NSS

Neutrosophic Sets and Systems, Vol. 59, 2023

University of New Mexico

Solution of System of Symbolic 2-Plithogenic Linear Equations using Cramer's Rule

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Abstract. In this article, the concept of system of symbolic 2-plithogenic linear equations and its solutions are introduced and studied. The Cramer's rule was applied to solve the system of symbolic 2-plithogenic linear equations. Also, provided enough examples for each case to enhance understanding.

Keywords: Symbolic 2-plithogenic linear equations; Cramer's rule; solution of the symbolic 2-plithogenic linear equations.

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1. **Introduction**

The concept of refined neutrosophic structure was studied by many authors in [1–5]. Symbolic plithogenic algebraic structures are introduced by Smarandache, that are very similar to refined neutrosophic structures with some differences in the definition of the multiplication operation [10].

In [7], the algebraic properties of symbolic 2-plithogenic rings generated from the fusion of symbolic plithogenic sets with algebraic rings, and some of the elementary properties and substructures of symbolic 2-plithogenic rings such as AH-ideals, AH-homomorphisms, and AHS-isomorphisms are studied. In [11], some more algebraic properties of symbolic 2-plithogenic rings are studied. Further, Taffach [8,9] studied the concepts of symbolic 2-plithogenic vector spaces and modules.

zenodo.10031260/10.5281

In [14], the concept of symbolic 2-plithogenic matrices with symbolic 2-plithogenic entries, determinants, eigen values and vectors, exponents, and diagonalization are studied. Hamiyet Merkepci et.al [13], studied the the symbolic 2-plithogenic number theory and integers. Ahmad Khaldi et.al [12], studied the different types of algebraic symbolic 2-plithogenic equations and its solutions.

In [6], Yaser Ahmad Alhasan studied the types of the nuetrosophic linear equations and Cramer's rule to solve the system of nuetrosophic linear equations. Motivated by this work, in this article the symbolic 2-plithogenic linear equations and its solutions are introduced and studied. Also, enough examples are given for all the cases to enhance understanding.

2. Preliminaries

Definition 2.1. [7] Let R be a ring, the symbolic 2-plithogenic ring is defined as follows:

$$2-SP_R = \left\{a_0 + a_1P_1 + a_2P_2; a_i \in R, P_j^2 = P_j, P_1 \times P_2 = P_{max(1,2)} = P_2\right\}$$

Smarandache has defined algebraic operations on $2-SP_R$ as follows:

Addition:

$[a_0 + a_1P_1 + a_2P_2] + [b_0 + b_1P_1 + b_2P_2] = (a_0 + b_0) + (a_1 + b_1)P_1 + (a_2 + b_2)P_2$

Multiplication:

$[a_0 + a_1P_1 + a_2P_2].[b_0 + b_1P_1 + b_2P_2] = a_0b_0 + a_0b_1P_1 + a_0b_2P_2 + a_1b_0P_1^2 + a_1b_2P_1P_2 + a_2b_0P_2 + a_2b_1P_1P_2 + a_2b_2P_2^2 + a_1b_1P_1P_1 = (a_0b_0) + (a_0b_1 + a_1b_0 + a_1b_1)P_1 + (a_0b_2 + a_1b_2 + a_2b_0 + a_2b_1 + a_2b_2)P_2$.

It is clear that $2-SP_R$ is a ring. If R is a field, then $2-SP_R$ is called a symbolic 2-plithogenic field. Also, if R is commutative, then $2-SP_R$ is commutative, and if R has a unity (1), than $2-SP_R$ has the same unity (1).

Theorem 2.2. *[7] Let $2-SP_R$ be a 2-plithogenic symbolic ring, with unity (1). Let $X = x_0 + x_1P_1 + x_2P_2$ be an arbitrary element, then:*

(1) *X is invertible if and only if $x_0, x_0 + x_1, x_0 + x_1 + x_2$ are invertible.*
(2) $X^{-1} = x_0^{-1} + [(x_0 + x_1)^{-1} - x_0^{-1}]P_1 + [(x_0 + x_1 + x_2)^{-1} - (x_0 + x_1)^{-1}]P_2$

Definition 2.3. [14] A symbolic 2-plithogenic square real matrix is a matrix with symbolic 2-plithogenic real entries.

Theorem 2.4. *[14] Let $S = S_0 + S_1P_1 + S_2P_2$ be a symbolic 2-plithogenic square real matrix, then*

(1) *S is invertible if and only if $S_0, S_0 + S_1, S_0 + S_1 + S_2$ are invertible.*
(2) *If S is invertible then*

$$S^{-1} = S_0^{-1} + [(S_0 + S_1)^{-1} - S_0^{-1}]P_1 + [(S_0 + S_1 + S_2)^{-1} - (S_0 + S_1)^{-1}]$$

3. The Symbolic 2-Plithogenic Linear Equations

We begin this section with the following definition.

Definition 3.1. The symbolic 2-plithogenic linear equation of n variables $x_1, x_2, x_3, ..., x_n$, is each equation that takes the form:

$$(a_{01} + a_{11}P_1 + a_{21}P_2)x_1 + (a_{02} + a_{12}P_1 + a_{22}P_2)x_2 + (a_{03} + a_{13}P_1 + a_2P_{23})x_3 + \cdots + (a_{0n} + a_{1n}P_1 + a_{2n}P_2)x_n = b_0 + b_1P_1 + b_2P_2$$

where $a_{0i}, a_{1i}, a_{2i},\ i = 1, 2, ..., n$ are real coefficients. We call $(a_{01} + a_{11}P_1 + a_{21}P_2), (a_{02} + a_{12}P_1 + a_{22}P_2), (a_{03} + a_{13}P_1 + a_{23}P_2)$ symbolic 2-plithogenic coefficients of the borders of the equation, and $b_0 + b_1P_1 + b_2P_2$ constant symbolic 2-plithogenic border of the equation.

Remark 3.2.

(1) We call each equation of the form:

$$(a_0 + a_1P_1 + a_2P_2)x + (b_0 + b_1P_1 + b_2P_2)y = c_0 + c_1P_1 + c_2P_2$$

the two-variable symbolic 2-plithogenic linear equation, where, $a_0, a_1, a_2, b_0, b_1, b_2$, and c_0, c_1, c_2 are real coefficients.

(2) We call each equation of the form:

$$(a_0 + a_1P_1 + a_2P_2)x + (b_2 + b_1P_1 + b_2P_2)y + (c_0 + c_1P_1 + c_2P_2)z = d_0 + d_1P_1 + d_2P_2$$

the three-variable symbolic 2-plithogenic linear equation, where, $a_0, a_1, a_2, b_0, b_1, b_2$, c_0, c_1, c_2, and d_0, d_1, d_2 are real coefficients.

Example 3.3.

(1) $(1 + P_2)x + (3 - P_1)y + (1 + P_1 - P_2)z = 5$
(2) $P_2x + P_1y + (P_1 - P_2)z = 2P_1 + 2P_2$
(3) $(1 + P_1 - P_2)x + (4 + P_1 - P_2)y = 11 + 4P_2$

Definition 3.4. Solution of the symbolic 2-plithogenic linear equation,

$$(a_{01} + a_{11}P_1 + a_{21}P_2)x_1 + (a_{02} + a_{12}P_1 + a_{22}P_2)x_2 + (a_{03} + a_{13}P_1 + a_2P_{23})x_3 + \cdots + (a_{0n} + a_{1n}P_1 + a_{2n}P_2)x_n = b_0 + b_1P_1 + b_2P_2$$

is finding the values of the variables $x_1, x_2, x_3, ..., x_n$ that satisfies the equation, where $a_{0i}, a_{1i}, a_{2i},\ i = 1, 2, ..., n$ are real coefficients.

Example 3.5. Consider the following the two-variable symbolic 2-plithogenic linear equation:

$$(1 + P_1 - P_2)x + (4 + P_1 - P_2)y = 4 + \frac{11}{2}P_1 - \frac{15}{2}P_2$$

The solution of this equation is

$$x = 2 + P_1 + 2P_2, \quad y = \frac{1}{2} - P_2.$$

Definition 3.6. For the two variable symbolic 2-plithogenic linear equation

$$(a_0 + a_1P_1 + a_2P_2)x + (b_0 + b_1P_1 + b_2P_2)y = c_0 + c_1P_1 + c_2P_2$$

the infinite number of solutions defined by

$$y = -\frac{a_0 + a_1P_1 + a_2P_2}{b_0 + b_1P_1 + b_2P_2}x + \frac{c_0 + c_1P_1 + c_2P_2}{b_0 + b_1P_1 + b_2P_2}$$

where, $a_0, a_1, a_2, b_0, b_1, b_2, c_0, c_1, c_2$ are real coefficients, $b_0 \neq 0, b_0 + b_1 \neq 0$ and $b_0 + b_1 + b_2 \neq 0$ by given a value for one of the two variables, we obtain a value for the other variable.

Example 3.7. Consider the two variable symbolic 2-plithogenic linear equation

$$(1 + P_1 - P_2)x + (4 + P_1 - P_2)y = 1 + P_1 - 2P_2.$$

This implies that,

$$y = -\left(\frac{1 + P_1 - P_2}{4 + P_1 - P_2}\right)x + \left(\frac{1 + P_1 - 2P_2}{4 + P_1 - P_2}\right)$$

Then the set of solution is:

$$S = \left\{x, y \in 2 - SP_R : y = -\left(\frac{1 + P_1 - P_2}{4 + P_1 - P_2}\right)x + \left(\frac{1 + P_1 - 2P_2}{4 + P_1 - P_2}\right)\right\}$$

$$i.e., S = \left\{x, y \in 2 - SP_R : y = \left(-\frac{1}{4} - \frac{3}{20}P_1 + \frac{3}{20}P_2\right)x + \left(\frac{1}{4} - \frac{3}{20}P_1 - \frac{2}{5}P_2\right)\right\}$$

By given any value for the variable x, we obtain a value of the variable y.

Definition 3.8. For the n-variable symbolic 2-plithogenic linear equation

$$(a_{01} + a_{11}P_1 + a_{21}P_2)x_1 + (a_{02} + a_{12}P_1 + a_{22}P_2)x_2 + (a_{03} + a_{13}P_1 + a_2P_{23})x_3 + \cdots + (a_{0n} + a_{1n}P_1 + a_{2n}P_2)x_n = b_0 + b_1P_1 + b_2P_2$$

where $a_{0i}, a_{1i}, a_{2i},\ i = 1, 2, ..., n$ are real coefficients, the infinite number of solutions are the unknown values $x_1, x_2, ..., x_n$ that satisfies the equation.

Definition 3.9. A non-homogeneous system of n-variable symbolic 2-plithogenic linear equations is given by the form:

$$(a^1_{01} + a^1_{11}P_1 + a^1_{21}P_2)x_1 + (a^1_{02} + a^1_{12}P_1 + a^1_{22}P_2)x_2 + (a^1_{03} + a^1_{13}P_1 + a^1_{23}P_2)x_3 + \cdots + (a^1_{0n} + a^1_{1n}P_1 + a^1_{2n}P_2)x_n = b^1_0 + b^1_1P_1 + b^1_2P_2$$

$$(a^2_{01} + a^2_{11}P_1 + a^2_{21}P_2)x_1 + (a^2_{02} + a^2_{12}P_1 + a^2_{22}P_2)x_2 + (a^2_{03} + a^2_{13}P_1 + a^2_{23}P_2)x_3 + \cdots + (a^2_{0n} + a^2_{1n}P_1 + a^2_{2n}P_2)x_n = b^2_0 + b^2_1P_1 + b^2_2P_2$$

$$\cdots$$

$$(a^m_{01} + a^m_{11}P_1 + a^m_{21}P_2)x_1 + (a^m_{02} + a^m_{12}P_1 + a^m_{22}P_2)x_2 + (a^m_{03} + a^m_{13}P_1 + a^m_{23}P_2)x_3 + \cdots + (a^m_{0n} + a^m_{1n}P_1 + a^m_{2n}P_2)x_n = b^m_0 + b^m_1P_1 + b^m_2P_2$$

where, $a^j_{0i}, a^j_{1i}, a^j_{2i}, b^j_0, b^j_1, b^j_2$ are real coefficients, $i = 1, 2, ..., n,\ j = 1, 2, ..., m.$

Definition 3.10. The solution of non-homogeneous system of n-variable symbolic 2-plithogenic linear equations:

$$(a_{01}^1 + a_{11}^1 P_1 + a_{21}^1 P_2)x_1 + (a_{02}^1 + a_{12}^1 P_1 + a_{22}^1 P_2)x_2 + (a_{03}^1 + a_{13}^1 P_1 + a_{23}^1 P_2)x_3 + \cdots + (a_{0n}^1 + a_{1n}^1 P_1 + a_{2n}^1 P_2)x_n = b_0^1 + b_1^1 P_1 + b_2^1 P_2$$
$$(a_{01}^2 + a_{11}^2 P_1 + a_{21}^2 P_2)x_1 + (a_{02}^2 + a_{12}^2 P_1 + a_{22}^2 P_2)x_2 + (a_{03}^2 + a_{13}^2 P_1 + a_{23}^2 P_2)x_3 + \cdots + (a_{0n}^2 + a_{1n}^2 P_1 + a_{2n}^2 P_2)x_n = b_0^2 + b_1^2 P_1 + b_2^2 P_2$$
$$\cdots$$
$$(a_{01}^m + a_{11}^m P_1 + a_{21}^m P_2)x_1 + (a_{02}^m + a_{12}^m P_1 + a_{22}^m P_2)x_2 + (a_{03}^m + a_{13}^m P_1 + a_{23}^m P_2)x_3 + \cdots + (a_{0n}^m + a_{1n}^m P_1 + a_{2n}^m P_2)x_n = b_0^m + b_1^m P_1 + b_2^m P_2$$

where, $a_{0i}^j, a_{1i}^j, a_{2i}^j, b_0^j, b_1^j, b_2^j$ are real coefficients, $i = 1, 2, ..., n,\ j = 1, 2, ..., m$, is the values of the variables $x_1, x_2, x_3, ..., x_n$ that satisfies the system of equations.

Remark 3.11. We distinguish three cases to solve the system given in Definition 3.10:

(1) The system is consistent, it has unique solution.
(2) The system is consistent, it has infinite number of solutions.
(3) The system is inconsistent, it has no solution.

4. Solving System of Symbolic 2-Plithogenic Linear Equations using Cramer's Rule

Consider the non-homogeneous system of n symbolic 2-plithogenic linear equations with n-variables:

$$(a_{01}^1 + a_{11}^1 P_1 + a_{21}^1 P_2)x_1 + (a_{02}^1 + a_{12}^1 P_1 + a_{22}^1 P_2)x_2 + (a_{03}^1 + a_{13}^1 P_1 + a_{23}^1 P_2)x_3 + \cdots + (a_{0n}^1 + a_{1n}^1 P_1 + a_{2n}^1 P_2)x_n = b_0^1 + b_1^1 P_1 + b_2^1 P_2$$
$$(a_{01}^2 + a_{11}^2 P_1 + a_{21}^2 P_2)x_1 + (a_{02}^2 + a_{12}^2 P_1 + a_{22}^2 P_2)x_2 + (a_{03}^2 + a_{13}^2 P_1 + a_{23}^2 P_2)x_3 + \cdots + (a_{0n}^2 + a_{1n}^2 P_1 + a_{2n}^2 P_2)x_n = b_0^2 + b_1^2 P_1 + b_2^2 P_2$$
$$\cdots$$
$$\cdots$$
$$(a_{01}^n + a_{11}^n P_1 + a_{21}^n P_2)x_1 + (a_{02}^n + a_{12}^n P_1 + a_{22}^n P_2)x_2 + (a_{03}^n + a_{13}^n P_1 + a_{23}^n P_2)x_3 + \cdots + (a_{0n}^n + a_{1n}^n P_1 + a_{2n}^n P_2)x_n = b_0^n + b_1^n P_1 + b_2^n P_2$$

Let $AX = B$ be the matrix form of this system and let, $\Delta = det(A) = a_0 + a_1 P_1 + a_2 P_2$. We distinguish the following cases:

(1) If $a_0 \neq 0$, $a_0 + a_1 \neq 0$ and $a_0 + a_1 + a_2 \neq 0$ then the system is consistent and the system has unique solution given by the formula:

$$x_i = \frac{\Delta_{x_i}}{\Delta}, \quad i = 1, 2, 3, ..., n$$

where Δ_{x_i} is the determinant of the matrix A where the ith column is replaced by the column matrix B.

(2) If $a_0 = 0$, or $a_0 + a_1 = 0$ or $a_0 + a_1 + a_2 = 0$. Then the system is inconsistent or it have infinite number of solutions.

Remark 4.1. Consider a system of 2 linear symbolic 2-plithogenic equations with 2 unknowns x and y with

$$\Delta = det(A) = a_0 + a_1 P_1 + a_2 P_2$$
$$\Delta_x = det(A_x) = a_0^x + a_1^x P_1 + a_2^x P_2$$
$$\Delta_y = det(A_y) = a_0^y + a_1^y P_1 + a_2^y P_2$$

We distinguish the following cases:

(1) If $a_0 \neq 0, \quad a_0 + a_1 \neq 0 \quad$ and $\quad a_0 + a_1 + a_2 \neq 0$ then the system is consistent and the system has unique solution given by the formula:

$$x = \frac{\Delta_x}{\Delta}, \quad y = \frac{\Delta_y}{\Delta}$$

(2) If the determinants satisfies any one of the following conditions, then the system is inconsistent and it has no solutions.

(i). $a_0 = 0$ and a_0^x, a_0^y are not all zero.

(ii). $a_0 + a_1 = 0$ and $a_0^x + a_1^x, a_0^y + a_1^y$ are not all zero.

(iii). $a_0 + a_1 + a_3 = 0$ and $a_0^x + a_1^x + a_2^x, a_0^y + a_1^y + a_2^y$ are not all zero.

(3) In all other cases the system is consistent with infinite number of solutions.

Example 4.2. Consider the system of equations:

$$(2 + P_1 + 3P_2)x + (1 - P_1 - P_2)y = 5 + P_1 + 11P_2$$
$$(3 + 4P_2)x + (1 + P_1)y = 7 + 3P_1 + 13P_2.$$

The coefficient matrix is, $A = \begin{pmatrix} 2 + P_1 + 3P_2 & 1 - P_1 - P_2 \\ 3 + 4P_2 & 1 + P_1 \end{pmatrix}$.

$$\Delta = det(A) = \begin{vmatrix} 2 + P_1 + 3P_2 & 1 - P_1 - P_2 \\ 3 + 4P_2 & 1 + P_1 \end{vmatrix} = -1 + 7P_1 + 13P_2$$

Here, $a_0 \neq 0, a_0 + a_1 \neq 0$ and $a_0 + a_1 + a_2 \neq 0$, the system is consistent and its has unique solution.

$$\Delta_x = det(A_x) = \begin{vmatrix} 5 + P_1 + 11P_2 & 1 - P_1 - P_2 \\ 7 + 3P_1 + 13P_2 & 1 + P_1 \end{vmatrix} = -2 + 14P_1 + 45P_2$$

$$\Delta_y = det(A_y) = \begin{vmatrix} 2 + P_1 + 3P_2 & 5 + P_1 + 11P_2 \\ 3 + 4P_2 & 7 + 3P_1 + 13P_2 \end{vmatrix} = -1 + 13P_1 + 7P_2$$

So the solution of the given system is,

$$x = \frac{\Delta_x}{\Delta} = \frac{-2 + 14P_1 + 45P_2}{-1 + 7P_1 + 13P_2} = 2 + P_2,$$

$$y = \frac{\Delta_y}{\Delta} = \frac{-1 + 13P_1 + 7P_2}{-1 + 7P_1 + 13P_2} = 1 + P_1 - P_2$$

Example 4.3. Consider the system of equations:

$$(1+P_1+P_2)x+(3-P_1+2P_2)y=5+3P_1+5P_2$$
$$P_1x+(P_1+P_2)y=4P_1+P_2.$$

The coefficient matrix is, $A=\begin{pmatrix}1+P_1+P_2 & 3-P_1+2P_2\\ P_1 & P_1+P_2\end{pmatrix}$.

$$\Delta=det(A)=\begin{vmatrix}1+P_1+P_2 & 3-P_1+2P_2\\ P_1 & P_1+P_2\end{vmatrix}=0+0P_1+2P_2$$

$$\Delta=det(A_x)=\begin{vmatrix}5+3P_1+5P_2 & 3-P_1+2P_2\\ 4P_1+P_2 & P_1+P_2\end{vmatrix}=0+0P_1+6P_2$$

$$\Delta=det(A_y)=\begin{vmatrix}1+P_1+P_2 & 5+3P_1+5P_2\\ P_1 & 4P_1+P_2\end{vmatrix}=0+0P_1+2P_2$$

Hence, by condition (3) of Remark 4.1 the system is consistent with infinite number of solutions and the solutions are given by:
For all $x=a_0+a_1P_1+a_2P_2\in 2-SP_R$ with $a_0+a_1+a_2=3$,

$$y=-\left(\frac{1+P_1+P_2}{3-P_1+2P_2}\right)x+\left(\frac{5+3P_1+5P_2}{3-P_1+2P_2}\right)$$

That is, for all $x=a_0+a_1P_1+a_2P_2\in 2-SP_R$ with $a_0+a_1+a_2=3$,

$$y=\left(-\frac{1}{3}-\frac{2}{3}P_1+\frac{1}{4}P_2\right)x+\left(\frac{5}{3}+\frac{7}{3}P_1-\frac{3}{4}P_2\right)$$

Example 4.4. Consider the system of equations:

$$(2+P_1+3P_2)x+(1+P_1+P_2)y=5+P_1+11P_2$$
$$(4+2P_1+6P_2)x+(2+2P_1+2P_2)y=10+2P_1+22P_2$$

The coefficient matrix is, $A=\begin{pmatrix}2+P_1+3P_2 & 1+P_1+P_2\\ 4+2P_1+6P_2 & 2+2P_1+2P_2\end{pmatrix}$.

$$\Delta=det(A)=\begin{vmatrix}2+P_1+3P_2 & 1+P_1+P_2\\ 4+2P_1+6P_2 & 2+2P_1+2P_2\end{vmatrix}=0+0P_1+0P_2$$

Also,

$$\Delta_x=det(A_x)=\begin{vmatrix}5+P_1+11P_2 & 1+P_1+P_2\\ 10+2P_1+22P_2 & 2+2P_1+2P_2\end{vmatrix}=0+0P_1+0P_2$$

$$\Delta_y=det(A_y)=\begin{vmatrix}2+P_1+3P_2 & 5+P_1+11P_2\\ 4+2P_1+6P_2 & 10+2P_1+22P_2\end{vmatrix}=0+0P_1+0P_2$$

Hence, by condition (3) of Remark 4.1 the system is consistent with infinite number of solutions and the solutions are given by:

$$S = \left\{ x, y \in 2 - SP_R : y = -\left(\frac{2 + P_1 + 3P_2}{1 + P_1 + P_2}\right) x + \left(\frac{5 + P_1 + 11P_2}{1 + P_1 + P_2}\right) \right\}$$

$$i.e, \quad S = \left\{ x, y \in 2 - SP_R : y = \left(-2 + \frac{1}{2}P_1 - \frac{1}{2}P_2\right) x + \left(5 - 2P_1 + \frac{8}{3}P_2\right) \right\}$$

Example 4.5. Consider the system of equations:

$$(1 + P_1 + P_2)x + (1 - P_1 + P_2)y = 1 + P_1$$
$$(2 + 2P_1 + 2P_2)x + (2 - 2P_1 + 2P_2)y = 3 + P_2$$

The coefficient matrix is, $A = \begin{pmatrix} 1 + P_1 + P_2 & 1 - P_1 + P_2 \\ 2 + 2P_1 + 2P_2 & 2 - 2P_1 + 2P_2 \end{pmatrix}$.

$$\Delta = det(A) = \begin{vmatrix} 1 + P_1 + P_2 & 1 - P_1 + P_2 \\ 2 + 2P_1 + 2P_2 & 2 - 2P_1 + 2P_2 \end{vmatrix} = 0 + 0P_1 + 0P_2$$

Also,

$$\Delta_x = det(A_x) = \begin{vmatrix} 1 + P_1 & 1 - P_1 + P_2 \\ 3 + P_2 & 2 - 2P_1 + 2P_2 \end{vmatrix} = -1 + P_1$$

$$\Delta_y = det(A_y) = \begin{vmatrix} 1 + P_1 + P_2 & 1 + P_1 \\ 2 + 2P_1 + 2P_2 & 3 + P_2 \end{vmatrix} = 1 - 3P_1 - 2P_2$$

Here, $a_0 = 0$ and $a_0^x \neq 0$, hence the system is inconsistent.

Remark 4.6. Consider a system of 3 linear symbolic 2-plithogenic equations with 3 unknowns x, y and z with

$$\Delta = det(A) = a_0 + a_1P_1 + a_2P_2$$
$$\Delta_x = det(A_x) = a_0^x + a_1^xP_1 + a_2^xP_2$$
$$\Delta_y = det(A_y) = a_0^y + a_1^yP_1 + a_2^yP_2$$
$$\Delta_z = det(A_z) = a_0^z + a_1^zP_1 + a_2^zP_2$$

We distinguish the following cases:

(1) If $a_0 \neq 0,\quad a_0 + a_1 \neq 0$ and $a_0 + a_1 + a_2 \neq 0$ then the system is consistent and the system has unique solution given by the formula:

$$x = \frac{\Delta_x}{\Delta}, \quad y = \frac{\Delta_y}{\Delta} \quad \& \quad z = \frac{\Delta_z}{\Delta}$$

(2) If the determinants satisfies any one of the following conditions, then the system is inconsistent and it has no solutions.

(i). $a_0 = 0$ and a_0^x, a_0^y, a_0^z are not all zero.

(ii). $a_0 + a_1 = 0$ and $a_0^x + a_1^x, a_0^y + a_1^y, a_0^z + a_1^z$ are not all zero.

(iii). $a_0 + a_1 + a_3 = 0$ and $a_0^x + a_1^x + a_2^x, a_0^y + a_1^y + a_2^y, a_0^z + a_1^z + a_2^z$ are not all zero.

(3) If $\Delta = 0+0P_1+0P_2$, $\Delta_x = 0+0P_1+0P_2$, $\Delta_y = 0+0P_1+0P_2$ and $\Delta_z = 0+0P_1+0P_2$ then by solving two equations of the system we will obtain the equation $0 = \alpha$. If $\alpha = 0$, then the system is consistent with infinite number of solutions and if $\alpha \neq 0$, then the system is inconsistent.

Example 4.7. Consider the system of equations:

$$(1+P_1)x + (1-P_1)y + (1+P_1-P_2)z = 1+5P_1-P_2$$
$$(1+P_2)x + (-1+P_1+P_2)y + (2+P_1)z = 1+4P_1+3P_2$$
$$(1-P_1+P_2)x + (-1+P_2)y + (1+P_1)z = 1+P_1+2P_2.$$

The coefficient matrix is, $A = \begin{pmatrix} 1+P_1 & 1-P_1 & 1+P_1-P_2 \\ 1+P_2 & -1+P_1+P_2 & 2+P_1 \\ 1-P_1+P_2 & -1+P_2 & 1+P_1 \end{pmatrix}$.

$$\Delta = det(A) = \begin{vmatrix} 1+P_1 & 1-P_1 & 1+P_1-P_2 \\ 1+P_2 & -1+P_1+P_2 & 2+P_1 \\ 1-P_1+P_2 & -1+P_2 & 1+P_1 \end{vmatrix} = 2+2P_1-P_2$$

$$\Delta_x = det(A_x) = \begin{vmatrix} 1+5P_1-P_2 & 1-P_1 & 1+P_1-P_2 \\ 1+4P_1+3P_2 & -1+P_1+P_2 & 2+P_1 \\ 1+P_1+2P_2 & -1+P_2 & 1+P_1 \end{vmatrix} = 2+6P_1-2P_2$$

$$\Delta_y = det(A_y) = \begin{vmatrix} 1+P_1 & 1+5P_1-P_2 & 1+P_1-P_2 \\ 1+P_2 & 1+4P_1+3P_2 & 2+P_1 \\ 1-P_1+P_2 & 1+P_1+2P_2 & 1+P_1 \end{vmatrix} = 3P_2$$

$$\Delta_z = det(A_z) = \begin{vmatrix} 1+P_1 & 1-P_1 & 1+5P_1-P_2 \\ 1+P_2 & -1+P_1+P_2 & 1+4P_1+3P_2 \\ 1-P_1+P_2 & -1+P_2 & 1+P_1+2P_2 \end{vmatrix} = 4P_1-P_2$$

Here, $a_0 \neq 0$, $a_0+a_1 \neq 0$, & $a_0+a_1+a_2 \neq 0$. Hence, the system is consistent with unique solution given by:

$$x = \frac{\Delta_x}{\Delta} = \frac{2+6P_1-2P_2}{2+2P_1-P_2} = 1+P_1,$$

$$y = \frac{\Delta_y}{\Delta} = \frac{3P_2}{2+2P_1-P_2} = P_2,$$

$$z = \frac{\Delta_z}{\Delta} = \frac{4P_1-P_2}{2+2P_1-P_2} = P_1.$$

Example 4.8. Consider the system of equations:

$$(1+P_2)x+(3-P_1)y+(1+P_1-P_2)z=5$$
$$P_2x+P_1y+(P_1+P_2)z=2P_1+2P_2$$
$$(2+P_1-P_2)x+(4+3P_1-P_2)y+(5+2P_2)z=11+4P_1.$$

The coefficient matrix is, $A=\begin{pmatrix} 1+P_2 & 3-P_1 & 1+P_1-P_2 \\ P_2 & P_1 & P_1+P_2 \\ 2+P_1-P_2 & 4+3P_1-P_2 & 5+2P_2 \end{pmatrix}$.

$$\Delta = det(A) = \begin{vmatrix} 1+P_2 & 3-P_1 & 1+P_1-P_2 \\ P_2 & P_1 & P_1+P_2 \\ 2+P_1-P_2 & 4+3P_1-P_2 & 5+2P_2 \end{vmatrix} = 0-2P_1-10P_2$$

$$\Delta_x = det(A_x) = \begin{vmatrix} 5 & 3-P_1 & 1+P_1-P_2 \\ 2P_1+2P_2 & P_1 & P_1+P_2 \\ 11+4P_1 & 4+3P_1-P_2 & 5+2P_2 \end{vmatrix} = 0-2P_1-10P_2$$

$$\Delta_y = det(A_y) = \begin{vmatrix} 1+P_2 & 5 & 1+P_1-P_2 \\ P_2 & 2P_1+2P_2 & P_1+P_2 \\ 2+P_1-P_2 & 11+4P_1 & 5+2P_2 \end{vmatrix} = 0-2P_1-10P_2$$

$$\Delta_z = det(A_z) = \begin{vmatrix} 1+P_2 & 3-P_1 & 5 \\ P_2 & P_1 & 2P_1+2P_2 \\ 2+P_1-P_2 & 4+3P_1-P+2 & 11+4P_1 \end{vmatrix} = 0-2P_1-10P_2$$

Here, $a_0=0,\ a_0^x=0,\ a_0^y=0\ \ \&\ \ a_0^z=0$. Hence, the system is consistent with infinite number of solutions and the solutions are given by:

$$S=\Bigg\{x,y,z\in 2-SP_R : x=\left(-\frac{11}{2}+\frac{3}{2}P_1+5P_2\right)z+\left(\frac{13}{2}-\frac{3}{2}P_1-5P_2\right),$$
$$y=\left(\frac{3}{2}-\frac{1}{2}P_1-\frac{5}{2}P_2\right)z+\left(-\frac{1}{2}+\frac{1}{2}P_1+\frac{5}{2}P_2\right),\quad z\Bigg\}$$

Example 4.9. Consider the system of equations:

$$(1+P_1)x+(1-P_2)y+(1+P_1-P_2)z=2+P_1$$
$$(2+2P_1)x+(2-2P_2)y+(2+2P_1-2P_2)z=4+2P_1$$
$$(3+3P_1)x+(3-3P_2)y+(3+3P_1-3P_2)z=6+3P_1$$

Here, $\Delta=0+0P_1+0P_2$, $\Delta_x=0+0P_1+0P_2$, $\Delta_y=0+0P_1+0P_2$, $\Delta_z=0+0P_1+0P_2$. By solving first two equations of the given system we will get $0=0$. Hence the system consistent with infinite number of solutions. In this case the system is reduced to a single equation. To solve we can assign arbitrary values to any two variables and can determine the value of the third variable.

Example 4.10. Consider the system of equations:

$$(1+P_1)x+(1-P_2)y+(1+P_1-P_2)z=2+P_1$$
$$(2+2P_1)x+(2-2P_2)y+(2+2P_1-2P_2)z=1-P_2$$
$$(3+3P_1)x+(3-3P_2)y+(3+3P_1-3P_2)z=3+P_2$$

Here, $\Delta=0+0P_1+0P_2$, $\Delta_x=0+0P_1+0P_2$, $\Delta_y=0+0P_1+0P_2$, $\Delta_z=0+0P_1+0P_2$. By solving first two equations of the given system we will get $0=3+2P_1+P_2$. Hence the system inconsistent and it has no solution.

Other than the above mentioned cases there are some special cases in which the system of linear symbolic 2-plithogenic equations is inconsistent.

Remark 4.11. If all coefficients of a the system of n linear symbolic 2-plithogenic equations with n variables are non invertible the the system is inconsistent. For example,

$$(1-P_1+P_2)x+(2+2P_1-4P_2)y+(1-P_1)z=3+P_1$$
$$(1-P_2)x+P_1y+p_2z=3P_2$$
$$(2-P-1-P_2)x+P_2y+(1-P_1)z=2+P_1+P_2$$

5. System of Homogeneous Symbolic 2-Plithogenic Linear Equations

Definition 5.1. Consider the homogeneous system of n-variable symbolic 2-plithogenic linear equations:

$$(a^1_{01}+a^1_{11}P_1+a^1_{21}P_2)x_1+(a^1_{02}+a^1_{12}P_1+a^1_{22}P_2)x_2+(a^1_{03}+a^1_{13}P_1+a^1_{23}P_2)x_3+...+(a^1_{0n}+a^1_{1n}P_1+a^1_{2n}P_2)x_n=0+0P_1+0P_2$$
$$(a^2_{01}+a^2_{11}P_1+a^2_{21}P_2)x_1+(a^2_{02}+a^2_{12}P_1+a^2_{22}P_2)x_2+(a^2_{03}+a^2_{13}P_1+a^2_{23}P_2)x_3+...+(a^2_{0n}+a^2_{1n}P_1+a^2_{2n}P_2)x_n=0+0P_1+0P_2$$
$$...$$
$$(a^n_{01}+a^n_{11}P_1+a^n_{21}P_2)x_1+(a^n_{02}+a^n_{12}P_1+a^n_{22}P_2)x_2+(a^n_{03}+a^n_{13}P_1+a^n_{23}P_2)x_3+...+(a^n_{0n}+a^n_{1n}P_1+a^n_{2n}P_2)x_n=0+0P_1+0P_2$$

Remark 5.2. Let $AX=B$ be the coefficient matrix of this system and let $\Delta=det(A)=a_0+a_1P_1+a_2P_2$. We distinguish the following cases:

(1) If $a_0\neq 0$ and $a_0+a_1\neq 0$ and $a_0+a_1+a_2\neq 0$ then the system is consistent and the system has unique solution $x_i=0,\quad i=1,2,...,n$.
(2) If $a_0=0$ or $a_0+a_1=0$ or $a_0+a_1+a_2=0$, then the system is consistent with infinite number of solutions.

Example 5.3. Consider the system of equations:

$$(2+P_1+3P_2)x+(1-P_1-P_2)y=0$$
$$(3+4P_2)x+(1+P_1)y=0.$$

The coefficient matrix is, $A = \begin{pmatrix} 2+P_1+3P_2 & 1-P_1-P_2 \\ 3+4P_2 & 1+P_1 \end{pmatrix}$.

$$\Delta = det(A) = \begin{vmatrix} 2+P_1+3P_2 & 1-P_1-P_2 \\ 3+4P_2 & 1+P_1 \end{vmatrix} = -1+7P_1+13P_2$$

Here, $a_0 \neq 0, a_0+a_1 \neq 0$ and $a_0+a_1+a_2 \neq 0$, the system is consistent with unique solution.

$$\Delta_x = det(A_x) = \begin{vmatrix} 0 & 1-P_1-P_2 \\ 0 & 1+P_1 \end{vmatrix} = 0$$

$$\Delta_y = det(A_y) = \begin{vmatrix} 2+P_1+3P_2 & 0 \\ 3+4P_2 & 0 \end{vmatrix} = 0$$

So the solution of the given system is,

$$x = \frac{\Delta_x}{\Delta} = \frac{0}{-1+7P_1+13P_2} = 0,$$

$$y = \frac{\Delta_y}{\Delta} = \frac{0}{-1+7P_1+13P_2} = 0$$

Example 5.4. Consider the system of equations:

$$(1+P_2)x + (3-P_1)y + (2+P_2)z = 0$$
$$P_2x + P_1y + (P_1 - \tfrac{1}{2}P_2)z = 0$$
$$(2+P_1-P_2)x + (4+3P_1-P_2)y + (3+4P_1+2P_2)z = 0.$$

The coefficient matrix is, $A = \begin{pmatrix} 1+P_2 & 3-P_1 & 2+P_2 \\ P_2 & P_1 & P_1-\frac{1}{2}P_2 \\ 2+P_1-P_2 & 4+3P_1-P_2 & 3+4P_1+2P_2 \end{pmatrix}$.

$$\Delta = det(A) = \begin{vmatrix} 1+P_2 & 3-P_1 & 2+P_2 \\ P_2 & P_1 & P_1-\frac{1}{2}P_2 \\ 2+P_1-P_2 & 4+3P_1-P_2 & 3+4P_1+2P_2 \end{vmatrix} = 0+0P_1+8P_2$$

Here, $a_0 = 0$, the system is consistent with infinite number of solutions and the solutions are given by:

$$S = \left\{ x, y, z \in 2-SP_R : x = \left(-\frac{1}{2}+\frac{1}{2}P_1\right)z, \quad y = \left(-\frac{1}{2}-\frac{1}{2}P_1-\frac{1}{2}P_2\right)z, \quad z \right\}$$

6. **Conclusion**

In this article, the solutions of symbolic 2-plithogenic linear equations are studied using Cramer's rule. The conditions are given for a system of symbolic 2-plithogenic linear equations to be consistent with unique solution, consistent with infinite solutions, and inconsitent. Further, many examples are given for the case of system of two symbolic 2-plithogenic linear equations with two variables and for the case of system of three symbolic 2-plithogenic linear equations with three variables.

Funding: This research received no external funding.

References

1. Adeleke, E. O., Agboola, A. A. A., and Smarandache, F., "Refined neutrosophic rings II", International Journal of Neutrosophic Science, vol. 2, pp. 89?4, 2020.
2. A. A. A. Agboola, "On refined neutrosophic algebraic structures", Neutrosophic Sets and Systems, vol. 10, pp. 99?01, 2015.
3. M. Abobala, "On some special elements in neutrosophic rings and refined neutrosophic rings", Journal of New Theory, vol. 33, 2020.
4. Abobala, M., "On Refined Neutrosophic Matrices and Their Applications In Refined Neutrosophic Algebraic Equations", Journal Of Mathematics, Hindawi, 2021.
5. Abobala, M., "On Some Algebraic Properties of n-Refined Neutrosophic Elements and n-Refined Neutrosophic Linear Equations", Mathematical Problems in Engineering, Hindawi, 2021.
6. Yaser Ahmad Alhasan, "Types of system of the nuetrosophic linear equations and Cramer's rule", Neutrosophic Sets and Systems, vol. 45, 2021.
7. Merkepci, H., and Abobala, M., "On The Symbolic 2-Plithogenic Rings", International Journal of Neutrosophic Science, 2023.
8. Taffach, N., "An Introduction to Symbolic 2-Plithogenic Vector Spaces Generated from The Fusion of Symbolic Plithogenic Sets and Vector Spaces", Neutrosophic Sets and Systems, Vol 54, 2023.
9. Taffach, N., and Ben Othman, K., "An Introduction to Symbolic 2-Plithogenic Modules Over Symbolic 2-Plithogenic Rings", Neutrosophic Sets and Systems, Vol 54, 2023.
10. Smarandache, F., "Introduction to the Symbolic Plithogenic Algebraic Structures (revisited)", Neutrosophic Sets and Systems, vol. 53, 2023.
11. Albasheer, O., Hajjari., A., and Dalla., R., "On The Symbolic 3-Plithogenic Rings and Their Algebraic Properties", Neutrosophic Sets and Systems, Vol 54, 2023.
12. Khaldi, A., Ben Othman, K., Von Shtawzen, O., Ali, R., and Mosa, S., "On Some Algorithms for Solving Different Types of Symbolic 2-Plithogenic Algebraic Equations", Neutrosophic Sets and Systems, Vol 54, 2023.
13. Merkepci, H., and Rawashdeh, A., "On The Symbolic 2-Plithogenic Number Theory and Integers ", Neutrosophic Sets and Systems, Vol 54, 2023.
14. Abuobida Mohammed A. Alfahal, Yaser Ahmad Alhasan, Raja Abdullah Abdulfatah, Arif Mehmood, Mustafa Talal Kadhim, "On Symbolic 2-Plithogenic Real Matrices and Their Algebraic Properties", International Journal of Neutrosophic Science, Vol. 21, PP. 96-104, 2023.

Received: 18/6/2023 / **Accepted:** 29/9/2023