Hongyan Sun[1], Mohamad Farooq[2]

[1,2]Department of Electrical & Computer, Engineering Royal Military College of Canada
Kingston, ON, Canada, K7K 7B4

On conjunctive and disjunctive combination rules of evidence

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Abstract: *In this chapter, the Dempster-Shafer (DS) combination rule is examined based on the multi-valued mapping (MVM) and the product combination rule of multiple independent sources of information. The shortcomings in DS rule are correctly interpreted via the product combination rule of MVM. Based on these results, a new justification of the disjunctive rule is proposed. This combination rule depends on the logical judgment of OR and overcomes the shortcomings of DS rule, especially, in the case of the counter-intuitive situation. The conjunctive, disjunctive and hybrid combination rules of evidence are studied and compared. The properties of each rule are also discussed in details. The role of evidence of each source of information, the comparison of the combination judgment belief and ignorance of each rule, the treatment of conflicting judgments given by sources, and the applications of combination rules are discussed. The new results yield valuable theoretical insight into the rules that can be applied to a given situation. Zadeh's example is also included in this chapter for the evaluation of the performance and the efficiency of each combination rule of evidence in case of conflicting judgments.*

9.1 Introduction

Combination theory of multiple sources of information is always an important area of research in information processing of multiple sources. The initial important contribution in this area is due to Dempster in terms of Dempster's rule [1]. Dempster derived the combination rule for multiple independent sources of information based on the product space of multiple sources of information and multi-valued mappings. In the product space, combination-mapping of multiple multi-valued mappings is defined as the intersection of each multi-valued mapping, that is, an element can be judged by combination sources of information if and only if it can be judged by each source of information simultaneously, irrespective of the magnitude of the basic judgment probability. Shafer extended Dempster's theory to the space with all the subsets of a given set (i.e. the power set) and defined the frame of discernment, degree of belief, and, furthermore, proposed a new combination rule of the multiple independent sources of information in the form of Dempster-Shafer's (DS) combination rule [2]. However, the interpretation, implementation, or computation of the technique are not described in a sufficient detail in [2]. Due to the lack of details in [2], the literature is full of techniques to arrive at DS combination rule. For example, compatibility relations [3, 4], random subsets [5, 6, 7], inner probability [8, 9], joint (conjunction) entropy [10] etc. have been utilized to arrive at the results in [2]. In addition, the technique has been applied in various fields such as engineering, medicine, statistics, psychology, philosophy and accounting [11], and multi-sensor information fusion [12, 13, 14, 15, 16] etc. DS combination rule is more efficient and effective than the Bayesian judgment rule because the former does not require a priori probability and can process ignorance. A number of researchers have documented the drawbacks of DS techniques, such as the counter-intuitive results for some pieces of evidence [17, 18, 19], computational expenses and independent sources of information [20, 21].

One of the problems in DS combination rule of evidence is that the measure of the basic probability assignment of combined empty set is not zero, i.e. $m(\emptyset) \neq 0$, however, it is supposed to be zero, i.e. $m(\emptyset) = 0$. In order to overcome this problem, the remaining measure of the basic probability assignment is reassigned via the orthogonal technique [2]. This has created a serious problem for the combination of the two sharp sources of information, especially, when two sharp sources of information have only one of the same focal elements (i.e. two sources of information are in conflict), thus resulting in a counter-intuitive situation as demonstrated by Zadeh. In addition, DS combination rule cannot be applied to two sharp sources of information that have none of the same focal elements. These problems are not essentially due to the orthogonal factor in DS combination rule (see references [22, 23]).

In general, there are two main techniques to resolve the Shafer problem. One is to suppose $m(\emptyset) \neq 0$ or $m(\emptyset) > 0$ as it is in reality. The Smets transferable belief model (TBM), and Yager, Dubois &

Prade and Dezert-Smarandache (DSm) combination rules are the ones that utilize this fact in references [20, 24, 25, 26, 27, 28]. The other technique is that the empty set in the combined focal elements is not allowed and this idea is employed in the disjunctive combination rule [22, 23, 29, 30, 31]. Moreover, E. Lefèvre et al. propose a general combination formula of evidence in [32] and further conjunctive combination rules of evidence can been derived from it.

In this chapter, we present some of work that we have done in the combination rules of evidence. Based on a multi-valued mapping from a probability space (X, Ω, μ) to space S, a probability measure over a class 2^S of subsets of S is defined. Then, using the product combination rule of multiple information sources, Dempster-Shafer's combination rule is derived. The investigation of the two rules indicates that Dempster's rule and DS combination rule are for different spaces. Some problems of DS combination rule are correctly interpreted via the product combination rule that is used for multiple independent information sources. An error in multi-valued mappings in [11] is pointed out and proven.

Furthermore, a novel justification of the disjunctive combination rule for multiple independent sources of information based on the redefined combination-mapping rule of multiple multi-valued mappings in the product space of multiple independent sources of information is being proposed. The combination rule reveals a type of logical inference in the human judgment, that is, the OR rule. It overcomes the shortcoming of DS combination rule with the AND rule, especially, the one that is counter-intuitive, and provides a more plausible judgment than DS combination rule over different elements that are judged by different sources of information.

Finally, the conjunctive and disjunctive combination rules of evidence, namely, DS combination rule, Yager's combination rule, Dubois and Prade's (DP) combination rule, DSm's combination rule and the disjunctive combination rule, are studied for the two independent sources of information. The properties of each combination rule of evidence are discussed in detail, such as the role of evidence of each source of information in the combination judgment, the comparison of the combination judgment belief and ignorance of each combination rule, the treatment of conflict judgments given by the two sources of information, and the applications of combination rules. The new results yield valuable theoretical insight into the rules that can be applied to a given situation. Zadeh's example is included in the chapter to evaluate the performance as well as efficiency of each combination rule of evidence for the conflict judgments given by the two sources of information.

9.2 Preliminary

9.2.1 Source of information and multi-valued mappings

Consider n sources of information and corresponding multi-valued mappings [1]. They are mathematically defined by n basic probability spaces (X_i, Ω_i, μ_i) and multi-valued mappings Γ_i which assigns a subset $\Gamma_i x_i \subset S$ to every $x_i \in X_i$, $i = 1, 2, \ldots, n$. The space S into which Γ_i maps is the same for each i, namely: n different sources yield information about the same uncertain outcomes in S.

Let n sources be independent. Then based on the definition of the statistical independence, the combined sources (X, Ω, μ) can be defined as

$$X = X_1 \times X_2 \times \ldots \times X_n \tag{9.1}$$

$$\Omega = \Omega_1 \times \Omega_2 \times \ldots \times \Omega_n \tag{9.2}$$

$$\mu = \mu_1 \times \mu_2 \times \ldots \times \mu_n \tag{9.3}$$

for all $x \in X$, Γ is defined as

$$\Gamma x = \Gamma_1 x \cap \Gamma_2 x \cap \ldots \cap \Gamma_n x \tag{9.4}$$

The definition of Γ implies that $x_i \in X_i$ is consistent with a particular $s \in S$ if and only if $s \in \Gamma_i x_i$, for $i = 1, 2, \ldots, n$, and consequently $x = (x_1, x_2, \ldots, x_n) \in X$ is consistent with s if and only if $s \in \Gamma_i x_i$ for all $i = 1, 2, \ldots, n$ [1].

For finite $S = \{s_1, s_2, \ldots, s_n\}$, suppose $S_{\delta_1 \delta_2 \ldots \delta_n}$ denotes the subset of S which contains s_j if $\delta_j = 1$ and excludes s_j if $\delta_j = 0$, for $j = 1, 2, \ldots, n$. Then the 2^n subsets of S so defined are possible for all $\Gamma_i x_i$ $(i = 1, 2, \ldots, n)$, and partition X_i into

$$X_i = \bigcup_{\delta_1 \delta_2 \ldots \delta_m} X^{(i)}_{\delta_1 \delta_2 \ldots \delta_m} \tag{9.5}$$

where

$$X^{(i)}_{\delta_1 \delta_2 \ldots \delta_n} = \{x_i \in X_i, \Gamma_i x_i = S_{\delta_1 \delta_2 \ldots \delta_n}\} \tag{9.6}$$

and define [1]

$$p^{(i)}_{\delta_1 \delta_2 \ldots \delta_n} = \mu(X^{(i)}_{\delta_1 \delta_2 \ldots \delta_n}) \tag{9.7}$$

9.2.2 Dempster's combination rule of independent information sources

Based on (9.1) - (9.7), the combination of probability judgments of multiple independent information sources is characterized by [1] $p^{(i)}_{\delta_1\delta_2\ldots\delta_n}$, $i = 1, 2, \ldots, n$. That is

$$p_{\delta_1\delta_2\ldots\delta_n} = \sum_{\delta_i=\delta_i^{(1)}\delta_i^{(2)}\ldots\delta_i^{(n)}} p^{(1)}_{\delta_1^{(1)}\delta_2^{(1)}\ldots\delta_n^{(1)}} p^{(2)}_{\delta_1^{(2)}\delta_2^{(2)}\ldots\delta_n^{(2)}} \cdots p^{(n)}_{\delta_1^{(n)}\delta_2^{(n)}\ldots\delta_n^{(n)}} \tag{9.8}$$

Equation (9.8) indicates that the combination probability judgment of n independent information sources for any element $S_{\delta_1\delta_2\ldots\delta_n}$ of S equals the sum of the product of simultaneously doing probability judgment of each independent information source for the element. It emphasizes the common role of each independent information source. That is characterized by the product combination rule.

9.2.3 Degree of belief

Definition 1:

If Θ is a frame of discernment, then function $m : 2^\Theta \to [0, 1]$ is called[1] a *basic belief assignment* whenever

$$m(\emptyset) = 0 \tag{9.9}$$

and

$$\sum_{A\subseteq\Theta} m(A) = 1 \tag{9.10}$$

The quantity $m(A)$ is called the belief mass of A (or *basic probability number* in [2]).

Definition 2:

A function $\text{Bel} : 2^\Theta \to [0, 1]$ is called a *belief function* over Θ [2] if it is given by

$$\text{Bel}(A) = \sum_{B\subseteq A} m(B) \tag{9.11}$$

for some basic probability assignment $m : 2^\Theta \to [0, 1]$.

Definition 3:

A subset A of a frame Θ is called a *focal element* of a belief function Bel over Θ [2] if $m(A) > 0$. The union of all the focal elements of a belief function is called its *core*.

Theorem 1:

If Θ is a frame of discernment, then a function $\text{Bel} : 2^\Theta \to [0, 1]$ is a belief function if and only if it satisfies the three following conditions [2]:

[1] also called *basic probability assignment* in [2].

1.
$$\text{Bel}(\emptyset) = 0 \tag{9.12}$$

2.
$$\text{Bel}(\Theta) = 1 \tag{9.13}$$

3. For every positive integer n and every collection $A_1, \ldots, A_n$ of subsets of Θ,

$$\text{Bel}(A_1 \cup \ldots \cup A_n) = \sum_{\substack{I \subset \{1,\ldots,n\} \\ I \neq \emptyset}} (-1)^{|I|+1} \text{Bel}(\cap_{i \in I} A_i) \tag{9.14}$$

Definition 4:

The function $\text{Pl} : 2^\Theta \to [0,1]$ defined by

$$\text{Pl}(A) = 1 - \text{Bel}(\bar{A}) \tag{9.15}$$

is called the *plausibility function* for Bel. $\bar{A}$ denotes the complement of A in 2^Θ.

Definition 5:

If Θ is a frame of discernment, then a function $\text{Bel} : 2^\Theta \to [0,1]$ is called *Bayesian belief* [2] if and only if

1. $\text{Bel}(\emptyset) = 0$ (9.16)
2. $\text{Bel}(\Theta) = 1$ (9.17)
3. If $A, B \subset \Theta$ and $A \cap B = \emptyset$, then $\text{Bel}(A \cup B) = \text{Bel}(A) + \text{Bel}(B)$ (9.18)

Theorem 2:

If $\text{Bel} : 2^\Theta \to [0,1]$ is a belief function over Θ, Pl is a plausibility corresponding to it, then the following conclusions are equal [2]

1. The belief is a Bayesian belief.

2. Each focal element of Bel is a single element set.

3. $\forall A \subset \Theta$, $\text{Bel}(A) + \text{Bel}(\bar{A}) = 1$.

9.2.4 The DS combination rule

Theorem 3:

Suppose Bel_1 and Bel_2 are belief functions over the same frame of discernment $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$ with basic belief assignments m_1 and m_2, and focal elements $A_1, A_2, \ldots, A_k$ and $B_1, B_2, \ldots, B_l$, respectively. Suppose

$$\sum_{\substack{i,j \\ A_i \cap B_j = \emptyset}} m_1(A_i) m_2(B_j) < 1 \tag{9.19}$$

Then the function $m : 2^\Theta \to [0,1]$ defined by $m(\emptyset) = 0$ and

$$m(A) = \frac{\displaystyle\sum_{\substack{i,j \\ A_i \cap B_j = A}} m_1(A_i)m_2(B_j)}{1 - \displaystyle\sum_{\substack{i,j \\ A_i \cap B_j = \emptyset}} m_1(A_i)m_2(B_j)} \tag{9.20}$$

for all non-empty $A \subseteq \Theta$ is a basic belief assignment [2]. The core of the belief function given by m is equal to the intersection of the cores of Bel_1 and Bel_2. This defines Dempster-Shafer's rule of combination (denoted as the DS combination rule in the sequel).

9.3 The DS combination rule induced by multi-valued mapping

9.3.1 Definition of probability measure over the mapping space

Given a probability space (X, Ω, μ) and a space S with a multi-valued mapping:

$$\Gamma : X \to S \tag{9.21}$$

$$\forall x \in X, \Gamma x \subset S \tag{9.22}$$

The problem here is that if the uncertain outcome is known to correspond to an uncertain outcome $s \in \Gamma x$, then the probability judgement of the uncertain outcome $s \in \Gamma x$ needs to be determined.

Assume S consists of n elements, i.e. $S = \{s_1, s_2, \ldots, s_n\}$. Let's denote $S_{\delta_1\delta_2\ldots\delta_n}$ the subsets of S, where $\delta_i = 1$ or 0, $i = 1, 2, \ldots, n$, and

$$S_{\delta_1\delta_2\ldots\delta_n} = \bigcup_{i \neq j, \delta_i = 1, \delta_j = 0} s_i \tag{9.23}$$

then from mapping (9.21)-(9.22) it is evident that $S_{\delta_1\delta_2\ldots\delta_n}$ is related to Γx. Therefore, the 2^S subsets such as in equation (9.23) of S yield a partition of X

$$X = \bigcup_{\delta_1\delta_2\ldots\delta_n} X_{\delta_1\delta_2\ldots\delta_n} \tag{9.24}$$

where

$$X_{\delta_1\delta_2\ldots\delta_n} = \{x \in X, \Gamma x = S_{\delta_1\delta_2\ldots\delta_n}\} \tag{9.25}$$

Define a probability measure over $2^S = \{S_{\delta_1\delta_2\ldots\delta_n}\}$ as $M : 2^S = \{S_{\delta_1\delta_2\ldots\delta_n}\} \to [0,1]$ with

$$M(S_{\delta_1\delta_2\ldots\delta_n}) = \begin{cases} 0, S_{\delta_1\delta_2\ldots\delta_n} = \emptyset \\ \frac{\mu(X_{\delta_1\delta_2\ldots\delta_n})}{1-\mu(X_{00\ldots0})}, S_{\delta_1\delta_2\ldots\delta_n} \neq \emptyset \end{cases} \tag{9.26}$$

where M is the probability measure over a class $2^S = \{S_{\delta_1\delta_2\ldots\delta_n}\}$ of subsets of space S which Γ maps X into.

9.3.2 Derivation of the DS combination rule

Given two $n = 2$ independent information sources, then from equation (9.8), we have

$$\mu(X_{\delta_1\delta_2\ldots\delta_n}) = \sum_{\Gamma X_{\delta_1\delta_2\ldots\delta_n}=\Gamma^{(1)}X^{(1)}_{\delta'_1\delta'_2\ldots\delta'_n}\cap\Gamma^{(2)}X^{(2)}_{\delta''_1\delta''_2\ldots\delta''_n}} \mu^{(1)}(X^{(1)}_{\delta'_1\delta'_2\ldots\delta'_n})\mu^{(2)}(X^{(2)}_{\delta''_1\delta''_2\ldots\delta''_n}) \tag{9.27}$$

From equation (9.26), if $S_{\delta_1\delta_2\ldots\delta_n} \neq \emptyset$, we have for $i = 1, 2$

$$\mu^{(i)}(X_{\delta_1\delta_2\ldots\delta_n}) = M^{(i)}(S_{\delta_1\delta_2\ldots\delta_n})(1 - \mu^{(i)}(X_{00\ldots0})) \tag{9.28}$$

and

$$\mu(X_{\delta_1\delta_2\ldots\delta_n}) = M(S_{\delta_1\delta_2\ldots\delta_n})(1 - \mu(X_{00\ldots0})) \tag{9.29}$$

where equations (9.28) and (9.29) correspond to information source i, ($i = 1, 2$) and their combined information sources, respectively. Substituting equations (9.28)-(9.29) into equation (9.27), we have

$$M(S_{\delta_1\delta_2\ldots\delta_n}) = \frac{\sum_{\delta=\delta'\delta''} M^{(1)}(S_{\delta'_1\delta'_2\ldots\delta'_n})M^{(2)}(S_{\delta''_1\delta''_2\ldots\delta''_n})[1 - \mu^{(1)}(X^{(1)}_{00\ldots0})[1 - \mu^{(2)}(X^{(2)}_{00\ldots0})]}{1 - \mu(X_{00\ldots0})} \tag{9.30}$$

and

$$\begin{aligned}\frac{[1 - \mu^{(1)}(X^{(1)}_{00\ldots0})][1 - \mu^{(2)}(X^{(2)}_{00\ldots0})]}{1 - \mu(X_{00\ldots0})} &= \frac{[1 - \mu^{(1)}(X^{(1)}_{00\ldots0})][1 - \mu^{(2)}(X^{(2)}_{00\ldots0})]}{\sum_{\Gamma_1X^{(1)}_{\delta'_1\delta'_2\ldots\delta'_n}\cap\Gamma_2X^{(2)}_{\delta''_1\delta''_2\ldots\delta''_n}\neq\emptyset} \mu^{(1)}(X^{(1)}_{\delta'_1\delta'_2\ldots\delta'_n})\mu^{(2)}(X^{(2)}_{\delta''_1\delta''_2\ldots\delta''_n})} \\ &= \frac{1}{\sum_{S_{\delta'_1\delta'_2\ldots\delta'_n}\cap S_{\delta''_1\delta''_2\ldots\delta''_n}\neq\emptyset} M^{(1)}(S_{\delta'_1\delta'_2\ldots\delta'_n})M^{(2)}(S_{\delta''_1\delta''_2\ldots\delta''_n})}\end{aligned} \tag{9.31}$$

Substitute (9.31) back into (9.30), hence we have

$$M(S_{\delta_1\delta_2\ldots\delta_n}) = \frac{\sum_{S_{\delta'_1\delta'_2\ldots\delta'_n}\cap S_{\delta''_1\delta''_2\ldots\delta''_n}=S_{\delta_1\delta_2\ldots\delta_n}} M^{(1)}(S_{\delta'_1\delta'_2\ldots\delta'_n})M^{(2)}(S_{\delta''_1\delta''_2\ldots\delta''_n})}{1 - \sum_{S_{\delta'_1\delta'_2\ldots\delta'_n}\cap S_{\delta''_1\delta''_2\ldots\delta''_n}=\emptyset} M^{(1)}(S_{\delta'_1\delta'_2\ldots\delta'_n})M^{(2)}(S_{\delta''_1\delta''_2\ldots\delta''_n})} \tag{9.32}$$

when $S_{\delta_1\delta_2\ldots\delta_n} = \emptyset$,

$$M(S_{\delta_1\delta_2\ldots\delta_n}) \triangleq 0 \tag{9.33}$$

Thus, equations (9.32) and (9.33) are DS combination rule. Where space $S = \{s_1, s_2, \ldots, s_n\}$ is the frame of discernment.

The physical meaning of equations (9.8) and (9.32)-(9.33) is different. Equation (9.8) indicates the probability judgement combination in the combination space (X, Ω, μ) of n independent information sources, while equations (9.32)-(9.33) denotes the probability judgement combination in the mapping space $(S, 2^S, M)$ of n independent information sources. The mappings of Γ and Γ_i, $(i = 1, 2, \ldots, n)$ relate equations (9.8) and (9.32)-(9.33). This shows the difference between Dempster's rule and DS combination rule.

9.3.3 New explanations for the problems in DS combination rule

From the above derivation, it can be seen that DS combination rule is mathematically based on the product combination rule of multiple independent information sources as evident from equations (9.1)-(9.8). For each of the elements in the space, the combination probability judgement of independent information sources is the result of the simultaneous probability judgement of each independent information source. That is, if each information source yields simultaneously its probability judgement for the element, then the combination probability judgement for the element can be obtained by DS combination rule, regardless of the magnitude of the judgement probability of each information source. Otherwise, it is the opposite. This gives raise to the following problems:

1. *The counter-intuitive results*

   Suppose a frame of discernment is $S = \{s_1, s_2, s_3\}$, the probability judgments of two independent information sources, (X_i, Ω_i, μ_i), $i = 1, 2$, are m_1 and m_2, respectively. That is:

   $$(X_1, \Omega_1, \mu_1): \quad m_1(s_1) = 0.99, \quad m_1(s_2) = 0.01$$

   and

   $$(X_2, \Omega_2, \mu_2): \quad m_2(s_2) = 0.01, \quad m_2(s_3) = 0.99$$

   Using DS rule to combine the above two independent probability judgements, results in

   $$m(s_2) = 1, m(s_1) = m(s_3) = 0 \tag{9.34}$$

   This is counter-intuitive. The information source (X_1, Ω_1, μ_1) judges s_1 with a very large probability measure, 0.99, and judges s_2 with a very small probability measure, 0.01, while the information source (X_2, Ω_2, μ_2) judges s_3 with a very large probability measure, 0.99, and judges s_2 with a very small probability measure, 0.01. However, the result of DS combination rule is that s_2 occurs with probability measure, 1, and others occur with zero probability measure. The reason for this result is that the two information sources simultaneously give their judgement only for an element s_2 of space $S = \{s_1, s_2, s_3\}$ although the probability measures from the two information sources for the element are very small and equal to 0.01, respectively. The elements s_1 and s_3 are not judged by the two information sources simultaneously. According to the product combination rule, the result in equation (9.34) is as expected.

It should be pointed out that this counter-intuitive result is not completely due to the normalization factor in highly conflicting evidence [17, 18, 19] of DS combination rule. This can be proven by the following example.

Suppose for the above frame of discernment, the probability judgments of another two independent information sources, (X_i, Ω_i, μ_i), $i = 3, 4$, are m_2 and m_4, are chosen. That is:

$$(X_3, \Omega_3, \mu_3): \quad m_3(s_1) = 0.99, \quad m_3(S) = 0.01$$

and

$$(X_4, \Omega_4, \mu_4): \quad m_4(s_3) = 0.99, \quad m_4(S) = 0.01$$

The result of DS combination rule is

$$m'(s_1) = 0.4975, m'(s_3) = 0.4975, m'(S) = 0.0050$$

This result is very different from that in equation (9.34) although the independent probability judgements of the two information sources are also very conflicting for elements s_1 and s_3. That is, the information source, (X_3, Ω_3, μ_3), judges s_1 with a very large probability measure, 0.99, and judges S with a very small probability measure, 0.01, while the information source (X_4, Ω_4, μ_4) judges s_3 with a very large probability measure, 0.99, and judges S with a very small probability measure, 0.01.

This is due to the fact that the same element $S = \{s_1, s_2, s_3\}$ of the two information sources includes elements s_1 and s_3. So, the element s_1 in the information source, (X_3, Ω_3, μ_3), and the element $S = \{s_1, s_2, s_3\}$ in the information source, (X_4, Ω_4, μ_4) have the same information, and the element $S = \{s_1, s_2, s_3\}$ in information source, (X_3, Ω_3, μ_3), and the element s_3 in information source, (X_4, Ω_4, μ_4) have the same information. Thus, the two independent information sources can simultaneously give information for the same probability judgement element $S = \{s_1, s_2, s_3\}$, and also simultaneously yield the information for the conflicting elements s_1 and s_3, respectively. That is required by the product combination rule.

2. *The combination of Bayesian (sensitive) information sources*

If two Bayesian information sources cannot yield the information about any element of the frame of discernment simultaneously, then the two Bayesian information sources cannot be combined by DS combination rule. For example, there are two Bayesian information sources (X_1, Ω_1, μ_1)

and (X_2, Ω_2, μ_2) over the frame of discernment, $S = \{s_1, s_2, s_3, s_4\}$, and the basic probability assignments are, respectively,

$$(X_1, \Omega_1, \mu_1): \quad m_1(s_1) = 0.4, \quad m_1(s_2) = 0.6$$

and

$$(X_2, \Omega_2, \mu_2): \quad m_2(s_3) = 0.8, \quad m_2(s_4) = 0.2$$

then their DS combination rule is

$$m(s_1) = m(s_2) = m(s_3) = m(s_4) = 0$$

This indicates that every element of the frame of discernment occurs with zero basic probability after DS combination rule is applied. This is a conflict. This is because the source (X_1, Ω_1, μ_1) gives probability judgements for elements s_1 and s_2 of the frame of discernment, $S = \{s_1, s_2, s_3, s_4\}$, while the source (X_2, Ω_2, μ_2) gives probability judgements for elements s_3 and s_4 of the frame of discernment, $S = \{s_1, s_2, s_3, s_4\}$. The two sources cannot simultaneously give probability judgements for any element of the frame of discernment, $S = \{s_1, s_2, s_3, s_4\}$. Thus, the product combination rule does not work for this case.

Based on the above analysis, a possible solution to the problem is to relax the conditions used in the product combination rule (equations (9.1)-(9.4)) for practical applications, and establish a new theory for combining information of multiple sources (see sections 9.4 and 9.5).

9.3.4 Remark about "multi-valued mapping" in Shafer's paper

On page 331 of [11] where G. Shafer explains the concept of multi-valued mappings of DS combination rule, the Dempter's rule is considered as belief, $\text{Bel}(T) = P\{x|\Gamma(x) \subseteq T, \forall T \subset S\}$, combination. The following proof shows this is incorrect.

Proof: Given the two independent information sources, equations (9.1)-(9.4) become as the followings:

$$X = X_1 \times X_2 \tag{9.35}$$

$$\Omega = \Omega_1 \times \Omega_2 \tag{9.36}$$

$$\mu = \mu_1 \times \mu_2 \tag{9.37}$$

$$\Gamma x = \Gamma_1 x \cap \Gamma_2 x \tag{9.38}$$

then

$$\text{Bel}(T) \neq \text{Bel}_1(T) \oplus \text{Bel}_2(T)$$

in fact, $\forall T \subset S$,

$$\{\Gamma(x) \subseteq T\} \nRightarrow \{\Gamma(x_1) \subseteq T\} \cap \{\Gamma(x_2) \subseteq T\}$$

hence,

$$\{x \in X | \Gamma(x) \subseteq T\} \neq \{x_1 \in X_1 | \Gamma(x_1) \subseteq T\} \times \{x_2 \in X_2 | \Gamma(x_2) \subseteq T\}$$

i.e. the product combination rule in equations (9.35)-(9.38) is not satisfied by the defined belief $\text{Bel}(T) = P\{x|\Gamma(x) \subseteq T, \forall T \subset S\}$. Therefore, the combination belief cannot be obtained from equations (9.35)-(9.38) with the belief, $\text{Bel}(T) = P\{x|\Gamma(x) \subseteq T, \forall T \subset S\}$. When we examine the product combination rule in equations (9.1)-(9.4), it is known that the combination rule is neither for upper probabilities, nor for lower probabilities (belief), nor for probabilities of the type, $p_{\delta_1\delta_2\ldots\delta_n} = \mu(X_{\delta_1\delta_2\ldots\delta_n})$ [1]. It is simply for probability spaces of multiple independent information sources with multi-valued mappings.

9.4 A new combination rule of probability measures over mapping space

It has been demonstrated in section 9.3 that DS combination rule is mathematically based on the product combination rule of multiple independent information sources. The combination probability judgment of n independent information sources for each element is the result of the simultaneous probability judgment of each independent information source. That is, if each information source yields simultaneously its probability judgment for the element, then the combination probability judgment for the element can be obtained by DS combination rule regardless of the magnitude of the judgment probability of each information source. Otherwise, such results are not plausible. This is the main reason that led to the counter-intuitive results in [17, 18, 19]. We will redefine the combination-mapping rule Γ using n independent mapping Γ_i, $i = 1, 2, \ldots, n$ in order to relax the original definition in equation (9.4) in section 9.2.1. The combination of probabilities of type $p^{(i)}_{\delta_2\delta_1\ldots\delta_n}$ in the product space (X, Ω, μ) will then be realized, and, furthermore, the combination rule of multiple sources of information over mapping space S will also be established.

9.4.1 Derivation of combination rule of probabilities $p^{(i)}_{\delta_1\delta_2\ldots\delta_n}$

Define a new combination-mapping rule for multiple multi-valued mappings as

$$\Gamma x = \Gamma_1 x \cup \Gamma_2 x \cup \ldots \cup \Gamma_n x \tag{9.39}$$

It shows that $x_i \in X$ is consistent with a particular $s \in S$ if and only if $s \in \Gamma_i x_i$, for $i = 1, 2, \ldots, n$, and consequently $x = \{x_1, x_2, \ldots, x_n\} \in X$ is consistent with that s if and only if there exist certain $i \in \{1, 2, \ldots, n\}$, such that $s \in \Gamma_i x_i$.

For any $T \subset S$, we construct sets

$$\bar{T} = \{x \in X, \Gamma x \subset T\} \tag{9.40}$$

$$\bar{T}_i = \{x_i \in X_i, \Gamma_i x_i \subset T\} \tag{9.41}$$

and let

$$\lambda(T) = \mu(\bar{T}) \tag{9.42}$$

$$\lambda^{(i)}(T) = \mu_i(\bar{T}_i) \tag{9.43}$$

Hence,

$$\bar{T} = \bar{T}_1 \times \bar{T}_2 \times \ldots \times \bar{T}_n \tag{9.44}$$

and

$$\lambda(T) = \lambda^{(1)}(T) \times \lambda^{(2)}(T) \times \ldots \times \lambda^{(n)}(T) \tag{9.45}$$

Consider a finite $S = \{s_1, s_2, s_3\}$ and two independent sources of information characterized by $p^{(i)}_{000}$, $p^{(i)}_{100}$, $p^{(i)}_{010}$, $p^{(i)}_{001}$, $p^{(i)}_{110}$, $p^{(i)}_{101}$, $p^{(i)}_{011}$ and $p^{(i)}_{111}$, $i = 1, 2$. Suppose $\lambda^{(i)}(T)$, $(i = 1, 2)$ corresponding to $T = \emptyset$, $\{s_1\}$, $\{s_2\}$, $\{s_3\}$, $\{s_1, s_2\}$, $\{s_2, s_3\}$, $\{s_1, s_3\}$, $\{s_1, s_2, s_3\}$ is expressed as $\lambda^{(i)}_{000}$, $\lambda^{(i)}_{100}$, $\lambda^{(i)}_{010}$, $\lambda^{(i)}_{001}$, $\lambda^{(i)}_{110}$, $\lambda^{(i)}_{101}$, $\lambda^{(i)}_{011}$ and $\lambda^{(i)}_{111}$, $i = 1, 2$. Then for $i = 1, 2$,

$$\lambda^{(i)}_{000} = p^{(i)}_{000} \tag{9.46}$$

$$\lambda^{(i)}_{100} = p^{(i)}_{000} + p^{(i)}_{100} \tag{9.47}$$

$$\lambda^{(i)}_{010} = p^{(i)}_{000} + p^{(i)}_{010} \tag{9.48}$$

$$\lambda^{(i)}_{001} = p^{(i)}_{000} + p^{(i)}_{001} \tag{9.49}$$

$$\lambda^{(i)}_{110} = p^{(i)}_{000} + p^{(i)}_{100} + p^{(i)}_{010} + p^{(i)}_{110} \tag{9.50}$$

$$\lambda^{(i)}_{101} = p^{(i)}_{000} + p^{(i)}_{100} + p^{(i)}_{001} + p^{(i)}_{101} \tag{9.51}$$

$$\lambda^{(i)}_{011} = p^{(i)}_{000} + p^{(i)}_{010} + p^{(i)}_{001} + p^{(i)}_{011} \tag{9.52}$$

$$\lambda^{(i)}_{111} = p^{(i)}_{000} + p^{(i)}_{100} + p^{(i)}_{010} + p^{(i)}_{001} + p^{(i)}_{110} + p^{(i)}_{101} + p^{(i)}_{011} + p^{(i)}_{111} \tag{9.53}$$

If $\lambda_{\delta_1\delta_2\delta_3}$ and $p_{\delta_1\delta_2\delta_3}$ ($\delta_i = 1$ or 0, $i = 1, 2, 3$) are used to express the combined probability measure of two independent sources of information in spaces $S = \{s_1, s_2, s_3\}$ and (X, Ω, μ), respectively, then based on equation (9.45) and through equations (9.46)-(9.53), the following can be obtained

$$p_{000} = p^{(1)}_{000}p^{(2)}_{000} \tag{9.54}$$

$$p_{100} = p^{(1)}_{000}p^{(2)}_{100} + p^{(1)}_{100}p^{(2)}_{000} + p^{(1)}_{100}p^{(2)}_{100} \tag{9.55}$$

$$p_{010} = p^{(1)}_{000}p^{(2)}_{010} + p^{(1)}_{010}p^{(2)}_{000} + p^{(1)}_{010}p^{(2)}_{010} \tag{9.56}$$

$$p_{001} = p^{(1)}_{000}p^{(2)}_{001} + p^{(1)}_{001}p^{(2)}_{000} + p^{(1)}_{001}p^{(2)}_{001} \tag{9.57}$$

$$\begin{aligned} p_{110} = {} & p^{(1)}_{000}p^{(2)}_{110} + p^{(1)}_{100}p^{(2)}_{010} + p^{(1)}_{100}p^{(2)}_{110} + p^{(1)}_{010}p^{(2)}_{100} \\ & + p^{(1)}_{010}p^{(2)}_{110} + p^{(1)}_{110}p^{(2)}_{000} + p^{(1)}_{110}p^{(2)}_{100} + p^{(1)}_{110}p^{(2)}_{010} + p^{(1)}_{110}p^{(2)}_{110} \end{aligned} \tag{9.58}$$

$$\begin{aligned} p_{101} = {} & p^{(1)}_{000}p^{(2)}_{101} + p^{(1)}_{100}p^{(2)}_{001} + p^{(1)}_{100}p^{(2)}_{101} + p^{(1)}_{001}p^{(2)}_{100} \\ & + p^{(1)}_{001}p^{(2)}_{101} + p^{(1)}_{101}p^{(2)}_{000} + p^{(1)}_{101}p^{(2)}_{100} + p^{(1)}_{101}p^{(2)}_{001} + p^{(1)}_{101}p^{(2)}_{101} \end{aligned} \tag{9.59}$$

$$\begin{aligned} p_{011} = {} & p^{(1)}_{000}p^{(2)}_{011} + p^{(1)}_{010}p^{(2)}_{001} + p^{(1)}_{010}p^{(2)}_{011} + p^{(1)}_{001}p^{(2)}_{010} \\ & + p^{(1)}_{001}p^{(2)}_{011} + p^{(1)}_{011}p^{(2)}_{000} + p^{(1)}_{011}p^{(2)}_{010} + p^{(1)}_{011}p^{(2)}_{001} + p^{(1)}_{011}p^{(2)}_{011} \end{aligned} \tag{9.60}$$

$$\begin{aligned} p_{111} = {} & p^{(1)}_{000}p^{(2)}_{111} + p^{(1)}_{100}p^{(2)}_{011} + p^{(1)}_{100}p^{(2)}_{111} + p^{(1)}_{010}p^{(2)}_{101} \\ & + p^{(1)}_{010}p^{(2)}_{111} + p^{(1)}_{001}p^{(2)}_{110} + p^{(1)}_{001}p^{(2)}_{111} + p^{(1)}_{011}p^{(2)}_{100} + p^{(1)}_{011}p^{(2)}_{101} \\ & + p^{(1)}_{011}p^{(2)}_{110} + p^{(1)}_{011}p^{(2)}_{111} + p^{(1)}_{101}p^{(2)}_{010} + p^{(1)}_{101}p^{(2)}_{011} + p^{(1)}_{101}p^{(2)}_{110} \\ & + p^{(1)}_{101}p^{(2)}_{111} + p^{(1)}_{110}p^{(2)}_{001} + p^{(1)}_{110}p^{(2)}_{011} + p^{(1)}_{110}p^{(2)}_{101} + p^{(1)}_{110}p^{(2)}_{111} \\ & + p^{(1)}_{111}p^{(2)}_{000} + p^{(1)}_{111}p^{(2)}_{100} + p^{(1)}_{111}p^{(2)}_{010} + p^{(1)}_{111}p^{(2)}_{001} + p^{(1)}_{111}p^{(2)}_{011} \\ & + p^{(1)}_{111}p^{(2)}_{101} + p^{(1)}_{111}p^{(2)}_{110} + p^{(1)}_{111}p^{(2)}_{111} \end{aligned} \tag{9.61}$$

For the case of $S = \{s_1, s_2, \ldots, s_n\}$, the general combination rule is

$$p_{\delta_1\delta_2\ldots\delta_n} = \sum_{\substack{\delta_i=\delta'_i\cup\delta''_i \\ i=1,2,\ldots,m}} p^{(1)}_{\delta'_1\delta'_2\ldots\delta'_n} p^{(2)}_{\delta''_1\delta''_2\ldots\delta''_n} \tag{9.62}$$

for all $(\delta'_1, \delta'_2, \ldots, \delta'_m, \delta''_1, \delta''_2, \ldots, \delta''_n)$.

9.4.2 Combination rule of probability measures in space S

Define a probability measure over $2^S = \{S_{\delta_1\delta_2\ldots\delta_n}\}$ as $M : 2^S = \{S_{\delta_1\delta_2\ldots\delta_m}\} \rightarrow [0, 1]$ with

$$M(S_{\delta_1\delta_2\ldots\delta_n}) = \begin{cases} 0, & S_{\delta_1\delta_2\ldots\delta_n} = S_{00\ldots0} \\ \frac{\mu(X_{\delta_1\delta_2\ldots\delta_n})}{1-\mu(X_{00\ldots0})}, & S_{\delta_1\delta_2\ldots\delta_n} \neq S_{00\ldots\delta_0} \end{cases} \tag{9.63}$$

where M is the probability measure over a class $2^S = \{S_{\delta_1\delta_2\ldots\delta_n}\}$ of subsets of space S and Γ maps X into S.

The combination rule:

Given two independent sources of information (X_i, Ω_i, μ_i), $i = 1, 2$, and the corresponding mapping space, $S = \{s_1, s_2, \ldots, s_n\} = \{S_{\delta_1\delta_2\ldots\delta_n}\}$, where Γ_i maps X_i into S. Based on equation (9.62), we have

$$\mu(X_{\delta_1\delta_2\ldots\delta_n}) = \sum_{\substack{\delta_i=\delta'_i\cup\delta''_i\\ i=1,2,\ldots,n}} \mu^{(1)}(X^{(1)}_{\delta'_1\delta'_2\ldots\delta'_n})\mu^{(2)}(X^{(2)}_{\delta''_1\delta''_2\ldots\delta''_n}) \tag{9.64}$$

From equation (9.63), for any $S_{\delta_1\delta_2\ldots\delta_m} \neq S_{00\ldots0}$, there exists

$$\mu^{(1)}(X^{(1)}_{\delta'_1\delta'_2\ldots\delta'_n}) = M^{(1)}(S_{\delta_1\delta_2\ldots\delta_n})(1-\mu^{(1)}(X^{(1)}_{00\ldots0})) \tag{9.65}$$

$$\mu^{(2)}(X^{(2)}_{\delta'_1\delta''_2\ldots\delta''_n}) = M^{(2)}(S_{\delta_1\delta_2\ldots\delta_n})(1-\mu^{(2)}(X^{(2)}_{00\ldots0})) \tag{9.66}$$

and

$$\mu(X_{\delta_1\delta_2\ldots\delta_n}) = M(S_{\delta_1\delta_2\ldots\delta_n})(1-\mu(X_{00\ldots0})) \tag{9.67}$$

such that equation (9.64) becomes

$$M(S_{\delta_1\delta_2\ldots\delta_n}) = \frac{\displaystyle\sum_{\substack{\delta_i=\delta'_i\cup\delta''_i\\ i=1,2,\ldots,n}} M^{(1)}(S_{\delta_1\delta_2\ldots\delta_n})M^{(2)}(S_{\delta_1\delta_2\ldots\delta_n})[1-\mu^{(1)}(X^{(1)}_{00\ldots0})[1-\mu^{(2)}(X^{(2)}_{00\ldots0})]}{1-\mu(X_{00\ldots0})} \tag{9.68}$$

and

$$\frac{[1-\mu^{(1)}(X^{(1)}_{00\ldots0})][1-\mu^{(2)}(X^{(2)}_{00\ldots0})]}{1-\mu(X_{00\ldots0})} = \frac{1}{\displaystyle\sum_{\substack{\delta'_i\cup\delta''_i\neq0\\ i=1,2,\ldots,n}} M^{(1)}(S_{\delta'_1\delta'_2\ldots\delta'_n})M^{(2)}(S_{\delta''_1\delta''_2\ldots\delta''_n})} \tag{9.69}$$

Substitute (9.69) into (9.68),

$$\begin{aligned} M(S_{\delta_1\delta_2\ldots\delta_n}) &= \frac{\displaystyle\sum_{\substack{\delta_i=\delta'_i\cup\delta''_i\\ i=1,2,\ldots,n}} M^{(1)}(S_{\delta'_1\delta'_2\ldots\delta'_n})M^{(2)}(S_{\delta''_1\delta''_2\ldots\delta''_n})}{1-\displaystyle\sum_{\substack{\delta'_i\cup\delta''_i\neq0\\ i=1,2,\ldots,n}} M^{(1)}(S_{\delta'_1\delta'_2\ldots\delta'_n})M^{(2)}(S_{\delta''_1\delta''_2\ldots\delta''_n})} \\ &= \sum_{\substack{\delta_i=\delta'_i\cup\delta''_i\\ i=1,2,\ldots,n}} M^{(1)}(S_{\delta'_1\delta'_2\ldots\delta'_n})M^{(2)}(S_{\delta''_1\delta''_2\ldots\delta''_n}) \end{aligned} \tag{9.70}$$

If $S_{\delta_1\delta_2\ldots\delta_n} = S_{00\ldots0}$, we define

$$M(S_{\delta_1\delta_2\ldots\delta_n}) \triangleq 0 \tag{9.71}$$

Hence, equations (9.70)- (9.71) express the combination of two sources of information, (X_i, Ω_i, μ_i), $i = 1, 2$, for the mapping space, $S = \{s_1, s_2, \ldots, s_n\} = S_{\delta_1\delta_2\ldots\delta_n}$, where Γ_i maps X_i into S.

9.5 The disjunctive combination rule

Based on the results in section 9.4, the disjunctive combination rule for two independent sources of information is obtained as follows:

Theorem 4:

Suppose $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$ is a frame of discernment with n elements. The basic probability assignments of the two sources of information, (X_1, Ω_1, μ_2) and (X_2, Ω_2, μ_2) over the same frame of discernment are m_1 and m_2, and focal elements A_1, A_2, …, A_k and B_1, B_2, …, B_l, respectively. Then the combined basic probability assignment of the two sources of information can be defined as

$$m(C) = \begin{cases} 0, & C = \emptyset \\ \displaystyle\sum_{C = A_i \cup B_j} m_1(A_i) m_2(B_j), & C \neq \emptyset \end{cases} \tag{9.72}$$

Proof: Since $m(\emptyset) = 0$ by definition, m is a basic probability assignment provided only that the $m(C)$ sum to one. In fact,

$$\begin{aligned}
\sum_{C \subseteq \Theta} m(C) &= m(\emptyset) + \sum_{\substack{C \subset \Theta \\ C \neq \emptyset}} m(C) \\
&= \sum_{\substack{C \subset \Theta \\ C \neq \emptyset}} \sum_{\substack{C = A_i \cup B_j \\ i \in \{1,2,\ldots,k\}, j \in \{1,2,\ldots,l\}}} m_1(A_i) m_2(B_j) \\
&= \sum_{\substack{A_i \cup B_j \neq \emptyset \\ i \in \{1,2,\ldots,k\}, j \in \{1,2,\ldots,l\}}} m_1(A_i) m_2(B_j) \\
&= \sum_{\substack{A_i \subset \Theta \\ A_i \neq \emptyset}} m_1(A_i) \sum_{\substack{B_j \subset \Theta \\ B_j \neq \emptyset}} m_2(B_j)
\end{aligned}$$

Hence, m is a basic probability assignment over the frame of discernment $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$. Its focal elements are

$$C = \Big(\bigcup_{i=1,2,\ldots,k} A_i\Big) \bigcup \Big(\bigcup_{j=1,2,\ldots,l} B_l\Big)$$

Based on theorem 4, theorem 5 can be stated as follows. A similar result can be found in [29, 31].

Theorem 5:

If Bel_1 and Bel_2 are belief functions over the same frame of discernment $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$ with basic probability assignments m_1 and m_2, and focal elements A_1, A_2, …, A_k and B_1, B_2, …, B_l, respectively, then the function $m : 2^\Theta \to [0, 1]$ defined as

$$m(C) = \begin{cases} 0, & C = \emptyset \\ \displaystyle\sum_{C = A_i \cup B_j} m_1(A_i) m_2(B_j), & C \neq \emptyset \end{cases} \tag{9.73}$$

yields a basic probability assignment. The core of the belief function given by m is equal to the union of the cores of Bel_1 and Bel_2.

Physical interpretations of the combination rule for two independent sources of information are:

1. The combination rule in theorem 4 indicates a type of logical inference in human judgments, namely: the OR rule. That is, for a given frame of discernment, the elements that are simultaneously judged by each source of information will also be judgment elements of the combined source of information; otherwise, it will result in uncertainty so the combination judgments of the elements will be ignorance.

2. The essential difference between the new combination rule and DS combination rule is that the latter is a type of logical inference with AND or conjunction, while the former is based on OR or disjunction. The new combination rule (or the OR rule) overcomes the shortcomings of DS combination rule with AND, such as in the counter-intuitive situation and in the combination of sharp sources of information.

3. The judgment with OR has the advantage over that with AND in treating elements that are not simultaneously judged by each independent source of information. The OR rule gives more plausible judgments for these elements than the AND rule. The judgment better fits to the logical judgment of human beings.

Example 1

Given the frame of discernment $\Theta = \{\theta_1, \theta_2\}$, the judgments of the basic probability from two sources of information are m_1 and m_2 as follows:

$$m_1(\theta_1) = 0.2, \quad m_1(\theta_2) = 0.4, \quad m_1(\theta_1, \theta_2) = 0.4$$

$$m_2(\theta_1) = 0.4, \quad m_2(\theta_2) = 0.4, \quad m_2(\theta_1, \theta_2) = 0.2$$

Then through theorem 4, the combination judgment is

$$m(\theta_1) = 0.08, \quad m(\theta_2) = 0.16, \quad m(\theta_1, \theta_2) = 0.76$$

Comparing the combined basic probabilities of θ_1 and θ_2, the judgment of θ_2 occurs more often than θ_1, but the whole combination doesn't decrease the uncertainty of the judgments, which is evident from the above results.

Example 2 (the counter-intuitive situation)

Zadeh's example:

The frame of discernment about the patient is $\Theta = \{M, C, T\}$where M denotes meningitis, C represents contusion and T indicates tumor. The judgments of two doctors about the patient are

$$m_1(M) = 0.99, \quad m_1(T) = 0.01$$

$$m_2(C) = 0.99, \quad m_2(T) = 0.01$$

Combining these judgments through theorem 4, results in

$$m(M \cup C) = 0.9801, \quad m(M \cup T) = 0.0099, \quad m(C \cup T) = 0.0099, \quad m(T) = 0.0001$$

From $m(M \cup T) = 0.0099$ and $m(C \cup T) = 0.0099$, it is clear that there are less uncertainties between T and M, as well as T and C; which implies that T can easily be distinguished from M and C. Also, T occurs with the basic probability $m(T) = 0.0001$, i.e. T probably will not occur in the patient. The patient may be infected with M or C. Furthermore, because of $m(M \cup C) = 0.9801$, there is a bigger uncertainty with 0.9801 between M and C, so the two doctors cannot guarantee that the patient has meningitis (M) or contusion (C) except that the patient has no tumor (T). The patient needs to be examined by more doctors to assure the diagnoses.

We see the disjunctive combination rule can be used to this case very well. It fits to the human intuitive judgment.

9.6 Properties of conjunctive and disjunctive combination rules

In the section, the conjunctive and disjunctive combination rules, namely, Dempster-Shafer's combination rule, Yager's combination rule, Dubois and Prade's (DP) combination rule, DSm's combination rule and the disjunctive combination rule, are studied. The properties of each combination rule of evidence are discussed in detail, such as the role of evidence of each source of information in the combination judgment, the comparison of the combination judgment belief and ignorance of each combination rule, the treatment of conflict judgments given by the two sources of information, and the applications of combination rules. Zadeh's example is included in this section to evaluate the performance as well as efficiency of each combination rule of evidence for the conflict judgments given by the two sources of information.

9.6.1 The combination rules of evidence

9.6.1.1 Yager's combination rule of evidence

Suppose Bel_1 and Bel_2 are belief functions over the same frame of discernment $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$ with basic probability assignments m_1 and m_2 , and focal elements A_1, A_2, …, A_k and B_1, B_2, …, B_l, respectively. Then Yager's combined basic probability assignment of the two sources of information can be defined as [20]

$$m_Y(C) = \begin{cases} \displaystyle\sum_{\substack{i,j \\ C=A_i\cap B_j}} m_1(A_i)m_2(B_j), & C \neq \Theta, \emptyset \\ m_1(\Theta)m_2(\Theta) + \displaystyle\sum_{\substack{i,j \\ A_i\cap B_j=\emptyset}} m_1(A_i)m_2(B_j), & C = \Theta \\ 0, & C = \emptyset \end{cases} \tag{9.74}$$

9.6.1.2 Dubois & Prade (DP)'s combination rule of evidence

Given the same conditions as in Yager's combination rule, Dubois and Prade's combined basic probability assignment of the two sources of information can be defined as [26]

$$m_{DP}(C) = \begin{cases} \displaystyle\sum_{\substack{i,j \\ C=A_i\cap B_j}} m_1(A_i)m_2(B_j) + \sum_{\substack{i,j \\ C=A_i\cup B_j \\ A_i\cap B_j=\emptyset}} m_1(A_i)m_2(B_j), & C \neq \emptyset \\ 0, & C = \emptyset \end{cases} \tag{9.75}$$

9.6.1.3 DSm combination rules of evidence

These rules are presented in details in chapters 1 and 4 and are just recalled briefly here for convenience for the two independent sources of information.

- *The classical DSm combination rule for free DSm model* [27]

$$\forall C \in D^\Theta, \qquad m(C) = \sum_{\substack{A,B\in D^\Theta \\ A\cap B=C}} m_1(A)m_2(B) \tag{9.76}$$

  where D^Θ denotes the hyper-power set of the frame Θ (see chapters 2 and 3 for details).

- *The general DSm combination rule for hybrid DSm model $\mathcal{M}$*

  We consider here only the two sources combination rule.

$$\forall A \in D^\Theta, \qquad m_{\mathcal{M}(\Theta)}(A) \triangleq \phi(A)\Big[S_1(A) + S_2(A) + S_3(A)\Big] \tag{9.77}$$

where $\phi(A)$ is the characteristic non emptiness function of a set A, i.e. $\phi(A) = 1$ if $A \notin \boldsymbol{\emptyset}$ and $\phi(A) = 0$ otherwise, where $\boldsymbol{\emptyset} \triangleq \{\boldsymbol{\emptyset}_{\mathcal{M}}, \emptyset\}$. $\boldsymbol{\emptyset}_{\mathcal{M}}$ is the set of all elements of D^Θ which have been forced to be empty through the constraints of the model $\mathcal{M}$ and $\emptyset$ is the classical/universal empty set. $S_1(A) \equiv m_{\mathcal{M}^f(\Theta)}(A)$, $S_2(A)$, $S_3(A)$ are defined by (see chapter 4)

$$S_1(A) \triangleq \sum_{\substack{X_1,X_2 \in D^\Theta \\ X_1 \cap X_2 = A}} \prod_{i=1}^{2} m_i(X_i) \tag{9.78}$$

$$S_2(A) \triangleq \sum_{\substack{X_1,X_2 \in \boldsymbol{\emptyset} \\ [\mathcal{U}=A] \vee [(\mathcal{U} \in \boldsymbol{\emptyset}) \wedge (A = I_t)]}} \prod_{i=1}^{2} m_i(X_i) \tag{9.79}$$

$$S_3(A) \triangleq \sum_{\substack{X_1,X_2 \in D^\Theta \\ X_1 \cup X_2 = A \\ X_1 \cap X_2 \in \boldsymbol{\emptyset}}} \prod_{i=1}^{2} m_i(X_i) \tag{9.80}$$

with $\mathcal{U} \triangleq u(X_1) \cup u(X_2)$ where $u(X)$ is the union of all singletons θ_i that compose X and $I_t \triangleq \theta_1 \cup \theta_2$ is the total ignorance. $S_1(A)$ corresponds to the classic DSm rule of combination based on the free DSm model; $S_2(A)$ represents the mass of all relatively and absolutely empty sets which is transferred to the total or relative ignorances; $S_3(A)$ transfers the sum of relatively empty sets to the non-empty sets.

9.6.1.4 The disjunctive combination rule of evidence

This rule has been presented and justified previously in this chapter and can be found also in [22, 23, 29, 30, 31].

Suppose $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$ is a frame of discernment with n elements (it is the same as in theorem 3). The basic probability assignments of the two sources of information over the same frame of discernment are m_1 and m_2, and focal elements A_1, A_2, ..., A_k and B_1, B_2, ..., B_l, respectively. Then the combined basic probability assignment of the two sources of information can be defined as

$$m_{Dis}(C) = \begin{cases} \sum\limits_{\substack{i,j \\ C = A_i \cup B_j}} m_1(A_i) m_2(B_j), & C \neq \emptyset \\ 0, & C = \emptyset \end{cases} \tag{9.81}$$

for any $C \subset \Theta$. The core of the belief function given by m is equal to the union of the cores of Bel_1 and Bel_2.

9.6.2 Properties of combination rules of evidence

Given two independent sources of information defined over the frame of discernment $\Theta = \{\theta_1, \theta_2\}$, their basic probability assignments or basic belief masses over Θ are

$$S1: \quad m_1(\theta_1) = 0.4, \quad m_1(\theta_2) = 0.3, \quad m_1(\theta_1 \cup \theta_2) = 0.3$$

$$S2: \quad m_2(\theta_1) = 0.5, \quad m_2(\theta_2) = 0.3, \quad m_2(\theta_1 \cup \theta_2) = 0.2$$

Then the results of each combination rule of evidence for the two independent sources of information are as follows. For the frame of discernment with n elements, similar results can be obtained.

$S2\ (m_2)\ \setminus\ S1\ (m_1)$	$\{\theta_1\}$ (0.4)	$\{\theta_2\}$ (0.3)	$\{\theta_1, \theta_2\}$ (0.3)
$\{\theta_1\}$ (0.5)	$\{\theta_1\}$ (0.2)	$\{\theta_1\} \cap \{\theta_2\} \Rightarrow k$ (0.15)	$\{\theta_1\}$ (0.15)
$\{\theta_2\}$ (0.3)	$\{\theta_1\} \cap \{\theta_2\} \Rightarrow k$ (0.12)	$\{\theta_2\}$ (0.09)	$\{\theta_2\}$ (0.09)
$\{\theta_1, \theta_2\}$ (0.2)	$\{\theta_1\}$ (0.08)	$\{\theta_2\}$ (0.06)	$\{\theta_1, \theta_2\}$ (0.06)

Table 9.1: The conjunctive combination of evidence (DS)

$S2\ (m_2)\ \setminus\ S1\ (m_1)$	$\{\theta_1\}$ (0.4)	$\{\theta_2\}$ (0.3)	$\{\theta_1, \theta_2\}$ (0.3)
$\{\theta_1\}$ (0.5)	$\{\theta_1\}$ (0.2)	$\{\theta_1\} \cap \{\theta_2\} \Rightarrow \Theta$ (0.15)	$\{\theta_1\}$ (0.15)
$\{\theta_2\}$ (0.3)	$\{\theta_1\} \cap \{\theta_2\} \Rightarrow \Theta$ (0.12)	$\{\theta_2\}$ (0.09)	$\{\theta_2\}$ (0.09)
$\{\theta_1, \theta_2\}$ (0.2)	$\{\theta_1\}$ (0.08)	$\{\theta_2\}$ (0.06)	$\{\theta_1, \theta_2\}$ (0.06)

Table 9.2: The conjunctive and disjunctive combination of evidence (Yager)

$S2\ (m_2)\ \setminus\ S1\ (m_1)$	$\{\theta_1\}$ (0.4)	$\{\theta_2\}$ (0.3)	$\{\theta_1, \theta_2\}$ (0.3)
$\{\theta_1\}$ (0.5)	$\{\theta_1\}$ (0.2)	$\{\theta_1\} \cap \{\theta_2\} \Rightarrow \{\theta_1\} \cup \{\theta_2\}$ (0.15)	$\{\theta_1, \theta_2\}$ (0.15)
$\{\theta_2\}$ (0.3)	$\{\theta_1\} \cap \{\theta_2\} \Rightarrow \{\theta_1\} \cup \{\theta_2\}$ (0.12)	$\{\theta_2\}$ (0.09)	$\{\theta_2\}$ (0.09)
$\{\theta_1, \theta_2\}$ (0.2)	$\{\theta_1\}$ (0.08)	$\{\theta_2\}$ (0.06)	$\{\theta_1, \theta_2\}$ (0.06)

Table 9.3: The conjunctive and disjunctive combination of evidence (Dubois-Prade)

Property 1: the role of evidence of each source of information in the combination judgment:

1. With DS combination rule of evidence [2], the combined judgment for element θ_i $(i = 1, 2)$ consists of two parts. One is from the simultaneous support judgment of two sources of information for the element θ_i $(i = 1, 2)$ and the other is that one of two sources of information yields a support judgment, while the second source is ignorant for the element θ_i $(i = 1, 2)$ (i.e. ignorance). The

$A \in D^\Theta$	m_1	m_2	$\Phi(A)$	$S_1(A)$	$S_2(A)$	$S_3(A)$	$m_{\mathcal{M}(\Theta)}(A)$
$\emptyset$	0	0	0	0	0	0	0
$\{\theta_1\}$	0.4	0.5	1	0.43	0	0	0.43
$\{\theta_2\}$	0.3	0.3	1	0.24	0	0	0.24
$\{\theta_1 \cap \theta_2\} \overset{\mathcal{M}(\Theta)}{=} \emptyset$	0	0	0	0.27	0	0	0
$\{\theta_1 \cup \theta_2\}$	0.3	0.2	1	0.06	0	0.27	0.33

Table 9.4: The hybrid DSm combination of evidence

$S2\ (m_2) \ \setminus \ S1\ (m_1)$	$\{\theta_1\}$ (0.4)	$\{\theta_2\}$ (0.3)	$\{\theta_1, \theta_2\}$ (0.3)
$\{\theta_1\}$ (0.5)	$\{\theta_1\}$ (0.2)	$\{\theta_1\} \cup \{\theta_2\}$ (0.15)	$\{\theta_1, \theta_2\}$ (0.15)
$\{\theta_2\}$ (0.3)	$\{\theta_1\} \cup \{\theta_2\}$ (0.12)	$\{\theta_2\}$ (0.09)	$\{\theta_1, \theta_2\}$ (0.09)
$\{\theta_1, \theta_2\}$ (0.2)	$\{\theta_1, \theta_2\}$ (0.08)	$\{\theta_1, \theta_2\}$ (0.06)	$\{\theta_1, \theta_2\}$ (0.06)

Table 9.5: The disjunctive combination of evidence

combined total ignorance is from the total ignorance of both sources of information. The failure combination judgment for some element is from the conflict judgments given by two sources of information for the element.

2. The difference between Yager's combination rule of evidence [20] and DS combination rule of evidence [2] is that the conflict judgments of combination given by two sources of information for some element is considered to be a part of combined ignorance i.e. it is added into the total ignorance.

3. Dubois and Prade's combination rule of evidence [26] is different from that of Yager's combination rule [20] in that when two sources of information give the conflict judgments for an element in the frame of discernment, one of two judgments is at least thought as a reasonable judgment. The conflict judgments of combination for the two conflict elements are distributed to the judgment corresponding to union of the two conflict elements.

4. The classical DSm combination rule of evidence [27] is different from those of Dubois and Prade's [26], Yager's [20] and DS [2]. The conflict judgments given by two sources of information for an element in the frame of discernment are considered as paradox. These paradoxes finally support the combination judgment of each element θ_i $(i = 1, 2)$. For the hybrid DSm combination rule, see chapter 4, it consists of three parts. The first one is from the classic DSm rule of combination based on the free-DSm model;the second one is the mass of all relatively and absolutely empty sets which are transferred to the total or relative ignorance, while the third one is the mass that transfers the all relatively empty sets to union of the elements that are included in the sets.

5. With the disjunctive combination rule of evidence [22, 23, 29, 30, 31], the combination judgment for each element is only from the simultaneous support judgment of each source of information for the element θ_i ($i = 1, 2$). The combined ignorance consists of the combination of conflict judgments given by two sources of information, the combination of the ignorance given by one source of information and the support judgment for any element given by another source, and the combination of the ignorance from both sources of information simultaneously. There is no failure combination judgment. However, the combined belief is decreased and the ignorance is increased.

6. The combination rules of evidence of DS and the classical DSm are the conjunctive rule, the disjunctive combination rule of evidence is the disjunctive rule, while the combination rule of evidence of Yager, Dubois & Prade, and the hybrid DSm are hybrid of the conjunctive and disjunctive rules.

Property 2: the comparison of combination judgment belief (Bel(.)) and ignorance (Ign(.) = Pl(.) − Bel(.)) of each combination rule is:

$$\text{Bel}_{DS}(\theta_i) > \text{Bel}_{DSm}(\theta_i) > \text{Bel}_{DP}(\theta_i) > \text{Bel}_{Y}(\theta_i) > \text{Bel}_{Dis}(\theta_i), \quad i = 1, 2 \tag{9.82}$$

$$\text{Ign}_{DS}(\theta_i) < \text{Ign}_{DSm}(\theta_i) > \text{Ign}_{DP}(\theta_i) < \text{Ign}_{Y}(\theta_i) < \text{Ign}_{Dis}(\theta_i), \quad i = 1, 2 \tag{9.83}$$

In fact, for the above two sources of information, the results from each combination rule are as the following:

Combination rule	$m(\theta_1)$	$m(\theta_2)$	$m(\Theta)$	Bel(θ_1)	Bel(θ_2)	Bel(Θ)	Ign(θ_1)	Ign(θ_2)
DS	0.589	0.329	0.082	0.589	0.329	1	0.082	0.082
Yager	0.43	0.24	0.33	0.43	0.24	1	0.33	0.33
DP	0.43	0.24	0.33	0.43	0.24	1	0.33	0.33
Hybrid DSm	0.43	0.24	0.33	0.43	0.24	1	0.33	0.33
Disjunctive	0.20	0.09	0.71	0.20	0.09	1	0.71	0.71

From the results in the above table, it can be observed that the hybrid DSm's, Yager's and Dubois & Prade's combination judgments are identical for the two independent sources of information. However, for more than two independent sources of information, the results of combination judgments are as in equations (9.82) and (9.83) (i.e. the results are different, the hybrid DSm model is more general than Dubois-Prade's and Yager's, while Dubois-Prade's model has less total ignorance than Yager's).

Property 3: The conflict judgments given by two sources of information for the frame of discernment:

Under DS combination rule, the combined conflict judgments are thought as failures and are deducted from the total basic probability assignment of combination, while under Yager's combination rule, they are thought as the total ignorance; under Dubois & Prade's combination rule; they are distributed to the union of the two conflict elements. That means one of conflict judgments is at least reasonable. Under the classical DSm combination rule, they constitute paradoxes to support the combined judgment belief of each element, and are also thought as a new event that takes part in the subsequent judgment when new evidences occur. While for the hybrid DSm combination rule, the treatment of conflict evidence is similar to Dubois & Prade's approach. For the disjunctive combination rule, the conflict judgments of combination constitute ignorance, and take part in the subsequent judgment when the new evidences occur.

Property 4: using them in applications:

Based on properties 1-3, when the two independent sources of information are not very conflict, the disjunctive combination rule is more conservative combination rule. The combined results are uncertain when conflict judgments of two sources of information occur and hence the final judgment is delayed until more evidence comes into the judgment systems. Also, the combined judgment belief for each element in the frame of discernment is decreased, and ignorance is increased as the new evidences come. Hence, the disjunctive combination rule is not more efficient when we want the ignorance be decreased in the combination of evidence. It is fair to assume that for the case when the two (conflict) judgments are not exactly known which one is more reasonable, however, at least one of them should provide a reasonable judgment. But DS combination rule is contrary to the disjunctive combination rule. It can make the final judgment faster than other rules (see equations (9.82)-(9.83)), but the disjunctive combination rule will make less erroneous judgments than other rules. The cases for the combination rules of the hybrid DSm, Dubois & Prade, and Yager's combination rule fall between the above two. For the other properties, for instance, the two conflict independent sources of information, see the next section and the example that follows.

9.6.3 Example

In this section, we examine the efficiency of each combination rule for conflict judgments via Zadeh's famous example. Let the frame of discernment of a patient be $\Theta = \{M, C, T\}$ where M denotes meningitis, C represents contusion and T indicates tumor. The judgments of two doctors about the patient are

$$m_1(M) = 0.99, m_1(T) = 0.01 \qquad \text{and} \qquad m_2(C) = 0.99, m_2(T) = 0.01$$

The results from each combination rule of evidence are:

Rules	$m(T)$	$m(M \cup C)$	$m(C \cup T)$	$m(M \cup T)$	$m(\Theta)$
DS	1	0	0	0	0
Yager	0.0001	0	0	0	0.9999
DP	0.0001	0.9801	0.0099	0.0099	0
Hybrid DSm	0.0001	0.9801	0.0099	0.0099	0
Disjunctive	0.0001	0.9801	0.0099	0.0099	0

The basic belief masses $m(M \cap C)$, $m(C \cap T)$ and $m(M \cap T)$ equal zero with all five rules of combination and the belief of propositions $M \cap C$, $C \cap T$, $M \cap T$, $M \cup C$, $C \cup T$, $M \cup T$, M, C, T and $M \cup C \cup T$ are given in the next tables:

Rules	Bel$(M \cap C)$	Bel$(C \cap T)$	Bel$(M \cap T)$	Bel$(M \cup C)$	Bel$(C \cup T)$	Bel$(M \cup T)$
DS	0	0	0	0	0	0
Yager	0	0	0	0	0	0
DP	0	0	0	0.9801	0.01	0.01
Hybrid DSm	0	0	0	0.9801	0.01	0.01
Disjunctive	0	0	0	0.9801	0.01	0.01

Rules	Bel(M)	Bel(C)	Bel(T)	Bel$(M \cup C \cup T)$
DS	0	0	1	1
Yager	0	0	0.0001	1
DP	0	0	0.0001	1
Hybrid DSm	0	0	0.0001	1
Disjunctive	0	0	0.0001	1

Comparison and analysis of the fusion results:

1. DS combination judgment belief of each element is:

$$\text{Bel}_{DS}(T) = 1, \qquad \text{Bel}_{DS}(M) = \text{Bel}_{DS}(C) = 0$$

It means that the patient must have disease T with a degree of belief of 1 and must not have diseases M and C, because their degrees of belief are 0, respectively. It is a counter-intuitive situation with $\text{Bel}_{DS,1}(M) = \text{Bel}_{DS,2}(C) = 0.99$, $\text{Bel}_{DS,1}(T) = \text{Bel}_{DS,2}(T) = 0.01$. Moreover, in spite of the basic probability assignment values over diseases T, M and C, the judgment of the two doctors for DS combination rule will always be T with the degree of belief of 1, and each M and C with degree of belief of 0. It shows DS combination rule is not effective in this case. The main reason for this situation has been presented in sections 9.3-9.5.

2. Yager's combination judgment belief of each element is:

$$\text{Bel}_Y(T) = 0.0001, \qquad \text{Bel}_Y(M) = \text{Bel}_Y(C) = 0$$

This degree of belief is too small to make the final judgment. Therefore, Yager's combination rule of evidence will wait for the new evidence to come in order to obtain more accurate judgment. The reason for this result is that the rule transforms all conflict judgments into the total ignorance.

3. For Dubois & Prade's combination rule, there is

$$\text{Bel}_{DP}(T) = 0.0001, \qquad \text{Bel}_{DP}(M \cup C) = 0.9801, \qquad \text{Bel}_{DP}(M \cup T) = \text{Bel}_{DP}(C \cup T) = 0.01$$

This result is the same as that of the disjunctive combination rule and the hybrid DSm combination rule. With a belief of T, $\text{Bel}_{DP}(T) = 0.0001$, we can judge that the patient having disease T is less probable event. Furthermore, $\text{Bel}_{DP}(M \cup T) = \text{Bel}_{DP}(C \cup T) = 0.01$, hence the patient may have disease M or C. Also, $\text{Bel}_{DP}(M \cup C) = 0.9801$, this further substantiates the fact that the patient has either M or C, or both. For the final judgment, one needs the new evidence or diagnosis by the third doctor.

Based on the judgments of two doctors, the different judgment results of each combination rules are clearly demonstrated. For this case, the results from Dubois & Prade's rulr, the hybrid DSm rule and from the disjunctive combination rule are more suitable to human intuitive judgment; the result from Yager's combination rule, can't make the final judgment immediately because of less degree of judgment belief and more ignorance, while the results of DS combination rule is counter-intuitive. These results demonstrate the efficiency of each combination rule for the conflict judgments given by two sources of information for the element in the frame of discernment.

9.7 Conclusion

In this chapter, DS combination rule is examined based on multi-valued mappings of independent information sources and the product combination rule of multiple independent information sources. It is obtained that Dempster's rule is different from DS combination rule and shortcomings in DS combination rule are due to the result of the product combination rule. The drawback in the explanation of multi-valued mappings when applied to Dempster's rule were pointed out and proven. Furthermore, based on these results, a novel justification of the disjunctive combination rule for two independent sources of information based on the redefined combination-mapping rule of multiple multi-valued mappings in the product space of multiple sources of information mappings has been proposed. The combination rule depends on the logical judgment of OR. It overcomes the shortcomings of Dempster-Shafer's combination rule, especially, in resolving the counter-intuitive situation. Finally, the conjunctive and disjunctive combination rules of evidence, namely, Dempster-Shafer's (DS) combination rule, Yager's combination rule, Dubois & Prade's (DP) combination rule, DSm's combination rule and the disjunctive combination rule, are studied for the two independent sources of information. The properties of each combination rule of evidence are discussed in detail, such as the role of evidence of each source of information in the combination judgment, the comparison of the combination judgment belief and ignorance of each combination rule, the treatment of conflict judgments given by the two sources of information, and the applications of combination rules. The new results yield valuable theoretical insight into the rules that can be applied to a given situation. Zadeh's typical example is included in this chapter to evaluate the performance as well as efficiency of each combination rule of evidence for the conflict judgments given by the two sources of information.

9.8 References

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