Neutrosophic Hyper BCK-Ideals

S. Khademan[1], M. M. Zahedi [2,*], R. A. Borzooei [3], Y. B. Jun [3,4]

[1]Department of Mathematics, Tarbiat Modares University, Tehran, Iran.

E-mail: somayeh.khademan@modares.ac.ir, Khademans@gmail.com

[2]Department of Mathematics, Graduate University of Advanced Technology, Kerman, Iran.

E-mail: zahedi_mm@kgut.ac.ir, zahedi_mm@yahoo.com

[3]Department of Mathematics, Shahid Beheshti University, Tehran 1983963113, Iran.

E-mail: borzooei@sbu.ac.ir

[4]Department of Mathematics Education, Gyeongsang National University, Jinju 52828, Korea.

E-mail: skywine@gmail.com

[*]Correspondence: M. M. Zahedi (zahedi_mm@kgut.ac.ir, zahedi_mm@yahoo.com)

Abstract: In this paper we introduced the notions of neutrosophic (strong, weak, s-weak) hyper BCK-ideal and reflexive neutrosophic hyper BCK-ideal. Some relevant properties and their relations are indicated. Characterization of neutrosophic (weak) hyper BCK-ideal is considered. Conditions for a neutrosophic set to be a (reflexive) neutrosophic hyper BCK-ideal and a neutrosophic strong hyper BCK-ideal are discussed. Also, conditions for a neutrosophic weak hyper BCK-ideal to be a neutrosophic s-weak hyper BCK-ideal, and conditions for a neutrosophic strong hyper BCK-ideal to be a reflexive neutrosophic hyper BCK-ideal are provided.

Keywords: Hyper BCK-algebra; hyper BCK-ideals; neutrosophic (strong, weak, s-weak) hyper BCK-ideal; reflexive neutrosophic hyper BCK-ideal.

1 Introduction

Algebraic hyperstructures represent a natural extension of classical algebraic structures and they were introduced in 1934 by the French mathematician F. Marty [17] when Marty defined hypergroups, began to analyze their properties, and applied them to groups and relational algebraic functions (See [17]). Since then, many papers and several books have been written on this topic. Hyperstructures have many applications to several sectors of both pure and applied sciences. (See [4, 5, 8, 11, 14, 19, 25]). In [16], Jun et al. applied the hyperstructures to BCK-algebras, and introduced the concept of a hyper BCK-algebra which is a generalization of a BCK-algebra. Since then, Jun et al. studied more notions and results in [12] and [15]. Also, several fuzzy versions of hyper BCK-algebras have been considered in [10] and [13]. The neutrosophic set, which is developed by Smarandache ([20], [21] and [22]), is a more general platform that extends the notions of classic set, (intuitionistic) fuzzy set and interval valued (intuitionistic) fuzzy set. Borzooei et al. [6] studied neutrosophic deductive filters on

BL-algebras. Zhang et al. [26] applied the notion of neutrosophic set to pseudo-BCI algebras, and discussed neutrosophic regular filters and fuzzy regular filters. Neutrosophic set theory is applied to varios part and received attentions from many researches were proceed to develop, improve and expand the neutrosophic theory ([1], [2], [3], [7], [9], [18], [23] and [24]).

Our purpose is to introduce the notions of neutrosophic (strong, weak, s-weak) hyper BCK-ideal, and reflexive neutrosophic hyper BCK-ideal. We consider their relations and related properties. We discuss characterizations of neutrosophic (weak) hyper BCK-ideal. We give conditions for a neutrosophic set to be a (reflexive) neutrosophic hyper BCK-ideal and a neutrosophic strong hyper BCK-ideal. We are interested in finding some provisions for a neutrosophic strong hyper BCK-ideal to be a reflexive neutrosophic hyper BCK-ideal. We discuss conditions for a neutrosophic weak hyper BCK-ideal to be a neutrosophic s-weak hyper BCK-ideal.

2 Preliminaries

In this section, we give the basic definitions of hyper BCK-ideals and neutrosophic set.

For a nonempty set H a function $\circ : H \times H \to \mathcal{P}^*(H)$ is called a hyper operation on H. If $A, B \subseteq H$, then $A \circ B = \cup\{a \circ b \mid a \in A,\ b \in B\}$.

A nonempty set H with a hyper operation "$\circ$" and a constant 0 is called a hyper BCK-algebra (See [16]), if it satisfies the following conditions: for any $x, y, z \in H$,

$(HBCK1)$ $(x \circ z) \circ (y \circ z) \ll x \circ y$,

$(HBCK2)$ $(x \circ y) \circ z = (x \circ z) \circ y$,

$(HBCK3)$ $x \circ H \ll \{x\}$,

$(HBCK4)$ $x \ll y$ and $y \ll x$ imply $x = y$,

where $x \ll y$ is defined by $0 \in x \circ y$. Also for any $A, B \subseteq H$, $A \ll B$ is defined by $\forall a \in A$, $\exists b \in B$ such that $a \ll b$.

Lemma 2.1. ([16]) In a hyper BCK-algebra H, the condition (HBCK3) is equivalent to the following condition:

$$(\forall x, y \in H)\,(x \circ y \ll \{x\})\,. \tag{2.1}$$

Lemma 2.2. ([16]) Let H be a hyper BCK-algebra. Then

(i) $x \circ 0 \ll \{x\}$, $0 \circ x \ll \{0\}$ and $0 \circ 0 \ll \{0\}$, for all $x \in H$

(ii) $(A \circ B) \circ C = (A \circ C) \circ B$, $A \circ B \ll A$ and $0 \circ A \ll \{0\}$, for any nonempty subsets A, B and C of H.

Lemma 2.3. ([16]) In any hyper BCK-algebra H, we have:

$$0 \circ 0 = \{0\},\ 0 \ll x,\ x \ll x \text{ and } A \ll A, \tag{2.2}$$
$$A \subseteq B \text{ implies } A \ll B, \tag{2.3}$$
$$0 \circ x = \{0\} \text{ and } 0 \circ A = \{0\}, \tag{2.4}$$
$$A \ll \{0\} \text{ implies } A = \{0\}, \tag{2.5}$$
$$x \in x \circ 0, \tag{2.6}$$

for all $x, y, z \in H$ and for all nonempty subsets A, B and C of H.

Let $I \subseteq H$ be such that $0 \in I$. Then I is said to be (See [16] and [15])

- hyper BCK-ideal of H if

$$(\forall x, y \in H)\,(x \circ y \ll A,\ y \in A \ \Rightarrow\ x \in A)\,. \tag{2.7}$$

- weak hyper BCK-ideal of H if

$$(\forall x, y \in H)\,(x \circ y \subseteq A,\ y \in A \ \Rightarrow\ x \in A)\,. \tag{2.8}$$

- strong hyper BCK-ideal of H if

$$(\forall x, y \in H)\,((x \circ y) \cap A \neq \emptyset,\ y \in A \ \Rightarrow\ x \in A)\,. \tag{2.9}$$

A subset I of a hyper BCK-algebra H is said to be reflexive if $(x \circ x) \subseteq I$ for all $x \in H$.

Let H be a non-empty set. A neutrosophic set (NS) in H (See [21]) is a structure of the form:

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in H\}$$

where $A_T : H \to [0, 1]$ is a truth membership function, $A_I : H \to [0, 1]$ is an indeterminate membership function, and $A_F : H \to [0, 1]$ is a false membership function. For abbreviation, we continue to write $A = (A_T, A_I, A_F)$ for the neutrosophic set

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in H\}.$$

Given a neutrosophic set $A = (A_T, A_I, A_F)$ in a hyper BCK-algebra H and a subset S of H, by ${}_*A_T$, ${}^*A_T$, ${}_*A_I$, ${}^*A_I$, ${}_*A_F$ and ${}^*A_F$ we mean

$$\begin{aligned}
{}_*A_T(S) &= \inf_{a\in S} A_T(a) \text{ and } {}^*A_T(S) = \sup_{a\in S} A_T(a),\\
{}_*A_I(S) &= \inf_{a\in S} A_I(a) \text{ and } {}^*A_I(S) = \sup_{a\in S} A_I(a),\\
{}_*A_F(S) &= \inf_{a\in S} A_F(a) \text{ and } {}^*A_F(S) = \sup_{a\in S} A_F(a),
\end{aligned}$$

respectively.

Notation. From now on,in this paper, we assume that H is a hyper BCK-algebra.

3 Neutrosophic hyper BCK-ideals

In this section, we introduced the notions of neutrosophic (strong, weak, s-weak) hyper BCK-ideal, reflexive neutrosophic hyper BCK-ideal and discuss their properties.

Definition 3.1. Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in H. Then A is said to be a neutrosophic hyper BCK-ideal of H if it satisfies the following assertions for all $x, y \in H$,

$$\left(x \ll y \;\Rightarrow \begin{cases} A_T(x) \geq A_T(y) \\ A_I(x) \geq A_I(y) \\ A_F(x) \leq A_F(y) \end{cases} \right), \tag{3.1}$$

$$\left(\begin{array}{l} A_T(x) \geq \min\{{}_*A_T(x \circ y), A_T(y)\} \\ A_I(x) \geq \min\{{}_*A_I(x \circ y), A_I(y)\} \\ A_F(x) \leq \max\{{}^*A_F(x \circ y), A_F(y)\} \end{array} \right). \tag{3.2}$$

Example 3.2. Let $H = \{0, a, b\}$ be a hyper BCK-algebra. The hyper operation "$\circ$" on H described by Table 1.

Table 1: Cayley table for the binary operation "$\circ$"

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0, a\}$	$\{0, a\}$
b	$\{b\}$	$\{a, b\}$	$\{0, a, b\}$

We define a neutrosophic set $A = (A_T, A_I, A_F)$ on H by Table 2.

Table 2: Tabular representation of $A = (A_T, A_I, A_F)$

H	$A_T(x)$	$A_I(x)$	$A_F(x)$
0	0.77	0.65	0.08
a	0.55	0.47	0.57
b	0.11	0.27	0.69

It is easy to check that $A = (A_T, A_I, A_F)$ is a neutrosophic hyper BCK-ideal of H.

Proposition 3.3. For any neutrosophic hyper BCK-ideal $A = (A_T,\, A_I,\, A_F)$ of H, the following assertions are valid.

(1) $A = (A_T,\, A_I,\, A_F)$ satisfies

$$(\forall x \in H) \left(\begin{array}{l} A_T(0) \geq A_T(x) \\ A_I(0) \geq A_I(x) \\ A_F(0) \leq A_F(x) \end{array} \right). \tag{3.3}$$

(2) If $A = (A_T, A_I, A_F)$ satisfies

$$(\forall S \subseteq H)(\exists\, a, b, c \in S) \begin{pmatrix} A_T(a) = {}_*A_T(S) \\ A_I(b) = {}_*A_I(S) \\ A_F(c) = {}^*A_F(S) \end{pmatrix}, \tag{3.4}$$

then the following assertion is valid.

$$(\forall x, y \in H)(\exists\, a, b, c \in x \circ y) \begin{pmatrix} A_T(x) \geq \min\{A_T(a), A_T(y)\} \\ A_I(x) \geq \min\{A_I(b), A_I(y)\} \\ A_F(x) \leq \max\{A_F(c), A_F(y)\} \end{pmatrix}. \tag{3.5}$$

Proof. By (2.2) and (3.1) we have

$$A_T(0) \geq A_T(x),\ A_I(0) \geq A_I(x) \text{ and } A_F(0) \leq A_F(x).$$

Assume that $A = (A_T, A_I, A_F)$ satisfies the condition (3.4). For all $x, y \in H$, there exists $a_0, b_0, c_0 \in x \circ y$ such that

$$A_T(a_0) = {}_*A_T(x \circ y),\ A_I(b_0) = {}_*A_I(x \circ y) \text{ and } A_F(c_0) = {}^*A_F(x \circ y).$$

Now condition (3.2) implies that

$$\begin{aligned} A_T(x) &\geq \min\{{}_*A_T(x \circ y), A_T(y)\} = \min\{A_T(a_0), A_T(y)\} \\ A_I(x) &\geq \min\{{}_*A_I(x \circ y), A_I(y)\} = \min\{A_I(b_0), A_I(y)\} \\ A_F(x) &\leq \max\{{}^*A_F(x \circ y), A_F(y)\} = \max\{A_F(c_0), A_F(y)\}. \end{aligned}$$

This completes the proof. □

We define the following sets:

$$\begin{aligned} U(A_T, \varepsilon_T) &:= \{x \in H \mid A_T(x) \geq \varepsilon_T\}, \\ U(A_I, \varepsilon_I) &:= \{x \in H \mid A_I(x) \geq \varepsilon_I\}, \\ L(A_F, \varepsilon_F) &:= \{x \in H \mid A_F(x) \leq \varepsilon_F\}, \end{aligned}$$

where $A = (A_T, A_I, A_F)$ is a neutrosophic set in H and $\varepsilon_T, \varepsilon_I, \varepsilon_F \in [0, 1]$.

Lemma 3.4 ([12]). Let A be a subset of H. If I is a hyper BCK-ideal of H such that $A \ll I$, then A is contained in I.

Theorem 3.5. A neutrosophic set $A = (A_T, A_I, A_F)$ is a neutrosophic hyper BCK-ideal of H if and only if the nonempty sets $U(A_T, \varepsilon_T)$, $U(A_I, \varepsilon_I)$ and $L(A_F, \varepsilon_F)$ are hyper BCK-ideals of H for all ε_T, ε_I, $\varepsilon_F \in [0, 1]$.

Proof. Assume that $A = (A_T, A_I, A_F)$ is a neutrosophic hyper BCK-ideal of H and suppose that $U(A_T, \varepsilon_T)$, $U(A_I, \varepsilon_I)$ and $L(A_F, \varepsilon_F)$ are nonempty for all ε_T, ε_I, $\varepsilon_F \in [0, 1]$. It is easy to see that

$0 \in U(A_T, \varepsilon_T)$, $0 \in U(A_I, \varepsilon_I)$ and $0 \in L(A_F, \varepsilon_F)$. Let $x, y \in H$ be such that $x \circ y \ll U(A_T, \varepsilon_T)$ and $y \in U(A_T, \varepsilon_T)$. Then $A_T(y) \geq \varepsilon_T$ and for any $a \in x \circ y$ there exists $a_0 \in U(A_T, \varepsilon_T)$ such that $a \ll a_0$. We conclude from (3.1) that $A_T(a) \geq A_T(a_0) \geq \varepsilon_T$ for all $a \in x \circ y$. Hence ${}_*A_T(x \circ y) \geq \varepsilon_T$, and so

$$A_T(x) \geq \min\{{}_*A_T(x \circ y), A_T(y)\} \geq \varepsilon_T,$$

that is, $x \in U(A_T, \varepsilon_T)$. Similarly, we show that if $x \circ y \ll U(A_I, \varepsilon_I)$ and $y \in U(A_I, \varepsilon_I)$, then $x \in U(A_I, \varepsilon_I)$. Hence $U(A_T, \varepsilon_T)$ and $U(A_I, \varepsilon_I)$ are hyper BCK-ideals of H. Let $x, y \in H$ be such that $x \circ y \ll L(A_F, \varepsilon_F)$ and $y \in L(A_F, \varepsilon_F)$. Then $A_F(y) \leq \varepsilon_F$. Let $b \in x \circ y$. Then there exists $b_0 \in L(A_F, \varepsilon_F)$ such that $b \ll b_0$, which implies from (3.1) that $A_F(b) \leq A_F(b_0) \leq \varepsilon_F$. Thus ${}^*A_F(x \circ y) \leq \varepsilon_F$, and so

$$A_F(x) \leq \max\{{}^*A_F(x \circ y), A_F(y)\} \leq \varepsilon_F.$$

Hence $x \in L(A_F, \varepsilon_F)$, and therefore $L(A_F, \varepsilon_F)$ is a hyper BCK-ideal of H.

Conversely, suppose that the nonempty sets $U(A_T, \varepsilon_T)$, $U(A_I, \varepsilon_I)$ and $L(A_F, \varepsilon_F)$ are hyper BCK-ideals of H for all ε_T, ε_I, $\varepsilon_F \in [0, 1]$. Let $x, y \in H$ be such that $x \ll y$. Then

$$y \in U(A_T, A_T(y)) \cap U(A_I, A_I(y)) \cap L(A_F, A_F(y)),$$

and thus $x \ll U(A_T, A_T(y))$, $x \ll U(A_I, A_I(y))$ and $x \ll L(A_F, A_F(y))$. According to Lemma 3.4 we have $x \in U(A_T, A_T(y))$, $x \in U(A_I, A_I(y))$ and $x \in L(A_F, A_F(y))$ which imply that $A_T(x) \geq A_T(y)$, $A_I(x) \geq A_I(y)$ and $A_F(x) \leq A_F(y)$. For any $x, y \in H$, let $\varepsilon_T := \min\{{}_*A_T(x \circ y), A_T(y)\}$, $\varepsilon_I := \min\{{}_*A_I(x \circ y), A_I(y)\}$ and $\varepsilon_F := \max\{{}^*A_F(x \circ y), A_F(y)\}$. Then

$$y \in U(A_T, \varepsilon_T) \cap U(A_I, \varepsilon_I) \cap L(A_F, \varepsilon_F),$$

and for each $a_T, b_I, c_F \in x \circ y$ we have

$$A_T(a_T) \geq {}_*A_T(x \circ y) \geq \min\{{}_*A_T(x \circ y), A_T(y)\} = \varepsilon_T,$$

$$A_I(b_I) \geq {}_*A_I(x \circ y) \geq \min\{{}_*A_I(x \circ y), A_I(y)\} = \varepsilon_I$$

and

$$A_F(c_F) \leq {}^*A_F(x \circ y) \leq \max\{{}^*A_F(x \circ y), A_F(y)\} = \varepsilon_F.$$

Hence $a_T \in U(A_T, \varepsilon_T)$, $b_I \in U(A_I, \varepsilon_I)$ and $c_F \in L(A_F, \varepsilon_F)$, and so $x \circ y \subseteq U(A_T, \varepsilon_T)$, $x \circ y \subseteq U(A_I, \varepsilon_I)$ and $x \circ y \subseteq L(A_F, \varepsilon_F)$. By (2.3), we have $x \circ y \ll U(A_T, \varepsilon_T)$, $x \circ y \ll U(A_I, \varepsilon_I)$ and $x \circ y \ll L(A_F, \varepsilon_F)$. It follows from (2.7) that

$$x \in U(A_T, \varepsilon_T) \cap U(A_I, \varepsilon_I) \cap L(A_F, \varepsilon_F).$$

Hence

$$A_T(x) \geq \varepsilon_T = \min\{{}_*A_T(x \circ y), A_T(y)\},$$

$$A_I(x) \geq \varepsilon_I = \min\{{}_*A_I(x \circ y), A_I(y)\}$$

and

$$A_F(x) \leq \varepsilon_F = \max\{{}^*A_F(x \circ y), A_F(y)\}.$$

Therefore $A = (A_T, A_I, A_F)$ is a neutrosophic hyper BCK-ideal of H. □

Theorem 3.6. If $A = (A_T, A_I, A_F)$ is a neutrosophic hyper BCK-ideal of H, then the set

$$J := \{x \in H \mid A_T(x) = A_T(0), A_I(x) = A_I(0), A_F(x) = A_F(0)\} \tag{3.6}$$

is a hyper BCK-ideal of H.

Proof. It is easy to check that $0 \in J$. Let $x, y \in H$ be such that $x \circ y \ll J$ and $y \in J$. Then $A_T(y) = A_T(0)$, $A_I(y) = A_I(0)$ and $A_F(y) = A_F(0)$. Let $a \in x \circ y$. Then there exists $a_0 \in J$ such that $a \ll a_0$, and thus by (3.1), $A_T(a) \geq A_T(a_0) = A_T(0)$, $A_I(a) \geq A_I(a_0) = A_I(0)$ and $A_F(a) \leq A_F(a_0) = A_F(0)$. It follows from (3.2) that

$$A_T(x) \geq \min\{{}_*A_T(x \circ y), A_T(y)\} \geq A_T(0),$$

$$A_I(x) \geq \min\{{}_*A_I(x \circ y), A_I(y)\} \geq A_I(0)$$

and

$$A_F(x) \leq \max\{{}^*A_F(x \circ y), A_F(y)\} \leq A_F(0).$$

Hence $A_T(x) = A_T(0)$, $A_I(x) = A_I(0)$ and $A_F(x) = A_F(0)$, that is, $x \in J$. Therefore J is a hyper BCK-ideal of H. □

We provide conditions for a neutrosophic set $A = (A_T, A_I, A_F)$ to be a neutrosophic hyper BCK-ideal of H.

Theorem 3.7. Let H satisfy $|x \circ y| < \infty$ for all $x, y \in H$, and let $\{J_t \mid t \in \Lambda \subseteq [0, 0.5]\}$ be a collection of hyper BCK-ideals of H such that

$$H = \bigcup_{t \in \Lambda} J_t, \tag{3.7}$$

$$(\forall s, t \in \Lambda)(s > t \iff J_s \subset J_t). \tag{3.8}$$

Then a neutrosophic set $A = (A_T, A_I, A_F)$ in H defined by

$$A_T : H \to [0,1],\ x \mapsto \sup\{t \in \Lambda \mid x \in J_t\},$$
$$A_I : H \to [0,1],\ x \mapsto \sup\{t \in \Lambda \mid x \in J_t\},$$
$$A_F : H \to [0,1],\ x \mapsto \inf\{t \in \Lambda \mid x \in J_t\}$$

is a neutrosophic hyper BCK-ideal of H.

Proof. We first shows that

$$q \in [0,1] \Rightarrow \bigcup_{p\in\Lambda, p\geq q} J_p \text{ is a hyper BCK-ideal of } H. \tag{3.9}$$

It is clear that $0 \in \bigcup_{p\in\Lambda,p\geq q} J_p$ for all $q \in [0,1]$. Let $x, y \in H$ be such that $x \circ y = \{a_1, a_2, \cdots, a_n\}$, $x \circ y \ll \bigcup_{p\in\Lambda,p\geq q} J_p$ and $y \in \bigcup_{p\in\Lambda,p\geq q} J_p$. Then $y \in J_r$ for some $r \in \Lambda$ with $q \leq r$, and for any $a_i \in x \circ y$ there exists $b_i \in \bigcup_{p\in\Lambda,p\geq q} J_p$, and so $b_i \in J_{t_i}$ for some $t_i \in \Lambda$ with $q \leq t_i$, such that $a_i \ll b_i$. If we let $t := \min\{t_i \mid i \in \{1, 2, \cdots, n\}\}$, then $J_{t_i} \subset J_t$ for all $i \in \{1, 2, \cdots, n\}$ and so $x \circ y \ll J_t$ with $q \leq t$. We may assume that $r > t$ without loss of generality, and so $J_r \subset J_t$. By (2.7), we have $x \in J_t \subset \bigcup_{p\in\Lambda,p\geq q} J_p$. Hence $\bigcup_{p\in\Lambda,p\geq q} J_p$ is a hyper BCK-ideal of H. Next, we consider the following two cases:

$$\text{(i) } t = \sup\{q \in \Lambda \mid q < t\}, \;\; \text{(ii) } t \neq \sup\{q \in \Lambda \mid q < t\}. \tag{3.10}$$

If the first case is valid, then

$$x \in U(A_T, t) \;\Leftrightarrow\; x \in J_q \text{ for all } q < t \;\Leftrightarrow\; x \in \bigcap_{q<t} J_q,$$

and so $U(A_T, t) = \bigcap_{q<t} J_q$ which is a hyper BCK-ideal of H. Similarly, we know that $U(A_I, t)$ is a hyper BCK-ideal of H. For the second case, we will show that $U(A_T, t) = \bigcup_{q\geq t} J_q$. If $x \in \bigcup_{q\geq t} J_q$, then $x \in J_q$ for some $q \geq t$. Thus $A_T(x) \geq q \geq t$, and so $x \in U(A_T, t)$ which shows that $\bigcup_{q\geq t} J_q \subseteq U(A_T, t)$. Assume that $x \notin \bigcup_{q\geq t} J_q$. Then $x \notin J_q$ for all $q \geq t$, and so there exist $\delta > 0$ such that $(t - \delta, t) \cap \Lambda = \emptyset$. Thus $x \notin J_q$ for all $q > t - \delta$, that is, if $x \in J_q$ then $q \leq t - \delta < t$. Hence $x \notin U(A_T, t)$. This shows that $U(A_T, t) = \bigcup_{q\geq t} J_q$ which is a hyper BCK-ideal of H by (3.9). Similarly we can prove that $U(A_I, t)$ is a hyper BCK-ideal of H. Now we consider the following two cases:

$$s = \inf\{r \in \Lambda \mid s < r\} \text{ and } s \neq \inf\{r \in \Lambda \mid s < r\}. \tag{3.11}$$

The first case implies that

$$x \in L(A_F, s) \;\Leftrightarrow\; x \in J_r \text{ for all } s < r \;\Leftrightarrow\; x \in \bigcap_{s<r} J_r,$$

and so $L(A_F, s) = \bigcap_{s<r} J_r$ which is a hyper BCK-ideal of H. For the second case, there exists $\delta > 0$ such that $(s, s + \delta) \cap \Lambda = \emptyset$. If $x \in \bigcup_{s\geq r} J_r$, then $x \in J_r$ for some $s \geq r$. Thus $A_F(x) \leq r \leq s$, that is, $x \in L(A_F, s)$. Hence $\bigcup_{s\geq r} J_r \subseteq L(A_F, s)$. If $x \notin \bigcup_{s\geq r} J_r$, then $x \notin J_r$ for all $r \leq s$ and thus $x \notin J_r$ for all $r < s + \delta$. This shows that if $x \in J_r$ then $r \geq s + \delta$. Hence $A_F(x) \geq s + \delta > s$, i.e., $x \notin L(A_F, s)$.

Therefore $L(A_F, s) \subseteq \bigcup_{s \geq r} J_r$. Consequently, $L(A_F, s) = \bigcup_{s \geq r} J_r$ which is a hyper BCK-ideal of H by (3.9). It follows from Theorem 3.5 that $A = (A_T,\, A_I,\, A_F)$ is a neutrosophic hyper BCK-ideal of H. □

Definition 3.8. A neutrosophic set $A = (A_T, A_I, A_F)$ in H is called a neutrosophic strong hyper BCK-ideal of H if it satisfies the following assertions.

$$\begin{aligned} {}_*A_T(x \circ x) \geq A_T(x) \geq \min\left\{ \sup_{a_0 \in x \circ y} A_T(a_0), A_T(y) \right\}, \\ {}_*A_I(x \circ x) \geq A_I(x) \geq \min\left\{ \sup_{b_0 \in x \circ y} A_I(b_0), A_I(y) \right\}, \\ {}^*A_F(x \circ x) \leq A_F(x) \leq \max\left\{ \inf_{c_0 \in x \circ y} A_F(c_0), A_F(y) \right\} \end{aligned} \tag{3.12}$$

for all $x, y \in H$.

Example 3.9. Consider a hyper BCK-algebra $H = \{0, a, b\}$ with the hyper operation "$\circ$" which is given by Table 3.

Table 3: Cayley table for the binary operation "$\circ$"

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{a\}$
b	$\{b\}$	$\{b\}$	$\{0, b\}$

Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in H which is described in Table 4.

Table 4: Tabular representation of $A = (A_T, A_I, A_F)$

H	$A_T(x)$	$A_I(x)$	$A_F(x)$
0	0.86	0.75	0.09
a	0.65	0.57	0.17
b	0.31	0.37	0.29

It is routine to verify that $A = (A_T,\, A_I,\, A_F)$ is a neutrosophic strong hyper BCK-ideal of H.

Theorem 3.10. For any neutrosophic strong hyper BCK-ideal $A = (A_T,\, A_I,\, A_F)$ of H, the following assertions are valid.

(1) $A = (A_T,\, A_I,\, A_F)$ satisfies the conditions (3.1) and (3.3).

(2) $A = (A_T,\, A_I,\, A_F)$ satisfies

$$(\forall x, y \in H)(\forall a, b, c \in x \circ y) \left(\begin{array}{l} A_T(x) \geq \min\{A_T(a), A_T(y)\} \\ A_I(x) \geq \min\{A_I(b), A_I(y)\} \\ A_F(x) \leq \max\{A_F(c), A_F(y)\} \end{array} \right). \tag{3.13}$$

Proof. (1) Since $x \ll x$, i.e., $0 \in x \circ x$ for all $x \in H$, we get

$$\begin{aligned} A_T(0) &\geq {}_*A_T(x \circ x) \geq A_T(x), \\ A_I(0) &\geq {}_*A_I(x \circ x) \geq A_I(x), \\ A_F(0) &\leq {}^*A_F(x \circ x) \leq A_F(x), \end{aligned}$$

which shows that (3.3) is valid. Let $x, y \in H$ be such that $x \ll y$. Then $0 \in x \circ y$, and so

$${}^*A_T(x \circ y) \geq A_T(0),\ {}^*A_I(x \circ y) \geq A_I(0) \text{ and } {}_*A_F(x \circ y) \leq A_F(0).$$

It follows from (3.3) that

$$\begin{aligned} A_T(x) &\geq \min\{{}^*A_T(x \circ y), A_T(y)\} \geq \min\{A_T(0), A_T(y)\} = A_T(y), \\ A_I(x) &\geq \min\{{}^*A_I(x \circ y), A_I(y)\} \geq \min\{A_I(0), A_I(y)\} = A_I(y), \\ A_F(x) &\leq \max\{{}_*A_F(x \circ y), A_F(y)\} \leq \max\{A_F(0), A_F(y)\} = A_F(y). \end{aligned}$$

Hence $A = (A_T,\, A_I,\, A_F)$ satisfies the condition (3.1) .

(2) Let $x, y, a, b, c \in H$ be such that $a, b, c \in x \circ y$. Then

$$\begin{aligned} A_T(x) &\geq \min\left\{ \sup_{a_0 \in x \circ y} A_T(a_0), A_T(y) \right\} \geq \min\{A_T(a), A_T(y)\}, \\ A_I(x) &\geq \min\left\{ \sup_{b_0 \in x \circ y} A_I(b_0), A_I(y) \right\} \geq \min\{A_I(b), A_I(y)\}, \\ A_F(x) &\leq \max\left\{ \inf_{c_0 \in x \circ y} A_F(c_0), A_F(y) \right\} \leq \max\{A_F(c), A_F(y)\}. \end{aligned}$$

This completes the proof. □

Theorem 3.11. If a neutrosophic set $A = (A_T,\, A_I,\, A_F)$ is a neutrosophic strong hyper BCK-ideal of H, then the nonempty sets $U(A_T, \varepsilon_T)$, $U(A_I, \varepsilon_I)$ and $L(A_F, \varepsilon_F)$ are strong hyper BCK-ideals of H for all $\varepsilon_T,\, \varepsilon_I,\, \varepsilon_F \in [0, 1]$.

Proof. Let $A = (A_T,\, A_I,\, A_F)$ be a neutrosophic strong hyper BCK-ideal of H. Then $A = (A_T,\, A_I,\, A_F)$ is a neutrosophic hyper BCK-ideal of H. Assume that $U(A_T, \varepsilon_T)$, $U(A_I, \varepsilon_I)$ and $L(A_F, \varepsilon_F)$ are nonempty for all $\varepsilon_T,\, \varepsilon_I,\, \varepsilon_F \in [0, 1]$. Then there exist $a \in U(A_T, \varepsilon_T)$, $b \in U(A_I, \varepsilon_I)$ and $c \in L(A_F, \varepsilon_F)$, that is, $A_T(a) \geq \varepsilon_T$, $A_I(b) \geq \varepsilon_I$ and $A_F(c) \leq \varepsilon_F$. It follows from (3.3) that $A_T(0) \geq A_T(a) \geq \varepsilon_T$, $A_I(0) \geq A_I(b) \geq \varepsilon_I$ and $A_F(0) \leq A_F(c) \leq \varepsilon_F$. Hence

$$0 \in U(A_T, \varepsilon_T) \cap U(A_I, \varepsilon_I) \cap L(A_F, \varepsilon_F).$$

Let $x, y, a, b, u, v \in H$ be such that $(x \circ y) \cap U(A_T, \varepsilon_T) \neq \emptyset$, $y \in U(A_T, \varepsilon_T)$, $(a \circ b) \cap U(A_I, \varepsilon_I) \neq \emptyset$, $b \in U(A_I, \varepsilon_I)$, $(u \circ v) \cap L(A_F, \varepsilon_F) \neq \emptyset$ and $v \in L(A_F, \varepsilon_F)$. Then there exist $x_0 \in (x \circ y) \cap U(A_T, \varepsilon_T)$, $a_0 \in (a \circ b) \cap U(A_I, \varepsilon_I)$ and $u_0 \in (u \circ v) \cap L(A_F, \varepsilon_F)$. It follows that

$$A_T(x) \geq \min\{{}^*A_T(x \circ y), A_T(y)\} \geq \min\{A_T(x_0), A_T(y)\} \geq \varepsilon_T,$$

$$A_I(a) \geq \min\left\{\sup_{d \in a \circ b} A_I(d), A_I(b)\right\} \geq \min\{A_I(a_0), A_I(b)\} \geq \varepsilon_I$$

and

$$A_F(u) \leq \max\left\{\inf_{e \in u \circ v} A_F(e), A_F(v)\right\} \leq \max\{A_F(u_0), A_F(v)\} \leq \varepsilon_F.$$

Hence $x \in U(A_T, \varepsilon_T)$, $a \in U(A_I, \varepsilon_I)$ and $u \in L(A_F, \varepsilon_F)$. Therefore $U(A_T, \varepsilon_T)$, $U(A_I, \varepsilon_I)$ and $L(A_F, \varepsilon_F)$ are strong hyper BCK-ideals of H. □

Theorem 3.12. For any neutrosophic set $A = (A_T, A_I, A_F)$ in H satisfying the condition

$$(\forall S \subseteq H)(\exists\, a, b, c \in S) \begin{pmatrix} A_T(a) = {}^*A_T(S) \\ A_I(b) = {}^*A_I(S) \\ A_F(c) = {}_*A_F(S) \end{pmatrix}, \tag{3.14}$$

if the nonempty sets $U(A_T, \varepsilon_T)$, $U(A_I, \varepsilon_I)$ and $L(A_F, \varepsilon_F)$ are strong hyper BCK-ideals of H for all ε_T, ε_I, $\varepsilon_F \in [0, 1]$, then $A = (A_T, A_I, A_F)$ is a neutrosophic strong hyper BCK-ideal of H.

Proof. Assume that $U(A_T, \varepsilon_T)$, $U(A_I, \varepsilon_I)$ and $L(A_F, \varepsilon_F)$ are nonempty and strong hyper BCK-ideals of H for all ε_T, ε_I, $\varepsilon_F \in [0, 1]$. For any $x, y, z \in H$, such that $x \in U(A_T, A_T(x))$, $y \in U(A_I, A_I(y))$ and $z \in L(A_F, A_F(z))$, since $x \circ x \ll x$, $y \circ y \ll y$ and $z \circ z \ll z$ by (2.1), we have $x \circ x \ll U(A_T, A_T(x))$, $y \circ y \ll U(A_I, A_I(y))$ and $z \circ z \ll L(A_F, A_F(z))$. By Lemma 3.4, $x \circ x \subseteq U(A_T, A_T(x))$, $y \circ y \subseteq U(A_I, A_I(y))$ and $z \circ z \subseteq L(A_F, A_F(z))$. Hence $a \in U(A_T, A_T(x))$, $b \in U(A_I, A_I(y))$ and $c \in L(A_F, A_F(z))$ for all $a \in x \circ x$, $b \in y \circ y$ and $c \in z \circ z$. Therefore ${}_*A_T(x \circ x) \geq A_T(x)$, ${}_*A_I(y \circ y) \geq A_I(y)$ and ${}^*A_F(z \circ z) \leq A_F(z)$. Now, let $\varepsilon_T := \min\{{}^*A_T(x \circ y), A_T(y)\}$, $\varepsilon_I := \min\{{}^*A_I(x \circ y), A_I(y)\}$ and $\varepsilon_F := \max\{{}_*A_F(x \circ y), A_F(y)\}$. By (3.14), we have

$$A_T(a_0) = {}^*A_T(x \circ y) \geq \min\{{}^*A_T(x \circ y), A_T(y)\} = \varepsilon_T,$$

$$A_I(b_0) = {}^*A_I(x \circ y) \geq \min\{{}^*A_I(x \circ y), A_I(y)\} = \varepsilon_I$$

and

$$A_F(c_0) = {}_*A_F(x \circ y) \leq \max\{{}_*A_F(x \circ y), A_F(y)\} = \varepsilon_F$$

for some $a_0, b_0, c_0 \in x \circ y$. Hence $a_0 \in U(A_T, \varepsilon_T)$, $b_0 \in U(A_I, \varepsilon_I)$ and $c_0 \in L(A_F, \varepsilon_F)$ which imply that

$$(x \circ y) \cap U(A_T, \varepsilon_T),\ (x \circ y) \cap U(A_I, \varepsilon_I) \text{ and } (x \circ y) \cap L(A_F, \varepsilon_F)$$

are nonempty. Since $y \in U(A_T, \varepsilon_T) \cap U(A_I, \varepsilon_I) \cap L(A_F, \varepsilon_F)$, it follows from (2.9) that $x \in U(A_T, \varepsilon_T) \cap U(A_I, \varepsilon_I) \cap L(A_F, \varepsilon_F)$. Thus

$$A_T(x) \geq \varepsilon_T = \min\left\{{}^{*}A_T(x \circ y), A_T(y)\right\},$$

$$A_I(x) \geq \varepsilon_I = \min\left\{{}^{*}A_I(x \circ y), A_I(y)\right\}$$

and

$$A_F(x) \leq \varepsilon_F = \max\left\{{}_{*}A_F(x \circ y), A_F(y)\right\}.$$

Consequently, $A = (A_T,\ A_I,\ A_F)$ is a neutrosophic strong hyper BCK-ideal of H. □

Since any neutrosophic set $A = (A_T,\ A_I,\ A_F)$ satisfies the condition (3.14) in a finite hyper BCK-algebra, we have the following corollary.

Corollary 3.13. Let $A = (A_T,\ A_I,\ A_F)$ be a neutrosophic set in a finite hyper BCK-algebra H. Then $A = (A_T,\ A_I,\ A_F)$ is a neutrosophic strong hyper BCK-ideal of H if and only if the nonempty sets $U(A_T, \varepsilon_T)$, $U(A_I, \varepsilon_I)$ and $L(A_F, \varepsilon_F)$ are strong hyper BCK-ideals of H for all $\varepsilon_T,\ \varepsilon_I,\ \varepsilon_F \in [0, 1]$.

Definition 3.14. A neutrosophic set $A = (A_T, A_I, A_F)$ in H is called a neutrosophic weak hyper BCK-ideal of H if it satisfies the following assertions.

$$\begin{array}{l} A_T(0) \geq A_T(x) \geq \min\left\{{}_{*}A_T(x \circ y), A_T(y)\right\}, \\ A_I(0) \geq A_I(x) \geq \min\left\{{}_{*}A_I(x \circ y), A_I(y)\right\}, \\ A_F(0) \leq A_F(x) \leq \max\left\{{}^{*}A_F(x \circ y), A_F(y)\right\} \end{array} \tag{3.15}$$

for all $x, y \in H$.

Definition 3.15. A neutrosophic set $A = (A_T, A_I, A_F)$ in H is called a neutrosophic s-weak hyper BCK-ideal of H if it satisfies the conditions (3.3) and (3.5).

Example 3.16. Consider a hyper BCK-algebra $H = \{0, a, b, c\}$ with the hyper operation "$\circ$" which is given by Table 5.

Table 5: Cayley table for the binary operation "$\circ$"

$\circ$	0	a	b	c
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{b\}$	$\{0\}$	$\{0\}$
c	$\{c\}$	$\{c\}$	$\{b, c\}$	$\{0, b, c\}$

Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in H which is described in Table 6.
It is routine to verify that $A = (A_T, A_I, A_F)$ is a neutrosophic weak hyper BCK-ideal of H.

Table 6: Tabular representation of $A = (A_T, A_I, A_F)$

H	$A_T(x)$	$A_I(x)$	$A_F(x)$
0	0.98	0.85	0.02
a	0.81	0.69	0.19
b	0.56	0.43	0.32
c	0.34	0.21	0.44

Theorem 3.17. Every neutrosophic s-weak hyper BCK-ideal is a neutrosophic weak hyper BCK-ideal.

Proof. Let $A = (A_T, A_I, A_F)$ be a neutrosophic s-weak hyper BCK-ideal of H and let $x, y \in H$. Then there exist $a, b, c \in x \circ y$ such that

$$A_T(x) \geq \min\{A_T(a), A_T(y)\} \geq \min\left\{\inf_{a_0 \in x \circ y} A_T(a_0), A_T(y)\right\},$$
$$A_I(x) \geq \min\{A_I(b), A_I(y)\}, \geq \min\left\{\inf_{b_0 \in x \circ y} A_I(b_0), A_I(y)\right\},$$
$$A_F(x) \leq \max\{A_F(c), A_F(y)\}. \leq \max\left\{\sup_{c_0 \in x \circ y} A_F(c_0), A_F(y)\right\}.$$

Hence $A = (A_T, A_I, A_F)$ is a neutrosophic weak hyper BCK-ideal of H. □

We can conjecture that the converse of Theorem 3.17 is not true. But it is not easy to find an example of a neutrosophic weak hyper BCK-ideal which is not a neutrosophic s-weak hyper BCK-ideal.

Now we provide a condition for a neutrosophic weak hyper BCK-ideal to be a neutrosophic s-weak hyper BCK-ideal.

Theorem 3.18. If $A = (A_T, A_I, A_F)$ is a neutrosophic weak hyper BCK-ideal of H which satisfies the condition (3.4), then $A = (A_T, A_I, A_F)$ is a neutrosophic s-weak hyper BCK-ideal of H.

Proof. Let $A = (A_T, A_I, A_F)$ be a neutrosophic weak hyper BCK-ideal of H in which the condition (3.4) is true. Then there exist $a_0, b_0, c_0 \in x \circ y$ such that $A_T(a_0) = {}_*A_T(x \circ y)$, $A_I(b_0) = {}_*A_I(x \circ y)$ and $A_F(c_0) = {}^*A_F(x \circ y)$. Hence

$$A_T(x) \geq \min\{{}_*A_T(x \circ y), A_T(y)\} = \min\{A_T(a_0), A_T(y)\},$$
$$A_I(x) \geq \min\{{}_*A_I(x \circ y), A_I(y)\} = \min\{A_I(b_0), A_I(y)\},$$
$$A_F(x) \leq \max\{{}^*A_F(x \circ y), A_F(y)\} = \max\{A_F(c_0), A_F(y)\}.$$

Therefore $A = (A_T, A_I, A_F)$ is a neutrosophic s-weak hyper BCK-ideal of H. □

Remark 3.19. In a finite hyper BCK-algebra, every neutrosophic set satisfies the condition (3.4). Hence the concept of neutrosophic s-weak hyper BCK-ideal and neutrosophic weak hyper BCK-ideal coincide in a finite hyper BCK-algebra.

Theorem 3.20. A neutrosophic set $A = (A_T,\, A_I,\, A_F)$ is a neutrosophic weak hyper BCK-ideal of H if and only if the nonempty sets $U(A_T, \varepsilon_T)$, $U(A_I, \varepsilon_I)$ and $L(A_F, \varepsilon_F)$ are weak hyper BCK-ideals of H for all $\varepsilon_T,\, \varepsilon_I,\, \varepsilon_F \in [0, 1]$.

Proof. The proof is similar to the proof of Theorem 3.5. □

Definition 3.21. A neutrosophic set $A = (A_T, A_I, A_F)$ in H is called a reflexive neutrosophic hyper BCK-ideal of H if it satisfies

$$(\forall x, y \in H) \left(\begin{array}{l} {}_*A_T(x \circ x) \geq A_T(y) \\ {}_*A_I(x \circ x) \geq A_I(y) \\ {}^*A_F(x \circ x) \leq A_F(y) \end{array} \right), \tag{3.16}$$

and

$$(\forall x, y \in H) \left(\begin{array}{l} A_T(x) \geq \min\{{}^*A_T(x \circ y), A_T(y)\} \\ A_I(x) \geq \min\{{}^*A_I(x \circ y), A_I(y)\} \\ A_F(x) \leq \max\{{}_*A_F(x \circ y), A_F(y)\} \end{array} \right). \tag{3.17}$$

Theorem 3.22. Every reflexive neutrosophic hyper BCK-ideal is a neutrosophic strong hyper BCK-ideal.

Proof. Straightforward. □

Theorem 3.23. If $A = (A_T,\, A_I,\, A_F)$ is a reflexive neutrosophic hyper BCK-ideal of H, then the nonempty sets $U(A_T, \varepsilon_T)$, $U(A_I, \varepsilon_I)$ and $L(A_F, \varepsilon_F)$ are reflexive hyper BCK-ideals of H for all $\varepsilon_T,\, \varepsilon_I,\, \varepsilon_F \in [0, 1]$.

Proof. Assume that $U(A_T, \varepsilon_T)$, $U(A_I, \varepsilon_I)$ and $L(A_F, \varepsilon_F)$ are nonempty for all $\varepsilon_T,\, \varepsilon_I,\, \varepsilon_F \in [0, 1]$. Let $a \in U(A_T, \varepsilon_T)$, $b \in U(A_I, \varepsilon_I)$ and $c \in L(A_F, \varepsilon_F)$. If $A = (A_T,\, A_I,\, A_F)$ is a reflexive neutrosophic hyper BCK-ideal of H, then by Theorem 3.22, $A = (A_T,\, A_I,\, A_F)$ is a neutrosophic strong hyper BCK-ideal of H, and so it is a neutrosophic hyper BCK-ideal of H. It follows from Theorem 3.5 that $U(A_T, \varepsilon_T)$, $U(A_I, \varepsilon_I)$ and $L(A_F, \varepsilon_F)$ are hyper BCK-ideals of H. For each $x \in H$, let $a_0, b_0, c_0 \in x \circ x$. Then

$$A_T(a_0) \geq \inf_{u \in x \circ x} A_T(u) \geq A_T(a) \geq \varepsilon_T,$$
$$A_I(b_0) \geq \inf_{v \in x \circ x} A_I(v) \geq A_I(b) \geq \varepsilon_I,$$
$$A_F(c_0) \leq \sup_{w \in x \circ x} A_F(w) \leq A_F(c) \leq \varepsilon_F,$$

and so $a_0 \in U(A_T, \varepsilon_T)$, $b_0 \in U(A_I, \varepsilon_I)$ and $c_0 \in L(A_F, \varepsilon_F)$. Hence $x \circ x \subseteq U(A_T, \varepsilon_T)$, $x \circ x \subseteq U(A_I, \varepsilon_I)$ and $x \circ x \subseteq L(A_F, \varepsilon_F)$. Therefore $U(A_T, \varepsilon_T)$, $U(A_I, \varepsilon_I)$ and $L(A_F, \varepsilon_F)$ are reflexive hyper BCK-ideals of H. □

Lemma 3.24 ([15]). Every reflexive hyper BCK-ideal is a strong hyper BCK-ideal.

We consider the converse of Theorem 3.23 by adding a condition.

Theorem 3.25. Let $A = (A_T,\, A_I,\, A_F)$ be a neutrosophic set in H satisfying the condition (3.14). If the nonempty sets $U(A_T, \varepsilon_T)$, $U(A_I, \varepsilon_I)$ and $L(A_F, \varepsilon_F)$ are reflexive hyper BCK-ideals of H for all $\varepsilon_T,\, \varepsilon_I,$ $\varepsilon_F \in [0, 1]$, then $A = (A_T,\, A_I,\, A_F)$ is a reflexive neutrosophic hyper BCK-ideal of H.

Proof. If the nonempty sets $U(A_T, \varepsilon_T)$, $U(A_I, \varepsilon_I)$ and $L(A_F, \varepsilon_F)$ are reflexive hyper BCK-ideals of H, then by Lemma 3.24 they are strong hyper BCK-ideals of H. By Theorem 3.12 that $A = (A_T,\ A_I,\ A_F)$ is a neutrosophic strong hyper BCK-ideal of H. Hence the condition (3.17) is valid. Let $x, y \in H$. Then the sets $U(A_T, A_T(y))$, $U(A_I, A_I(y))$ and $L(A_F, A_F(y))$ are reflexive hyper BCK-ideals of H, and so $x \circ x \subseteq U(A_T, A_T(y))$, $x \circ x \subseteq U(A_I, A_I(y))$ and $x \circ x \subseteq L(A_F, A_F(y))$. Hence $A_T(a) \geq A_T(y)$, $A_I(b) \geq A_I(y)$ and $A_F(c) \leq A_F(y)$ for all $a, b, c \in x \circ x$ and so ${}_*A_T(x \circ x) \geq A_T(y)$, ${}_*A_I(x \circ x) \geq A_I(y)$ and ${}^*A_F(x \circ x) \leq A_F(y)$. Therefore $A = (A_T,\ A_I,\ A_F)$ is a reflexive neutrosophic hyper BCK-ideal of H. □

We provide conditions for a neutrosophic strong hyper BCK-ideal to be a reflexive neutrosophic hyper BCK-ideal.

Theorem 3.26. Let $A = (A_T,\ A_I,\ A_F)$ be a neutrosophic strong hyper BCK-ideal of H which satisfies the condition (3.14). Then $A = (A_T,\ A_I,\ A_F)$ is a reflexive neutrosophic hyper BCK-ideal of H if and only if the following assertion is valid.

$$(\forall x \in H) \left(\begin{array}{c} {}_*A_T(x \circ x) \geq A_T(0) \\ {}_*A_I(x \circ x) \geq A_I(0) \\ {}^*A_F(x \circ x) \leq A_F(0) \end{array} \right). \tag{3.18}$$

Proof. It is clear that if $A = (A_T,\ A_I,\ A_F)$ is a reflexive neutrosophic hyper BCK-ideal of H, then the condition (3.18) is valid.

Conversely, assume that $A = (A_T,\ A_I,\ A_F)$ is a neutrosophic strong hyper BCK-ideal of H which satisfies the conditions (3.14) and (3.18). Then $A_T(0) \geq A_T(y)$, $A_I(0) \geq A_I(y)$ and $A_F(0) \leq A_F(y)$ for all $y \in H$. Hence

$${}_*A_T(x \circ x) \geq A_T(y),\ {}_*A_I(x \circ x) \geq A_I(y) \text{ and } {}^*A_F(x \circ x) \leq A_F(y).$$

For any $x, y \in H$, let

$$\begin{aligned} \varepsilon_T &:= \min\left\{{}^*A_T(x \circ y), A_T(y)\right\}, \\ \varepsilon_I &:= \min\left\{{}^*A_I(x \circ y), A_I(y)\right\}, \\ \varepsilon_F &:= \max\left\{{}_*A_F(x \circ y), A_F(y)\right\}. \end{aligned}$$

Then $U(A_T, \varepsilon_T)$, $U(A_I, \varepsilon_I)$ and $L(A_F, \varepsilon_F)$ are strong hyper BCK-ideals of H by Theorem 3.11. Since $A = (A_T,\ A_I,\ A_F)$ satisfies the condition (3.14), there exist $a_0, b_0, c_0 \in x \circ y$ such that

$$A_T(a_0) = {}^*A_T(x \circ y),\ A_I(b_0) = {}^*A_I(x \circ y),\ A_F(c_0) = {}_*A_F(x \circ y).$$

Hence $A_T(a_0) \geq \varepsilon_T$, $A_I(b_0) \geq \varepsilon_I$ and $A_F(c_0) \leq \varepsilon_F$, that is, $a_0 \in U(A_T, \varepsilon_T)$, $b_0 \in U(A_I, \varepsilon_I)$ and $c_0 \in L(A_F, \varepsilon_F)$. Hence $(x \circ y) \cap U(A_T, \varepsilon_T) \neq \emptyset$, $(x \circ y) \cap U(A_I, \varepsilon_I) \neq \emptyset$ and $(x \circ y) \cap L(A_F, \varepsilon_F) \neq \emptyset$. Since $y \in U(A_T, \varepsilon_T) \cap U(A_I, \varepsilon_I) \cap L(A_F, \varepsilon_F)$, by (2.9), $x \in U(A_T, \varepsilon_T) \cap U(A_I, \varepsilon_I) \cap L(A_F, \varepsilon_F)$. Thus

$$\begin{aligned} A_T(x) &\geq \varepsilon_T = \min\left\{{}^*A_T(x \circ y), A_T(y)\right\}, \\ A_I(x) &\geq \varepsilon_I = \min\left\{{}^*A_I(x \circ y), A_I(y)\right\}, \\ A_F(x) &\leq \varepsilon_F = \max\left\{{}_*A_F(x \circ y), A_F(y)\right\}. \end{aligned}$$

Therefore $A = (A_T, A_I, A_F)$ is a reflexive neutrosophic hyper BCK-ideal of H. □

4 Conclusions

We have introduced the notions of neutrosophic (strong, weak, s-weak) hyper BCK-ideal and reflexive neutrosophic hyper BCK-ideal. We have considered their relations and related properties. We have discussed characterizations of neutrosophic (weak) hyper BCK-ideal, and have given conditions for a neutrosophic set to be a (reflexive) neutrosophic hyper BCK-ideal and a neutrosophic strong hyper BCK-ideal. We have provided conditions for a neutrosophic weak hyper BCK-ideal to be a neutrosophic s-weak hyper BCK-ideal, and have provided conditions for a neutrosophic strong hyper BCK-ideal to be a reflexive neutrosophic hyper BCK-ideal.

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