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Equiform Spacelike Smarandache Curves of Anti-Eqiform Salkowski Curve According to Equiform Frame

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Abstract

In this paper, we construct the equiform spacelike Smarandache curves of spacelike anti-equiform Salkowski curves with timelike binormal according to equiform frame in R_1^3. Furthermore, we calculate the equiform Frenet apparatus of these curves. Finally, the latter curves were plotted.

Mathematics Subject Classification: 53B30, 53C40, 53C50

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1 Introduction

In 1909, Erich Salkowski defined families of curve with constant torsion τ but non-constant curvature κ which is called anti-Salkowski curve [8]. Recently, Şenyurt investegated the Smarandache curves of anti-Salkowski curve according to Sabban frame and Frenet frame in Minkowski 3-space see [12, 13]. Smarandache curves are constructed when the Frenet vectors of curve are taken as the position vector [1]. Later many authors are studied Smarandache curves see [9, 10, 11, 14, 15].

Analogously, in this work, we introduce the spacelike anti-equiform Salkowski curve with timelike binormal according to equiform frame in $\mathbb{R}^3_1$ and constructing the Frenet apparatus of the equiform spacelike Smarandache curves of these curve.

2 Basic concepts

The Lorentzian metric in 3-dimentional Minkowski space $\mathbb{R}^3_1$ is defined as

$$\mathcal{H} = -du_1^2 + du_2^2 + du_3^2,$$

where $(u_1, u_2, u_3) \in \mathbb{R}^3_1$. Any arbitrary curve $\zeta = \zeta(\theta) : I \subset \mathbb{R} \to \mathbb{R}^3_1$ can be spacelike if $\mathcal{H}(\zeta'(\theta), \zeta'(\theta)) > 0$, timelike if $\mathcal{H}(\zeta'(\theta), \zeta'(\theta)) < 0$, and lightlike if $\mathcal{H}(\zeta'(\theta), \zeta'(\theta)) = 0$ [6, 7].

Let $\zeta(\theta)$ be C^3-spacelike curve with timelike binormal and have non vanishing curvature $\kappa(\theta) \neq 0$ in $\mathbb{R}^3_1$. The Frenet frame of ζ given as [6]:

$$\begin{pmatrix} t'(\theta) \\ n'(\theta) \\ b'(\theta) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(\theta) & 0 \\ -\kappa(\theta) & 0 & \tau(\theta) \\ 0 & \tau(\theta) & 0 \end{pmatrix} \begin{pmatrix} t(\theta) \\ n(\theta) \\ b(\theta) \end{pmatrix}, \tag{1}$$

where $\left(' = \dfrac{d}{d\theta}\right)$, $\mathcal{H}(t,t) = \mathcal{H}(n,n) = -\mathcal{H}(b,b) = 1$ and $\mathcal{H}(t,n) = \mathcal{H}(t,b) = \mathcal{H}(n,b) = 0$. Also $\{t, n, b, \kappa, \tau\}$ of ζ is defined as [2]

$$\begin{aligned} t(\theta) &= \frac{\zeta'(\theta)}{\|\zeta'(\theta)\|}, \qquad b(\theta) = \frac{\zeta'(\theta) \times \zeta''(\theta)}{\|\zeta'(\theta) \times \zeta''(\theta)\|}, \qquad n(\theta) = b(\theta) \times t(\theta), \\ \kappa(\theta) &= \frac{\|\zeta'(\theta) \times \zeta''(\theta)\|}{\|\zeta'(\theta)\|^3}, \qquad \tau(\theta) = \frac{det\big(\zeta'(\theta), \zeta''(\theta), \zeta'''(\theta)\big)}{\|\zeta'(\theta) \times \zeta''(\theta)\|^2}. \end{aligned} \tag{2}$$

Now, let s be the equiform parameter of ζ by $s = \displaystyle\int \kappa(\theta) d\theta$. Then, the radius of curvature of ζ is given by $\rho = \dfrac{d\theta}{ds}$. Recall $\{T, N_1, N_2\}$ be the moving equiform frame where $T(s) = \rho\, t(\theta)$, $N_1(s) = \rho\, n(\theta)$ and $N_2(s) = \rho\, b(\theta)$. Furthermore, the equiform curvatures of the curve ζ are defined by $k_1(\sigma) = \rho'$ and $k_2(\sigma) = \left(\dfrac{\tau}{\kappa}\right)$. Also, $\{T, N_1, N_2\}$ satisfying [4]:

$$\begin{pmatrix} \dot{T}(s) \\ \dot{N}_1(s) \\ \dot{N}_2(s) \end{pmatrix} = \begin{pmatrix} k_1(s) & 1 & 0 \\ -1 & k_1(s) & k_2(s) \\ 0 & k_2(s) & k_1(s) \end{pmatrix} \begin{pmatrix} T(s) \\ N_1(s) \\ N_2(s) \end{pmatrix}, \tag{3}$$

where $\left(\cdot = \dfrac{d}{ds}\right)$, $\mathcal{H}(T,T) = \rho^2$, $\mathcal{H}(N_1, N_1) = \rho^2$, $\mathcal{H}(N_2, N_2) = -\rho^2$, and $\mathcal{H}(T, N_1) = \mathcal{H}(T, N_2) = \mathcal{H}(N_1, N_2) = 0$.

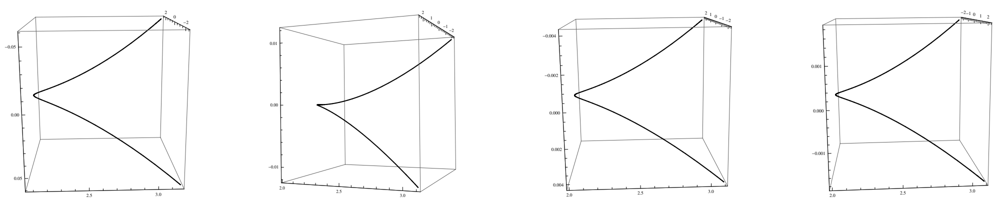

Figure 1: Anti-spacelike Salkowski curve for $q = \{3, 6, 10, 15\}$.

Definition 1. For any $m \in \mathbb{R}^+$, we define the spacelike curve (anti-spacelike Salkowski curve [13])

$$\delta_q(\theta) = \frac{p}{4q}\Bigg(\frac{1-p}{1+2p}\sinh\big((1+2p)\theta\big) - \frac{1+p}{1-2p}\sinh\big((1-2p)\theta\big) + 2p\sinh(\theta), \frac{1-p}{1+2p}\cosh\big((1+2p)\theta\big) - \frac{1+p}{1-2p}\cosh\big((1-2p)\theta\big) + 2p\cosh(\theta), \frac{1}{q}\Big(2p\theta - \sinh(2p\theta)\Big)\Bigg),$$

where $p = \frac{q}{\sqrt{q^2-1}}$ (see Figure 1). Moreover, the geometric characteristic of $\delta_q(\theta)$ are

1. The arc-length $\varsigma = \dfrac{\sinh(p\theta)}{q}$.
2. The curvature functions are $\kappa_\delta(\theta) = \coth p\theta$ and $\tau_\delta(\theta) = 1$.
3. The frenet frame are

$$T_\delta(\theta) = \begin{pmatrix} -p\cosh(\theta)\sinh(p\theta) + \sinh(\theta)\cosh(p\theta), \\ -p\sinh(\theta)\sinh(p\theta) + \cosh(\theta)\cosh(p\theta), \\ -\dfrac{p}{q}\sinh(p\theta) \end{pmatrix},$$

$$N_\delta(\theta) = \left(-\sqrt{1-p^2}\cosh(\theta), -\sqrt{1-p^2}\sinh(\theta), -\frac{p\sqrt{1-p^2}}{\sqrt{p^2-1}}\sinh(p\theta)\right),$$

$$B_\delta(s) = \begin{pmatrix} \frac{p}{q\sqrt{1-p^2}}\Big(\sinh(\theta)\sinh(p\theta) - p\cosh(\theta)\cosh(p\theta)\Big), \\ \frac{p}{q\sqrt{1-p^2}}\Big(\cosh(\theta)\sinh(p\theta) - p\sinh(\theta)\cosh(p\theta)\Big), \\ \sqrt{1-p^2}\cosh(p\theta) \end{pmatrix}.$$

Now, the equiform parameter $s = \dfrac{1}{p}\ln\big(\sinh(p\theta)\big)$, so we have $\theta = \dfrac{1}{p}\ln\big(e^{ps} + \sqrt{e^{2ps}+1}\big)$ and $\rho = \Big(\dfrac{e^{2ps} + e^{ps}\sqrt{e^{2ps}+1} + 1}{e^{2ps} + e^{ps}\sqrt{e^{2ps}+1}}\Big)$. Furthermore,

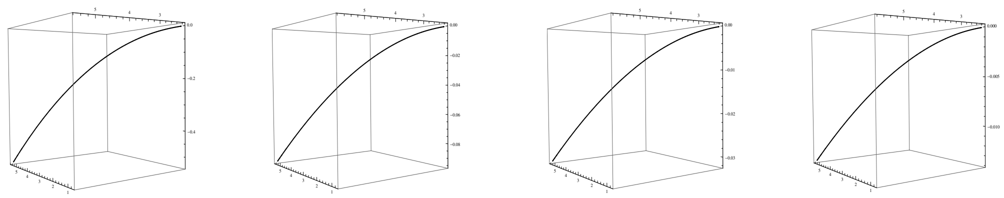

Figure 2: Spacelike anti-equiform Salkowski curve for $q = \{3, 6, 10, 15\}$.

the equiform curvature are $k_1(s) = \frac{-1}{p}\, sech^2\big(\ln\left(e^{ps} + \sqrt{e^{2ps}+1}\right)\big)$ and $k_2(s) = \tanh\big(\ln\left(e^{ps} + \sqrt{e^{2ps}+1}\right)\big)$ So, the spacelike anti-equiform Salkowski $\delta_q(s)$ defined as (see Figure 2):

$$\begin{aligned}\delta_q(s) =& \frac{p}{4q}\Bigg(\frac{1-p}{1+2p}\sinh\left(\frac{1+2p}{p}\ln\left(e^{ps}+\sqrt{e^{2ps}+1}\right)\right) - \frac{1+p}{1-2p}\\ &\sinh\left(\frac{1-2p}{p}\ln\left(e^{ps}+\sqrt{e^{2ps}+1}\right)\right) + 2p\sinh\left(\frac{1}{p}\ln\left(e^{ps}+\sqrt{e^{2ps}+1}\right)\right)\\ &,\frac{1-p}{1+2p}\cosh\left(\frac{1+2p}{p}\ln\left(e^{ps}+\sqrt{e^{2ps}+1}\right)\right) - \frac{1+p}{1-2p}\cosh\Bigg(\frac{1-2p}{p}\\ &\ln\left(e^{ps}+\sqrt{e^{2ps}+1}\right)\Bigg) + 2p\cosh\left(\frac{1}{p}\ln\left(e^{ps}+\sqrt{e^{2ps}+1}\right)\right), \frac{1}{q}\Bigg(2p\ln\Big(e^{ps}\\ &+\sqrt{e^{2ps}+1}\Big) - \sinh\left(2\ln\left(e^{ps}+\sqrt{e^{2ps}+1}\right)\right)\Bigg)\Bigg),\end{aligned} \tag{4}$$

also the equiform Frenet frame are defined as

$$T_q(s) = \rho\begin{pmatrix} -p\cosh\left(\frac{1}{p}\ln\left(e^{ps}+\sqrt{e^{2ps}+1}\right)\right)\sinh\left(\ln\left(e^{ps}+\sqrt{e^{2ps}+1}\right)\right)\\ +\sinh\left(\frac{1}{p}\ln\left(e^{ps}+\sqrt{e^{2ps}+1}\right)\right)\cosh\left(\ln\left(e^{ps}+\sqrt{e^{2ps}+1}\right)\right),\\ -p\sinh\left(\frac{1}{p}\ln\left(e^{ps}+\sqrt{e^{2ps}+1}\right)\right)\sinh\left(\ln\left(e^{ps}+\sqrt{e^{2ps}+1}\right)\right)\\ +\cosh\left(\frac{1}{p}\ln\left(e^{ps}+\sqrt{e^{2ps}+1}\right)\right)\cosh\left(\ln\left(e^{ps}+\sqrt{e^{2ps}+1}\right)\right),\\ \frac{-p}{q}\sinh\left(\ln\left(e^{ps}+\sqrt{e^{2ps}+1}\right)\right)\end{pmatrix} \tag{5}$$

$$\begin{aligned}N_q(s) =& -\rho\Bigg(\sqrt{1-p^2}\cosh\left(\frac{1}{p}\ln\left(e^{ps}+\sqrt{e^{2ps}+1}\right)\right), \sqrt{1-p^2}\sinh\Bigg(\frac{1}{p}\ln\Big(e^{ps}\\ &+\sqrt{e^{2ps}+1}\Big)\Bigg), \frac{p\sqrt{1-p^2}}{\sqrt{p^2-1}}\Bigg),\end{aligned} \tag{6}$$

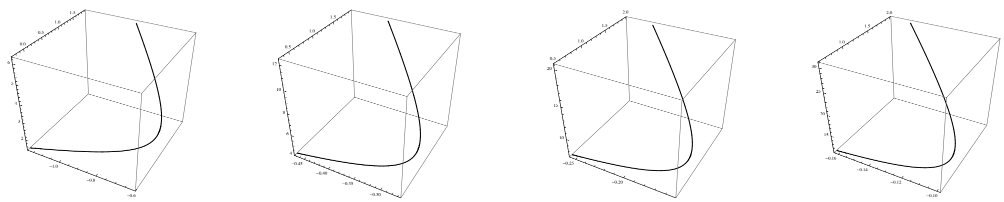

Figure 3: The equiform spacelike Γ_{TN}-Smarandache curve for $q = \{3, 6, 10, 15\}$.

$$B_q(s) = \frac{p\,\rho}{q\sqrt{1-p^2}} \begin{pmatrix} \sinh\left(\frac{1}{p}\ln\left(e^{ps}+\sqrt{e^{2ps}+1}\right)\right)\sinh\left(\ln\left(e^{ps}+\sqrt{e^{2ps}+1}\right)\right) \\ -p\cosh\left(\frac{1}{p}\ln\left(e^{ps}+\sqrt{e^{2ps}+1}\right)\right)\cosh\left(\ln\left(e^{ps}+\sqrt{e^{2ps}+1}\right)\right), \\ \cosh\left(\frac{1}{p}\ln\left(e^{ps}+\sqrt{e^{2ps}+1}\right)\right)\sinh\left(\ln\left(e^{ps}+\sqrt{e^{2ps}+1}\right)\right) \\ -p\sinh\left(\frac{1}{p}\ln\left(e^{ps}+\sqrt{e^{2ps}+1}\right)\right)\cosh\left(\ln\left(e^{ps}+\sqrt{e^{2ps}+1}\right)\right), \\ \frac{q(1-p^2)}{p}\cosh\left(\ln\left(e^{ps}+\sqrt{e^{2ps}+1}\right)\right) \end{pmatrix} \tag{7}$$

3 Main results

In this section, we investigate the equiform spacelike Smarandache curves of spacelike anti-equiform Salkowski curve with timelike binormal via equiform frame in Minkowski 3-space $\mathbb{R}^3_1$. Also, we obtain the equiform Frenet apparatus of these curves.

Definition 2. Let $\delta_q(s)$ be spacelike anti-equiform Salkowski curve with timelike binormal in $\mathbb{R}^3_1$. The equiform spacelike Γ_{TN}-Smarandache curve of $\delta_q(s)$ is defined by

$$\Gamma_{TN}(s) = \frac{1}{\sqrt{2}\,\rho}\Big(\ell\, T_\delta(s) + m\, N_\delta(s)\Big), \quad \ell^2 + m^2 = 2. \tag{8}$$

We can write Eq. (8) in the form (see Figure 3)

$$\Gamma_{TN}(s) = \frac{1}{\sqrt{2}} \begin{pmatrix} -\ell\,p\cosh\left(\frac{1}{p}\ln\left(e^{ps}+\sqrt{e^{2ps}+1}\right)\right)\sinh\left(\ln\left(e^{ps}+\sqrt{e^{2ps}+1}\right)\right)+\ell \\ \sinh\left(\frac{1}{p}\ln\left(e^{ps}+\sqrt{e^{2ps}+1}\right)\right)\cosh\left(\ln\left(e^{ps}+\sqrt{e^{2ps}+1}\right)\right)-m\sqrt{1-p^2} \\ \cosh\left(\frac{1}{p}\ln\left(e^{ps}+\sqrt{e^{2ps}+1}\right)\right), -\ell\,p\sinh\left(\frac{1}{p}\ln\left(e^{ps}+\sqrt{e^{2ps}+1}\right)\right) \\ \sinh\left(\ln\left(e^{ps}+\sqrt{e^{2ps}+1}\right)\right)+\ell\cosh\left(\frac{1}{p}\ln\left(e^{ps}+\sqrt{e^{2ps}+1}\right)\right) \\ \cosh\left(\ln\left(e^{ps}+\sqrt{e^{2ps}+1}\right)\right)-m\sqrt{1-p^2}\sinh\left(\frac{1}{p}\ln\left(e^{ps}+\sqrt{e^{2ps}+1}\right)\right), \\ \frac{-\ell\,p}{q}\sinh\left(\ln\left(e^{ps}+\sqrt{e^{2ps}+1}\right)\right)-\frac{m\,p\sqrt{1-p^2}}{\sqrt{p^2-1}} \end{pmatrix} \tag{9}$$

Theorem 1. Let $\Gamma_{TN}(s)$ be equiform spacelike Smarandache curve of spacelike anti-equiform Salkowski curve $\delta_q(s)$ with timelike binormal in $\mathbb{R}^3_1$. Then $\Gamma_{TN}(s)$ is equiform spacelike curve with timelike binormal. Additionally, its equiform frame $\{T_\Gamma, N_\Gamma, B_\Gamma\}$ satisfying

$$\begin{pmatrix} T_\Gamma \\ N_\Gamma \\ B_\Gamma \end{pmatrix} = \begin{pmatrix} \frac{\ell k_1 - m}{\rho\sqrt{2k_1^2+(m^2+1)k_2^2+2(m-\ell)k_1}} & \frac{mk_1+1}{\rho\sqrt{2k_1^2+(m^2+1)k_2^2+2(m-\ell)k_1}} & \frac{mk_2}{\rho\sqrt{2k_1^2+(m^2+1)k_2^2+2(m-\ell)k_1}} \\ \frac{m(\varepsilon_2 k_2 - \varepsilon_3 k_1)+\varepsilon_3}{\Theta_1} & \frac{m(\varepsilon_1 k_2 - \varepsilon_3) - \ell\varepsilon_3 k_1}{\Theta_1} & \frac{(\ell\varepsilon_2 - m\varepsilon_1)k_1 - m\varepsilon_2 - \varepsilon_1}{\Theta_1} \\ \frac{\varepsilon_1}{\rho\sqrt{\varepsilon_1^2+\varepsilon_2^2-\varepsilon_3^2}} & \frac{\varepsilon_2}{\rho\sqrt{\varepsilon_1^2+\varepsilon_2^2-\varepsilon_3^2}} & \frac{\varepsilon_3}{\rho\sqrt{\varepsilon_1^2+\varepsilon_2^2-\varepsilon_3^2}} \end{pmatrix} \begin{pmatrix} T_\delta \\ N_\delta \\ B_\delta \end{pmatrix}, \tag{10}$$

where

$$\begin{aligned}
\varepsilon_1 &= \frac{1}{2\rho^2}\left[m(mk_1+1)(2k_1k_2+k_2'-1) - mk_2\big(m(k_1^2+k_2^2+k_1'-1)+(\ell+1)k_1\big)\right], \\
\varepsilon_2 &= \frac{1}{2\rho^2}\left[m(\ell k_1-m)(2k_1k_2+k_2'-1) - mk_2\big(\ell(k_1^2+k_1')-2mk_1+1\big)\right], \\
\varepsilon_3 &= \frac{1}{2\rho^2}\Big[(mk_1+1)\big(\ell(k_1^2+k_1')-2mk_1+1\big) - (\ell k_1-m)\big(m(k_1^2+k_2^2+k_1'-1) \\
&\quad + (\ell+1)k_1\big)\Big], \\
\Theta_1 &= \rho^2\sqrt{\varepsilon_1^2+\varepsilon_2^2-\varepsilon_3^2}\sqrt{2k_1^2+(m^2+1)k_2^2+2(m-\ell)k_1}.
\end{aligned}$$

Proof. Differentiationg Eq. (8) and using Eq. (3), we get

$$\dot{\Gamma}_{TN}(s) = \frac{1}{\sqrt{2}\,\rho}\Big((\ell k_1-m)T_\delta(s) + (mk_1+1)N_\delta(s) + mk_2\,B_\delta(s)\Big). \tag{11}$$

Then

$$T_\Gamma(s) = \frac{(\ell k_1-m)T_\delta(s)+(mk_1+1)N_\delta(s)+mk_2\,B_\delta(s)}{\rho\sqrt{2k_1^2+(m^2+1)k_2^2+2(m-\ell)k_1}}, \tag{12}$$

where

$$\|\dot{\Gamma}_{TN}(s)\| = \frac{\sqrt{2k_1^2+(m^2+1)k_2^2+2(m-\ell)k_1}}{\sqrt{2}}. \tag{13}$$

Now, from Eq. (11) we have

$$\begin{aligned}
\ddot{\Gamma}_{TN}(s) =& \frac{1}{\sqrt{2}\,\rho}\Big(\big[\ell(k_1^2+k_1')-2mk_1+1\big]T_\delta(s) + \big[m(k_1^2+k_2^2+k_1'-1) \\
&+ (\ell+1)k_1\big]N_\delta(s) + \big[m(2k_1k_2+k_2'-1)\big]B_\delta(s)\Big),
\end{aligned} \tag{14}$$

and

$$\dddot{\Gamma}_{TN}(s) = \frac{1}{\sqrt{2}\,\rho}\Big(\alpha_1\,T_\delta(s) + \alpha_2\,N_\delta(s) + \alpha_3\,B_\delta(s)\Big), \tag{15}$$

where

$$\begin{aligned}\alpha_1 &= k_1\big[\ell(k_1^2+k_1')-2mk_1+1\big]-\big[m(k_1^2+k_2^2+k_1')+(\ell+1)k_1-m\big]+2k_1'(\ell k_1-m)+\ell k_1'',\\ \alpha_2 &= k_1\big[m(k_1^2+k_2^2+k_1'-1)+(\ell+1)k_1\big]+mk_2(2k_1k_2+k_2'-1)+2m(k_1k_1'+k_2k_2'-k_1)\\ &\quad+(2\ell+1)k_1'+\ell k_1^2+mk_1''+1,\\ \alpha_3 &= k_2\big[m(k_1^2+k_2^2+k_1'-1)+(\ell+1)k_1\big]+mk_1(2k_1k_2+k_2'-1)+2m(k_1'k_2+k_1k_2')+mk_2''.\end{aligned} \tag{16}$$

From Eqs. (11) and (14), we have

$$\dot{\Gamma}_{TN}(s)\times\ddot{\Gamma}_{TN}(s)=\varepsilon_1\,T_\delta(s)+\varepsilon_2\,N_\delta(s)+\varepsilon_3\,B_\delta(s), \tag{17}$$

where

$$\begin{aligned}\varepsilon_1 &= \frac{1}{2\rho^2}\left[m(mk_1+1)(2k_1k_2+k_2'-1)-mk_2\big(m(k_1^2+k_2^2+k_1'-1)+(\ell+1)k_1\big)\right],\\ \varepsilon_2 &= \frac{1}{2\rho^2}\left[m(\ell k_1-m)(2k_1k_2+k_2'-1)-mk_2\big(\ell(k_1^2+k_1')-2mk_1+1\big)\right],\\ \varepsilon_3 &= \frac{1}{2\rho^2}\left[(mk_1+1)\big(\ell(k_1^2+k_1')-2mk_1+1\big)-(\ell k_1-m)\big(m(k_1^2+k_2^2+k_1'-1)+(\ell+1)k_1\big)\right],\end{aligned} \tag{18}$$

so we obtain

$$B_\Gamma(s)=\frac{1}{\rho\sqrt{\varepsilon_1^2+\varepsilon_2^2-\varepsilon_3^2}}\Big(\varepsilon_1\,T_\delta(s)+\varepsilon_2\,N_\delta(s)+\varepsilon_3\,B_\delta(s)\Big), \tag{19}$$

From Eqs. (12) and (19), we found

$$\begin{aligned}N_\Gamma(s)=&\frac{1}{\Theta_1}\Big[\big[m(\varepsilon_2k_2-\varepsilon_3k_1)+\varepsilon_3\big]T_\delta(s)+\big[m(\varepsilon_1k_2-\varepsilon_3)-\ell\varepsilon_3k_1\big]N_\delta(s)\\ &+\big[(\ell\varepsilon_2-m\varepsilon_1)k_1-m\varepsilon_2-\varepsilon_1\big]B_\delta(s)\Big],\end{aligned} \tag{20}$$

where $\Theta_1=\rho^2\sqrt{\varepsilon_1^2+\varepsilon_2^2-\varepsilon_3^2}\sqrt{2k_1^2+(m^2+1)k_2^2+2(m-\ell)k_1}$. □

Theorem 2. Let $\Gamma_{TN}(s)$ be equiform spacelike Smarandache curve of spacelike anti-equiform Salkowski curve $\delta_q(s)$ with timelike binormal in $\mathbb{R}_1^3$. Then the curvature functions of Γ_{TN} are given by

$$\begin{aligned}\kappa_\Gamma(s) &= \frac{2\sqrt{2}\sqrt{\varepsilon_1^2+\varepsilon_2^2-\varepsilon_3^2}}{\big[2k_1^2+(m^2+1)k_2^2+2(m-\ell)k_1\big]^{\frac{3}{2}}},\\ \tau_\Gamma(s) &= \frac{1}{2\sqrt{2}\rho^3(\varepsilon_1^2+\varepsilon_2^2-\varepsilon_3^2)}\Big\{(\ell k_1-m)\big[m\alpha_3\big((k_1^2+k_2^2+k_1'-1)+(\ell+1)k_1\big)\\ &\quad-m\alpha_2(2k_1k_2+k_2'-1)\big]+(mk_1+1)\big[m\alpha_1(2k_1k_2+k_2'-1)-\alpha_3\big(\ell(k_1^2+k_1')\\ &\quad-2mk_1+1\big)\big]+mk_2\big[\alpha_2\big(\ell(k_1^2+k_1')-2mk_1+1\big)-m\alpha_1\big((k_1^2+k_2^2+k_1'-1)\\ &\quad+(\ell+1)k_1\big)\big]\Big\},\end{aligned} \tag{21}$$

where $\{\alpha_1,\alpha_2,\alpha_3\}$ and $\{\varepsilon_1,\varepsilon_2,\varepsilon_3\}$ are given by Eqs. (16) and (18) respectively.

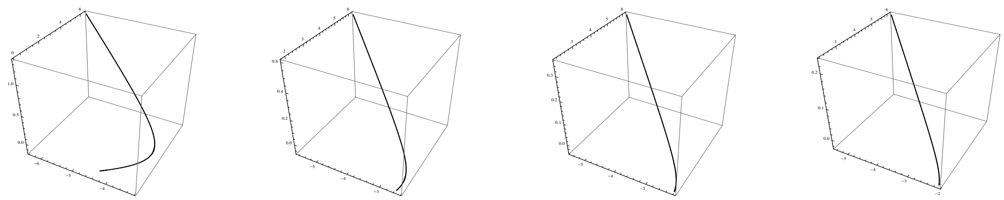

Figure 4: The equiform spacelike Ψ_{TB}-Smarandache curve for $q = \{3, 6, 10, 15\}$.

Proof. From Eqs. (2), (13) and (17) directly we consider $\kappa_\Gamma(s)$. Similarly, from Eqs. (2), (11), (14), (15) and (17) once can obtain $\tau_\Gamma(s)$. □

Definition 3. Let $\delta_q(s)$ be spacelike anti-equiform Salkowski curve with timelike binormal in $\mathbb{R}^3_1$. The equiform spacelike Ψ_{TB}-Smarandache curve of $\delta_q(s)$ is defined by

$$\Psi_{TB}(s) = \frac{1}{\sqrt{2}\,\rho}\Big(\ell\, T_\delta(s) + m\, B_\delta(s)\Big), \quad \ell^2 - m^2 = 2, \tag{22}$$

or equivalently (see Figure 4)

$$\Psi_{TB}(s) = \frac{1}{\sqrt{2}} \begin{pmatrix} -\ell\, p\, \cosh\left(\frac{1}{p}\ \ln\left(e^{ps} + \sqrt{e^{2ps}+1}\right)\right) \sinh\left(\ln\left(e^{ps} + \sqrt{e^{2ps}+1}\right)\right) + \ell \\ \sinh\left(\frac{1}{p}\ \ln\left(e^{ps} + \sqrt{e^{2ps}+1}\right)\right) \cosh\left(\ln\left(e^{ps} + \sqrt{e^{2ps}+1}\right)\right) \\ +\frac{mp}{q\sqrt{1-p^2}}\Big(\sinh\left(\frac{1}{p}\ \ln\left(e^{ps} + \sqrt{e^{2ps}+1}\right)\right) \sinh\left(\ln\left(e^{ps} + \sqrt{e^{2ps}+1}\right)\right) \\ -p\cosh\left(\frac{1}{p}\ \ln\left(e^{ps} + \sqrt{e^{2ps}+1}\right)\right) \cosh\left(\ln\left(e^{ps} + \sqrt{e^{2ps}+1}\right)\right)\Big), \\ -\ell\, p\, \sinh\left(\frac{1}{p}\ \ln\left(e^{ps} + \sqrt{e^{2ps}+1}\right)\right) \sinh\left(\ln\left(e^{ps} + \sqrt{e^{2ps}+1}\right)\right) \\ +\ell\, \cosh\left(\frac{1}{p}\ \ln\left(e^{ps} + \sqrt{e^{2ps}+1}\right)\right) \cosh\left(\ln\left(e^{ps} + \sqrt{e^{2ps}+1}\right)\right) \\ +\frac{mp}{q\sqrt{1-p^2}}\Big(\cosh\left(\frac{1}{p}\ \ln\left(e^{ps} + \sqrt{e^{2ps}+1}\right)\right) \sinh\left(\ln\left(e^{ps} + \sqrt{e^{2ps}+1}\right)\right) \\ -p\sinh\left(\frac{1}{p}\ \ln\left(e^{ps} + \sqrt{e^{2ps}+1}\right)\right) \cosh\left(\ln\left(e^{ps} + \sqrt{e^{2ps}+1}\right)\right)\Big), \\ \frac{-\ell\, p}{q}\sinh\left(\ln\left(e^{ps} + \sqrt{e^{2ps}+1}\right)\right) + m\sqrt{1-p^2}\cosh\left(\ln\left(e^{ps} + \sqrt{e^{2ps}+1}\right)\right) \end{pmatrix} \tag{23}$$

Theorem 3. Let $\Psi_{TB}(s)$ be equiform spacelike Smarandache curve of spacelike anti-equiform Salkowski curve $\delta_q(s)$ with timelike binormal in $\mathbb{R}^3_1$. Then $\Psi_{TB}(s)$ is equiform spacelike curve with timelike binormal. Additionally, its equiform frame $\{T_\Psi, N_\Psi, B_\Psi\}$ satisfying

$$\begin{pmatrix} T_\Psi \\ N_\Psi \\ B_\Psi \end{pmatrix} = \begin{pmatrix} \frac{\ell k_1}{\rho\sqrt{2k_1^2+(mk_2+\ell)^2}} & \frac{mk_2+\ell}{\rho\sqrt{2k_1^2+(mk_2+\ell)^2}} & \frac{mk_1}{\rho\sqrt{2k_1^2+(mk_2+\ell)^2}} \\ \frac{m(\mu_2 k_1-\mu_3 k_2)+\ell\mu_3}{\Theta_2} & \frac{k_1(m\mu_1-\ell\mu_3)}{\Theta_2} & \frac{m\mu_1 k_2+\ell(\mu_1-\mu_2 k_1)}{\Theta_2} \\ \frac{\mu_1}{\rho\sqrt{\mu_1^2+\mu_2^2-\mu_3^2}} & \frac{\mu_2}{\rho\sqrt{\mu_1^2+\mu_2^2-\mu_3^2}} & \frac{\mu_3}{\rho\sqrt{\mu_1^2+\mu_2^2-\mu_3^2}} \end{pmatrix} \begin{pmatrix} T_\delta \\ N_\delta \\ B_\delta \end{pmatrix}, \tag{24}$$

where

$$\mu_1 = \frac{1}{2\rho^2}\left[(mk_2+\ell)\big(m(k_1^2+k_1')+k_2(mk_2+\ell)\big)-mk_1\big(m(k_1k_2+k_2')+k_1(mk_2+2\ell)\big)\right],$$
$$\mu_2 = \frac{1}{2\rho^2}\left[\ell k_1\big(m(k_1^2+k_1')+k_2(mk_2+\ell)\big)-mk_1\big(\ell(k_1^2+k_1')-m(mk_2+\ell)\big)\right],$$
$$\mu_3 = \frac{1}{2\rho^2}\left[\ell k_1\big(m(k_1k_2+k_2')+k_1(mk_2+2\ell)\big)-(mk_2+\ell)\big(\ell(k_1^2+k_1')-m(mk_2+\ell)\big)\right],$$
$$\Theta_2 = \rho^2\sqrt{\mu_1^2+\mu_2^2-\mu_3^2}\sqrt{2k_1^2+(mk_2+\ell)^2}.$$

Proof. Using Eq. (3) in differentiationg Eq. (22), we get

$$\dot{\Psi}_{TB}(s) = \frac{1}{\sqrt{2}\,\rho}\Big(\ell k_1\,T_\delta(s)+(mk_2+\ell)N_\delta(s)+mk_1\,B_\delta(s)\Big).$$

Then

$$\|\dot{\Psi}_{TB}(s)\| = \frac{\sqrt{2k_1^2+(mk_2+\ell)^2}}{\sqrt{2}},$$

$$\ddot{\Psi}_{TB}(s) = \frac{1}{\sqrt{2}\,\rho}\Big(\big[\ell(k_1^2+k_1')-m(mk_2+\ell)\big]T_\delta(s)+\big[m(k_1k_2+k_2')+k_1(mk_2 + 2\ell)\big]N_\delta(s)+\big[m(k_1^2+k_1')+k_2(mk_2+\ell)\big]B_\delta(s)\Big),$$

and

$$\dddot{\Psi}_{TB}(s) = \frac{1}{\sqrt{2}\,\rho}\Big(\sigma_1\,T_\delta(s)+\sigma_2\,N_\delta(s)+\sigma_3\,B_\delta(s)\Big),$$

where

$$\sigma_1 = \ell(k_1^3+3k_1k_1')-m(k_1+3k_2')+\ell k_1'',$$
$$\sigma_2 = k_1^2(3mk_2+2\ell)+k_2^2(mk_2+\ell)+(2m+1)(k_1'k_2+k_1k_2')+2\ell k_1'+mk_2'',$$
$$\sigma_3 = mk_1(k_1^2+3k_2^2+k_1'+3k_2')+(m+1)k_2k_2'+\ell(3k_1k_2+k_2')+mk_1''.$$

$$T_\Psi(s) = \frac{\ell k_1\,T_\delta(s)+(mk_2+\ell)N_\delta(s)+mk_1\,B_\delta(s)}{\rho\sqrt{2k_1^2+(mk_2+\ell)^2}}, \tag{25}$$

and

$$B_\Psi(s) = \frac{1}{\rho\sqrt{\mu_1^2+\mu_2^2-\mu_3^2}}\Big(\mu_1\,T_\delta(s)+\mu_2\,N_\delta(s)+\mu_3\,B_\delta(s)\Big), \tag{26}$$

where

$$\mu_1 = \frac{1}{2\rho^2}\left[(mk_2+\ell)\big(m(k_1^2+k_1')+k_2(mk_2+\ell)\big)-mk_1\big(m(k_1k_2+k_2')+k_1(mk_2+2\ell)\big)\right],$$
$$\mu_2 = \frac{1}{2\rho^2}\left[\ell k_1\big(m(k_1^2+k_1')+k_2(mk_2+\ell)\big)-mk_1\big(\ell(k_1^2+k_1')-m(mk_2+\ell)\big)\right],$$
$$\mu_3 = \frac{1}{2\rho^2}\left[\ell k_1\big(m(k_1k_2+k_2')+k_1(mk_2+2\ell)\big)-(mk_2+\ell)\big(\ell(k_1^2+k_1')-m(mk_2+\ell)\big)\right],$$

So, we have

$$N_\Psi(s) = \frac{1}{\Theta_2}\Big[\big[m(\mu_2 k_1 - \mu_3 k_2) + \ell\mu_3\big]T_\delta(s) + (m\mu_1 - \ell\mu_3)k_1\, N_\delta(s) + \big[m\mu_1 k_2 + \ell(\mu_1 - \mu_2 k_1)\big]B_\delta(s)\Big]. \tag{27}$$

where $\Theta_2 = \rho^2\sqrt{\mu_1^2+\mu_2^2-\mu_3^2}\,\sqrt{2k_1^2+(mk_2+\ell)^2}$.

Consequently, the curvature and torsion of $\Psi_{TB}(s)$ are given by

$$\kappa_\Psi(s) = \frac{2\sqrt{2}\sqrt{\mu_1^2+\mu_2^2-\mu_3^2}}{\big[2k_1^2+(mk_2+\ell)^2\big]^{\frac{3}{2}}},$$
$$\tau_\Psi(s) = \frac{1}{2\sqrt{2}\rho^3(\mu_1^2+\mu_2^2-\mu_3^2)}\Big\{\ell k_1\big[\sigma_3\big(m(k_1k_2+k_2')+k_1(mk_2+2\ell)\big) - \sigma_2\big(m(k_1^2+k_1')+k_2(mk_2+\ell)\big)\big] + (mk_2+\ell)\big[\sigma_1\big(m(k_1^2+k_1') + k_2(mk_2+\ell)\big) - \sigma_3\big(m(k_1k_2+k_2')+k_1(mk_2+2\ell)\big)\big] + mk_1\big[\sigma_2\big(m(k_1k_2 + k_2')+k_1(mk_2+2\ell)\big) - \sigma_1\big(m(k_1k_2+k_2')+k_1(mk_2+2\ell)\big)\big]\Big\}, \tag{28}$$

□

Definition 4. Let $\delta_q(s)$ be spacelike anti-equiform Salkowski curve with timelike binormal in $\mathbb{R}^3_1$. The equiform spacelike Φ_{NB}-Smarandache curve of $\delta_q(s)$ is defined by

$$\Phi_{NB}(s) = \frac{1}{\sqrt{2}\,\rho}\Big(\ell\, N_\delta(s) + m\, B_\delta(s)\Big), \quad \ell^2 - m^2 = 2, \tag{29}$$

or equivalently (see Figure 5)

$$\Phi_{NB}(s) = \frac{1}{\sqrt{2}}\begin{pmatrix} -\ell\sqrt{1-p^2}\cosh\left(\frac{1}{p}\ln\left(e^{ps}+\sqrt{e^{2ps}+1}\right)\right) \\ +\frac{mp}{q\sqrt{1-p^2}}\Big(\sinh\left(\frac{1}{p}\ln\left(e^{ps}+\sqrt{e^{2ps}+1}\right)\right)\sinh\left(\ln\left(e^{ps}+\sqrt{e^{2ps}+1}\right)\right) \\ -p\cosh\left(\frac{1}{p}\ln\left(e^{ps}+\sqrt{e^{2ps}+1}\right)\right)\cosh\left(\ln\left(e^{ps}+\sqrt{e^{2ps}+1}\right)\right)\Big), \\ -\ell\sqrt{1-p^2}\sinh\left(\frac{1}{p}\ln\left(e^{ps}+\sqrt{e^{2ps}+1}\right)\right) \\ +\frac{mp}{q\sqrt{1-p^2}}\Big(\cosh\left(\frac{1}{p}\ln\left(e^{ps}+\sqrt{e^{2ps}+1}\right)\right)\sinh\left(\ln\left(e^{ps}+\sqrt{e^{2ps}+1}\right)\right) \\ -p\sinh\left(\frac{1}{p}\ln\left(e^{ps}+\sqrt{e^{2ps}+1}\right)\right)\cosh\left(\ln\left(e^{ps}+\sqrt{e^{2ps}+1}\right)\right)\Big), \\ \frac{-\ell p\sqrt{1-p^2}}{q\sqrt{p^2-1}} + m\sqrt{1-p^2}\cosh\left(\ln\left(e^{ps}+\sqrt{e^{2ps}+1}\right)\right) \end{pmatrix} \tag{30}$$

Theorem 4. Let $\Phi_{NB}(s)$ be equiform spacelike Smarandache curve of spacelike anti-equiform Salkowski curve $\delta_q(s)$ with timelike binormal in $\mathbb{R}^3_1$. Then $\Phi_{NB}(s)$

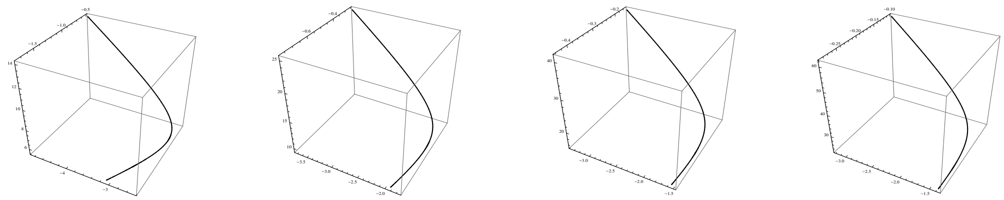

Figure 5: The equiform spacelike Φ_{NB}-Smarandache curve for $q = \{3, 6, 10, 15\}$.

is equiform spacelike curve with timelike binormal. Additionally, its equiform frame $\{T_\Phi, N_\Phi, B_\Phi\}$ satisfying

$$\begin{pmatrix} T_\Phi \\ N_\Phi \\ B_\Phi \end{pmatrix} = \begin{pmatrix} \frac{-\ell}{\rho\sqrt{2(k_1^2-k_2^2)+\ell^2}} & \frac{\ell k_1+mk_2}{\rho\sqrt{2(k_1^2-k_2^2)+\ell^2}} & \frac{\ell k_2+mk_1}{\rho\sqrt{2(k_1^2-k_2^2)+\ell^2}} \\ \frac{\ell(\gamma_2k_2-\gamma_3k_1)-m(\gamma_2k_1-\gamma_3k_2)}{\Theta_3} & \frac{\ell(\gamma_1k_2+\gamma_3)+m\gamma_1k_1}{\Theta_3} & \frac{\ell(\gamma_1k_1+\gamma_2)+m\gamma_1k_2}{\Theta_3} \\ \frac{\gamma_1}{\rho\sqrt{\gamma_1^2+\gamma_2^2-\gamma_3^2}} & \frac{\gamma_2}{\rho\sqrt{\gamma_1^2+\gamma_2^2-\gamma_3^2}} & \frac{\gamma_3}{\rho\sqrt{\gamma_1^2+\gamma_2^2-\gamma_3^2}} \end{pmatrix} \begin{pmatrix} T_\delta \\ N_\delta \\ B_\delta \end{pmatrix}, \tag{31}$$

where

$$\begin{aligned} \gamma_1 &= \frac{1}{2\rho^2}\Big[-(\ell k_1+mk_2)\big(\ell(k_1k_2+k_2')+m(k_1^2+k_1')\big)-(\ell k_2+mk_1)\big(\ell(k_1^2+k_1'-1) \\ &\quad + m(k_1k_2+k_2')\big)\Big], \\ \gamma_2 &= \frac{1}{2\rho^2}\left[\ell\big(\ell(k_1k_2+k_2')+m(k_1^2+k_1')\big)-(\ell k_2+mk_1)\big(2\ell k_1+mk_2\big)\right], \\ \gamma_3 &= \frac{1}{2\rho^2}\left[\ell\big(\ell(k_1^2+k_1'-1)+m(k_1k_2+k_2')\big)-(\ell k_1+mk_2)\big(2\ell k_1+mk_2\big)\right], \\ \Theta_3 &= \rho^2\sqrt{\gamma_1^2+\gamma_2^2-\gamma_3^2}\sqrt{2(k_1^2-k_2^2)+\ell^2}. \end{aligned}$$

Proof. Differentiationg Eq. (29) and using Eq. (3), we get

$$\dot\Phi_{NB}(s) = \frac{1}{\sqrt{2}\,\rho}\Big(-\ell\,T_\delta(s)+(\ell k_1+mk_2)N_\delta(s)+(\ell k_2+mk_1)B_\delta(s)\Big).$$

Then

$$\|\dot\Phi_{NB}(s)\| = \frac{\sqrt{2(k_1^2-k_2^2)+\ell^2}}{\sqrt{2}},$$

$$\begin{aligned} \ddot\Phi_{NB}(s) =& \frac{1}{\sqrt{2}\,\rho}\Big(-\big[2\ell k_1+mk_2\big]T_\delta(s)+\big[\ell(k_1^2+k_1'-1)+m(k_1k_2+k_2')\big]N_\delta(s) \\ &+\big[\ell(k_1k_2+k_2')+m(k_1^2+k_1')\big]B_\delta(s)\Big), \end{aligned}$$

and

$$\dddot\Phi_{NB}(s) = \frac{1}{\sqrt{2}\,\rho}\Big(r_1\,T_\delta(s)+r_2\,N_\delta(s)+r_3\,B_\delta(s)\Big),$$

where

$$\begin{aligned}
r_1 &= mk_2' - 2\ell k_1 + k_1(2\ell k_1 + mk_2) - \big[\ell(k_1^2 + k_1' - 1) + m(k_1k_2 + k_2')\big],\\
r_2 &= \ell(2k_1k_1' + k_1'') + m(k_1'k_2 + k_1k_2' + k_2'') + k_1\big[\ell(k_1^2 + k_1' - 1) + m(k_1k_2 + k_2')\big]\\
&\quad + k_2\big[\ell(k_1k_2 + k_2') + m(k_1^2 + k_1')\big] + 2\ell k_1 + mk_2,\\
r_3 &= \ell(k_1'k_2 + k_1k_2' + k_2'') + m(2k_1k_1' + k_1'')k_2\big[\ell(k_1^2 + k_1' - 1) + m(k_1k_2 + k_2')\big]\\
&\quad + k_1\big[\ell(k_1k_2 + k_2') + m(k_1^2 + k_1')\big].
\end{aligned}$$

$$T_\Phi(s) = \frac{-\ell\, T_\delta(s) + (\ell k_1 + mk_2)N_\delta(s) + (\ell k_2 + mk_1)B_\delta(s)}{\rho\sqrt{2(k_1^2 - k_2^2) + \ell^2}}, \tag{32}$$

and

$$B_\Phi(s) = \frac{1}{\rho\sqrt{\gamma_1^2 + \gamma_2^2 - \gamma_3^2}}\Big(\gamma_1\, T_\delta(s) + \gamma_2\, N_\delta(s) + \gamma_3\, B_\delta(s)\Big), \tag{33}$$

where

$$\begin{aligned}
\gamma_1 &= \frac{1}{2\rho^2}\Big[-(\ell k_1 + mk_2)\big(\ell(k_1k_2 + k_2') + m(k_1^2 + k_1')\big) - (\ell k_2 + mk_1)\big(\ell(k_1^2 + k_1' - 1)\\
&\quad + m(k_1k_2 + k_2')\big)\Big],\\
\gamma_2 &= \frac{1}{2\rho^2}\left[\ell\big(\ell(k_1k_2 + k_2') + m(k_1^2 + k_1')\big) - (\ell k_2 + mk_1)\big(2\ell k_1 + mk_2\big)\right],\\
\gamma_3 &= \frac{1}{2\rho^2}\left[\ell\big(\ell(k_1^2 + k_1' - 1) + m(k_1k_2 + k_2')\big) - (\ell k_1 + mk_2)\big(2\ell k_1 + mk_2\big)\right].
\end{aligned}$$

So, we have

$$\begin{aligned}
N_\Phi(s) =& \frac{1}{\Theta_3}\Big[\big[\ell(\gamma_2k_2 - \gamma_3k_1) - m(\gamma_2k_1 - \gamma_3k_2)\big]T_\delta(s) + \big[\ell(\gamma_1k_2 + \gamma_3)\\
&+ m\gamma_1k_1\big]N_\delta(s) + \big[\ell(\gamma_1k_1 + \gamma_2) + m\gamma_1k_2\big]B_\delta(s)\Big].
\end{aligned} \tag{34}$$

where $\Theta_3 = \rho^2\sqrt{\gamma_1^2 + \gamma_2^2 - \gamma_3^2}\,\sqrt{2(k_1^2 - k_2^2) + \ell^2}$. Therefore, the curvature functions of $\Phi_{NB}(s)$ are given by

$$\begin{aligned}
\kappa_\Phi(s) &= \frac{2\sqrt{2}\sqrt{\gamma_1^2 + \gamma_2^2 - \gamma_3^2}}{\big[2(k_1^2 - k_2^2) + \ell^2\big]^{\frac{3}{2}}},\\
\tau_\Phi(s) &= \frac{1}{2\sqrt{2}\rho^3(\gamma_1^2 + \gamma_2^2 - \gamma_3^2)}\Big\{\ell\big[r_2\big(\ell(k_1k_2 + k_2') + m(k_1^2 + k_1')\big) - r_3\big(\ell(k_1^2 + k_1' - 1)\\
&\quad + m(k_1k_2 + k_2')\big)\big] + (\ell k_1 + mk_2)\big[r_1\big(\ell(k_1k_2 + k_2') + m(k_1^2 + k_1')\big) - r_3\big(2\ell k_1 + mk_2\big)\big]\\
&\quad + (\ell k_2 + mk_1)\big[r_2\big(2\ell k_1 + mk_2\big) - r_1\big(\ell(k_1^2 + k_1' - 1) + m(k_1k_2 + k_2')\big)\big]\Big\},
\end{aligned} \tag{35}$$

□

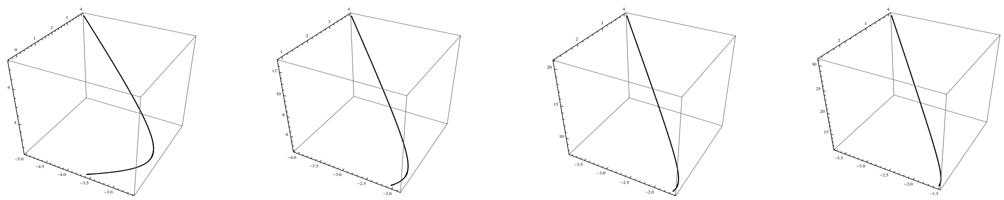

Figure 6: The equiform spacelike Ω_{TNB}-Smarandache curve for $q = \{3, 6, 10, 15\}$.

Definition 5. Let $\delta_q(s)$ be spacelike anti-equiform Salkowski curve with timelike binormal in $\mathbb{R}^3_1$. The equiform spacelike Ω_{TNB}-Smarandache curve of $\delta_q(s)$ is defined by

$$\Omega_{TNB}(s) = \frac{1}{\sqrt{3}\,\rho}\Big(\ell\, T_\delta(s) + m\, N_\delta(s) + n\, B_\delta(s)\Big), \quad \ell^2 + m^2 - n^2 = 2, \tag{36}$$

or equivalently (see Figure 6)

$$\Omega_{TNB}(s) = \frac{1}{\sqrt{3}} \left(\begin{array}{c} -\ell\, p \cosh\left(\frac{1}{p}\ln\left(e^{ps} + \sqrt{e^{2ps}+1}\right)\right)\sinh\left(\ln\left(e^{ps} + \sqrt{e^{2ps}+1}\right)\right) + \ell \\ \sinh\left(\frac{1}{p}\ln\left(e^{ps} + \sqrt{e^{2ps}+1}\right)\right)\cosh\left(\ln\left(e^{ps} + \sqrt{e^{2ps}+1}\right)\right) \\ +\frac{np}{q\sqrt{1-p^2}}\Big(\sinh\left(\frac{1}{p}\ln\left(e^{ps} + \sqrt{e^{2ps}+1}\right)\right)\sinh\left(\ln\left(e^{ps} + \sqrt{e^{2ps}+1}\right)\right) \\ -p\cosh\left(\frac{1}{p}\ln\left(e^{ps} + \sqrt{e^{2ps}+1}\right)\right)\cosh\left(\ln\left(e^{ps} + \sqrt{e^{2ps}+1}\right)\right)\Big) \\ -m\sqrt{1-p^2}\cosh\left(\frac{1}{p}\ln\left(e^{ps} + \sqrt{e^{2ps}+1}\right)\right), \\ -\ell\, p \sinh\left(\frac{1}{p}\ln\left(e^{ps} + \sqrt{e^{2ps}+1}\right)\right)\sinh\left(\ln\left(e^{ps} + \sqrt{e^{2ps}+1}\right)\right) \\ +\ell \cosh\left(\frac{1}{p}\ln\left(e^{ps} + \sqrt{e^{2ps}+1}\right)\right)\cosh\left(\ln\left(e^{ps} + \sqrt{e^{2ps}+1}\right)\right) \\ +\frac{np}{q\sqrt{1-p^2}}\Big(\cosh\left(\frac{1}{p}\ln\left(e^{ps} + \sqrt{e^{2ps}+1}\right)\right)\sinh\left(\ln\left(e^{ps} + \sqrt{e^{2ps}+1}\right)\right) \\ -p\sinh\left(\frac{1}{p}\ln\left(e^{ps} + \sqrt{e^{2ps}+1}\right)\right)\cosh\left(\ln\left(e^{ps} + \sqrt{e^{2ps}+1}\right)\right)\Big), \\ -m\sqrt{1-p^2}\sinh\left(\frac{1}{p}\ln\left(e^{ps} + \sqrt{e^{2ps}+1}\right)\right), \\ \frac{-\ell\, p}{q}\sinh\left(\ln\left(e^{ps} + \sqrt{e^{2ps}+1}\right)\right) + n\sqrt{1-p^2}\cosh\left(\ln\left(e^{ps} + \sqrt{e^{2ps}+1}\right)\right) \\ -\frac{mp\sqrt{1-p^2}}{\sqrt{p^2-1}} \end{array} \right) \tag{37}$$

Theorem 5. Let $\Omega_{TNB}(s)$ be equiform spacelike Smarandache curve of spacelike anti-equiform Salkowski curve $\delta_q(s)$ with timelike binormal in $\mathbb{R}^3_1$. Then $\Omega_{TNB}(s)$ is equiform spacelike curve with timelike binormal. Additionally, its equiform frame $\{T_\Omega, N_\Omega, B_\Omega\}$ satisfying

$$\begin{aligned}
T_\Omega(s) &= \frac{(\ell k_1 - m)T_\delta(s) + (mk_1 + nk_2 + \ell)N_\delta(s) + (mk_2 + nk_1)B_\delta(s)}{\rho\sqrt{\ell^2(k_1^2+1) + m^2(k_1^2 - k_2^2 + 1) - n^2(k_1^2 - k_2^2) + 2n\ell k_2}},\\
N_\Omega(s) &= \frac{1}{\Theta_4}\Big\{\big[m(\nu_2 k_2 - \nu_3 k_1) + n(\nu_2 k_1 - \nu_3 k_2) - \ell\nu_3\big]T_\delta(s) + \big[m(\nu_1 k_2 + \nu_3) - n\nu_1 k_1\\
&\quad - \ell\nu_3 k_1\big]N_\delta(s) + \big[\ell(\nu_2 k_1 - \nu_1) - m(\nu_1 k_1 + \nu_2) - n\nu_1 k_2\big]B_\delta(s)\Big\},\\
B_\Omega(s) &= \frac{\nu_1\, T_\delta(s) + \nu_2\, N_\delta(s) + \nu_3\, B_\delta(s)}{\rho\sqrt{\nu_1^2 + \nu_2^2 - \nu_3^2}},
\end{aligned} \tag{38}$$

where

$$\begin{aligned}
\nu_1 &= \frac{1}{3\rho^2}\Big[(mk_1 + nk_2 + \ell)\big(\ell k_2 + m(2k_1k_2 + k_2') + n(k_1^2 + k_2^2 + k_1')\big) - (mk_2 + nk_1)\big(2\ell k_1\\
&\quad + m(k_1^2 + k_1' - 1) + n(2k_1k_2 + k_2')\big)\Big],\\
\nu_2 &= \frac{1}{3\rho^2}\Big[(\ell k_1 - m)\big(\ell k_2 + m(2k_1k_2 + k_2') + n(k_1^2 + k_2^2 + k_1')\big) - (mk_2 + nk_1)\big(\ell(k_1^2 + k_1' - 1)\\
&\quad - 2mk_1 - nk_2\big)\Big],\\
\nu_3 &= \frac{1}{3\rho^2}\Big[(mk_1 + nk_2 + \ell)\big(\ell(k_1^2 + k_1' - 1) - 2mk_1 - nk_2\big) - (\ell k_1 - m)\big(2\ell k_1 + m(k_1^2 + k_1' - 1)\\
&\quad + n(2k_1k_2 + k_2')\big)\Big],\\
\Theta_4 &= \rho^2\sqrt{\nu_1^2 + \nu_2^2 - \nu_3^2}\,\sqrt{\ell^2(k_1^2+1) + m^2(k_1^2 - k_2^2 + 1) - n^2(k_1^2 - k_2^2) + 2n\ell k_2}.
\end{aligned}$$

Proof. Taking the derivative of Eq. (36) and using Eq. (3), we get

$$\dot{\Omega}_{TNB}(s) = \frac{1}{\sqrt{3}\,\rho}\Big((\ell k_1 - m)T_\delta(s) + (mk_1 + nk_2 + \ell)N_\delta(s) + (mk_2 + nk_1)B_\delta(s)\Big). \tag{39}$$

Then

$$\|\dot{\Omega}_{TNB}(s)\| = \frac{\sqrt{\ell^2(k_1^2+1) + m^2(k_1^2 - k_2^2 + 1) - n^2(k_1^2 - k_2^2) + 2n\ell k_2}}{\sqrt{3}}, \tag{40}$$

$$\begin{aligned}
\ddot{\Omega}_{TNB}(s) = &\frac{1}{\sqrt{3}\,\rho}\Big(\big[\ell(k_1^2 + k_1' - 1) - 2mk_1 - nk_2\big]T_\delta(s) + \big[2\ell k_1 + m(k_1^2 + k_1' - 1)\\
&+ n(2k_1k_2 + k_2')\big]N_\delta(s) + \big[\ell k_2 + m(2k_1k_2 + k_2') + n(k_1^2 + k_2^2 + k_1'\big]B_\delta(s)\Big),
\end{aligned} \tag{41}$$

and

$$\dddot{\Omega}_{TNB}(s) = \frac{1}{\sqrt{3}\,\rho}\Big(\xi_1\, T_\delta(s) + \xi_2\, N_\delta(s) + \xi_3\, B_\delta(s)\Big), \tag{42}$$

where

$$
\begin{aligned}
\xi_1 &= \ell(2k_1k_1' + k_1'') + 2mk_1' - nk_2' + k_1\big(\ell(k_1^2 + k_1' - 1) - 2mk_1 - nk_2\big),\\
\xi_2 &= 2\ell k_1' + m(2k_1'k_2 + 2k_1k_2' + k_1'') + n(2k_1'k_2 + 2k_1k_2' + k_2'') + k_1\big(2\ell k_1 + m(k_1^2 + k_1' - 1)\\
&\quad + n(2k_1k_2 + k_2')\big) + k_2\big(\ell k_2 + m(2k_1k_2 + k_2') + n(k_1^2 + k_2^2 + k_1')\big) + \ell(k_1^2 + k_1' - 1)\\
&\quad - 2mk_1 - nk_2,\\
\xi_3 &= \ell k_2' + m(2k_1'k_2 + 2k_1k_2' + k_2'') + n(2k_1k_1' + 2k_2k_2' + k_1'') + k_2\big(2\ell k_1 + m(k_1^2 + k_1' - 1)\\
&\quad + n(2k_1k_2 + k_2')\big) + k_1\big(\ell k_2 + m(2k_1k_2 + k_2') + n(k_1^2 + k_2^2 + k_1')\big).
\end{aligned}
$$

Then, from Eqs. (39) and (40), we have

$$
T_\Omega(s) = \frac{(\ell k_1 - m)T_\delta(s) + (mk_1 + nk_2 + \ell)N_\delta(s) + (mk_2 + nk_1)B_\delta(s)}{\rho\sqrt{\ell^2(k_1^2+1) + m^2(k_1^2 - k_2^2 + 1) - n^2(k_1^2 - k_2^2) + 2n\ell k_2}}\,, \tag{43}
$$

and

$$
B_\Omega(s) = \frac{1}{\rho\sqrt{\nu_1^2 + \nu_2^2 - \nu_3^2}}\Big(\nu_1\, T_\delta(s) + \nu_2\, N_\delta(s) + \nu_3\, B_\delta(s)\Big), \tag{44}
$$

where

$$
\begin{aligned}
\nu_1 &= \frac{1}{3\rho^2}\Big[(mk_1 + nk_2 + \ell)\big(\ell k_2 + m(2k_1k_2 + k_2') + n(k_1^2 + k_2^2 + k_1')\big) - (mk_2 + nk_1)\big(2\ell k_1\\
&\quad + m(k_1^2 + k_1' - 1) + n(2k_1k_2 + k_2')\big)\Big],\\
\nu_2 &= \frac{1}{3\rho^2}\Big[(\ell k_1 - m)\big(\ell k_2 + m(2k_1k_2 + k_2') + n(k_1^2 + k_2^2 + k_1')\big) - (mk_2 + nk_1)\big(\ell(k_1^2 + k_1' - 1)\\
&\quad - 2mk_1 - nk_2\big)\Big],\\
\nu_3 &= \frac{1}{3\rho^2}\Big[(mk_1 + nk_2 + \ell)\big(\ell(k_1^2 + k_1' - 1) - 2mk_1 - nk_2\big) - (\ell k_1 - m)\big(2\ell k_1 + m(k_1^2 + k_1' - 1)\\
&\quad + n(2k_1k_2 + k_2')\big)\Big].
\end{aligned}
$$

Now, from Eqs. (43) and (44), we get

$$
\begin{aligned}
N_\Omega(s) =& \frac{1}{\Theta_4}\Big\{\big[m(\nu_2k_2 - \nu_3k_1) + n(\nu_2k_1 - \nu_3k_2) - \ell\nu_3\big]T_\delta(s) + \big[m(\nu_1k_2 + \nu_3)\\
&- n\nu_1k_1 - \ell\nu_3k_1\big]N_\delta(s) + \big[\ell(\nu_2k_1 - \nu_1) - m(\nu_1k_1 + \nu_2) - n\nu_1k_2\big]B_\delta(s)\Big\},
\end{aligned} \tag{45}
$$

where $\Theta_4 = \rho^2\sqrt{\nu_1^2 + \nu_2^2 - \nu_3^2}\,\sqrt{\ell^2(k_1^2+1) + m^2(k_1^2 - k_2^2 + 1) - n^2(k_1^2 - k_2^2) + 2n\ell k_2}$.
Also, from Eqs. (39), (40), (41) and (42), we found

$$
\kappa_\Omega(s) = \frac{3\sqrt{3}\sqrt{\nu_1^2 + \nu_2^2 - \nu_3^2}}{\big[\ell^2(k_1^2+1) + m^2(k_1^2 - k_2^2 + 1) - n^2(k_1^2 - k_2^2) + 2n\ell k_2\big]^{\frac{3}{2}}}, \tag{46}
$$

$$\tau_\Omega(s) = \frac{1}{3\sqrt{3}\rho^3(\nu_1^2+\nu_2^2-\nu_3^2)}\Big\{(\ell k_1 - m)\big[\xi_3\big(2\ell k_1 + m(k_1^2+k_1'-1)+n(2k_1k_2+k_2')\big) - \xi_2\big(\ell k_2 + m(2k_1k_2+k_2') + n(k_1^2+k_2^2+k_1')\big)\big] + (mk_1+nk_2+\ell)\big[\xi_1\big(\ell k_2 + m(2k_1k_2 + k_2') + n(k_1^2+k_2^2+k_1')\big) - \xi_3\big(\ell(k_1^2+k_1'-1) - 2mk_1 - nk_2\big)\big] + (mk_2+nk_1)\big[\xi_2\big(\ell(k_1^2 + k_1'-1) - 2mk_1 - nk_2\big) - \xi_1\big(2\ell k_1 + m(k_1^2+k_1'-1)+n(2k_1k_2+k_2')\big)\big]\Big\}. \tag{47}$$

□

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