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Smarandache Curves of Curves lying on Lightlike Cone in $\mathbb{R}^3_1$

Tanju Kahraman[1] and Hasan Hüseyin Uğurlu[2]

1. Celal Bayar University, Department of Mathematics, Manisa-Turkey.
2. Gazi University, Gazi Faculty of Education, Mathematics Teaching Program, Ankara-Turkey.

E-mail: tanju.kahraman@cbu.edu.tr, hugurlu@gazi.edu.tr

Abstract: In this paper, we consider the notion of the Smarandache curves by considering the asymptotic orthonormal frames of curves lying fully on lightlike cone in Minkowski 3-space $\mathbb{R}^3_1$. We give the relationships between Smarandache curves and curves lying on lightlike cone in $\mathbb{R}^3_1$.

Key Words: Minkowski 3-space, Smarandache curves, lightlike cone, curvatures.

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§1. Introduction

In the study of the fundamental theory and the characterizations of space curves, the related curves for which there exist corresponding relations between the curves are very interesting and an important problem. The most fascinating examples of such curves are associated curves and special curves. Recently, a new special curve is called Smarandache curve is defined by Turgut and Yılmaz in Minkowski space-time [9]. These curves are called Smarandache curves: If a regular curve in Euclidean 3-space, whose position vector is composed by Frenet vectors on another regular curve, then the curve is called a Smarandache Curve. Then, Ali have studied Smarandache curves in the Euclidean 3-space E^3[1]. Kahraman and Uğurlu have studied dual Smarandache curves of curves lying on unit dual sphere $\tilde{S}^2$ in dual space D^3 [3] and they have studied dual Smarandache curves of curves lying on unit dual hyperbolic sphere $\tilde{H}^2_0$ in D^3_1 [4]. Also, Kahraman, Önder and Uğurlu have studied Blaschke approach to dual Smarandache curves [2]

In this paper, we consider the notion of the Smarandache curves by means of the asymptotic orthonormal frames of curves lying fully on Lightlike cone in Minkowski 3-space $\mathbb{R}^3_1$. We show the relationships between frames and curvatures of Smarandache curves and curves lying on lightlike cone in $\mathbb{R}^3_1$.

§2. Preliminaries

The Minkowski 3-space $\mathbb{R}^3_1$ is the real vector space $\mathbb{R}^3$ provided with the standart flat metric

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given by

$$\langle\ ,\ \rangle = -dx_1^2 + dx_2^2 + dx_3^2$$

where (x_1, x_2, x_3) is a rectangular coordinate system of IR_1^3. An arbitrary vector $\vec{v} = (v_1, v_2, v_3)$ in $\mathbb{R}_1^3$ can have one of three Lorentzian causal characters; it can be spacelike if $\langle \vec{v}, \vec{v} \rangle > 0$ or $\vec{v} = 0$, timelike if $\langle \vec{v}, \vec{v} \rangle < 0$ and null (lightlike) if $\langle \vec{v}, \vec{v} \rangle = 0$ and $\vec{v} \neq 0$. Similarly, an arbitrary curve $\vec{x} = \vec{x}(s)$ can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors $\vec{x}'(s)$ are respectively spacelike, timelike or null (lightlike) [6, 7]. We say that a timelike vector is future pointing or past pointing if the first compound of the vector is positive or negative, respectively. For any vectors $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ in $\mathbb{R}_1^3$, in the meaning of Lorentz vector product of $\vec{a}$ and $\vec{b}$ is defined by

$$\vec{a} \times \vec{b} = \begin{vmatrix} e_1 & -e_2 & -e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - a_3 b_2,\ a_1 b_3 - a_3 b_1,\ a_2 b_1 -\ a_1 b_2).$$

Denote by $\left\{\vec{T},\ \vec{N},\ \vec{B}\right\}$ the moving Frenet along the curve $x(s)$ in the Minkowski space $\mathbb{R}_1^3$. For an arbitrary spacelike curve $x(s)$ in the space $\mathbb{R}_1^3$, the following Frenet formulae are given ([8]),

$$\begin{bmatrix} \vec{T}' \\ \vec{N}' \\ \vec{B}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\varepsilon\kappa & 0 & \tau \\ 0 & \varepsilon\tau & 0 \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{bmatrix}$$

where $\left\langle \vec{T}, \vec{T} \right\rangle = 1$, $\left\langle \vec{N}, \vec{N} \right\rangle = \varepsilon = \pm 1$, $\left\langle \vec{B}, \vec{B} \right\rangle = -\varepsilon$, $\left\langle \vec{T}, \vec{N} \right\rangle = \left\langle \vec{T}, \vec{B} \right\rangle = \left\langle \vec{N}, \vec{B} \right\rangle = 0$ and κ and τ are curvature and torsion of the spacelike curve $x(s)$ respectively [10]. Here, ε determines the kind of spacelike curve $x(s)$. If $\varepsilon = 1$, then $x(s)$ is a spacelike curve with spacelike first principal normal $\vec{N}$ and timelike binormal $\vec{B}$. If $\varepsilon = -1$, then $x(s)$ is a spacelike curve with timelike principal normal $\vec{N}$ and spacelike binormal $\vec{B}$ [8]. Furthermore, for a timelike curve $x(s)$ in the space $\mathbb{R}_1^3$, the following Frenet formulae are given in as follows,

$$\begin{bmatrix} \vec{T}' \\ \vec{N}' \\ \vec{B}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{bmatrix}$$

where $\left\langle \vec{T}, \vec{T} \right\rangle = -1$, $\left\langle \vec{N}, \vec{N} \right\rangle = \left\langle \vec{B}, \vec{B} \right\rangle = 1$, $\left\langle \vec{T}, \vec{N} \right\rangle = \left\langle \vec{T}, \vec{B} \right\rangle = \left\langle \vec{N}, \vec{B} \right\rangle = 0$ and κ and τ are curvature and torsion of the timelike curve $x(s)$ respectively [10].

Curves lying on lightlike cone are examined using moving asymptotic frame which is denoted by $\{\vec{x},\ \vec{\alpha},\ \vec{y}\}$ along the curve $x(s)$ lying fully on lightlike cone in the Minkowski space $\mathbb{R}_1^3$.

For an arbitrary curve $x(s)$ lying on lightlike cone in $\mathbb{R}_1^3$, the following asymptotic frame formulae are given by

$$\begin{bmatrix} \vec{x}' \\ \vec{\alpha}' \\ \vec{y}' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \kappa & 0 & -1 \\ 0 & -\kappa & 0 \end{bmatrix} \begin{bmatrix} \vec{x} \\ \vec{\alpha} \\ \vec{y} \end{bmatrix}$$

where $\langle \vec{x}, \vec{x} \rangle = \langle \vec{y}, \vec{y} \rangle = \langle \vec{x}, \vec{\alpha} \rangle = \langle \vec{y}, \vec{\alpha} \rangle = 0, \quad \langle \vec{x}, \vec{y} \rangle = \langle \vec{\alpha}, \vec{\alpha} \rangle = 1$ and κ is curvature function of curve $\alpha(s)$ [5].

§3. Smarandache Curves of Curves lying on Lightlike Cone in Minkowski 3-Space $\mathbb{R}_1^3$

In this section, we first define the four different type of the Smarandache curves of curves lying fully on lightlike cone in $\mathbb{R}_1^3$. Then, by the aid of asymptotic frame, we give the characterizations between reference curve and its Smarandache curves.

3.1 Smarandache $\vec{x}\vec{\alpha}$-curves of curves lying on lightlike cone in $\mathbb{R}_1^3$

Definition 3.1 *Let $x = x(s)$ be a unit speed regular curve lying fully on lightlike cone and $\{\vec{x}, \vec{\alpha}, \vec{y}\}$ be its moving asymptotic frame. The curve α_1 defined by*

$$\vec{\alpha}_1(s) = \vec{x}(s) + \vec{\alpha}(s) \tag{3.1}$$

is called the Smarandache $\vec{x}\vec{\alpha}$-curve of x and α_1 fully lies on Lorentzian sphere S_1^2.

Now, we can give the relationships between x and its Smarandache $\vec{x}\vec{\alpha}$-curve α_1 as follows.

Theorem 3.1 *Let $x = x(s)$ be a unit speed regular curve lying on Lightlike cone in $\mathbb{R}_1^3$. Then the relationships between the asymptotic frame of x and Frenet of its Smarandache $\vec{x}\vec{\alpha}$-curve α_1 are given by*

$$\begin{pmatrix} \vec{T}_1 \\ \vec{N}_1 \\ \vec{B}_1 \end{pmatrix} = \begin{pmatrix} \frac{\kappa}{\sqrt{1-2\kappa}} & \frac{1}{\sqrt{1-2\kappa}} & \frac{-1}{\sqrt{1-2\kappa}} \\ \frac{\kappa'+\kappa(-\kappa'-2\kappa+1)}{(1-2\kappa)^2\sqrt{A}} & \frac{\kappa'+2\kappa-4\kappa^2}{(1-2\kappa)^2\sqrt{A}} & \frac{2\kappa-\kappa'-1}{(1-2\kappa)^2\sqrt{A}} \\ \frac{-1}{\sqrt{1-2\kappa}\sqrt{A}} & \frac{\kappa'}{(1-2\kappa)^{3/2}\sqrt{A}} & \frac{\kappa'+\kappa-2\kappa^2}{(1-2\kappa)^{3/2}\sqrt{A}} \end{pmatrix} \begin{pmatrix} \vec{x} \\ \vec{\alpha} \\ \vec{y} \end{pmatrix} \tag{3.2}$$

where κ is curvature function of $x(s)$ and A is

$$A = \frac{(2\kappa-1)(\kappa')^2 + (8\kappa - 8\kappa^2 - 2)\kappa' - 16\kappa^4 - 24\kappa^3 + 4\kappa^2 - 2\kappa}{(1-2\kappa)^4}.$$

Proof Let the Frenet of Smarandache $\vec{x}\vec{\alpha}$-curve be $\left\{\vec{T}_1, \vec{N}_1, \vec{B}_1\right\}$. Since $\vec{\alpha}_1(s) = \vec{x}(s) + \vec{\alpha}(s)$

and $\vec{T}_1 = \vec{\alpha}_1' / \|\vec{\alpha}_1'\|$, we have

$$\vec{T}_1 = \frac{\kappa}{\sqrt{1-2\kappa}}\vec{x}(s) + \frac{1}{\sqrt{1-2\kappa}}\vec{\alpha}(s) - \frac{1}{\sqrt{1-2\kappa}}\vec{y}(s) \tag{3.3}$$

where

$$\frac{ds}{ds_1} = \frac{1}{\sqrt{1-2\kappa}} \quad \text{and} \quad \kappa < \frac{1}{2}.$$

Since $\vec{N}_1 = \vec{T}_1' \Big/ \left\|\vec{T}_1'\right\|$, we get

$$\vec{N}_1 = \frac{\kappa' - \kappa\kappa' + \kappa - 2\kappa^2}{(1-2\kappa)^2\sqrt{A}}\vec{x}(s) + \frac{\kappa' + 2\kappa - 4\kappa^2}{(1-2\kappa)^2\sqrt{A}}\vec{\alpha}(s) + \frac{2\kappa - \kappa' - 1}{(1-2\kappa)^2\sqrt{A}}\vec{y}(s) \tag{3.4}$$

where

$$A = \frac{(2\kappa-1)(\kappa')^2 + (8\kappa - 8\kappa^2 - 2)\kappa' - 16\kappa^4 - 24\kappa^3 + 4\kappa^2 - 2\kappa}{(1-2\kappa)^4}.$$

Then from $\vec{B}_1 = \vec{T}_1 \times \vec{N}_1$, we have

$$\vec{B}_1 = \frac{-1}{\sqrt{1-2\kappa}\sqrt{A}}\vec{x}(s) + \frac{\kappa'}{(1-2\kappa)^{3/2}\sqrt{A}}\vec{\alpha}(s) + \frac{\kappa' - 2\kappa^2 + \kappa}{(1-2\kappa)^{3/2}\sqrt{A}}\vec{y}(s). \tag{3.5}$$

From (3.3)-(3.5) we have (3.2). □

Theorem 3.2 *The curvature function κ_1 of Smarandache $\vec{x}\vec{\alpha}$-curve α_1 according to curvature function of curve x is given by*

$$\kappa_1 = \frac{\sqrt{A}}{(1-2\kappa)^2}. \tag{3.6}$$

Proof Since $\kappa_1 = \left\|\vec{T}_1'\right\|$. Using the equation (3.3), we get the desired equality (3.6). □

Corollary 3.1 *If curve x is a line. Then Smarandache $\vec{x}\vec{\alpha}$-curve α_1 is line.*

Theorem 3.3 *Torsion τ_1 of Smarandache $\vec{x}\vec{\alpha}$-curve α_1 according to curvature function of curve x is as follows*

$$\begin{aligned}\tau_1 \quad = \quad & \frac{A - A' + 2\kappa A' - \kappa\kappa' A - \kappa' A' - 2\kappa\kappa' A' + 4\kappa^2 A - A\kappa' - 2A\kappa + 2\kappa\kappa'' A}{(1-2\kappa)^3 A^2} \\ & + \frac{\kappa' A - A\kappa\kappa'^2 + \kappa'\kappa'' A - \kappa'^2 A' - 6A\kappa\kappa' - \frac{3A\kappa'^2}{1-2\kappa}}{(1-2\kappa)^4 A^2}\end{aligned}$$

Proof Since $\tau_1 = \left\langle \frac{dB_1}{ds_1}, N_1 \right\rangle$. Using derivation of the equation (3.5), we obtain the wanted equation. □

Corollary 3.2 *If Smarandache $\vec{x}\vec{\alpha}$-curve α_1 is a plane curve. Then, we obtain*

$$\begin{aligned}&\big(A - A' + 2\kappa A' - \kappa\kappa' A - \kappa' A' - 2\kappa\kappa' A' + 4\kappa^2 A - A\kappa' - 2A\kappa + 2\kappa\kappa'' A\big)\\&+ (1-2\kappa) + \kappa' A - A\kappa\kappa'^2 + \kappa'\kappa'' A - \kappa'^2 A' - 6A\kappa\kappa' - \frac{3A\kappa'^2}{1-2\kappa} = 0.\end{aligned}$$

3.2 Smarandache $\vec{x}\vec{y}$-curves of curves lying on Lightlike cone in $\mathbb{R}^3_1$

In this section, we define the second type of Smarandache curves that is called Smarandache $\vec{x}\vec{y}$-curve. Then, we give the relationships between the curve lying on lightlike cone and its Smarandache $\vec{x}\vec{y}$-curve.

Definition 3.2 *Let $x = x(s)$ be a unit speed regular curve lying fully on lightlike cone and $\{\vec{x}, \vec{\alpha}, \vec{y}\}$ be its moving asymptotic frame. The curve α_2 defined by*

$$\vec{\alpha}_2(s) = \frac{1}{\sqrt{2}}\left(\vec{x}(s) + \vec{y}(s)\right)$$

is called the Smarandache $\vec{x}\vec{y}$-curve of x and fully lies on Lorentzian sphere S_1^2.

Now, we can give the relationships between x and its Smarandache $\vec{x}\vec{y}$-curve α_2 as follows.

Theorem 3.4 *Let $x = x(s)$ be a unit speed regular curve lying on lightlike cone in $\mathbb{R}^3_1$. Then the relationships between the asymptotic frame of x and Frenet of its Smarandache $\vec{x}\vec{y}$-curve α_2 are given by*

$$\begin{pmatrix}\vec{T}_2\\ \vec{N}_2\\ \vec{B}_2\end{pmatrix} = \begin{pmatrix}0 & 1 & 0\\ \frac{\kappa}{\sqrt{-2\kappa}} & 0 & \frac{-1}{\sqrt{-2\kappa}}\\ \frac{1}{\sqrt{-2\kappa}} & 0 & \frac{-\kappa}{\sqrt{-2\kappa}}\end{pmatrix}\begin{pmatrix}\vec{x}\\ \vec{\alpha}\\ \vec{y}\end{pmatrix}$$

where κ is curvature function of $x(s)$.

Theorem 3.5 *The curvature function κ_2 of Smarandache $\vec{x}\vec{y}$-curve α_2 according to curvature function of curve x is given*

$$\kappa_2 = \sqrt{-2\kappa}.$$

Corollary 3.3 *Curve x is a line if and only if Smarandache $\vec{x}\vec{y}$-curve α_2 is line.*

Theorem 3.6 *Torsion τ_2 of Smarandache $\vec{x}\vec{y}$-curve α_2 according to curvature function of curve x is as follows*

$$\tau_2 = \frac{-2\sqrt{2}\kappa'}{4\kappa^2 - 4\kappa^3}.$$

Corollary 3.4 *If Smarandache $\vec{x}\vec{y}$-curve α_2 is a plane curve. Then, curvature κ of curve x is constant.*

3.3 Smarandache $\vec{\alpha}\vec{y}$-curves of curves lying on lightlike cone in $\mathbb{R}^3_1$

In this section, we define the third type of Smarandache curves that is called Smarandache $\vec{\alpha}\vec{y}$-curve. Then, we give the relationships between the curve lying on lightlike cone and its Smarandache $\vec{\alpha}\vec{y}$-curve.

Definition 3.3 *Let $x = x(s)$ be a unit speed regular curve lying fully on lightlike cone and $\{\vec{x}, \vec{\alpha}, \vec{y}\}$ be its moving asymptotic frame. The curve α_3 defined by*

$$\vec{\alpha}_3(s) = \vec{\alpha}(s) + \vec{y}(s)$$

is called the Smarandache $\vec{\alpha}\vec{y}$-curve of x and fully lies on Lorentzian sphere S^2_1.

Now we can give the relationships between x and its Smarandache $\vec{\alpha}\vec{y}$-curve α_3 as follows.

Theorem 3.7 *Let $x = x(s)$ be a unit speed regular curve lying on lightlike cone in $\mathbb{R}^3_1$. Then the relationships between the asymptotic frame of x and Frenet of its Smarandache $\vec{\alpha}\vec{y}$-curve α_3 are given by*

$$\begin{pmatrix} \vec{T}_3 \\ \vec{N}_3 \\ \vec{B}_3 \end{pmatrix} = \begin{pmatrix} \frac{\kappa}{\sqrt{\kappa^2-2\kappa}} & \frac{-\kappa}{\sqrt{\kappa^2-2\kappa}} & \frac{-1}{\sqrt{\kappa^2-2\kappa}} \\ \frac{-\kappa^4+2\kappa^3}{\sqrt{A}} & \frac{-2\kappa^2\kappa'+4\kappa\kappa'+2\kappa^3-4\kappa^2}{\sqrt{A}} & \frac{\kappa\kappa'-\kappa'+\kappa^3-2\kappa^2}{\sqrt{A}} \\ \frac{-3\kappa^2\kappa'+5\kappa\kappa'-\kappa^4+4\kappa^3-4\kappa^2}{\sqrt{A}\sqrt{\kappa^2-2\kappa}} & \frac{\kappa^2\kappa'-\kappa\kappa'}{\sqrt{A}\sqrt{\kappa^2-2\kappa}} & \frac{\kappa^5-4\kappa^4+4\kappa^3+2\kappa^3\kappa'-4\kappa^2\kappa'}{\sqrt{A}\sqrt{\kappa^2-2\kappa}} \end{pmatrix} \begin{pmatrix} \vec{x} \\ \vec{\alpha} \\ \vec{y} \end{pmatrix}$$

where κ is curvature function of $x(s)$ and A is

$$A = (4\kappa^4 - 16\kappa^3 + 16\kappa^2)(\kappa')^2 + \kappa'(-10\kappa^5 + 38\kappa^4 - 36\kappa^3) - 2\kappa^7 + 12\kappa^6 - 24\kappa^5 + 16\kappa^4.$$

Theorem 3.8 *The curvature function κ_3 of Smarandache $\vec{\alpha}\vec{y}$-curve α_3 according to curvature function of curve x is given*

$$\kappa_3 = \frac{\sqrt{(4\kappa^4 - 16\kappa^3 + 16\kappa^2)(\kappa')^2 + \kappa'(-10\kappa^5 + 38\kappa^4 - 36\kappa^3) - 2\kappa^7 + 12\kappa^6 - 24\kappa^5 + 16\kappa^4}}{(\kappa^2 - 2\kappa)^2}.$$

Corollary 3.5 *If curve x is a line. Then, Smarandache $\vec{\alpha}\vec{y}$-curve α_3 is line.*

Theorem 3.9 *Torsion τ_3 of Smarandache $\vec{\alpha}\vec{y}$-curve α_3 according to curvature function of curve x is as follows*

$$\tau_3 = \frac{b_1\left(\kappa\kappa' - \kappa' + \kappa^3 - 2\kappa^2\right)}{A(\kappa^2 - 2\kappa)} + \frac{2b_2\left(\kappa - \kappa'\right)}{A} - \frac{b_3\kappa^2}{A}$$

where b_1, b_2 *and* b_3 *are*

$$\begin{aligned} b_1 &= -6\kappa(\kappa')^2+9\kappa^2\kappa'+5(\kappa')^2+5\kappa\kappa''-4\kappa^3\kappa'-8\kappa\kappa' \\ &\quad +(3\kappa^2\kappa'-5\kappa\kappa'+\kappa^4-4\kappa^3+4\kappa^2)\left(\frac{A'}{2A}+\frac{\kappa\kappa'-\kappa'}{\kappa^2-2\kappa}\right) \\ b_2 &= (\kappa^2-\kappa)\kappa''+(2\kappa-1)(\kappa')^2-(\kappa^2\kappa'-\kappa\kappa')\left(\frac{A'}{2A}+\frac{\kappa\kappa'-\kappa'}{\kappa^2-2\kappa}\right) \\ b_3 &= (5\kappa^2-16\kappa+12)\kappa^2\kappa'+(2\kappa-8)\kappa(\kappa')^2+2\kappa^3\kappa'' \\ &\quad -(2\kappa^3\kappa'-4\kappa^2\kappa'+\kappa^5-4\kappa^4+4\kappa^3)\left(\frac{A'}{2A}+\frac{\kappa\kappa'-\kappa'}{\kappa^2-2\kappa}\right). \end{aligned}$$

Corollary 3.6 *If Smarandache* $\vec{\alpha}\vec{y}$*-curve* α_3 *is a plane curve. Then, we obtain*

$$b_1\left(\kappa\kappa'-\kappa'+\kappa^3-2\kappa^2\right)+(\kappa^2-2\kappa)\left(2b_2\left(\kappa-\kappa'\right)-b_3\kappa^2\right)=0.$$

3.4 Smarandache $\vec{x}\vec{\alpha}\vec{y}$-curves of curves lying on lightlike cone in $\mathbb{R}^3_1$

In this section, we define the fourth type of Smarandache curves that is called Smarandache $\vec{x}\vec{\alpha}\vec{y}$-curve. Then, we give the relationships between the curve lying on lightlike cone and its Smarandache $\vec{x}\vec{\alpha}\vec{y}$-curve.

Definition 3.4 *Let* $x=x(s)$ *be a unit speed regular curve lying fully on Lightlike cone and* $\{\vec{x},\vec{\alpha},\vec{y}\}$ *be its moving asymptotic frame. The curves* α_4 *and* α_5 *defined by*

(i) $\vec{\alpha}_4(s)=\frac{1}{\sqrt{3}}\left(\vec{x}(s)+\vec{\alpha}(s)+\vec{y}(s)\right)$;

(ii) $\vec{\alpha}_5(s)=-\vec{x}(s)+\vec{\alpha}(s)+\vec{y}(s)$

are called the Smarandache $\vec{x}\vec{\alpha}\vec{y}$*-curves of* x *and fully lies on Lorentzian sphere* S^2_1 *and hyperbolic sphere* H^3_0.

Now, we can give the relationships between x and its Smarandache $\vec{x}\vec{\alpha}\vec{y}$-curve α_4 on Lorentzian sphere S^2_1 as follows.

Theorem 3.10 *Let* $x=x(s)$ *be a unit speed regular curve lying on lightlike cone in* $\mathbb{R}^3_1$*. Then the relationships between the asymptotic frame of* x *and Frenet of its Smarandache* $\vec{x}\vec{\alpha}\vec{y}$*-curve* α_4 *are given by*

$$\begin{pmatrix}\vec{T}_4\\ \vec{N}_4\\ \vec{B}_4\end{pmatrix}=\begin{pmatrix}\frac{\kappa}{\sqrt{\kappa^2-4\kappa+1}} & \frac{1-\kappa}{\sqrt{\kappa^2-4\kappa+1}} & \frac{-1}{\sqrt{\kappa^2-4\kappa+1}}\\ \frac{a_1}{\sqrt{2a_1c_1+b_1^2}} & \frac{b_1}{\sqrt{2a_1c_1+b_1^2}} & \frac{c_1}{\sqrt{2a_1c_1+b_1^2}}\\ \frac{c_1(1-\kappa)+b_1}{\sqrt{\kappa^2-4\kappa+1}\sqrt{2a_1c_1+b_1^2}} & \frac{c_1\kappa+a_1}{\sqrt{\kappa^2-4\kappa+1}\sqrt{2a_1c_1+b_1^2}} & \frac{a_1(\kappa-1)-b_1\kappa}{\sqrt{\kappa^2-4\kappa+1}\sqrt{2a_1c_1+b_1^2}}\end{pmatrix}\begin{pmatrix}\vec{x}\\ \vec{\alpha}\\ \vec{y}\end{pmatrix}$$

where κ is curvature function of $x(s)$ and a_1, b_1, c_1 are

$$\begin{aligned} a_1 &= \frac{\sqrt{3}(\kappa' - 2\kappa\kappa' - 5\kappa^2 + 5\kappa^3 - \kappa^4 + \kappa)}{(\kappa^2 - 4\kappa + 1)^2} \\ b_1 &= \frac{\sqrt{3}(2\kappa + \kappa' - 8\kappa^2 + 2\kappa\kappa' + 2\kappa^3)}{(\kappa^2 - 4\kappa + 1)^2} \\ c_1 &= \frac{\sqrt{3}(-2\kappa' + \kappa\kappa' + 5\kappa - 5\kappa^2 + \kappa^3 - 1)}{(\kappa^2 - 4\kappa + 1)^2}. \end{aligned}$$

Theorem 3.11 *The curvature function κ_4 of Smarandache $\vec{x}\vec{\alpha}\vec{y}$-curve α_4 according to curvature function of curve x is given*

$$\kappa_4 = \sqrt{2a_1c_1 + b_1^2}.$$

Corollary 3.7 *If curve x is a line. Then, Smarandache $\vec{x}\vec{\alpha}\vec{y}$-curve α_4 is line.*

Theorem 3.12 *Torsion τ_4 of Smarandache $\vec{x}\vec{\alpha}\vec{y}$-curve α_4 according to curvature function of curve x is as follows*

$$\tau_4 = \frac{a_2c_1 + b_1b_2 + a_1c_2}{\sqrt{2a_1c_1 + b_1^2}}$$

where a_2, b_2 and c_2 are

$$\begin{aligned} a_2 &= \frac{\sqrt{3}(c_1' - c_1'\kappa - c_1\kappa' + b_1' + c_1\kappa^2 + a_1\kappa) - \sqrt{3}(c_1 - c_1\kappa + b_1)\left(\frac{\kappa\kappa' - 2\kappa'}{\kappa^2 - 4\kappa + 1} + \frac{a_1'c_1 + a_1c_1' + b_1b_1'}{2a_1c_1 + b_1^2}\right)}{(\kappa^2 - 4\kappa + 1)\sqrt{2a_1c_1 + b_1^2}} \\ b_2 &= \frac{\sqrt{3}(c_1'\kappa + c_1\kappa' + a_1' + c_1 - c_1\kappa + b_1 + a_1\kappa - a_1\kappa^2 + b_1\kappa^2) - \sqrt{3}(c_1\kappa + a_1)\left(\frac{\kappa\kappa' - 2\kappa'}{\kappa^2 - 4\kappa + 1} + \frac{a_1'c_1 + a_1c_1' + b_1b_1'}{2a_1c_1 + b_1^2}\right)}{(\kappa^2 - 4\kappa + 1)\sqrt{2a_1c_1 + b_1^2}} \\ c_2 &= \frac{\sqrt{3}(a_1'\kappa - a_1' + a_1\kappa' - b_1'\kappa - b_1\kappa' - c_1\kappa - a_1) - \sqrt{3}(a_1\kappa - a_1 - b_1\kappa)\left(\frac{\kappa\kappa' - 2\kappa'}{\kappa^2 - 4\kappa + 1} + \frac{a_1'c_1 + a_1c_1' + b_1b_1'}{2a_1c_1 + b_1^2}\right)}{(\kappa^2 - 4\kappa + 1)\sqrt{2a_1c_1 + b_1^2}}. \end{aligned}$$

Corollary 3.8 *If Smarandache $\vec{x}\vec{\alpha}\vec{y}$-curve α_4 is a plane curve. Then, we obtain*

$$a_2c_1 + b_1b_2 + a_1c_2 = 0.$$

Results of statement (ii) can be given by using the similar ways used for the statement (i).

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