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SOME REMARKS ON THE SMARANDACHE FUNCTION

by

M. Andrei, C. Dumitrescu, V. Seleacu, L. Tutescu, St. Zanfir

1. On the method of calculus proposed by Florentin Smarandache. In [6] is defined a numerical function $S: \mathbb{N} \to \mathbb{N}$, as follows:

$S(n)$ is the smallest nonnegative integer such that $S(n)!$ is divisible by $n$.

For example $S(1) = 0$, $S(2^{12}) = 16$.

This function characterizes the prime numbers in the sense that $p > 4$ is prime if and only if $S(p) = p$. As it is showed in [6] this function may be extended to all integers by defining $S(-n) = S(n)$. If $a$ and $b$ are relatively prime then $S(a \cdot b) = \max\{S(a), S(b)\}$

More general, if $[a, b]$ is the last common multiple of $a$ and $b$ then

$$S([a, b]) = \max\{|S(a), S(b)|$$

(1)

So, if $n = p_1^{a_1} \cdot p_2^{a_2} \cdot \ldots \cdot p_t^{a_t}$ is the factorization of $n$ into primes, then

$$S(n) = \max\{|S(p_i^{a_i})| \mid i = 1, \ldots, t\}$$

(2)

For the calculus of $S(p_i^{a_i})$ in [6] it is used the fact that if $a = (p^n - 1)/(p - 1)$ then $S(p_i^{a_i}) = p^n$.

This equality results from the fact, if $\alpha_p(n)$ is the exponent of the prime $p$ in the decomposition of $n!$ into primes then

$$\alpha_p(n) = \sum_{i=1}^{\left\lfloor \frac{n}{p^i} \right\rfloor} \left\lfloor \frac{n}{p^i} \right\rfloor$$

(3)

From (3) is results that $S(p_i^{a_i}) \leq p \cdot a$.

Now, if we note $a_p(n) = (p^n - 1)/(p - 1)$ then

$$S(p^m \cdot p_2^{a_2} \cdot \ldots \cdot p_t^{a_t}) = k_{a_1} p_1^{a_1} - k_{a_2} p_2^{a_2} - \ldots - k_{a_t} p_t^{a_t}$$

(4)

for $k_{a_1}, k_{a_2}, \ldots, k_{a_t} \in \{1, p - 1\}$ and $k_{a_i} \in \{1, 2, \ldots, p\}$.
That is, if we consider the generalized scale

\[ [p] : a_1(p), a_2(p), \ldots, a_n(p), \ldots \]

and the standard scale

\[ (p) : 1, p, p^2, \ldots, p^n, \ldots \]

and we express the exponent \( a \) in the scale \([p]\), \( a_{[p]} = k_m k_{m-1} \ldots k_1 \), then the left hand of the equality (4) is \( s(p^n) \) and the right hand becomes \( p(a_{[p]})_{(p)} \). In other words, the right hand of (4) is the number obtained multiplying by \( p \) the exponent \( a \) written in the scale \([p]\), readed it in the scale \((p)\).

So, (4) may be written as

\[ s(p^n) = p(a_{[p]})_{(p)} \quad (5) \]

For example, to calculate \( S(3^{89}) \) we write the exponent \( a = 89 \) in the scale

\[ [3] : 1, 4, 13, 40, 121, \ldots \]

and so

\[ a_\alpha(p) \leq a \leq (p-1) \cdot (p-1) \cdot a = m_1 \leq \log_p((p-1) \cdot a + 1) \].

It results that \( m_1 \) is the integer part of \( \log_p((p-1) \cdot a + 1) \).

For our example \( m_1 = \lfloor \log_3(2a_1+1) \rfloor = \log_3 179 = 4 \). Then first digit of \( a_{[3]} \) is \( k_4 = \lfloor a/a_4(3) \rfloor = 2 \). So, 89 = 2a_4(3) - 9.

For \( p_1 = 9 \) it results \( m_2 = \lfloor \log_3(2a_2+1) \rfloor = 2 \), \( k_2 = \lfloor a_1/a_2(3) \rfloor = 2 \) and so \( a_1 = 2a_2(3) + 1 \). Then \( 89 = 2a_4(3) - 2a_2(3) - a_3(3) = 2021_{[3]} \). and \( S(3^{89}) = 32021_{[3]} = 183 \).

Indeed, \( \sum_{j=3}^{183} \frac{183}{3^j} = 61 + 20 + 6 + 2 = 89 \).

Let us observe that the calculus in the generalized scale \([p]\) is essentially different from the calculus in the standard scale \((p)\). That because if we note \( b_\alpha(p) = p^n \) then it results

\[ a_{n+1}(p) = p a_n(p) + 1 \quad \text{and} \quad a_{n+1}(p) = p a_n(p) - 1 \quad (6) \]

For this, to add some numbers in the scale \([p]\) we do as follows. We start to add from the digits of "decimals", that is from the column of \( a_\alpha(p) \). If adding some digits it is obtained \( p a_\alpha(p) \) then it is utilized a unit from the class of units (coefficients of \( a_\alpha(p) \)) to obtain \( p a_\alpha(p) + 1 = a_\alpha(p) \). Continuing to add, if agains it is obtained \( p a_\alpha(p) \), then a new unit must be used, from the class of units, etc.
For example if \( m_{13} = 442 \), \( n_{13} = 412 \) and \( r_{13} = 44 \) then
\[
m + n + r = 442 + 412 + 44 = 1450.
\]

We start to add from the column corresponding to \( a_2(5) \):
\[
4a_2(5) - a_2(5) - 4a_2(5) = 5a_2(5) + 4a_2(5).
\]
Now utilizing a unit from the first column we obtain
\[
5a_2(5) + 4a_2(5) = a_2(5) + 4a_2(5), \text{ so } b = 4.
\]
Continuing, \( 4a_2(5) + 4a_2(5) + a_2(5) = 5a_2(5) + 4a_2(5) \) and using a new unit it results
\[
4a_2(5) - a_2(5) = a_4(5) - 4a_3(5), \text{ so } c = 4 \text{ and } d = 1.
\]
Finally, adding the remained units \( 4a_2(5) + 2a_1(5) = 5a_2(5) + a_1(5) = 5a_2(5) + 5\) \( a_3(5) + 1 = a_4(5) \) it results that \( b \) must be modified and \( a = 0 \). So \( m + n + r = 1450 \).

We have applied the formula (5) to the calculus of the values of \( S \) for any integer between \( N_1 = 31,000,000 \) and \( N_2 = 31,001,000 \). A program has been designed to generate the factorization of every integer \( n \in [N_1, N_2] \) (TIME (minutes): START: 40:8:93, STOP: 56:38:85, more than 16 minutes).

Afterwards, the Smarandache function has been calculated for every \( n = p_1^{a_1} \cdot p_2^{a_2} \cdot \ldots \cdot p_i^{a_i} \) as follows:

1) Max \( p_i \cdot a_i \) is determined

2) \( S_0 = S(p_i^{a_i}) \), for \( i \) determined above

3) Because \( S(p_i^{a_i}) \leq p_i \cdot a_i \), we ignore the factors for which \( p_i \cdot a_i \leq S_0 \).

4) Are calculated \( S(p_i^{a_i}) \) for \( p_i \cdot a_i > S_0 \) and is determined the greatest of these values.


2. Some diofantine equations concerning the function \( S \).

In this section we shall apply the formula (5) for the study of the solutions of some diofantine equations proposed in (6).

a) Using (5) it can be proved that the diofantine equation
\[
S(x \cdot y) = S(x) + S(y)
\]
has infinitely many solutions. Indeed, let us observe that from (2) every relatively prime integers \( x_0 \) and \( y_0 \) can’t be a solution from (7). Let now \( x = p^a \cdot A \), \( y = p^b \cdot B \) be such that \( S(x) = S(p^a) \) and \( S(y) = S(p^b) \).

Then \( S(x \cdot y) = S(p^{a+b}) \) and (7) becomes
\[
p^a(a - b)(p^a) = p^b(a - b)(p^b) + p(a - b)(p^a)
\]
or

3
\[
\left( (a + b)_{(p)} \right)_{(p)} = \left( a_{(p)} \right)_{(p)} + \left( b_{(p)} \right)_{(p)}
\]

There exists infinitely many values for \( a \) and \( b \) satisfying this equality. For example \( a = a_1(p) = 100_{(p)} \), \( b = a_2(p) = 10_{(p)} \) and (8) becomes \( 110_{(p)} \left( a_{(p)} \right)_{(p)} - \left( 10_{(p)} \right)_{(p)} \).

b) We shall prove now that the equation

\[
S(x \cdot y) = S(x) \cdot S(y)
\]

has no solution \( x, y \geq 1 \).

Let \( m = S(x) \) and \( n = S(y) \). It is sufficient to prove that \( S(x \cdot y) = m \cdot n \). But it is said that \( m! \cdot n! \) divide \( (m - n)! \), so

\[
(m \cdot n)! \mid (m - n)! \mid m! \cdot n! \mid x \cdot y
\]

c) If we note by \( (x, y) \) the greatest common divisor of \( x \) and \( y \), then the equation

\[
(x, y) = (S(x), S(y))
\]

has infinitely many solutions. Indeed, because \( x \geq S(x) \), the equality holding if and only if \( x \) is a prime it results that (9) has as solution every pair \( x, y \) of prime numbers and also every pair of product of prime numbers.

Let now \( S(x) = p_1(a_{(p)})_{(p)} \), \( S(y) = q_1(b_{(p)})_{(p)} \) be such that \( (x, y) = d > 1 \). Then because \( (p, q) = 1 \), if

\[
\begin{align*}
(a_{(p)})_{(p)} &= a_1, \\
(b_{(p)})_{(p)} &= b_1, \\
(p, b_1) &= (a_1, q) = 1.
\end{align*}
\]

it result that the equality (9) becomes

\[
\left[ (a_{(p)})_{(p)}, (b_{(p)})_{(p)} \right] = d
\]

and it is satisfied for various positive integers \( a \) and \( b \). For example if \( x = 2 \cdot 3^3 \) and \( y = 2 \cdot 5^3 \) it results \( d = 2 \) and the equality \( \left[ (a_{(p)})_{(p)}, (b_{(p)})_{(p)} \right] = 2 \) is satisfied for many values of \( a, b \in N \).

d) If \( [x, y] \) is the least common multiple of \( x \) and \( y \) then the equation

\[
[x, y] = [S(x), S(y)]
\]

has as solutions every pair of prime numbers. Now, if \( x \) and \( y \) are composite numbers such that \( S(x) = S(p_i^a) \) and \( S(y) = S(p_j^b) \) with \( p_i = p_j \) then the pair \( x, y \) can't be solution of the equation because in this case we have

\[
[x, y] > p_i^a \cdot p_i^b > S(x) \cdot S(y) \geq S(x), S(y)
\]

and if \( x = p_i^a \cdot A \) and \( y = p_j^b \cdot B \) with \( S(x) = S(p_i^a), S(y) = S(p_j^b) \) then

\[
[x, y] = [S(x), S(y)] = p_i^a \cdot [a_{(p)}]_{(p)} \cdot [b_{(p)}]_{(p)}
\]
and \([x, y] = p^{\max(x, y)} \cdot [A, B]\) so (10) is satisfied also for many values of non relatively prime integers.

e) Finally we consider the equation

\[ S(x) + y = x - S(y) \]

which has as solution every pair of prime numbers, but also the composite numbers \(x = y\).

It can be found other composite numbers as solutions. For example if \(p\) and \(q\) are consecutive prime numbers such that

\[ q - p = h > 0 \quad (11) \]

and \(x = p \cdot A\), \(y = q \cdot B\) then our equation is equivalent to

\[ y - x = S(y) - S(x) \quad (12) \]

If we consider the diophantine equation \(qB - pA = h\) it results from (11) that \(A_0 = B_0 = 1\) is a particular solution, so the general solution is \(A = 1 + rp\), \(B = 1 + rq\), for arbitrary integer \(r\). Then for \(r = 1\) it results \(x = p(1 + q)\), \(y = q(1 + p)\) and \(y - x = h\). In addition, because \(p\) and \(q\) are consecutive primes it results that \(p + 1\) and \(q - 1\) are composite and so

\[ S(x) = p, \; S(y) = q, \; S(y) - S(x) = h \]

and (12) holds.

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SMARANDACHE NUMERICAL FUNCTIONS

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F. Smarandache defines [1] a numerical function

\[ S: \mathbb{N}^* \rightarrow \mathbb{N} \] is the smallest integer \( m \) such that \( m! \) is divisible by \( n \). Using certain results on standardised structures, three kinds of Smarandache functions are defined and are established some compatibility relations between these functions.

1. Standardising functions. Let \( X \) be a nonvoid set, \( r \subseteq X \times X \) an equivalence relation, \( \hat{X} \) the corresponding quotient set and \( (I,\leq) \) a totally ordered set.

1.1 Definition. If \( g: \hat{X} \rightarrow I \) is an arbitrarily injective function, then \( f: X \rightarrow I \) defined by \( f(x) = g(x) \) is a standardising function. In this case the set \( X \) is said to be \( (r,(I,\leq),f) \) standardised.

If \( r_1 \) and \( r_2 \) are two equivalence relations on \( X \), then \( r = r_1 \wedge r_2 \) is defined as \( x r y \) if and only if \( x r_1 y \) and \( x r_2 y \). Of course \( r \) is an equivalence relation.

In the following theorem we consider functions having the same monotonicity. The functions \( f_i: X \rightarrow I \), \( i = 1, s \) are of the same monotonicity if for every \( x, y \) from \( X \) it results

\[ f_k(x) \leq f_k(y) \] if and only if \( f_j(x) \leq f_j(y) \) for \( k, j = 1, s \)
1.2 Theorem. If the standardising functions $f_i : X \rightarrow I$ corresponding to the equivalence relations $r_i$, $i = 1, s$, are of the same monotonicity then $f = \max_i (f_i)$ is a standardising function corresponding to $r = \bigwedge_i r_i$, having the same monotonicity as $f_i$.

Proof. We give the proof of theorem in case $s = 2$. Let $\hat{x}_i, \hat{x}_2, \hat{x}_r$ be the equivalence classes of $x$ corresponding to $r_1, r_2$ and to $r = r_1 \bigwedge r_2$ respectively and $\hat{X}_1, \hat{X}_2, \hat{X}_r$ the quotient sets on $X$.

We have $f_1(x) = g_1(\hat{x}_1)$ and $f_2(x) = g_2(\hat{x}_2)$, where $g_1 : \hat{X}_1 \rightarrow I$ and $g_2 : \hat{X}_2 \rightarrow I$ are injective functions. The function $g : \hat{X}_r \rightarrow I$ defined by $g(\hat{x}_r) = \max(g_1(\hat{x}_1), g_2(\hat{x}_2))$ is injective.

Indeed, if $\hat{x}_1 \neq \hat{x}_2$ and $\max(g_1(\hat{x}_1), g_2(\hat{x}_2)) = \max(g_1(\hat{x}_2), g_2(\hat{x}_2))$, then be cause of the injectivity of $g_1$ and $g_2$, we have for example $\max(g_1(\hat{x}_1), g_2(\hat{x}_2)) = g_1(\hat{x}_1) = g_2(\hat{x}_2) = \max(g_1(\hat{x}_2), g_2(\hat{x}_2))$ and we obtain a contradiction because $f_1(x_2) = g_1(\hat{x}_2) < g_1(\hat{x}_1) = f_1(x_1)$ and $f_2(x_2) = g_2(\hat{x}_2) < g_2(\hat{x}_1) = f_2(x_1)$, that is $f_1$ and $f_2$ are not of the same monotonicity from the injectivity of $g$ it results that $f : X \rightarrow I$ defined by $f(x) = g(\hat{x}_r)$ is a standardising function. In addition we have $f_1(x_1) \leq f_1(x_2) \leftrightarrow g_1(\hat{x}_1) \leq g_1(\hat{x}_2) \leftrightarrow \max(g_1(\hat{x}_1), g_2(\hat{x}_1)) \leq \max(g_1(\hat{x}_2), g_2(\hat{x}_2)) \leftrightarrow \max(f_1(x_1), f_2(x_1)) \leq \max(f_1(x_2), f_2(x_2)) \leftrightarrow f_1(x_1) \leq f_1(x_2)$ and $f_2(x_1) \leq f_2(x_2)$ because $f_1$ and $f_2$ are of the same monotonicity.
Let us suppose now that \( \top \) and \( \perp \) are two algebraic laws on \( X \) and \( I \) respectively.

1.3. Definition. The standardising function \( f: X \longrightarrow I \) is said to be \( \Sigma \)-compatible with \( \top \) and \( \perp \) if for every \( x,y \) in \( X \) the triplet \( (f(x), f(y), f(x \perp y)) \) satisfies the condition \( \Sigma \). In this case it is said that the function \( f \) \( \Sigma \)-standardise the structure \( (X, \top) \) in the structure \( (I, \leq, \perp) \).

For example, if \( f \) is the Smarandache function \( S: \mathbb{N}^* \longrightarrow \mathbb{N}^* \) \( (SC(n)) \) is the smallest integer such that \( (SC(n))! \) is divisible by \( n \) then we get the following \( \Sigma \)-standardisations:

a) \( S \) \( \Sigma_1 \)-standardise \( (\mathbb{N}^*, \cdot) \) in \( (\mathbb{N}^*, \leq, +) \) because we have

\[
\Sigma_1: SC(a, b) \leq SC(a) + SC(b)
\]

b) but \( S \) verifies also the relation

\[
\Sigma_2: \max(SC(a), SC(b)) \leq SC(a, b) \leq SC(a) \cdot SC(b)
\]

so \( S \) \( \Sigma_2 \)-standardise the structure \( (\mathbb{N}^*, \cdot) \) in \( (\mathbb{N}^*, \leq, +) \).

2. Smarandache functions of first kind. The Smarandache function \( S \) is defined by means of the following functions \( S_p \); for every prime number \( p \) let \( S_p: \mathbb{N}^* \longrightarrow \mathbb{N}^* \) having the property that \( (S_p(n))! \) is divisible by \( p^n \) and is the smallest positive integer with this property. Using the notion of standardising functions in this section we give some generalisation of \( S_p \).

2.1. Definition. For every \( n \in \mathbb{N}^* \) the relation \( \equiv_n \in \mathbb{N}^* \times \mathbb{N}^* \) is defined as follows: i) if \( n = u^i \cdot v^j \) or \( v = p \) number prime, \( i, v \in \mathbb{N}^* \) and \( a, b \in \mathbb{N}^* \) then \( a \equiv_n b \) if and only if it exists \( k \in \mathbb{N}^* \) such that \( k! = M^a u^i \), \( k! = M^b u^j \) and \( k \) is the smallest positive integer with this property.
\(I.\) \(i.\) if \(n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}, \) then
\[ n = \prod_{i=1}^{r} p_i^{a_i} \]

2.2. Definition. For each \(n \in \mathbb{N}^*\) the Smarandache function of first kind is the numerical function \(S_n : \mathbb{N}^* \to \mathbb{N}^*\) defined as follows

i) if \(n = u^t (u=1 \text{ or } u=p \text{ number prime})\) then \(S_n(a) = k, \) \(k \) being the smallest positive integer with the property that \(k! = M_u^a\)

ii) if \(n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}, \) then \(S_n(a) = \max\{S_{p_1}^i(a)\}_{1 \leq i \leq r}\)

Let us observe that:

a) the functions \(S_n\) are standardising functions corresponding to the equivalence relations \(r_n\) and for \(n=1\) we get \(x_1 = \mathbb{N}^*\) for every \(x \in \mathbb{N}^*\) and \(S_1(n) = 1\) for every \(n.\)

b) if \(n=p\) then \(S_n\) is the function \(S_p\) defined by Smarandache.

c) the functions \(S_n\) are increasing and so, are of the same monotonicity in the sense given in the above section.

2.3. Theorem. The functions \(S_n,\) for \(n \in \mathbb{N}^*, \) \(\Sigma_i\)-standardise \((\mathbb{N}^*, +)\) in \((\mathbb{N}^*, \leq, +)\) by \(\Sigma_i\): max\(\{S_n(a), S_n(b)\} \leq S_n(a+b) \leq S_n(a) + S_n(b)\) for every \(a, b \in \mathbb{N}^*\) and \(\Sigma_2\)-standardise \((\mathbb{N}^*, +)\) in \((\mathbb{N}^*, \leq, \cdot)\) by \(\Sigma_2: \) max\(\{S_n(a), S_n(b)\} \leq S_n(a+b) \leq S_n(a) \cdot S_n(b)\) for every \(a, b \in \mathbb{N}^*\)

Proof. Let, for instance, \(p\) be a prime number, \(n=p^i, i \in \mathbb{N}^*\) and \(a^* = S_p(a), b^* = S_p(b), k = S_p(a+b)\). Then by the definition of \(S_n\)

(Definition 2.2.) the numbers \(a^*, b^*, k\) are the smallest positive integers such that \(a^*! = M_p^a, b^*! = M_p^b, k! = M_p^i(a+b).\)

Because \(k! = M_p^i(a+b)\) we get \(a^* \leq k, b^* \leq k\), so max\(\{a^*, b^*\} \leq k\).

That is the first inequalities in \(\Sigma_i\) and \(\Sigma_2\) holds.

Now, \((a^* + b^*)! = a^*!(a^* + 1). \ldots (a^* + b^*) = M_p^i(a+b) = M_p^i(a^* + b^*)\) and
so \( k \leq a^* + b^* \) which implies that \( \Sigma_1 \) is valid.

If \( n = p_1 \cdot \cdots \cdot p_s^* \), from the first case we have

\[
\Sigma_1: \quad \max\{S_j(a), S_j(b)\} \leq S_j(a + b) \leq S_j(a) + S_j(b), j = 1, \ldots, s
\]

in consequence

\[
\max\{\max\{S_j(a), S_j(b)\}\} \leq \max\{S_j(a + b)\} \leq \max\{S_j(a)\} + \max\{S_j(b)\}
\]

\[
\max\{S_j(b)\}, j = 1, \ldots, s. \quad \text{That is}
\]

\[
\max\{S_j(a), S_j(b)\} \leq S_n(a + b) \leq S_n(a) + S_n(b)
\]

For the proof of the second part in \( \Sigma_2 \) let us notice that

\((a + b)! \leq (ab)! \iff a + b \leq ab \iff a > 1 \text{ and } b > 1 \) and that

our inequality is satisfied for \( n = 1 \) because \( S_1(a + b) = S_1(a) = S_1(b) = 1 \).

Let now \( n > 1 \). It results that for \( a = S_n(a) \) we have \( a^* > 1 \). Indeed, if \( n = p_1 \cdot p_2^* \cdot \cdots \cdot p_s^* \) then \( a^* = 1 \) if and only if \( S_n(a) = \max\{S_j(a)\} = 1 \) which implies that \( p_1 = p_2 = \ldots = p_s = 1 \),

so \( n = 1 \). It results that for every \( n > 1 \) we have \( S_n(a) = a^* > 1 \) and

\( S_n(b) = b^* > 1 \). Then \((a^* + b^*)! \leq (a^* \cdot b^*)! \) we obtain

\( S_n(a + b) \leq S_n(a) + S_n(b) \leq S_n(a) \cdot S_n(b) \) from \( n > 1 \).

3. Smarandache functions of the second kind. For every \( n \in N^* \), let

\( S_n \) by the Smarandache function of the first kind defined above.

3.1. Definition. The Smarandache functions of the second kind are

the functions \( S^k : N^* \rightarrow N^* \) defined by \( S^k(n) = S_n(k) \), for \( k \in N^* \).

We observe that for \( k = 1 \) the function \( S^1 \) is the Smarandache function \( S \) defined in [1], with the modify \( S(1) = 1 \). Indeed for.

\( n > 1 \)

\[
S^i(n) = S^i(1) = \max\{S_{p_j}(1)\} = \max\{S_{p_j}(i_j)\} = S(n).
\]
3.2. Theorem. The Smarandache functions of the second kind $\Sigma_3$-standardise $(\mathbb{N}^*,\cdot)$ in $(\mathbb{N}^*,\leq,+)$ by

$\Sigma_3: \quad \max(s^k(a),s^k(b)) \leq s^k(a.b) \leq s^k(a) + s^k(b)$, for every $a,b \in \mathbb{N}^*$

and $\Sigma_4$-standardise $(\mathbb{N}^*,\cdot)$ in $(\mathbb{N}^*,\leq,\cdot)$ by

$\Sigma_4: \quad \max(s^k(a),s^k(b)) \leq s^k(a.b) \leq s^k(a).s^k(b)$, for every $a,b \in \mathbb{N}^*$

Proof. The equivalence relation corresponding to $s^k$ is $r^k$, defined by $a r^k b$ if and only if there exists $a^* \in \mathbb{N}^*$ such that $a^*! = Ma^k$, $b^*! = Mb^k$ and $a^*$ is the smallest integer with this property.

That is, the functions $s^k$ are standardising functions attached to the equivalence relations $r^k$.

This functions are not of the same monotonicity because, for example, $s^2(a) \leq s^2(b) \iff s(a^2) \leq s(b^2)$ and from these inequalities $s^2(a) \leq s^2(b)$ does not result.

Now for every $a,b \in \mathbb{N}^*$ let $s^k(a) = a^*, s^k(b) = b^*, s^k(a.b) = s$.

Then $a^*, b^*, s$ are respectively these smallest positive integers such that $a^*! = Ma^k$, $b^*! = Mb^k$, $s! = M(a^k b^k)$ and so $s! = Ma^k = Mb^k$, that is, $a^* \leq s$ and $b^* \leq s$, which implies that $\max(a^*, b^*) \leq s$

or

\[ \max\{s^k(a),s^k(b)\} \leq s^k(a.b) \]  \hspace{1cm} (3.1)

Because of the fact that $(a^* + b^*)! = M(a^*! b^*!) = M(a^k b^k)$, it results that $s \leq a^* + b^*$, so

\[ s^k(a.b) \leq s^k(a) + s^k(b) \]  \hspace{1cm} (3.2)

From (3.1) and (3.2) it results that

\[ \max\{s^k(a),s^k(b)\} \leq s^k(a) + s^k(b) \]  \hspace{1cm} (3.3)

which is the relation $\Sigma_3$.

From $(a^*b^*)! = M(a^*! b^*!)$ it results that $s^k(a.b) \leq s^k(a).s^k(b)$ and thus the relation $\Sigma_4$. 

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4. The Smarandache functions of the third kind.

We consider two arbitrary sequences

(a) \( 1 = a_1, a_2, \ldots, a_n, \ldots \)

(b) \( 1 = b_1, b_2, \ldots, b_n, \ldots \)

with the properties that \( a_n = a_k \cdot a_n, b_n = b_k \cdot b_n \). Obviously, there are infinitely many such sequences; because choosing an arbitrary value for \( a_z \), the next terms in the sequence can be easily determined by the imposed condition.

Let now the function \( f_a : \mathbb{N}^* \rightarrow \mathbb{N}^* \) defined by \( f_a(n) = S_a(b_n) \), \( S_a \) is the Smarandache function of the first kind. Then it is easily to see that:

(i) for \( a_n = 1 \) and \( b_n = n, n \in \mathbb{N}^* \) it results that \( f_a^b = S_1 \)

(ii) for \( a_n = n \) and \( b_n = 1, n \in \mathbb{N}^* \) it results that \( f_a^b = S^i \)

4.1 Definition. The Smarandache functions of the third kind are the functions \( S_a^b = f_a^b \) in the case that the sequences (a) and (b) are different from those concerned in the situation (i) and (ii) from above.

4.2 Theorem. The functions \( f_a^b \in \Sigma_a \)-standardise \( (\mathbb{N}^*, \cdot) \) in

\( (\mathbb{N}^*, \leq, +, \cdot) \) by

\[ \Sigma_a : \quad \max\{f_a^b(k), f_a^b(n)\} \leq f_a^b(k, n) \leq b_n f_a^b(k) + b_k f_a^b(n) \]

Proof. Let \( f_a^b(k) = S_a(b_k) = k \), \( f_a^b(n) = S_a(b_n) = n \) and \( f_a^b(kn) = S_a(b_{kn}) = t \). Then \( k^*, n^* \) and \( t \) are the smallest positive integers such that \( k^* = M a_k, n^* = M a_n \) and \( t = M a_{kn} \)

\[ = M(a_k \cdot a_n)^{b_{kn}} \] 

Of course,

\[ \max(k^*, n^*) \leq t \]  

(4.1)
Now, because \((b^n \cdot k^*)! = \mathcal{M}(n^*)^b_k^*\), \((b^n \cdot k^*)! = \mathcal{M}(k^*)^b_n^*\) and
\((b^n \cdot b^n k^*)! = \mathcal{M}((b^n \cdot n^*)! \cdot (b^n \cdot k^*)!) = \mathcal{M}((n^*)^b_k^*, (k^*)^b_n^*)\)

\[
\begin{align*}
&= \mathcal{M}[(a^n_k)^* \cdot (a^n_k)^*] = \mathcal{M}[(a^n_k \cdot a^n_k)^*] \quad \text{it results that} \\
&t \leq b^n_k + b^n_k \quad (4.2)
\end{align*}
\]
From (4.1) and (4.2) we obtain
\[
\max(k^*, n^*) \leq t \leq b^n_k + b^n_k \quad (4.3)
\]

From (4.3) we get \(\Sigma_3\), so the Smarandache functions of the third kind satisfy
\[
\Sigma_3: \max(S^b_a(k), S^b_a(n)) \leq S^b_a(kn) \leq S^b_a(k) + S^b_a(n), \text{ for every } k, n \in \mathbb{N}^*
\]

4.3. Example. Let the sequences (a) and (b) defined by \(a_n = b_n = n\) \(\forall n \in \mathbb{N}^*\).

The corresponding Smarandache function of the third kind is
\[
S^a_3: \mathbb{N}^* \rightarrow \mathbb{N}^*, \quad S^a_3(n) = S^a_3(n) \quad \text{and} \quad \Sigma_3 \text{ becomes}
\]
\[
\max(S^a_3(k), S^a_3(n)) \leq S^a_3(kn) \leq S^a_3(k) + S^a_3(n), \text{ for every } k, n \in \mathbb{N}^*
\]

This relation is equivalent with the following relation written by means with the Smarandache function:
\[
\max(S(k^n), S(n^k)) \leq S((kn)^n) \leq n \cdot S(k^n) + k \cdot S(n^k).
\]

References


THE SOLUTION OF THE DIOPHANTINE EQUATION $\sigma_\eta(n) = n$ ($\Omega$)

by Pál Gronás

This problem is closely connected to Problem 29916 in the first issue of the "Smarandache Function Journal" (see page 47 in [1]). The question is: "Are there an infinity of nonprimes $n$ such that $\sigma_\eta(n) = n$?". My calculations will show that the answer is negative.

Let us move on to the first step in deriving the solution of ($\Omega$). As the wording of Problem 29916 indicates, ($\Omega$) is satisfied if $n$ is a prime. This is not the case for $n = 1$ because $\sigma_\eta(1) = 0$.

Suppose $\prod_{i=1}^{k} p_i^{r_i}$ is the prime factorization of a composite number $n \geq 4$, where $p_1, \ldots, p_k$ are distinct primes, $r_i \in \mathbb{N}$ and $p_i r_i \geq p_i r_i$ for all $i \in \{1, \ldots, k\}$ and $p_i < p_{i+1}$ for all $i \in \{2, \ldots, k-1\}$ whenever $k \geq 3$.

First of all we consider the case where $k = 1$ and $r_1 \geq 2$. Using the fact that $\eta(p_i^{r_i}) \geq p_i s_i$ we see that $p_i^{r_i} = n = \sigma_\eta(n) = \sigma_\eta(p_i^{r_i}) = \sum_{d=0}^{r_i} \eta(p_i^d) \leq \sum_{d=0}^{r_i} p_i s_i = p_i r_i (r_i+1) / 2$. Therefore $2 p_i^{r_i-1} \leq r_1 (r_1 + 1) (\Omega_1)$ for some $r_1 \geq 2$. For $p_i \geq 5$ this inequality ($\Omega_1$) is not satisfied for any $r_1 \geq 2$. So $p_1 < 5$, which means that $p_1 \in \{2, 3\}$. By the help of ($\Omega_1$) we can find a supremum for $r_1$ depending on the value of $p_1$. For $p_1 = 2$ the actual candidates for $r_1$ are 2, 3, 4 and for $p_1 = 3$ the only possible choice is $r_1 = 2$. Hence there are maximum 4 possible solution of ($\Omega$) in this case, namely $n = 4, 8, 9$ and 16. Calculating $\sigma_\eta(n)$ for each of these 4 values, we get $\sigma_\eta(4) = 6, \sigma_\eta(8) = 10, \sigma_\eta(9) = 9$ and $\sigma_\eta(16) = 16$. Consequently the only solutions of ($\Omega$) are $n = 9$ and $n = 16$.

Next we look at the case when $k \geq 2$:

$$n = \sigma_\eta(n)$$

Substituting $n$ with it's prime factorization we get

$$\prod_{i=1}^{k} p_i^{r_i} = \sigma_\eta(\prod_{i=1}^{k} p_i^{r_i}) = \sum_{d=0}^{r_1} \cdots \sum_{d=0}^{r_k} \eta(\prod_{i=1}^{k} p_i^{r_i})

= \sum_{s_1=0}^{r_1} \cdots \sum_{s_k=0}^{r_k} \max\{\eta(p_1^{s_1}), \ldots, \eta(p_k^{s_k})\}

\leq \sum_{s_1=0}^{r_1} \cdots \sum_{s_k=0}^{r_k} \max\{p_1 s_1, \ldots, p_k s_k\} \text{ since } \eta(p_i^{s_i}) \leq p_i s_i

< \sum_{s_1=0}^{r_1} \cdots \sum_{s_k=0}^{r_k} \max\{p_1 r_1, \ldots, p_k r_k\} \text{ because } s_i \leq r_i

= \sum_{s_1=0}^{r_1} \cdots \sum_{s_k=0}^{r_k} p_1 r_1 \ldots (p_1 r_1 \geq p_i r_i \text{ for } i \geq 2)

\leq p_1 r_1 \prod_{i=1}^{k} (r_i + 1).$$
which is equivalent to
\[
\prod_{i=2}^{k} \frac{p_i r_i}{r_i + 1} < \frac{p_1 r_1 (r_1 + 1)}{p_1^2} = \frac{r_1 (r_1 + 1)}{p_1 - 1} \quad (\Omega_2)
\]

This inequality motivates a closer study of the functions \( f(x) = \frac{x^2}{x+1} \) and \( g(x) = \frac{2(x+1)}{x^2} \) for \( x \in [1, \infty) \), where \( a \) and \( b \) are real constants \( \geq 2 \). The derivatives of these two functions are
\[
f'(x) = \frac{x^2}{(x+1)^2} \quad \text{and} \quad g'(x) = \frac{(2-\ln b)x^2 + (2-\ln b)x - 1}{x^4}.
\]
Hence \( f'(x) > 0 \) for \( x \geq 1 \) since \((x+1)\ln b - 1 \geq (1+1)\ln 2 - 1 = 2\ln 2 - 1 > 0\). So \( f \) is increasing on \([1, \infty)\). Moreover \( g(x) \) reaches its absolute maximum value for \( x = \max\{1, \frac{2-\ln b + \sqrt{(\ln b)^2 - 4}}{2\ln b}\} \). Now \( (\ln b)^2 + 1 < \ln b + 2 \) for \( b \geq 2 \), which implies that \( x < \frac{(2-\ln b) + \sqrt{(\ln b)^2 - 4}}{2\ln b} = \frac{2}{\ln b} \leq \frac{2}{2} < 3 \).

Futhermore it is worth mentioning that \( f(x) \to \infty \) and \( g(x) \to 0 \) as \( x \to \infty \).

Applying this to our situation means that \( \frac{p_i}{r_i+1} (i \geq 2) \) is strictly increasing from \( \frac{p_2}{2} \) to \( \infty \). Besides \( \frac{r_1 (r_1 + 1)}{p_1} \leq \max\{2, \frac{4}{p_1}, \frac{12}{p_1^2}\} = \max\{\frac{4}{p_1}, \frac{12}{p_1^2}\} \leq 3 \) because \( \frac{4}{p_1} \geq \frac{12}{p_1^2} \) whenever \( p_1 \geq 2 \).

Combining this knowledge with (\( \Omega_2 \)) we get that \( \prod_{i=2}^{k} \frac{p_i}{r_i+1} \leq \prod_{i=2}^{k} \frac{p_i}{r_i+1} \leq \frac{r_1 (r_1 + 1)}{p_1} \leq \frac{r_1 (r_1 + 1)}{2} \leq 3 \) (\( \Omega_3 \)) for all \( r_i \in \mathbb{N} \). In other words, \( \prod_{i=2}^{k} \frac{p_i}{r_i+1} \leq 3 \). Now \( \prod_{i=2}^{k} \frac{p_i}{r_i+1} \leq \frac{3}{2} \cdot \frac{3}{2} \cdot \frac{3}{2} = 18 \leq 3 \), which implies that \( k \leq 3 \).

Let us assume \( k = 2 \). Then (\( \Omega_2 \)) and (\( \Omega_3 \)) state that \( \frac{p_2}{r_2+1} < \frac{r_1 (r_1 + 1)}{2} \) and \( \frac{p_2}{r_2+1} \leq 3 \), i.e. \( p_2 < 6 \). Next we suppose \( r_2 \geq 3 \). It is obvious that \( p_1 p_2 \geq 2 \cdot 3 = 6 \), which is equivalent to \( p_2 \geq \frac{6}{p_1} \). Using this fact we get \( \frac{3}{2} \leq \frac{p_2}{r_2+1} < \frac{r_1 (r_1 + 1)}{2} \leq \max\{\frac{6}{p_1}, \frac{12}{p_1^2}\} \leq \max\{2, p_2\} = p_2 \), so \( p_2 < 4 \). Accordingly \( p_2 < 2 \), a contradiction which implies that \( r_2 \leq 2 \). Hence \( p_2 \in \{2, 3, 5\} \) and \( r_2 \in \{1, 2\} \).

Futhermore \( 1 \leq \frac{p_1}{r_1+1} = \frac{r_1 (r_1 + 1)}{2 r_1 + 1} \), which implies that \( r_1 \leq 6 \). Consequently by fixing the values of \( p_2 \) and \( r_2 \), the inequalities \( \frac{r_1 (r_1 + 1)}{2 r_1 + 1} > \frac{p_2}{r_2+1} \) and \( p_1 r_1 \geq p_2 r_2 \) give us enough information to determine a supremum (less than 7) for \( r_1 \) for each value of \( p_1 \).

This is just what we have done, and the result is as follows:

<table>
<thead>
<tr>
<th>( p_2 )</th>
<th>( r_2 )</th>
<th>( p_1 )</th>
<th>( r_1 )</th>
<th>( n = p_1^2 p_2^2 )</th>
<th>( \sigma_n(n) )</th>
<th>IF ( \sigma_n(n) = n ) THEN</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>1 \leq r_1 \leq 3</td>
<td>2 \cdot 3^{r_1}</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>5</td>
<td>1</td>
<td>5 \leq r_1 \leq 12</td>
<td>2 \cdot 5^{r_1}</td>
<td>2 \cdot 5^{r_1}</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>( p_1 \geq 7 )</td>
<td>1</td>
<td>2p_1</td>
<td>2p_1</td>
<td>2p_1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>36</td>
<td>36</td>
<td>36</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>( p_1 \geq 5 )</td>
<td>1</td>
<td>4p_1</td>
<td>4p_1</td>
<td>4p_1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2 \leq r_1 \leq 5</td>
<td>3 \cdot 2^{r_1}</td>
<td>2 \cdot 2^{r_1} - 2r_1 + 5</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>( p_1 \geq 5 )</td>
<td>1</td>
<td>3p_1</td>
<td>2p_1 + 3</td>
<td>2p_1 + 3</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>40</td>
<td>30</td>
<td>40</td>
</tr>
</tbody>
</table>

By looking at the rightmost column in the table above, we see that there are only contradictions except in the case where \( n = 3 \cdot 2^3 \) and \( r_1 = 3 \). So \( n = 3 \cdot 2^3 = 24 \) and \( \sigma_n(24) = 24 \).

In other words, \( n = 24 \) is the only solution of (\( \Omega \)) when \( k = 2 \).
Finally, suppose \( k = 3 \). Then we know that \( \frac{p_2}{2} \cdot \frac{p_3}{2} < 3 \), i.e. \( p_2 p_3 < 12 \). Hence \( p_2 = 2 \) and \( p_3 \geq 3 \). Therefore \( \frac{r_i(t_i+1)}{p_i(t_i+1)} \leq \frac{r_i(t_i+1)}{p_i(t_i+1)} \leq 2 \) (\( \Omega_4 \)) and by applying (\( \Omega_3 \)) we find that \( \prod_{i=1}^{3} \frac{p_i}{2} = \frac{p_3}{2} < 2 \), giving \( p_3 = 3 \).

Combining the two inequalities (\( \Omega_3 \)) and (\( \Omega_4 \)) we get that \( \frac{p_2}{r_2+1} \cdot \frac{p_3}{r_3+1} < 2 \). Knowing that the left side of this inequality is a product of two strictly increasing functions on \( [1, \infty) \), we see that the only possible choices for \( r_2 \) and \( r_3 \) are \( r_2 = r_3 = 1 \). Inserting these values in (\( \Omega_3 \)), we get \( \frac{2}{i+1} \cdot \frac{3}{i+1} = \frac{3}{i+1} < \frac{r_i(t_i+1)}{p_i(t_i+1)} \). This implies that \( r_1 = 1 \). Accordingly (\( \Omega \)) is satisfied only if \( n = 2 \cdot 3 \cdot p_1 = 6 p_1 \):

\[
6 p_1 = \sigma_6(6 p_1)
\]

\[
= \eta(1) + \eta(2) + \eta(3) + \eta(6) + \sum_{i=0}^{1} \sum_{j=0}^{1} \eta(2^i 3^j p_1)
\]

\[
= 0 + 2 + 3 + 3 + \sum_{i=0}^{1} \sum_{j=0}^{1} \max\{ \eta(p_1), \eta(2^i 3^j) \}
\]

\[
= 8 + \sum_{i=0}^{1} \sum_{j=0}^{1} \max\{ \eta(p_1), \eta(2^i 3^j) \}
\]

\[
= 8 + 4 p_1 \text{ because } \eta(2^i 3^j) \leq 3 < p_1 \text{ for all } i, j \in \{0, 1\}
\]

\[
p_1 = 4
\]

which contradicts the fact that \( p_1 \geq 5 \). Therefore (\( \Omega \)) has no solution for \( k = 3 \).

**Conclusion:** \( \sigma_n(n) = n \) if and only if \( n \) is a prime, \( n = 9, n = 16 \) or \( n = 24 \).

**REMARK:** A consequence of this work is the solution of the inequality \( \sigma_n(n) > n \) (\( * \)). This solution is based on the fact that (\( * \)) implies (\( \Omega_2 \)).

So \( \sigma_n(n) > n \) if and only if \( n = 8, 12, 18, 20 \) or \( n = 2p \) where \( p \) is a prime. Hence \( \sigma_n(n) \leq n + 4 \) for all \( n \in \mathbb{N} \).

Moreover, since we have solved the inequality \( \sigma_n(n) \geq n \), we also have the solution of \( \sigma_n(n) < n \).

**References**


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ON THE SUMATORY FUNCTION ASSOCIATED TO THE SMARANDACHE FUNCTION

E. Radescu, N. Radescu, C. Dumitrescu

It is said that for every numerical function \( f \) it can be attached the sumatory function:

\[
F(n) = \sum_{d | n} f(d)
\]

(1)

The function \( f \) is expressed as:

\[
f(n) = \sum_{u,v} \mu(u) \cdot F_r(v)
\]

(2)

Where \( \mu \) is the Möbius function \( (\mu(1)=1, \mu(n)=0 \text{ if } n \text{ is divisible by the square of a prime number, } \mu(n)=(-1)^k \text{ if } n \text{ is the product of } k \text{ different prime numbers}) \)

If \( f \) is the Smarandache function and \( n = p^\alpha \) then:

\[
F_s(p^\alpha) = \sum_{j=1}^{\alpha} S(p^j)
\]

In [2] it is proved that

\[
S(p^j) = p - 1 \cdot j - a_{(p^j)}(j)
\]

(3)

where \( a_{(p^j)}(j) \) is the sum of the digits of the integer \( j \), written in the generalised scale:

\[
[p] = a_1(p), a_2(p), \ldots, a_\chi(p), \ldots
\]

with \( a_\chi(p) = (p^n - 1)/(p - 1) \)

So

\[
F_s(p^\alpha) = \sum_{j=1}^{\alpha} S(p^j) = (p - 1) \frac{a(a + 1)}{2} + \sum_{j=1}^{\alpha} a_{(p^j)}(j)
\]

(4)

Using the expression of \( a \) given by (3) it results

\[
(a + 1)(S(p^\alpha) - a_{(p^\alpha)}(\alpha)) = 2(F_s(p^\alpha) - \sum_{j=1}^{\alpha} a_{(p^j)}(j))
\]
In the following we give an algorithm to calculate the sum in the right hand of (4). For this, let \( a_{(p)} = k_s k_{s-1} \ldots k_1 \) the expression of \( a \) in the scale \([p]\) and \( j_{(p)} = k_s k_{s-1} \ldots k_1 \). We shall say that \( k_i \) are the digits of order \( i \), for \( j = 1, 2, \ldots, \alpha \).

To calculate the sum of all the digits of order \( i \), let \( \nu_i = \alpha - a_i(p) + 1 \). Now we consider two cases:

(i) if \( k_i \neq 0 \), let :

\[
\nu_i(a) = \left( k_s k_{s-1} \ldots k_{i-1} \right)_{z=a(p)},
\]

the equality \( u = a_i(p) \) denoting that for the number written between parantheses, the clase of units is \( a_i(p) \).

Then \( \nu_i(a) \) is the number of all zeros of order \( i \) for the integers \( j \leq \alpha \) and \( \alpha_i = \nu_i(a) - \nu_i(a) \) is the number of the non-null digits.

(ii) if \( k_i = 0 \), let \( \beta \) the greatest number, less then \( \alpha \), having a non-null digit of order \( i \). Then \( \beta \) is of the form :

\[
\beta_{(p)} = k_s k_{s-1} \ldots k_{i-2} (k_{i-1} - 1)p00\ldots 0
\]

and of course \( s_i(a) = s_i(\beta) \). It results that there exist \( a_i(\beta) \) non-null digits of order \( i \).

Let \( A_i, B_i, \Gamma_i, \varphi_i \) given by equalities :

\[
\begin{align*}
A_i &= A_i((p - 1)a_i(p) + 1) + \Gamma_i = A_i(a_{i+t}(p) - a_i(p)) + \Gamma_i \\
\Gamma_i &= B_i a_i(p) + \varphi_i 
\end{align*}
\]

Then

\[
s_i(a) = A_i a_i(p) \frac{p(\beta - 1)}{z} - A_i p + a_i(p) \frac{p(\beta - 1)}{z} - \varphi_i (B_i + 1)
\]

and

\[
\sum_{j=1}^{\alpha} \nu_{i,j} = \sum_{i=1}^{\alpha} s_i(a) = \frac{p(p - 1)}{z} \sum_{i=1}^{\alpha} A_i a_i(p) + p \sum_{i=1}^{\alpha} A_i + \frac{1}{z} \sum_{i=1}^{\alpha} a_i(p) B_i (B_i + 1) + \sum_{i=1}^{\alpha} a_i(B_i + 1)
\]

For example if \( \alpha = 149 \) and \( p = 3 \) it results :

\[
[3] \quad 1, 4, 13, 40, 121, \ldots
\]
$\alpha = 10202, \nu (\alpha) = (1020)_{\mu=\alpha (3)} = 48, \alpha = \nu (\alpha) - z (\alpha) = 101$

For $\beta = 10130 = 146$ it results $\nu (\beta) = 143, z (\beta) =$

$(101)_{\mu=\alpha (3)} = u + u = 3u + 1 + u = 3(3u + 1) + 1 + u = 44$, 
$\alpha = 99, \nu (\alpha) = 137, z (\alpha) = (10)_{\mu=\alpha (3)} = 40, \alpha = 97$.

For $\beta = 3000 = 120$ it results $\nu (\beta) = 81, z (\beta) = 0, \alpha = 108$.
$\nu (\alpha) = 29, z (\alpha) = 0, \alpha = 29$, and

$A_1 = \left[ \frac{\alpha}{\alpha + 1} \right] = 33, F_1 = 2, B_1 = \left[ \frac{z}{\alpha} \right], \rho = 0, s = 201$.

Analogously $s_2 = 165, s_3 = 145, s_4 = 123$ and $s_5 = 129$, so

$\sum_{i=1}^{149} \alpha (i) = 633, F_5 (3^{149}) = 22983$.

Now let us consider $n = p_1 \cdot p_2 \cdot \ldots \cdot p_k$, with

$p_1 < p_2 < \ldots < p_k$ prime numbers. Of course, $S(n) = p_k$ and from

$F_1 (1) = S(1) = 0$

$F_1 (p_1) = S(1 + S(p_1)) = p_1$

$F_1 (p_1 \cdot p_2) = p_1 + 2p_2 = F(p_1) + 2p_2$

$F_3 (p_1 \cdot p_2 \cdot p_3) = p_1 + 2p_2 + 2^2 p_3 = F(p_1 \cdot p_2) + 2^2 p_3$

it results:

$F_k (p_1 \cdot p_2 \cdot \ldots \cdot p_k) = F(p_1 \cdot p_2 \cdot \ldots \cdot p_{k-1}) + 2^{k-1} p_k$

That is:

$F(p_1 \cdot p_2 \cdot \ldots \cdot p_k) = \sum_{i=1}^{k} 2^{k-i} p_i$

The equality (2) becomes:

$p_k = S(n) = \sum_{n=1}^{n} \mu(n) F_S (n) = F(n) - \sum_{i} F (\frac{n}{p_i}) - \sum_{p_i, p_j} F (\frac{n}{p_i \cdot p_j}) + \ldots + \sum_{i=1}^{k} F(p_i)$

and become $F(p_i) = p_i$, it results:
\[ F\left( \frac{n}{p_i} \right) = F(p_1 \cdot p_2 \cdots p_{i-1} \cdot p_i \cdots p_k) = \sum_{j=1}^{i-1} 2^{-j} p_j + \sum_{j=i+1}^{k} 2^{-j} p_j \]

\[ = F(p_1 \cdot p_2 \cdots p_{i-1}) + 2^{i-1} F(p_{i+1} \cdot p_{i+2} \cdots p_k). \]

Analogously,

\[ F\left( \frac{n}{p_i} \right) = F(p_1 \cdot p_2 \cdots p_{i-1}) + 2^{i-1} F(p_{i+1} \cdot p_{i+2} \cdots p_k) + \]

\[ + 2^{i-1} F(p_{j+1} \cdots p_k) \]

Finally, we point out as an open problem that, by the Shapiro's theorem, if it exist a numerical function \( g : \mathbb{N} \rightarrow \mathbb{R} \) such that

\[ g(n) = \sum_{d|n} P(n) S\left( \frac{n}{d} \right) \]

were \( P \) is a totally multiplicative function and \( P(1) = 1 \), then

\[ S(n) = \sum_{d|n} \mu(d) P(d) g\left( \frac{n}{d} \right) \]

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Str. A.I. Cuza, Nr. 13, Craiova (1100), Romania.
A proof of the non-existence of "Samma".

by Pål Grønås

Introduction: If $\prod_{i=1}^{k} p_i^{r_i}$ is the prime factorization of the natural number $n \geq 2$, then it is easy to verify that

$S(n) = S(\prod_{i=1}^{k} p_i^{r_i}) = \max\{S(p_i^{r_i})}\}_{i=1}^{k}$.

From this formula we see that it is essential to determine $S(p^r)$, where $p$ is a prime and $r$ is a natural number.

Legendre's formula states that

\begin{equation}
 n! = \prod_{i=1}^{k} p_i \sum_{m=1}^{\infty} \left[ \frac{n}{p_i^m} \right].
\end{equation}

The definition of the Smarandache function tells us that $S(p^r)$ is the least natural number such that $p^r \mid (S(p^r))!$. Combining this definition with (1), it is obvious that $S(p^r)$ must satisfy the following two inequalities:

\begin{equation}
\sum_{k=1}^{\infty} \left\lfloor \frac{S(p^r)-1}{p^k} \right\rfloor < r \leq \sum_{k=1}^{\infty} \left\lfloor \frac{S(p^r)}{p^k} \right\rfloor.
\end{equation}

This formula (2) gives us a lower and an upper bound for $S(p^r)$, namely

\begin{equation}
(p-1)r + 1 \leq S(p^r) \leq pr.
\end{equation}

It also implies that $p$ divides $S(p^r)$, which means that

$S(p^r) = p(r - i)$ for a particular $0 \leq i \leq \left[ \frac{r-1}{p} \right]$.

"Samma": Let $T(n) = 1 - \log(S(n)) + \sum_{i=2}^{\infty} \frac{1}{S(i)}$ for $n \geq 2$. I intend to prove that $\lim_{n \to \infty} T(n) = \infty$. i.e. "Samma" does not exists.

First of all we define the sequence $p_1 = 2$, $p_2 = 3$, $p_3 = 5$ and $p_n = the \ n$th prime.
Next we consider the natural number $p_n$. Now (3) gives us that

$$S(p_i^t) \leq p_i \quad \forall i \in \{1, \ldots, m\} \text{ and } \forall k \in \{1, \ldots, n\}$$

$$\downarrow$$

$$\frac{1}{S(p_i^t)} \geq \frac{1}{p_i k}$$

$$\downarrow$$

$$\sum_{i=1}^{m} \sum_{k=1}^{n} \frac{1}{S(p_i^t)} \geq \sum_{i=1}^{m} \sum_{k=1}^{n} \frac{1}{p_i k} = \left( \sum_{i=1}^{m} \frac{1}{p_i} \right) \cdot \left( \sum_{k=1}^{n} \frac{1}{k} \right)$$

$$\downarrow$$

$$\sum_{k=2}^{n} \frac{1}{S(k)} \geq \left( \sum_{k=1}^{m} \frac{1}{p_k} \right) \cdot \left( \sum_{k=1}^{n} \frac{1}{k} \right)$$

(4)

since $S(k) > 0$ for all $k \geq 2$, $p_i^t \leq p_n^m$ whenever $a \leq m$ and $b \leq n$ and $p_i^t = p_n^t$ if and only if $a = c$ and $b = d$.

Furthermore $S(p_n^m) \leq p_n$, which implies that $-\log S(p_n^m) \geq -\log(p_n n)$ because $\log x$ is a strictly increasing function in the interval $[2, \infty)$. By adding this last inequality and (4), we get

$$T(p_n^m) = 1 - \log(S(p_n^m)) + \sum_{i=2}^{m} \frac{1}{S(i)} \geq 1 - \log(p_n n) + \left( \sum_{k=1}^{m} \frac{1}{p_k} \right) \cdot \left( \sum_{k=1}^{n} \frac{1}{k} \right)$$

$$\downarrow$$

$$T(p_n^m) \geq 1 - \log(p_n^m) + \left( \sum_{k=1}^{m} \frac{1}{p_k} \right) \cdot \left( \sum_{k=1}^{n} \frac{1}{k} \right) \quad (n = p_m)$$

$$\downarrow$$

$$T(p_n^m) \geq 1 + 2 \left( -\log p_m + \sum_{k=1}^{m} \frac{1}{k} \right) + \left( -2 + \sum_{k=1}^{m} \frac{1}{p_k} \right) \cdot \left( \sum_{k=1}^{m} \frac{1}{k} \right)$$

$$\downarrow$$

$$\lim_{m \to \infty} T(p_n^m) \geq 1 + 2 \cdot \lim_{m \to \infty} \left( -\log p_m + \sum_{k=1}^{m} \frac{1}{k} \right) + \lim_{m \to \infty} \left[ \left( -2 + \sum_{k=1}^{m} \frac{1}{p_k} \right) \cdot \left( \sum_{k=1}^{m} \frac{1}{k} \right) \right]$$

$$= 1 + 2 \gamma + \lim_{m \to \infty} \left( -2 + \sum_{k=1}^{m} \frac{1}{p_k} \right) \cdot \lim_{m \to \infty} \left( \sum_{k=1}^{m} \frac{1}{k} \right) \quad (\gamma = \text{Euler's constant})$$

$$= \infty$$

since both $\sum_{k=1}^{m} \frac{1}{k}$ and $\sum_{k=1}^{m} \frac{1}{p_k}$ diverges as $t \to \infty$. In other words, $\lim_{n \to \infty} T(n) = \infty$. $\square$
Introduction

The Smarandache function is an integer function, S, of an integer variable, n. S is the smallest integer such that S! is divisible by n. If the prime factorisation of n is known

\[ n = \prod m_i \cdot p_i \]

where the \( p_i \) are primes then it has been shown that

\[ S(n) = \text{Max} \left( \left( \frac{\log n}{\log p_i} \right) \right) \]

so a method of calculating S for prime powers will be useful in calculating \( S(n) \).

The inverse function

It is easier to start with the inverse problem. For a given prime, p, and a given value of S, a multiple of p, what is the maximum power, m, of p which is a divisor of S! ? If we consider the case p=2 then all even numbers in the factorial contribute a factor of 2, all multiples of 4 contribute another, all multiples of 8 yet another and so on.

\[ m = S \text{ DIV2} + (S \text{ DIV2}) \text{ DIV2} + ((S \text{ DIV2}) \text{ DIV2}) \text{ DIV2} + \ldots \]

In the general case

\[ m = S \text{ DIV}p + (S \text{ DIV}p) \text{ DIV}p + ((S \text{ DIV}p) \text{ DIV}p) \text{ DIV}p + \ldots \]

The series terminates by reaching a term equal to zero. The Pascal program at the end of this paper contains a function invSpp to calculate this function.
Using the inverse function

If we now look at the values of $S$ for successive powers of a prime, say $p=3$,

\[
\begin{array}{cccccccccccc}
  m & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \ldots \\
  \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
  S(3^m) & 3 & 6 & 9 & 9 & 12 & 15 & 18 & 18 & 21 & 24 & \ldots \\
\end{array}
\]

where the asterisked values of $m$ are those found by the inverse function, we can see that these latter determine the points after which $S$ increases by $p$. In the Pascal program the procedure tabsmarpp fills an array with the values of $S$ for successive powers of a prime.

The Pascal program

The program tests the procedure by accepting a prime input from the keyboard, calculating $S$ for the first 1000 powers, reporting the time for this calculation and entering an endless loop of accepting a power value and reporting the corresponding $S$ value as stored in the array.

The program was developed and tested with Acornsoft ISO-Pascal on a BBC Master. The function 'time' is an extension to standard Pascal which delivers the timelapse since last reset in centi-seconds. On a computer with a 65C12 processor running at 2 MHz the 1000 $S$ values are calculated in about 11 seconds, the exact time is slightly larger for small values of the prime.

program TestabSpp(input,output);
var t,p,x: integer;
Smarpp:array[1..1000] of integer;

function invSpp(prime,smar:integer):integer;
var m,x:integer;
begin
  m:=0;
  x:=smar;
  repeat
    x:=x div prime;
    m:=m+x;
  until x<prime;
  invSpp:=m;
end; {invSpp}
procedure tabsmarpp(prime,tabsize:integer);
var i,s,is:integer;
exit:boolean;
begin
exit:=false;
i:=1;
is:=1;
s:=prime;
repeat
repeat
Smarpp[1]:=s;
i:=i+1;
if i>tabsize then exit:=true;
until (i>is) or exit;
s:=s+prime;
is:=invSpp(prime,s);
until exit;
end; {tabsmarpp}

begin
read(p);
t:=time;
tabsmarpp(p,1000);
writeln((time-t)/100);
repeat
read(x);
writeln('Smarandache for ',p,' to power ',x,' is ',Smarpp[x]);
until false;
end. {testtabspp}
CALCULATING THE SMARANDACHE FUNCTION WITHOUT FACTORISING

J. R. SUTTON
(16A Overland Road, Mumbles, SWANSEA SA3 4LP, UK)

Introduction

The usual way of calculating the Smarandache function $S(n)$ is to factorise $n$, calculate $S$ for each of the prime powers in the factorisation and use the equation

$$S(n) = \max \left\{ S\left( m^{p} \right) \right\}$$

This paper presents an alternative algorithm for use when $S$ is to be calculated for all integers up to $n$. The integers are synthesised by combining all the prime powers in the range up to $n$.

The Algorithm

The Pascal program at the end of this paper contains a procedure tabsmarand which fills a globally declared array, Smaran, with the values of $S$ for the integers from 2 to the limit specified by a parameter. The calculation is carried out in four stages.

Powers of 2

The first stage calculates $S$ for those powers of 2 that fall within the limit and stores them in the array Smaran at the subscript which corresponds to the value of that power of 2. At the end of this stage the array contains $S$ for:

$$2, 4, 8, 16, 32, \ldots$$

interspersed with zeros for all the other entries.

General case

The next stage uses successive primes from 3 upwards. For each prime the $S$ values of the relevant powers of the prime, and also the values of the prime powers are calculated, and stored in the arrays Smarpp and Prpwr, by the procedure tabsmarpp. This procedure is essentially the same as that in a previous paper except that:

a) the calculation stops when the last prime power exceeds the limit

and b) the prime powers are also calculated and stored.
Then for each non-zero entry in Smarand that entry is multiplied by successive powers of the prime and the $S$ values calculated and stored in Smarand. Both of these loops terminate on reaching the limit value. Finally the $S$ values for the prime powers are copied into Smarand.

After the prime 3 the array contains:

\[ 2, 3, 4, 0, 3, 0, 4, 6, 0, 0, 4 \ldots \]

This process is followed for each prime up to the square root of the limit. This general case could be continued up to the limit but it is more efficient to stop at the square root and treat the larger primes as separate cases.

Largest primes

The largest primes, those greater than half the limit, contribute only themselves, $S(\text{prime}) = \text{prime}$, to the array of Smarandache values.

Multiples of prime only

The intermediate case between the last two is for primes larger than the square root but smaller than half the limit. In this case no powers of the prime are needed, only multiples of those entries already in Smarand by the prime itself. The prime is then copied into the array.

The Pascal program

The main program calls tabsmarand to calculate $S$ values then enters a loop in which two integers are input from the keyboard which specify a range of values for which the contents of the array are displayed for checking.

The program was developed and tested with Acornsoft ISO-Pascal on a BBC Master computer. The function ‘time’ delivers the time lapse (in centiseconds) since last reset. On a computer with a 65C12 processor running at 2MHz the following timings were obtained:

<table>
<thead>
<tr>
<th>limit</th>
<th>seconds</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>6.56</td>
</tr>
<tr>
<td>2000</td>
<td>12.87</td>
</tr>
<tr>
<td>3000</td>
<td>19.19</td>
</tr>
<tr>
<td>4000</td>
<td>25.64</td>
</tr>
<tr>
<td>5000</td>
<td>31.80</td>
</tr>
</tbody>
</table>

In this range the times appear almost linear. It would be useful to have this confirmed or disproved on a larger, faster computer.
program Testsmarand(input,output);
const limit=5000;
var count,sti,fin:integer;
Smaran:array[1..5001] of integer;

procedure tabsmarand(limit:integer);
var count,t,i,s,ls,pp,prime,pwcount,mcount,multiple: integer;
extit: boolean;
Prpwr:array[1..12] of integer;
Smarpp:array[1..12] of integer;

function max(x,y: integer):integer;
begin
  if x>y then max:=x else max:=y;
end; {max}

function invSpp(prime,smar:integer):integer;
var n,x:integer;
begin
  n:=0;
x:=smar;
repeat
  x:=x div prime;
n:=n+x;
until x<prime;
invSpp:=n;
end; {invSpp}

procedure tabsmarpp(prime,limit:integer);
var i,s,ls,pp:integer;
extit:boolean;
begin
  exit:=false;
  pp:=1;
i:=1;
is:=1;
s:=prime;
repeat
  repeat
    Smarpp[i]:=s;
    pp:=pp*prime;
    Prpwr[i]:=pp;
i:=i+1;
  until (i>is) or exit;
  s:=s+prime;
is:=invSpp(prime,s);
until exit;
end; {tabsmarpp}
begin writeln('Calculate Smarandache function for all integers up to \nlimit');
for count:=1 to limit do Smaran[count]:=0;
Smaran[limit+1]:=limit+1;
t:=time;

{powers of 2}
s:=2;
1:=1;
is:=1;
pp:=1;
exit:=false;
repeat
repeat
pp:=pp*2;
Smaran[pp]:=s;
1:=1+1;
if 2*pp>limit then exit:=true;
until (is=1s) or exit;
s:=s+2;
is:=invSpp(2,s);
until exit;

{general case}
prime:=3;
repeat
tabsmarpp(prime,limit);
mcount:=1;
repeat
pwcount:=1;
multiple:=mcount*prime;
repeat
if multiple<=limit then
    if Smaran[multiple]=0 then
        Smaran[multiple]:=max(Smaran[mcount],Smarpp[pwcount]);
pwcount:=pwcount+1;
multiple:=mcount*Prpwr[pwcount];
until multiple>limit;
repeat
mcount:=mcount+1;
until Smaran[mcount]<0;
until mcount*prime>limit;
pwcount:=1;
repeat
Smaran[Prpwr[pwcount]]:=Smarpp[pwcount];
pwcount:=pwcount+1;
until Prpwr[pwcount]>limit;
repeat
prime:=prime+1;
until Smaran[prime]=0;
until prime*prime>limit;
{multiple case}
repeat mcount:=1;
multiple:=prime;
repeat
if multiple<=limit then
  if Smaran[multiple]=0 then
    Smaran[multiple]:=max(Smaran[mcount],prime);
repeat
mcount:=mcount+1;
until Smaran[mcount]<0;
multiple:=mcount*prime;
until multiple>limit;
Smaran[prime]:=prime;
repeat
prime:=prime+1;
until Smaran[prime]=0;
until prime>limit/2;
{largest primes}
count:=1;
repeat
if Smaran[count]=0 then Smaran[count]:=count;
count:=count+1;
until count>limit;
writeln((time/t)/100,' seconds');
end; {tabsmarand}

begin
  tabsmarand(limit);
repeat
  writeln('Enter start and finish integers for display of results');
  read(st,fin);
if (st>1) and (st<=limit) and (fin<=limit) then
  for count := st to fin do writeln(count,Smaran[count]);
until fin=1;
end. {Testsamarand}
A BRIEF HISTORY OF THE "SMARANDACHE FUNCTION" (II)

by Dr. Constantin Dumitrescu

We apologize, but the following conjecture that:
the equation \( S(x) = S(x+1) \), where \( S \) is the Smarandache Function, has no solutions,
was not completely solved.
Any idea about it is welcome.

See the previous issue of the journal for the first part of this article.

ADDENDA:

New References concerning this function (got by the editorial board after January 1, 1994):

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An Illustration of the Distribution of the Smarandache Function

by Henry Ibstedt

The cover illustration is a representation of the values of the Smarandache function for $n \leq 53$. The group at the back of the diagram essentially corresponds to $S(p) = p$, the middle group to $S(2p) = p$ ($p \neq 2$) while the front group represents all the other values of $S(n)$ for $n \leq 53$.

Diagram 1. Distribution of $S(n)$ up to $n = 32000$ (not to scale)

It may be interesting to take this graphical presentation a bit further. All the values of $S(n)$ for $n \leq 32000$ (conveniently chosen in order to use short integers only) have been sorted as shown in table 1. Of the 19114 points $(n, S(n))$ situated above the line $y = x/50$ only 61 points fall between lines. All of these of course correspond to cases where $n$ is not square free. Diagram 1 illustrates this for the lines $y = x$, $y = x/2$, $y = x/3$, $y = x/4$, $y = x/5$ and $y = x/6$. The top line contains 3433 points $(n, S(n))$ although there are only 3432 primes less than 32000. This is because $(4, S(4))$ belongs to this line.
TABLE 1. On the distribution of the Smarandache Function $S(n)$ for $n \leq 32000$.

$N =$ number of values of $S(n)$ on the line $y=x/k$, i.e. $S(n)=n/k$. The points $(n,S(n))$ are the only ones between lines $y=x/k$ and $y=x/(k+1)$ for $k<50$.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$N$</th>
<th>Points $(n,S(n))$ between lines:</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3433</td>
<td>( 9, 6)</td>
</tr>
<tr>
<td>2</td>
<td>1862</td>
<td>(16, 6) (25, 10)</td>
</tr>
<tr>
<td>3</td>
<td>1302</td>
<td>(49, 14)</td>
</tr>
<tr>
<td>4</td>
<td>1006</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>832</td>
<td>(121, 22)</td>
</tr>
<tr>
<td>6</td>
<td>707</td>
<td>(169, 26)</td>
</tr>
<tr>
<td>7</td>
<td>616</td>
<td>(45, 6) (75, 10)</td>
</tr>
<tr>
<td>8</td>
<td>550</td>
<td>(125, 15) (289, 34)</td>
</tr>
<tr>
<td>9</td>
<td>495</td>
<td>(361, 38)</td>
</tr>
<tr>
<td>10</td>
<td>450</td>
<td>(147, 14)</td>
</tr>
<tr>
<td>11</td>
<td>417</td>
<td>(529, 46)</td>
</tr>
<tr>
<td>12</td>
<td>387</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>359</td>
<td>(80, 6)</td>
</tr>
<tr>
<td>14</td>
<td>336</td>
<td>(841, 58)</td>
</tr>
<tr>
<td>15</td>
<td>321</td>
<td>(961, 62)</td>
</tr>
<tr>
<td>16</td>
<td>301</td>
<td>(250, 15) (343, 21) (363, 22)</td>
</tr>
<tr>
<td>17</td>
<td>283</td>
<td>(175, 10) (245, 14)</td>
</tr>
<tr>
<td>18</td>
<td>273</td>
<td>(1369, 74)</td>
</tr>
<tr>
<td>19</td>
<td>256</td>
<td>(507, 26)</td>
</tr>
<tr>
<td>20</td>
<td>250</td>
<td>(243, 12) (1681, 82)</td>
</tr>
<tr>
<td>21</td>
<td>239</td>
<td>(1849, 86)</td>
</tr>
<tr>
<td>22</td>
<td>227</td>
<td>(225, 10)</td>
</tr>
<tr>
<td>23</td>
<td>213</td>
<td>(2209, 94)</td>
</tr>
<tr>
<td>24</td>
<td>218</td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>204</td>
<td>(256, 10) (867, 34)</td>
</tr>
<tr>
<td>26</td>
<td>196</td>
<td>(2809, 106)</td>
</tr>
<tr>
<td>27</td>
<td>190</td>
<td>(605, 22)</td>
</tr>
<tr>
<td>28</td>
<td>187</td>
<td>(1083, 38)</td>
</tr>
<tr>
<td>29</td>
<td>176</td>
<td>(3481, 118)</td>
</tr>
<tr>
<td>30</td>
<td>179</td>
<td>(3721, 122)</td>
</tr>
<tr>
<td>31</td>
<td>163</td>
<td>(441, 14) (625, 20)</td>
</tr>
<tr>
<td>32</td>
<td>164</td>
<td>(686, 21) (845, 26)</td>
</tr>
<tr>
<td>33</td>
<td>159</td>
<td>(500, 15) (4489, 134)</td>
</tr>
<tr>
<td>34</td>
<td>154</td>
<td>(1587, 46)</td>
</tr>
<tr>
<td>35</td>
<td>154</td>
<td>(5041, 142)</td>
</tr>
<tr>
<td>36</td>
<td>153</td>
<td>(5329, 146)</td>
</tr>
<tr>
<td>37</td>
<td>139</td>
<td></td>
</tr>
<tr>
<td>38</td>
<td>139</td>
<td>(539, 14) (847, 22)</td>
</tr>
<tr>
<td>39</td>
<td>136</td>
<td>(6241, 158)</td>
</tr>
<tr>
<td>40</td>
<td>139</td>
<td>(486, 12) (1331, 33)</td>
</tr>
<tr>
<td>41</td>
<td>125</td>
<td>(6889, 166)</td>
</tr>
<tr>
<td>42</td>
<td>133</td>
<td>(512, 12) (1445, 34)</td>
</tr>
<tr>
<td>43</td>
<td>119</td>
<td>(2523, 58)</td>
</tr>
<tr>
<td>44</td>
<td>125</td>
<td>(7921, 178)</td>
</tr>
<tr>
<td>45</td>
<td>126</td>
<td>(637, 14) (1183, 26)</td>
</tr>
<tr>
<td>46</td>
<td>117</td>
<td>(2883, 62)</td>
</tr>
<tr>
<td>47</td>
<td>109</td>
<td>(1805, 38)</td>
</tr>
<tr>
<td>48</td>
<td>120</td>
<td>(729, 15) (9409, 194)</td>
</tr>
<tr>
<td>49</td>
<td>114</td>
<td>(1089, 22)</td>
</tr>
<tr>
<td>50</td>
<td>112</td>
<td></td>
</tr>
</tbody>
</table>

Number of elements below $y = x/50$: 12774.
Find a strictly increasing infinite series of integer numbers such that for any consecutive three of them the Smarandache Function is neither increasing nor decreasing.  
*Find the largest strictly increasing series of integer numbers for which the Smarandache Function is strictly decreasing.

a) To solve the first part of this problem, we construct the following series:

\[ P_3, P_3+1, P_4, P_4+1, \ldots, P_n, P_n+1, \ldots \]

where \( P_3, P_4, P_5, \ldots \) are the series of prime odd numbers \( 5, 7, 11 \ldots \)

Of course, \( S(P_i) = P_i \) and \( S(P_i+1) < P_i \), for any \( i \geq 3 \).

b) A way to look at this unsolved question is the following: 
Because \( S(p) = p \), for any prime number, we should get a large interval in between two prime numbers. A bigger chance is when \( p \) and \( q \), the primes with that propriety, are very large (and \( q \neq p + c \), where \( c = 2, 4, \) or \( 6 \)). In this case the series is finite. But this is not the optimum method!

The Smarandache Function is, generally speaking, increasing (we mean that for any positive integer \( k \) there is another integer \( j > k \) such that \( S(j) > S(k) \)). This property makes us to think that our series should be finite.

Calculating at random, for example, the series' width is at least seven, because:

for \( n = 43, 46, 57, 68, 70, 72, 120 \) then

\[ S(n) = 43, 23, 19, 17, 10, 6, 5 \text{ respectively.} \]

We are sure it's possible to find a larger series, but we worry if a maximum width does exist, and if this does: how much is it?  
[Sorry, the author is not able to solve it!]

Solution of a problem by J. Rodriguez

by Pál Grønås

Problem: "Find the largest strictly increasing series of integer numbers for which the Smarandache Function is strictly decreasing".

My intention is to prove that there exists series of arbitrary finite length with the properties described above.

To begin with, define $p_1 = 2$, $p_2 = 3$, $p_3 = 5$ and more generally, $p_n$ is the $n$th prime. Now we have the following lemma:

Lemma: $p_k < p_{k+1} < 2p_k$ for all $k \in \mathbb{N}$. ($\Delta$).

Proof: A theorem conjectured by Bertrand Russell and proven by Tchebycheff states that for all natural numbers $n \geq 2$, there exists a prime $p$ such that $n < p < 2n$. Using this theorem for $n = p_k$, we get $p_k < p < 2p_k$ (i) for at least one prime $p$. The smallest prime $> p_k$ is $p_{k+1}$, so $p \geq p_{k+1}$. But then it is obvious that (i) is satisfied by $p = p_{k+1}$. Hence $p_k < p_{k+1} < 2p_k$. □

This lemma plays an important role in the proof of the following theorem:

Theorem: Let $n$ be a natural number $\geq 2$ and define the series $\{x_k\}_{k=0}^{n-1}$ of length $n$ by $x_k = 2^k p_{2n-k}$ for $k \in \{0, \ldots, n-1\}$. Then $x_k < x_{k+1}$ and $S(x_k) > S(x_{k+1})$ for all $k \in \{0, \ldots, n-2\}$. ($\Omega$)

Proof: For $k \in \{0, \ldots, n-2\}$ we have the following equivalences: $x_k < x_{k+1} \iff 2^k p_{2n-k} < 2^{k+1} p_{2n-(k+1)} \iff p_{2n-k} < 2 p_{2n-(k+1)}$ according to Lemma ($\Delta$).

Furthermore $p_{2n-k} \geq p_{2n-(n-1)} = p_{n+1} \geq p_3 = 5 > 2$, so $(p_{2n-k}, 2) = 1$ for all $k \in \{0, \ldots, n-1\}$. Hence $S(x_k) = S(2^k p_{2n-k}) = \max\{S(2^k), S(p_{2n-k})\} = \max\{S(2^k), p_{2n-k}\}$. Consequently $p_{2n-k} \leq S(x_k) \leq \max\{2k, p_{2n-k}\}$ (i) since $S(2^k) \leq 2k$.

Moreover we know that $p_{k+1} - p_k \geq 2$ for all $k \geq 2$ because both $p_k$ and $p_{k+1}$ are odd integers. This inequality gives us the following result:

$$\sum_{k=2}^{n-1} (p_{k+1} - p_k) = p_n - p_2 = p_n - 3 \geq \sum_{k=2}^{n-1} 2 = 2(n-2),$$

so $p_n \geq 2n - 1$ for all $n \geq 3$. In other words, $p_{n+1} \geq 2n + 1 > 2(n-1)$ for $n \geq 2$, i.e. $p_{2n-k} > 2k$ for $k = n-1$. The fact that $p_{2n-k}$ increases and $2k$ decreases as $k$ decreases from $n - 1$ to 0 implies that $p_{2n-k} > 2k$ for all $k \in \{0, \ldots, n-1\}$. From this last inequality (i) it follows that $S(x_k) = p_{2n-k}$. This formula brings us to the conclusion: $S(x_k) = p_{2n-k} > p_{2n-k-1} = S(x_{k+1})$ for all $k \in \{0, \ldots, n-2\}$. □

Example: For $n = 10$ Theorem ($\Omega$) generates the following series:

<table>
<thead>
<tr>
<th>$k$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_k$</td>
<td>71</td>
<td>134</td>
<td>244</td>
<td>472</td>
<td>848</td>
<td>1504</td>
<td>2752</td>
<td>5248</td>
<td>9472</td>
<td>15872</td>
</tr>
<tr>
<td>$S(x_k)$</td>
<td>71</td>
<td>67</td>
<td>61</td>
<td>59</td>
<td>53</td>
<td>47</td>
<td>43</td>
<td>41</td>
<td>37</td>
<td>31</td>
</tr>
</tbody>
</table>
*Is it possible to extend the Smarandache Function from the integer numbers to the rational numbers (by finding then a rational approach to the factorials, i.e. $(3/2)! = \, ?$)?

*More intriguing is to extend this function to the real numbers (by finding then a real approach to the factorials, i.e. $(\sqrt{5})! = \, ?$)?

*Idem for the complex numbers (i.e. $(4 + 6i)! = \, ?$)?

For example, we know that the Smarandache Function is defined as follows:

$$S : \mathbb{Z} \setminus \{0\} \rightarrow \mathbb{N}, \, S(n) \text{ is as the smallest integer such that } (S(n))! = 1 \times 2 \times 3 \times \ldots \times S(n) \text{ is divisible by } n.$$  

But what about $S(1/2)$, or $S(\pi)$, or $S(-i)$ are they equal to what? It's interesting to try enlarging this function adopting in the same time new definitions for division and factorial, respectively.

Reference:
Let \( \eta(n) \) be Smarandache Function: the smallest integer \( m \) such that \( m! \) is divisible by \( n \). Calculate \( \eta(p^{p+1}) \), where \( p \) is an odd prime number.

Solution.

The answer is \( p^2 \), because:

\[
p^2! = 1 \cdot 2 \cdot \ldots \cdot p \cdot \ldots \cdot (2p) \cdot \ldots \cdot ((p-1)p) \cdot \ldots \cdot (pp),
\]
which is divisible by \( p^{p+1} \).

Any another number less than \( p^2 \) will have the property that its factorial is divisible by \( p^k \), with \( k < p + 1 \), but not divisible by \( p^{p+1} \).
PROPOSED PROBLEM (4)

Let \( m \) be a fixed positive integer. Calculate:

\[
\lim_{i \to \infty} \eta(p_i^m)/p_i
\]

where \( \eta(n) \) is Smarandache Function defined as the smallest integer \( m \) such that \( m! \) is divisible by \( n \), and \( p_i \) the prime series.

Solution:

We note by \( p_j \) a prime number greater than \( m \). We show that

\[
\eta(p_j^m) = mp_j, \text{ for any } i > j:
\]

if by absurd \( \eta(p_j^m) = a < mp_j \) then

\[
a! = 1 \cdot 2 \cdot \ldots \cdot p_j \cdot \ldots \cdot (2p_j) \cdot \ldots \cdot ((m-k)p_j) \cdot \ldots \cdot a, \text{ with } k > 0, \text{ will be divisible by } p_j^{m-k} \text{ but not by } p_j^m.
\]

Then this limit is equal to \( m \).
Prove that the equation
\[ S(x) = p, \] where \( p \) is a given prime number,
has just \( D((p-1)!) \) solutions, all of them in between \( p \) and \( p! \)
\[ (S(n) \text{ is the Smarandache Function: the smallest integer such that } S(n)! \text{ is divisible by } n, \]
and \( D(n) \) is the number of positive divisors of \( n \)].

PROOF (inspired by a remark of D. W. Sharpe):

Of course the smallest solution is \( x = p \), and the largest one is \( x = p! \).
Any other solution should be an integer number divided by \( p \), but not by \( p^2 \) (because \( S(kp^2) \geq S(p^2) = 2p \), where \( k \) is a positive integer).
Therefore \( x = pq \), where \( q \) is a divisor of \((p-1)!\).


Examples (of D. W. Sharpe):
\[ S(x) = 5, \text{ then } x \in \{ 5, 10, 15, 20, 30, 40, 60, 120 \} \] (eight solutions).
\[ S(x) = 7 \] has just 30 solutions, because \( 6! = 2^4 \times 3^2 \times 5^1 \) and \( 6! \) has just \( 5 \times 3 \times 2 = 30 \) positive divisors.
A PROBLEM CONCERNING THE FIBONACCI RECURRENCE (6)
by T. Yau, student, Pima Community College

Let \( S(n) \) be defined as the smallest integer such that \( (S(n))! \) is divisible by \( n \) (Smarandache Function). For what triplets this function verifies the Fibonacci relationship, i.e. find \( n \) such that
\[
S(n) + S(n+1) = S(n+2) \, ?
\]

Solution:
Checking the first 1200 numbers, I found just two triplets for which this function verifies the Fibonacci relationship:
\[
S(9) + S(10) = S(11) \Leftrightarrow 6 + 5 = 11,
\]
and
\[
S(119) + S(120) = S(121) \Leftrightarrow 17 + 5 = 22.
\]
How many other triplets with the same property do exist? (I can't find a theoretical proof ...)

Reference:
GENERALISATION DU PROBLEME 1075' (7)

Soit n un nombre positif entier > 1. Trouver Card\{x, \eta(x) = n\}. L'on a noté par \eta(x) la Fonction Smarandache: qui est définie pour tout entier x comme le plus petit nombre m tel que m! est divisible par x.

M. Costewitz, Bordeaux, France

SOLUTION DU PROBLEME"":
(Ce problème est dans un sens une généralisation du problème 1075, publié dans l'Elemente der Mathematik'.
Soit n = r_1^{d_1} \ldots r_s^{d_s}, la décomposition factorielle unique de ce nombre.

Calculons pour tout 1 \leq i \leq s,
\[ \sum_{j=1}^{\infty} \left\lfloor \frac{n}{r_i^j} \right\rfloor = e_i \geq d_i \geq 1, \text{ où } \left\lfloor a \right\rfloor \text{ signifie la partie entière de } a. \]

C'est-à-dire: n! se divise par r_i^m, pour tout 1 \leq i \leq s.
Nous nottons par M l'ensemble demande.
Biensur,
\[ \bigcup_{i=1}^{s} \{ r_i, r_i^{e_i-1}, \ldots, r_i^{e_i-d_i+1} \} \subseteq M. \]
Nous nottons par R le membre gauche de l'inclusion antérieure, et par
\[ R_i = \{ r_i, r_i^{e_i-1}, \ldots, r_i^{e_i-d_i+1} \}, \]
\[ R'_i = \{ r_i^{e_i-d_i}, \ldots, r_i, 1 \}, \text{ pour tous les } i. \]
Soient q_1, \ldots, q_t tous les nombres premiers différents entre eux, plus petits que n, et non-diviseurs de n. Il est clair que ceux-ci sont tous différents de r_1, \ldots, r_s.

Construisons les suivantes suites finies:
\[ q_1, q_1^2, \ldots, q_1^{e_1}, \text{ tels que } \eta(q_1^{e_1}) < n < \eta(q_1^{e_1+1}); \]
\[ \vdots \]
\[ q_t, q_t^2, \ldots, q_t^{e_1}, \text{ tels que } \eta(q_t^{e_1}) < n < \eta(q_t^{e_1+1}); \]
\[ q_t > n; \]

et

\[ f_k = \sum_{j=1}^{\infty} \left\lfloor \frac{n}{q_k^j} \right\rfloor, \text{ pour tous les } k. \]

Nous formons \( q = \prod_{x=1}^{t} (1 + f_k) \) de combinaisons entre les nombres (éléments) de ces suites, que nous réunissons dans un ensemble noté par \( Q \).

Il est évident que chaque solution de l'équation \( \eta(x) = n \) doit être de la forme: \( a,b,c \), pour tous les \( i \),

où \( a_i \in R_i, b \in \left( \bigcup_{j=1}^{s} R_i \bigcup \bigcup_{j=1}^{s} R'_j \right), c \in Q. \)

Donc, le nombre des solutions pour l'équation demandée est égal à

\[ q \sum_{i=1}^{s} d_i \prod_{j=1}^{s} \left( e_j + 1 \right) \]


["Solution complétée par les éditeurs (C. Dumitrescu)"]
A PROBLEM OF MAXIMUM (8)
by T. Yau, student, Pima Community College

Let $S(n)$ be defined as the smallest integer such that $(S(n))!$ is divisible by $n$ (Smarandache Function). Find:
\[
\max \left\{ \frac{S(n)}{n} \right\},
\]
over all composite integers $n \neq 4$.

Solution:

Let $n = p_1 \ldots p_s$, its canonical factorial decomposition.

Because $S(n) = \max \left\{ S(p_i) \right\} = S(p_1) \leq p_i r_i$,

it's easy to see that $n$ should have only a prime divisor for $S(n)/n$ to become maximum. Therefore $s = 1$.

Then

\[ n = p^r, \text{ where: } p, r \text{ are integers, and } p \text{ is prime}. \]

\[ S(n)/n \leq p r / p^r. \]

Hence $p$ and $r$ should be as small as possible, i.e.

\[ p = 2 \text{ or } 3 \text{ or } 5, \text{ and } r = 2 \text{ or } 3. \]

By checking these combinations, we find

\[ n = 3^2 = 9, \text{ whence } \max \left\{ \frac{S(n)}{n} \right\} = 2/3 \]

over all composite integers $n \neq 4$.

Reference:

Prove that if $N = 0$ there are neither an operation $\, \cdot \, \cdot \, \cdot$ nor integers replacing the letters, for which the following statement:

\[
\text{SMARANDACHE} \quad \cdot \\
\text{FUNCTION} \quad \cdot \\
\text{IN} \\
= \text{NUMBERTHEORY}
\]

is available.

Solution:
Of course $\, \cdot \, \cdot \, \cdot$ may not be an addition, because in that case "S" (as a digit) should be equal to "U", which involves $N = 0$. Contradiction.
[Same for a subtraction.]
Nor a multiplication, because the product should have more than 12 digits.
Not a division, because the quotient should have less than 12 digits.
For other kind of operation, I think it's not necessary to check anymore.

Reference:
are related to the Smarandache Function in the Analytic Number Theory:

\[ S : \mathbb{Z}^* \rightarrow \mathbb{N}, \quad S(n) \text{ is defined as the smallest integer such that } S(n)! \text{ is divisible by } n. \]

The number of these unsolved problems concerning the function is equal to ... an infinity !! Therefore, they will never be all solved!

The Smarandache Function is defined: for all non-null integers, \( n \), to be the smallest integer such that \((S(n))!\) is divisible by \( n \) [see 1, 2, 3].

In order to make students from the American competitions to learn and understand better this notion, used in many east - european national mathematical competions, the author: calculates it for some small numbers, establishes a few proprieties of it, and involves it in relations with other famous functions in the number theory.

It’s important for the teachers to familiarize American students with the work done in other countries. (I would call it: multi - scientifical exchange.)

References:
2. Debra Austin, "The Smarandache Function featured" in <Honeywell Pride>, Phoenix, June 22. 1993, p.8;
1) Proprietatea fundamenta ce contribuie la recunoașterea numerelor prime cu ajutorul acestei funcții (verb);
2) Prezumriii autorului funcției;
3) "O infinitate de probleme... referitoare la o funcție în Teoria Numerelor", articol stărmind interesul matematicienilor (vezi "Proceedings of the International Congress of Mathematicians 1986", Berkeley, CA);
4) Multimi de numere pe care este definită funcția;
5) n pentru \( \{S(n)\}\) în definiția funcției;
6) "... Universiții din Timișoara", revista în care s-a publicat prima dată articolul "O funcție în Teoria Numerelor", 1980 (near.);
7) Numele de familie al autoriului funcției;
8) Număr întreg care nu apartINE domeniul de definiție al funcției;
9) Județul în care se tipărește revista "Analele Universității" din Timișoara;
10) Conf. dr. V. Seleacu, lect. dr. C. Dumitrescu, etc. care au format un grup de cercetare în cadrul Universității din Craiova privind proprietățile și aplicabilitatea acestei funcții (pl.);
12) Ramura științifică inclusă zând Teoria numerelor;
13) Fascicul matematic;
14) De pildă \( \{S(n)\}\);

Author: G. Dincu - Drăgășani, ROMANIA.

(din revista "Abracadabra", Salinas, California, Noiembrie 1993, pp. 14-5, Editor: Ion Bludea)
A collection of papers concerning Smarandache type functions, numbers, sequences, integer algorithms, paradoxes, experimental geometries, algebraic structures, neutrosophic probability, set, and logic, etc.

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Department of Mathematics
University of Craiova, Romania;