+q^2+\sigma'^2}}T+\frac{\eta\sqrt{2}\sigma'}{(\eta-1)\sqrt{p^2+q^2+\sigma'^2}}N$$
$$+\frac{\eta\sqrt{2}\sqrt{p^2+q^2}\sin\sigma+\sqrt{2p^2+q^2+2\sigma'^2}\cos\sigma}{(\eta-1)\sqrt{p^2+q^2+\sigma'^2}}B, \quad (2.18)$$

where, $\kappa_g^{\beta_2}$ geodesic curvature for β_2 is

$$\kappa_g^{\beta_2}=\frac{1+\eta}{\eta-1}. \quad (2.19)$$

2.3 The β_3-*Smarandache curves* can be defined by

$$\beta_3(s)=\frac{1}{\sqrt{2}}(T_{C_1}+C_1\wedge T_{C_1}) \quad (2.20)$$

or from the equation (2.1), (1.4) and (1.2) we can write

$$\beta_3(s) = \frac{\sigma'\cos\sigma+\sqrt{p^2+q^2+\sigma'^2}\sin\sigma}{\sqrt{2\sigma'^2+2p^2+q^2}}T+\frac{\sqrt{p^2+q^2}}{\sqrt{2\sigma'^2+2p^2+q^2}}N$$
$$+\frac{\sqrt{p^2+q^2+\sigma'^2}\cos\sigma-\sigma'\sin\sigma}{\sqrt{2\sigma'^2+2p^2+q^2}}B. \quad (2.21)$$

Differentiating (2.21) we reach

$$
\begin{aligned}
T_{\beta_3} &= \frac{\sqrt{p^2+q^2+\sigma'^2}\sin\sigma-(\sigma'+\eta\sqrt{p^2+q^2})\cos\sigma}{\sqrt{2+\eta^2}\sqrt{p^2+q^2+\sigma'^2}}T+\frac{\sqrt{p^2+q^2}-\eta\sigma'}{\sqrt{2+\eta^2}\sqrt{p^2+q^2+\sigma'^2}}N \\
&+\frac{(\eta\sqrt{p^2+q^2}+\sigma')\sin\sigma+\sqrt{p^2+q^2+\sigma'^2}\cos\sigma}{\sqrt{2+\eta^2}\sqrt{p^2+q^2+\sigma'^2}}B. \qquad (2.22)
\end{aligned}
$$

Considering the equations (2.21) and (2.22) it is easily seen

$$
\begin{aligned}
\beta_3\wedge T_{\beta_3} &= \frac{(2\sqrt{p^2+q^2}-\eta\sigma')\cos\sigma+\eta\sqrt{p^2+q^2+\sigma'^2}\sin\sigma}{\sqrt{4+2\eta^2}\sqrt{p^2+q^2+\sigma'^2}}T \\
&+\frac{2\sigma'-\eta\sqrt{p^2+q^2}}{\sqrt{4+2\eta^2}\sqrt{p^2+q^2+\sigma'^2}}N \\
&+\frac{(\eta\sigma'-2\sqrt{p^2+q^2})\sin\sigma+\eta\sqrt{p^2+q^2+\sigma'^2}\cos\sigma}{\sqrt{4+2\eta^2}\sqrt{p^2+q^2+\sigma'^2}}B. \qquad (2.23)
\end{aligned}
$$

Differentiating (2.22) where

$$
\begin{cases}
\overline{\delta}_1=\frac{1}{\eta}+2\frac{1}{\eta^3}+2\frac{1}{\eta'}\frac{1}{\eta} \\
\overline{\delta}_2=-1-3\frac{1}{\eta^2}-2\frac{1}{\eta^4}-\frac{1}{\eta'} \\
\overline{\delta}_3=-\frac{1}{\eta^2}-2\frac{1}{\eta^4}+\frac{1}{\eta'}
\end{cases} \qquad (2.24)
$$

including we have

$$
\begin{aligned}
T'_{\beta_3} &= \frac{(\overline{\delta}_2\sqrt{p^2+q^2}-\overline{\delta}_1\sigma')\eta^4\sqrt{2}\cos\sigma+\overline{\delta}_3\eta^4\sqrt{2p^2+q^2+2\sigma'^2}\sin\sigma}{(2+\eta^2)^2\sqrt{\sigma'^2+p^2+q^2}}T \\
&+\frac{\eta^4\sqrt{2}(\overline{\delta}_2\sigma'+\overline{\delta}_1\sqrt{p^2+q^2})}{(2+\eta^2)^2\sqrt{\sigma'^2+p^2+q^2}}N \\
&+\frac{(\overline{\delta}_1\sigma'-\overline{\delta}_2\sqrt{p^2+q^2})\eta^4\sqrt{2}\sin\sigma+\overline{\delta}_3\eta^4\sqrt{2p^2+q^2+2\sigma'^2}\cos\sigma}{(2+\eta^2)^2\sqrt{\sigma'^2+p^2+q^2}}B, \qquad (2.25)
\end{aligned}
$$

where, $\kappa_g^{\beta_3}$ geodesic curvature for Mannheim curve β_3 is

$$
\kappa_g^{\beta_3}=\frac{1}{(2+\eta^2)^{\frac{5}{2}}}\big(2\eta^5\overline{\delta}_1-\eta^4\overline{\delta}_2+\eta^4\overline{\delta}_3\big). \qquad (2.26)
$$

2.4 The β_4-*Smarandache curves* can be defined by

$$
\beta_4(s)=\frac{1}{\sqrt{3}}(C_{_1}+T_{C_{_1}}+C_{_1}\wedge T_{C_{_1}}) \qquad (2.27)
$$

or from the equation (1.2), (1.4) and (2.1) we can write

$$\begin{aligned}\beta_4(s) &= \frac{(\sigma' + \sqrt{p^2+q^2})\cos\sigma + \sqrt{\sigma'^2+p^2+q^2}\sin\sigma}{\sqrt{3\sigma'^2+3p^2+q^2}}T + \frac{\sigma' - \sqrt{p^2+q^2}}{\sqrt{3\sigma'^2+3p^2+q^2}}N \\ &+ \frac{\sqrt{\sigma'^2+p^2+q^2}\cos\sigma - (\sigma' + \sqrt{p^2+q^2})\sin\sigma}{\sqrt{3\sigma'^2+3p^2+q^2}}B. \end{aligned} \tag{2.28}$$

Differentiating (2.28), we reach

$$\begin{aligned}T_{\beta_4} &= \frac{\big((\eta-1)\sigma' - \eta\sqrt{p^2+q^2}\big)\cos\sigma + \sqrt{p^2+q^2+\sigma'^2}\sin\sigma}{\sqrt{2(1-\eta+\eta^2)}\sqrt{p^2+q^2+\sigma'^2}}T \\ &+ \frac{\eta\sigma' - (1-\eta)\sqrt{p^2+q^2}}{\sqrt{2(1-\eta+\eta^2)}\sqrt{p^2+q^2+\sigma'^2}}N \\ &+ \frac{\big(\eta\sqrt{p^2+q^2} + (1-\eta)\sigma'\big)\sin\sigma + \sqrt{p^2+q^2+\sigma'^2}\cos\sigma}{\sqrt{2(1-\eta+\eta^2)}\sqrt{p^2+q^2+\sigma'^2}}B. \end{aligned} \tag{2.29}$$

Considering the equations (2.28) and (2.29) it is easily seen

$$\begin{aligned}\beta_4 \wedge T_{\beta_4} &= \frac{((2-\eta)\sqrt{p^2+q^2} - (1+\eta)\sigma')\cos\sigma + (2\eta 1)\sqrt{p^2+q^2+\sigma'^2}\sin\sigma}{\sqrt{6-6\eta+6\eta^2}\sqrt{p^2+q^2+\sigma'^2}}T \\ &+ \frac{(2-\eta)\sigma' + (1+\eta)\sqrt{p^2+q^2}}{\sqrt{6-6\eta+6\eta^2}\sqrt{p^2+q^2+\sigma'^2}}N \\ &+ \frac{((\eta-2)\sqrt{p^2+q^2} - (1+\eta)\sigma')\sin\sigma + (2\eta-1)\sqrt{p^2+q^2+\sigma'^2}\cos\sigma}{\sqrt{6-6\eta+6\eta^2}\sqrt{p^2+q^2+\sigma'^2}}B. \end{aligned} \tag{2.30}$$

Differentiating (2.29) where

$$\begin{cases} \overline{\rho}_1 = -2 + 4\frac{1}{\eta} - 4\frac{1}{\eta^2} + 2\frac{1}{\eta^3} + 2\frac{1}{\eta'}\big(2\frac{1}{\eta} - 1\big) \\ \overline{\rho}_2 = -2 + 2\frac{1}{\eta} - 4\frac{1}{\eta^2} + 2\frac{1}{\eta^3} - 2\frac{1}{\eta^4} - \frac{1}{\eta'}\big(1 + \frac{1}{\eta}\big) \\ \overline{\rho}_3 = 2\frac{1}{\eta} - 4\frac{1}{\eta^2} + 4\frac{1}{\eta^3} - 2\frac{1}{\eta^4} + \frac{1}{\eta'}\big(2 - \frac{1}{\eta}\big) \end{cases} \tag{2.31}$$

including we can write

$$\begin{aligned}T'_{\beta_4} &= \frac{(\overline{\rho}_1\sqrt{p^2+q^2} + \overline{\rho}_2\sigma')\eta^4\sqrt{3}\cos\sigma + \overline{\rho}_3\eta^4\sqrt{3p^2+q^2+3\sigma'^2}\sin\sigma}{4(1-\eta+\eta^2)^2\sqrt{\sigma'^2+p^2+q^2}}T \\ &+ \frac{(\overline{\rho}_1\sigma' - \overline{\rho}_2\sqrt{p^2+q^2})\eta^4\sqrt{3}}{4(1-\eta+\eta^2)^2\sqrt{\sigma'^2+p^2+q^2}}N \\ &+ \frac{\overline{\rho}_3\eta^4\sqrt{3p^2+q^2+3\sigma'^2}\cos\sigma - (\overline{\rho}_1\sqrt{p^2+q^2} + \overline{\rho}_2\sigma')\eta^4\sqrt{3}\sin\sigma}{4(1-\eta+\eta^2)^2\sqrt{\sigma'^2+p^2+q^2}}B, \end{aligned} \tag{2.32}$$

where, $\kappa_g^{\beta_4}$ geodesic curvature for Mannheim curve $\beta_4(s_{\beta_4})$ is

$$\kappa_g^{\beta_4} = \frac{(2\eta^4 - \eta^5)\overline{\rho}_1 - (\eta^5 + \eta^4)\overline{\rho}_2 + (2\eta^5 - \eta^4)\overline{\rho}_3}{4\sqrt{2}(1 - \eta + \eta^2)^{\frac{5}{2}}}. \tag{2.33}$$

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