Neutrosophic Sets and Systems, Vol. 50, 20222
University of New Mexico

Fixed Point Theorem of Weak Compatible Maps of Type (γ) in Neutrosophic Metric Space

A.N. Mangayarkkarasi[1] V. Jeyanthi[2], M. Jeyaraman[3] and V.B. Shakila[4]

[1] Department of Mathematics, Nachiappa Swamigal Arts & Science College, Karaikudi.
Affiliated to Alagappa University, Karaikudi, Tamilnadu, India; E-mail: murugappan.mangai@gmail.com
[2] Government Arts College for Women, Sivagangai. Affiliated to Alagappa University, Karaikudi, Tamilnadu, India;
E-mail: jeykaliappa@gmail.com.
[3] P.G. and Research Department of Mathematics, Raja Doraisingam Government Arts College, Sivagangai.
Affiliated to Alagappa University, Karaikudi, Tamilnadu, India; E-mail : jeya.math@gmail.com.
[4]Department of Mathematics with CS (SF), Sourashtra College, Madurai, Tamilnadu, India.
E-mail: vbshakila1977@gmail.com.

Abstract: In this paper, we give definitions of compatible mappings of type (γ) in neutrosophic metric space, and obtain a common fixed point theorem under the conditions of weakly compatible mappings of type (γ) in complete neutrosophic metric spaces. Our research generalizes, extends and improves the results given by Sedghi et al.[19].

Keywords: Fixed point, Neutrosophic metric Space, Compatible Mappings, Weak Compatible Mappings of Type (γ).

1. Introduction :

Fuzzy set was presented by Zadeh [30] as a class of elements with a grade of membership. Kramosil and Michalek [9] defined new notion called Fuzzy Metric Space (FMS). Later on, many authors have examined the concept of fuzzy metric in various aspects. Since then, many authors have obtained fixed point results in fuzzy metric space using these compatible notions. Also, Kutukcu et al.[11] obtained the common fixed points of compatible maps of type(β) on fuzzy metric spaces, and Sedghi et.al.[19] studied the common fixed point of compatible maps of type (γ) in complete fuzzy metric spaces.

Atanassov [1] introduced and studied the notion of intuitionistic fuzzy set by generalizing the notion of fuzzy set. Recently, Park[14] and Park et al. [17] defined the intuitionistic fuzzy metric space. Many authors [15, 16, 17] obtained a fixed point theorems in this space. Also, Park et al. [17] introduced the concept of compatible mappings of type(α) and type(β), and obtained common fixed point theorems in intuitionistic fuzzy metric space.

In 1998, Smarandache [20, 21, 22] characterized the new concept called neutrosophic logic and neutrosophic set and explored many results in it. In the idea of neutrosophic sets, there is T degree of membership, I degree of indeterminacy and F degree of non-membership. Basset et al. [3] Explored the neutrosophic applications in dif and only iferent fields such as model for sustainable supply chain risk management, resource levelling problem in construction projects, Decision Making. In 2020, Kirisci et al [10] defined NMS as a generalization of IFMS and brings about fixed point theorems in complete NMS. In 2020, Sowndrarajan et al. [23] proved some fixed point results for contraction theorems in neutrosophic metric spaces.

In this paper, we give definitions of compatible mappings of type (γ) in neutrosophic metric space and obtain common fixed point theorem under the conditions of weak compatible mappings of type (γ) in complete neutrosophic metric space.

2. Some Relevent Results:

Definition: 2.1.[18]

A binary operation * : [0, 1] x [0, 1] → [0, 1] is a continuous t-norm [CTN] if it satisfies the following conditions :

1. * is commutative and associative,
2. * is continuous,
3. $\varepsilon_1 * 1 = \varepsilon_1$ for all $\varepsilon_1 \in [0, 1]$,
4. $\varepsilon_1 * \varepsilon_2 \le \varepsilon_3 * \varepsilon_4$ whenever $\varepsilon_1 \le \varepsilon_3$ and $\varepsilon_2 \le \varepsilon_4$, for each $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in [0, 1]$.

Definition: 2.2.[18]

A binary operation ⋄ : [0, 1] x [0, 1] → [0, 1] is a continuous t-conorm [CTC] if it satisfies the following conditions:

1. ⋄ is commutative and associative,
2. ⋄ is continuous,
3. $\varepsilon_1 \diamond 0 = \varepsilon_1$ for all $\varepsilon_1 \in [0, 1]$,
4. $\varepsilon_1 \diamond \varepsilon_2 \le \varepsilon_3 \diamond \varepsilon_4$ whenever $\varepsilon_1 \le \varepsilon_3$ and $\varepsilon_2 \le \varepsilon_4$, for each $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and $\varepsilon_4 \in [0, 1]$.

Definition: 2.3.[23]

A 6-tuple $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ is said to be an Neutrosophic Metric Space (shortly NMS), if Σ is an arbitrary non empty set, $*$ is a neutrosophic CTN, $\diamond$ is a neutrosophic CTC and Ξ, Θ *and* Υ are neutrosophic on $\Sigma^2 \times \mathbb{R}^+$ satisfying the following conditions:

For all $\zeta, \eta, \delta, \omega \in \Sigma$, $\lambda \in \mathbb{R}^+$.

1. $0 \le \Xi(\zeta, \eta, \lambda) \le 1$; $0 \le \Theta(\zeta, \eta, \lambda) \le 1$; $0 \le \Upsilon(\zeta, \eta, \lambda) \le 1$;
2. $\Xi(\zeta, \eta, \lambda) + \Theta(\zeta, \eta, \lambda) + \Upsilon(\zeta, \eta, \lambda) \le 3$;
3. $\Xi(\zeta, \eta, \lambda) = 1$ if and only if $\zeta = \eta$;
4. $\Xi(\zeta, \eta, \lambda) = \Xi(\eta, \zeta, \lambda)$,
5. $\Xi(\zeta, \eta, \lambda) * \Xi(\eta, \delta, \mu) \le \Xi(\zeta, \delta, \lambda + \mu)$, for all $\lambda, \mu > 0$;
6. $\Xi(\zeta, \eta, .) : (0, \infty) \to (0, 1]$ is neutrosophic continuous ;
7. $\lim_{\lambda \to \infty} \Xi(\zeta, \eta, \lambda) = 1$ for all $\lambda > 0$;
8. $\Theta(\zeta, \eta, \lambda) = 0$ if and only if $\zeta = \eta$;
9. $\Theta(\zeta, \eta, \lambda) = \Theta(\eta, \zeta, \lambda)$;
10. $\Theta(\zeta, \eta, \lambda) \diamond \Theta(\eta, \delta, \mu) \ge \Theta(\zeta, \delta, \lambda + \mu)$, for all $\lambda, \mu > 0$;

11. $\Theta(\zeta, \eta, .) : (0, \infty) \to (0, 1]$ is neutrosophic continuous;
12. $\lim_{\lambda\to\infty} \Theta(\zeta, \eta, \lambda) = 0$ for all $\lambda > 0$;
13. $\Upsilon(\zeta, \eta, \lambda) = 0$ if and only if $\zeta = \eta$;
14. $\Upsilon(\zeta, \eta, \lambda) = \Upsilon(\eta, \zeta, \lambda)$;
15. $\Upsilon(\zeta, \eta, \lambda) \diamond \Upsilon(\eta, \delta, \mu) \geq \Upsilon(\zeta, \delta, \lambda + \mu)$, for all $\lambda, \mu > 0$;
16. $\Upsilon(\zeta, \eta, .) : (0, \infty) \to (0, 1]$ is neutrosophic continuous;
17. $\lim_{\lambda\to\infty} \Upsilon(\zeta, \eta, \lambda) = 0$ for all $\lambda > 0$;
18. If $\lambda \leq 0$ then $\Xi(\zeta, \eta, \delta, \lambda) = 0$; $\Theta(\zeta, \eta, \delta, \lambda) = 1$; $\Upsilon(\zeta, \eta, \delta, \lambda) = 1$.

Then, (Ξ, Θ, Υ) is called an NMS on Σ. The functions Ξ, Θ and Υ denote degree of closedness, neturalness and non-closedness between ζ and η with respect to λ respectively.

Example: 2.4.[23]

Let (Σ, d) be a metric space. Define $\omega * \tau = \min\{\omega, \tau\}$ and $\omega \diamond \tau = \max\{\omega, \tau\}$ and $\Xi, \Theta, \Upsilon : \Sigma^2 \times \mathbb{R}^+ \to [0, 1]$ defined by,we define $\Xi(\zeta, \eta, \lambda) = \frac{\lambda}{\lambda + d(\zeta, \eta)}$; $\Theta(\zeta, \eta, \lambda) = \frac{d(\zeta,\eta)}{\lambda + d(\zeta,\eta)}$; $\Upsilon(\zeta, \eta, \lambda) = \frac{d(\zeta,\eta)}{\lambda}$, for all $\zeta, \eta \in \Sigma$ and $\lambda > 0$. Then $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond)$ is called NMS induced by a metric d the standard neutrosophic metric.

Definition 2.5.[23]

Let Σ be an NMS. Then Ξ, Θ are said to be continuous on $\Sigma^2 \times \mathbb{R}^+$ if $\lim_{n\to\infty} \Xi(\zeta_n, \eta_n, \lambda_n) = \Xi(\zeta, \eta, \lambda)$; $\lim_{n\to\infty} \Theta(\zeta_n, \eta_n, \lambda_n) = \Theta(\zeta, \eta, \lambda)$; $\lim_{n\to\infty} \Upsilon(\zeta_n, \eta_n, \lambda_n) = \Upsilon(\zeta, \eta, \lambda)$,

Whenever a sequence $\{(\zeta_n, \eta_n, \lambda_n)\} \subset \Sigma^2 \times \mathbb{R}^+$ converges to a point $(\zeta, \eta, \lambda) \in \Sigma^2 \times \mathbb{R}^+$.

Definition 2.6.

Let Γ and Ω be mappings from an NMS Σ into itself. Then the mappings are said to be compatible if $\lim_{n\to\infty} \Xi(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 1$, $\lim_{n\to\infty} \Theta(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 0$, $\lim_{n\to\infty} \Upsilon(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 0$, $\forall\lambda > 0$, whenever $\{\zeta_n\}$ is a sequence in Σ such that $\lim_{n\to\infty} \Gamma\zeta_n = \lim_{n\to\infty} \Omega\zeta_n = \zeta \in \Sigma$.

Example 2.7.

Let Σ be an NMS, where $\Sigma = [0, 2]$, $*$, $\Diamond$ defined $a * b = \min\{a, b\}$, $a \Diamond b = \max\{a, b\}$ for all $a, b \in [0, 1]$ and $\Xi(\zeta, \eta, \lambda) = \frac{\lambda}{\lambda + d(\zeta, \eta)}$; $\Theta(\zeta, \eta, \lambda) = \frac{d(\zeta,\eta)}{\lambda + d(\zeta,\eta)}$; $\Upsilon(\zeta, \eta, \lambda) = \frac{d(\zeta,\eta)}{\lambda}$, for all $\lambda > 0$ and $\zeta, \eta \in \Sigma$. Define self maps Γ and Ω on Σ as follows:

$$\Gamma\zeta = \begin{cases} 2 \ if \ 0 \leq \zeta \leq 1 \\ \frac{\zeta}{2} \ if \ 1 < \zeta \leq 2 \end{cases} \quad \text{and} \quad \Omega\zeta = \begin{cases} 2 \ if \ \zeta = 1 \\ \frac{\zeta+3}{3} \ otherewise \end{cases}$$

and $\zeta_n = 2 - \frac{1}{2n}$. Then we have $\Omega 1 = 2 = \Gamma 1$ and $\Omega 2 = 1 = \Gamma 2$.

Also, $\Omega\Gamma 2 = \Omega 1 = 2$, $\Gamma\Omega 2 = \Gamma 1 = 2$($\Omega\Gamma 2 = \Gamma\Omega 2 = 2$), thus Γ and Ω are weak compatible.

Also, since $\Gamma\zeta_n = \frac{1}{2}(2 - \frac{1}{2n}) = 1 - \frac{1}{2n}$, $\Omega\zeta_n = \frac{1}{2}(2 - \frac{1}{2n} + 3) = 1 - \frac{1}{10n}$. Thus $\lim_{n\to\infty} \Gamma\zeta_n = 1 = \lim_{n\to\infty} \Omega\zeta_n$.

Furthermore, $\Omega\Gamma\zeta_n = \Omega(1 - \frac{1}{4n}) = \frac{1}{5}(1 - \frac{3}{4n} + 3) = \frac{4}{5} - \frac{1}{20n}$, $\Gamma\Omega\zeta_n = \Gamma(1 - \frac{1}{10n}) = 2$.

Now,

$\lim_{n\to\infty} \Xi(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \lambda) = \lim_{n\to\infty} \Xi(2, \frac{4}{5} - \frac{1}{20n}, \lambda) = \frac{5t}{5t+6}$,

$\lim_{n\to\infty} \Theta(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \lambda) = \lim_{n\to\infty} \Theta(2, \frac{4}{5} - \frac{1}{20n}, \lambda) = \frac{6}{5t+6}$,

$\lim_{n\to\infty} \Upsilon(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \lambda) = \lim_{n\to\infty} \Upsilon(2, \frac{4}{5} - \frac{1}{20n}, \lambda) = \frac{6}{5t}$, Hence Γ and Ω is not compatible.

3. Weak Compatible mappings of type (γ):

Definition 3.1.

Let Γ and Ω be mappings from an NMS Σ into itself. Then the mappings Γ and Ω are said to be compatible maps of type (γ) if satisfying:

1. Γ and Ω are compatible, that is, $\lim_{n\to\infty} \Xi(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 1$, $\lim_{n\to\infty} \Theta(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 0$, $\lim_{n\to\infty} \Upsilon(\Gamma\Omega\zeta_n, \Omega\Gamma\zeta_n, \lambda) = 0$, $\forall\lambda > 0$.
   Whenever $\{\zeta_n\} \subset \Sigma$ such that $\lim_{n\to\infty} \Gamma\zeta_n = \lim_{n\to\infty} \Omega\zeta_n = \zeta \in \Sigma$.
2. They are continuous at ζ. On the other hand, we have
   $\Gamma\zeta = \Gamma(\lim_{n\to\infty} \Gamma\zeta_n) = \Gamma(\lim_{n\to\infty} \Omega\zeta_n) = (\lim_{n\to\infty} \Omega\Gamma\zeta_n) = \Omega(\lim_{n\to\infty} \Gamma\zeta_n) = \Omega\zeta$.

Definition 3.2.

Let Γ and Ω be mappings from an NMS Σ into itself. The mappings Γ and Ω are said to be weak - compatible of type (γ) if $\lim_{n\to\infty} \Gamma\zeta_n = \lim_{n\to\infty} \Omega\zeta_n = \zeta$ for some $\zeta \in \Sigma$ implies that $\Gamma\zeta = \Omega\zeta$.

Remark 3.3.

If self maps Γ and Ω of an NMS Σ are compatible of type(γ), then they are weak compatible type(γ). But the converse is not true.

Lemma 3.5.

Let Σ be an NMS,

1. If we define $E_\alpha : \Sigma^2 \times \mathbb{R}^+$ by
   $E_\alpha(\zeta, \eta) = \inf\{\lambda > 0 ; \Xi(\zeta, \eta, \lambda) > 1-\lambda, \Theta(\zeta, \eta, \lambda) < \lambda$ and $\Upsilon(\zeta, \eta, \lambda) < \lambda\}$ for each $\mu \in (0,1)$ there exists $\alpha \in (0,1)$ such that $E_\alpha(\zeta_1, \zeta_n) \le E_\alpha(\zeta_1, \zeta_2) + E_\alpha(\zeta_2, \zeta_3) + \ldots E_\alpha(\zeta_{n-1}, \zeta_n)$ for any $\zeta_1, \zeta_2 \ldots \zeta_n \in \Sigma$.
2. The sequence $\{\zeta_n\}_{n\in N}$ is convergent in NMS Σ if and only if $E_\alpha(\zeta_n, \zeta) \to 0$.
   Also, the sequence $\{\zeta_n\}_{n\in N}$ is Cauchy sequence if and only if it is Cauchy sequence with E_α.

Lemma 3.6.

Let Σ be an NMS. $\Xi(\zeta_n, \zeta_{n+1}, \lambda) \ge \Xi(\zeta_0, \zeta_1, k^n\lambda)$, $\Theta(\zeta_n, \zeta_{n+1}, \lambda) \le \Theta(\zeta_0, \zeta_1, k^n\lambda)$ and $\Upsilon(\zeta_n, \zeta_{n+1}, \lambda) \le \Upsilon(\zeta_0, \zeta_1, k^n\lambda)$ for some $k > 1$ and for every $n \in N$. Then sequence $\{\zeta_n\}$ is a Cauchy sequence.

Lemma 3.7.

Let Σ be an NMS. If there exists a number $k \in (0, 1)$ such that for all $\zeta, \eta \in \Sigma$ and $\lambda > 0$. $\Xi(\zeta, \eta, k\lambda) \ge \Xi(\zeta, \eta, \lambda)$, $\Theta(\zeta, \eta, k\lambda) \le \Theta(\zeta, \eta, \lambda)$ and $\Upsilon(\zeta, \eta, k\lambda) \le \Upsilon(\zeta, \eta, \lambda)$ then $\zeta = \eta$.

4. Main Results

Lemma 4.1.

Let Γ and Ω are self – mappings of a complete NMS Σ satisfying:

There exists a constant $k \in (0, 1)$ such that

$\Xi^2(\Gamma\zeta, \Omega\eta, k\lambda) * [\Xi(\zeta, \Gamma\zeta, k\lambda)\, \Xi(\eta, \Omega\eta, k\lambda)] * \Xi^2(\eta, \Omega\eta, k\lambda) + a\, \Xi(\eta, \Omega\eta, k\lambda)$
$\Xi(\zeta, \Omega\eta, 2k\lambda) \geq [p\, \Xi(\zeta, \Gamma\zeta, \lambda) + q\, \Xi(\zeta, \eta, \lambda)]\, \Xi(\zeta, \Omega\eta, 2k\lambda)$ (4.1.1)
$\Theta^2(\Gamma\zeta, \Omega\eta, k\lambda) \lozenge [\Theta(\zeta, \Gamma\zeta, k\lambda)\, \Theta(\eta, \Omega\eta, k\lambda)] \lozenge \Theta^2(\eta, \Omega\eta, k\lambda) + a\, \Theta(\eta, \Omega\eta, k\lambda)$
$\Theta(\zeta, \Omega\eta, 2k\lambda) \leq [p\, \Theta(\zeta, \Gamma\zeta, \lambda) + q\, \Theta(\zeta, \eta, \lambda)]\, \Theta(\zeta, \Omega\eta, 2k\lambda)$ (4.1.2)
$\Upsilon^2(\Gamma\zeta, \Omega\eta, k\lambda) \lozenge [\Upsilon(\zeta, \Gamma\zeta, k\lambda)\, \Upsilon(\eta, \Omega\eta, k\lambda)] \lozenge \Upsilon^2(\eta, \Omega\eta, k\lambda) + a\, \Upsilon(\eta, \Omega\eta, k\lambda)$
$\Upsilon(\zeta, \Omega\eta, 2k\lambda) \leq [p\, \Upsilon(\zeta, \Gamma\zeta, \lambda) + q\, \Upsilon(\zeta, \eta, \lambda)]\, \Upsilon(\zeta, \Omega\eta, 2k\lambda)$ (4.1.3)

for every $\zeta, \eta \in \Sigma$ and $\lambda > 0$, where $0 < p, q < 1$, $0 \leq a < 1$ such that $p + q - a = 1$. Then Γ and Ω have a unique common fixed point in Σ.

Proof: Let $\zeta_0 \in \Sigma$ be an arbitrary point, there exist $\zeta_1 \in \Sigma$ such that $\Gamma\zeta_0 = \zeta_1$, $\Omega\zeta_1 = \zeta_2$. Inductively, construct the sequences $\{\zeta_n\} \subset \Sigma$ such that $\zeta_{2n+1} = \Gamma\zeta_{2n}$, $\zeta_{2n+2} = \Omega\zeta_{2n+1}$
for n = 0, 1, 2, ... Then we prove that $\{\zeta_n\}$ is a Cauchy sequence.
For $\zeta = \zeta_{2n}$, $\eta = \zeta_{2n+1}$ by we have

$\Xi^2(\Gamma\zeta_{2n}, \Omega\zeta_{2n+1}, k\lambda) *[\Xi(\zeta_{2n}, \Gamma\zeta_{2n}, k\lambda)\, \Xi(\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda)] * \Xi^2(\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda)$
$+ a\, \Xi(\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda)\, \Xi(\zeta_{2n}, \Omega\zeta_{2n+1}, 2k\lambda) \geq [p\, \Xi(\zeta_{2n}, \Gamma\zeta_{2n}, \lambda) + q\, \Xi(\zeta_{2n}, \zeta_{2n+1}, \lambda)] \times$
$\Xi(\zeta_{2n}, \Omega\zeta_{2n+1}, 2k\lambda)$ and
$\Xi^2(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) *[\Xi(\zeta_{2n}, \zeta_{2n+1}, k\lambda)\, \Xi(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda)] * \Xi^2(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda)$
$+ a\, \Xi(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda)\, \Xi(\zeta_{2n}, \zeta_{2n+2}, 2k\lambda) \geq [p\, \Xi(\zeta_{2n}, \zeta_{2n+1}, \lambda) + q\, \Xi(\zeta_{2n}, \zeta_{2n+1}, \lambda)] \times$
$\Xi(\zeta_{2n}, \zeta_{2n+2}, 2k\lambda)$,
$\Theta^2(\Gamma\zeta_{2n}, \Omega\zeta_{2n+1}, k\lambda) \lozenge[\Theta(\zeta_{2n}, \Gamma\zeta_{2n}, k\lambda)\, \Theta(\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda)] \lozenge \Theta^2(\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda)$
$+ a\, \Theta(\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda)\, \Theta(\zeta_{2n}, \Omega\zeta_{2n+1}, 2k\lambda) \leq [p\, \Theta(\zeta_{2n}, \Gamma\zeta_{2n}, \lambda) + q\, \Theta(\zeta_{2n}, \zeta_{2n+1}, \lambda)] \times$
$\Theta(\zeta_{2n}, \Omega\zeta_{2n+1}, 2k\lambda)$ and
$\Theta^2(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) \lozenge [\Theta(\zeta_{2n}, \zeta_{2n+1}, k\lambda)\, \Theta(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda)] \lozenge \Theta^2(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda)$
$+ a\, \Theta(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda)\, \Theta(\zeta_{2n}, \zeta_{2n+2}, 2k\lambda) \leq [p\, \Theta(\zeta_{2n}, \zeta_{2n+1}, \lambda) + q\, \Theta(\zeta_{2n}, \zeta_{2n+1}, \lambda)] \times$
$\Theta(\zeta_{2n}, \zeta_{2n+2}, 2k\lambda)$.
$\Upsilon^2(\Gamma\zeta_{2n}, \Omega\zeta_{2n+1}, k\lambda) \lozenge[\Upsilon(\zeta_{2n}, \Gamma\zeta_{2n}, k\lambda)\, \Upsilon(\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda)] \lozenge \Upsilon^2(\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda)$
$+ a\, \Upsilon(\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda)\, \Upsilon(\zeta_{2n}, \Omega\zeta_{2n+1}, 2k\lambda) \leq [p\, \Upsilon(\zeta_{2n}, \Gamma\zeta_{2n}, \lambda) + q\, \Upsilon(\zeta_{2n}, \zeta_{2n+1}, \lambda)] \times$
$\Upsilon(\zeta_{2n}, \Omega\zeta_{2n+1}, 2k\lambda)$ and
$\Upsilon^2(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) \lozenge [\Upsilon(\zeta_{2n}, \zeta_{2n+1}, k\lambda)\, \Upsilon(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda)] \lozenge \Upsilon^2(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda)$
$+ a\, \Upsilon(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda)\, \Upsilon(\zeta_{2n}, \zeta_{2n+2}, 2k\lambda) \leq [p\, \Upsilon(\zeta_{2n}, \zeta_{2n+1}, \lambda) + q\, \Upsilon(\zeta_{2n}, \zeta_{2n+1}, \lambda)] \times$
$\Upsilon(\zeta_{2n}, \zeta_{2n+2}, 2k\lambda)$.

Then

$\Xi^2(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) *[\Xi(\zeta_{2n}, \zeta_{2n+1}, k\lambda)\, \Xi(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda)] + a\, \Xi(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda)$
$\Xi(\zeta_{2n}, \zeta_{2n+2}, 2k\lambda) \geq (p + q)\, \Xi(\zeta_{2n}, \zeta_{2n+1}, \lambda)\, \Xi(\zeta_{2n}, \zeta_{2n+2}, 2k\lambda)$,
$\Theta^2(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) \lozenge[\Theta(\zeta_{2n}, \zeta_{2n+1}, k\lambda)\, \Theta(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda)] + a\, \Theta(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda)$
$\Theta(\zeta_{2n}, \zeta_{2n+2}, 2k\lambda) \leq (p + q)\, \Theta(\zeta_{2n}, \zeta_{2n+1}, \lambda)\, \Theta(\zeta_{2n}, \zeta_{2n+2}, 2k\lambda)$,
$\Upsilon^2(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) \lozenge[\Upsilon(\zeta_{2n}, \zeta_{2n+1}, k\lambda)\, \Upsilon(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda)] + a\, \Upsilon(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda)$
$\Upsilon(\zeta_{2n}, \zeta_{2n+2}, 2k\lambda) \leq (p + q)\, \Upsilon(\zeta_{2n}, \zeta_{2n+1}, \lambda)\, \Upsilon(\zeta_{2n}, \zeta_{2n+2}, 2k\lambda)$.

So,

$\Xi(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) + a\, \Xi(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) \geq (p + q)\, \Xi(\zeta_{2n}, \zeta_{2n+1}, \lambda)$
$\Theta(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) + a\, \Theta(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) \leq (p + q)\, \Theta(\zeta_{2n}, \zeta_{2n+1}, \lambda)$
$\Upsilon(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) + a\, \Upsilon(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) \leq (p + q)\, \Upsilon(\zeta_{2n}, \zeta_{2n+1}, \lambda)$.

Therefore

$\Xi(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) \geq \Xi(\zeta_{2n}\, \zeta_{2n+1}, \lambda)$, $\Theta(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) \leq \Theta(\zeta_{2n}, \zeta_{2n+1}, \lambda)$ and
$\Upsilon(\zeta_{2n+1}, \zeta_{2n+2}, k\lambda) \leq \Upsilon(\zeta_{2n}, \zeta_{2n+1}, \lambda)$.

Similarly, we also have

$\Xi(\zeta_{2n+2}, \zeta_{2n+3}, k\lambda) \geq \Xi(\zeta_{2n+1}, \zeta_{2n+2}, \lambda)$, $\Theta(\zeta_{2n+2}, \zeta_{2n+3}, k\lambda) \leq \Theta(\zeta_{2n+1}, \zeta_{2n+2}, \lambda)$,

$\Upsilon(\zeta_{2n+2}, \zeta_{2n+3}, k\lambda) \leq \Upsilon(\zeta_{2n+1}, \zeta_{2n+2}, \lambda)$.

For $k \in (0, 1)$ if $k_1 = \frac{1}{k} > 1$ and $\lambda = k_1 \lambda_1$, then we have

$\Xi(\zeta_n, \zeta_{n+1}, \lambda) \geq \Xi(\zeta_0, \zeta_1, k_1^n \lambda_1)$, $\Theta(\zeta_n, \zeta_{n+1}, \lambda) \leq \Theta(\zeta_0, \zeta_1, k_1^n \lambda_1)$and

$\Upsilon(\zeta_n, \zeta_{n+1}, \lambda) \leq \Upsilon(\zeta_0, \zeta_1, k_1^n \lambda_1)$.

By Lemma 3.6, since $\{\zeta_n\}$ is a Cauchy sequence in Σ which is complete, $\{\zeta_n\}$ converges to ω in Σ. Hence $\lim_{n\to\infty} \Gamma\zeta_{2n} = \lim_{n\to\infty} \zeta_{2n+1} = \lim_{n\to\infty} \zeta_{2n+2} = \lim_{n\to\infty} \Omega\zeta_{2n+1} = \omega$.

Now, taking $\zeta = \omega$ and $\eta = \zeta_{2n+1}$ in (i), we have as $n\to\infty$,

$$\Xi^2(\Gamma\omega, \omega, k\lambda) * [\Xi(\omega, \Gamma\omega, k\lambda)\, \Xi(\omega, \omega, k\lambda)] * \Xi^2(\omega, \omega, k\lambda) + a\, \Xi(\omega, \omega, k\lambda)\, \Xi(\omega, \omega, 2k\lambda) \geq [p\Xi(\omega, \Gamma\omega, \lambda) + q\Xi(\omega, \omega, \lambda)]\, \Xi(\omega, \omega, 2k\lambda),$$

$$\Theta^2(\Gamma\omega, \omega, k\lambda) * [\Theta(\omega, \Gamma\omega, k\lambda)\, \Theta(\omega, \omega, k\lambda)] \lozenge \Theta^2(\omega, \omega, k\lambda) + a\, \Theta(\omega, \omega, k\lambda)\, \Theta(\omega,\omega, 2k\lambda) \leq [p\, \Theta(\omega, \Gamma\omega, \lambda) + q\, \Theta(\omega, \omega, \lambda)]\, \Theta(\omega, \omega, 2k\lambda),$$

$$\Upsilon^2(\Gamma\omega, \omega, k\lambda) * [\Upsilon(\omega, \Gamma\omega, k\lambda)\, \Upsilon(\omega, \omega, k\lambda)] \lozenge \Upsilon^2(\omega, \omega, k\lambda) + a\, \Upsilon(\omega, \omega, k\lambda)\, \Theta(\omega,\omega, 2k\lambda) \leq [p\, \Upsilon(\omega, \Gamma\omega, \lambda) + q\, \Upsilon(\omega, \omega, \lambda)]\, \Upsilon(\omega, \omega, 2k\lambda).$$

Therefore

$\Xi(\Gamma\omega, \omega, k\lambda) + a \geq p\Xi(\omega, \Gamma\omega, \lambda) + q$, $\Theta(\Gamma\omega, \omega, k\lambda) \leq 0$, $\Upsilon(\Gamma\omega, \omega, k\lambda) \leq 0$,

for all $\lambda > 0$, so $\Gamma\omega = \omega$. Taking $\zeta = \zeta_{2n}$ and $\eta = \omega$ in (i), we have as $n \to \infty$,

$\Xi(\omega, \Omega\omega, \lambda) + a \geq p + q$, $\Theta(\omega, \Omega\omega, \lambda) + a\, \Theta(\omega, \Omega\omega, \lambda) \leq 0$ and $\Upsilon(\omega, \Omega\omega, \lambda) + a\, \Upsilon(\omega, \Omega\omega, \lambda) \leq 0$,

for all $\lambda > 0$, so $\Omega\omega = \omega$. Thus ω is a common fixed point of Γ and Ω,

Let β be another common fixed point of Γ and Ω. Then using (i), we have

$\Xi^2(\omega, \beta, k\lambda) + a\, \Xi(\omega, \beta, 2k\lambda) \geq [p + q\, \Xi(\omega, \beta, \lambda)]\, \Xi(\omega, \beta, 2k\lambda)$,

$\Theta^2(\omega, \beta, k\lambda) \leq q\, \Theta(\omega, \beta, \lambda)\, \Theta(\omega, \beta, 2k\lambda)$ and $\Upsilon^2(\omega, \beta, k\lambda) \leq q\, \Upsilon(\omega, \beta, \lambda)\, \Upsilon(\omega, \beta, 2k\lambda)$

and

$\Xi(\omega, \beta, \lambda)\, \Xi(\omega, \beta, 2k\lambda) + a\Xi(\omega, \beta, 2k\lambda) \geq [p + q\, \Xi(\omega, \beta, \lambda)]\Xi(\omega, \beta, 2k\lambda)$,

$\Theta(\omega, \beta, \lambda)\, \Theta(\omega, \beta, 2k\lambda) \leq q\, \Theta(\omega, \beta, \lambda)\, \Theta(\omega, \beta, 2k\lambda)$,

$\Upsilon(\omega, \beta, \lambda)\, \Upsilon(\omega, \beta, 2k\lambda) \leq q\, \Upsilon(\omega, \beta, \lambda)\, \Upsilon(\omega, \beta, 2k\lambda)$.

Thus, it follows that

$\Xi(\omega, \beta, \lambda) \geq \frac{p-a}{1-q} = 1$, $\Theta(n, \beta, \lambda) \leq 0$, $\Upsilon(n, \beta, \lambda) \leq 0$,

for all $\lambda > 0$, so $\omega = \beta$. Hence Γ and Ω have a unique common fixed point in Σ.

Theorem 4.2.

Let Γ, Ω, Λ and $\vee$ be self mappings of a complete NMS Σ satisfying

1. $\Gamma(\Sigma) \subseteq \vee(\Sigma)$, $\Omega(\Sigma) \subseteq \Lambda(\Sigma)$,
2. There exists a constant $k \in (0, 1)$ such that

$$\Xi^2(\Gamma\zeta, \Omega\eta, k\lambda) * [\Xi(\Lambda\zeta, \Gamma\zeta, k\lambda)\, \Xi(\vee\eta, \Omega\eta, k\lambda)] * \Xi^2(\vee\eta, \Omega\eta, k\lambda) + a\Xi(\vee\eta, \Omega\eta, k\lambda)\, \Xi(\Lambda\zeta, \Omega\eta, 2k\lambda) \geq [p\Xi(\Lambda\zeta, \Gamma\zeta, \lambda) + q\Xi(\Lambda\zeta, \vee\eta, \lambda)]\, \Xi(\Lambda\zeta, \Omega\eta, 2k\lambda) \quad (4.2.1)$$

$$\Theta^2(\Gamma\zeta, \Omega\eta, k\lambda) \lozenge [\Theta(\Lambda\zeta, \Gamma\zeta, k\lambda)\, \Theta(\vee\eta, \Omega\eta, k\lambda)] \lozenge \Theta^2(\vee\eta, \Omega\eta, k\lambda) + a\, \Theta(\vee\eta, \Omega\eta, k\lambda)\, \Theta(\Lambda\zeta, \Omega\eta, 2k\lambda) \leq [p\, \Theta(\Lambda\zeta, \Gamma\zeta, \lambda) + q\, \Theta(\Lambda\zeta, \vee\eta, \lambda)]\, \Theta(\Lambda\zeta, \Omega\eta, 2k\lambda) \quad (4.2.2)$$

$$\Upsilon^2(\Gamma\zeta, \Omega\eta, k\lambda) \lozenge [\Upsilon(\Lambda\zeta, \Gamma\zeta, k\lambda)\, \Upsilon(\vee\eta, \Omega\eta, k\lambda)] \lozenge \Upsilon^2(\vee\eta, \Omega\eta, k\lambda) + a\, \Upsilon(\vee\eta, \Omega\eta, k\lambda)\, \Upsilon(\Lambda\zeta, \Omega\eta, 2k\lambda) \leq [p\, \Upsilon(\Lambda\zeta, \Gamma\zeta, \lambda) + q\, \Upsilon(\Lambda\zeta, \vee\eta, \lambda)]\, \Upsilon(\Lambda\zeta, \Omega\eta, 2k\lambda) \quad (4.2.3)$$

for every $\zeta, \eta \in \Sigma$ and $\lambda > 0$, where $0 < p, q < 1$, $0 \leq a < 1$ such that $p + q - a = 1$,

3. The pairs (Γ, Λ) and $(\Omega, \vee)$ are weak compatible of type (γ).

Then Γ, Ω, Λ and $\vee$ have a unique common fixed point in Σ.

Proof: Let $\zeta_0 \in \Sigma$ be an arbitrary point. Since $\Gamma(\Sigma) \subseteq \vee(\Sigma)$ and $\Omega(\Sigma) \subseteq \wedge(\Sigma)$, there exists $\zeta_1, \zeta_2 \in \Sigma$ such that $\Gamma\zeta_0 = \vee\zeta_1 = \eta_1$, $\Omega\zeta_1 = \wedge\zeta_2 = \eta_2$. Because we can construct the sequences $\{\zeta_n\}$, $\{\eta_n\} \subset \Sigma$ such that $\eta_{2n+1} = \Gamma\zeta_{2n} = \vee\zeta_{2n+1}$, $\eta_{2n+2} = \Omega\zeta_{2n+1} = \wedge\zeta_{2n+2}$, for n = 0, 1, 2, . . . , we prove $\{\eta_n\}$ is Cauchy sequence.

For $\zeta = \zeta_{2n}$, $\eta = \zeta_{2n+1}$ by (ii), we have

$$\Xi^2(\Gamma\zeta_{2n}, \Omega\zeta_{2n+1}, k\lambda) * [\Xi(\wedge\zeta_{2n}, \Gamma\zeta_{2n}, k\lambda)\, \Xi(\vee\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda) * \Xi^2(\vee\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda) + a\,\Xi(\vee\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda)\, \Xi(\wedge\zeta_{2n}, \Omega\zeta_{2n+1}, 2k\lambda) \geq [p\Xi(\wedge\zeta_{2n}, \Gamma\zeta_{2n}, \lambda) + q\Xi(\wedge\zeta_{2n}, \vee\zeta_{2n+1}, \lambda)] \times \Xi(\wedge\zeta_{2n}, \Omega\zeta_{2n+1}, 2k\lambda),$$

$$\Theta^2(\Gamma\zeta_{2n}, \Omega\zeta_{2n+1}, k\lambda) \Diamond [\Theta(\wedge\zeta_{2n}, \Gamma\zeta_{2n}, k\lambda)\, \Theta(\vee\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda) \Diamond \Theta^2(\vee\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda) + a\,\Theta(\vee\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda)\, \Theta(\wedge\zeta_{2n}, \Omega\zeta_{2n+1}, 2k\lambda) \leq [p\,\Theta(\wedge\zeta_{2n}, \Gamma\zeta_{2n}, \lambda) + q\,\Theta(\wedge\zeta_{2n}, \vee\zeta_{2n+1}, \lambda)] \times \Theta(\wedge\zeta_{2n}, \Omega\zeta_{2n+1}, 2k\lambda),$$

$$\Upsilon^2(\Gamma\zeta_{2n}, \Omega\zeta_{2n+1}, k\lambda) \Diamond [\Upsilon(\wedge\zeta_{2n}, \Gamma\zeta_{2n}, k\lambda)\, \Upsilon(\vee\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda) \Diamond \Upsilon^2(\vee\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda) + a\,\Upsilon(\vee\zeta_{2n+1}, \Omega\zeta_{2n+1}, k\lambda)\, \Upsilon(\wedge\zeta_{2n}, \Omega\zeta_{2n+1}, 2k\lambda) \leq [p\,\Upsilon(\wedge\zeta_{2n}, \Gamma\zeta_{2n}, \lambda) + q\,\Upsilon(\wedge\zeta_{2n}, \vee\zeta_{2n+1}, \lambda)] \times \Upsilon(\wedge\zeta_{2n}, \Omega\zeta_{2n+1}, 2k\lambda).$$

Hence

$$\Xi(\eta_{2n+1}, \eta_{2n+2}, k\lambda)\, \Xi(\eta_{2n}, \eta_{2n+2}, 2k\lambda) + a\Xi(\eta_{2n+1}, \eta_{2n+2}, k\lambda)\, \Xi(\eta_{2n}, \eta_{2n+2}, 2k\lambda) \geq (p + q)\, \Xi(\eta_{2n}, \eta_{2n+1}, \lambda)\, \Xi(\eta_{2n}, \eta_{2n+2}, 2k\lambda),$$

$$\Theta(\eta_{2n+1}, \eta_{2n+2}, k\lambda)\, \Theta(\eta_{2n}, \eta_{2n+2}, 2k\lambda) + a\,\Theta(\eta_{2n+1}, \eta_{2n+2}, k\lambda)\, \Theta(\eta_{2n}, \eta_{2n+2}, 2k\lambda) \leq (p + q)\, \Theta(\eta_{2n}, \eta_{2n+1}, \lambda)\, \Theta(\eta_{2n}, \eta_{2n+2}, 2k\lambda),$$

$$\Upsilon(\eta_{2n+1}, \eta_{2n+2}, k\lambda)\, \Upsilon(\eta_{2n}, \eta_{2n+2}, 2k\lambda) + a\,\Upsilon(\eta_{2n+1}, \eta_{2n+2}, k\lambda)\, \Upsilon(\eta_{2n}, \eta_{2n+2}, 2k\lambda) \leq (p + q)\, \Upsilon(\eta_{2n}, \eta_{2n+1}, \lambda)\, \Upsilon(\eta_{2n}, \eta_{2n+2}, 2k\lambda).$$

So, we have

$$\Xi(\eta_{2n+1}, \eta_{2n+2}, k\lambda) \geq \Xi(\eta_{2n}, \eta_{2n+1}, \lambda),\ \Theta(\eta_{2n+1}, \eta_{2n+2}, k\lambda) \leq \Theta(\eta_{2n}, \eta_{2n+1}, \lambda) \text{ and}$$
$$\Upsilon(\eta_{2n+1}, \eta_{2n+2}, k\lambda) \leq \Upsilon(\eta_{2n}, \eta_{2n+1}, \lambda).$$

Similarly, also we have

$\Xi(\eta_{2n+2}, \eta_{2n+3}, k\lambda) \geq \Xi(\eta_{2n+1}, \eta_{2n+2}, \lambda)$, $\Theta(\eta_{2n+2}, \eta_{2n+3}, k\lambda) \leq \Theta(\eta_{2n+1}, \eta_{2n+2}, \lambda)$,

$\Upsilon(\eta_{2n+2}, \eta_{2n+3}, k\lambda) \leq \Upsilon(\eta_{2n+1}, \eta_{2n+2}, \lambda)$, for $k \in (0, 1)$, if $k_1 = \frac{1}{k} > 1$ and $\lambda = k_1 \lambda_1$, then

$$\Xi(\eta_n, \eta_{n+1}, \lambda) \geq \Xi(\eta_{n-1}, \eta_n, k_1 \lambda_1) \geq \ldots \geq \Xi(\eta_0, \eta_1, k_1^n \lambda_1),$$
$$\Theta(\eta_n, \eta_{n+1}, \lambda) \leq \Theta(\eta_{n-1}, \eta_n, k_1, \lambda_1) \leq \ldots \leq \Theta(\eta_0, \eta_1, k_1^n \lambda_1),$$
$$\Upsilon(\eta_n, \eta_{n+1}, \lambda) \leq \Upsilon(\eta_{n-1}, \eta_n, k_1, \lambda_1) \leq \ldots \leq \Upsilon(\eta_0, \eta_1, k_1^n \lambda_1).$$

Thus $\{\eta_n\}$ is a Cauchy sequence and completeness of Σ, $\{\eta_n\}$ converges to $\omega \in \Sigma$.

Hence

$$\lim_{n\to\infty} \Gamma\zeta_{2n} = \lim_{n\to\infty} \eta_{2n+1} = \lim_{n\to\infty} \vee\zeta_{2n+1} = \lim_{n\to\infty} \eta_{2n+2} = \lim_{n\to\infty} \Omega\zeta_{2n+1} = \lim_{n\to\infty} \wedge\zeta_{2n+2} = \lim_{n\to\infty} \wedge\zeta_{2n} = \omega.$$

Since Γ, $\wedge$ are weak compatible of type (γ), $A\omega = \wedge\omega$.

Now, taking $\zeta = \omega$ and $\eta = \zeta_{2n+1}$ in (ii), we have as $n \to \infty$.

$$\Xi^2(\Gamma\omega, \omega, k\lambda) * [\Xi(\wedge\omega, \Gamma\omega, k\lambda)\, \Xi(\omega, \omega, k\lambda)] * \Xi^2(\omega, \omega, k\lambda) + a\Xi(\omega, \omega, k\lambda)\, \Xi(\wedge\omega, \omega, 2k\lambda) \geq [p\Xi(\wedge\omega, \Gamma\omega, \lambda) + q\Xi(\wedge\omega, \omega, \lambda)]\Xi(\wedge\omega, \omega, 2k\lambda),$$

$$\Theta^2(\Gamma\omega, \omega, k\lambda) \Diamond [\Theta(\wedge\omega, \Gamma\omega, k\lambda)\, \Theta(\omega, \omega, k\lambda)] \Diamond \Theta^2(\omega, \omega, k\lambda) + a\,\Theta(\omega, \omega, k\lambda)\, \Theta(\wedge\omega, \omega, 2k\lambda) \leq [p\,\Theta(\wedge\omega, \Gamma\omega, \lambda) + q\,\Theta(\wedge\omega, \omega, \lambda)]\, \Theta(\wedge\omega, \omega, 2k\lambda),$$

$$\Upsilon^2(\Gamma\omega, \omega, k\lambda) \Diamond [\Upsilon(\wedge\omega, \Gamma\omega, k\lambda)\, \Upsilon(\omega, \omega, k\lambda)] \Diamond \Upsilon^2(\omega, \omega, k\lambda) + a\,\Upsilon(\omega, \omega, k\lambda)\, \Upsilon(\wedge\omega, \omega, 2k\lambda) \leq [p\,\Upsilon(\wedge\omega, \Gamma\omega, \lambda) + q\,\Upsilon(\wedge\omega, \omega, \lambda)]\, \Upsilon(\wedge\omega, \omega, 2k\lambda).$$

It follows that

$$\Xi^2(\Gamma\omega, \omega, k\lambda) + a\Xi(\Gamma\omega, \omega, 2k\lambda) \geq [p + q\Xi(\Gamma\omega, \omega, \lambda)]\Xi(\Gamma\omega, \omega, 2k\lambda),$$
$$\Theta^2(\Gamma\omega, \omega, k\lambda) \leq q\,\Theta(\Gamma\omega, \omega, \lambda)\, \Theta(\Gamma\omega, \omega, 2k\lambda),\ \Upsilon^2(\Gamma\omega, \omega, k\lambda) \leq q\,\Upsilon(\Gamma\omega, \omega, \lambda)\, \Upsilon(\Gamma\omega, \omega, 2k\lambda).$$

Since $\Xi(\zeta, \eta, .)$ is nondecreasing , $\Theta(\zeta, \eta, .)$ is nonincreasing and $\Upsilon(\zeta, \eta, .)$ is nonincreasing for all $\zeta, \eta \in \Sigma$, we have

$\Xi(\Gamma\omega, \omega, \lambda) \geq \frac{p-a}{1-q} = 1$, $\Theta(\Gamma\omega, \omega, \lambda) \leq \frac{0}{1-q} = 0$, $\Upsilon(\Gamma\omega, \omega, \lambda) \leq \frac{0}{1-q} = 0$.

for all $\lambda > 0$, So $\Gamma\omega = \omega$. Hence $\Gamma\omega = \wedge\omega = \omega$.

Similarly, since Ω, $\vee$ are weak compatible of type (γ), we get $\Omega\omega = \vee\omega$.

For taking $\zeta = \zeta_{2n}$ and $\eta = \omega$ in (ii), we have as $n \to \infty$,

$\Xi^2(\omega, \Omega\omega, k\lambda) * [\Xi(\omega, \omega, k\lambda)\, \Xi(\vee\omega, \Omega\omega, k\lambda)] * \Xi^2(\vee\omega, \Omega\omega, k\lambda) + a\Xi(\vee\omega, \Omega\omega, k\lambda)$
$\Xi(\omega, \Omega\omega, 2k\lambda) \geq [p\Xi(\omega, \omega, \lambda) + q\Xi(\omega, \vee\omega, \lambda)]\, \Xi(\omega, \Omega\omega, 2k\lambda)$,

$\Theta^2(\omega, \Omega\omega, k\lambda) \diamond [\Theta(\omega, \omega, k\lambda)\, \Theta(\vee\omega, \Omega\omega, k\lambda)] \diamond \Theta^2(\vee\omega, \Omega\omega, k\lambda) + a\,\Theta(\vee\omega, \Omega\omega, k\lambda)$
$\Theta(\omega, \Omega\omega, 2k\lambda) \leq [p\,\Theta(\omega, \omega, \lambda) + q\,\Theta(\omega, \vee\omega, \lambda)]\, \Theta(\omega, \Omega\omega, 2k\lambda)$,

$\Upsilon^2(\omega, \Omega\omega, k\lambda) \diamond [\Upsilon(\omega, \omega, k\lambda)\, \Upsilon(\vee\omega, \Omega\omega, k\lambda)] \diamond \Upsilon^2(\vee\omega, \Omega\omega, k\lambda) + a\,\Upsilon(\vee\omega, \Omega\omega, k\lambda)$
$\Upsilon(\omega, \Omega\omega, 2k\lambda) \leq [p\,\Upsilon(\omega, \omega, \lambda) + q\,\Upsilon(\omega, \vee\omega, \lambda)]\, \Upsilon(\omega, \Omega\omega, 2k\lambda)$.

Then,

$\Xi^2(\omega, \Omega\omega, k\lambda) + a\Xi(\omega, \Omega\omega, 2k\lambda) \geq [p + q\Xi(\omega, \vee\omega, \lambda)]\, \Xi(\omega, \Omega\omega, 2k\lambda)$,

$\Theta^2(\omega, \Omega\omega, k\lambda) \leq q\,\Theta(\omega, \vee\omega, \lambda)\, \Theta(\omega, \Omega\omega, 2k\lambda)$, $\Upsilon^2(\omega, \Omega\omega, k\lambda) \leq q\,\Upsilon(\omega, \vee\omega, \lambda)\, \Upsilon(\omega, \Omega\omega, 2k\lambda)$.

Thus it follows that

$\Xi(\omega, \Omega\omega, \lambda) \geq \frac{p-a}{1-q} = 1$, $\Theta(\omega, \Omega\omega, \lambda) \leq \frac{0}{1-q} = 0$ and $\Upsilon(\omega, \Omega\omega, \lambda) \leq 0$, for all $\lambda > 0$, so $\Omega\omega = \omega$.

Hence $\Omega\omega = \vee\omega = \omega$.

Therefore ω is a common fixed point of Γ, Ω, $\wedge$ and $\vee$.

Let β be another common fixed point of Γ, Ω, $\wedge$ and $\vee$. Then we have

$\Xi^2(\Gamma\omega, \Omega\beta, k\lambda) * [\Xi(\wedge\omega, \Gamma\omega, k\lambda)\, \Xi(\vee\beta, \Omega\beta, k\lambda)] * \Xi^2(\vee\beta, \Omega\beta, k\lambda)$
$+ a\Xi(\vee\beta, \Omega\beta, k\lambda)\, \Xi(\wedge\beta, \Omega\beta, 2k\lambda) \geq [p\Xi(\wedge\omega, \Gamma\omega, \lambda) + q\Xi(\wedge\omega, \vee\beta, \lambda)]\Xi(\wedge\omega, \Omega\beta, 2k\lambda)$,

$\Theta^2(\Gamma\omega, \Omega\beta, k\lambda) \diamond [\Theta(\wedge\omega, \Gamma\omega, k\lambda)\, \Theta(\vee\beta, \Omega\beta, k\lambda)] \diamond \Theta^2(\vee\beta, \Omega\beta, k\lambda)$
$+ a\,\Theta(\vee\beta, \Omega\beta, k\lambda)\, \Theta(\wedge\beta, \Omega\beta, 2k\lambda) \leq [p\,\Theta(\wedge\omega, \Gamma\omega, \lambda) + q\,\Theta(\wedge\omega, \vee\beta, \lambda)]\, \Theta(\wedge\omega, \Omega\beta, 2k\lambda)$,

$\Upsilon^2(\Gamma\omega, \Omega\beta, k\lambda) \diamond [\Upsilon(\wedge\omega, \Gamma\omega, k\lambda)\, \Upsilon(\vee\beta, \Omega\beta, k\lambda)] \diamond \Upsilon^2(\vee\beta, \Omega\beta, k\lambda)$
$+ a\,\Upsilon(\vee\beta, \Omega\beta, k\lambda)\, \Upsilon(\wedge\beta, \Omega\beta, 2k\lambda) \leq [p\,\Upsilon(\wedge\omega, \Gamma\omega, \lambda) + q\,\Upsilon(\wedge\omega, \vee\beta, \lambda)]\, \Upsilon(\wedge\omega, \Omega\beta, 2k\lambda)$,

So,

$\Xi^2(\omega, \beta, k\lambda) + a\Xi(\omega, \beta, 2k\lambda) \geq [p + q\,\Xi(\omega, \beta, \lambda)]\Xi(\omega, \beta, 2k\lambda)$,

$\Theta^2(\omega, \beta, k\lambda) \leq q\,\Theta(\omega, \beta, \lambda)\, \Theta(\omega, \beta, 2k\lambda)$ and $\Upsilon^2(\omega, \beta, k\lambda) \leq q\,\Upsilon(\omega, \beta, \lambda)\, \Upsilon(\omega, \beta, 2k\lambda)$.

Therefore

$\Xi(\omega, \beta, \lambda) \geq \frac{p-a}{1-q} = 1$, $\Theta(\omega, \beta, \lambda) \leq \frac{0}{1-q} = 0$, $\Upsilon(\omega, \beta, \lambda) \leq 0$,

for all $\lambda > 0$, so $\omega = \beta$, hence Γ, Ω, $\wedge$ and $\vee$ have unique common fixed point on Σ.

Example 4.3.

Let (Σ, d) be a metric space with $\Sigma = [0,1]$. Denote $\omega * \tau = \min\{\omega, \tau\}$ and $\omega \diamond \tau = \max\{\omega, \tau\}$ for $\omega, \tau \in [0,1]$ and let Ξ_d, Θ_d, Υ_d be neutroshopic sets on $\Sigma^2 \times [0, \infty]$ defined as follows;

$\Xi_d(\zeta, \eta, \lambda) = \frac{\lambda}{\lambda + d(\zeta, \eta)}$; $\Theta_d(\zeta, \eta, \lambda) = \frac{d(\zeta,\eta)}{\lambda + d(\zeta,\eta)}$; $\Upsilon_d(\zeta, \eta, \lambda) = \frac{d(\zeta,\eta)}{\lambda}$.

Then $(\Xi_d, \Theta_d, \Upsilon_d)$ is an NMS on Σ and $(\Sigma, \Xi_d, \Theta_d, \Upsilon_d, *, \diamond)$ is an NMS.

Define self mappings Γ, Ω, $\wedge$ and $\vee$ by

$\Gamma(\Sigma) = 1$; $\Omega(\Sigma) = 1$; $\wedge(\Sigma) = \begin{cases} 1 \text{ if } \zeta \text{ is rational} \\ 0 \text{ if } \zeta \text{ is irrational} \end{cases}$; $\vee(\Sigma) = \frac{\zeta+1}{2}$

If we define $\{\zeta_n\} \subset \Sigma$ by $\zeta_n = 1 - \frac{1}{n}$, then we have for $\lim_{n\to\infty} \Gamma\zeta_n = \lim_{n\to\infty} \Omega\zeta_n = 1$ and $\Gamma 1 = 1 = \wedge 1$.

$\lim_{n\to\infty} \Xi(\wedge\Gamma\zeta_n, 1, \lambda) \leq \Xi(\Gamma 1, 1, \lambda) = 1$; $\lim_{n\to\infty} \Theta(\wedge\Gamma\zeta_n, 1, \lambda) \geq \Theta(\Gamma 1, 1, \lambda) = 0$;

$\lim_{n\to\infty} \Upsilon(\wedge\Gamma\zeta_n, 1, \lambda) \geq \Upsilon(\Gamma 1, 1, \lambda) = 0$.

Also, for $\lim_{n\to\infty} \Omega\zeta_n = \lim_{n\to\infty} \vee\zeta_n = 1$ and $\Omega 1 = 1 = \vee 1$.

$\lim_{n\to\infty} \Xi(\vee\Omega\zeta_n, 1, \lambda) \leq \Xi(\Omega 1, 1, \lambda) = 1$; $\lim_{n\to\infty} \Theta(\vee\Omega\zeta_n, 1, \lambda) \geq \Theta(\Omega 1, 1, \lambda) = 0$;

$\lim_{n\to\infty} \Upsilon(\vee\Omega\zeta_n, 1, \lambda) \geq \Upsilon(\Omega 1, 1, \lambda) = 0$.

Therefore, (Γ, $\wedge$) and (Ω, $\vee$) are weak compatible of type (γ). Then all the conditions of Theorem 4.2. are satisfied and 1 is a unique common fixed point of Γ, Ω, $\wedge$ and $\vee$ on Σ.

Conclusion: In this study, we have made common fixed point results for weak compatiple maps of type in neutrosophic metric Space. There is a degree to set up many fixed point brings about the spaces like fuzzy metric, generalized fuzzy metric, bipolar and partial fuzzy metric spaces by utilizing the idea of Neutrosophic Set**.**

References:

1. Atanassov K. (1986). Intuitionistic Fuzzy Sets. *Fuzzy Sets and System, 20*, 87 - 96.
2. Alaca, C., Turkoglu D., Yildiz C.(2006). Fixed points in intuitionistic fuzzy metric spaces. *Chaos Solitons and Fractals* , 29, 1073 - 1078.
3. Basset, M., Gamal, A., Son, L. H. and Smarandache, F.(2020). A Bipolar Neutrosophic Multi Criteria Decision Making Framework for Professional Selection. *Applied Sciences*, 10,4.
4. Das. S, Das. R and Tripathy. B.C.(2020). Multi criteria group decision making model using single valued neutrosophic set. *LogForum*, 16(3), 421 - 429.
5. Schweizer B and Sklar A.(1960). Statistical metric spaces. *Pacific Journal of Mathematics*., 10, 314 - 334.
6. Jeyaraman. M, Jeyanthi. V, Mangayarkkarasi. A.N, Smarandache. F.(2021). Some New Structures in Neutrosophic Metric Spaces. *Neutrosophic sets and systems*, 42, 49 - 64**.**
7. Jungck. G.(1976). Commuting maps and fixed points. *The American Mathematical Monthly*, 83 , 261 - 263.
8. Jungck. G.(1986). Compatible mappings and common fixed points. *International Journal of Mathematics and Mathematical Sciences,* 9 , 771 - 774.
9. Kramosil O, Michalek J.(1975). Fuzzy metric and statistical metric spaces. *Kybernetika*, 11, 326 - 334.
10. Kirisci, M., Simsek, N.(2020). Neutrosophic metric spaces. *Mathematical* Sciences, 14, 241 - 248.
11. Kutukcu S., Turkoglu D., Yildiz C.(2006). Common fixed points of compatible maps of type (β) on fuzzy metric spaces. *Communications* of the *Korean Mathematical Society*, 21, 1, 89 - 100.
12. Menger, K.M.(1942). Statistical Metrics. *Proceedings of the National Academy of Sciences*, 28, 535 - 537.
13. Memet Sahin, Abdullah Kargin and Murat Yucel.(2020). Neutrosophic Triplet Partial g - Metric Space. *Neutrosophic Sets and Systems*, 33, 116 - 134.
14. Park J.H.(2004). Intuitionistic fuzzy metric spaces. *Chaos Solitons and Fractals*, 22, 1039 -1046.
15. Park J.S., Kim S.Y.(2008). Common fixed point theorem and example in intuitionistic fuzzy metric space. *K.I.I.S*., vol. 18, 4, 524 - 552.
16. Park J.S.(2007). Kwun Y.C., Some fixed point theorems in the intuitionistic fuzzy metric spaces. *F.J.M.S.*, vol. 24, 2, 227 - 239.
17. Park J.S., Kwun Y.C., Park J.H.(2005). A fixed point theorem in the intuitionistic fuzzy metric spaces. *F.J.M.S.*, vol. 16, 2, 137 - 149.
18. Schweizer. B and Sklar. A.(1960). Statistical metric spaces. *Pacific Journal of Mathematics*, vol. 10, 313 – 334.
19. Sedghi S, Turkoglu. D, Shobe.N.(2009). Common fixed point of compatible maps of type(γ) on complete fuzzy metric spaces. *Communications* of the *Korean Mathematical Society*, *24*, *4*, 581 - 594.

20. Smarandache, F. A.(1998). Unifying Field in Logics, Neutrosophy: Neutrosophic Probability, Set and Logic. *American Research Press: Reheboth*, MA, USA.
21. Smarandache, F. (1998). Neutrosophy: Neutrosophic Probability, Set and Logic; ProQuest Information and Learning: Ann Arbor, MI, USA, pp. 105
22. Smarandache, F.(2005). Neutrosophic set, a generalisation of the intuitionistic fuzzy sets. *International Journal of Pure* and *Applied Mathematics*, 24, 287 - 297.
23. Sowndrarajan. S, Jeyaraman. M.(2020). Fixed Point theorems in neutrosophic metric spaces. *Neutrosophic Sets and Systems*, 36, 251 - 268.
24. Sowndrarajan. S, Jeyaraman. M.(2020). Fixed point Results for Contraction Theorems in Neutrosophic Metric Spaces. *Neutrosophic sets and systems, 36*, 308 – 318.
25. Suganthi. M, Jeyaraman. M.(2021). A Generalized Neutrosophic Metric Space and Coupled Coincidence point Results. *Neutrosophic sets and systems,* 42, 253 – 269.
26. Simsek N, Kiri_sci M.(2019). Fixed Point Theorems in Neutrosophic Metric Spaces. *Sigma* Journal of *Engineering and Natural Sciences*., 10(2), 221 - 230.
27. Tripathy. B. C., Paul. S. and Das. N. R.(2013). Banach's and Kannan's fixed point results in fuzzy 2 – metric spaces. *Proyecciones J. Math.,* 32(4), 359 – 375.
28. Tripathy. B. C., Paul. S. and Das. N. R.(2014). A fixed point theorem in a generalized fuzzy metric space. *Boletim da Sociedade Paranaense de Matematica*, 32(2), 221 – 227.
29. Turkoglu D., Alaca C., Cho Y.J., Yildiz C.(2006). Common fixed point theorems in intuitionistic fuzzy metric spaces. *J Appl Math Comput,* 22, 411 - 424.
30. Zadeh LA.(1965). Fuzzy sets. *Information and* Control, *8*, 338 - 353.

Received: Feb18, 2022. Accepted: Jun 4, 2022