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Isotropic Smarandache Curves in Complex Space C^3

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Abstract: A regular curve in complex space, whose position vector is composed by Cartan frame vectors on another regular curve, is called a isotropic Smarandache curve. In this paper, I examine isotropic Smarandache curve according to Cartan frame in Complex 3-space and give some differential geometric properties of Smarandache curves. We define type-1 e_1e_3-isotropic Smarandache curves, type-2 e_1e_3-isotropic Smarandache curves and $e_1e_2e_3$-isotropic Smarandache curves in Complex space C^3.

Key Words: Complex space C^3, isotropic Smarandache curves, isotropic cubic.

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§1. Introduction

It is observe that the imaginary curve in complex space were pioneered by E. Cartan. Cartan defined his moving frame and his special equations in C^3. In [6], the Cartan equations of isotropic curve is extended to space C^4. Moreover U. Pekmen [2] wrote some characterizations of minimal curves by means of E. Cartan equations in C^3.

A regular curve in Euclidean 3-space, whose position vector is composed by Frenet frame vectors on another regular curve, is called Smarandache curve. M. Turgut and S. Yilmaz have defined a special case of such curves and call it Smarandache TB_2 curves in the space E_1^4 [7]. A.T. Ali has introduced some special Smarandache curves in the Euclidean space [9]. Moreover, special Smarandache curves have been investigated by using Bishop frame in Euclidean space [10]. Special Smarandache curves according to Sabban frame have been studied by [11]. Besides some special Smarandache curves have been obtained in E_1^3 by [12]. Apart from M. Turgut defined Smarandache breadth curves [8].

It is given that complex elements and complex curves to real space $\mathbb{R}^3$ which are mentioned by Ferruh Semin, see [1]. In complex space C^3 helices are characterized in [5]. In complex space C^4, S. Yilmaz characterized the isotropic curves with constant pseudo curvature which is called the slant isotropic helix. Yılmaz and Turgut give some characterization of isotropic helices in C^3 [3].

Several authors introduce different types of helices and investigated their properties. For instance, Barros et. al. studied general helices in 3- dimensional Lorentzian space. Izumiya and

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Takeuchi defined slant helices by the property that principal normal mekes a constant angle with a fixed direction [14]. Kula and Yayli studied spherical images of tangent and binormal indicatrices of slant helices and they have shown that spherical images are spherical helix [15]. Ali and Lopez gave some characterization of slant helices in Minkowski 3-space E_1^3 [13].

In this work, using not common vector field know as Cartan frame, I introduce a new Smarandache curves in C^3. Also, Cartan apparatus of Smarandache curves have been formed by Cartan apparatus of given curve $\alpha = \alpha(s)$.

§2. Preliminaries

Let x_p be a complex analytic function of a complex variable t. Then the vector function

$$\overrightarrow{x}(t) = \sum_{p=1}^{4} x_p(t)\overrightarrow{k}_p,$$

is called an imaginary curve, where $\overrightarrow{x} : C \to C^4$, $\overrightarrow{k}_p$ are standard basis unit vectors of E^3 [6].

An isotropic curve $x = x(s)$ in C^3 is called an isotropic cubic if pseudo curvature of $x(s)$ is congruent to zero. A direction (b_1, b_2, b_3) is a minimal direction if and only if

$$\sum_{p=1}^{3} b_p^2 = 0.$$

A vector which has a minimal direction is called an isotropic vector or minimal vector. A vector $\overrightarrow{\vartheta}$ is a minimal vector if and only if $\overrightarrow{\vartheta}^2 = 0$. Common points of a complex plane and absolute are called *siklik* points of the plane. A plane which is tangent to the absolute is called a minimal plane, see [6]. The curves, of which the square of the distance between the two points equal to zero, are called minimal or isotropic curves [3]. Let s denote pseudo arc-length A curve is an minimal (isotropic) curve if and only if ([4,5])

$$\left[\overrightarrow{x}'(t)\right]^2 = 0 \tag{2.2}$$

where $\frac{d\overrightarrow{x}}{dt} = \overrightarrow{x}'(t) \neq 0$. Let be each point $\overrightarrow{x}$ of the isotropic curve. E. Cartan frame is defined (for well-known complex number $i^2 = -1$) as follows, (see [1,4])

$$\begin{aligned} \overrightarrow{e}_1 &= \overrightarrow{x}' \\ \overrightarrow{e}_2 &= i\overrightarrow{x}'' \\ \overrightarrow{e}_3 &= -\tfrac{\beta}{2}\overrightarrow{x}' + \overrightarrow{x}''' \end{aligned} \tag{2.3}$$

where $\beta = (\overrightarrow{x}''')^2$, equation (2.3) denote by $\{\overrightarrow{e}_1, \overrightarrow{e}_2, \overrightarrow{e}_3\}$ the moving E. Cartan frame along the isotropic curve $\overrightarrow{x}$ in the space C^3.

The inner products of these frame vectors are given by

$$\overrightarrow{e}_i \cdot \overrightarrow{e}_j = \left\{ \begin{array}{ll} 0 \ \text{if} \ i+j \equiv 1,2,3, \ (\text{mod}4) \\ 1 \ \text{if} \ i+j=4 \end{array} \right\} \tag{2.4}$$

The cross (vectoral) and fixed products of these frame vectors are given by

$$\begin{array}{l} \overrightarrow{e}_j \wedge \overrightarrow{e}_k = i\overrightarrow{e}_{j+k-2} \\ < \overrightarrow{e}_1, \overrightarrow{e}_2 \wedge \overrightarrow{e}_3 >= i \end{array} \tag{2.5}$$

for $j,k = 1,2,3$, $s = \int\limits_{t_0}^{t} -[\overrightarrow{x}^{\shortmid}(t)]^{\frac{1}{4}} dt$ is a pseudo arc length, also invariant with respect to parameter t. Thus the vector $\overrightarrow{e}_1$ and $\overrightarrow{e}_3$ are isotropic vector, $\overrightarrow{e}_2$ is real vector E. Cartan derivative formulas can be deduced from equation (2.3) as follows

$$\begin{array}{l} \overrightarrow{e}^{\shortmid}_1 = i\overrightarrow{e}_2 \\ \overrightarrow{e}^{\shortmid}_2 = i(k\overrightarrow{e}_1 + \overrightarrow{e}_3) \\ \overrightarrow{e}^{\shortmid}_3 = ik\overrightarrow{e}_2 \end{array} \tag{2.6}$$

where $k = \frac{\beta}{2}$ is called pseudo curvature of isotropic curve $x = x(s)$. These equations can be used if the minimal curve is at least of class C^4. Here $(\shortmid)$ denotes derivative according to pseudo arc length s. In the rest of the paper, we will suppose pseudo curvature is non-vanishing expect in the case of an isotropic cubic. Isotropic sphere with center $\overrightarrow{m}$ and radius $r > 0$ in C^3 is defined by

$$S^2 = \left\{ \overrightarrow{p} = (p_1, p_2, p_3) \in C^3 : (\overrightarrow{p} - \overrightarrow{m})^2 = 0 \right\}.$$

§3. Type-1 $e_1^\alpha e_3^\alpha$−Isotropic Smarandache Curves

Definition 3.1 *Let $\alpha = \alpha(s)$ be a unit speed regular isotropic curve in C^3 and $\{e_1^\alpha, e_2^\alpha, e_3^\alpha\}$ be its moving Cartan frame. Type-1 $e_1^\alpha e_3^\alpha$-isotropic Smarandache curves can be defined by*

$$\vartheta(s^*) = \frac{1}{\sqrt{2}}(e_1^\alpha + e_3^\alpha). \tag{3.1}$$

Now, we can investigate Cartan invariants of $e_1^\alpha e_3^\alpha$-isotropic Smarandache curves according to $\alpha = \alpha(s)$. Differentiating equation (3.1) with respect to pseudo arc length s, we obtain

$$\vartheta^{\shortmid} = \frac{d\vartheta}{ds^*}\frac{ds^*}{ds} = -\frac{i}{\sqrt{2}}(1+k^\alpha)e_2^\alpha \tag{3.2}$$

where

$$\frac{ds^*}{ds} = \frac{(1+k^\alpha)i}{\sqrt{2}}. \tag{3.3}$$

The tangent isotropic vector of curve ϑ can be expressed as follow

$$e_1^{\vartheta} = -\sqrt{1+k^{\alpha}}e_2^{\alpha} \tag{3.4}$$

Differentiating equation (3.4) with respect to pseudo arc length s, we obtain

$$\left(e_1^{\vartheta}\right)' \frac{ds^*}{ds} = 2(1+k^{\alpha})ie_1^{\alpha} + (k^{\alpha})' e_2^{\alpha} + 2(1+k^{\alpha})ie_3^{\alpha}. \tag{3.5}$$

Substituting equation (3.3) into equation (3.5), we find

$$\left(e_1^{\vartheta}\right)' = \left(2\sqrt{2}k^{\alpha}\right) e_1^{\alpha} - \left(\frac{\sqrt{2}\,(k^{\alpha})'}{1+k^{\alpha}}i\right) e_2^{\alpha} + 2\sqrt{2}e_3^{\alpha}.$$

Since $\left(e_1^{\vartheta}\right)' = -ie_2^{\vartheta}$, the principal vector field of curve ϑ

$$e_2^{\vartheta} = \left(2\sqrt{2}k^{\alpha}\right) e_1^{\alpha} - \left(\frac{\sqrt{2}\,(k^{\alpha})'}{1+k^{\alpha}}\right) ie_2^{\alpha} + 2\sqrt{2}e_3^{\alpha}. \tag{3.6}$$

Using Cartan equation $(2.6)_3$, we have

$$e_3^{\vartheta} = i\int k^{\vartheta}\left[2\sqrt{2}k^{\alpha}e_1^{\alpha} + \frac{\sqrt{2}\,(k^{\alpha})'}{1+k^{\alpha}}e_2^{\alpha} + 2\sqrt{2}ie_3^{\alpha}\right] ds \tag{3.7}$$

and

$$k^{\vartheta} = -\frac{\left(e_3^{\vartheta}\right)'}{e_2^{\vartheta}}i. \tag{3.8}$$

Substituting equations (3.6) and (3.7) into equation (3.8), we obtain

$$k^{\vartheta} = \frac{\left\{i\int k^{\vartheta}\left[2\sqrt{2}k^{\alpha}e_1^{\alpha} + \frac{\sqrt{2}\,(k^{\alpha})'}{1+k^{\alpha}}e_2^{\alpha} + 2\sqrt{2}ie_3^{\alpha}\right] ds\right\}'}{2\sqrt{2}k^{\alpha}e_1^{\alpha} + \frac{\sqrt{2}(k^{\alpha})'}{1+k^{\alpha}}e_2^{\alpha} + 2\sqrt{2}ie_3^{\alpha}}i. \tag{3.9}$$

Proposition 3.1 *If ϑ a isotropic Smarandache curves in C^3, then $k^{\alpha} = -1$.*

Proof Using equation (3.4) and definition isotropic curves, it is seen straightforwardly. □

Proposition 3.2 *Let $\alpha = \alpha(s)$ be a unit speed regular isotropic curve in C^3, If δ a isotropic cubic in C^3, then pseudo curvature of α satisfies e_3^{ϑ} =constant and $e_2^{\vartheta} \neq 0$.*

Proof It is seen straightforwardly from definition isotrobic cubic. □

§4. Type-2 $e_1^{\alpha}e_3^{\alpha}$−Isotropic Smarandache Curves

Definition 4.1 *Let $\alpha = \alpha(s)$ be a unit speed regular isotropic curve in C^3 and $\{e_1^{\alpha}, e_2^{\alpha}, e_3^{\alpha}\}$ be* *its moving Cartan frame. Type-2 $e_1^\alpha e_3^\alpha$-isotropic Smarandache curves can be defined by*

$$\delta(s^*) = \frac{i}{\sqrt{2}}(e_1^\alpha - e_3^\alpha). \tag{4.1}$$

Now, we can investigate Cartan invariants of type-2 $e_1^\alpha e_3^\alpha$-isotropic Smarandache curves according to $\alpha = \alpha(s)$. Differentiating equation (4.1) with respect to pseudo arc length s, we obtain

$$\delta' = \frac{d\delta}{ds^*}\frac{ds^*}{ds} = -\frac{1}{\sqrt{2}}(k^\alpha - 1)e_2^\alpha \tag{4.2}$$

and

$$e_1^\delta \frac{ds^*}{ds} = -\frac{1}{\sqrt{2}}(k^\alpha - 1)e_2^\alpha$$

where

$$\frac{ds^*}{ds} = \frac{\sqrt{k^\alpha - 1}}{\sqrt{2}}. \tag{4.3}$$

The tangent isotropic vector of curve δ can be expressed as follow

$$e_1^\delta = -\sqrt{k^\alpha - 1}e_2^\alpha \tag{4.4}$$

Differentiating equation (4.4) with respect to pseudo arc length s, we obtain

$$e_2^\delta = \sqrt{k^\alpha - 1}k^\alpha e_1^\alpha - \frac{i\,(k^\alpha)'}{2\sqrt{k^\alpha - 1}}e_2^\alpha + \sqrt{k^\alpha - 1}e_3^\alpha. \tag{4.5}$$

Using definition, binormal vector field and pseudo curvature of isotropic Smarandache curve δ are respectively,

$$e_3^\delta = i\int k^\delta \left[\sqrt{k^\alpha - 1}k^\alpha e_1^\alpha - \frac{i\,(k^\alpha)'}{2\sqrt{k^\alpha - 1}}e_2^\alpha + \sqrt{k^\alpha - 1}e_3^\alpha\right] ds \tag{4.6}$$

and

$$k^\delta = \frac{\left\{-i\int k^\delta \left[\sqrt{k^\alpha - 1}k^\alpha e_1^\alpha - \frac{i\,(k^\alpha)'}{2\sqrt{k^\alpha - 1}}e_2^\alpha + \sqrt{k^\alpha - 1}e_3^\alpha\right] ds\right\}'}{\sqrt{k^\alpha - 1}k^\alpha e_1^\alpha - \frac{i(k^\alpha)'}{2\sqrt{k^\alpha - 1}}e_2^\alpha + \sqrt{k^\alpha - 1}e_3^\alpha} i. \tag{4.7}$$

Proposition 4.1 *If δ a isotropic Smarandache curves in C^3, then $k^\alpha = 1$.*

Proof Using equation (4.4) and definition isotropic curves, it is seen straightforwardly. □

Proposition 4.2 *Let $\alpha = \alpha(s)$ be a unit speed regular isotropic curve in C^3, If δ a isotropic cubic in C^3, then pseudo curvature of α satisfies e_3^δ =constant and $e_2^\delta \neq 0$.*

Proof It is seen straightforwardly from definition isotrobic cubic. □

§5. $e_1^\alpha e_2^\alpha e_3^\alpha-$Isotropic Smarandache Curves

Definition 5.1 *Let $\alpha = \alpha(s)$ be a unit speed regular isotropic curve in C^3 and $\{e_1^\alpha, e_2^\alpha, e_3^\alpha\}$ be its moving Cartan frame. Type-1 $e_1^\alpha e_3^\alpha$-isotropic Smarandache curves can be defined by*

$$\eta(s^*) = \frac{1}{\sqrt{3}}(e_1^\alpha + e_2^\alpha + e_3^\alpha). \tag{5.1}$$

Now, we can investigate Cartan invariants of $e_1^\alpha e_2^\alpha e_3^\alpha$-isotropic Smarandache curves according to $\alpha = \alpha(s)$. Differentiating equation (5.1) with respect to pseudo arc length s, we have

$$\eta' = \frac{d\eta}{ds^*}\frac{ds^*}{ds} = \frac{1}{\sqrt{3}}\left[ik^\alpha e_1^\alpha - i(k^\alpha+1)e_2^\alpha + ie_3^\alpha\right] \tag{5.2}$$

and

$$\eta' = e_1^\eta \frac{ds^*}{ds} = \frac{1}{\sqrt{3}}\left[ik^\alpha e_1^\alpha - i(k^\alpha+1)e_2^\alpha + ie_3^\alpha\right]$$

where

$$\frac{ds^*}{ds} = \frac{\sqrt{1+k^\alpha}}{\sqrt{3}}. \tag{5.3}$$

The tangent isotropic vector of curve η can be written as follow:

$$e_1^\eta = \frac{1}{\sqrt{1+k^\alpha}}\left[ik^\alpha e_1^\alpha - i(k^\alpha+1)e_2^\alpha + ie_3^\alpha\right] \tag{5.4}$$

Differentiating equation (5.4) with respect to pseudo arc length s, we obtain

$$\begin{aligned} e_2^\eta = \;& \left\{\left(\tfrac{-\sqrt{3}}{1+k^\alpha}\right)\left[i\,(k^\alpha)' + (k^\alpha+1)k\right] - \left(\tfrac{-\sqrt{3}}{1+k^\alpha}\right)' ik^\alpha\right\} e_1^\alpha \\ & - \left\{\left(\tfrac{\sqrt{3}}{1+k^\alpha}\right)\left[-2k^\alpha + (k^\alpha+1)'\right] - \left(\tfrac{-\sqrt{3}}{1+k^\alpha}\right)'(k^\alpha+1)\right\} e_2^\alpha \\ & - \left\{\left(\tfrac{-\sqrt{3}}{1+k^\alpha}\right)' + \left(\tfrac{\sqrt{3}}{1+k^\alpha}\right)\right\} e_3^\alpha \end{aligned}$$

Using definition, binormal vector field and pseudo curvature of isotropic Smarandache curve η are respectively

$$\begin{aligned} e_3^\eta = \;& -i\int k^\eta \{\left\{\left(\frac{-\sqrt{3}}{1+k^\alpha}\right)\left[(k^\alpha)' + (k^\alpha+1)k^\alpha\right] - \left(\frac{-\sqrt{3}}{1+k^\alpha}\right)' ik^\alpha\right\} e_1^\alpha \\ & - \left\{\left(\tfrac{\sqrt{3}}{1+k^\alpha}\right)\left[2k^\alpha + (k^\alpha+1)'\right] - \left(\tfrac{-\sqrt{3}}{1+k^\alpha}\right)'(k^\alpha+1)\right\} e_2^\alpha \\ & - \left\{\left(\tfrac{\sqrt{3}}{1+k^\alpha}\right)' + \left(\tfrac{\sqrt{3}}{1+k^\alpha}\right)\right\} e_3^\alpha\} ds \end{aligned}$$

Let $e_3^\eta = H(s)$ and $e_2^\eta = G(s)$ ın this case, we have

$$k^\eta = \frac{(H(s))'}{G(s)}. \tag{5.5}$$

Proposition 5.1 *If η a isotropic Smarandache curves in C^3, then $k^\alpha \neq -1$.*

Proof Using equation (5.4) and definition isotropic curves, it is seen straightforwardly. □

Proposition 5.2 *Let $\alpha = \alpha(s)$ be a unit speed regular isotropic curve in C^3, If η a isotropic cubic in C^3, then pseudo curvature of α satisfies e_3^η =constant and $e_2^\eta \neq 0$.*

Proof It is seen straightforwardly from definition isotrobic cubic. □

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