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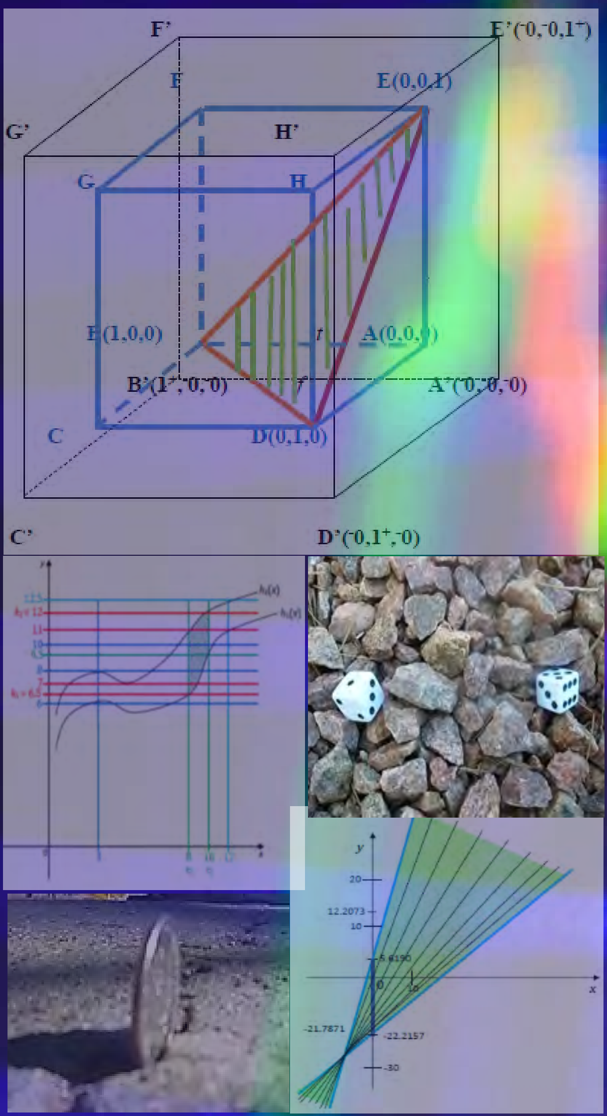

<A> <neutA> <antiA>

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The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]^-0, 1^+[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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Google Dictionaries have translated the neologisms "**neutrosophy**" (1) and"**neutrosophic**" (2), coined in 1995 for the first time, into about 100 languages.
FOLDOC Dictionary of Computing (**1, 2**), Webster Dictionary (**1, 2**), Wordnik (**1**),Dictionary.com, The Free Dictionary (**1**), Wiktionary (**2**), YourDictionary (**1, 2**),OneLook Dictionary (**1, 2**), Dictionary / Thesaurus (**1**), Online Medical Dictionary (**1,2**), Encyclopedia (**1, 2**), Chinese Fanyi Baidu Dictionary (**2**), Chinese Youdao Dictionary (**2**) etc. have included these scientific neologisms.

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The Neutrosophic Regular and Most Important Properties that Bind Neutrosophic Ring Elements

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Abstract: This research has broadened the definition of the neutrosophic regular in neutrosophic rings, similar to what is known in classical rings. We have studied the properties of neutrosophic regular elements and the most important properties that link them to the neutrosophic (zero divisor, idempotent, unit and nilpotent) elements in neutrosophic rings, and we reached several important results linking these elements to each other, as some of them are different from what is known in classical rings. The most important of which are:
If $R(\mathrm{I})$ is a neutrosophic right (left) strongly regular neutrosophic ring, then $NSNil_{R(\mathrm{I})} = \{0\}$. In any neutrosophic field $R(\mathrm{I})$ is achieved: $R(\mathrm{I}) = NReg_{R(I)}$, although there are some elements that are not neutrosophic unit, $NU_{R(I)} \cap NZ_{R(I)} = \{bI\ , b \in U_R\}$, and $NU_{R(I)} \cap NId_{R(I)} = \{1, I\}$.

Keywords: Neutrosophic ring, Neutrosophic field, Neutrosophic regular, strongly regular, unit, simple nilpotent, zero divisor.

1. Introduction

The concept of the regular element in the rings appeared in the hands of researcher J.Von Neumann [1]. Regular rings and their properties have been extensively examined by many authors and researchers [2-3-4-5]. Neutosophy is a comprehensive perspective on intuitionistic fuzzy logic that represents a fresh expansion of fuzzy ideas. This method has an intriguing influence on applied science [6-7-8-9]. More neutrosophical applications in many areas may be found in [10-11-12-13-14].

Pure mathematics has various applications, including neutrosophic groups [15], metric spaces [16], and rings [17-18-19-20]. In 1980, Smarandache first introduced the neutrosophic theory.
This idea has created a new notion in algebraic structures, known as neutrosophic structures.

Kandasamy and her colleague, Smarandache, introduced the idea of neutrosophic algebraic structures in [21]. Vasantha Kandasamy and Smarandache introduced the notion of neutrosophic zero divisors, idempotent, and unit elements in neutrosophic rings and fields [22].
Agboola, Akinola, and Oyebolain conducted further research on neutrosophic rings [23-24].
Chalapathi and Kiran examined the enumeration of neutrosophic units in neutrosophic rings and fields [25]. A novel multiplication operation based on neutrosophic theory has been developed to enhance the algebraic structures of classical rings and enable easier derivation of elementary structural theorems for indeterminate situations. Therefore, when a real-world problem involves indeterminacy, the use of neutrosophic algebraic theory is necessary.

This paper explores the concepts of neutrosophic regularity in neutrosophic rings using relevant examples of key points. In addition, we analyzed the properties of specific elements of the neutrosophic rings to determine the properties that bind these elements together.

2. Definitions and notations

Given that researchers interested in the subject are well aware of classical rings and other fields, in this section, we provide various definitions and key findings of neutrosophic rings. For those interested in delving deeper into the topic of neutrosophic rings, we recommend referring to references.

Definition.2.1 [22] Assume that we have a ring denoted by R. The set $\langle \mathrm{R} \cup I\rangle = \{a + bI\ ; a, b \in R \text{ and } I^2 = I\}$ is called the neutrosophic ring. $\langle \mathrm{R} \cup I\rangle$ is referred to as a neutrosophic field when R is a field.

Properties.2.2 [19-22]

1. If R is a unity ring, then $\langle R \cup I\rangle$ is a unity neutrosophic ring with neutrosophic unity I.
2. $I^n = I$ for each $n \in \mathbb{Z}^+$
3. $aI = Ia\ \forall a \in R$.
4. $0I = 0\ ,\ I + I + \cdots + I = nI$

Definition.2.3 [22] If $\langle \mathrm{R} \cup I\rangle$ is a neutrosophic ring, then $x \in \langle \mathrm{R} \cup I\rangle$, where $x \neq 0$ is considered a neutrosophic zero divisor if found $y \neq 0$ of $\langle \mathrm{R} \cup I\rangle$, such that $xy = yx = 0$.

Definition.2.4 [22-23-25] Assume that $\langle \mathrm{R} \cup I\rangle$ is a neutrosophic ring, then

1. If $e \in \langle \mathrm{R} \cup I\rangle$ satisfies $e^2 = e$, it is considered to be a neutrosophic idempotent.
2. Any element $x \in \langle \mathrm{R} \cup I\rangle$ is considered neutrosophic nilpotent if it satisfies the condition $x^n = 0$, where $n \in \mathbb{Z}^+$.
3. Any element $x \in \langle \mathrm{R} \cup I\rangle$ is considered a unit if there is y in $\langle \mathrm{R} \cup I\rangle$, where, $xy = yx = 1$.
4. Any element $x \in \langle \mathrm{R} \cup I\rangle$ is considered a neutrosophic unit if there is y in $\langle \mathrm{R} \cup I\rangle$, where $xy = yx =$I.

Theorem.2.5 [19] If $\langle K \cup\ I\rangle$ is a neutrosophic field, then each element in the form of $a + bI$ is unit $\Leftrightarrow a \neq 0\ \mathit{and}\ a \neq -b$.

To represent the neutrosophic (field) ring, we use the symbol $R(I)$ instead of $\langle R \cup\ I\rangle$.

3. Results

We present the idea of regularity and its effects on the components of neutrosophic rings in this section, and we explain the most important properties that link the elements of the neutrosophic ring to each other.

In a neutrosophic ring $R(I)$, we indicate by $NZ_{R(I)}$ the collection of neutrosophic zero divisor elements, $NId_{R(I)}$ the collection of neutrosophic idempotent elements, $U_{R(I)} = \{x \in R(I);\ \exists y \in R(I); xy = yx = 1\}$ the collection of unit elements, $NU_{R(I)} = \{x \in R(I);\ \exists y \in R(I); xy = yx = I\}$ the collection of neutrosophic unit elements, $NNil_{R(I)}$ the collection of neutrosophic nilpotent elements, $NReg_{R(I)}$ the collection of neutrosophic regular elements. In addition, in classical ring R we are going indicate by Z_R the set of zero divisors, Id_R is the collection of idempotent, U_R is the collection of unit, Nil_R is the collection of nilpotent, Reg_R is the collection of regular. We will indicate by $\mathbb{R}$ to the collection of real numbers, $\mathbb{Z}$ is the collection of integers.

Definition.3.1 Assume that R(I) is a neutrosophic ring and let x be its element. We can say that in R(I), if there is an element y where $x = xyx$, then x is a neutrosophic regular element. We call R(I) a regular neutrosophic ring if $NReg_{R(\mathrm{I})} = R(\mathrm{I})$.

Example.3.2 The element $3 + 4I \in NReg_{\mathbb{Z}_7(I)}$ because it achieves $3 + 4I = (3 + 4I)(5 + 6I)(3 + 4I)$.

Definition.3.3 Assume that R(I) is a neutrosophic ring. $x \in R(\mathrm{I})$ is a neutrosophic right (left) neutrosophic strongly regular if found y in R(I) where $x = yxx$ $(x = xxy)$. If each element in R(I) is a right (left) neutrosophic regular element, we call R(I) a right (left) neutrosophic strongly regular. If R(I) is a right and left neutrosophic strongly regular, we call it a neutrosophic strongly regular. It is clear that R(I) is a neutrosophic strongly regular, when R(I) is a commutative neutrosophic regular.

Definition.3.4 In R(I), we define the set of neutrosophic simple nilpotent elements, which we denote by $NSNil_{R(\mathrm{I})}$ as follows $NSNil_{R(\mathrm{I})} = \{x \in R(\mathrm{I}); x^2 = 0\}$. clear that $NSNil_{R(\mathrm{I})} \subseteq NNil_{R(\mathrm{I})}$

Example.3.5 In $\mathbb{Z}_4$(I), we have $(2 + 2I)^2 = 0 \Rightarrow 2 + 2I \in NSNil_{\mathbb{Z}_4(\mathrm{I})}$.

Corollary.3.6 Assume that R(I) is a neutrosophic ring.

1. If $x \in NReg_{R(\mathrm{I})}$, then there is z of R(I) where $x = xzx$ and there is $y = zxz$ of R(I) where $x = xyx \ and \ y = yxy$.

2. If $x \in NReg_{R(\mathrm{I})}$, then there is $y \in R(\mathrm{I})$ where $x = xyx$. Now if we put $f = xy$ and $g = yx$, then f and g are neutrosophic idempotent. (It can be easily verified that $f^2 = f$ and $g^2 = g$).

Example.3.7 In $\mathbb{Z}_{11}$(I), we have $3 + 8I \in NReg_{\mathbb{Z}_{11}(I)}$; $3 + 8I = (3 + 8I)4(3 + 8I)$. If we put $f = (3 + 8I)4 = 1 + 10I$ and $g = 4(3 + 8I) = 1 + 10I$, then we note that $f^2 = g^2 = (1 + 10I)^2 = 1 + 10I = f = g$.

Example.3.8 In $\mathbb{Z}_7$(I), we have $5 + 6I \in NReg_{\mathbb{Z}_7(I)}$; $5 + 6I = (5 + 6I)(3 + 6I)(5 + 6I)$.
If we put $f = (5 + 6I)(3 + 6I) = 1$ and $g = (3 + 6I)(5 + 6I) = 1$, we note that $f^2 = g^2 = 1 = f = g$.

Theorem.3.9 If R(I) is an infinite (finite) neutrosophic field, then every element of the form aI ; $a \neq 0$ it has an infinite (finite) number of neutrosophic inverses of the shape $b + cI \in R(\mathrm{I})$ where $b \neq -c$.

Proof.

We have $aI(b + cI) = I \Rightarrow (ab + ac)I = I \Rightarrow a(b + c) = 1$. Since $a \neq 0$ so $b + c = a^{-1}$. Therefore $\forall b \in R$, $c = a^{-1} - b \in R$.

Corollary.3.10 In any neutrosophic field is achieved. Every element of the form $a + bI$, where, $a \neq$

$-b$ it has a neutrosophic inverse dI such that $c = \frac{1}{a+b}$. Since $a + bI \in U_{R(I)}$ according to the theorem.2.5, therefore $U_{R(I)} \subset NU_{R(I)}$.

Example.3.11

1. In the neutrosophic field $\mathbb{R}(I)$, the neutrosophic inverse of $3 + 5I$ is $\frac{1}{8}I$.

2. In the neutrosophic field $\mathbb{R}(I)$, the neutrosophic inverse of $aI = 3I$ is $b + cI$; $\forall b \in \mathbb{R}$, $c = a^{-1} - b$.

$$\text{Suppose that,} \quad b = 3 \Rightarrow c = \frac{-8}{3} \quad ; \quad 3I\left(3 - \frac{8}{3}I\right) = I$$

$$\text{Suppose that,} \quad b = \sqrt{2} \Rightarrow c = \frac{1}{3} - \sqrt{2} = \frac{1 - 3\sqrt{2}}{3} \quad ; \quad 3I\left(\sqrt{2} + (\frac{1 - 3\sqrt{2}}{3})I\right) = I$$

3. In the neutrosophic field $\mathbb{Z}_3(I)$, the element $2I$ has a finite number of neutrosophic inverses of the shape $b + cI$; $\forall b \in \mathbb{Z}_3, c = 2^{-1} - b$.

If $b = 0$ *then* $c = 2^{-1} - 0 = 2,$ *thus* $b + cI = 2I$

If $b = 1$ *then* $c = 2^{-1} - 1 = 1,$ *thus* $b + cI = 1 + I$

If $b = 2$ *then* $c = 2^{-1} - 2 = 2 + 1 = 0,$ *thus* $b + cI = 2$

Theorem.3.12 Let R(I) be unity. If $x \neq 0$ has a right inverse (right neutrosophic inverse) and let it be y and has a left inverse (left neutrosophic inverse) and let it be z then we can distinguish the following cases:

$$\text{If} \quad x.y = 1 \ \ and \ \ z.x = 1 \ \ then \ \ y = z$$
$$\text{If} \quad x.y = I \ \ and \ \ z.x = 1 \ \ then \ \ y = zI$$
$$\text{If} \quad x.y = 1 \ and \ z.x = I \ \ then \ \ z = yI$$
$$\text{If} \quad x.y = I \ and \ z.x = I \ then \ \ yI = zI$$

Proof.

In the first case, it is clear.

In the rest of the cases

$$\text{If} \quad x.y = I \ \ and \ \ z.x = 1, \text{then we note} \quad y = 1.y = (zx).y = z.(xy) = zI$$
$$\text{If} \quad x.y = 1 \ and \ \ z.x = I, \text{then we note} \quad z = z.1 = z.(xy) = (zx).y = Iy = yI$$
$$\text{If} \quad x.y = I \ and \ \ z.x = I, \text{then we note} \quad zI = z.(xy) = (zx).y = Iy = yI$$

Example.3.13

1. In $\mathbb{Z}_8$(I), the element $4 + I$ is a neutrosophic unit and achieves $(4 + I)(4 + I) = I$ and $5I(4 + I) = I$. And we note $(5I)I = (4 + I)I = 5I$

2. In the neutrosophic ring $\mathbb{R}\,(I)$ we have $(3 + 5I)\left(\frac{1}{3} - \frac{5}{24}I\right) = 1$ and also $(3 + 5I)\left(\frac{1}{8}I\right) = I$. And we note $\left(\frac{1}{3} - \frac{5}{24}I\right)I = \frac{1}{8}I$.

3. In the neutrosophic ring $\mathbb{Z}_8(I)$, the element $4 + 3I$ is a neutrosophic unit and achieves $(4 + 3I)7I = I$ and also $(4 + 3I)(4 + 3I) = I$ And we note $(4 + 3I)I = (7I)I = 7I$.

Theorem.3.14 In any neutrosophic field R(I) is achieved $NU_{R(I)} \cap NZ_{R(I)} = \{bI \ , b \in U_R\}$.

Proof. We have a first $b \in U_R \Rightarrow bI \in NU_{R(I)}$

and also $bI(b^{-1} - b^{-1}I) = (b^{-1} - b^{-1})bI = 0 \Rightarrow bI \in NZ_{R(I)}$, therefore $bI \in NU_{R(I)} \cap NZ_{R(I)} \neq \emptyset$.

On the other hand, $\forall x = a + bI \in NU_{R(I)} \cap NZ_{R(I)}$ where $a \neq 0$ *or* $b \neq 0$ thus $a + bI \in NU_{R(I)}$ *and* $a + bI \in NZ_{R(I)}$. Since $a + bI \in NU_{R(I)}$ so $a \neq -b$ and since $a + bI \in NZ_{R(I)}$, there is $c + dI \in R(I)$ where $c \neq 0$ *or* $d \neq 0$ such that $(a + bI)(c + dI) = 0$. In fact $a = 0$ *and* $b \neq 0$ because if we suppose $a \neq 0$, then we distinguish two cases, *if* $a \neq 0$ and $b = 0$ then $a(c + dI) = 0$ thus $ac + adI = 0$, since $a \neq 0$ hence $c = d = 0$ and this is contradictory to that $x = a + bI \in NZ_{R(I)}$.
Now if $a \neq 0$ and $b \neq 0$ then $(a + bI)(c + dI) = 0 \Rightarrow ac + (ad + bc + bd)I = 0 \Rightarrow ac = 0$ and $ad + bc + bd = 0 \underset{a \neq 0}{\Longrightarrow} c = 0$ and $(a + b)d = 0$. Since $a \neq -b$ so $d = 0$. This is contradictory to that $x \in NZ_{R(I)}$.Therefore, $x = bI\,; b \neq 0$.

Corollary.3.15 In general, it is not necessarily only that $NU_{R(I)} \cap NZ_{R(I)} = \{bI\,, b \in U_R\}$, when $R(\mathrm{I})$ is a unity neutrosophic ring.

Example.3.16 In $\mathbb{Z}_8(\mathrm{I})$, the element $(4 + I) \in NU_{\mathbb{Z}_8(\mathrm{I})} \cap NZ_{\mathbb{Z}_8(\mathrm{I})}$, where it achieves $5I(4 + I) = I$ *and* $(4 + I)(4 + 4I) = 0$.

Theorem.3.17 In any neutrosophic field $R(\mathrm{I})$ is achieved $\mathrm{NU}_{R(I)} \cap \mathrm{NId}_{R(I)} = \{1, \mathrm{I}\}$.

Proof.

We note $1, I \in NU_{R(I)}$ and $1, I \in NId_{R(I)}$, thus $1, I \in NU_{R(I)} \cap NId_{R(I)}$.

At other hand, $\forall x = a + bI \in NU_{R(I)} \cap NId_{R(I)}$ where $(a \neq 0$ *or* $b \neq 0)$ and $a \neq -b$.

$$\Rightarrow (\exists x^{-1} \in R(I); x^{-1}\mathrm{x} = \mathrm{x}x^{-1} = 1 \text{ or } \mathrm{I}) \text{ and } x^2 = \mathrm{x}$$

$$\text{Now if } \quad x^{-1}\mathrm{x} = 1 \text{ and } x^2 = x, \text{then } x^{-1}\mathrm{x} = x^{-1}x^2 = 1. \text{Subsequently } x = 1.$$

$$\text{If } \quad x^{-1}\mathrm{x} = \mathrm{I}, x^2 = x, \qquad \text{then } x^{-1}x^2 = x^{-1}\mathrm{x} \Rightarrow (x^{-1}\mathrm{x})\mathrm{x} = \mathrm{I} \Rightarrow Ix = I$$

$$\text{Since } I(a + bI) = I, \text{so } a + b = 1$$

We have $x^2 = \mathrm{x} \Rightarrow (a + bI)^2 = a + bI \Rightarrow a^2 + (2ab + b^2)I = a + bI \Rightarrow a^2 = a$ and $2ab + b^2 = b$.

Now we have $a^2 = a$ and $2ab + b^2 = b$ and $a + b = 1$.

If $a \neq 0$, we have $a^2 = a$ so $a(a - 1) = 0$ *thus* $a - 1 = 0$. Therefore, $a = 1$. And since $a + b = 1$ *thus* $b = 0$. Therefore $x = 1$.

Now if $b \neq 0$, we have $2ab + b^2 = b$. Since $a = 1 - b$ so $2(1 - b)b + b^2 = b$
$\Rightarrow 2b - 2b^2 - b = 0 \Rightarrow b(1 - b) = 0 \Rightarrow 1 - b = 0 \Rightarrow b = 1$. Since $a + b = 1$, so $a = 0$. Therefore, $x = I$. So $\mathrm{NU}_{R(I)} \cap \mathrm{NId}_{R(I)} = \{1, \mathrm{I}\}$

Example.3.18 In the neutrosophic field $\mathbb{Z}_3(I)$, we have $\mathrm{NId}_{\mathbb{Z}_3(I)} = \{0,1,I,1 + 2I\}$, $\mathrm{NU}_{\mathbb{Z}_3(I)} = \{1,2,I,2I,1 + I\}$. Clear that $\mathrm{NId}_{\mathbb{Z}_3(I)} \cap \mathrm{NU}_{\mathbb{Z}_3(I)} = \{1, \mathrm{I}\}$

Corollary.3.19 In general, it is not necessarily only that $\mathrm{NU}_{R(I)} \cap \mathrm{NId}_{R(I)} = \{1, \mathrm{I}\}$, when $R(\mathrm{I})$ is unity.

Example.3.20 In the neutrosophic ring $\mathbb{Z}_6(\mathrm{I})$, the element $(3 + 4I) \in NU_{\mathbb{Z}_6(\mathrm{I})} \cap NId_{\mathbb{Z}_6(\mathrm{I})}$, where $I(3 + 4I) = I$ and $(3 + 4I)^2 = 3 + 4I$

Theorem.3.21 Assume that $R(\mathrm{I})$ is unity. Therefore, every unit element is a neutrosophic regular.

Proof. $\forall x \in U_{R(I)} \Rightarrow \exists x^{-1} \in R(I);\; xx^{-1} = 1 \Rightarrow xx^{-1}x = x \in NReg_{R(I)}$.

Theorem.3.22 Assume that $R(\mathrm{I})$ is unity. Then, for every neutrosophic unit element of shape $bI\,, b \neq 0$ is a neutrosophic regular.

Proof.

Since $bI \in NU_{R(I)}$ thus $\exists x \in R(I)$ such that $(bI)x = x(bI) = I$. So $(bI)x(bI) = bI \in NReg_{R(I)}$.

Theorem.3.23 In any neutrosophic field $R(\mathrm{I})$, every neutrosophic unit element of the shape $a+bI$; $a \neq 0$ and $a \neq -b$ is a neutrosophic regular.

Proof. Using theorem.2.5. We have $a+bI$ is a unit. Therefore, it is a neutrosophic regular according to the theorem.3.21.

Corollary.3.24 In general, in a unity neutrosophic ring, every neutrosophic unit element of the shape $a+bI$; $a \neq 0 \neq b$ is not necessarily a neutrosophic regular.

Example.3.25 In the neutrosophic ring $\mathbb{Z}_8(\mathrm{I})$, the element $4+I$ is a neutrosophic unit and achieves $(4+I)(4+I)=I$. We note $4+I \notin NReg_{\mathbb{Z}_8(\mathrm{I})}$.

Theorem.3.26 Assume that $R(\mathrm{I})$ is unity. If $x \in NNil_{R(I)}$, then $1-x$, $I(1-x)=I-Ix$, $1+x$, $I(1+x)=I+Ix \in NReg_{R(I)}$.

Proof. We have $x \in NNil_{R(\mathrm{I})} \Rightarrow \exists n \in \mathbb{Z}^+;\ x^n=0$. On the other hand, we note $(I-Ix)(I+x+x^2+\ldots.+x^{n-1})=I+Ix+Ix^2+\ldots.+Ix^{n-1}-Ix-Ix^2-\ldots.-Ix^{n-1}-Ix^n=1-Ix^n=I$. Therefore, $I-Ix=I(1-x) \in NU_{R(\mathrm{I})}$. Using theorem.3.22, $I-Ix \in NReg_{R(I)}$.

Finally, $(I+Ix)(I-x+x^2-x^3+\ldots.+(-1)^{n-1}x^{n-1})=I-Ix+Ix^2-I\,x^3+\ldots\ldots+I(-1)^{n-1}x^{n-1}+Ix-Ix^2+Ix^3-\ldots.+I(-1)^{n-2}x^{n-1}+I(-1)^{n-1}x^n=I+I(-1)^{n-1}x^n=I \Rightarrow I+Ix=(1+x)I \in NU_{R(\mathrm{I})}$. Using theorem.3.22, $I+Ix \in NReg_{R(I)}$.

Similarly, we prove that if $x \in NNil_{R(\mathrm{I})}$, then $1-x, x-1,\ x+1,\ Ix-I \in NReg_{R(\mathrm{I})}$.

Example.3.27 In the neutrosophic ring $\mathbb{Z}_9(\mathrm{I})$, the element $(3+3I)$ is a neutrosophic simple nilpotent,,and we have $1-(3+3I)=1+6+6I=7+6I$. We note

$$(7+6I)(4+3I)(7+6I)=7+6I. \text{ Therefore, } 7+6I \in NReg_{\mathbb{Z}_9(\mathrm{I})}.$$

$$I-I(3+3I)=I+3I=4I. \text{ We note } (4I)(7)(4I)=4I \in NReg_{\mathbb{Z}_9(\mathrm{I})}.$$

$$3+3I-1=2+3\mathrm{I}. \text{ We have } (2+3\mathrm{I})(5+6\mathrm{I})(2+3\mathrm{I})=2+3I \in NReg_{\mathbb{Z}_9(\mathrm{I})}.$$

$$I(3+3I)-I=5I;\ (5I)(2)(5I)=5I \in NReg_{\mathbb{Z}_9(\mathrm{I})}.$$

On the other hand, $3+3I+1=4+3I$, where $(4+3I)(7+6I)(4+3I)=4+3I \in NReg_{\mathbb{Z}_9(\mathrm{I})}$. $I(3+3I)+I=7I; (7I)4(7I)=7I \in NReg_{\mathbb{Z}_9(\mathrm{I})}$.

Corollary.3.28 Assume that $R(\mathrm{I})$ is unity. Now if $x=bI \in NNil_{R(I)}$, then $I-x$ and $I+x \in NReg_{R(I)}$.

Proof. We have $x=bI \in NNil_{R(I)} \Rightarrow \exists n \in \mathbb{Z}^+;\ x^n=(bI)^n=b^nI^n=0 \underset{I^n \neq 0}{\Longrightarrow} b^n=0 \Rightarrow\ b \in Nil_{R(I)}$. On the other hand, $I-x=I-bI=I(1-b)$. Using theorem.3.26, we have $I(1-b) \in NReg_{R(I)}$.

Similarly, we prove that $I+x \in NReg_{R(I)}$.

Corollary.3.29 In general, if $R(\mathrm{I})$ is a unity neutrosophic ring and $x=a+bI \in NNil_{R(I)}$, then it is not necessarily that $I-x, I+x \in\ NReg_{R(I)}$.

Example.3.30 In the neutrosophic ring $\mathbb{Z}_8(\mathrm{I})$, the element $(4+4I)$ is a neutrosophic simple nilpotent, while $I-(4+4I)=I+4+4I=4+5I \notin NReg_{\mathbb{Z}_8(\mathrm{I})}$, because if $a+bI \in \mathbb{Z}_8(\mathrm{I})$; $a \neq 0$ *or* $b \neq 0$. We have $(4+5I)(a+bI)(4+5I)=(4+5I)(4+5I)(a+bI)=I(a+bI)=(a+b)I \neq 4+5I\ \ \forall a,b \in \mathbb{Z}_8$.

Corollary.3.31 Assume that $R(\mathrm{I})$ is unity. Now, if $x_1=a+bI \in NU_{R(I)}$ *and* $x_2=c+dI \in NNil_{R(I)}$, then it is not necessarily that $x_1-\ x_2, x_1+x_2 \in\ NReg_{R(I)}$.

Example.3.32 In the neutrosophic ring $\mathbb{Z}_9(\mathrm{I})$, the element $(3+3I) \in NSNil_{\mathbb{Z}_9(\mathrm{I})}$,and $8I \in NU_{R(I)}$, we have $8I-(3+3I)=8I+6+6I=6+5I \notin NReg_{\mathbb{Z}_9(\mathrm{I})}$, because if $a+bI \in \mathbb{Z}_9(\mathrm{I})$; $a \neq 0$ *or* $b \neq 0$. We have $(6+5I)(a+bI)(6+5I)=(6+5I)(6+5I)(a+bI)=4I(a+bI)=(4a+4b)I\neq 6+5I\ \ \forall a,b\in\mathbb{Z}_9$.

Theorem.3.33 In any neutrosophic field, $R(\mathrm{I})$ is achieved $R(\mathrm{I})=NReg_{R(I)}$.

Proof.

$\forall x=a+bI\in R(I)$. If $x=0$, then $x\in NReg_{R(I)}$, because $\forall y\in R(I)$ so $0=0.y.0$

If $x\neq 0$, then $a\neq 0\ or\ b\neq 0$.

Now if $a\neq 0\ and\ b=0\Rightarrow x=a\in U_{R(I)}$. Using theorem. 3.21, $x\in NReg_{R(I)}$

If $a=0\ and\ b\neq 0\Rightarrow x=bI\in NU_{R(I)}$. Using theorem. 3.22, $x\in NReg_{R(I)}$

If $a\neq 0\ and\ b\neq 0\ and\ a\neq -b$. Using theorem. 2.6, $x\in U_{R(I)}$. Using theorem. 3.21, $x\in NReg_{R(I)}$

If $a\neq 0\ and\ b\neq 0\ and\ a=-b\Rightarrow x=a-aI$; $(a-aI)a^{-1}(a-aI)=(1-I)(a-aI)=a-aI=x$

$$\Rightarrow x\in NReg_{R(I)}$$

Another way to prove (in case $a\neq 0\ and\ b\neq 0$).

If $a\neq 0\ and\ b\neq 0\Rightarrow \exists\ x+yI\in R(\mathrm{I})$; $x\neq 0\ or\ y\neq 0\ and\ x,y\in R$.

And it is achieved $a+bI=(a+bI)(x+yI)(a+bI)=[a^2+(2ab+b^2)I](x+yI)$

Suppose that $c=a^2$, $d=2ab+b^2\Rightarrow a+bI=(c+dI)(x+yI)$

It's clear $c\neq 0$ in $R(\mathrm{I})$ and that $d=0\ or\ d\neq 0$.

If $d=0$ then $a+bI=c(x+yI)=cx+cyI\Rightarrow\ a=cx, b=cy\Rightarrow x=c^{-1}a\ , y=c^{-1}b\in R(\mathrm{I})$

If $d\neq 0\Rightarrow a+bI=(c+dI)(x+yI)=cx+(cy+dx+dy)I\Rightarrow\ a=cx\ , b=cy+dx+dy$

$$\Rightarrow x=c^{-1}a\ \in R(\mathrm{I})\Rightarrow b=cy+dc^{-1}a\ +dy\Rightarrow b=(c+d)y+dc^{-1}a$$

If $c+d=0\Rightarrow b=0y+dc^{-1}a$, $\forall y\ \in R(\mathrm{I})$, in this case we will consider $y=0$ for ease.

If, $c+d\neq 0\Rightarrow b-dc^{-1}a=(c+d)y\Rightarrow y=(c+d)^{-1}(b-dc^{-1}a)\in R(\mathrm{I})$.

Corollary.3.34 Let $R(\mathrm{I})$ be a neutrophilic field, then every element $a+bI$; $a\neq 0, a=-b$ is neutrosophic regular, so there is $a^{-1}+cI$; $\forall c\in R(I)$ where $(a+bI)(a^{-1}+cI)(a+bI)=a+bI$.

Proof. Since $a\neq 0\ and\ a=-b$, so $\forall c\in R(I)$. We note that

$$(a-aI)(a^{-1}+cI)(a-aI)=(1+acI-I-acI)(a-aI)=(1-I)(a-aI)=a-aI.$$

Example.3.35 In the neutrosophic field $\mathbb{Z}_7(I)$, the element $5+6I$ is a neutrosophic regular, where $a=5\ , b=6$. Using theorem.3.33, we can find the element $x+yI$ that achieves neutrosophic regularity, $c=\overline{a^2}=4$, $d=\overline{2ab+b^2}=\overline{60+36}=5$; $c+d=4+5=2\ \Rightarrow x=c^{-1}a=2(5)=3$

$$\Rightarrow y=(c+d)^{-1}(b-dc^{-1}a)=(2)^{-1}[6-(5.2.5)]=4[6-(1)]=4[6+6]=4(5)=6$$

Now easily it can be verified that $5+6I=(5+6I)(3+6I)(5+6I)$

Example.3.36 In the neutrosophic field $\mathbb{Z}_7(I)$, the element $3+4I$ is a neutrosophic regular element where $a=3\ , b=4$. Using theorem.3.33, we can find the element $x+yI$ that achieves neutrosophic regularity

$$c=\overline{a^2}=2\ \ , d=\overline{2ab+b^2}=\overline{24+16}=5\ \ ; c+d=2+5=0\ \Rightarrow y\in\mathbb{Z}_7$$
$$\Rightarrow x=c^{-1}a=4(3)=5$$

Now easily it can be verified that $\forall y\in\mathbb{Z}_7$; $3+4I=(3+4I)(5+yI)(3+4I)$

Example.3.37 In the neutrosophic field $\mathbb{Z}_{11}(I)$, the element $3+8I$ is a neutrosophic regular element where $a=3\ , b=8$. Using theorem.3.33, we can find the element $x+yI$ that achieves neutrosophic regularity

$$c=\overline{a^2}=9\ \ , d=\overline{2ab+b^2}=\overline{48+64}=2\ \ ; c+d=9+2=0\ \Rightarrow y\in\mathbb{Z}_{11}\Rightarrow x=c^{-1}a=5(3)=4$$

Now easily it can be verified that

$3+8I=(3+8I)(4+yI)(3+8I)\ \forall y\in\mathbb{Z}_{11}$. Suppose that $y=0\ \Rightarrow 3+8I=(3+8I)4(3+8I)$

Corollary.3.38 Assume that $R(\mathrm{I})$ is a unity and $a+bI\in R(I)$ where $a,b\in Reg_{R(I)}$, then it is not necessarily that $a+bI\in NReg_{R(I)}$, and also if $a+bI\in NReg_{R(I)}$, then it is not necessarily $a,b\in Reg_R$.

Example.3.39 In the neutrosophic ring $\mathbb{Z}_4(I)$, the element $3+3I$ is neutrosophic irregular, although $a=b=3=3.3.3\in Reg_{\mathbb{Z}_4}$, because if we assume that

$3+3I=(3+3I)(x+yI)(3+3I)=(3+3I)(3+3I)(x+yI)=(1+3I)(x+yI)$

$=x+yI+3xI+3yI=x+3xI\ ; x,y\in\mathbb{Z}_4$

$\Rightarrow 3+3I=x+3xI\Rightarrow x=3$ and $3x=3$ $thus\ x=3$ and $x=1$, but this is a contradiction. Therefore, $3+3I$ is a neutrosophic irregular.

Example.3.40 In the neutrosophic ring $\mathbb{Z}_8(I)$, the element $x=3+2I$ is a neutrosophic regular although $2\notin Reg_{\mathbb{Z}_8}$, where $x=xxx$.

Theorem.3.41 If $R(I)$ is a unity neutrosophic regular neutrosophic ring, then $R(I)=NU_{R(I)}\cup NZ_{R(I)}$.

Proof. Always be an investigator $NU_{R(I)}\cup NZ_{R(I)}\subseteq R(I)$.

On other hand, $\forall\ x\in R(I)$ where $x\notin NZ_{R(I)}$. Now since $x\in R(I)$ so there is y belonging to $R(I)$ that achieves

$x=xyx$ so $x-xyx=0$ thus $x(1-yx)=0\xRightarrow[x\notin NZ_{R(I)}]{}1-yx=0$ thus $yx=1$

$x=xyx\Rightarrow\ x-xyx=0\Rightarrow\ (1-xy)x=0\xRightarrow[x\notin NZ_{R(I)}]{}1-xy=0$ thus $xy=1$

Since $x\in NU_{R(I)}$ so $x\in NU_{R(I)}\cup NZ_{R(I)}$. Therefore, $R(I)\subseteq NU_{R(I)}\cup NZ_{R(I)}$.

Example.3.42 We have $\mathbb{Z}_3(I)$ is a neutrosophic regular, which $NZ_{\mathbb{Z}_3(I)}=\{0,I,2I,1+2I,2+I\}$, $NU_{\mathbb{Z}_3(I)}=\{1,2,I,2I,1+I,2+2I\}$, and we note $\mathbb{Z}_3(I)=NU_{\mathbb{Z}_3(I)}\cup NZ_{\mathbb{Z}_3(I)}$.

Note.3.43 The condition of neutrosophic regularity in the unity neutrosophic ring is necessary for it to satisfy $R(I)=NU_{R(I)}\cup NZ_{R(I)}$.

Example.3.44 We have $\mathbb{Z}(\mathrm{I})$ is a unity neutrosophic irregular neutrosophic ring and is not achieved $\mathbb{Z}(\mathrm{I})=NU_{\mathbb{Z}(\mathrm{I})}\cup NZ_{\mathbb{Z}(\mathrm{I})}$.

Theorem.3.45 If $R(\mathrm{I})$ is a neutrosophic right (left) strongly regular neutrosophic ring, then $NSNil_{R(\mathrm{I})}=\{0\}$.

Proof. $\forall\ x\in NSNil_{R(\mathrm{I})}\Rightarrow x^2=xx=0$. Since $x\in R(\mathrm{I})$, there is y belongs to $R(I)$ that achieves $x=yxx$, therefore $x=yxx=y0=0\Rightarrow NSN_{R(\mathrm{I})}=\{0\}$.

Similarly, we prove that $NSNil_{R(\mathrm{I})}=\{0\}$ in the case of $R(\mathrm{I})$ is a neutrosophic left strongly regular.

Table 1. key distinctions between the classical and neutrosophic rings.

	unity classical ring R	unity neutrosophic ring R(I)
1	$U_R\cap Z_R=\emptyset$	$NU_{R(I)}\cap NZ_{R(I)}\neq\emptyset$

2	$\mathrm{U}_R \cap \mathrm{Id}_R = \{1\}$	If $R(\mathrm{I})$ be a neutrosophic field, then $\mathrm{NU}_{R(I)} \cap \mathrm{NId}_{R(I)} = \{1, \mathrm{I}\}$
3	Suppose R is a field then then every element $x \neq 0$ has inverse.	If $R(\mathrm{I})$ be an infinite (finite) neutrosophic field, then there are elements that have an infinite (finite) number of neutrosophic inverse.
		Suppose $R(\mathrm{I})$ is a field then there are elements that non unit.
4	If $u \in U_R$ and $a \in Nil_R$, then $u - a,\ u + a \in Reg_R$	If $x_1 \in NU_{R(I)}$ and $x_2 \in NNil_{R(I)}$ then it is not necessarily that $x_1 - x_2,\ x_1 + x_2 \in NReg_{R(I)}$
5	Every unit element is regular.	Every neutrosophic unit element is not necessarily a neutrosophic regular.

4. Conclusion and Future Works

This study broadened the idea of neutrosophic regularity in neutrosophic rings. We investigated the qualities of neutrosophic regular elements and the most significant properties that connect them to neutrosophic elements in neutrosophic rings. We discovered numerous key findings that connect these components, some of which differ from what is known about classic rings. Furthermore, various examples were constructed to demonstrate the reliability of the study.

We intend to investigate the characteristics of ideals in regular neutrosophic rings in the future.

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Data availability

The datasets generated during and/or analyzed during the current study are not publicly available due to the privacy-preserving nature of the data but are available from the corresponding author upon reasonable request.

Conflicts of Interest

The author declare that there is no conflict of interest in the research.

Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

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Analysis of drug diffusion in human connective tissue in neutrosophic environment

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Abstract: Neutrosophic sets play a crucial role in handling uncertainty, ambiguity, and indeterminacy in numerous theories. They are a kind of extension of the two types of fuzzy sets and intuitionistic fuzzy sets. In the context of modeling drug diffusion within human connective tissues, a differential equation is employed within the neutrosophic framework, utilizing Hukuhara differentiability. We establish the initial conditions and parameters in the form of Type 2 triangular single-valued neutrosophic numbers. This study explores the stability and existence of equilibrium points, providing precise solutions. To conduct numerical simulations across various values of the (α, β, γ)-cut of the triangular neutrosophic number, we utilize MATLAB 2018a.

Keywords: Drug diffusion human model, Single valued triangular neutrosophic number of type 2, Hukuhara differentiability, (α, β, γ)-cut, Neutrosophic differential equation, Stability Analysis, Numerical study.

1. Introduction

The importance of mathematical modeling in the context of drug delivery within human tissues is increasing due to substantial advancements in information technology. This field has now become a consistently explored area in both academic research and commercial applications. Much like other scientific disciplines, the development of pharmaceutical technology involves the prediction of how the delivered substance flows and behaves kinematically. The design, distribution, dose, and delivery of numerous pharmaceuticals inside the human body can all be optimized using mathematical formulas. Within a neutrosophic framework, the dynamics of drug release and transport processes can be elucidated with significantly greater precision than in classical contexts. The neutrosophic environment is aptly utilized to represent dynamic systems that may have inherent uncertainty. The concept of a fuzzy set, where each element is associated with a membership degree, was first introduced by L. Zadeh [1]. Subsequently, K. T. Atanassov extended this idea to intuitionistic fuzzy sets (IFS) [3], and F. Smarandache further expanded it to neutrosophic sets (NS) [6, 8, 9]. Apart from membership degrees, Intuitionistic Fuzzy Sets (IFS) also incorporate degrees of non-membership. In recent years, there has been extensive research on fuzzy differential equations, which are characterized by imprecise parameters [4, 5, 7, 10, 11, 12, 13, 18]. Subsequently, these differential equations were explained in an intuitive context [14, 18, 38]. Neutrosophic differential equations (NDE) [17, 23, 24] were developed to accommodate the uncertainty associated with the parameters. In contrast to Kaleva [4], who first introduced the idea of differential equations inside a fuzzy framework, Hukuhara [16] adapted the idea of differences and differentiation in interval-valued functions in order to solve the problem of unsolvable boundary value problems. Dey et al. [33] described topological subspace and produced several significant findings based on single valued neutrosophic numbers. Karak et al. [37] has applied the theory of single valued neutrosophic numbers to transportation problems. Acharya et al. [38] explored a prey refuge harvesting model employing intuitionistic fuzzy sets. In a different study [30], it was found that multi-criteria group decision-making problems could be applied to assess pollution characteristics in megacities using a trapezoidal neutrosophic set. In the realm of mathematical research, neutrosophic integral calculus plays a pivotal role. Biswas et al. [29] have used the concept of neutrosophic Riemann integral at (α,β,γ)-levels. Biswas et al. [31] have alsostudied the Gaussian quadrature methods to evaluate numerical integration of netrosophic valued function. Gahlot et al. [25] have developed several distinctive types of single-valued neutrosophic numbers and employed them in multi-criteria decision-making. Sumanth and his team [21] found that a first-order neutrosophic differential equation, incorporating neutrosophic numbers, can be applied in bacterial culture models. Subsequently, Sumathi et al. [22] discussed methods for solving a second-order neutrosophic differential equation with a boundary condition, utilizing a trapezoidal neutrosophic number. In the article [39], they have studied the fractional order derivative in neutrosophic number and discussed nonlinear ecological model with Allee effect. As an extension of the Z-number, Borah [40] introduced the quadric partitioned single neutrosophic Z-number and investigated the operator and score function in the context of three multi-criteria decision-making scenarios during the COVID-19 pandemic. In our research, we focused on modeling drug diffusion in a neutrosophic environment within human tissues. We considered single-valued triangular neutrosophic numbers of type 2 to represent both the initial conditions for the drug diffusion quantity in the bloodstream at time t (g(t)) and the drug concentration in the bloodstream (). Additionally, we conducted stability analyses for the equilibrium points of the Neutrosophic Differential Equations (NDE) and obtained precise solutions for them. To validate our findings, we employed Matlab numerical simulations (version 2018a).

1.1. Arrangement of the article:

Section 2 presents essential prerequisites. In Section 3, we outline the model for drug transportation within neutrophilic human connective tissues. Section 4 delves into the precise solutions of the model and conducts a stability analysis. Section 5 focuses on numerical simulations for different (α, β, γ)-cut values of the type 2 single-valued triangular neutrosophic number. Finally, Section 6 provides a summary of the paper.

1.2. Motivation and novelty:

Several natural factors or those related to human activities may affect the parameters of a biological model and this may lead to certain vagueness, impreciseness or indeterminacy in the values of the parameters. Various approaches are considered to tackle such situations, including interval differential equations (IDE), fuzzy differential equations (FDE), intuitionistic fuzzy differential equations (IFDE), and more. In IDE, parameters are constrained to specific intervals. Conversely, in the FDE method, parameters are assigned precise membership values. IFDE takes into account both membership and non-membership values of the parameters. However, the neutrosophic differential equation (NDE) is essential for addressing the inherent uncertainty in parameter values.

2. Preliminaries:

Definition 2.1: "Single valued neutrosophic set [19]:
A neutrosophic set $\widetilde{X}_{ne}$ on the universe U is defined as

$$\widetilde{X}_{ne} = \{x: \left(\xi_{\widetilde{X}_{ne}}(x), \eta_{\widetilde{X}_{ne}}(x), \varsigma_{\widetilde{X}_{ne}}(x)\right); x \in U\}$$

where $\xi_{\widetilde{X}_{ne}}(x): U \to [0,1]$, $\eta_{\widetilde{X}_{ne}}(x): U \to [0,1]$ and $\varsigma_{\widetilde{X}_{ne}}(x): U \to [0,1]$ represent the truth membership function, indeterminacy membership function and falsity membership function respectively such that $0 \le \xi_{\widetilde{X}_{ne}}(x) + \eta_{\widetilde{X}_{ne}}(x) + \varsigma_{\widetilde{X}_{ne}}(x) \le 3$."

Definition 2.2: "(α, β, γ-cut) neutrosophic set [21]: The (α, β, γ-cut) neutrosophic set $\widetilde{X}_{ne_{(\alpha,\beta,\gamma)}}$ is defined as $\widetilde{X}_{ne_{(\alpha,\beta,\gamma)}} = \{\left(\xi_{\widetilde{X}_{ne}}(x), \eta_{\widetilde{X}_{ne}}(x), \varsigma_{\widetilde{X}_{ne}}(x)\right): x \in U, \xi_{\widetilde{X}_{ne}}(x) \ge \alpha, \eta_{\widetilde{X}_{ne}}(x) \le \beta, \varsigma_{\widetilde{X}_{ne}}(x) \le \gamma\}$ where α, β, γ are fixed numbers in [0,1] such that $\alpha + \beta + \gamma \le 3$."

Definition 2.3: Triangular single valued neutrosophic number of type 2 (TrSVNN type 2) [19]:
Let us consider a TrSVNN of type 2 as $\widetilde{X}_{ne} = \langle [n_{11}, n_{12}, n_{13}; m_{11}, m_{12}, m_{13}; \rho, \sigma] \rangle$ whose truth, indeterminacy and falsity membership function are as follows

$$\xi_{\widetilde{X}_{ne}}(x) = \begin{cases} \dfrac{x - n_{11}}{n_{12} - n_{11}}; & n_{11} \le x < n_{12} \\ 1; & x = n_{12} \\ \dfrac{n_{13} - x}{n_{13} - n_{12}}; & n_{12} < x \le n_{13} \\ 0 & \text{otherwise} \end{cases}$$

$$\eta_{\widetilde{X}_{ne}}(x) = \begin{cases} \dfrac{m_{12} - x + \rho(x - m_{11})}{m_{12} - m_{11}}; & m_{11} \le x < m_{12} \\ \rho; & x = m_{12} \\ \dfrac{x - m_{12} + \rho\,(m_{13} - x)}{m_{13} - m_{12}}; & m_{12} < x \le m_{13} \\ 0; & \text{otherwise} \end{cases}$$

$$\varsigma_{\widetilde{X}_{ne}}(x) = \begin{cases} \dfrac{m_{12} - x + \sigma(x - m_{11})}{m_{12} - m_{11}} ; & m_{11} \le x < m_{12} \\ \rho \quad ; & x = m_{12} \\ \dfrac{x - m_{12} + \sigma\,(m_{13} - x)}{m_{13} - m_{12}} ; & m_{12} < x \le m_{13} \\ 0 \quad ; & \text{otherwise} \end{cases}$$

where,

$0 \le \xi_{\widetilde{X}_{ne}}(x) + \eta_{\widetilde{X}_{ne}}(x) + \varsigma_{\widetilde{X}_{ne}}(x) \le 2, \quad x \in \widetilde{X}_{neu}\ , \ \rho,\ \sigma \in (0,1]$

The parametric form of TrSVNN type 2 is

$$(\widetilde{X}_{ne})_{\alpha,\beta,\gamma} = [\xi_{ne1}(\alpha), \xi_{ne2}(\alpha); \eta_{ne1}(\beta), \eta_{ne2}(\beta); \varsigma_{ne1}(\gamma), \varsigma_{ne2}(\gamma)]$$

where

$$\xi_{ne1}(\alpha) = n_{11} + \alpha(n_{12} - n_{11}),$$

$\xi_{ne2}(\alpha) = n_{13} - \alpha(n_{13} - n_{12}),$

$\eta_{ne1}(\beta) = \frac{m_{12} - \rho m_{11} - \beta(m_{12} - m_{11})}{1-\rho},$

$\eta_{ne2}(\beta) = \frac{m_{12} - \rho m_{13} + \beta(m_{13} - m_{12})}{1-\rho},$

$\varsigma_{ne1}(\gamma) = \frac{m_{12} - \sigma m_{11} - \gamma(m_{12} - m_{11})}{1-\sigma},$

$\varsigma_{ne2}(\gamma) = \frac{m_{12} - \sigma m_{13} + \gamma(m_{13} - m_{12})}{1-\sigma}.$

Example 1: Consider the TrSVNN of type 2, $\widetilde{X}_{ne} = (20,25,28; 24,26,32; 0.4,0.5)$.

The parametric form of TrSVNN of type 2 is represented as,

$\xi_{ne1}(\alpha) = 20 + 5\alpha, \xi_{ne2}(\alpha) = 28 - 3\alpha, \eta_{ne1}(\beta) = \frac{8.2-\beta}{0.3}, \eta_{ne2}(\beta) = \frac{13.2+6\beta}{0.6}, \varsigma_{ne1}(\gamma) = \frac{14-2\beta}{0.5}, \varsigma_{ne2}(\gamma) = \frac{10+6\beta}{0.5}.$

Definition 2.4 Hukuhara derivative on neutrosophic function [15]: "Let $g_{ne}: (a,b) \to NF(R)$ be a neutrosophic valued function and $l_0, l_0 + h \in (a,b)$. gis hukuhara differentiable at l_0, if $\exists$ an element $g'_{ne}(l_0) \in NF(R)$ such that for all $h > 0$,

$\lim\limits_{h\to 0} \frac{g_{ne}(l_0+h)\ominus g_{ne}(l_0)}{h} = \lim\limits_{h\to 0} \frac{g_{ne}(l_0)\ominus g_{ne}(l_0-h)}{h} = g'_{ne}(l_0)$is satisfied."

Remark [24]: "Let $g_{ne}: (a,b) \to NF(R)$ be a neutrosophic valued function. Let$g_{ne}(t,\alpha,\beta,\gamma) = \langle [g_{ne1}(t,\alpha), g_{ne2}(t,\alpha)], [g'_{ne1}(t,\beta), g'_{ne2}(t,\beta)], [g''_{ne1}(t,\gamma), g''_{ne2}(t,\gamma)] \rangle$is its α,β,γ-cut . If$g_{ne}$is hukuhara differentiable at l_0 such that

$\dot{g}_{ne}\,(l_0, \alpha) = [\min\{\dot{g}_{neu\,1}(l_0{:}\,\alpha), \dot{g}_{neu\,2}(l_0{:}\,\alpha)\}, \max\{\dot{g}_{neu\,1}(l_0{:}\,\alpha), \{\dot{g}_{neu\,2}(l_0{:}\,\alpha)\}]$ if

$\dot{g}_{neu\,1}(l_0{:}\,\alpha), \dot{g}_{neu\,2}(l_0{:}\,\alpha)$ exist.

$\dot{g}'(l_0{:}\,\beta) = [\min\{\dot{g}'_{ne\,1}(l_0{:}\,\beta), \dot{g}'_{ne\,2}(l_0{:}\,\beta)\}, \max\{\dot{g}'_{ne1}(l_0{:}\,\beta), \dot{g}'_{ne\,2}(l_0{:}\,\beta)\}]$,if$\dot{g}'_{ne\,1}(l_0{:}\,\beta), \dot{g}'_{ne\,2}(l_0{:}\,\beta)$ exist.

$\dot{g}''(l_0{:}\,\gamma) = [\min\{g''_{ne1}(l_0{:}\,\gamma), g''_{ne2}(l_0{:}\,\gamma)\}, \max\{g''_{ne1}(l_0{:}\,\gamma), g''_{ne2}(l_0{:}\,\gamma)\}]$,if$g''_{ne1}(l_0{:}\,\gamma), g''_{ne2}(l_0{:}\,\gamma)$ exist.

Definition 2.5. Neutrosophic differential equation (NDE) [22]: "A first order initial value problem of the form $\frac{df(t)}{dx} = kf_1(t)$, $f(t_0) = f_{1_0}$ is called a neutrosophic differential equation if any one or both of k and f_0 are neutrosophic numbers.

Let the solution of the above neutrosophic differential equation be f(x) and its (α,β,γ)- cut be

$$f_1(t,\alpha,\beta,\gamma) = \langle [f_{11}(t,\alpha), f_{12}(t,\alpha)], [f'_{11}(t,\beta), f'_{12}(t,\beta)], [f''_{11}(t,\gamma), f''_{12}(t,\gamma)] \rangle$$

In general the solution is considered to be strong if

i) $\frac{df_{11}(t,\alpha)}{d\alpha} > 0, \frac{df_{12}(t,\alpha)}{d\alpha} < 0, \forall\ \alpha \in [0,1],\ f_{11}(t,1) \leq f_{12}(t,1).$

ii) $\frac{df'_{11}(t,\beta)}{d\beta} < 0, \frac{df'_{12}(t,\beta)}{d\beta} > 0, \forall\ \beta \in [0,1],\ f'_{11}(t,0) \leq f'_{12}(t,0).$

iii) $\frac{df''_{11}(t,\gamma)}{d\gamma} < 0, \frac{df''_{12}(t,\gamma)}{d\gamma} > 0, \forall\ \gamma \in [0,1],\ f''_{11}(t,0) \leq f''_{12}(t,0).$

Otherwise, the solution is a weak solution."

2.1 Properties on neutrosophic number [23]:

Proposition 2.1.1 Let ũ and ũ be two neutrosophic numbers then,

(i) $(\tilde{u} \oplus \tilde{v})_{(\alpha,\beta,\gamma)} = \tilde{u}_{(\alpha,\beta,\gamma)} \oplus \tilde{v}_{(\alpha,\beta,\gamma)}$.

(ii) $(\tilde{u} \ominus \tilde{v})_{(\alpha,\beta,\gamma)} = \tilde{u}_{(\alpha,\beta,\gamma)} \ominus \tilde{v}_{(\alpha,\beta,\gamma)}$.

(iii) $(\tilde{u} \otimes \tilde{v})_{(\alpha,\beta,\gamma)} = \tilde{u}_{(\alpha,\beta,\gamma)} \otimes \tilde{v}_{(\alpha,\beta,\gamma)}$.

(iv) $(\lambda\tilde{u})_{(\alpha,\beta,\gamma)} = \lambda\tilde{u}_{(\alpha,\beta,\gamma)}$, for $\lambda \neq 0 \in \mathbb{R}$.

Example 2: If $\tilde{p}_{ne} = (14,18,22; 17,21,25; 0.5,0.4)$ and $\tilde{q}_{ne} = (12,15,18; 13,20,26; 0.5,0.4)$ are two TrSVNN of type 2, then following properties are given as,

i) Addition: $\tilde{p}_{ne} + \tilde{q}_{ne} = (26,33,40; 30,41,51; 0.5,0.4)$.

ii) Substraction: $\tilde{p}_{ne} - \tilde{q}_{ne} = (2,3,4; 4,1,1; 0.5,0.4)$.

iii) Multiplication: :$\tilde{p}_{ne} \times \tilde{q}_{ne} = (168,270,396; 221,420,650; 0.5,0.4)$.

iv) Multiplication by a constant: $\lambda\tilde{p}_{ne} = (42,54,66; 51,63,75; 0.5,0.4)$,where λ=3.

3. Mathematical formulation:

In biological processes, the relation between the amount of drug intake and concentration of drug in human body at different sites through various compartments has substantial impact on the drug diffusion process. Thus, owing to its necessity to study the dynamics of the quatmtity of drug diffusion within blood at time t along with its concentration we consider the following differential equation

$$u\frac{dg(t)}{dt} = -\lambda g(t), \qquad g(t_0) = g_0,\ \ t \in [t_0, \infty) \qquad \text{……………………….(1)}$$

Where, g(t) is the amount of drug diffusion at time t, u is the body's blood volume and λ ($>$ 0) is the rate of concentration of drug.$g_0$is the amount of drug diffusion at initial time $t = t_0$.

4. In neutrosophic environment the analysis of the drug diffusion human tissues model system:

From (1), we have considered the following three cases:

i) The amount of drug diffusion in blood at initial time t_0i.e.$\tilde{g}_0$ is TrSVNN Type 2.

ii) The concentration of drug in the blood stream $\tilde{\lambda}$is neutrosophic number TrSVNN Type 2

iii) Both $\tilde{g}_0$ and $\tilde{\lambda}$ are TrSVNN Type 2.

4.1 Case 1: Amount of drug diffusion in blood at initial time t_0i.e. $\tilde{g}_0$ is TrSVNN Type 2.

The system of NDDE (1) can be written as

$u\frac{dg_1(t,\alpha)}{dt} = -\lambda g_2(t,\alpha),$

$u\frac{dg_2(t,\alpha)}{dt} = -\lambda g_1(t,\alpha),$

$u\frac{dg'_1(t,\beta)}{dt} = -\lambda g'_2(t,\beta),$

$u\frac{dg'_2(t,\beta)}{dt} = -\lambda g'_1(t,\beta),$

$u\frac{dg''_1(t,\gamma)}{dt} = -\lambda g''_2(t,\gamma),$

$u\frac{dg''_2(t,\gamma)}{dt} = -\lambda g''_1(t,\gamma),$ …………………..(2)

where α,β,γ –cut of g(t) is $\langle[g_1(t,\alpha), g_2(t,\alpha)], [g'_1(t,\beta), g'_2(t,\beta)], [g''_1(t,\gamma), g''_2(t,\gamma)]\rangle$ and the initial conditions $\tilde{g}_0$are: $g_i(0,\alpha) = g_{0i}(\alpha)$; $g'_i(0,\beta) = g'_{0i}(\beta)$; $g''_i(0,\gamma) = g''_{0i}(\gamma)$; $i = 1, 2$.

The particular solution of (2), we have

$$g_1(t,\alpha) = \frac{1}{2}\left(g_{01}(\alpha) + g_{02}(\alpha)\right)e^{-\frac{\lambda t}{u}} - \frac{1}{2}\left(g_{02}(\alpha) - g_{01}(\alpha)\right)e^{\frac{\lambda t}{u}},$$
$$g_2(t,\alpha) = \frac{1}{2}\left(g_{02}(\alpha) - g_{01}(\alpha)\right)e^{\frac{\lambda t}{u}} + \frac{1}{2}\left(g_{01}(\alpha) + g_{02}(\alpha)\right) e^{-\frac{\lambda t}{u}},$$
$$g'_1(t,\beta) = \frac{1}{2}\left(g'_{01}(\beta) + g'_{02}(\beta)\right)e^{-\frac{\lambda t}{u}} - \frac{1}{2}\left(g'_{02}(\beta) - g'_{01}(\beta)\right)e^{\frac{\lambda t}{u}},$$
$$g'_2(t,\beta) = \frac{1}{2}\left(g'_{02}(\beta) - g'_{01}(\beta)\right)e^{\frac{\lambda t}{u}} + \frac{1}{2}\left(g'_{01}(\beta) + g'_{02}(\beta)\right) e^{-\frac{\lambda t}{u}},$$
$$g''_1(t,\gamma) = \frac{1}{2}\left(g''_{01}(\gamma) + g''_{02}(\gamma)\right)e^{-\frac{\lambda t}{u}} - \frac{1}{2}\left(g''_{02}(\gamma) - g''_{01}(\gamma)\right)e^{\frac{\lambda t}{u}},$$
$$g''_2(t,\gamma) = \frac{1}{2}\left(g''_{02}(\gamma) - g''_{01}(\gamma)\right)e^{\frac{\lambda t}{u}} + \frac{1}{2}\left(g''_{01}(\gamma) + g''_{02}(\gamma)\right) e^{-\frac{\lambda t}{u}}. \quad ……………(3)$$

Equilibrium point: We get one equilibrium point say $E_1^* = (0,0,0,0,0,0)$ for the model (2).

Stability Analysis:

Lemma 1: E_1^* is unstable.

Proof: The variational matrix V_{11} at E_1^* which is given by

$$V_{11} = \begin{pmatrix} 0 & -\frac{\lambda}{u} & 0 & 0 & 0 & 0 \\ -\frac{\lambda}{u} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\lambda}{u} & 0 & 0 \\ 0 & 0 & -\frac{\lambda}{u} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{\lambda}{u} \\ 0 & 0 & 0 & 0 & -\frac{\lambda}{u} & 0 \end{pmatrix}.$$

The characteristic equation becomes,

$y_1^6 - 3\frac{\lambda^2}{u^2}y_1^4 + 3\frac{\lambda^4}{u^4}y_1^2 - \frac{\lambda^6}{u^6} = 0$ where$y_1$is the eigenvalue ofV_{11}.

Obviously, the eigenvalues of the matrixV_{11} are$\frac{\lambda}{u}, -\frac{\lambda}{u}, \frac{\lambda}{u}, -\frac{\lambda}{u}, \frac{\lambda}{u}, -\frac{\lambda}{u}$, the equilibrium point E_1^* is saddle node and the system is unstable at E_1^*.

4.2 Case 2: The concentration of the drug in the bloodstream is represented as a Neutrosophic number of TrSVNN Type 2.

Considering the NDDE (1) becomes the following system as

$u\frac{dg_1(t,\alpha)}{dt} = -\lambda_2(\alpha)g_1(t,\alpha),$

$u\frac{dg_2(t,\alpha)}{dt} = -\lambda_1(\alpha)g_2(t,\alpha),$

$u\frac{dg''_1(t,\beta)}{dt} = -\lambda'_2(\beta)g''_1(t,\beta),$

$u\frac{dg_2''(t,\beta)}{dt} = -\lambda'_1(\beta)g'_2(t,\beta),$

$u\frac{dg''_1(t,\gamma)}{dt} = -\lambda''_2(\gamma)g''_1(t,\gamma),$

$u\frac{dg''_2(t,\gamma)}{dt} = -\lambda''_1(\gamma)g''_2(t,\gamma).$ ………………………..(4)

With initial condition,

$g_i(0,\alpha) = g'_i(0,\beta) = g''_i(0,\gamma) = g_0, i = 1,2.$

Solving the above differential equation (4), we get the exact solution as

$g_1(t,\alpha) = g_0 e^{-\frac{\lambda_2(\alpha)}{u}t}; g_2(t,\alpha) = g_0 e^{-\frac{\lambda_1(\alpha)}{u}t},$

$g'_1(t,\beta) = g_0 e^{-\frac{\lambda'_2(\beta)}{u}t};\quad g'_2(t,\beta) = g_0 e^{-\frac{\lambda'_1(\beta)}{u}t},$

$g''_1(t,\gamma) = g_0 e^{-\frac{\lambda''_2(\gamma)}{u}t};\quad g''_2(t,\gamma) = g_0 e^{-\frac{\lambda''_1(\gamma)}{u}t}.$ ……………….(5)

Equilibrium point: We get one equilibrium point say $E_2^* = (0,0,0,0,0,0)$ of the system (4).

Stability Analysis:

Lemma – 2: E_2^* is LAS (Locally asymptotically stable).

Proof: The variational matrix V_{12} at E_2^* is given by,

$$V_{12} = \begin{pmatrix} -\frac{\lambda_2(\alpha)}{u} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\lambda_1(\alpha)}{u} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\lambda'_2(\beta)}{u} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\lambda'_1(\beta)}{u} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{\lambda''_2(\gamma)}{u} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{\lambda''_1(\gamma)}{u} \end{pmatrix}.$$

The eigenvalue of the matrix V_{12} are $-\frac{\lambda_2(\alpha)}{u}, -\frac{\lambda_1(\alpha)}{u}, -\frac{\lambda'_2(\beta)}{u}, -\frac{\lambda'_1(\beta)}{u}, -\frac{\lambda''_2(\gamma)}{u}, -\frac{\lambda''_1(\gamma)}{u}.$

Obviously, the equilibrium point E_2^* is LAS.

4.3 Case 3: Both $\tilde{g}_0$ and $\tilde{\lambda}$ are TrSVNN Type 2.

In this case the system (1) as follows,

$u\frac{dg_1(t,\alpha)}{dt} = -\lambda_2(\alpha)g_1(t,\alpha),$

$u\frac{dg_2(t,\alpha)}{dt} = -\lambda_1(\alpha)g_2(t,\alpha),$

$u\frac{dg_1'(t,\beta)}{dt} = -\lambda'_2(\beta)g'_1(t,\beta),$

$u\frac{dg_2'(t,\beta)}{dt} = -\lambda'_1(\beta)g'_2(t,\beta),$

$u\frac{dg''_1(t,\gamma)}{dt} = -\lambda''_2(\gamma)g''_1(t,\gamma),$

$u\frac{dg''_2(t,\gamma)}{dt} = -\lambda''_1(\gamma)g''_2(t,\gamma).$ …………………………..(6)

Considering the initial situation,,

$g_i(0,\alpha) = g_{0i}(\alpha);\ g_i'(0,\beta) = g_{0i}'(\beta);\ g''_i(0,\gamma) = g''_{0i}(\gamma);\ i = 1,2.$

We obtain the precise respond to as by solving the differential equations of model system (6).

$g_1(t,\alpha) = g_{01}(\alpha)e^{-\frac{\lambda_2(\alpha)}{u}t}; g_2(t,\alpha) = g_{02}(\alpha)e^{-\frac{\lambda_1(\alpha)}{u}t};$

$g'_1(t,\beta) = g'_{01}(\beta)e^{-\frac{\lambda'_2(\beta)}{u}t};\ g_2'(t,\beta) = g'_{02}(\beta)e^{-\frac{\lambda'_1(\beta)}{u}t}$

$$g''_1(t,\gamma) = g''_{01}(\gamma)e^{-\frac{\lambda''_2(\gamma)}{u}t} \; ; \quad g''_2(t,\gamma) = g''_{02}(\gamma)e^{-\frac{\lambda''_1(\gamma)}{u}t}. \qquad \ldots\ldots\ldots\ldots\ldots(7)$$

Equilibrium point: We get one equilibrium point say $E_3^* = (0,0,0,0,0,0)$ for the model (6).

Stability Analysis:

Lemma – 3: E_3^* is LAS.

Proof: Proof similar to that in Lemma - 2.

5. Numerical study: We have conducted extensive numerical simulations to substantiate and verify the outcomes of our analysis regarding the drug diffusion model NDDE. These simulations have been carried out using Matlab (version 2018a) and Matcont.

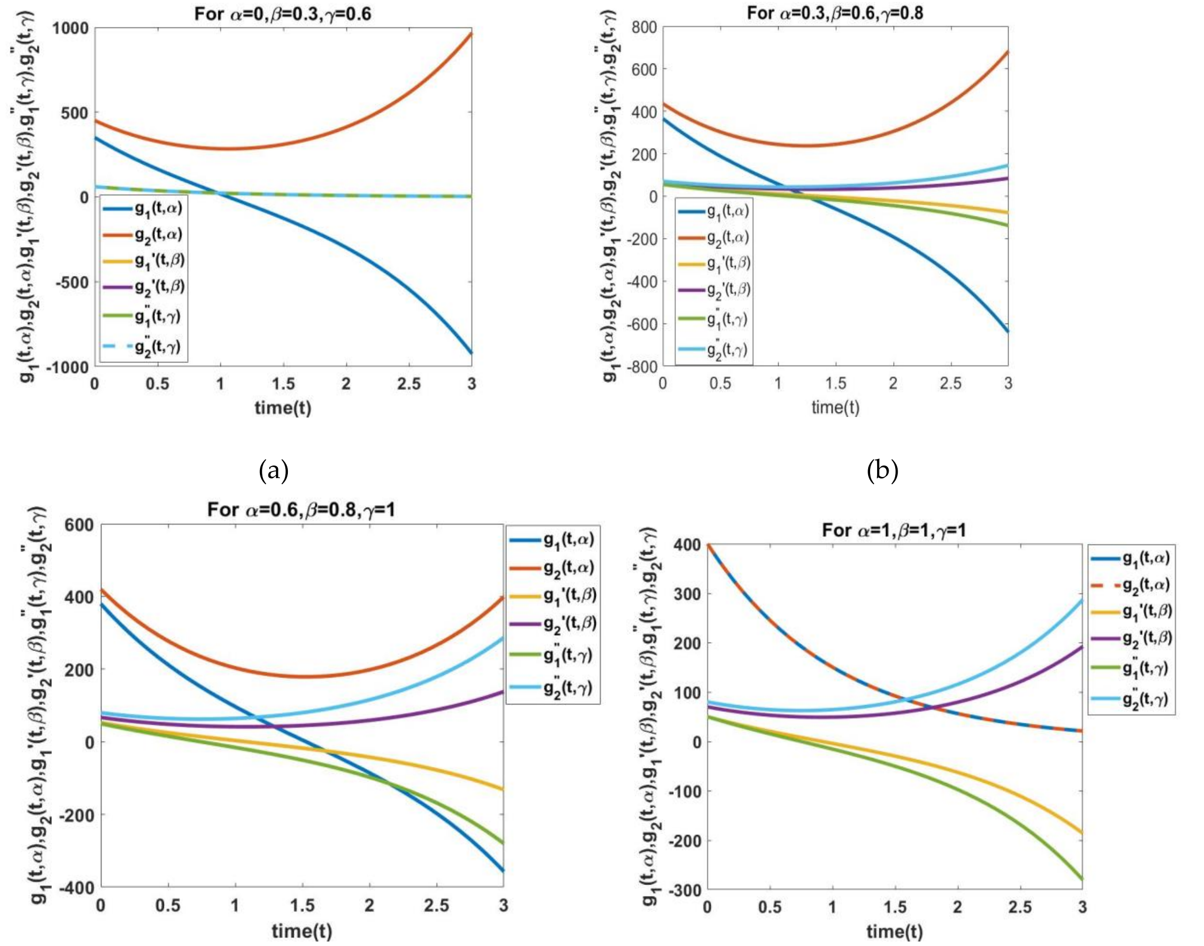
Part A: Study of the nature of NDDE when initial conditions are TrSVNN-Type 2

Let us assume the TrSVNN type 2initial condition $\widetilde{g_0} = (350,400,450; 50,60,40; 0.3,0.6)$..The parametric representation of the triangular single valued Neutrosophic number of type2 can be formulated as follows:

$g_{01}(\alpha) = 350 + 50\alpha$, $g_{02}(\alpha) = 400 - 50\alpha$;$g'_{01}(\beta) = \frac{450-100\beta}{7}$,$g'_{02}(\beta) = \frac{390+100\beta}{7}$;

$g''_{01}(\gamma) = 75 - 25\gamma$;$g''_{02}(\gamma) = 30 + 25\gamma$; ……………. (8)

Applying the value provided in equation (8) with $u = 1$ we construct Figure 1(a), (b), (c), (d), for$\alpha = 0, \beta = 0.3, \gamma = 0.6, \alpha = 0.3, \beta = 0.6, \gamma = 0.8, \alpha = 0.6, \beta = 0.8\ \gamma = 1$, and $\alpha = 1, \beta = 1, \gamma = 1$ respectively.



(a) (b)

(c) (d)

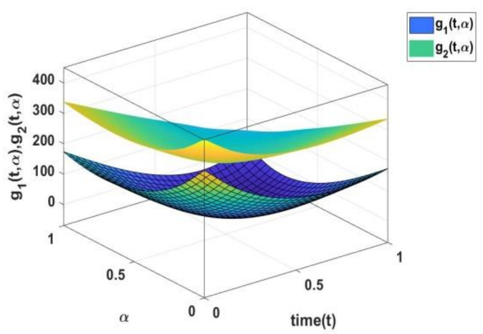
Figure 1. Neutrosophic fuzzy solution: Figure 1(a) for α=0,β=0.3,γ=0.6; Figure 1(b) for α=0.3,β=0.6,γ=0.8; Figure 1(c) for α=0.6,β=0.8,γ=1; Figure 1(d) for α=1,β=1,γ=1 where $t \in [0,3]$.

Here, in Figure 1(a) we see that$g_1(t,\alpha) \le g_2(t,\alpha)$; $g'_1(t,\beta) = g'_2(t,\beta)$; $g''_1(t,\gamma) = g''_2(t,\gamma)$.
In Figure 1(b), Figure 1(c), Figure 1(d) we observe that $g_1(t,\alpha) \le g_2(t,\alpha)$; $g'_1(t,\beta) \le g'_2(t,\beta)$; $g''_1(t,\gamma) \le g''_2(t,\gamma)$for $t \in [0,3]$.Clearly, Figure1 shows the dynamical behaviour of $g_1(t,\alpha), g_2(t,\alpha), g'_1(t,\beta), g'_2(t,\beta)$; $g''_1(t,\gamma), g''_2(t,\gamma)$ relative to time (t) for $t \in [0,3]$, for$\alpha = 0, \beta = 0.3, \gamma = 0.6, \alpha = 0.3,\ \beta = 0.6, \gamma = 0.8,\ \alpha = 0.6, \beta = 0.8\ \gamma = 1$, and $\alpha = 1, \beta = 1, \gamma = 1$
Now, setting $u = 1$ and $t = 1$ with the initial conditions specified in equation (8),we list the solution of (2) in Table 1 where $\alpha \in [0,1], \beta \in [0.3,1]$ and $\gamma \in [0.6,1]$.

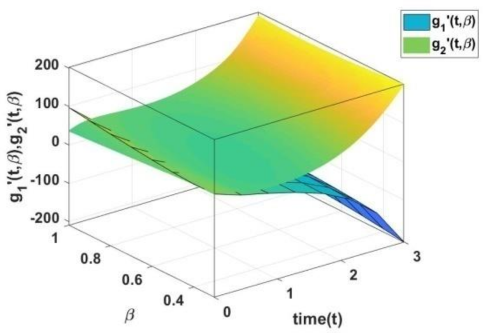
α	$g_1(t,\alpha)$	$g_2(t,\alpha)$	β	$g_1'(t,\beta)$	$g_2'(t,\beta)$	γ	$g_1''(t,\gamma)$	$g_2''(t,\gamma)$
0	16.9016	283.3473						
0.1	30.2239	270.0250						
0.2	43.5462	256.7027						
0.3	56.8685	243.3804	0.3	22.5187	22.5187			
0.4	70.1908	230.0581	0.4	18.7123	26.3250			
0.5	83.5130	216.7358	0.5	14.9059	30.1314			
0.6	96.8353	203.4136	0.6	11.0996	33.9378	0.6	22.5187	22.5187
0.7	110.1576	190.0913	0.7	7.2932	37.7441	0.7	12.9961	32.9795
0.8	123.4799	176.7690	0.8	3.4868	41.5505	0.8	3.4735	43.4404
0.9	136.8022	163.4467	0.9	-0.3195	45.3569	0.9	-6.0491	53.9012
1	150.1244	150.1244	1	-4.1259	49.1632	1	-15.5716	64.3621

Table 1 displays Neutrosophic fuzzy solution for system (2) at time t=1.

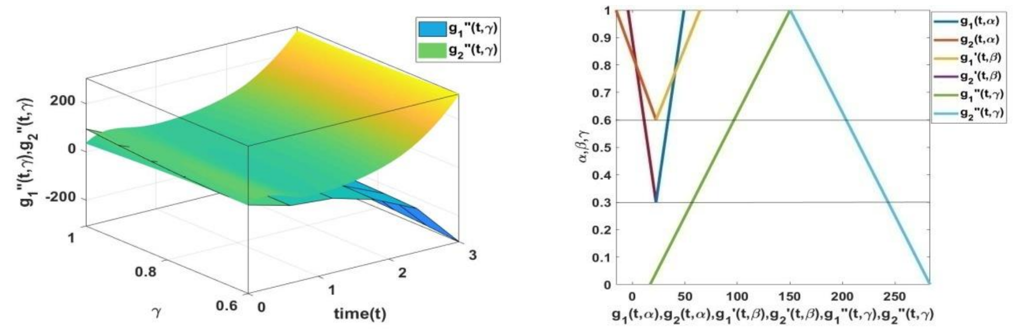
Table 1 reflects that $g_1(t,\alpha)$ is increasing, $g_2(t,\alpha)$ is decreasing; $g'_1(t,\beta)$ exhibits decreasing while $g'_2(t,\beta)$ displays an increasing; $g''_1(t,\gamma)$ exhibits decreasing whereas $g''_2(t,\gamma)$ demonstrate increasing for $\alpha \in [0,1], \beta \in [0.3,1]$and $\gamma \in [0.6,1]$ at t=1.



(a)



(b)



(c) (d)

Figure 2: (3D plot)

Figure 2(a): Pictorial diagram of 3D plot of $g_1(t,\alpha), g_2(t,\alpha)$ with respect to time(t) and α where $t \in [0,3]$, $\alpha \in [0,1]$;

Figure 2(b): Pictorial diagram of 3D plot of $g'_1(t,\beta), g'_2(t,\beta)$ with respect to time(t) and β where $t \in [0,3]$, $\beta \in [0.3,1]$;

Figure 2(c): Pictorial diagram of 3D plot of $g''_1(t,\gamma), g''_2(t,\gamma)$ with respect to time(t) and γ where $t \in [0,3]$, $\gamma \in [0.6,1]$;

Figure 2(d): Pictorial diagram of 3D plot of $g_1(t,\alpha), g_2(t,\alpha), g'_1(t,\beta), g'_2(t,\beta), g''_1(t,\gamma), g''_2(t,\gamma)$ with respect to time(t) and α,β,γ where $\alpha \in [0,1], \beta \in [0.3,1], \gamma \in [0.6,1]$.

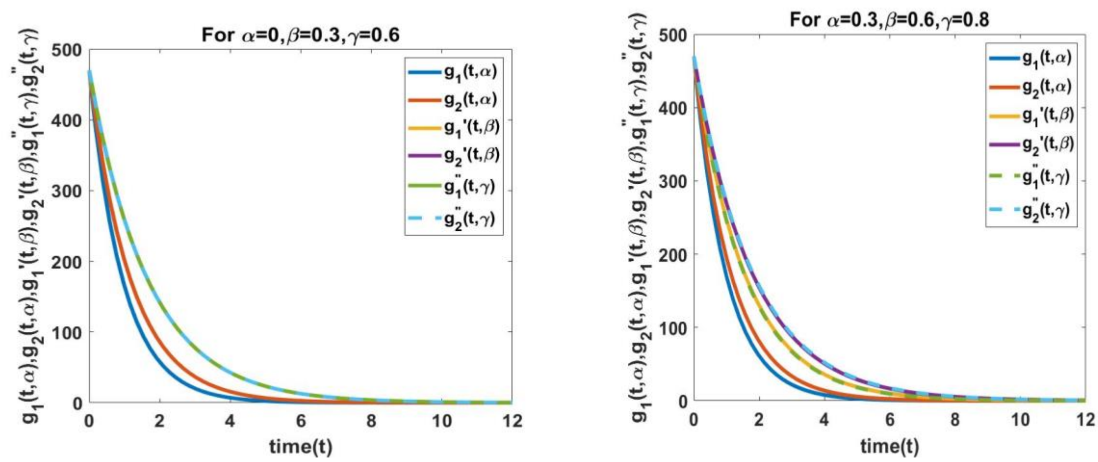
Figure 1, Figure 2 and Table 1 clearly depicts that, for all values of α within the interval [0,1], $g_1(t,\alpha)$ exhibits strictly increasing whereas $g_2(t,\alpha)$ displays strictly decreasing i.e. $g_1(t,1) \le g_2(t,1)$; for all values of β within the interval[0.3,1], $g'_1(t,\beta)$ demonstrate strictly decreasing whereas $g'_2(t,\beta)$ exhibits strictly increasing i.e $g'_1(t,0.3) \le g'_2(t,0.3)$; for all values of γ within the interval [0.6,1], $g''_1(t,\gamma)$ display strictly decreasing whereas $g''_2(t,\gamma)$ exhibits strictly increasing, $g''_1(t,0.6) \le g''_2(t,0.6)$.

Part B: Study of the nature of NDDE when $\tilde{\lambda}$ isTrSVNN-Type 2

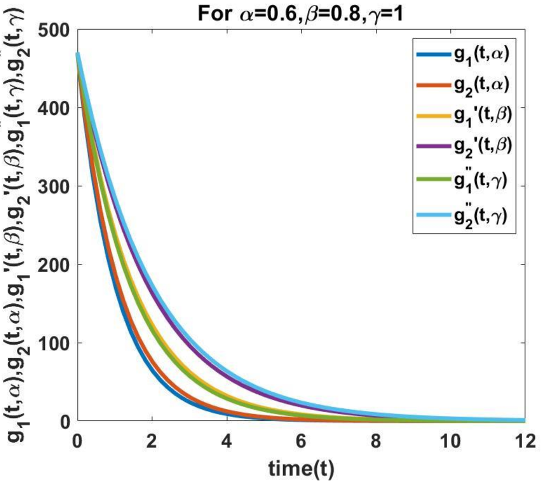
Let us assume the TrSVNN Type 2 values of $\tilde{\lambda}$ to be $\lambda_{neu} = (0.85,0.95,1.05; 0.5,0.6,0.7; 0.3,0.6)$

Its parametric form is given by,

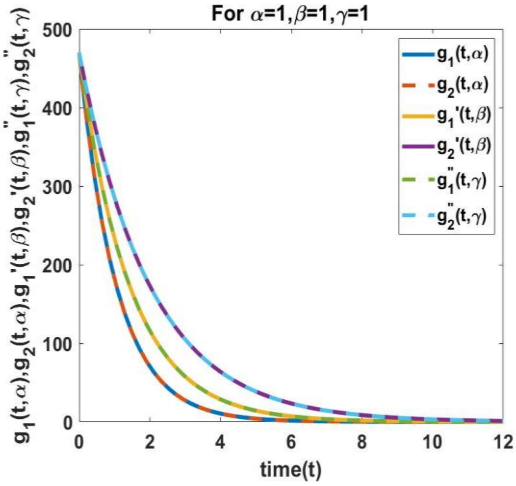
$$\lambda_1(\alpha) = 0.85 + 0.1\alpha;\ \lambda_2(\alpha) = 1.05 - 0.1\alpha;\ \lambda'_1(\beta) = \frac{4.5-\beta}{7};$$
$$\lambda'_2(\beta) = \frac{3.9+\beta}{7};\ \lambda''_1(\gamma) = \frac{3-\gamma}{4};\ \lambda''_2(\gamma) = \frac{1.8+\gamma}{4} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(9)$$



(a) (b)



(c)



(d)

Figure 3. Neutrosophic fuzzy solution: Figure 3(a) for α=0,β=0.3,γ=0.6; Figure 3(b) for α=0.6,β=0.8,γ=1; Figure 3(c) for α=0.3,β=0.5,γ=0.8; Figure 3(d) for α=1,β=1,γ=1 where $t \in [0,12]$.

In Figure 3(a), we see that $g_1(t,\alpha) \le g_2(t,\alpha)$; $g'_1(t,\beta) = g'_2(t,\beta)$; $g''_1(t,\gamma) = g''_2(t,\gamma)$.
Figure 3(b), Figure 3(c) shows that $g_1(t,\alpha) \le g_2(t,\alpha)$; $g'_1(t,\beta) \le g'_2(t,\beta)$; $g''_1(t,\gamma) \le g''_2(t,\gamma)$ and Figure 3(d) shows that$g_1(t,\alpha) = g_2(t,\alpha)$, $g'_1(t,\beta) = g'_2(t,\beta)$;$g''_1(t,\gamma) = g''_2(t,\gamma)$; for $t \in [0,12]$. From Figure 3 we observe that the system (4) is LAS at E_3^*.

Taking u=1, t=6 with TrSVNN Type 2 values of $\tilde{\lambda}$ described in (9), the solutions for equation (4) are exhibited in Table 2 where $\alpha \in [0,1], \beta \in [0.3,1]$ and $\gamma \in [0.6,1]$.

α	$g_1(t,\alpha)$	$g_2(t,\alpha)$	β	$g_1'(t,\beta)$	$g_2'(t,\beta)$	γ	$g_1''(t,\gamma)$	$g_2''(t,\gamma)$
0	0.8634	2.8658						
0.1	0.9168	2.6989						
0.2	0.9735	2.5417						
0.3	1.0336	2.3937	0.3	12.8423	12.8423			
0.4	1.0975	2.2544	0.4	11.7874	13.9916			
0.5	1.1654	2.1231	0.5	10.8191	15.2437			
0.6	1.2374	1.9995	0.6	9.9304	16.6080	0.6	12.8423	12.8423
0.7	1.3139	1.8831	0.7	9.1147	18.0943	0.7	11.0535	14.9206
0.8	1.3951	1.7734	0.8	8.3660	19.7136	0.8	9.5139	17.3352
0.9	1.4814	1.6702	0.9	7.6789	21.4778	0.9	8.1887	20.1406
1	1.5729	1.5729	1	7.0481	23.4000	1	7.0481	23.4000

Table 2 displays Neutrosophic fuzzy solution for the system described by (4) when t=6.

Table 2 reflects that $g_1(t,\alpha)$ exhibits increasing,$g_2(t,\alpha)$ demonstrate decreasing; $g'_1(t,\beta)$ display decreasing whereas $g'_2(t,\beta)$ exhibits increasing;$g''_1(t,\gamma)$ display decreasing and $g''_2(t,\gamma)$ demonstrate increasing for $\alpha \in [0,1], \beta \in [0.3,1]and\gamma \in [0.6,1]$ and t=6.

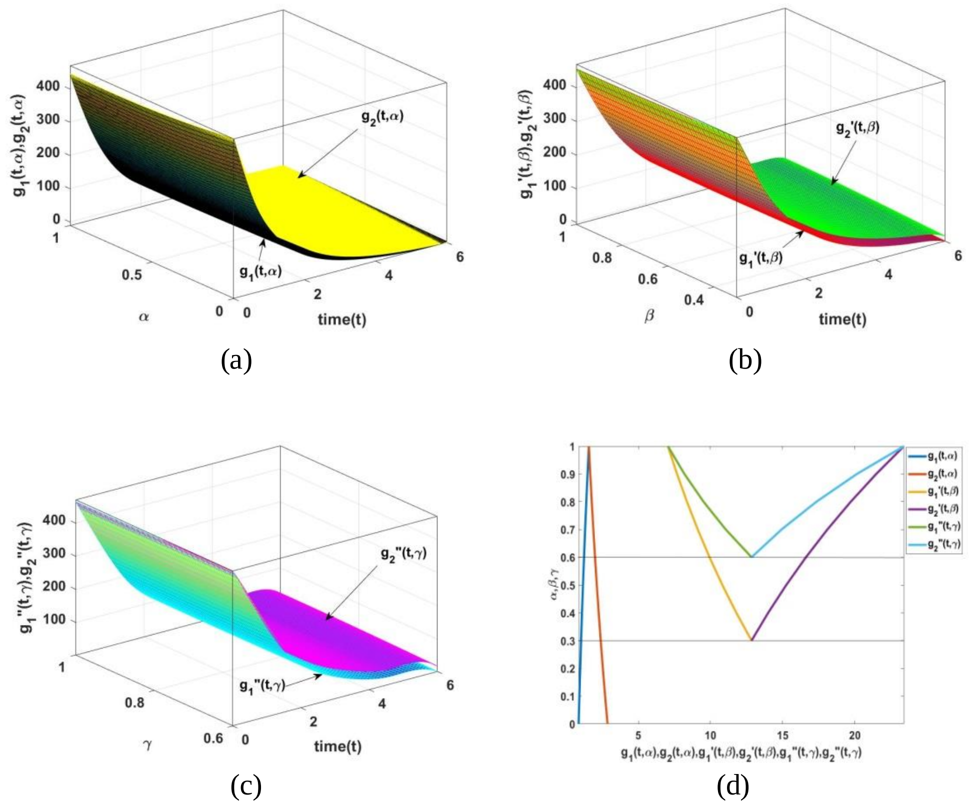


Figure. 4: (3D plot)

Figure 4(a): Pictorial diagram of 3D plot of $g_1(t,\alpha), g_2(t,\alpha)$ with respect to time(t) and α where $t \in [0,6]$, $\alpha \in [0,1]$;

Figure 4(b): Pictorial diagram of 3D plot of $g'_1(t,\beta), g'_2(t,\beta)$ with respect to time(t) and β where $t \in [0,6]$, $\beta \in [0.3,1]$;

Figure 4(c): Pictorial diagram of 3D plot of $g''_1(t,\gamma), g''_2(t,\gamma)$ with respect to time(t) and γ where $t \in [0,6]$, $\gamma \in [0.6,1]$;

Figure 4(d): Pictorial diagram of 3D plot of $g_1(t,\alpha), g_2(t,\alpha), g'_1(t,\beta), g'_2(t,\beta), g''_1(t,\gamma), g''_2(t,\gamma)$ with respect to time(t) and α,β,γ where $\alpha \in [0,1], \beta \in [0.3,1], \gamma \in [0.6,1]$.

Figure 3, Figure 4 and Table 2 clearly depicts that for all values of α within the interval [0,1], $g_1(t,\alpha)$ exhibits strictly increasing whereas $g_2(t,\alpha)$ displays strictly decreasing i.e. $g_1(t,1) \le g_2(t,1)$; for all values of β within the interval[0.3,1],$g'_1(t,\beta)$ demonstrate strictly decreasing whereas $g'_2(t,\beta)$ exhibits strictly increasing i. e $g'_1(t,0.3) \le g'_2(t,0.3)$; for all values of γ∈ [0.6,1], $g''_1(t,\gamma)$ display strictly decreasing whereas $g''_2(t,\gamma)$ exhibits strictly increasing,$g''_1(t,0.6) \le g''_2(t,0.6)$.

Part C: Study the nature of NDDE when initial condition and $\tilde{\lambda}$ are TrSVNN-Type 2.

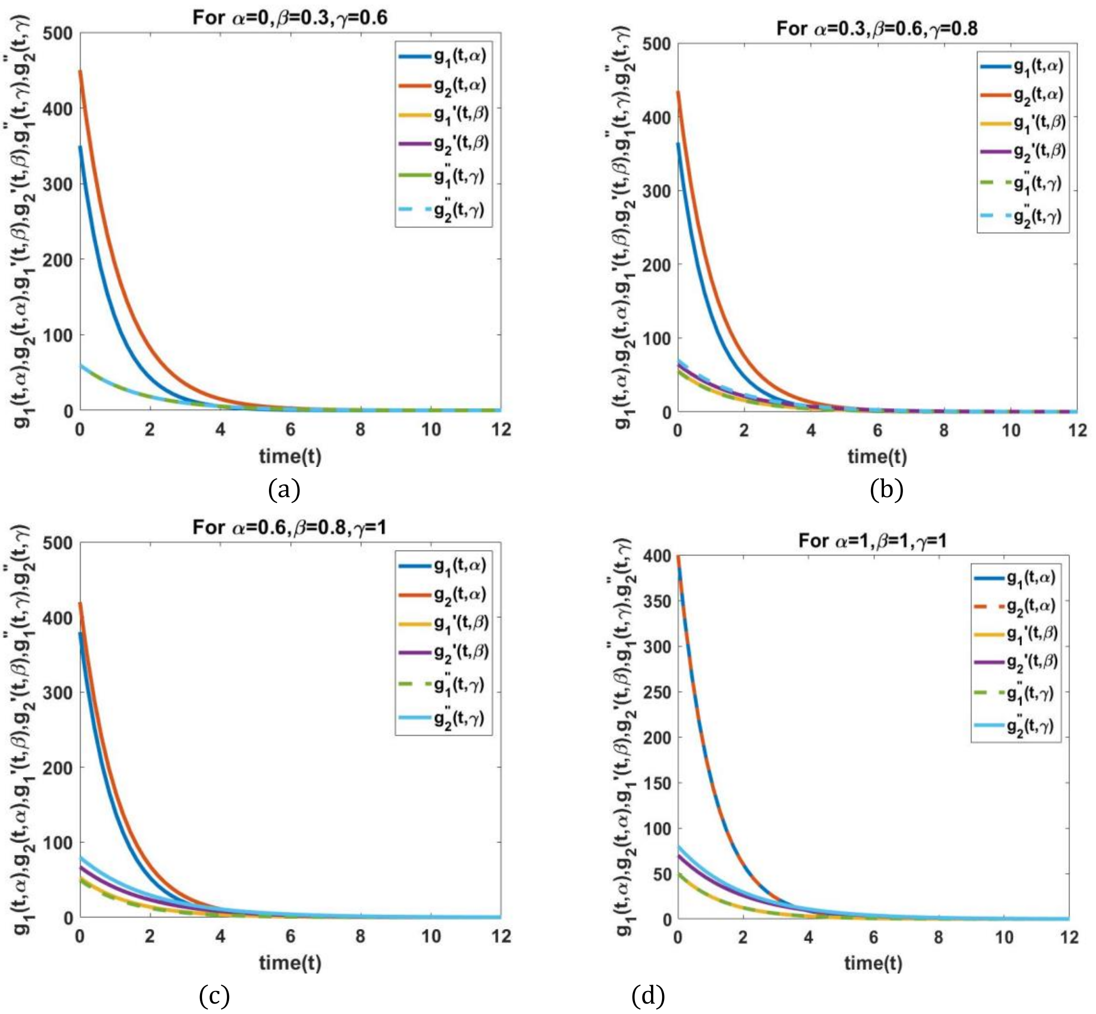


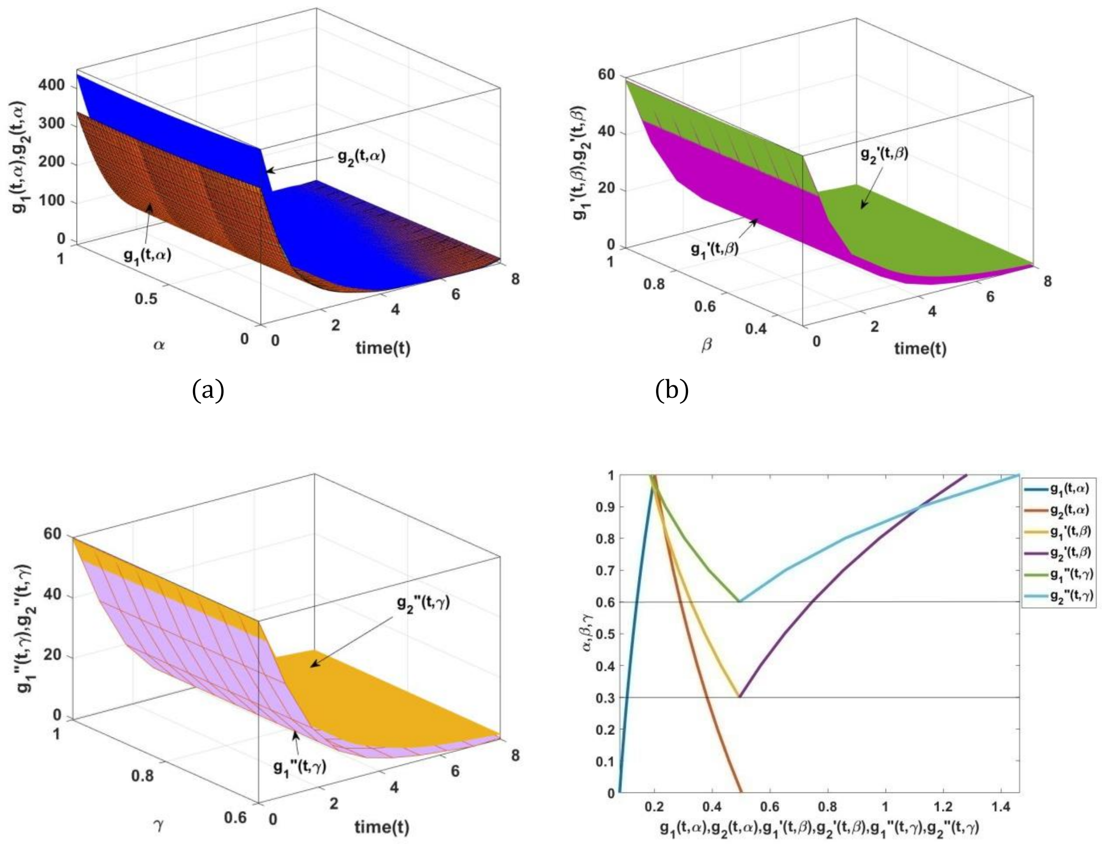
Figure 5: Neutrosophic fuzzy solution: Figure 5(a) for α=0,β=0.3,γ=0.6; Figure 5(b) for α=0.3,β=0.6,γ=0.8; Figure 5(c)for α=0.6,β=0.8,γ=1;Figure5(d) for α=1,β=1,γ=1 where $t \in [0,12]$

In Figure 5(a) we see that $g_1(t,\alpha) \le g_2(t,\alpha)$; $g'_1(t,\beta) = g'_2(t,\beta)$;$g''_1(t,\gamma) = g''_2(t,\gamma)$; Figure 5(b), Figure 5(c) and Figure 5(d) shows that $g_1(t,\alpha) \le g_2(t,\alpha)$; $g'_1(t,\beta) \le g'_2(t,\beta)$;$g''_1(t,\gamma) \le g''_2(t,\gamma)$ for $t \in [0,12]$. From Figure 5 we observe that E_3^* is LAS.
Considering u=1, t=8 with initial conditions stated in (8) as well as (9), the solutions for equation (6) are shown in Table 3 where $\alpha \in [0,1], \beta \in [0.3,1]$ and $\gamma \in [0.6,1]$.

α	$g_1(t,\alpha)$	$g_2(t,\alpha)$	β	$g'_1(t,\beta)$	$g'_2(t,\beta)$	γ	$g''_1(t,\gamma)$	$g''_2(t,\gamma)$
0	0.0789	0.5015						
0.1	0.0867	0.4578						
0.2	0.0952	0.4179						
0.3	0.1046	0.3814	0.3	0.4938	0.4938			
0.4	0.1148	0.3481	0.4	0.4300	0.5668			
0.5	0.1261	0.3176	0.5	0.3742	0.6502			
0.6	0.1384	0.2898	0.6	0.3255	0.7454	0.6	0.4938	0.4938
0.7	0.1519	0.2644	0.7	0.2829	0.8543	0.7	0.3875	0.6534
0.8	0.1666	0.2411	0.8	0.2457	0.9785	0.8	0.3034	0.8594
0.9	0.1828	0.2199	0.9	0.2132	1.1203	0.9	0.2372	1.1247
1	0.2005	0.2005	1	0.1849	1.2821	1	0.1849	1.4653

Table 3 displays Neutrosophic fuzzy solution for the system described by (6) when t=8.

Table 3 reflects that $g_1(t,\alpha)$ exhibits increasing, $g_2(t,\alpha)$ demonstrate decreasing; $g'_1(t,\beta)$ display decreasing whereas $g'_2(t,\beta)$ exhibits increasing; $g''_1(t,\gamma)$ display decreasing and $g''_2(t,\gamma)$ demonstrate increasing for $\alpha \in [0,1], \beta \in [0.3,1]$ and $\gamma \in [0.6,1]$ and t=8.



(a) (b)

(b) (d)

Figure 6. (3D plot).

Figure 6(a): Pictorial diagram of 3D plot of $g_1(t,\alpha), g_2(t,\alpha)$ with respect to time(t) and α where $t \in [0,8]$, $\alpha \in [0,1]$;

Figure 6(b): Pictorial diagram of 3D plot of $g'_1(t,\beta), g'_2(t,\beta)$ with respect to time(t) and β where $t \in [0,8]$, $\beta \in [0.3,1]$;

Figure 6(c): Pictorial diagram of 3D plot of $g''_1(t,\gamma), g''_2(t,\gamma)$ with respect to time(t) and γ where $t \in [0,8]$, $\gamma \in [0.6,1]$;

Figure 6(d): Pictorial diagram of 3D plot of $g_1(t,\alpha), g_2(t,\alpha), g'_1(t,\beta), g'_2(t,\beta), g''_1(t,\gamma), g''_2(t,\gamma)$ with respect to time(t) and α,β,γ where $\alpha \in [0,1], \beta \in [0.3,1], \gamma \in [0.6,1]$.

From Figure 5, Figure 6 and Table 3 clearly depicts, for all values of α within the interval [0,1], $g_1(t,\alpha)$ exhibits strictly increasing whereas $g_2(t,\alpha)$ displays strictly decreasing i.e. $g_1(t,1) \le g_2(t,1)$; for all values of β within the interval [0.3,1], $g'_1(t,\beta)$ demonstrate strictly decreasing whereas $g'_2(t,\beta)$ exhibits strictly increasing i.e $g'_1(t,0.3) \le g'_2(t,0.3)$; for all values of $\gamma \in [0.6,1]$, $g''_1(t,\gamma)$ display strictly decreasing whereas $g''_2(t,\gamma)$ exhibits strictly increasing, $g''_1(t,0.6) \le g''_2(t,0.6)$. Hence by definition 2.6, the solution to equation (1), $\tilde{g}(t,\alpha,\beta,\gamma)$ qualifies as a robust Neutrosophic fuzzy solution.

6. Comparison of the model in both environment:

The suggested system on drug diffusion in connective tissue plays an important role in determination of the amount of drug in blood stream with passage of time. However, due to certain parameters like environment factors, etc there arises an uncertainty in such biological models. To overcome such uncertainties, we adopt the concept of Neutrosophic environment and convert the given differential equations of the proposed biological model toneutrosophic differential equations (NDE) by taking three cases: (i) the quantity of drug diffusion within blood at initial time t_0i.e. $\tilde{g}_0$ is TrSVNN Type 2, (ii) the concentration of drug within bloodstream $\tilde{\lambda}$is neutrosophic number TrSVNN Type 2, (iii) both $\tilde{g}_0$ and $\tilde{\lambda}$ are TrSVNN Type 2.We have generated the exact solution and stability criteria for each of the three cases. The results and theorems have been verified numerically and graphically. From Table 1, Figure 1 and Figure 2 we observe that all the solutions are strong neutrosophic solutions of the converted system (2) when t=1. From Figure 3 as well as Figure 4 we see that all solutions are strong neutrosophic solutions of the converted system (3) when $t = 6$ using Table 2. Similar conclusions are drawn are for converted system (4) from Table 3, Figure 5 and Figure 6 when t=8.

7. Conclusion

In a neutrosophic situation, we have effectively handled the differential equation relating to medication dispersion in human connective tissues. The initial condition and the parameter are represented by single-valued triangular neutrosophic numbers. Neutrosophic values are considered for the initial state, aiming to capture the nuances of truth and falsity in the dynamics of drug diffusion within human tissues. As an example, when α is set to 0.3, the membership degree or truth value of $\tilde{g}(t,\alpha,\beta,\gamma)$ needs to exceed 0.3. In other words, the drug diffusion level $g(t)$ at time t, as defined in equation (7), should be true in more than 30 out of 1000 instances. To achieve this, the truth values of the initial condition, where $g(t_0) = g_0$, should be $g_{01} = 345$ within the range [350, 400) and $g_{02} = 385$ within the range (400, 450]. Similarly, when $\beta = 0.6$ the value of $\tilde{g}(t,\alpha,\beta,\gamma)$in case of indeterminacy must be less than 0.6 and when $\gamma = 0.8$ the falsity membership must be less than 0.8. Corresponding values of the initial conditions are evaluated from (8). Thus, in a neutrosophic environment, the values of the initial conditions are more appropriately applicable in all the three possibilities, i.e truth, indeterminacy and falsity. Analogous explanations hold for the TrSVNN Type 2 values of $\widetilde{g_0} = (350{,}400{,}450; 50{,}60{,}40; 0.3{,}0.6)$ and $\tilde{\lambda} =$ (0.85,0.95,1.05; 0.5,0.6,0.7;0.3,0.6). The above discussion enables us to determine whether a drug is potent enough to be used in regard to its therapeutic advantages. In a broader sense, this study enables us to understand the appropriateness of pharmaceutical intervention in various kinds of human and animal pathogenesis.In future more research works can be carried out on bio-mathematical modelling using several types of neutrosophic numbers.

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Linguistic Hypersoft Set with Application to Multi-Criteria Decision-Making to Enhance Rural Health Services

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Abstract: Language, as an abstract system and a creative act, possesses inherent complexity due to its contextual nature and the variability of its meaning. The context of language is shaped by an individual's empirical knowledge, derived from observation and experience. Decision-making challenges related to language encompass both quantitative and qualitative factors, which further contribute to the intricacy of the process. Decision-making challenges may involve both quantitative and qualitative aspects of further subdivided attributes. However, linguistic knowledge cannot be easily quantified by existing methods. Therefore, current methods are ineffective in handling linguistic knowledge. Using mathematical values, such as fuzzy, intuitionistic, and neutrosophic, in decision-making problems without following linguistic knowledge rules can result in vagueness and imprecision. To address these issues, this paper presents a comprehensive generic model. The model introduces the linguistic set structure of the hypersoft set (LHSS) as a solution for decision-making problems. The definition of fundamental operations, including AND, NOT, OR complement, and negation, is proposed alongside illustrative examples and their respective properties. Additionally, operational laws for the linguistic hypersoft set are introduced to effectively address decision-making challenges. By implementing the proposed aggregate operators and operational laws, linguistic quantifiers can be converted into numerical values, thereby enhancing the accuracy and precision of the hypersoft set structure in decision-making scenarios.

Keywords: Linguistic quantifiers; linguistic set; hypersoft set; aggregate operators; multi-criteria decision-making (MCDM).

1. Introduction

These influential 1975 papers, (Zadeh, 1975, 1975a, 1975b) introduces the concept of a linguistic variable and explores its application in approximate reasoning, specifically focusing on their use in decision-making. These work expands on the understanding of linguistic variables' potential in practical scenarios. To implement these concepts in real life problems the scientists explored the areas, and now these concepts are widely used in decision-making process i.e. multi-criteria decision-making (Hwang & Yoon, 1981). By considering multiple criteria, MCDM aims to enhance the decision-making process, improve transparency, and facilitate the selection of robust solutions that align with the desired goals and objectives. To address the challenges associated with decision-making and linguistic preferences (Delgado, Verdegay, & Vila, 1992) presented a paper focuses on linguistic decision-making models. It presents different approaches and techniques for modeling decision making in linguistic contexts, contributing to the understanding of decision-making processes. The method based on linguistic aggregation operators for decision-making with linguistic preference relations was proposed by (Xu, 2004). A semantic model for computing with flexible linguistic expressions was proposed by (Jiang et al., 2021), and (Wu et al., 2023) paper presents a multiple criteria decision-making method that incorporates heterogeneous linguistic expressions. it enhances the understanding of linguistic expressions and their use in decision-making processes. It was difficult to deal with the problems having doubt, uncertainty, vagueness, ambiguity, and indeterminacy. The concept of linguistic to mathematic was unclear, then to overcome the problem some set theories were proposed by the researchers. In next paragraph we present those theories with application to MCDM.

The groundwork for the concept of fuzzy set (membership values) to deal with uncertainty and its use in information and control systems was presented by (Zadeh, 1965). Ambiguity was another problem faced by the decision-makers then (Attanasov, 1986) came up with the concept of intuitionistic set, in which each alternative is assigned a membership and non-membership degree with the condition that their sum is not greater than 1. To deal with the problem having indeterminacy (Smarandache, 1998, 2002, 2002a, 2003, 2005, 2006) came up with the concept of neutrosophic set theory, which has membership, non-membership, and indeterminacy values. Many researchers came up with the concept of extensions and by merging linguistic with fuzzy, intuitionistic, neutrosophic sets, and other hybrid structures. The application in multi-criteria decision-making under fuzzy linguistic sets was proposed by (Joyce, 1976), the study illustrates the use of fuzzy in linguistic environment. The application in multi-criteria decision-making under hesitant fuzzy linguistic term sets was proposed by (Dinesh et al., 2022) The group decision-making process under linguistic intuitionistic fuzzy sets using aggregation operators was presented by (Garg & Kumar, 2018). The application of linguistic sets to group decision making and a method to handle complex decision scenarios was presented by (Wang, Ju, & Liu, 2019) based on q-rung orthopair fuzzy linguistic sets. The novel method for multi-attribute decision-making using interval-valued Pythagorean fuzzy linguistic information was published by (Du et al., 2017). The research addresses the challenges of decision making when dealing with interval-valued linguistic data. The new methods for addressing MCDM problems using linguistic neutrosophic sets in which the interrelationships among individual data are considered was proposed by (Li, Zhang & Wang, 2017).

In decision-making problems decision-makers deal with the alternatives having attributes, the mathematical notation given by (Molodtsov, 1999), the paper presents the foundational concepts of soft set theory. It establishes the groundwork for the study of soft sets and their applications in various domains, including decision-making and data analysis. (Maji & Roy, 2002) the paper applies soft sets to a decision-making problem. The research demonstrates the practical utility of soft sets in real-world decision-making scenarios, showcasing their effectiveness in capturing uncertainty. To further enhance the capabilities in decision-making and data analysis (Ali et al., 2009) came up with some new operations in soft set theory. The concept of the soft set was later extended to fuzzy soft set (Maji, Biswas, & Roy, 2001), intuitionistic soft set (Deli & Çağman, 2013), neutrosophic soft set (Amalini et al., 2020), and other hybrid structures.

To incorporate uncertainty a comprehensive framework for group decision-making was proposed by (Tao et al., 2015). The paper introduces uncertain linguistic fuzzy soft sets and their applications in group decision-making. The research of (Aiwu & Hongjun, 2016) proposes fuzzy-valued linguistic soft set theory and applies it to multi-attribute decision making. This work presents a novel approach to handle linguistic uncertainty and supports decision-making processes. The multi-attribute decision-making method using belief-based probabilistic linguistic term sets was proposed by (Liu, Fei, & Mi, 2023). For the selection of medical waste treatment stations based on linguistic q-rung orthopair fuzzy numbers (Ling, Li, & Lin, 2021) proposes a methodology. To handle linguistic uncertainty in group decision-making processes (Vijayabalaji & Ramesh, 2018) proposed a method to solve these problems.

Novel approaches have been demonstrated by recent studies that have advanced a variety of sectors (Saqlain, 2023). Decision-making utilizing Pythagorean fuzzy Hamacher aggregation operators has been extended by (Paul, Jana, & Pal, 2023), (Du, Wang, & Lu, 2023), maximized wireless power transmission with an improved approach, and (Haq & Saqlain, 2023) used machine learning for attendance tracking in a pandemic (Zulqarnain & Saqlain, 2023) using convolutional neural networks were used to evaluate text readability in higher education, while (Saqlain et al., 2023) presented a multi-polar interval-valued neutrosophic hypersoft set for uncertainty and decision-making. These projects demonstrate (Stević et al., 2023) a dedication to creativity and cross-domain problem-solving (Tešić et al., 2023). The strategic framework for leveraging artificial intelligence in future marketing decision-making has been explored by (Hicham, Nassera, & Karim, 2023). Furthermore, (Saqlain et al., 2023) introduced proportional distribution-based Pythagorean fuzzy fairly aggregation operators in multi-criteria decision-making .

In MCDM, if attributes are further sub-divided, then existing set structures cannot be applied, thus (Smarandache , 2018) proposed the concept of a hypersoft set, which is the generalization of soft set theory. Hypersoft set (HSS) theory tends to consider further divided attributes or attributes bifurcation. The theory of HSS has been applied to solve both, MCDM and MADM problems [23]. Another beauty of HSS, it can be molded as per the DM requirements. The hypersoft set structure have been extended to a fuzzy hypersoft set (Yolcu & Öztürk, 2021; Jafar & Saeed, 2021; Debnath, 2021) , intuitionistic hypersoft set (Yolcu, Smarandache & Öztürk, 2021) and neutrosophic hypersoft set (Smarandache, 2018) and (Saqlain et al., 2020). These papers represent a diverse range of research contributions in the field of linguistic variables, fuzzy sets, soft sets, hypersoft sets and their applications in decision-making and data analysis.

1.1. Novelty

Comprehending language as an abstract system and a creative process poses significant complexity due to its inherent reliance on context. This context is intricately influenced by an individual's empirical knowledge, which is acquired through keen observation and personal experience. When confronted with the need to make decisions involving further subdivided attributes, a combination of quantitative and qualitative factors comes into play. Nevertheless, the absence of a standardized methodology for assigning numerical values to language hinders existing approaches from effectively managing linguistic knowledge operations. The practice of indiscriminately assigning mathematical values (such as fuzzy, intuitionistic, and neutrosophic) to decision-making problems, without taking linguistic rules into account, leads to ambiguity and inaccuracy. Consequently, the primary objective of this paper is to propose an inclusive model that directly addresses these issues. The paper introduces the concept of the linguistic set structure of the hypersoft set (LHSS) as a proficient approach to tackle the challenges encountered in decision-making processes.

1.2. Contribution

This paper makes significant contributions to the field of decision-making by addressing the limitations of existing approaches in dealing with linguistic knowledge. By introducing the LHSS model, this research offers a novel solution to the challenges posed by the abstract and context-dependent nature of language. The definition of basic operations and the proposal of operational laws for the LHSS provide a systematic framework for converting linguistic quantifiers into numerical values. This framework increases the accuracy and precision of decision-making processes, enabling more reliable and effective outcomes. The implementation of the proposed aggregate operators and operational laws offers a practical tool for solving decision-making issues and improving the overall understanding and application of linguistic knowledge. This contribution has the potential to benefit various fields that rely on language-based decision-making, such as natural language processing, sentiment analysis, and artificial intelligence, among others.

1.3. Scientific Validity

The scientific validity of this paper's approach lies in its rigorous and systematic treatment of the challenges associated with language and decision-making. By acknowledging the abstract and context-dependent nature of language, the authors have developed a novel model, the linguistic set structure of the hypersoft set (LHSS), to address these complexities. The paper provides a clear definition of basic operations and operational laws for the LHSS, ensuring the consistency and reproducibility of the proposed framework. Additionally, the authors illustrate the application of the LHSS through examples and properties, further enhancing the scientific validity of their approach. The proposed model offers a systematic and mathematically grounded methodology to convert linguistic quantifiers into numerical values, thereby improving the accuracy and precision of decision-making processes. The scientific validity of this research is further supported by its potential applicability to various domains that rely on linguistic knowledge. Overall, the systematic approach, rigorous analysis, and practical examples presented in this paper contribute to its scientific validity and establish a foundation for further research in the field of decision-making with linguistic elements. operational laws and aggregate operators are indispensable in the development of Mechanics of advanced manufacturing and robotics. They provide the necessary tools and frameworks for decision-making, optimization, and performance evaluation. By leveraging these tools effectively, engineers and researchers can enhance the efficiency, effectiveness, and overall performance of advanced manufacturing processes and robotic systems, leading to advancements in these fields and enabling the realization of advanced technologies and automation.

The power of the proposed method explored in this research lies in its ability to effectively address the challenges posed by the abstract nature of language and its context-dependent meaning in decision-making processes. The method offers a systematic and mathematically grounded framework, the linguistic set structure of the hypersoft set (LHSS), which enables the conversion of linguistic quantifiers into numerical values. One of the key strengths of this method is its ability to handle complicated decision-making scenarios. By incorporating weighted linguistic quantifiers or linguistic variables, the method allows for the consideration of multiple factors and attributes with varying degrees of importance. This extension enhances the versatility of the proposed framework and enables a more comprehensive evaluation of complex decision criteria. Furthermore, the proposed method enhances the accuracy and precision of decision-making processes by providing operational laws and aggregate operators that facilitate the conversion of linguistic knowledge into numerical values. This conversion allows for quantitative analysis and comparison, leading to more reliable and informed decision outcomes.

Moreover, the generic nature of the proposed model makes it applicable to various domains that rely on language-based decision-making. From natural language processing to sentiment analysis and artificial intelligence, the method has the potential to contribute to a wide range of fields. In advanced manufacturing and robotics, operational laws establish the logical rules and principles for manipulating and transforming data, whether it is linguistic or numerical in nature. These laws provide a foundation for modeling and analyzing various aspects of manufacturing processes and robotic systems. By applying operational laws, engineers and researchers can develop algorithms, control strategies, and optimization techniques that ensure the efficient and effective operation of advanced manufacturing systems and robotic devices.

1.4. Layout of proposed research

The following shows that, how the work has been organized: The fundamental ideas of linguistic hypersoft set (LHSS) are broken down in detail in section 2. In section 3, we present a definition, notions, and examples of LHSS with basic properties and operations. Operational laws on LHSS has been proposed in section 4. The aggregate operator Linguistic Hypersoft Ordered Weighted Geometric Averaging Operator (LHSOWGAO) and Linguistic Hypersoft Weighted Geometric Averaging Operator (LHSWGAO) has been presented in section 5. In part 6, an MCDM framework is described for the "LHSS Algorithm to solve MCDM Problem" with a case study to demonstrate the benefits of the proposed algorithm. The findings of the study have been summarized, along with their significance, in section 7, and concluded with future directions. The layout of the paper also presented by Figure 1.

Figure 1. Layout of the paper

2. Preliminary section

In this section, we go through some basic definitions that support the construction of the framework of this paper: linguistic set, linguistic quantifiers, soft set, and hypersoft set (HSS).

Definition 2.1. Linguistic Set

Let $\mathrm{K} = \{\kappa^1, \kappa^2, \kappa^3, \dots, \kappa^t\}$ $where\ \ t = 2n+1 :\ \ n \geq 1\ \ and\ \ n\ \ \in \mathbb{R}^+$, be a finite strictly increasing set. For example, if n = 1 then,

$\mathrm{K} = \{\kappa^1, \kappa^2, \kappa^3\} = \{very\ \ bad, fair, very\ \ good\}$

For Linguistic set, which is under consideration, the relationship to its elements κ^t and the superscript t will be strictly increasing. To define the continuity this set is extended to $\mathrm{K} = \{\kappa^\beta : \beta \in \mathbb{R}\}$ $where\ \ \beta\ \ is\ \ also\ \ stricly\ \ incrasing$.

Definition 2.2. Linguistic Quantifiers

The linguistic quantifiers were introduced by Zadeh [48-51] also known as absolute quantifiers and are represented below in Table 1. Let $\mathrm{K} = \{\kappa^1, \kappa^2, \kappa^3, \dots, \kappa^t\}$ $where\ \ t = 2n+1 :\ \ n \geq 1\ \ and\ \ n\ \ \in \mathbb{R}^+$, be a finite strictly increasing set.

Table 1: linguistic quantifiers

Quantifiers						
Low		**Medium**			**High**	
None	**Very-Low**	**Low**	**Medium**	**High**	**Very-High**	**Perfect**

Definition 2.3. Soft Set

A pair $(\mathcal{F}, Å)$ is known as soft set (over $\mho$) iff $\mathcal{F}: Å \longrightarrow P(\mho)$. It means, soft set is the parametrized subset of the universe $\mho$.

Definition 2.4. Hypersoft Set

Let, $a^1, a^2, a^3, \dots, a^t$ for $t \geq 1$ be t distinct parameters, whose corresponding parametric values are respectively the sets $\mathcal{L}^1, \mathcal{L}^2, \mathcal{L}^3, \dots, \mathcal{L}^t$ with $\mathcal{L}^i \cap \mathcal{L}^j = \emptyset$, for $i \neq j$, and $i, j \in \{1, 2, \dots, t\}$.

Then the pair $(\mathcal{F}, \mathbb{L})$ where $\mathbb{L} = \{\mathcal{L}^1 \times \mathcal{L}^2 \times \mathcal{L}^3 \times \dots \times \mathcal{L}^t$: t is finite and real valued} is known as Hypersoft set over $\mho$ with mapping $\mathcal{F}: \mathbb{L} = \mathcal{L}^1 \times \mathcal{L}^2 \times \mathcal{L}^3 \times \dots \times \mathcal{L}^t \longrightarrow P(\mho)$.

3. Linguistic Hypersoft Set (LHSS)

In this section, we propose LHSS with its set structure properties.

Definition 3.1: Linguistic Hypersoft Set (LHSS)

Let, $\alpha^1, \alpha^2, \alpha^3, \dots, \alpha^t$ for $t \geq 1$ be t distinct parameters, whose corresponding parametric values are respectively the sets $\Upsilon^1, \Upsilon^2, \Upsilon^3, \dots, \Upsilon^t$ with $\Upsilon^i \cap \Upsilon^j = \emptyset$, for $i \neq j$, and $i, j \in \{1, 2, \dots, t\}$.

Then the pair (Γ, Λ) where $\Lambda = \{\Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \times \dots \times \Upsilon^t$: t is finite and real valued} is known as hypersoft set over Ω with mapping $\Gamma: \Lambda = \Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \times \dots \times \Upsilon^t \longrightarrow P(\Omega)$.

Then the linguistic hypersoft set will be,

$$\Gamma(\{\mathrm{M}(\Omega)(\imath)\}) : M \subseteq \Lambda \quad \& \quad \imath \in \mathrm{K} = \{\kappa^1, \kappa^2, \kappa^3, \dots, \kappa^t\} \; where \;\; t = 2n + 1 : \;\; n \geq 1, \qquad n \in \mathbb{R}^+\}$$

Numerical Example 3.1.1:

Let $\Omega = \{\sigma^1, \sigma^2, \sigma^3, \sigma^4\}$ and set $M = \{\sigma^2, \sigma^3\} \subset \Omega$.

Consider the parameters be: $\alpha^1 = nationality, \alpha^2 =$ gender, $\alpha^3 =$ color, and their respective parametric values are:

Nationality = $\Upsilon^1 = \{Pakistani, Chinese, American\}$

Gender = $\Upsilon^2 = \{Male, Female\}$

Color = $\Upsilon^3 = \{Pink, Black, Orange\}$

Then the function $\Gamma: \Lambda = \Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \longrightarrow P(\Omega)$ and assume the hypersoft set,

$\Gamma(\{Pakistani, Male, Orange\}) = \{\sigma^2, \sigma^3\} = M$

The linguistic hypersoft set (LHSS), $\Gamma(\{\mathrm{M}(\Omega)(\imath)\}) : M \subseteq \Lambda \quad \& \quad \imath \in \mathrm{K} = \{\kappa^1, \kappa^2, \kappa^3, \dots, \kappa^t\}$ $where \;\; t = 2n + 1 : \;\; n \geq 1, \; n \in \mathbb{R}^+\}$ Can be given as;

$\Gamma(\{Pakistani, Male, Orange\}) = \{\sigma^2, \sigma^3\} = \{\sigma^2(High), \sigma^3(None)\} = L.$

Similarly,

$\Gamma_1(\{Pakistani, Male, Pink\}) = \{\sigma^2(Perfect), \sigma^3(low)\} = L_1$

$\Gamma_2(\{Chinese, Female, Pink\}) = \{\sigma^1(high), \sigma^4(low)\} = L_2$

$\Gamma_3(\{American, male, black\}) = \{\sigma^1(medium), \sigma^3(none)\} = L_3$

Definition 3.2: Let $(\Gamma_1, \Lambda_1) = L_1$ be a LHSS, then the subset L_s can be defined as;

$\Gamma_2 : \Lambda_s = \Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \times \dots \times \Upsilon^f \longrightarrow P(\Omega) \;\; with \;\; s \leq n. Also, \;\; \Gamma_2(\{L_s(\Omega)(\imath)\}) : \Lambda_s \subseteq \Lambda \quad \& \quad \imath \in \mathrm{K} = \{\kappa^1, \kappa^2, \kappa^3, \dots, \kappa^t\}$

$$where \;\; t = 2n + 1 : \;\; n \geq 1, \qquad n \in \mathbb{R}^+\}$$

1. $L_s \subseteq L_1$;
2. $\forall \ell \in L_s, \Gamma_2(\ell) \subseteq \Gamma_1(\ell)$.

This holds only when linguistic variables K^i satisfy the property i.e. each K^i of $(\Gamma_s, \Lambda_s) = K^i$ of (Γ_1, Λ_1).

Example 3.2.1: Recall Example 1. The function $\Gamma_2 : \Lambda_s = \Upsilon^1 \times \Upsilon^2 \longrightarrow P(\Omega)$ and assume the hypersoft set, $\Gamma_2(\{Pakistani, Male\}) = \{\sigma^2(high)\} = L_s$. Where $\Lambda_s \subseteq \Lambda$ and $L_s \subseteq L_1$.

Definition 3.3: Empty linguistic hypersoft set (ELHSS) can be defined as;

$\Gamma_1 : \Lambda_E = \Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \times \ldots \times \Upsilon^n \longrightarrow P(\Omega)$

$such\ that\ each\ \Upsilon^i\ (i \leq n) is\ empty.$ $\Gamma_1(\{L_E(\Omega)(\imath)\}) : \Lambda_E \subseteq \Lambda$ & $\imath \in \mathrm{K} = \{\kappa^1, \kappa^2, \kappa^3, \ldots, \kappa^t\}$ $where\ t = 2n + 1 :\ n \geq 1,\ n \in \mathbb{R}^+\}$.

1. $(\Gamma_1, \Lambda_E)^{\phi} = L_E$ **if** $\forall \Gamma_1(\ell) = \phi : \forall \ell \in \Lambda_E$.

Example 3.3.1: Recall Example 1. The function $\Gamma_1 : \Lambda_E = \Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \longrightarrow P(\Omega)$ and assume the Hypersoft set, $\Gamma_1(\emptyset) = \emptyset = L_E$. Where $\Lambda_E \subseteq \Lambda$.

Definition 3.4: The AND operation on two $(\Gamma_1, \Lambda_1) = L_1$ and $(\Gamma_2, \Lambda_2) = L_2$ linguistic hypersoft set LHSS can be defined by;

1. $L_1 \wedge L_2 = (\Gamma_3, \Lambda_3) = L_3$; max of (K^i)
2. $(\ell_i, \ell_j) = \ell_k = L_3$ where $\ell_i \in L_1$ and $\ell_j \in L_2$ with $i \neq j$;
3. $\Gamma_3(\ell_i, \ell_j) = \Gamma_1(\ell_i) \cup \Gamma_2(\ell_j)$

Definition 3.5: The OR operation on two $(\Gamma_1, \Lambda_1) = L_1$ and $(\Gamma_2, \Lambda_2) = L_2$ linguistic hypersoft set LHSS can be defined by.

1. $L_1 \vee L_2 = (\Gamma_3, \Lambda_3) = L_3$;
2. $(\ell_i, \ell_j) = \ell_k = L_3$ where $\ell_i \in L_1$ and $\ell_j \in L_2$ with $i \neq j$;
3. $\Gamma_3(\ell_i, \ell_j) = \Gamma_1(\ell_i) \cap \Gamma_2(\ell_j)$

Definition 3.6: The NOT operation on (Γ, Λ) linguistic hypersoft set LHSS can be defined by;

1. $\sim L = \sim(\Gamma, \Lambda) = \sim \Upsilon^1 \times \sim \Upsilon^2 \times \sim \Upsilon^3 \times \ldots \times \sim \Upsilon^n$;
2. $\sim L = \sim \prod \ell_i : i = 1,2,3, \ldots, n$
3. $|\sim L| = n - Tuple$

Definition 3.7: The Complement on $(\Gamma, \Lambda) = L$ linguistic hypersoft set LHSS can be defined by;

1. $(\Gamma, \Lambda)^{\sim} = (\Gamma^{\sim}, \sim L)$; $\Gamma^{\sim} : \sim L \longrightarrow P(\Omega)$.
2. $\Gamma^{\sim}(\sim \ell) = \Omega \setminus \Gamma(\ell)$; $\forall \ell \in L$

Proposition 3.8: Let $(\Gamma, \Lambda) = L$, $(\Gamma_1, \Lambda_1) = L_1$, $(\Gamma_2, \Lambda_2) = L_2$ and $(\Gamma_3, \Lambda_3) = L_3$ be linguistic hypersoft set LHSS then following holds;

1. $(\Gamma_1, \Lambda_1) \subseteq (\Gamma_1, \Lambda_1)$
2. $(\Gamma_1, \Lambda_{\mathrm{E}})^{\Phi} \subseteq (\Gamma_1, \Lambda_1)$
3. $\sim(\sim L) = L$
4. $\sim(\Gamma_1, \Lambda_{\mathrm{E}})^{\Phi} = \Omega$
5. If $(\Gamma_1, \Lambda_1) \subseteq (\Gamma_2, \Lambda_2)$ and $(\Gamma_2, \Lambda_2) \subseteq (\Gamma_2, \Lambda_2)$ **then** $(\Gamma_1, \Lambda_1) = (\Gamma_2, \Lambda_2)$
   Iff each $K^i$of $(\Gamma_1, \Lambda_1) = K^i$ of (Γ_2, Λ_2).

This property holds only when linguistic variables satisfy the property i.e. each $K^i$of $(\Gamma_1, \Lambda_1) = K^i$of (Γ_2, Λ_2).

6. If $(\Gamma_1, \Lambda_1) \subseteq (\Gamma_2, \Lambda_2)$ *and* $(\Gamma_2, \Lambda_2) \subseteq (\Gamma_3, \Lambda_3)$ then $(\Gamma_1, \Lambda_1) \subseteq (\Gamma_3, \Lambda_3)$.

This property holds only when linguistic variables satisfy the property i.e. each K^i of $(\Gamma_1, \Lambda_1) = K^i$ of $(\Gamma_2, \Lambda_2) = K^i$of (Γ_3, Λ_3).

Proof: Recall L, L_1, L_2 *and* L_3 from example 3.3.1.

1. $\Gamma_1(\{Pakistani, Male, Pink\}) = \{\sigma^2, \sigma^3\} = \{\sigma^2(Perfect), \sigma^3(low)\} = L_1$ $\because$ $\sigma^2(Perfect) \in L_1$ *also* $\sigma^3(low) \in L_1$

$$\Rightarrow \sigma^2, \sigma^3 \in L_1$$

Thus $(\Gamma_1, \Lambda_1) \subseteq L_1 = (\Gamma_1, \Lambda_1)$.

2. Consider $L_1 = (\Gamma_1, \Lambda_1)$

$\because \quad \phi \in L_1 \quad \Rightarrow (\Gamma_1, \Lambda_E)^{\phi} \in L_1$

Thus $(\Gamma_1, \Lambda_E)^{\phi} \subseteq L_1 = (\Gamma_1, \Lambda_1)$ $(\Gamma_1, \Lambda_E)^{\phi} \subseteq (\Gamma_1, \Lambda_1)$.

3. Consider $L = \{\sigma^2(Perfect), \sigma^3(None)\}$, apply definition 6, *we get*, $(\sim L) = \{\sigma^1(none), \sigma^4(perfect)\}$ again apply definition 6, we get;

$\sim(\sim L) = \{\sigma^2(Perfect), \sigma^3(None)\} = L$

4. Consider $(\Gamma_1, \Lambda_E)^{\phi} = \phi \Rightarrow \phi \in L_E$ taking complement, $\sim(L_E) = \Omega \backslash\ \Gamma_1(\ell) = \phi$;

$\Rightarrow \sim(L_E) = \Omega$

hence $\sim(\Gamma_1, \Lambda_E)^{\phi} = \Omega$.

5. Consider, $(\Gamma_1, \Lambda_1) = \{\sigma^1(high), \sigma^3(low)\}$

$$(\Gamma_2, \Lambda_2) = \{\sigma^1(high), \sigma^3(low)\}$$

Each linguistic variable K^i of $(\Gamma_1, \Lambda_1) =$ linguistic variable K^i of (Γ_2, Λ_2) then this implies that $(\Gamma_1, \Lambda_1) \subseteq (\Gamma_2, \Lambda_2)$ *also* $(\Gamma_2, \Lambda_2) \subseteq (\Gamma_1, \Lambda_1)$

thus $(\Gamma_2, \Lambda_2) = (\Gamma_1, \Lambda_1)$.

Counter Example:

Consider,

$(\Gamma_1, \Lambda_1) = \{\sigma^2(high), \sigma^3(very\ \ low)\}$

and

$$(\Gamma_2, \Lambda_2) = \{\sigma^2(perfect), \sigma^3(low)\}$$

Each linguistic variable K^i of $(\Gamma_1, \Lambda_1) <$ linguistic variable K^i of (Γ_2, Λ_2) then this implies that $(\Gamma_1, \Lambda_1) \subseteq (\Gamma_2, \Lambda_2)$ *But* $(\Gamma_2, \Lambda_2) \nsubseteq (\Gamma_1, \Lambda_1)$ since linguistic variable of $(\Gamma_2, \Lambda_2) >$ linguistic variable of (Γ_1, Λ_1).

$$(\Gamma_2, \Lambda_2) \neq (\Gamma_1, \Lambda_1)$$

6. Same as 5.

4. Operational Laws on LHSS

In this section, we discuss the importance of operational laws and theorems and propose for LHSS. Let $(\Gamma_1, \Lambda_1) = L_1$ and $(\Gamma_2, \Lambda_2) = L_2$ be two LHSS and $\mu \geq 0$, where $\Lambda_1 = \{\Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \times \ldots \times \Upsilon^n$: n is finite and real valued} over Ω with mapping $\Gamma : \Lambda_1 = \Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \times \ldots \times \Upsilon^n \longrightarrow P(\Omega)$ and $\Lambda_2 = \{\Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \times \ldots \times \Upsilon^m$: m is finite and real valued} over Ω with mapping $\Gamma_2 : \Lambda_2 = \Upsilon^1 \times \Upsilon^2 \times$

$\Upsilon^3 \times \ldots \times \Upsilon^m \longrightarrow P(\Omega)$. Then the operational laws on LHSS can be defined with some necessary conditions;

Definition 4.1 Union of LHSS

Case 1: $L_1 \cup L_2 = \left\{\prod \alpha^i(K^i) \times \prod \alpha^j(K^j) \in \prod_{i=1}^n \Upsilon^i \times \prod_{j=1}^n \Upsilon^j\right\}$

Where, $\alpha^i(K^i) \in \prod_{i=1}^n \Upsilon^i$, *and* $\alpha^j(K^j) \in \prod_{j=1}^n \Upsilon^j$ *should be distinct* with $\Upsilon^i \cap \Upsilon^j = \emptyset$, for $i \neq j$, and $i, j \in \{1, 2, \ldots, t\}$.

Case 2: $L_1 \cup L_2 = \left\{\alpha^i(K^i) \in \prod_{i=1}^n \Upsilon^i \times \prod_{j=1}^n \Upsilon^j\right\}$

with $i = j$, *and linguistic* variable K^i *of* σ^i *should be same*.

Example: Consider,

Case 1;

$$\Gamma_1(\{Pakistani, male, black\}) = \{\sigma^2(Perfect), \sigma^3(low)\} = L_1$$

$$\Gamma_2(\{American, Female, Pink\}) = \{\sigma^1(high), \sigma^4(low)\} = L_2$$

$\because \Upsilon^i \cap \Upsilon^j = \emptyset$

$L_1 \cup L_2 = \{\sigma^2(Perfect), \sigma^3(low), \sigma^1(high),$
$\sigma^4(low)\}$.

Case 2;

$$\Gamma_1(\{Pakistani, male, black\}) = \{\sigma^2(Perfect), \sigma^3(low)\} = L_1$$

$$\Gamma_2(\{Pakistani, female, pink\}) = \{\sigma^2(perfect), \sigma^3(low)\} = L_2$$

$\because \Upsilon^i \cap \Upsilon^j \neq \emptyset$ with $i = j$

$L_1 \cup L_2 = \{\sigma^2(Perfect), \sigma^3(low)\}$.

Case 3; (Counter example) \Restriction

$$\Gamma_1(\{Pakistani, male, black\}) = \{\sigma^2(high), \sigma^3(verylow)\} = L_1$$

$$\Gamma_2(\{Pakistani, female, pink\}) = \{\sigma^2(perfect), \sigma^3(low)\} = L_2$$

$\because \Upsilon^i \cap \Upsilon^j \neq \emptyset$ with $i = j$

Each linguistic variable K^i of $L_1 <$ linguistic variable K^i of L_2 then this implies $L_1 \cup L_2$ can be defined with some restriction i.e. consider highest linguistic value K^i of each attribute.

Example: $L_1 = \{\sigma^2(high), \sigma^3(verylow)\}$

$$L_2 = \{\sigma^2(perfect), \sigma^3(low)\}$$

As, $\sigma^2(high) < \sigma^2(perfect)$, *and*

$$\sigma^3(verylow) < \sigma^3(low)$$

Then $L_1 \cup L_2 = \{\sigma^2(perfect), \sigma^3(low)\}$.

Definition 4.2 Intersection of LHSS

Let $(\Gamma_1, \Lambda_1) = L_1$ and $(\Gamma_2, \Lambda_2) = L_2$ be two LHSS and $\mu \geq 0$, then the intersection can be defined as;

$$L_1 \cap L_2 = \left\{\prod \alpha^i(K^i) \times \prod \alpha^j(K^j) \in \prod_{i=1}^n \Upsilon^i \times \prod_{j=1}^n \Upsilon^j\right\} = \emptyset$$

Where, $\alpha^i(K^i) \in \prod_{i=1}^{n} \Upsilon^i, and \quad \alpha^j(K^j) \in \prod_{j=1}^{n} \Upsilon^j$ *should be distinct* with $\Upsilon^i \cap \Upsilon^j = \emptyset$, for $\mathrm{i} = \mathrm{j}$, and $\mathrm{i}, \mathrm{j} \in \{1, 2, \dots, \mathrm{t}\}$.

Case 2: $L_1 \cap L_2 = \{\alpha^i(K^i) \in \prod_{i=1}^{n} \Upsilon^i \times \prod_{j=1}^{n} \Upsilon^j\}$

with $i = j$ *, and linguistic* variable K^i *of* σ^i Then $L_1 \cap L_2 = L_1$ *or* L_2

Example: Consider,

Case 1;

$$\Gamma_1(\{Pakistani, male, black\}) = \{\sigma^2(Perfect), \sigma^3(low)\} = L_1$$

$$\Gamma_2(\{American, Female, Pink\}) = \{\sigma^1(high), \sigma^4(low)\} = L_2$$

$$\because \Upsilon^i \cap \Upsilon^j = \emptyset$$

$$L_1 \cap L_2 = \{\emptyset\}$$

Case 2;

$$\Gamma_1(\{Pakistani, male, black\}) = \{\sigma^2(Perfect), \sigma^3(low)\} = L_1$$

$$\Gamma_2(\{Pakistani, female, pink\}) = \{\sigma^2(perfect), \sigma^3(low)\} = L_2$$

$\because \Upsilon^i \cap \Upsilon^j \neq \emptyset$ with $\mathrm{i} = \mathrm{j}$

$L_1 \cap L_2 = \{\sigma^2(Perfect), \sigma^3(low)\}$.

Case 3; (Counter example) \Restriction

$$\Gamma_1(\{Pakistani, male, black\}) = \{\sigma^2(high), \sigma^3(verylow)\} = L_1$$

$$\Gamma_2(\{Pakistani, female, pink\}) = \{\sigma^2(perfect), \sigma^3(low)\} = L_2$$

$\because \Upsilon^i \cap \Upsilon^j \neq \emptyset$ with $\mathrm{i} = \mathrm{j}$

Each linguistic variable K^i of $L_1 <$ linguistic variable K^i of L_2 then this implies $L_1 \cup L_2$ can be defined with some restriction i.e. consider highest linguistic value K^i of each attribute.

Example: $L_1 = \{\sigma^2(high), \sigma^3(verylow)\}$

$$L_2 = \{\sigma^2(perfect), \sigma^3(low)\}$$

As, $\sigma^2(high) < \sigma^2(perfect), and$

$$\sigma^3(verylow) < \sigma^3(low)$$

Then $L_1 \cap L_2 = \emptyset$.

Theorem 4.3: If $\mathbf{L_1}$, $\mathbf{L_2}$ and $\mathbf{L_3}$ be three LHSS then the following holds:

i. $L_1 \cup L_1 = L_1$
ii. $L_1 \cup \emptyset = L_1$
iii. $L_1 \cap L_1 = L_1$
iv. $L_1 \cap \emptyset = \emptyset$
v. $L_1 \cup L_2 = L_2 \cup L_1$
vi. $L_1 \cap L_2 = L_2 \cap L_1$
vii. $L_1 \cup (L_2 \cup L_3) = (L_1 \cup L_2) \cup L_3)$
viii. If $L_1 \subset L_2$ and $L_2 \subset L_1$ the $L_1 = L_2$.
ix. $\mu(L_1) = \mu L_1$; $\mu \geq 0$.
x. $\mu(L_1 \cup L_2) = \mu(L_2 \cup L_1)$

The proofs are straight forward. ■

Theorem 4.4

If L_1, L_2 be two LHSS then the operations are given as follows:

1. $\mu \times L_1 = L_{\mu\times 1}$; μ (linguistic variable);
2. $L_1 \oplus L_2 = L_{1\oplus 2}$;
3. $L_1 \otimes L_2 = L_{1\otimes 2}$;
4. $(L_1)^{\mu} = L_{1^{\mu}}$.

Proof:

1. Consider, $\Gamma_1(\{Pakistani, male, black\}) = \{\sigma^2(Perfect), \sigma^3(low)\} = L_1$ and $\mu =$ low,

The proofs are straight forward. ■

5. Some Aggregation Operators

Aggregate operators play a crucial role in decision-making processes, and their importance cannot be overstated. These operators are responsible for combining and aggregating individual linguistic quantifiers or numerical values to derive a comprehensive assessment of various factors and attributes. By employing aggregate operators, decision-makers can effectively analyze and evaluate complex information, facilitating informed decision-making. One of the key benefits of aggregate operators is their ability to handle multiple criteria simultaneously. Decision-making often involves considering various factors, such as cost, quality, reliability, and customer satisfaction. Aggregate operators enable decision-makers to combine and weigh these criteria appropriately, considering their relative importance. This allows for a comprehensive evaluation and comparison of different options or alternatives.

Aggregate operators also provide a means to summarize and condense large amounts of data into manageable and meaningful information. They enable decision-makers to reduce complex and diverse inputs into a single aggregated value or linguistic quantifier. This simplification aids in understanding and interpreting the information, making it easier to make decisions based on the aggregated results. Moreover, aggregate operators facilitate the integration of subjective or qualitative assessments into the decision-making process. They provide a mechanism to convert linguistic expressions, which often involve subjective opinions or judgments, into numerical values that can be analyzed and compared objectively. This enables decision-makers to incorporate both objective and subjective information, leading to more comprehensive and well-rounded decisions. Additionally, aggregate operators allow for flexibility and adaptability in decision-making. Different aggregate operators, such as weighted averages, minimum or maximum operations, and fuzzy logic operators, offer diverse ways to combine and aggregate data. This flexibility enables decision-makers to tailor the aggregation process to their specific needs and preferences, accommodating different decision contexts and requirements. In conclusion, aggregate operators are vital tools in decision-making processes, enabling the integration, evaluation, and comparison of diverse criteria and information. They enhance the ability to handle multiple factors, summarize complex data, incorporate subjective assessments, and provide flexibility in decision-making. By leveraging aggregate operators effectively, decision-makers can make more informed and well-founded decisions, leading to improved outcomes and increased overall effectiveness in various domains.

Aggregate operators are required by decision-makers (DMs) to rank the given alternatives. The ordered weighted averaging operator (OWAO) proposed by Yager [25] is the most widely used methodology for aggregating decision information. Later, various new OWAO were introduced [26]. The OWAO has been employed in an amazingly wide range of applications [24, 27]. The majority of these operators, on the other hand, can only be employed when the input arguments are exact values, and only a few of them can be used to aggregate linguistic preference data.

Decision-making, on the other hand, is influenced by personal psychological factors such as experience, learning, situation, mood, like-dislike, and so on. It is more appropriate to express their preferences using linguistic parameters rather than numerical variables. Thus, in this section the aggregation operators for hypersoft set have been proposed.

Definition 5.1 LHSWGAO

Consider, $\alpha^1, \alpha^2, \alpha^3, \dots, \alpha^t$ for t ≥ 1 be t distinct parameters, whose corresponding parametric values are respectively the sets $\Upsilon^1, \Upsilon^2, \Upsilon^3, \dots, \Upsilon^t$ with $\Upsilon^i \cap \Upsilon^j = \emptyset$, for $i \neq j$, and $i, j \in \{1, 2, \dots, t\}$.

Let $\mathfrak{A}$: $\Lambda = \Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \times \dots \times \Upsilon^t \longrightarrow P(\Omega) = \{\mathrm{M}(\Omega)(i)\} \subseteq \mathbb{R}^+$ (1)

if $\mathfrak{A}^{\omega}$ $(\alpha^1, \alpha^2, \alpha^3, \dots, \alpha^t) = \prod_{t=1}^{n} (\alpha^t(i))^{(\omega^t)}$

Such that

$\mathfrak{A}^{\omega}$ $(\alpha^1, \alpha^2, \alpha^3, \dots, \alpha^t) =$

$\alpha^1{}_i^{\omega^1} \otimes \alpha^2{}_i^{\omega^2} \otimes \alpha^3{}_i^{\omega^3} \otimes \dots \otimes \alpha^t{}_i^{\omega^t} = \sigma_i$

Where $\omega = (\omega^1, \omega^2, \omega^3, \dots, \omega^t)^T$ is the exponential weighting vector of the $\alpha^t(i) \in \{\mathrm{M}(\Omega)(i)\}$ and $\omega^t \in [0,1]$ with $\sum_{t=1}^{n} \omega^t = 1$, and $i \in \mathrm{K} = \{\kappa^1, \kappa^2, \kappa^3, \dots, \kappa^t\}$ Then $\mathfrak{A}$ is called Linguistic Hypersoft Weighted Geometric Averaging Operator (LHSWGAO).

Example: Assume $\omega = (0.4, 0.3, 0.3)^T$ then LHSWGAO $\{\sigma^2(Pakistani, Male, Orange),$ $\sigma^3(Pakistani, Male, Orange)\}$

The linguistic set of definition 2, is labeled as

Table 2. linguistic quantifiers

	Quantifiers					
Low		Medium			High	
None	Very-Low	Low	Medium	High	Very-High	Perfect
0	1	2	3	4	5	6

$$= \sigma^2 \begin{pmatrix} Pakistani(low), Male(medium), \\ Orange(high) \end{pmatrix}$$

$$\because \mathfrak{A}^{\omega}\ (\alpha^1, \alpha^2, \alpha^3, \dots, \alpha^t) = \prod_{t=1}^{n} (\alpha^t(i))^{(\omega^t)}$$

$= {\alpha^1}_i^{\omega^1} \otimes {\alpha^2}_i^{\omega^2} \otimes {\alpha^3}_i^{\omega^3} \otimes \dots \otimes {\alpha^t}_i^{\omega^t} = \sigma_i$

= {Pakistani$(low)^{0.4}$, Male$(medium)^{0.3}$,

Orange$(high)^{0.3}$}

$= \sigma^2\{(2)^{0.4} + (3)^{0.3} + (4)^{0.3}\}$

$= \sigma^2(i)^{4.22}$

$= \sigma^2(i)^{4.22} = \sigma^2(High)$ From Table 2, the linguistic value for $i = 4\ \ is\ \ high$.

Now,

$$= \sigma^3 \begin{pmatrix} Pakistani(none), Male(none), \\ Orange(none) \end{pmatrix}$$

$$\because \mathfrak{A}^{\omega}\ (\alpha^1, \alpha^2, \alpha^3, \dots, \alpha^t) = \prod_{t=1}^{n} (\alpha^t(i))^{(\omega^t)}$$

$= {\alpha^1}_i^{\omega^1} \otimes {\alpha^2}_i^{\omega^2} \otimes {\alpha^3}_i^{\omega^3} \otimes \dots \otimes {\alpha^t}_i^{\omega^t} = \sigma_i$

= {Pakistani$(none)^{0.4}$, Male$(none)^{0.3}$,

Orange$(none)^{0.3}$}

$= \sigma^2\{(0)^{0.4} + (0)^{0.3} + (0)^{0.3}\}$

$= \sigma^2(0)$

$= \sigma^2(none)$

From Table 2, the linguistic value for $i = 0\ \ is\ \ none$.

Definition 5.2 LHSOWGAO

Consider, $\alpha^1, \alpha^2, \alpha^3, \dots, \alpha^t$ for $t \geq 1$ be t distinct parameters, whose corresponding parametric values are respectively the sets $\Upsilon^1, \Upsilon^2, \Upsilon^3, \dots, \Upsilon^t$ with $\Upsilon^i \cap \Upsilon^j = \emptyset$, for $i \neq j$, and $i, j \in \{1, 2, \dots, t\}$.

Let $\mathfrak{D}$: $\Lambda = \Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \times \dots \times \Upsilon^t \longrightarrow P(\Omega) = \{\mathrm{M}(\Omega)(\imath)\} \subseteq \mathbb{R}^+$ (2)

If $\mathfrak{D}^\omega\ (\alpha^1, \alpha^2, \alpha^3, \dots, \alpha^t) = \prod_{t=1}^n (\alpha^t(i))^{(\omega^t)}$

Such that $\mathfrak{D}^\omega\ (\alpha^1, \alpha^2, \alpha^3, \dots, \alpha^t) =$

$$\alpha^1{}_i^{\omega^1} \otimes \alpha^2{}_i^{\omega^2} \otimes \alpha^3{}_i^{\omega^3} \otimes \dots \otimes \alpha^t{}_i^{\omega^t} = \sigma_i$$

Subject to the condition, the linguistic values of α_i should be in ascending order. Where $\omega = (\omega^1, \omega^2, \omega^3, \dots, \omega^t)^T$ is the exponential weighting vector of the $\alpha^t(i) \in \{\mathrm{M}(\Omega)(\imath)\}$ and $\omega^t \in [0,1]$ with $\sum_{t=1}^n \omega^t = 1$, and $\imath \in \mathrm{K} = \{\kappa^1, \kappa^2, \kappa^3, \dots, \kappa^t\}$ then $\mathfrak{D}$ is called Linguistic Hypersoft Ordered Weighted Geometric Averaging Operator (LHSOWGAO).

Example:

Assume $\omega = (0.4, 0.3, 0.3)^T$ then LHSOWGAO $\{\sigma^2(Pakistani, Male, Orange)$, $\sigma^3(Pakistani, Male, Orange)\}$

The linguistic set of definition 2, is labeled as;

$$= \sigma^2 \begin{pmatrix} Pakistani(low), Male(medium), \\ Orange(high) \end{pmatrix}$$

$$\because \mathfrak{D}^\omega\ (\alpha^1, \alpha^2, \alpha^3, \dots, \alpha^t) = \prod_{t=1}^n (\alpha^t(i))^{(\omega^t)}$$

$= \alpha^1{}_i^{\omega^1} \otimes \alpha^2{}_i^{\omega^2} \otimes \alpha^3{}_i^{\omega^3} \otimes \dots \otimes \alpha^t{}_i^{\omega^t} = \sigma_i$

$= \{\mathrm{Pakistani}(high)^{0.4}, \mathrm{Male}(medium)^{0.3},$

$\mathrm{Orange}(low)^{0.3}\}$

$= \sigma^2\{(4)^{0.4} + (3)^{0.3} + (2)^{0.3}\}$

$= \sigma^2(i)^{4.36}$

$= \sigma^2(i)^{4.36} = \sigma^2(High)$ From Table 2, the linguistic value for $i = 4$ *is high*.

Now,

$$= \sigma^3 \begin{pmatrix} Pakistani(none), Male(none), \\ Orange(none) \end{pmatrix}$$

$$\because \mathfrak{D}^\omega\ (\alpha^1, \alpha^2, \alpha^3, \dots, \alpha^t) = \prod_{t=1}^n (\alpha^t(i))^{(\omega^t)}$$

$= \alpha^1{}_i^{\omega^1} \otimes \alpha^2{}_i^{\omega^2} \otimes \alpha^3{}_i^{\omega^3} \otimes \dots \otimes \alpha^t{}_i^{\omega^t} = \sigma_i$

$= \{\text{Pakistani}(none)^{0.4}, \text{Male}(none)^{0.3},$
$\text{Orange}(none)^{0.3}\}$
$= \sigma^2\{(0)^{0.4} + (0)^{0.3} + (0)^{0.3}\}$
$= \sigma^2(0)$
$= \sigma^2(none)$

From Table 2, the linguistic value for $i = 0$ is $none$.

Theorem 5.1:

1. $\min_i(\alpha^t(i)) \leq \mathfrak{A}^{\omega}(\alpha^1, \alpha^2, .., \alpha^t) \leq \max_i(\alpha^t(i))$

2. $\min_i(\alpha^t(i)) \leq \mathfrak{D}^{\omega}(\alpha^1, \alpha^2, .., \alpha^t) \leq \max_i(\alpha^t(i))$

Proof: The proofs are straight forward. ■

Theorem 5.2:

1. $\mathfrak{D}^{\omega}(\alpha^t(i)) = \mathfrak{D}^{\omega}(\alpha^t(i'))$

Where $(\alpha^t(i'))$ is any permutation of $(\alpha^t(i))$

2. If $\forall(\alpha^t(i)) = (\alpha(i))$ for all *t*, then $\mathfrak{D}^{\omega}(\alpha^t(i)) = \boldsymbol{\sigma}_i$

3. If $(\alpha^t(i)) \leq (\hat{\alpha}^t(i))$ for all *t*, then

$$\mathfrak{D}^{\omega}(\alpha^t(i)) \leq \mathfrak{D}^{\omega}(\hat{\alpha}^t(i))$$

Proof: The proofs are straight forward. ■

6. Multi-Criteria Decision-Making Method (LHSS Algorithm to solve MCDM Problem)

A decision-making technique based on linguistic hypersoft weighted geometric averaging operator (LHSWGAO) has been used to construct an algorithm known as linguistic hypersoft set based multi-criteria group decision-making method (LHSS algorithm to solve MCGDM problem). The graphical representation of the proposed LHSS algorithm is presented in Figure 2.

Step1: *Consider,* $\alpha^1, \alpha^2, \alpha^3, \ldots, \alpha^t$ for $t \geq 1$ *be* t *distinct parameters, whose corresponding parametric values are respectively the sets* $\Upsilon^1, \Upsilon^2, \Upsilon^3, \ldots, \Upsilon^t$ *with* $\Upsilon^i \cap \Upsilon^j = \emptyset$, *for* $i \neq j$, *and* $i, j \in \{1, 2, \ldots, t\}$.
Let $\omega = (\omega^1, \omega^2, \omega^3, \ldots, \omega^t)^T$ *be the exponential weighting vector. Where* $\omega^t \geq 0$ *, and* $\sum_{t=1}^{n} \omega^t = 1$.
Let $\mathfrak{A}$: $\Lambda = \Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \times \ldots \times \Upsilon^t \longrightarrow P(\Omega) = \{M(\Omega)(i)\} \subseteq \mathbb{R}^+$
The decision-makers $\mathcal{D}^m$ *compare the values with the linguistic quantifiers and assign linguistic variable to each alternative as* $H_i = \{(\alpha^t(i) : \ i = 1, 2, \ldots, t\}$, *and construct a linguistic preference table for* $(\alpha^t(i))^{(\omega^t)}$.

Step2: *Construct a matrix* $[\sigma_i^j, s_i^j]_{\ i \times j}$ *for each* $\mathcal{D}^m$ *using linguistic hypersoft weighted geometric averaging operator (LHSWGAO),*

$$\sigma_i^t = \alpha^1{}_i^{\omega^1} \otimes \alpha^2{}_i^{\omega^2} \otimes \alpha^3{}_i^{\omega^3} \otimes \ldots \otimes \alpha^t{}_i^{\omega^t}$$

Construct a matrix individually for each $\mathcal{D}^m$ using linguistic hypersoft ordered weighted geometric averaging operator (LHSOWGAO)

$$\mathfrak{D}^{\omega}\ (\alpha^1, \alpha^2, \alpha^3, \ldots, \alpha^t) = \prod_{t=1}^{n} (\alpha^t(i))^{(\omega^t)}$$

Such that $\mathfrak{D}^{\omega}\ (\alpha^1, \alpha^2, \alpha^3, \ldots, \alpha^t) = {\alpha^1}_i^{\omega^1} \otimes {\alpha^2}_i^{\omega^2} \otimes {\alpha^3}_i^{\omega^3} \otimes \ldots \otimes {\alpha^t}_i^{\omega^t} = s_i$

Step3: *Construct a matrix using* $[min(\sigma_i^j, s_i^j)]_{\ i\ \times j}$ *for each* $\mathcal{D}^m$.

Step4: List *max value among all the decision-makers.*

$$\max\ [\mathcal{D}^1 \min(\sigma_i^j, s_i^j), \mathcal{D}^2 \min(\sigma_i^j, s_i^j), \ldots, \mathcal{D}^m \min(\sigma_i^j, s_i^j)]_{\ i\ \times j}$$

Step5: *Write value from linguistic table or reference table* known as total score.

Step6: *Finally, list the alternatives with total scores H_i and rank highest value.*

Figure 2. Graphical representation of Proposed LHSS algorithm

6.1 Illustrative example

Problem: Inequitable access to healthcare in rural areas

Rural areas often have limited access to healthcare services due to a shortage of healthcare providers and facilities. This can result in poorer health outcomes for rural residents compared to their urban counterparts. The current policy approach to addressing this problem includes initiatives such as the Sehat Card in Pakistan, which aims to increase access to healthcare services for all Pakistanis. However, the health service has faced challenges related to hospital affordability, accessibility, and political opposition.

1. *Increase funding for rural healthcare services and providers:*

This solution would involve providing more resources for rural healthcare services and providers, such as funding for healthcare facilities, equipment, and staff. It could potentially improve access to healthcare services in rural areas and address the issue of healthcare workforce shortages. However, it may be costly and could face political opposition.

2. *Incentivize healthcare providers to work in rural areas:*

This solution could involve offering financial incentives or loan forgiveness to healthcare providers who work in rural areas. It could help address the issue of healthcare workforce shortages in these areas and improve access to healthcare services. However, it may be difficult to implement and could face resistance from healthcare providers who prefer to work in urban areas.

3. *Invest in infrastructure improvements to support healthcare delivery in rural areas:*

There is no fixed percentage or rule for how much focus should be given to each solution when addressing a public policy problem. The appropriate mix of solutions will depend on various factors, such as the context of the problem, the goals of the policy, the available resources, and the political environment. In general, when developing policy solutions, it is important to consider a range of options and evaluate their feasibility, effectiveness, and potential impacts on equity. This can involve conducting research, consulting with stakeholders, and considering multiple perspectives. The goal should be to identify a set of solutions that are likely to achieve the desired outcomes, while minimizing any unintended negative consequences.

6.2 Demonstration of proposed example

Consider $H = \{H^1, H^2, H^3\}$ be three hospitals as alternatives in rural area, and we want to improve the health services. The services of the experts in this domain has been taken and known as decision-makers $\mathcal{D} = \{\mathcal{D}^{\mathrm{m}} \;\; ; m = 1,2,3\}$. The goal should be to identify a set of solutions that are likely to achieve the desired outcomes, while minimizing any unintended negative consequences. Consider the parameters be: $\alpha^1 =$ Increase in funding, $\alpha^2 =$ Incentivize healthcare, $\alpha^3 =$ Invest in infrastructure, and their respective parametric values are:

Increase in funding $= \Upsilon^1 = \{fascilities,\ equipment, staff\}$

Incentivize healthcare $= \Upsilon^2 = \{healthcare\ \ workforce, improve\ \ access\ \}$

Invest in infrastructure $= \Upsilon^3 = \{Transportation, telemedicine\}$

Then the function $\Gamma : \Lambda = \Upsilon^1 \times \Upsilon^2 \times \Upsilon^3 \ \longrightarrow P(\Omega)$ and assume the hypersoft set $M = \{H^1, H^2, H^3\} \subset \Omega$ where $\Omega = \{H^1, H^2, H^3\}$ be the universal set.

$\Gamma(\{fascilities, healthcare\ \ workforce, telemedicine\}) = \{H^1, H^2, H^3\} = M$ The linguistic hypersoft set (LHSS), $\Gamma(\{\mathrm{M}(\Omega)(\imath)\}) : M \subseteq \Lambda$ & $\imath \in \mathrm{K} = \{\kappa^1, \kappa^2, \kappa^3, \ldots, \kappa^t\}$ $where\ \ t = 2n + 1 :\ \ n \geq 1,$ $n \in \mathbb{R}^+\}$ can be given by three decision-makers $\mathcal{D} = \{\mathcal{D}^{\mathrm{m}} \;\; ; m = 1,2,3\}$.

$\mathcal{D}^1$ $define$ $\Gamma_1(\{fascilities, healthcare\ \ workforce, telemedicine\})$

$= \{H^1, H^2, H^3\}$

$$= \left\{\begin{matrix} H^1 < fascilities\ \ (low), healthcare\ \ workforce\ \ (high), telemedicine(medium) >, \\ H^2 < fascilities\ \ (v.high), healthcare\ \ workforce\ \ (medium), telemedicine(high) >, \\ H^3 < fascilities\ \ (medium), healthcare\ \ workforce(high), telemedicine(low) > \end{matrix}\right\} = L_1$$

$\mathcal{D}^2$ $define$ $\Gamma_2(\{equipment, improve\ \ access, transportation\}) = \{H^1, H^2, H^3\}$

$= \{H^1 < equipment(high), improve\ \ access\ \ (v.high), transportation(low) >,$

$H^2 \quad < equipment(low), improve \quad access(v.high), transportation(high) >,$

$H^3 < equipment(medium), improve \quad access(high), transportation(medium) >\}$

$= L_2$

$\mathcal{D}^3 \quad define \quad \Gamma_3(\{staff, healthcare \quad workforce, telemedicine\})$

$= \{H^1 < staff(medium), healthcare \quad workforce(high), telemedicine(v.low) >,$

$H^2 \quad < staff(low), healthcare \quad workforce(low), telemedicine(high) >,$

$H^3 < staff(high), healthcare \quad workforce(high), telemedicine(low) >\}$

$= L_3$

Step1: The decision matrix by decision-makers $\mathcal{D} = \{\mathcal{D}^{\mathrm{m}} \quad : \; m = 1,2,3\}$, presented below.

Refer Table 3.

Step2: Construct a matrix using LHSWGAO, and LHSOWGAO.

$$\begin{array}{c} \cdot \\ H^1 \\ H^2 \\ H^3 \end{array} = \begin{bmatrix} \mathcal{D}^1 & \mathcal{D}^2 & \mathcal{D}^3 \\ high, medium & medium, high & high, medium \\ v.high, \quad perfect & high, medium & high, v.high \\ high, medium & high, high & high, medium \end{bmatrix}$$

Step3: Find the min of matrix values of step2.

$$\begin{array}{c} \cdot \\ H^1 \\ H^2 \\ H^3 \end{array} = \begin{bmatrix} \mathcal{D}^1 & \mathcal{D}^2 & \mathcal{D}^3 \\ medium & medium & medium \\ v.high & medium & high \\ medium & high & medium \end{bmatrix}$$

Step4: *Write max value among all the decision-makers.*

$$S = \{H^1 < medium >, H^2 < v.high >, H^3 < high >\}$$

Step5: *Write value from linguistic table.*

$$S = \{H^1 < 4 >, H^2 < 5 >, H^3 < 4 >\}$$

Step6: *Finally, list the alternatives with total scores* $\mathcal{S}_i \quad and \quad rank$ *highest value.*

Alternative	*Score Value*	*Rank*
$\boldsymbol{H^1}$	*3*	*1*
$\boldsymbol{H^2}$	*5*	*3*
$\boldsymbol{H^3}$	*4*	*2*

This solution shows that $H^1 < H^3 < H^2$ involve investing in infrastructure improvements, such as broadband internet access and transportation infrastructure, to support healthcare delivery in rural areas. It could improve access to telemedicine and other remote healthcare services, as well as address transportation barriers to accessing healthcare services. The results are presented in Figure 3.

6.3 Result discussion comparison and future directions

The comparison analysis presented highlights the prioritization of infrastructure improvements for healthcare delivery in rural areas, with the order being $H^1 < H^3 < H^2$. The analysis suggests that investing in infrastructure improvements, such as broadband internet access and transportation infrastructure, can have significant benefits for healthcare accessibility in rural communities.

H^1 represents the highest priority, indicating that addressing healthcare infrastructure deficiencies in rural areas should be the primary focus. This may involve initiatives to improve broadband internet access, which can facilitate telemedicine and remote healthcare services. By enhancing connectivity, individuals in rural areas can access healthcare professionals and receive medical consultations without the need for in-person visits, thereby reducing barriers to healthcare access.

Alternatives	Linguistic Score	Score Value	Ranking
Alternative H^1	$medium$	3	1
Alternative H^2	$v.high$	5	3
Alternative H^3	$high$	4	2

Figure 3. Result and ranking of the alternatives.

H^3 denotes a relatively lower priority compared to H^1but higher than H^2This suggests that while health care infrastructure improvements are crucial, other factors may also need attention. These factors could include policy reforms, financial support, or workforce development to complement the infrastructure enhancements. A comprehensive approach that combines infrastructure improvements with these additional measures can yield a more effective and sustainable healthcare system in rural areas.

H^2 represents the lowest priority, indicating that while still important, addressing transportation barriers to healthcare access may be of lesser immediate significance compared to infrastructure enhancements. Transportation infrastructure improvements could include better roads, public transportation systems, or medical transport services to ensure that individuals can reach healthcare facilities conveniently and efficiently.

The power of the proposed method lies in its ability to overcome the limitations of existing approaches by providing a systematic and effective framework for dealing with linguistic knowledge in decision-making. By offering a mathematically grounded solution, the method enhances the accuracy, precision, and applicability of decision-making processes, contributing to advancements in the field.

To extend the proposed method for complicated cases, additional considerations and modifications can be made to the existing framework. This can be demonstrated through numerical examples that illustrate the application of the extended method. Here is an explanation of how the proposed method can be extended along with numerical examples:

1. Handling Complicated Cases: In complex decision-making scenarios, where multiple factors and attributes need to be considered, the proposed method can be expanded to accommodate these complexities. This can be achieved by incorporating weighted linguistic quantifiers or linguistic variables that represent the relative importance or degree of each factor.
2. Numerical Examples: Let's consider a decision-making problem involving the selection of a new supplier for a company. The decision criteria include factors such as price, quality, delivery time, and customer service. Each factor can be represented by linguistic quantifiers, such as "low," "medium," or "high." To extend the method, weights can be assigned to these linguistic quantifiers based on their relative importance.

For instance, let's assume the weight assigned to price is 0.4, quality is 0.3, delivery time is 0.2, and customer service is 0.1. The linguistic quantifiers for each factor can be mapped to numerical values using the proposed LHSS framework. Suppose "low" corresponds to 1, "medium" corresponds to 3, and "high" corresponds to 5. Now, let's assume we have three potential suppliers: Supplier A, Supplier B, and Supplier C. We can evaluate each supplier's performance on each factor and calculate an overall score based on the assigned weights. The scores can be computed by multiplying the numerical value of each linguistic quantifier by its weight and summing up the results. For example, Supplier A may have a price score of (1 * 0.4), a quality score of (3 * 0.3), a delivery time score of (3 * 0.2), and a customer service score of (5 * 0.1). Summing up these scores, we obtain an overall score for Supplier A. Similarly, we can calculate the overall scores for Supplier B and Supplier C. Based on the calculated overall scores, the company can then make an informed decision on selecting the most suitable supplier. These numerical examples demonstrate the extension of the proposed method to handle complicated decision-making scenarios by incorporating weighted linguistic quantifiers. By assigning weights and mapping linguistic quantifiers to numerical values, the method allows for a more comprehensive and precise evaluation of complex decision criteria.

7. Conclusion

This paper acknowledges the complexity of the relationship between language and meaning, highlighting the challenges in assigning numerical values to linguistic variables. To address this issue, the concept of LHSS (Linguistic Hypersoft Set Structure) along with operational laws, aggregate operators, and MCGDM (Multi-Criteria Group Decision Making) techniques has been proposed. The application of these concepts has been demonstrated through a real-life case study, yielding promising results. The findings indicate that assigning numerical values to linguistic variables enhances accuracy in decision-making. Future directions include extending the framework to complex decision-making scenarios, integrating it with machine learning and AI techniques, exploring hybrid set structures, conducting real-world case studies, focusing on human-computer interaction, addressing ethical and social implications, and advancing user-centric approaches. These future directions aim to enhance the applicability, effectiveness, and ethical use of linguistic set structures in decision-making across various domains.

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Neutrosophic Similarity Measure for Assessing Digital Watermarked Images

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Abstract: Digital watermarking is an essential tool for numerous applications, and the quality of watermarked images must be assessed using accurate criteria. Peak Signal-to-Noise Ratio (PSNR), a widely used image assessment metric, has limits when evaluating images containing noise, such as watermarks. To tackle such kind of issues this, this study investigates a different assessment metric, the Neutrosophic Similarity Measure, and assesses its performance in evaluating watermarked images when compared to PSNR. Similarities to ascertain whether the neutrosophic similarity Measure has a higher noise tolerance and offers a more accurate evaluation of watermarked images. The results show that Neutrosophic Similarity Measure overcomes PSNR in capturing the influence of additive watermarks and demonstrating superior noise tolerance through experimental evaluation on a dataset of watermarked images. These findings highlight the possibility of adopting new assessment metric, such as neutrosophic similarity measure, for assessing watermarked images, thereby enhancing the effectiveness of evaluating watermarked Images.

Keywords: Digital Image Assessment; Watermark; Neutrosophic Similarity Measure, PSNR.

1. Introduction

Digital watermarking is a commonly used method for adding undetectable data—also referred to as watermarks—to digital assets including images, sounds, and video. In addition to copyright protection, these watermarks also verify data integrity and authenticate material. For determining the efficacy of watermarking algorithms and guaranteeing the preservation of imagine fidelity, the ability to reliably assess the quality of watermarked images is essential [1].

Peak Signal-to-Noise Ratio (PSNR) has been used extensively as a metric to assess the quality of watermarked images. By evaluating the ratio of peak signal strength to mean square error, PSNR evaluates the distinction between original and watermarked images. However, PSNR has certain limitations [2], it does not consider the perceptual impact of noise or distortion introduced by watermarks, and its effectiveness diminishes in scenarios involving additive noise.

To overcome these limitations, an alternative assessment metric which is neutrosophic similarity measure [3]has utilized in this paper. The utilization of neutrosophic similarity measure (NSM) as an assessment metric offers several advantages. It enables a more comprehensive analysis of watermarked image quality by considering the perceptual aspects, indeterminacy, and ambiguity. By capturing the impact of additive watermarks more effectively, neutrosophic similarity Measure can provide a better assessment of the overall fidelity and visual quality of watermarked images.

Considering the importance of digital watermarking and the limitations in its evaluation using traditional metrics like PSNR, exploring alternative assessment metrics such as neutrosophic similarity measure becomes imperative. This paper aims to compare the performance of PSNR and neutrosophic similarity Measure in evaluating watermarked images and ascertain the advantages of adopting a more robust assessment metric for accurate and reliable quality assessment.

2. Materials and Methods

2.1 Theoretical Background

In this section, a brief description of digital image processing and neutrosophic systems is presented.

2.1.1 Description of Digital Images

A digital image can be described as a two-dimensional function, $f(x, y)$, where x and y represent spatial coordinates, and the intensity or gray level of the image at any given (x, y) coordinate is determined by the value of f [4]. When both x, y, and the intensity values of f are discrete and finite, the image is referred to as a digital image. Digital image processing involves manipulating digital images using a computer. It's important to note that a digital image consists of a finite number of elements, each with a specific location and value. These elements are commonly referred to as picture elements, image elements, pels, or pixels. The term "pixel" is widely used to describe the elements of a digital image.

In the early days of the newspaper industry, digital images found one of their initial uses in transmitting pictures between London and New York via submarine cables. The introduction of the Bartlane cable picture transmission system during the early 1920s significantly decreased the time needed to transport a picture across the Atlantic Ocean, reducing it from over a week to under three hours. This system involved specific printing equipment that encoded pictures for transmission through the cables and reconstructed them upon reaching the receiving destination.

The image in Figure 1, created in 1921, was generated from a coded tape using a telegraph printer equipped with a unique typeface [5].

Figure 1. Telegrapher Printer Image in 1921

2.1.2 Digital Image Representation

A digital image serves as a numerical representation of a real image that can be stored and processed by a digital computer. The process begins by dividing the image into small areas known as pixels or picture elements. Each pixel corresponds to a specific location within the image and is associated with a numerical value or a set of numbers that describe certain properties of the pixel, such as its brightness or color. These numerical values are organized in an array format, with rows and columns representing the vertical and horizontal positions of the pixels in the image.

Digital images possess several fundamental characteristics. One important aspect is the image type, which can vary. For instance, a black and white image records only the intensity of light falling on the pixels. Color images, on the other hand, can consist of three colors (typically RGB - Red, Green, Blue) or four colors (CMYK - Cyan, Magenta, Yellow, black). RGB images are commonly used in computer monitors and scanners, while CMYK images are utilized in color printers. There are also non-optical images, like ultrasound or X-ray, where the intensity of sound or X-rays is recorded. In range images, the distance of each pixel from the observer is captured.

Resolution is another key characteristic of digital images and is measured in pixels per inch (PPI). Higher resolution results in a more detailed image. Computer monitors generally have a resolution of around 100 PPI, while printers have resolutions ranging from 300 PPI to over 1440 PPI. Consequently, images tend to appear better in print due to the higher resolution compared to a monitor [6]. The color depth, applicable to color images, refers to the number of bits used to represent the brightness or color information. More bits allow for a greater range of shades of gray or colors. For example, an RGB image with 8 bits per color has a total of 24 bits per pixel, commonly referred to as "true color." Each bit can represent two possible colors, resulting in a total of 16,777,216 possible colors. The grayscale image is represented by brightness using 8 bits value. The brightness of a pixel value of a grayscale image ranges from 0 (black) to 255 (white) [7]. Binary images typically have only one bit or two "colors," representing black and white (Figure 2).

The format of an image provides additional details on how the numerical values are arranged within the image file, including information about compression techniques employed, if any. Various formats are available, with popular ones including BMP (is a format native to the Windows operating system, JPEG (recognized for lossy compressing and encoding high-resolution digital images), PNG (images with lossless compression), and GIF (animated images) [8].

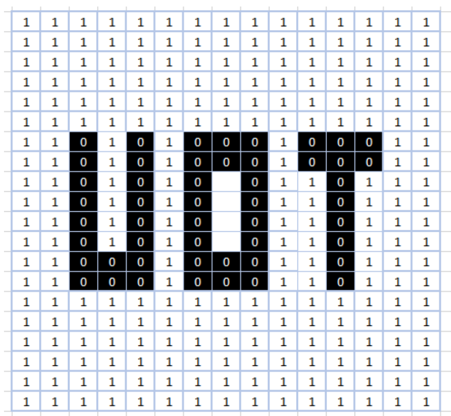

Figure 2. Binary Image

2.1.3 Digital Image Watermarking:

Image watermarking is the process of embedding a watermark signal (as a text or small binary image) into the cover image, which is the target that needs to be protected/tracked. Image watermarking can be considered as the basis for video watermarking as the video is a set of consecutive frames, where each frame can be considered as a separate image. A digital watermarking system consists of two main steps: embedding and extraction. In embedding, the watermark is embedded inside the host image, while in extraction, the watermark is retrieved from the host image. If the process of retrieving can be applied without the existing of the original image, then it is "blind Extraction", and if the host image is required for extraction, then it is non-blind extraction. Figures 3 shows the process of watermark embedding [1], [9]:

Generally, the watermarking process consists of the following major components.

- **Host** (Original) image: The target of the watermarking system that needs to be watermarked.
- **Watermark**: Information to be embedded, which might be the company logo, metadata, etc.
- **Key**: The encryption key that is used to encrypt the watermark before embedding to apply more security. The existence of the key is optional.
- **Watermarked Image**: Image that implicitly contains the watermark.

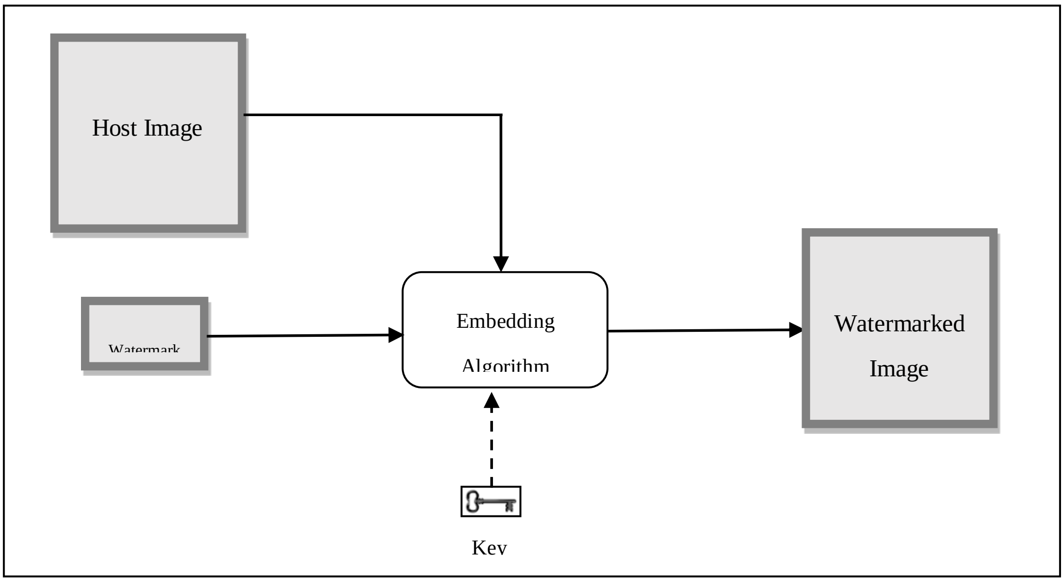

Figure 3. Watermark Embedding Process

2.1.4 Neutrosophic Sets

Neutrosophic sets, introduced by Smarandache [3], provide a novel approach for addressing uncertainty by incorporating truth-membership (T), indeterminacy-membership (I), and falsity-membership (F) values within the range of $0 \leq T + I + F \leq 3$. Compared to intuitionistic fuzzy sets, these values provide a more thorough and precise description of ambiguous information. The idea of neutrosophic sets has received a lot of attention from researchers and has been expanded into a number of different fields. These extensions have found use in decision-making, information measures, image processing, graph theory, and algebraic structures. Neutrosophic sets rapidly became a tool for handling vagueness in a variety of real-life scenarios [10].

Numerous studies highlight the neutrosophic sets' quick development and adaptability, which enable quantitative and qualitative analyses from a variety of angles[11] [12]. The field of image processing has benefited greatly from the use of neutrosophic theory, particularly in the areas of edge detection and image segmentation. Neutrosophic offsets have been used to segment images successfully, offering a solid framework to deal with the ambiguity and uncertainty that come with image analysis. Neutrosophic offsets (when some neutrosophic components are off the interval [0, 1], i.e., some neutrosophic component > 1 and some neutrosophic component < 0 [10]) enable a thorough characterization of image regions and boundaries by considering truth-membership, indeterminacy-membership, and falsity-membership values. Additionally, edge detection applications have been shown promise when using neutrosophic theory. The flexibility of these forms provides a method for capturing minute changes and transitions in edge information, improving the precision and dependability of edge detection algorithms [13].

Neurotrophic Similarity Measure: The neutrosophic similarity measure is a metric used within the neutrosophic framework to quantify the similarity between two neutrosophic sets or objects. Neutrosophic similarity measures consider the truth-membership, indeterminacy-membership, and falsity-membership values associated with the objects being compared. Similarity measure is calculated to identify the degree to the ideal object under intensity condition.

Neutrosophic similarity measure (NSM) calculation steps [14]**:**

1. Normalize the images to the range [0,1].
2. Calculate the positive, neutral, and negative memberships.
3. Calculate the numerator and denominator of the NSM.

$$Numerator = sum\ (sum\ (min\ (a_1, a_2) + min\ (ap_1, ap_2) + min\ (an_1, an_2))) \tag{1}$$

$$Denominator = sum\ (sum\ (max\ (a_1, a_2) + max\ (ap_1, ap_2) + max\ (an_1, an_2))) \tag{2}$$

Where a_1 is the host image, a_2 is the watermarked image; ap_1 is the positive membership of a_1, ap_2, is the negative membership of a_2. While an_1, $an_2$are the negative membership of a_1 and a_2, respectively.

4. Calculate the NSM between the two images.

$$NSM = Nominator\ /\ Denominator \tag{3}$$

2.1.5 Image Quality Assessment:

For assessing the quality of digital images, there are two available methods. The initial method uses judgment from humans and is known as subjective assessment. Human observations, however, can differ greatly between people due to perception differences. To get a range of opinions, this calls for involving multiple subjects. However, it can be inconvenient, time-consuming, and expensive to conduct subjective experiments. Hence, it is not usually employed.

On the other hand, objective assessment offers an alternative strategy for computing-based image quality evaluation. In the literature, a variety of objective metrics have been placed out to evaluate the quality of images that have undergone compression, transformation, or other image processing operations. A single metric could not be able to adequately address all types of distortions, so it's important to note that different distortion types may call for the use of multiple metrics[15].

For watermarked images, Peak signal to noise ratio ($PSNR$) is a common metric that is used in literature studies for watermarked image assessment [16][17], and its equation is based on calculating the mean square error as shown in the following equations:

$$\mathrm{PSNR} = 20\ \log_{10}\left(\frac{255}{\sqrt{\mathrm{MSE}}}\right) \tag{4}$$

And

$$\mathrm{MSE} = \frac{1}{\mathrm{m} \times \mathrm{n}} + \sum_{\mathrm{i}=1}^{\mathrm{m}} \sum_{\mathrm{j}=1}^{\mathrm{n}} \| \ \mathrm{X(i,j)} - \ \mathrm{Y} \ \mathrm{(i,j)} \ \| \ ^{2} \tag{5}$$

Where X and Y represent the original and altered images, respectively, with dimensions m and n. The indices i and j are used to denote individual pixels within the images.
However, the bias of Human Vision System (HVS) in observing the noise in different image structures is not considered by $PSNR$ [18] .
Hence, in this paper, neutrosophic similarity measure (NSM) will be utilized to assess the quality of watermarked images and to be compared with $PSNR$.

2.2 Literature Studies

Numerous studies and comparisons of various measures for assessing the quality of watermarked images have been conducted in literature. A comprehensive review of a number of quality metrics was carried out in study [15] to determine the best metric for assessing watermarked images. The performance of the metrics in evaluating the quality of watermarked images was examined and contrasted. The metrics "PSNR_wav2" and "Komparator" were discovered to be most relevant and useful for assessing the overall quality of watermarked images out of the many that were evaluated.
In another notable research by Kutter et.al [19] focus was on addressing the challenges associated with fair benchmarking and the evaluation of digital watermarking methods. This study not only aimed to identify suitable evaluation metrics but also proposed a novel metric specifically designed for the evaluation of watermarked images. The proposed metric aimed to provide a more comprehensive and accurate assessment of the visual quality of watermarked images, considering factors such as robustness, perceptual transparency, and resistance to attacks.
Additionally, in another experiment [20], in which an image quality metric based on singular value decomposition (SVD) was used to improve the evaluation of watermarked image visual quality. Several watermarking methods' performance was evaluated using the SVD-based metric. In order to evaluate the visual quality of watermarked images, fidelity, distortion, and robustness against typical image processing operations were taken into account.
These literature examples highlight the ongoing research efforts to improve the evaluation of watermarked images through the exploration, comparison, and refinement of various evaluation metrics. By identifying and utilizing suitable metrics, researchers aim to enhance the accuracy, reliability, and effectiveness of evaluating the quality and performance of watermarked images in different applications and scenarios.
E. F-Navarro et al. [21] proposed a set of assessment metrics for visible watermarking algorithms. These metrics consist of four components: visibility assessment, global obtrusiveness assessment, local obtrusiveness assessment, and global quality assessment. They are based on the characteristics of the Human Visual System (HVS) and utilize the concept of Just Noticeable Difference (JND) functions (JND is the maximum sensory distortion that human eye does not percieve [22]) . The mentioned metrics require the input of the host image, watermark pattern, and visible watermarked image for evaluation, and the existing of watermarking pattern and the original watermark may not be possible in all cases. These image evalaution metrics were found to be particularly useful in

evaluating the robustness of watermark removal and assessing the visibility and quality of attacked watermarked images.

To the best of our knowledge, the utilization of the neutrosophic framework with its ambiguity and uncertainty in the creation of a metric for digital image evaluation, specifically for watermarked images, has not been explored in literature and its usage may lead to promising results for evaluating watermarked images and it also can participate in developing watermarking algorithms.

2.3 Proposed Work

The methodology employed in this paper involves two main stages: watermark embedding and the assessment of the watermarked images. In the first stage, the binary watermark, as illustrated in Figure 4, was incorporated into ten standard images of size 512×512 pixels. Host image thumbnails, shown in Figure 5, provide a visual representation of the chosen images. To ensure consistency across different image sizes, the watermark was embedded four times in each host image. The binary watermark consists of pixels with binary values of either zero or one, representing the black and white colors, respectively. To prevent issues arising from multiplication by zero during the embedding process, each zero value in the watermark was transformed to -1, which was then multiplied by the embedding power (ep). The selection of the ep value directly influences the watermark's strength, enabling the analysis of various distortion levels. By varying the ep value, a range of distortion scenarios can be examined, providing valuable insights into the watermark's robustness under different conditions.

Figure 4: Binary Watermark

Figure 5: Tested Host Images (Numbered as I1-I10, starting from top left)

The watermark was embedded into the images at five different intensities: $2, 4, 6, 8$, and 10. This range of embedding intensities allowed for a comprehensive assessment of the watermark's performance under varying degrees of strength. For each embedded image, two evaluation metrics were used: the Peak Signal-to-Noise Ratio ($PSNR$) and the Neutrosophic Similarity Measure (NSM). The $PSNR$, a commonly used objective metric, quantifies the quality of the watermarked image by measuring the ratio of the peak signal power to the distortion caused by the watermark embedding process. Higher $PSNR$ values indicate better preservation of image quality, with less distortion introduced by the watermark.

NSM was employed as a new metric specifically designed for assessing the quality and similarity of watermarked images within the neutrosophic framework. The NSM takes into account the truth-membership, indeterminacy-membership, and falsity-membership values. Figure 6 shows the process of watermarked image evaluation.

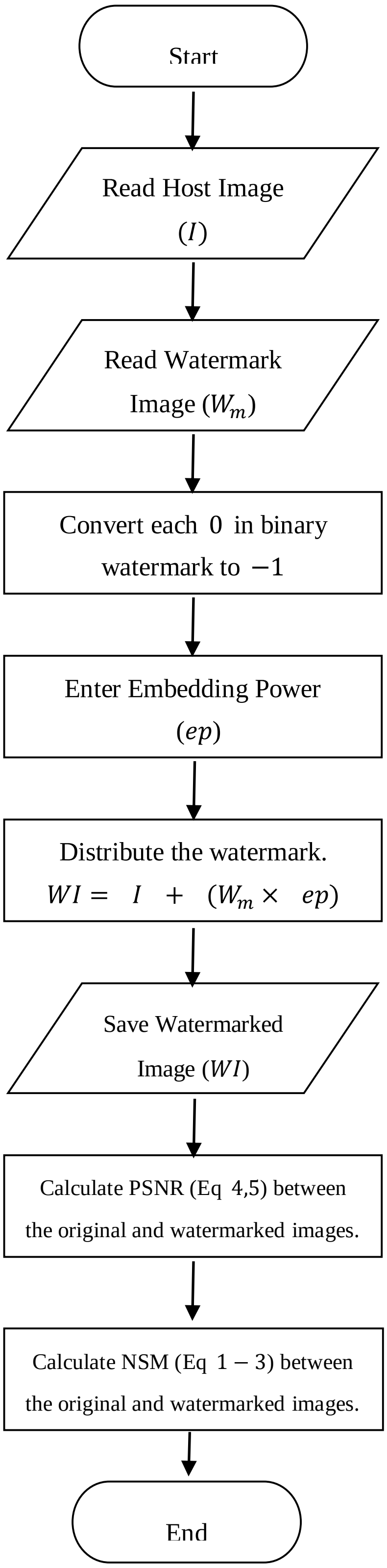

Figure 6: Process of Watermarked Image Evaluation

3. Results

In this section the results of embedding will be depicted, in addition to the results of $PSNR$ and NSM metrics. A comparison between the two metrics is presented at the end of the section.

3.1. Watermark Embedding Results

Watermark had been embedded four times in each host image. Figure 7 shows the results after embedding in different intensities.

Figure 7: Watermarked Images with Different Embedding Intensities (**Cont**.)

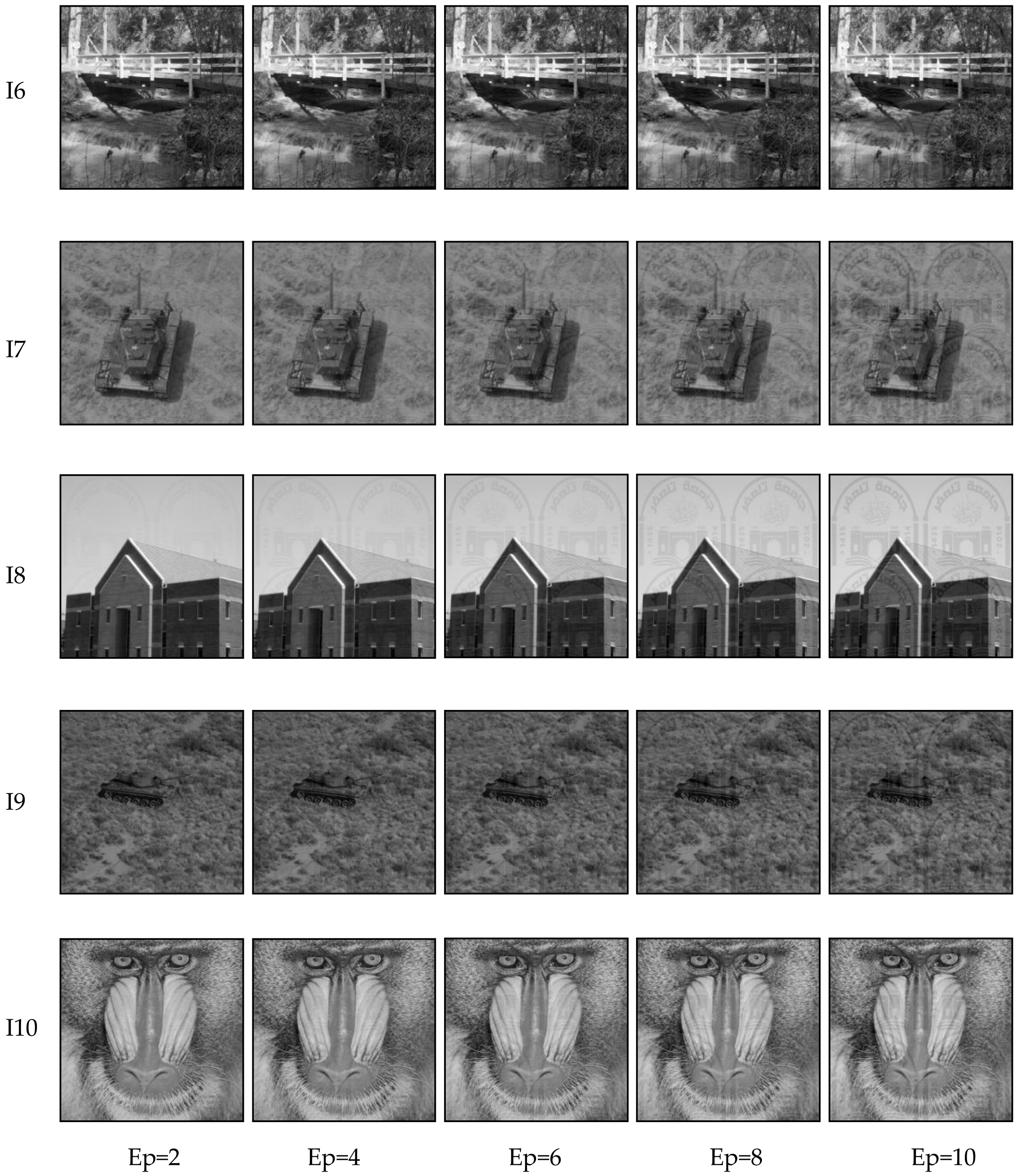

Figure 7: Watermarked Images with Different Embedding Intensities

Figure 7 makes clear that even at the same embedding intensities, the perceptibility of the watermark varies between the tested images. In Image I8, which has a lot of large smooth areas like the sky, it is important to note that even with an embedding intensity of 2, the watermark is still visible. In contrast, the watermark is less noticeable at intensities 4 and 6 in images with more complex content, such $I7$ and $I10$. This variation in watermark visibility can be related to the images' various characteristics and texture amount and distribution. Extensive smooth areas might make a watermark more noticeable, whereas complex textures and details can partially hide its appearance. These observations highlight the importance of considering image content and structure when assessing the perceptibility of watermarks at different intensities. However, there are other factors are involved in visual quality, as high intensities and low intensities, and the textures are appeared more visible to human eye when it positioned in the edges [16].

3.2. $PSNR$ and NSM Values:

Tables $1-5$ present the results of the evaluation conducted on the tested images, providing the values of both Peak Signal-to-Noise Ratio ($PSNR$) and Neutrosophic Similarity Measure (NSM) for various embedding intensities. Each table corresponds to a specific embedding intensity, namely $2,\ 4, 6, 8,$ and 10. The $PSNR$ values indicate the level of signal degradation caused by the watermark embedding process, with higher values indicating better image quality. On the other hand, the NSM values reflect the similarity between the watermarked images and their corresponding original counterparts, with higher values indicating a stronger resemblance. By examining these tables, it is possible to analyze the impact of different embedding intensities on both the signal quality and the similarity measure, providing valuable insights into the performance of the watermarking algorithm under different settings

Table 1. PSNR and NSM Values for Ep=2

Image	PSNR	NSM
I1	42.1129	0.98704
I2	42.1126	0.98706
I3	42.1102	0.98713
I4	42.1342	0.98692
I5	42.1102	0.98705
I6	42.1271	0.98695
I7	42.1102	**0.98789**
I8	42.1193	**0.9866**
I9	42.1102	0.98731
I10	42.1109	**0.98749**

Table 2. PSNR and NSM Values for Ep=4

Image	PSNR	NSM
I1	36.0934	0.97429
I2	36.0968	0.97436
I3	36.0899	0.97446
I4	36.1181	0.97405
I5	36.0896	0.9743
I6	36.1065	0.97413
I7	36.0896	**0.97621**
I8	36.1000	0.97339
I9	36.0896	**0.97484**
I10	36.0906	**0.97521**

Table 3. PSNR and NSM Values for Ep=6

Image	PSNR	NSM
I1	32.5731	0.96174
I2	32.5778	0.96189
I3	32.5687	0.96199
I4	32.5989	0.96139
I5	32.5678	0.96177
I6	32.5856	0.96152
I7	32.5678	**0.96486**
I8	32.5794	0.96037
I9	32.5678	**0.96257**
I10	32.5691	**0.96317**

Table 4. PSNR and NSM Values for Ep=8

Image	PSNR	NSM
I1	30.0770	0.9494
I2	30.0807	0.94961
I3	30.0716	0.94972
I4	30.1018	0.94893
I5	30.0690	0.94943
I6	30.0880	0.94909
I7	30.0690	**0.9538**
I8	30.0822	0.94753
I9	30.0690	**0.9505**
I10	30.0706	**0.95134**

Table 5. PSNR and NSM Values for Ep=10

Image	PSNR	NSM
I1	28.1530	0.93733
I2	28.1438	0.93754
I3	28.1373	0.93765
I4	28.1652	0.93666
I5	28.1308	0.93729
I6	28.1510	0.93687
I7	28.1308	**0.94295**
I8	28.1465	**0.93487**
I9	28.1314	0.93864
I10	28.1326	**0.93972**

The analysis of the obtained results from Tables $1-5$ reveals several key observations. Firstly, the $PSNR$ values exhibit variations ranging from $42dB$ to $28dB$ across the different embedding power levels of 2 to 10. It is worth noting that the $PSNR$ values remain relatively consistent among all the tested images, suggesting a consistent level of signal degradation caused by the watermark embedding process. On the other hand, the NSM values exhibit a better behavior. Despite minor variations and a relatively limited range between 0.98 and 0.93, the NSM values demonstrate higher perceptual quality in all embedding intensities for images with higher texture, such as $I7, I9$, and $I10$. Conversely, images like $I8$, characterized by smoother features and lower texture, consistently yield lower NSM values compared to the other images.

The values of NSM are normalized between $0-1$, while $PSNR$ values are results of logarithmic equations where changes can have more impact on the obtained results. Hence, the changes in embedding intensity have higher impact in $PSNR$ than NSM.

However, the NSM values require scaling to accurately reflect the observed changes, they can serve as a valuable assessment measurement for evaluating the quality of watermarked images. These findings suggest that the NSM metric is sensitive to the perceptual characteristics of the images and can provide insights into the effectiveness of the watermarking algorithm in preserving image quality and similarity to the original content.

Figure 8 shows the variation of changes in low textured image $I1$ and High textured image $I7$ for $PSNR$. And NSM for the same images is shown in Figure 9. Similar results will be obtained by using $I8$ with $I10$.

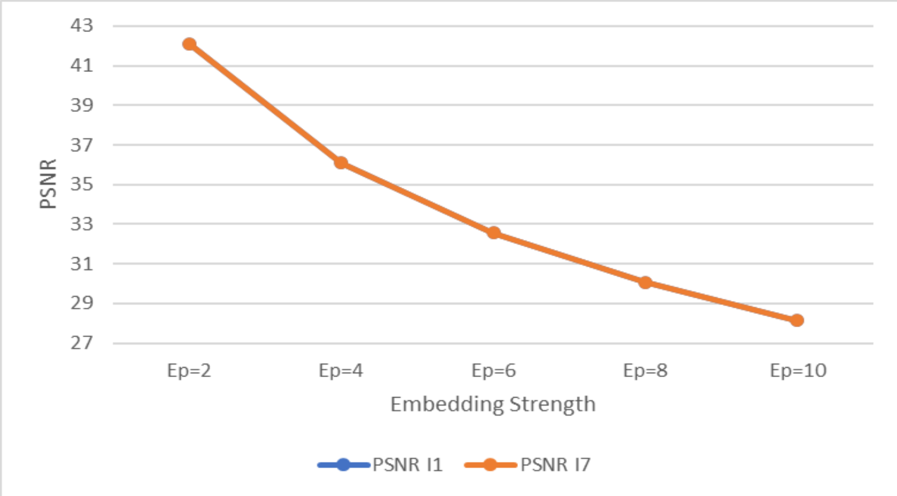

Figure 8: $PSNR$ values for low textured image $I1$ and high Textured Image $I7$

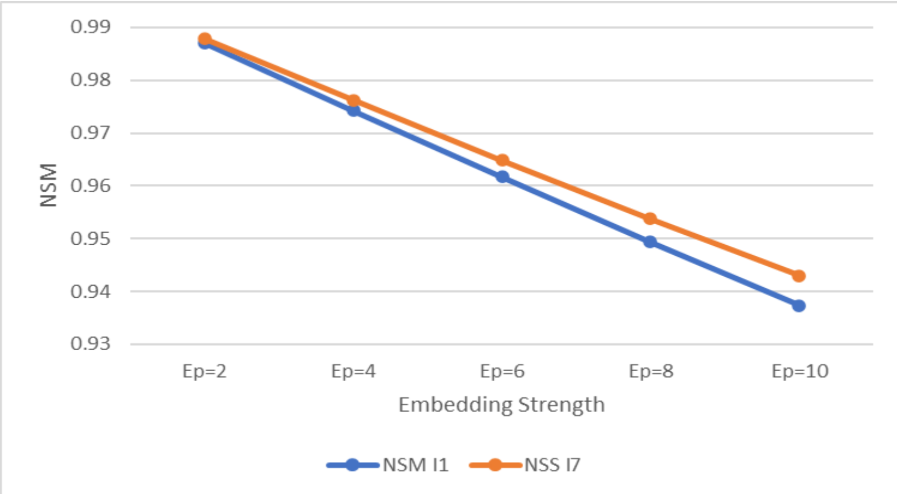

Figure 9: NSM values for low textured image $I1$ and high Textured Image $I7$

The comparative analysis between the $PSNR$ and NSM metrics reveals notable distinctions in their performance. It is observed that the $PSNR$ curves for both images are nearly identical, with one curve being consistently displayed above the other on the graph. This implies that the $PSNR$ metric assigns similar values to both images, irrespective of the texture or structure of the image. In contrast, the NSM metric demonstrates a different behavior, where an increase in the amount of noise (embedding strength) leads to greater dissimilarity between the two images. Notably, the NSM metric exhibits a preference for high-textured images, as they are less affected by noise and consequently yield higher NSM values. This highlights the superiority of the NSM metric over $PSNR$ in simulating the sensitivity of the Human Visual System (HVS) to noise. By capturing the perceptual aspects and incorporating image texture information, the NSM metric offers a more comprehensive and accurate evaluation of image quality, surpassing the limitations of $PSNR$.

4. Applications

These are some applications that highlight the practical implications of the study's findings in image processing field:

- Image Quality Assessment: The NSM metric can be utilized as a perceptual quality assessment tool for image processing algorithms, including watermarking techniques.

It can help determine the effectiveness of different watermark embedding strengths in preserving image quality.

- Watermarking Algorithm Optimization: By analyzing the performance of different embedding intensities and their corresponding NSM values, this study can aid in optimizing watermarking algorithms to achieve the best balance between robustness and perceptual quality.
- Creating a new Just Noticeable Distortion (JND) model to simulate human vision system in perceiving noise.
- Image Authentication and Forensics: The comparative analysis between $PSNR$ and NSM metrics provides insights into the sensitivity of watermarking algorithms to noise and image texture. This information can be applied to image authentication and forensic investigations to assess the integrity and authenticity of watermarked images.
- Content Protection and Copyright Verification: Watermarking is often used for copyright protection and content verification purposes. The findings of this study can contribute to the selection of appropriate watermark embedding strengths, ensuring optimal protection of intellectual property while maintaining acceptable visual quality.
- NSM can be combined with other image evaluation metrics as structural Similarity Index ($SSIM$) [23] to achieve better evaluation results.

5. Conclusion

This study has explored the limitations of the widely used Peak Signal-to-Noise Ratio ($PSNR$) metric in evaluating watermarked images and has introduced the Neutrosophic Similarity Measure (NSM) as an alternative assessment metric. The experimental evaluation conducted on a dataset of watermarked images has demonstrated that NSM surpasses $PSNR$ in capturing the influence of additive watermarks and exhibits superior noise tolerance. This was achieved because NSM values exhibited a better behavior, with minor variations and higher perceptual quality for images with higher texture. the study's findings underscore the importance of utilizing accurate assessment criteria for watermarked images, and the Neutrosophic Similarity Measure has demonstrated its potential to address the limitations of traditional metrics like $PSNR$, thereby advancing the field of digital watermarking. Future research can enhance watermarking algorithms by further exploring the impact of utilizing the NSM to find the best watermark embedding intensities.

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Neutrosophic cross entropy-based discrimination of antioxidant potential and polyphenolic contents of *Arisaema tortuosum*

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Abstract: The study presents the utilization of an innovative neutrosophic cross entropy-based technique for the discrimination of the antioxidant potential of *Arisaema tortuosum* leaf extract based on polyphenolic content. The effect of three extraction techniques, namely, Soxhlet extraction, ultrasound-assisted extraction, and maceration, have been analyzed on the percentage yield of the extract. The effect of solvents, namely, methanol, chloroform and hexane, on the percentage yield has also been explored. The leaf extract was found to exhibit significant antioxidant potential and polyphenolic content. Substantial discrimination was observed among antioxidant potential and polyphenolic contents with minimum neutrosophic cross entropy values designated to the extant higher amount of polyphenols in *A. tortuosum* leaf extract. The proposed neutrosophic cross entropy-based technique is beneficial for further mathematical treatments because of its symmetric nature in comparison with the existing methods, which may indicate vagueness in the evaluation information under certain situations and thus affects the prognosis analysis.

Keywords: Cross entropy, Fuzzy Sets, Neutrosophic Sets, *Arisaema tortuosum*, TPC, antioxidant potential.

1. Introduction

The neutrosophic sets (NSs) proposed by Smarandache have played an intelligent role in dealing with real-world problems containing imprecise, inconsistent and indeterminate information [1]. One of the characteristics of NSs is that they include a non-standard unit interval that includes membership gradations such as "true," "indeterminate," and "false" [2]. Since the indeterminacy inherited in the NSs depend upon the "true" and "false" values, the neutrosophic cross entropy measures (NCEMs) can handle real-world problems with imprecise, inconsistent and incomplete information [1]. Ishtiaq et al. (2021) initiated the ideas of orthogonal NMSs, investigated many fixed points results, and validated their findings by providing some non-trivial counterexamples [3]. Uddin et al. (2021) established the concepts of orthogonal controlled fuzzy metric-like spaces to establish some fixed point theorems and validated the main findings [4]. Also, the authors provided some counterexamples and an application to the fuzzy Fredholm IE of the second kind. Javed et al. (2021) idealized the ideas of fuzzy b-metric like spaces to prove some interesting fixed point theorems and validated their findings by applying the fuzzy Fredholm IE of the first kind [5]. Ishtiaq, Hussain, and Al Sulami (2022) introduced the concept of fuzzy rectangular metric-like spaces and proved some exciting results, combined with single and multi-valued mappings, of fixed point theory [6].

Ali et al. (2022) provided several unique solutions to non-linear fractional differential equations for weekly compatible and contractive functions under the environment of neutrosophic metric spaces (NMSs) [7]. Hussain, Al Sulami, and Ishtiaq (2022) introduced the concept of intuitionistic fuzzy rectangle metric and b-metric spaces as well as neutrosophic rectangular metric and b-metric spaces to prove some important fixed point results along with an application to the Fredholm Integral equation (IE) of the second kind [8]. Hussain et al. (2022) also confirmed some Banach fixed point results by generalizing the concept of pentagonal controlled fuzzy and fuzzy extended hexagonal metric spaces [9]. The authors provided some counterexamples in support of their findings and also gave an application to dynamic marketing. Farheen et al. (2022) gave authentic proof of the Banach fixed point theorem under intuitionistic fuzzy double-controlled metric spaces. To support their findings, the authors provided an application to the fuzzy Fredholm IE of the second kind [10]. Saleem et al. (2022) idealized the ideas of graphical fuzzy metric-like spaces and proved the Banach fixed point theorem. To validate their outcomes, the authors solved a non-linear fractional differential equation in the context of graphical fuzzy metric spaces [11]. Unfortunately, the existing literature on NCEMs mainly covers asymmetrical aspects ignoring the symmetrical and undefined parameters [12]. To overcome these shortcomings and limitations and to handle complex real-life problems with ambiguity and vagueness, it is necessary to develop an efficient methodology that can accurately discriminate the desired parameters.

Since ancient times, medicinal plants have been explored to heal various diseases because of the availability of vital volatile organic compounds or phytochemicals exhibiting medicinal properties [13]. The biomedical potential of any plant is explored based on phytochemical analysis of the extracts of its various parts, especially in terms of antioxidant potential [14]. Discrimination of antioxidant potentials and polyphenolic contents of bioactive plant extract with medicinal and therapeutic potential needs to be explored in scientific interest [15]. Castellano et al.

(2013) have determined the number of classes and classification levels for the flavonoids using information entropy [16]. However, there is no study available in the literature on the discrimination of the medicinal potential with the phytochemical constituents of medicinal plants based on fuzzy sets (FSs) theory. This observation reinforces the necessity to develop a superior and intelligent methodology that can optimize the phytochemical extraction of medicinal plants.

Arisaema tortuosum is a medicinal herb with multiple therapeutic uses because of the availability of volatile organic compounds, including flavonoids, terpenoids, polyphenols etc. [17]. So far, no neutrosophic cross entropy base methodology has been established and applied for studying the phytochemical analysis of *A. tortuosum*. The conventional Information Theory approaches based upon the theory of fuzzy sets (FSs) and neutrosophic sets (NSs) can be used in discriminating antioxidant potentials and polyphenolic contents of the extract obtained from aerial parts of bioactive plants. In the current study, a classy trigonometric neutrosophic cross entropy measure (NCEM) hinging on the two single-valued neutrosophic sets (SVNSs) is proposed and applied to discriminate the polyphenolic contents and antioxidant potential in the extracts obtained from the aerial parts of *A. tortuosum*. The subsequent development of the proposed work has been arranged in Figure 1 and described ahead.

Section 2 discusses the materials and methodology deployed for obtaining the extract and assessing the antioxidant potential and polyphenolic contents. The antioxidant potential has been analyzed in the following terms:

1. 1,1-diphenyl-2-picryl-hydrazyl (DPPH) assay
2. 2,2′-azino-bis (3-ethylbenzothiazoline-6-sulphonic acid) (ABTS) radical scavenging assay
3. Ferric reducing antioxidant power (FRAP) assay

The polyphenolic content has been obtained in the following terms:

1. Total polyphenolic content (TPC)
2. Total flavonoid content (TFC)

The section also provides information about establishing the proposed cross-entropy measures. Section 3 provides results for the experimental and numerical analysis to provide the discrimination among antioxidant potential and polyphenolic contents, with the findings summarized in section 4.

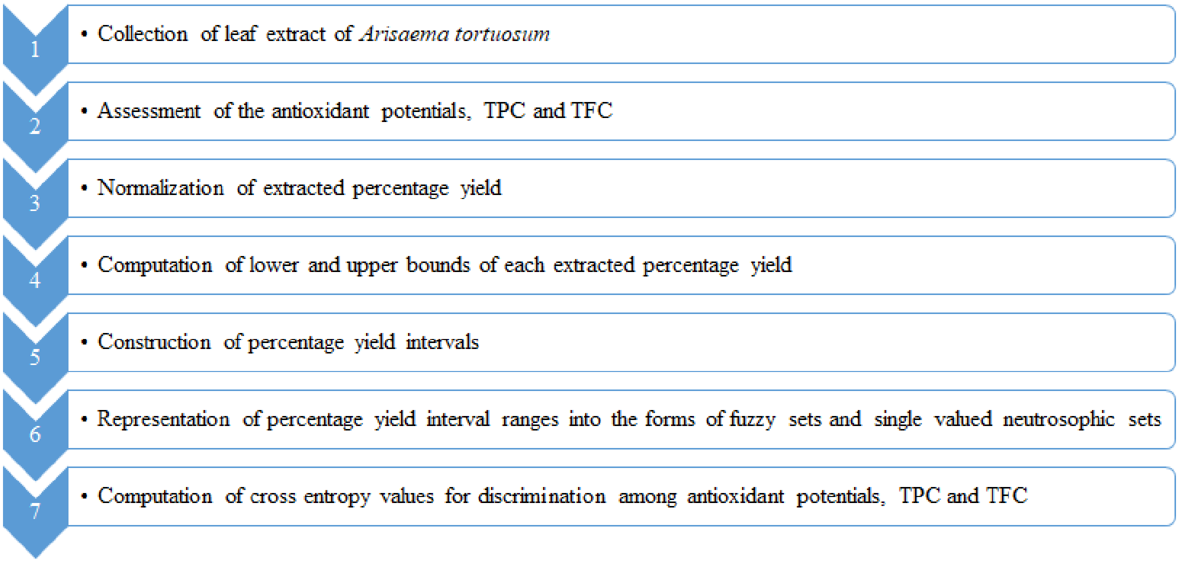


Figure 1. A detailed schematic flow chart of the underlying methodology

2. Materials and Methods

2.1. Materials

The plant was collected from the Himalayan region of Dharamshala (Naddi), Himachal Pradesh. Young leaves of *Arisaema tortuosum* were selected and washed with regular and double distilled water to expel the residue and dust particles over their surface. After draining free water, the obtained material was dried in a shady region at room temperature for 30 days and ground to get the powdered form. The various chemicals utilized in the present work were procured from Merck Ltd. Mumbai and used as such. Double distilled water was used to prepare all the formulations.

2.2. Leaf extract preparation

Three solvents of different polarities, chloroform, methanol and hexane, were used to extract the sample in the powdered form. The plant material was extracted in triplicate for 30 minutes with 100 mL of solvent using Soxhlet extraction (SE) [18], Ultrasound aided extraction (UAEM) [19], and Maceration extraction (ME) [20]. Through a rotary vacuum evaporator at 45 ^{o}C, the obtained extract was evaporated to dryness and then preserved for further investigations at a low temperature.

2.3. Determination of polyphenolic content

The Folin-Ciocalteu technique was deployed to measure the TPC. 20 µL of leaf extract (5 mg/mL) in DMSO (25% v/v) was amalgamated to diluted Folin–Ciocalteu reagent (100 µL) with agitation for 1 minute. 75 µL of Na_2CO_3 solution (100 mg/mL) was added and again agitated for 1 minute. The mixture was left for 2 hours, and the absorbance of the coloured solution was monitored at 750 nm through a UV-Vis spectrophotometer (Agilent Cary-60) with gallic acid as a calibration standard. TPC was estimated as the mg equivalents of gallic acid / g of leaf extract (mg GAE/g).

The aluminum chloride technique was deployed to access the TFC of the extracts. 20 μL of leaf extract (5 mg/mL) in DMSO (25% v/v) was added to 10 μL of $AlCl_3$ (10%) and 10 μL of CH_3COOK (1 M). The mixture was diluted with double distilled water to achieve a final volume of 200 μL and left for 30 minutes. The absorbance of the solution was monitored at 415 nm using quercetin as the reference. TFC was estimated as mg quercetin equivalents/ g of leaf extract (mg QE / g).

2.4. Determination of antioxidant potential

Standard methods were used to investigate the antioxidant potential of the extracts quantitatively. 3 ml of the freshly available DPPH solution (0.1 mM) was well mixed with 0.2 ml of extract (10-100 μg/mL) and incubated for thirty minutes in the dark. The absorbance was monitored at 517 nm. The scavenging impact of the extracts against DPPH free radicals was determined by employing Ascorbic acid as a standard.

3 mL of FRAP solution was mixed with leaf extract (10-100 μg/mL) and then incubated for 30 min. The absorbance of the obtained solution was measured at 593 nm. The ferrous sulphate was utilized as standard, and FRAP was expressed as ferrous II equivalents in mg per g of the leaf extract (mg Fe (II) /g).

180 μL of ABTS solution was added to 20 μL of leaf extract (10-100 μg/mL) and incubated for 30 min. The absorbance of the obtained solution was monitored at 734 nm using ascorbic acid as standard. The ABTS scavenging potential was determined in terms of ABTS radical scavenging %.

2.5. Neutrosophic cross entropy measure

2.5.1. A symmetric fuzzy cross entropy measure

The cross-entropy information measures in the reported literature face a major drawback because of their asymmetrical nature. They return undefined or meaningless when their membership functions conceive zero value in some mathematical treatments. The following **Theorem 1** overcomes the shortcomings mentioned above and its limitations.

Theorem 1 Set $l_1 = {}_A\mu(x_i)(1 - {}_A\mu(x_i))$. Let $A = (\prec x_i, {}_A\mu(x_i) \succ \forall x_i \in X)$ be any FS belonging to $\mathrm{X} = (x_1, x_2, x_3, .., x_n)$. Then

$$H_{\mathrm{FS}}^{\mu}(\mathrm{A}) = \sum_{i=1}^{n}\left[\tan\left(\frac{3}{2}\right) - \tan\left(\frac{3}{2+2\sqrt{l_1}}\right)\right] \quad (1)$$

is a reliable measure (**Def. 1**) where $H_{\mathrm{FS}}^{\mu}(\mathrm{A})$ indicates the mathematical fuzziness value of the FS *A*. Moreover, its minimum value is zero and $\mathrm{Max}.H_{\mathrm{FS}}^{\mu}(\mathrm{A}) = \left(\tan\left(\frac{3}{2}\right) - \tan(1)\right)n,$ where the cardinality of the FS A is represented by n.

Proof (i) Clearly $H^{\mu}_{\mathrm{FS}}(\mathrm{A})\geq 0\,\forall A\in X$ with equality if ${}_{A}\mu(x_i)=0$ or 1.

(ii) $H^{\mu}_{\mathrm{FS}}(\mathrm{A})$ doesn't change even if ${}_{A}\mu(x_i)$ is replaced with $1-{}_{A}\mu(x_i)$.

(iii) Concavity of $H^{\mu}_{\mathrm{FS}}(\mathrm{A})$ for each ${}_{A}\mu(x_i)$:

$$\frac{\partial H^{\mu}_{\mathrm{FS}}(\mathrm{A})}{\partial_{A}\mu(x_i)}=\frac{3\left(1-2\,{}_{A}\mu(x_i)\right)\sec^2\left(\frac{3}{2+2\sqrt{l_1}}\right)}{4\sqrt{l_1}\left(1+\sqrt{l_1}\right)^2} \tag{2}$$

$$\frac{\partial^2 H^{\mu}_{\mathrm{FS}}(\mathrm{A})}{\partial_{A}\mu^2(x_i)}=-\frac{3\sec^2\left(\frac{3}{2+2\sqrt{l_1}}\right)\begin{pmatrix}16\,{}_{A}\mu^3(x_i)+4\sqrt{l_1}+1\\ +3_{A}\mu(x_i)+8\,{}_{A}\mu^2(x_i)\sqrt{l_1}\\ +3\left(1-2_{A}\mu(x_i)\right)^2\sqrt{l_1}\tan\left(\frac{3}{2+2\sqrt{l_1}}\right)\end{pmatrix}}{8l_1^{\frac{3}{2}}\left(1+\sqrt{l_1}\right)^4}\leq 0 \quad \text{for each}$$

${}_{A}\mu(x_i)\in[0,1]$. This justifies the concavity.

(iv) There exists a maximum value of $H^{\mu}_{\mathrm{FS}}(\mathrm{A})$ with respect to each ${}_{A}\mu(x_i)$ owing to its concavity property. Using (2), this maximum value arises when $\frac{\partial H^{\mu}_{\mathrm{FS}}(\mathrm{A})}{\partial_{A}\mu(x_i)}=0$ which yields ${}_{A}\mu(x_i)=\frac{1}{2}$. From (1)

$$\mathrm{Max.}H^{\mu}_{\mathrm{FS}}(\mathrm{A})=H^{\mu}_{\mathrm{FS}}(\mathrm{A})\Big|_{{}_{A}\mu(x_i)=\frac{1}{2}}=\left(\tan\left(\frac{3}{2}\right)-\tan(1)\right)n. \tag{3}$$

The concavity of $H^{\mu}_{\mathrm{FS}}(\mathrm{A})$ is evident in Figure 2.

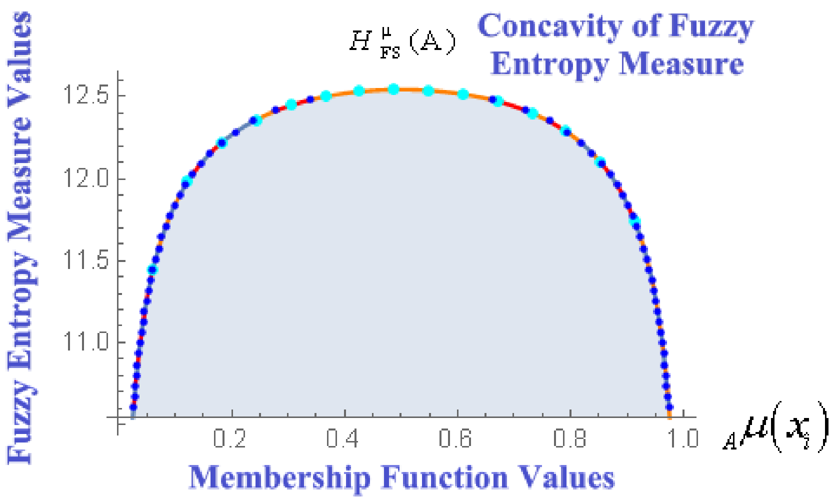


Figure 2. Concavity of fuzzy entropy measure $H^{\mu}_{\mathrm{FS}}(\mathrm{A})$ with respect to ${}_{A}\mu(x_i)$

Theorem 2 Set $l_2 = {}_A\mu(x_i) + {}_B\mu(x_i), l_3 = {}_A\mu^2(x_i) + {}_B\mu^2(x_i), l_4 = {}_A\mu(x_i)\,{}_B\mu(x_i)$, then $\mathrm{L}^{\mu}_{\mathrm{FS}}(\mathrm{A},\mathrm{B})$ represents a valid TSFCE measure that hinges on two FSs A and B where

$$\mathrm{L}^{\mu}_{\mathrm{FS}}(\mathrm{A},\mathrm{B}) = \sum_{i=1}^{n}\left[-6\tan(1) + (2+l_2)\tan\left(\frac{2+l_2}{2+2\sqrt{l_3}}\right) + (4-l_2)\tan\left(\frac{4-l_2}{2+2\sqrt{l_4}}\right)\right] \tag{4}$$

Proof. It is easy to verify the symmetric nature of $\mathrm{L}^{\mu}_{\mathrm{FS}}(\mathrm{A},\mathrm{B})$ as $\mathrm{L}^{\mu}_{\mathrm{FS}}(\mathrm{A},\mathrm{B}) = \mathrm{L}^{\mu}_{\mathrm{FS}}(\mathrm{B},\mathrm{A}) \forall A, B \in S(X)$. Further, $\mathrm{L}^{\mu}_{\mathrm{FS}}(\mathrm{A},\mathrm{B})$ remains unchanged on replacing ${}_A\mu(x_i), {}_B\mu(x_i)$ with $1-{}_A\mu(x_i), 1-{}_B\mu(x_i)$ into equation **(3)**. To establish $\mathrm{L}^{\mu}_{\mathrm{FS}}(\mathrm{A},\mathrm{B}) \ge 0$, we shall first establish the following **Lemma 1 .**

Lemma 1 Define $l_5 = {}_A\mu^2(x_i) + {}_B\mu^2(x_i)$. There exists the inequality $\sqrt{\frac{l_5}{2}} \ge \sqrt{l_3}$ with equality whenever ${}_A\mu(x_i) = {}_B\mu(x_i) \in [0,1]$.

Proof. In our notations,

$$\frac{l_5}{2} - \frac{l_2^2}{4} = \left(\frac{{}_A\mu(x_i) - {}_B\mu(x_i)}{2}\right)^2 \ge 0 \;\Rightarrow \sqrt{\frac{l_5}{2}} \ge \frac{l_2}{2} \forall\, {}_A\mu(x_i), {}_B\mu(x_i) \in [0,1] \tag{5}$$

with equality if ${}_A\mu(x_i) = {}_B\mu(x_i) \forall i = 1,2,...,n.$

Define $m_0 == \left(\frac{\sqrt{{}_A\mu(x_i)} + \sqrt{{}_B\mu(x_i)}}{2}\right)^2$ and consider

$$\frac{l_2}{2} - m_0 = \left(\frac{\sqrt{{}_A\mu(x_i)} - \sqrt{{}_B\mu(x_i)}}{2}\right)^2 \ge 0 \Rightarrow \frac{l_2}{2} \ge m_0 \forall\, {}_A\mu(x_i), {}_B\mu(x_i) \in [0,1] \tag{6}$$

with equality if ${}_A\mu(x_i) = {}_B\mu(x_i) \forall i = 1,2,...,n.$

Again $m_0 - \sqrt{l_3} = \left(\frac{\sqrt{{}_A\mu(x_i)} - \sqrt{{}_B\mu(x_i)}}{2}\right)^2 \ge 0$

$$\Rightarrow m_0 \ge \sqrt{l_3} \forall\, {}_A\mu(x_i), {}_B\mu(x_i) \in [0,1] \tag{7}$$

with equality if ${}_A\mu(x_i) = {}_B\mu(x_i) \forall i = 1,2,...,n.$

We can combine the resulting inequalities **(5-7)** to get the desired result.

The outcomings of **Lemma 1** can be re-designed as

$$\frac{l_5}{2} \ge l_3 \Rightarrow \frac{{}_A\mu^2(x_i) + {}_B\mu^2(x_i)}{2} \ge {}_A\mu(x_i)\,{}_B\mu(x_i)$$

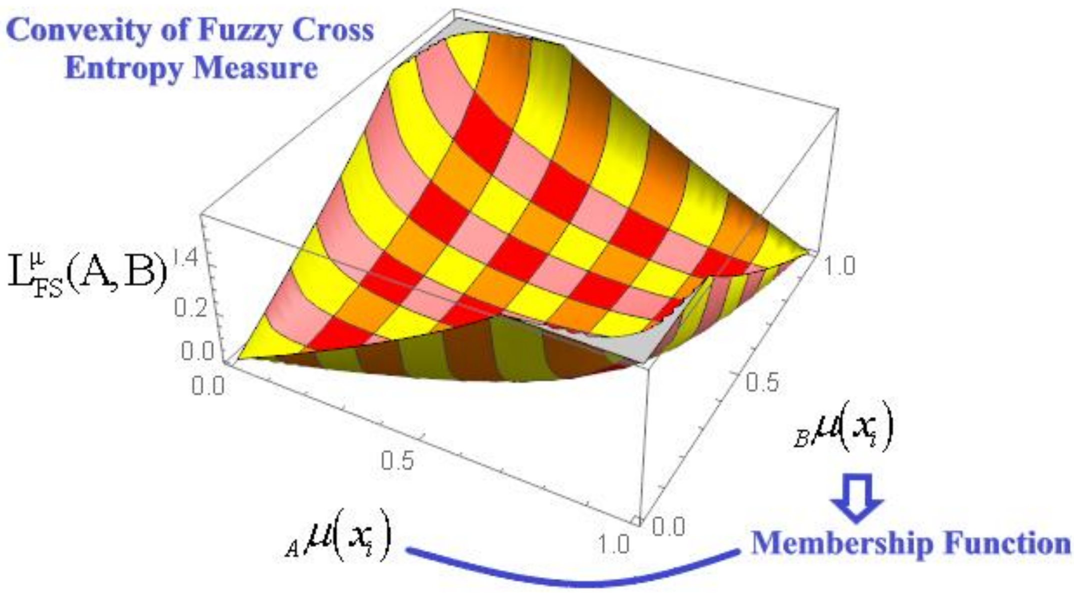
$$\Rightarrow \frac{\left({}_A\mu(x_i) + {}_B\mu(x_i)\right)^2 - 2\,{}_A\mu(x_i)\,{}_B\mu(x_i)}{2} \ge {}_A\mu(x_i)\,{}_B\mu(x_i)$$

$$\frac{l_2^2}{2} - l_3 \ge l_3 \Rightarrow \frac{l_2^2}{4} \ge l_3 \Rightarrow \frac{l_2}{2} \ge \sqrt{l_3} \Rightarrow \frac{l_2}{2} + 1 \ge \sqrt{l_3} + 1$$

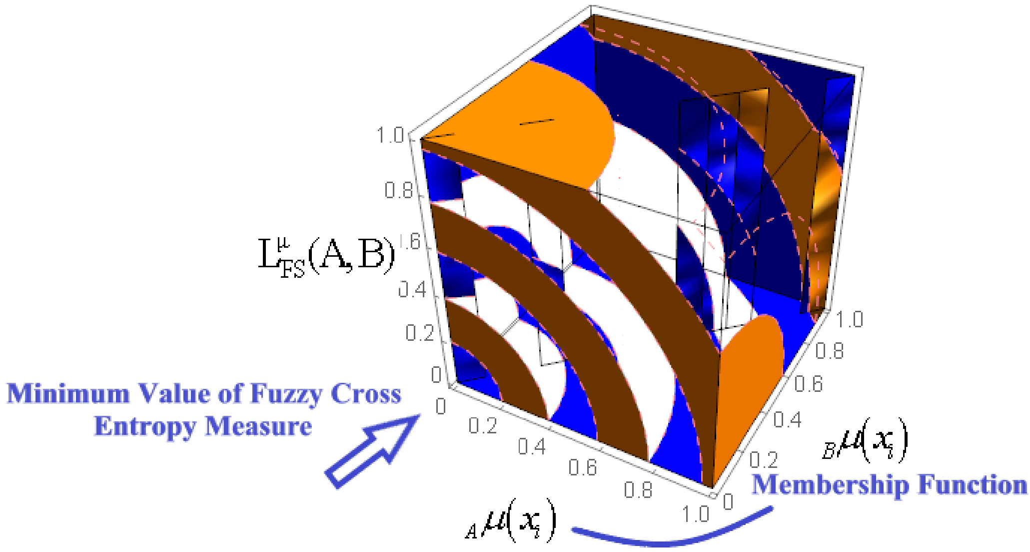
$$\Rightarrow \frac{2+l_2}{2+2\sqrt{l_3}} \geq 1 \tag{8}$$

After taking tangent of the undergoing inequality **(8)** yields

$$\left(2+l_2\right)\tan\left(\frac{2+l_2}{2+2\sqrt{l_3}}\right) \geq \left(2+l_2\right)\tan 1 \tag{9}$$



(a)



(b)

Figure 3. (a) Convexity and (b) maximum/minimum values $L^{\mu}_{FS}(A,B)$

On the similar pattern, the replacement of $_{A}\mu\left(x_i\right);{}_{B}\mu\left(x_i\right)$ with $1-{}_{A}\mu\left(x_i\right);1-{}_{B}\mu\left(x_i\right)$ yields

$$\left(4-l_2\right)\tan\left(\frac{4-l_2}{2+2\sqrt{l_4}}\right)\geq\left(4-l_2\right)\tan 1 \tag{10}$$

We can simply add **(9)** and **(10)** and then take summation over $i=1$ to n for getting the desired result. That is, $L^{\mu}_{FS}(A,B)\geq 0$ for each ${}_A\mu(x_i),\ {}_B\mu(x_i)\in[0,1]$ with equality whenever ${}_A\mu(x_i)={}_B\ \mu(x_i)\forall i=1,2,...,n.$

It will be informative to know that $L^{\mu}_{FS}(A,B)$ discloses its minimum and maximum values as proved in Theorem 3.

Theorem 3. $\exists$ the inequality: $0\leq L^{\mu}_{FS}(A,A^c)\leq 6\left(\tan\left(\frac{3}{2}\right)-\tan(1)\right)n$ where $n\in N.$

Proof. In our notations, if we replace B with its counterpart A^c into the resulting equation **(4)**, then $2+l_2$ changes to 3, $l_3\to l_1, 4-l_2\to 3, l_4\to l_1.$

With these restrictions, the resulting equation **(4)** be rescheduled as

$$L^{\mu}_{FS}(A,A^c)=\sum_{i=1}^{n}\left[-6\tan(1)+6\tan\left(\frac{3}{2+2\sqrt{l_1}}\right)\right]$$

$$=\sum_{i=1}^{n}\left[6\tan\left(\frac{3}{2}\right)-6\tan(1)-6\left[\tan\left(\frac{3}{2}\right)-\tan\left(\frac{3}{2+2\sqrt{l_1}}\right)\right]\right]=6\,Max.H^{\mu}_{FS}(A)-6H^{\mu}_{FS}(A)$$

$$\Rightarrow H^{\mu}_{FS}(A)=Max.H^{\mu}_{FS}(A)-\frac{1}{6}L^{\mu}_{FS}(A,A^c) \tag{11}$$

With the aid of non-negative condition $H^{\mu}_{FS}(A)\geq 0$, the expression **(11)** yields

$$0\leq L^{\mu}_{FS}(A,A^c)\leq 6\left(\tan\left(\frac{3}{2}\right)-\tan(1)\right)n \tag{12}$$

The inequality expression **(12)** suggests $L^{\mu}_{FS}(A,A^c)$ as finite since n is finite. Also, readers can easily establish that $0\leq L^{\mu}_{FS}(A,B)\leq 6\left(\tan\left(\frac{3}{2}\right)-\tan(1)\right)n$ which suggests that $Max.L^{\mu}_{FS}(A,B)=6\left(\tan\left(\frac{3}{2}\right)-\tan(1)\right)n$. Also, the plots displayed in **Figure 3 (a-b)** affirm that $Min.L^{\mu}_{FS}(A,B)$ is zero.

To predict the various antioxidant potentials and polyphenolic contents extracted from the aerial parts of *A. tortuosum,* it becomes essential for us to cultivate the following Theorem 4.

2.5.2. A neutrosophic cross entropy measure

Theorem 2 can be extended to establish another important cross entropy measure based on two SVNSs as follows.

Def. 2 Set $r_2={}_Ai(x_i)+{}_Bi(x_i), r_3={}_Ai(x_i)\,{}_Bi(x_i), r_4=\left(1-{}_Ai(x_i)\right)\left(1-{}_B\,i(x_i)\right).$ Let $A=\left(\prec x,{}_A\mu(x_i),{}_Ai(x_i),{}_Af(x_i)\succ\forall x_i\in X\right); B=\left(\prec x,{}_B\mu(x_i),{}_Bi(x_i),{}_Bf(x_i)\succ\forall x_i\in X\right)$

be any two SVNSs. The amount of fuzziness inherited by the truth membership degree of A and B is given by $\mathrm{L}^{\mu}_{\mathrm{FS}}(\mathrm{A,B})$ and represented by **(3).** Similarly, as per **Theorem 2,** the amount of fuzziness inherited by the degree of indeterminancy membership of A and B is

$$\mathrm{L}^{i}_{\mathrm{FS}}(\mathrm{A,B}) = \sum_{i=1}^{n}\left[-6\tan(1)+(2+r_2)\tan\left(\frac{2+r_2}{\sqrt{2+2\sqrt{r_3}}}\right)+(4-r_2)\tan\left(\frac{4-r_2}{\sqrt{2+2\sqrt{r_4}}}\right)\right] \tag{13}$$

If we set

$s_2 = {}_Af(x_i)+{}_Bf(x_i), s_3 = {}_Af(x_i)\,{}_Bf(x_i), s_4 = (1-{}_Af(x_i))(1-{}_Bf(x_i)),$ then the amount of fuzziness inherited by the falsity membership degree of A and B is

$$\mathrm{L}^{f}_{\mathrm{FS}}(\mathrm{A,B}) = \sum_{i=1}^{n}\left[-6\tan(1)+(2+s_2)\tan\left(\frac{2+s_2}{\sqrt{2+2\sqrt{s_3}}}\right)+(4-s_2)\tan\left(\frac{4-s_2}{\sqrt{2+2\sqrt{s_4}}}\right)\right] \tag{14}$$

Hence, $\mathrm{L}_{\mathrm{SV}}(\mathrm{A,B})$, the proclaimed trigonometric symmetric SVNCE measure, via two SVNSs, A and B, can be obtained as per following expression:

$$\mathrm{L}_{\mathrm{SV}}(\mathrm{A,B}) = \mathrm{L}^{\mu}_{\mathrm{FS}}(\mathrm{A,B}) + \mathrm{L}^{i}_{\mathrm{FS}}(\mathrm{A,B}) + \mathrm{L}^{f}_{\mathrm{FS}}(\mathrm{A,B}) \tag{15}$$

Theorem 4. $\exists$ the inequality: $0 \le \mathrm{L}_{\mathrm{SV}}(\mathrm{A,A}^c) \le 18\left(\tan\left(\frac{3}{2}\right)-\tan(1)\right)n$ where, the cardinality of the SVNSs, A and B is represented by $n \in N$

Proof. Set

$\eta_1 = {}_Ai(x_i)(1-{}_Ai(x_i)), \eta_2 = {}_A\mu(x_i)+{}_Af(x_i),$

$\eta_3 = {}_A\mu(x_i)\,{}_Af(x_i), \eta_4 = (1-{}_A\mu(x_i))(1-{}_Af(x_i)).$

In our notations, if we replace the SVNS B with its counterpart A^c into the resulting equation **(15)**, then

$$\begin{aligned}\mathrm{L}_{\mathrm{SV}}(\mathrm{A,A}^c) &= \sum_{i=1}^{n}\left[-18\tan(1)+6\tan\left(\frac{3}{2+2\sqrt{\eta_1}}\right)+2(2+\eta_2)\tan\left(\frac{2+\eta_2}{2+2\sqrt{\eta_3}}\right)+2(4-\eta_2)\tan\left(\frac{4-\eta_2}{2+2\sqrt{\eta_4}}\right)\right]\\
&= \sum_{i=1}^{n}\left[18\tan\left(\frac{3}{2}\right)-18\tan(1)-6\left[\begin{array}{l}3\tan\left(\frac{3}{2}\right)-\tan\left(\frac{3}{2+2\sqrt{\eta_1}}\right)-\left(\frac{2+\eta_2}{3}\right)\tan\left(\frac{2+\eta_2}{2+2\sqrt{\eta_3}}\right)\\ -\left(\frac{4-\eta_2}{3}\right)\tan\left(\frac{4-\eta_2}{2+2\sqrt{\eta_4}}\right)\end{array}\right]\right]\\
&= 6\,Max.L_{\mathrm{SV}}(\mathrm{A})-6L_{\mathrm{SV}}(\mathrm{A});\end{aligned}$$

$$L_{\mathrm{SV}}(\mathrm{A}) = \sum_{i=1}^{n}\left[3\tan\left(\frac{3}{2}\right)-\tan\left(\frac{3}{2+2\sqrt{\eta_1}}\right)-\left(\frac{2+\eta_2}{3}\right)\tan\left(\frac{2+\eta_2}{2+2\sqrt{\eta_3}}\right)-\left(\frac{4-\eta_2}{3}\right)\tan\left(\frac{4-\eta_2}{2+2\sqrt{\eta_4}}\right)\right] \tag{16}$$

The resulting expression **(16)** represents the desired SVNCE measure with following all the essential conditions:

1. $L_{SV}(\mathrm{A})$ exhibits the concavity property for each $L_{SV}(\mathrm{A})$
2. $L_{SV}(\mathrm{A})=0$ if either ${}_{A}\mu(x_i)=0, {}_{A}i(x_i)=0, {}_{A}f(x_i)=1.$ or ${}_{A}\mu(x_i)=1, {}_{A}i(x_i)=0, {}_{A}f(x_i)=0$
3. $L_{SV}(\mathrm{A})\geq 0 \forall A\in W(X)$
4. $L_{SV}(\mathrm{A}^c)=L_{SV}(\mathrm{A})$

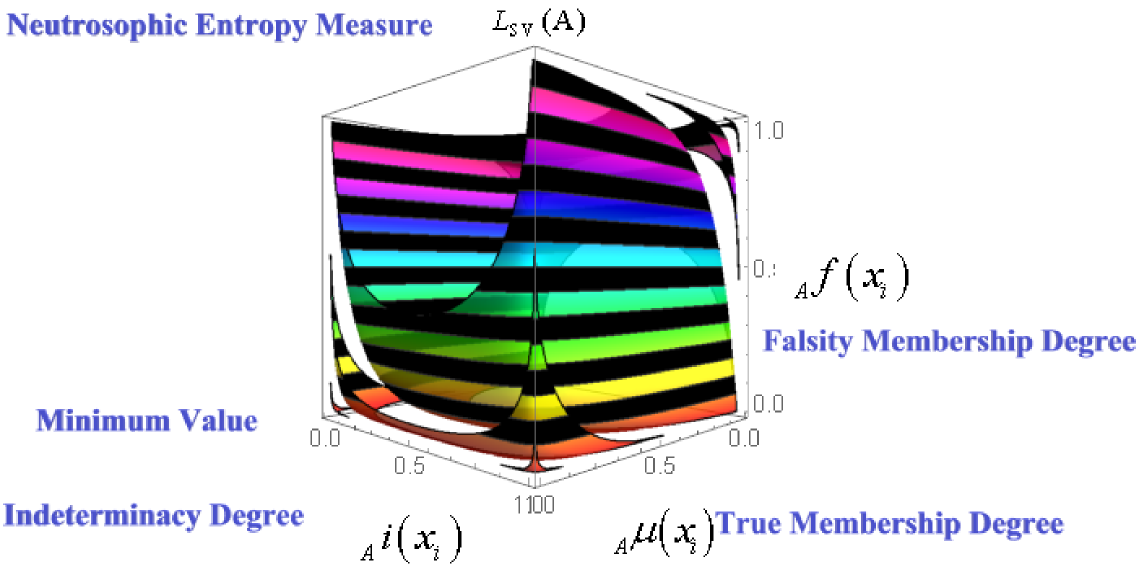


Figure 4. Neutrosophic entropy measure

The overall discussion in the outcoming theorems has brought us in a strong situation to apply the proposed measure (Figure 4) for discriminating the polyphenolic contents and antioxidant potentials of the *A. tortuosum* leaf extract.

2.5.3. Neutrosophic cross entropy-based methodology

Step 1. Normalization of Extracted Percentage Inhibitions of Various Assays

We first assume that the number of parameters (influencing factors) is "n". Also, "m". is the number of reaction sets. Let the maximum and minimum values of the percentage yields extracted using UAEM, SEM and MM as $l_{\max}$ and $l_{\min}$ respectively. To predict the highly efficient solvent needed for the extraction of polyphenolics, it is mandatory to normalize the percentage yields of each assay to be bounded in the interval [0, 1] which can be done by using the formula:

$$l^* = \frac{l - l_{\min}}{l_{\max} - l_{\min}} \tag{17}$$

Step 2. Extracting upper and lower bounds for various antioxidant potentials and polyphenolic contents

Suppose the knowledge of percentage yield of the studied antioxidant potentials can be represented by the discrete set $A=(A_1,A_2,A_3,A_4)$. Also, the knowledge of various known polyphenolic contents-TFC and TPC-can also be represented by the set $F = \left(F_{T_1}, F_{T_2}\right)$

Let $U^*_{A_K}(x)(K=1,2,3,4)$ and $\mu^*_{A_K}(x)$ be the upper and lower bounds of the K^{th} assay. Define A_K as

$$A_K = \left\{ < x, \left[\mu^*_{A_1}(x), U^*_{A_1}(x)\right] >, < x, \left[\mu^*_{A_2}(x), U^*_{A_2}(x)\right] >, \ldots, < x, \left[\mu^*_{A_4}(x), U^*_{A_4}(x)\right] > \right\}$$

Define the set $F_{T_j}\ (j=1,2)$ as

$$F_{T_j} = \left\{ < x, \left[\mu^*_{F_{T_1}}(x), U^*_{F_{T_1}}(x)\right] >, < x, \left[\mu^*_{F_{T_2}}(x), U^*_{F_{T_2}}(x)\right] > \right\}$$

Step 3. Extending Percentage Yield Interval into the Form of SVNSs

Let $f^*_{A_K}(x) = 1 - U^*_{A_K}(x)$ and $i^*_{A_K}(x) = 1 - f^*_{A_K}(x) - U^*_{A_K}(x)$ where $i^*_{A_K}(x)$ is restricted to $0\cdot 01$ if it assumes any value less than $0\cdot 001$.Then, $A_K\,(K=1,2,3,4)$ can be further extended into the following form:

$$A_K = \left\{ \begin{array}{l} \left(< x, \mu^*_{A_1}(x), i^*_{A_1}(x), f^*_{A_1}(x) >\right), \left(< x, \mu^*_{A_2}(x), i^*_{A_2}(x), f^*_{A_2}(x) >\right), \ldots, \\ \left(< x, \mu^*_{A_4}(x), i^*_{A_4}(x), f^*_{A_4}(x) >\right) \end{array} \right\}$$

Similarly, the set F_{T_j} can also take the form of SVNSs as

$$F_{T_j} = \left\{ \left(< x, \mu^*_{F_{T_1}}(x), i^*_{F_{T_1}}(x), f^*_{F_{T_1}}(x) >\right), \left(< x, \mu^*_{F_{T_2}}(x), i^*_{F_{T_2}}(x), f^*_{F_{T_2}}(x) >\right) \right\}$$

Step 4. Computing cross entropy values. The trigonometric SVNCE measure values between F_{T_j} and A_K can be found by substituting ${}_A\mu(x_i), {}_A i(x_i), {}_A f(x_i); {}_B\mu(x_i), {}_B i(x_i), {}_B f(x_i)$ with $\mu^*_{A_K}(x), i^*_{A_K}(x), f^*_{A_K}(x); \mu^*_{F_{T_j}}(x), i^*_{F_{T_j}}(x), f^*_{F_{T_j}}(x)$ into (15). Thus,

$$\begin{aligned} &L_{\text{SV}}(A_K, F_{T_j})(K=1,2,3,4;\ j=1,2) = -18\tan 1 \\ &+\left[\left(2 + i^*_{A_K}(x) + i^*_{F_{T_j}}(x)\right)\tan\left(\frac{2 + i^*_{A_K}(x) + i^*_{F_{T_j}}(x)}{2 + 2\sqrt{i^*_{A_K}(x_i)\, i^*_{F_{T_j}}(x_i)}} \right) + \left(4 - i^*_{A_K}(x) - i^*_{F_{T_j}}(x)\right)\tan\left(\frac{4 - i^*_{A_K}(x) - i^*_{F_{T_j}}(x)}{2 + 2\sqrt{\left(1 - i^*_{A_K}(x_i)\right)\left(1 - i^*_{F_{T_j}}(x_i)\right)}} \right) \right] \\ &+\left[\left(2 + f^*_{A_K}(x) + f^*_{F_{T_j}}(x)\right)\tan\left(\frac{2 + f^*_{A_K}(x) + f^*_{F_{T_j}}(x)}{2 + 2\sqrt{f^*_{A_K}(x)\, f^*_{F_{T_j}}(x)}} \right) + \left(4 - f^*_{A_K}(x) - f^*_{F_{T_j}}(x)\right)\tan\left(\frac{4 - f^*_{A_K}(x) - f^*_{F_{T_j}}(x)}{2 + 2\sqrt{\left(1 - f^*_{A_K}(x)\right)\left(1 - f^*_{F_{T_j}}(x)\right)}} \right) \right] \end{aligned}$$

Step 5. Identification of the Most Efficient Solvent Based Upon Cross Entropy Values

Smaller value of $\underset{w}{L_{\text{CE}}}(A_K, F_{T_j})$ indicates that the antioxidant potential A_K is closer to the known polyphenolic content F_{T_j}.

3. Results and Discussion

3.1. Extract yield

The extract yield of different dry extracts was found in the range of 35.21-61.62% for 100 g of the powdered dry matter. The highest yield was shown by methanol extract, then by chloroform and hexane extracts that was contributed to the polarity of the solvent. Thus, the essence of methanol promoted the solubilization of secondary metabolites [21]. Furthermore, the yield varied for the extraction techniques with the best results for SEM followed by UAEM and substantially lower for MM. Comparable findings have also been recorded previously, where the maximum yield was demonstrated by different polar extracts obtained by SEM [22]. The Higher yields by employing SEM can be obtained via exhaustive extraction of plant material followed by repeated washing with a hot solvent [13]. Whereas, acoustic vibrations provide better results as compared to MM by raising the solubility and diffusion coefficients of secondary metabolites as well as lowering the viscosity of the solvent [19].

3.2. Antioxidant Assay

The polyphenolic compounds present in medicinal plants are referred as antioxidant agents and used as radical scavengers or metal chelators [23]. The antioxidant effect of the plant depends on phytochemicals present as both major and minor constituents that play an effective role in radical scavenging. The FRAP assay is based on using antioxidants for reducing the 2,4,6-tripyridyl-s-triazine – Fe(III) complex [21]. The FRAP values of these extracts ranged from 146.18 to 439.95 g DPE. The highest FRAP values were exhibited by all the extracts resulted by SEM while the lowest values were obtained for those obtained by MM. A comparable pattern has also been found in earlier studies for medicinal plant extracts [15]. Further, a maximum FRAP value was found for the methanol extract and a minimum for the hexane extracts. It is apparent from the current findings that the FRAP values are highly dependent on the type of solvent and the extraction methods used for the study.

ABTS assay involves scavenging the pre-generated $ABTS^{\bullet+}$ radical cation by the antioxidant [23]. The results showed that extracts in various solvents had various method-dependent scavenging potentials for radicals. Therefore, based on their IC50 value, the methanol extract obtained by SEM has been found to be most active in comparison with the other extracts. The leaf extract of *A. tortuosum* was found to exhibit significant DPPH scavenging activity with IC_{50} value maximum for methanol extract followed by chloroform extract and hexane extract. Therefore, the high antioxidant potential of the *A.tortuosum* leaf extract observed in this study signifies its ethno-medicinal usage to treat oxidative stress-related diseases [24].

3.3. Polyphenolic content

TPC is the quantitative approach to determine the extent of polyphenols present in the plant. Phenols are one of the important plant constituents as the radical scavenging ability depends on the hydroxyl group [14]. The phenolic groups are present in the secondary metabolites with redox properties that allow them to function as a radical scavenger in various biotic activities such as antioxidant, antibacterial and antifungal [21]. TPC of *A. tortuosum* was also found to depend upon the nature of the extraction method and the solvent used. The extracts obtained by Soxhlet extraction exhibited the maximum amount, while the least amount was received by maceration. The methanol extract was found to provide the best yield of polyphenols.

Flavonoids, including flavanols and flavones, are secondary plant metabolites. These metabolites result in the colour of the plants as well as the antioxidant and antimicrobial activity depending upon the presence of -OH groups. Plant flavonoids exhibit antioxidant activity in both vivo and vitro studies [22]. The maximum yield of TFC for *A. tortuosum* leaf extract was obtained for the extracts obtained by Soxhlet extraction. Flavonoids exhibit more excellent solubility in polar solvents than non-polar solvents [21]. Hence, the methanol extract contained the highest TFC as compared to the other extracts [25]. Other researchers have also reported a significant amount of polyphenols in the tuber extract of *A. tortuosum* [17].

Table 1. Lower and upper bounds of each known antioxidant Potential, TPC and TFC of *A. tortuosum* leaf extract

Sets	SEM		
	Methanol	**Chloroform**	**Hexane**
DPPH	[0.997,1.000]	[0.995,1.000]	[0.994,1.000]
ABTS	[0.589,0.593]	[0.589,0.593]	[0.590,0.603]
FRAP	[0.464,0.468]	[0.461,0.465]	[0463,0.468]
TPC	[0.000,0.006]	[0.000,0.007]	[0.000,0.007]
TFC	[0.107,0.111]	[0.105,0.107]	[0.105,0.110]
	UAEM		
	Methanol	**Chloroform**	**Hexane**
DPPH	[0.998,1.000]	[0.989,1.000]	[0.986,1.000]
ABTS	[0.590,0.594]	[0.590,0.601]	[0.581,0.592]
FRAP	[0.462,0.469]	[0.464,0.474]	[0.447,0.465]
TPC	[0.000,0.003]	[0.000,0.006]	[0.000,0.012]
TFC	[0.110,0.114]	[0.107,0.114]	[0.106,0.114]
	MM		
	Methanol	**Chloroform**	**Hexane**
DPPH	[0.994,1.000]	[0.995,1.000]	[0.984,1.000]
ABTS	[0.585,0.591]	[0.592,0.605]	[0.580,0.591]
FRAP	[0.461,0.465]	[0.468,0.477]	[0.448,0.456]
TPC	[0.000,0.006]	[0.000,0.013]	[0.000,0.012]
TFC	[0.107,0.111]	[0.118,0.128]	[0.000,0.012]

3.4. Discrimination among antioxidant potential and polyphenolic content

The step-by-step procedure of the proclaimed methodology for the identification of the most effective solvent is already displayed in **Figure 1**. In the present case, we represent the various antioxidant potentials and polyphenolic contents extracted from the leaves and flowers of *Arisaema tortuosum* by the set $A=\left(A_1, A_2, A_3, A_4\right)$ where A_1 =DPPH radical scavenging A_2 = ABTS, A_3 = FRAP and the set $F_T=\left(F_{T_1}, F_{T_2}\right)$ where F_{T_1} =TPC and F_{T_2} =TFC, respectively.

Step 1. The lower and upper bounds of each known antioxidant potentials $A_K (K=1,2,3,4)$ and polyphenolic contents $F_{T_j} (j=1,2)$ can be extracted and represented in Table 1.

Step 2. Extending the percentage yield interval ranges of each known antioxidant potentials $A_K (K=1,2,3,4)$ and known polyphenolic contents $F_{T_j} (j=1,2)$ into the form of SVNSs and depicted in Table 2.

Table 2. Representation of extract yield interval range as SVNSs

Sets	**SEM**		
	Methanol	**Chloroform**	**Hexane**
DPPH	[0.997,0.003,0.000]	[0.995,0.005,0.000]	[0.994,0.006,0.000]
ABTS	[0.589,0.004,0.407]	[0.587,0.006,0.407]	[0.590,0.013,0.397]
FRAP	[0.464,0.004,0.532]	[0.461,0.004,0.535]	[0.463,0.003,0.534]
TPC	[0.000,0.006,0.994]	[0.000,0.007,0.993]	[0.000,0.007,0.993]
TFC	[0.107,0.004,0.889]	[0.105,0.002,0.893]	[0.105,0.005,0.890]
	UAEM		
	Methanol	**Chloroform**	**Hexane**
DPPH	[0.998,0.002,0.000]	[0.989,0.011,0.000]	[0.986,0.014,0.000]
ABTS	[0.590,0.004,0.406]	[0.590,0.011,0.399]	[0.581,0.011,0.408]
FRAP	[0.462,0.007,0.531]	[0.464,0.010,0.526]	[0.447,0.018,0.535]
TPC	[0.000,0.003,0.997]	[0.000,0.006,0.994]	[0.000,0.012,0.988]
TFC	[110,0.004,0.886]	[0.107,0.007,0.886]	[0.106,0.008,0.886]
	MM		
	Methanol	**Chloroform**	**Hexane**
DPPH	[0.994,0.006,0.000]	[0.995,0.005,0.000]	[0.984,0.016,0.000]
ABTS	[0.585,0.006,0.409]	[0.592,0.013,0.395]	[0.580,0.011,0.409]
FRAP	[0.461,0.004,0.535]	[0.468,0.009,0.523]	[0.448,0.008,0.544]
TPC	[0.000,0.006,0.994]	[0.000,0.013,0.987]	[0.000,0.012,0.988]
TFC	[0.107,0.004,0.889]	[0.118,0.010,0.872]	[0.000,0.012,0.988]

Step 3. The minimum cross entropy measure $L^{\mu}_{FS}(A_K, F_{T_1})$ values between various antioxidant potentials and TPC, for SEM, UAEM and MM, amount as 10.19, 9.99 and 10.04, respectively (Table 3). Similarly, The minimum cross entropy values $L^{\mu}_{FS}(A_K, F_{T_2})$ between various antioxidant potentials and TFC, for SEM, UAEM and

MM, amount as 1.96,1.88 and 4.45, respectively. These values indicate that the TFC followed by TPC are the most efficient solvents for extracting various antioxidant potentials viz FRAP, DPPH radical scavenging, TAC and NOS, respectively, extracted from the aerial parts of *Arisaema tortuosum* entropy measure based upon SVNSs. Furthermore, the minimum neutrosophic cross entropy measure $L_{SV}(A_K, F_{T_1})$ values between various antioxidant potentials and TPC, for SEM, UAEM and MM, amount as 16.98, 16.94 and 16.25, respectively (Table 4). Also, the minimum NCEM $L_{SV}(A_K, F_{2_1})$ values between various antioxidant potentials and TFC, for SEM, UAEM and MM, amount as 3.91, 3.78 and 7.56, respectively. These values also confirm that TFC followed by TPC are the most efficient solvents for extracting various antioxidant potentials.

Further, the best results were obtained for the Soxhlet extraction method followed by ultrasonication and maceration methods. The results reveal that the precision of the proposed methodology ranges from 86.60% to 173.21%/ , which justifies the accuracy of the proposed method.

Table 3. Fuzzy cross entropy value between each antioxidant potentials and TPC and TFC of *A. Tortuosum* leaf extract

Cross Entropy Values	SEM	UAEM	MM
$L_{FS}^{\mu}(A_1, F_{T_1})$	149.57	140.85	138.49
$L_{FS}^{\mu}(A_2, F_{T_1})$	16.74	16.63	16.55
$L_{FS}^{\mu}(A_3, F_{T_1})$	**10.19**	**9.99**	**10.04**
$L_{FS}^{\mu}(A_1, F_{T_2})$	35.21	31.56	62.99
$L_{FS}^{\mu}(A_2, F_{T_2})$	3.50	3.43	7.62
$L_{FS}^{\mu}(A_3, F_{T_2})$	**1.96**	**1.88**	**4.45**

Table 4. Neutrosophic cross entropy value between each antioxidant potentials and TPC and TFC of *A. Tortuosum* leaf extract

Neutrosophic Cross Entropy Values	SEM	UAEM	MM
$L_{SV}(A_1, F_{T_1})$	291.80	284.58	272.74
$L_{SV}(A_2, F_{T_1})$	27.70	27.73	26.55
$L_{SV}(A_3, F_{T_1})$	**16.98**	**16.94**	**16.25**
$L_{SV}(A_1, F_{T_2})$	97.86	92.59	145.92
$L_{SV}(A_2, F_{T_2})$	7.03	6.82	12.95
$L_{SV}(A_3, F_{T_2})$	**3.91**	**3.78**	**7.56**

5. Conclusion

The present study explores the use of a novel symmetric neutrosophic entropy measure for discrimination among antioxidant potential with TPC and TFC of *A. tortuosum* leaf extract obtained by using three different techniques viz UAE, SE and ME in presence of three solvents of different polarities (methanol, hexane and chloroform). The antioxidant potential was assessed through DPPH, ABTS and FRAP assay. The study revealed the impact of different solvents on the extraction of phytochemicals and antioxidant activity of *A. tortuosum*. The extracts possessed significant antioxidant activity due to high content of total polyphenols and flavonoids. However, the methanol extract was found to possess higher content of total polyphenols and flavonoids resulting in comparatively higher antioxidant activity. Similarly, SEM extraction was found to be more beneficial with best results. The quantitative analysis was carried out using neutrosophic cross-entropy based methodology. The studies confirmed a valid discrimination among TPC and TFC, DPPH, ABTS and FRAP parameters owing to high value of cross-entropy irrespective of the method of extraction used for the study. It shows that the antioxidant potential of the extract can be accredited to the rich TPC and TFC content. The study signifies that Soxhlet extraction method is the best method among the three methods for the phytochemical extraction from the medicinal plants.

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Introduction to the MultiNeutrosophic Set

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Abstract: In the real word, in most cases, everything (an attribute, event, proposition, theory, idea, person, object, action, etc.) is evaluated in general by many *sources* (called *experts*), not only one. The more sources evaluate a subject, the better accurate result (after fusioning all evaluations). That's why, in this paper, we straightforwardly extend the Refined Neutrosophic Set to the MultiNeutrosophic Set, and we show that the last two are isomorphic. A MultiNeutrosophic Set is a Neutrosophic Set whose all elements' degrees of truth/indeterminacy/falsehood are evaluated by many (Multi) sources.

Afterwards, we introduce a total order on the set of n-plets of the form (p, r, s), we build the operators on the (p, r, s)-plets, and show several applications of the MultiNeutrosophic Sets.

Several particular cases of the MultiNeutrosophic Sets are presented: such as MultiFuzzy Set, MultiIntuitionistic Fuzzy Set, MultiPicture Fuzzy Set, and other Multi(Fuzzy Extension) Set.

1. General Definition of the Neutrosophic Set (or Subset Neutrosophic Set - *SNS*)

Let $\mathcal{U}$ be a universe of discourse and a subset N of it.

Then:

$N = \{x, (T, I, F), x \ \epsilon \ \ U\}$

is called a *Neutrosophic Set*, where *T, I, F* are subsets of [0, 1], and they are called respectively degrees of Truth (*T*), Indeterminacy (*I*), and Falsehood (*F*) of the element x with respect to the set *A*. No other restrictions on *T, I,* and *F*. Of course, it implies that:

$0 \leq \inf T + \inf I + \inf F \quad \leq \sup T + \sup I + \sup F \quad \leq 3.$

The most used (particular cases are):

i) If *T, I, F* are all single-values (numbers) from [0, 1], then one has a *Singe-Valued Neutrosophic Set* (SVNS);
ii) If *T, I, F* are intervals included in [0, 1], then one has an *Interval-Valued Neutrosophic Set* (IVNS).

2. The MultiNeutrosophic Set

In the real word, in most cases, everything: an attribute, event, proposition, theory, idea, person, object, action, etc., is evaluated in general by many *sources* (called *experts*), let's denote them by $S_1, S_2, \ldots, S_n$, where the number of sources $n \quad \geq 2$ (to ensure the MultiSource). The more sources evaluate a subject, the better accurate result (after fusioning all evaluations).

Therefore, let's assume the degree of truth (or membership) of the generic element x with respect to the set N is evaluated by p sources of information, that give the following results,

respectively $T_1, \ T_2, \ldots, T_p$;

and the degree of indeterminacy (neither truth/membership, nor falsehood/nonmembership) of the element x with respect to the set N is evaluated by r sources of information, that give the following results, respectively: $I_1, \ I_2, \ldots, I_r$;

while the degree of falsehood (or nonmembership) of the element x with respect to the set N is also evaluated by s sources of information that give the following results, respectively: $F_1, \ F_2, \ldots, F_s$;

where all $T_1,\ T_2, \dots, T_p,\ I_1,\ I_2, \dots, I_r,\ F_1,\ F_2, \dots, F_s$ are subsets of [0,1], with p, r, s integers ≥ 0, and at least one of p, r, s is ≥ 2 (in order to ensure the multiplicity of at least one of: truth, indeterminacy, or falsehood), with $p + r + s = n \geq 2$.

All n sources may be either totally independent two by two, or partially independent and partially dependent, or totally dependent - according to the need of each specific application.

In the situation where there is some dependence between sources, we understand that either they communicate with each other and share information (influencing each other), or the same source may evaluate two or three of the components: truth, indeterminacy, falsehood of the same element.

3. General Definition of MultiNeutrosophic Set (or Subset MultiNeutrosophic Set - *SMNS*)

Let $\mathcal{U}$ be a universe of discourse and M a subset of it. Then, a *MultiNeutrosophic Set* is:

$M = \{x, x(T_1, T_2, \dots, T_p;\ \ I_1, I_2, \dots, I_r;\ \ F_1, F_2, \dots, F_s), x \in \mathcal{U},$

where p, r, s are integers $\geq 0,\ p + r + s = n \geq 2,$

and at least one of p, r, s is ≥ 2, in order to ensure the existence of multiplicity of at least one neutrosophic component: truth/membership, indeterminacy, or falsehood/nonmembership;

all subsets $T_1, T_2, \dots, T_p;\ \ I_1, I_2, \dots, I_r;\ \ F_1, F_2, \dots, F_s \subseteq [0,1];$

$0 \leq \sum_{j=1}^{p} infT_j + \sum_{k=1}^{r} infI_k + \sum_{l=1}^{s} infF_l \leq \sum_{j=1}^{p} supT_j + \sum_{k=1}^{r} supI_k + \sum_{l=1}^{s} supF_l \leq n\}.$

No other restrictions apply on these neutrosophic multicomponents.

$T_1, T_2, \dots, T_p$ are multiplicities of the truth, each one provided by a different source of information (expert).

Similarly, $I_1, I_2, \dots, I_r$ are multiplicities of the indeterminacy, each one provided by a different source.

And $F_1, F_2, \dots, F_s$ are multiplicities of the falsehood, each one provided by a different source.

The **Degree of MultiTruth** (MultiMembership), also called *MultiDegree of Truth*, of the element x with respect to the set M are $T_1, T_2, \dots, T_p$;

the **Degree of MultiIndeterminacy** (MultiNeutrality), also called *MultiDegree of Indeterminacy*, of the element x with respect to the set M are $I_1, I_2, \dots, I_r$;

and the **Degree of MultiFalsehood** (MultiNonmembership), also called *MultiDegree of Falsehood*, of element x with respect to the set M are $F_1, F_2, \dots, F_s$.

All these $p + r + s = n \geq 2$ are assigned by n sources (experts) that may be:

— either totally independent;

— or partially independent and partially dependent;

— or totally dependent;

according or as needed to each specific application.

A generic element x with regard to the MultiNeutrosophic Set A has the form:

$$x(\underbrace{T_1, T_2, \dots, T_p}_{\text{multi-truth}};\quad \underbrace{I_1, I_2, \dots, I_r}_{\text{multi-indeterminacy}};\quad \underbrace{F_1, F_2, \dots, F_s}_{\text{multi-falsehood}})$$

In many particular cases $p = r = s$, and a source (expert) assigns all three degrees of truth, indetermincay, and falsehood (T_j, I_j, F_j) for the same element.

4. Particular Cases of MultiNeutrosophic Set (MNS)

Upon the types of sets that the neutrosophic components are, one has:

i. *Single-Valued MultiNeutrosophic Set (SVMNS),*

when all neutrosophic components

$T_j,\ 1 \leq j \leq p;\ T_j,\ I_k, 1 \leq k \leq r,$ and $F_l, 1 \leq l \leq s,$

are single-values (numbers), such that all $T_j, I_k, F_l \in [0, 1]$.

ii. *Interval-Valued MultiNeutrosophic Set (IVMNS),*

when all neutrosophic components

$T_j,\ 1 \le j \le p;\ T_j,\ I_k, 1 \le k \le r$, and $F_l, 1 \le l \le s$,
are interval-values, such that all $T_j, I_k, F_l \subseteq [0, 1]$.

5. Particular Cases of Single-Valued MultiNeutrosophic Set (*SVMNS*)

a. *MultiFuzzy Set*, by setting $p \ge 2$, and $r = s = 0$, into the above SVMNS Definition.

b. *MultiIntuitionistic Set*, by setting $r = 0,\ p$ and $s \ge 1$, with $p + s \ge 3$, and $0 \le T_j + F_l \le 1$, for all $j \in \{1, 2, \dots p\}, l \in \{1, 2, \dots s\}$, into the SVMNS Definition.

c. *MultiPythagorean Fuzzy Set*, by letting $r = 0$, and $p, s \ge 3$, with $p + s \ge 3$, and $0 \le T_j^2 + F_l^2 \le 1$, for $j \in \{1, 2, \dots, p\},\ l \in \{1, 2, \dots, s\}$, into the SVMNS Definition.

d. *MultiFermatean Fuzzy Set*, by letting $r = 0$, and $p, s \ge 1$, with $p + s \ge 3$, and $0 \le T_j^3 + F_l^3 \le 1$, for $j \in \{1, 2, \dots, p\},\ l \in \{1, 2, \dots, s\}$, into SVMNS Definition.

e. *Multi q-Rung Orthopair Fuzzy Set*, by letting $r = 0$, and $p, s \ge 1$, with $p + s \ge 3$, and $0 \le T_j^q + F_l^q \le 1$, with $q \ge 1$, for $j \in \{1, 2, \dots, p\},\ l \in \{1, 2, \dots, s\}$, into SVMNS Definition.

f. *MultiPicture Fuzzy Set*, by letting $p, r, s \ge 1$, with $p + r + s \ge 4$, and $0 \le T_j + N_k + F_e \le 1$, for $j \in \{1, 2, \dots, p\},\ k \in \{1, 2, \dots, r\},\ l \in \{1, 2, \dots, s\}$, where N_k is considered neutral (as in neutrosophic set is ideterminacy) into SVMNS Definition.

g. *MultiSpherical Set*, by setting $p, r, s \ge 1$, with $p + r + s \ge 4$, and $0 \le T_j^2 + I_k^2 + F_e^2 \le 1$, and $R_{jkl} = \sqrt{1 - T_j^2 - I_k^2 - F_e^2}$, for all $j \in \{1, 2, \dots, p\},\ k \in \{1, 2, \dots, r\},\ l \in \{1, 2, \dots, s\}$, into the SVMNS Definition.

6. Particular Cases of Interval-Valued MultiNeutrosophic Set (*IVMNS*)

In an identical way we get the *Particular Cases of Interval-Valued MultiNeutrosophic Set*, as being Interval-Valued (fuzzy and fuzzy-extension) sets, replacing the single-valued components by interval components and using the operations of intervals:

For any $[a, b],\ [c, d] \subseteq [0, 1]$, where $a \le b$ and $c \le d$, one has:

$[a, b] + [c + d] = [min\{a + c, 1\}, min\{b + d, 1\}]$

$[a, b]^n = [min\{a^n, 1\}, min\{b^n, 1\}]$

$1 - [a, b] = [1 - b, 1 - a]$

$[a, b] - [c, d] = [max\{a - d, 0\}, max\{b - c, 0\}]$.

7. Application of Single-Valued MultiNeutrosophic Set

Let $M = \{A, B, C, D\}$ be a group of students.

Their performance in science is evaluated by several professors (= sources of information, experts).

Let's assume that three professors $P_1,\ P_2,\ P_3$ evaluate the degrees of positive knowledge (truth) of the students, and:

– Professor P_1 assigns the value T_1 respectively to all the students;
– Professor P_2 assigns the value T_2 respectively to all the students;
– Professor P_3 assigns the value T_3 respectively to all the students, as follows:

A($T_1 = 0.8,\ T_2 = 0.6,\ T_3 = 0.7$),
B($T_1 = 0.6,\ T_2 = 0.9,\ T_3 = 0.5$),
C($T_1 = 0.4,\ T_2 = 0.4,\ T_3 = 0.6$),
D($T_1 = 0.7,\ T_2 = 0.0,\ T_3 = 0.4$).

But two professors Q_1 and Q_2 are not very sure of the students' performances and assign indeterminate degrees (I_1 and I_2 respectively) to the students:

$A\ \ (I_1 = 0.2, I_2 = 0.3)$,
$B\ \ (I_1 = 0.5, I_2 = 0.4)$,
$C\ \ (I_1 = 0.1, I_2 = 0.0)$,
$D\ \ (I_1 = 0.3, I_2 = 0.1)$.

Further on, four professor $R_1, R_2, R_3,$ and R_4, dissatisfied with the students' performance, assign negative evaluations (falsehood degrees), $F_1, F_2, F_3,$ and F_4 respectively:

A $(F_1 = 0.7, F_2 = 0.4, F_3 = 0.5, F_4 = 0.4)$,

B $(F_1 = 0.6, F_2 = 0.3, F_3 = 0.5, F_4 = 0.1)$,

C $(F_1 = 0.2, F_2 = 0.1, F_3 = 0.2, F_4 = 0.3)$,

D $(F_1 = 0.5, F_2 = 0.2, F_3 = 0.1, F_4 = 0.2)$.

The students have been evaluated by $3 + 2 + 4$ sources of information. In the case that all sources were independent two by two, one has 9 sources. But, if there was some dependence (i.e. the same professor assigning, for example, not only the truth, but also the indeterminacy and/or the falsehood, the number of independent sources is < 9).

The more sources evaluate a subject, the better accurate result.

We got the following single-valued MultiNeutrosophic Set, where each element has the form:

$x(\{T_1, T_2, T_3\}, \{I_1, I_2\}, \{F_1, F_2, F_3, F_4\})$.

$M = \{A(\{0.8, 0.6, 0.7\}, \{0.2, 0.3\}, \{0.7, 0.4, 0.5, 0.4\}),$

$B(\{0.6, 0.9, 0.3\}, \{0.5, 0.4\}, \{0.6, 0.3, 0.5, 0.1\}),$

$C(\{0.4, 0.4, 0.6\}, \{0.1, 0.0\}, \{0.2, 0.1, 0.2, 0.3\}),$

$D(\{0.7, 0.0, 0.4\}, \{0.3, 0.1\}, \{0.5, 0.2, 0.1, 0.2\}).$

7.1. Remark on previous Application

The Single-Valued MultiNeutrosophic Set (SVMNS) coincides in form with the particular case of the Subset Neutrosophic Set (SNS) by taking the neutrosophic components as discrete subsets of the form $\{a_1, a_2, \dots, a_m\} \subset [0,1], m \geq 1$.

For example, considering the student A, his degree of truth (membership) is $T(A) = \{0..8, 0.6, 0.7\}$, his degree of indeterminacy-membership is $I(A) = \{0.2, 0.3\}$, and his degree of falsehood (nonmembership) is $F(A) = \{0.7, 0.4, 0.5, 0.4\}$, from the point of view of Subset Neutrosophic Set.

i. The first distinction is that in the case of Subset Neutrosophic Set, *only one source (expert)* provides information about let's say the student degree $T(A) = \{0.8, 0.6, 0.7\}$, while in the case of Single-Value Multi Neutrosophic Set, three sources provide information on $T(A)$, i.e. one source evaluates the student A degree of truth as 0.8, the second one as 0.6, and the third one as 0.7. The more experts evaluating, the better accuracy, whence the SVMNS better evaluates than the SNS. Similarly for the degree of indeterminacy $I(A)$, and the degree of falsehood $F(A)$.

ii. The second distinction is in applying the neutrosophic operators, since in general the operators for the Subset Neutrosophic Sets are different from the operators for the Single-Valued MultiNeutrosophic Set (we'll see it below on Section 13).

8. Ranking of n-valued MultiNeutrosophic types of the same (*p, r, s*)-form

$(T_1, T_2, \dots, T_p;\ \ I_1, I_2, \dots, I_r;\ \ F_1, F_2, \dots, F_s)$,

where p, r, s are integers ≥ 0, and $p + r + s = n \geq 2$, and at least one of $p, r, s \geq 2$ to be sure that we have multiplicity for at least one neutrosophic component (either truth, or indeterminacy, or falsehood).

The first research in n-ranking neutrosophic triplets was done in 2023 by V. Lakshmana Gomathi Nayagam, and Bharanidharan R. [3], using the dictionary ranking.

We propose an easier n-ranking, but this is rather an approximation. Let's compute the following.

1. Average Positivity:

$$\frac{\sum_{j=1}^{p} T_j + \sum_{k=1}^{r}(1 - I_k) + \sum_{e=1}^{s}(1 - F_e)}{p + r + s}$$

2. Average (Truth-Falsehood):

$$\frac{\sum_{j=1}^{p} T_j - \sum_{e=1}^{s} F_e}{p+s}$$

3. Average Truth

$$\frac{\sum_{j=1}^{p} T_j}{p}$$

Let's compare $(T_1, T_2, \dots, T_p;\ \ I_1, I_2, \dots, I_r;\ \ F_1, F_2, \dots, F_s) \equiv N$
with $(T_1', T_2', \dots, T_p';\ \ I_1', I_2', \dots, I_r';\ \ F_1', F_2', \dots, F_s') \equiv N'$.

If their *Average Positivity* is the same, one gets (1):

$$\sum_{j=1}^{p} T_j - \sum_{k=1}^{r} I_k - \sum_{e=1}^{s} F_e = \sum_{j=1}^{p} T_j' - \sum_{k=1}^{r} I_k' - \sum_{e=1}^{s} F_e'$$

If their *Average (Truth-Falsehood)* is the same, one gets (2):

$$\sum_{j=1}^{p} T_j - \sum_{e=1}^{s} F_e = \sum_{j=1}^{p} T_j' - \sum_{e=1}^{s} F_e'$$

whence, by combining (1) and (2), one gets (3):

$$\sum_{k=1}^{r} I_k = \sum_{k=1}^{r} I_k'$$

If their *Average Truth* is the same, one gets (4):

$$\sum_{j=1}^{p} T_j = \sum_{j=1}^{p} T_j'$$

Then, from (2) and (4), one gets:

$$\sum_{e=1}^{s} F_e = \sum_{e=1}^{s} F_e'$$

Therefore $N = N'$ means that their corresponding averages of truths, indeterminacies, and falsehoods respectively are equal:

$$\begin{cases} \frac{1}{p}\sum_{j=1}^{p} T_j = \frac{1}{p}\sum_{j=1}^{p} T_j' \\ \frac{1}{r}\sum_{k=1}^{r} I_k = \frac{1}{r}\sum_{k=1}^{r} I_k' \\ \frac{1}{s}\sum_{e=1}^{s} F_e = \frac{1}{s}\sum_{e=1}^{s} F_e' \end{cases}$$

9. Ranking *n*-valued MultiNeutrosophic tuples of different (*p, r, s*)–forms

Let's consider two n-valued multi neutrosophic tuples of the forms (p_1, r_1, s_1) and respectively (p_2, r_2, s_2), where $p_1, r_1, s_1, p_2, r_2, s_2$ are integers ≥ 0, and $p_1 + \ \ r_1 + \ \ s_1 = n_1 \geq 2$, and at least one of $p_1, r_1, s_1 \ \ is \geq 2$ to be sure that we have multiplicity for at least one neutrusophic component (either truth, or indeterminacy, or falsehood); similarly $p_2 + \ \ r_2 + \ \ s_2 \geq 2$, and at least one of $p_2, r_2, s_2 \geq 2$. Let's take the following Single-Valued Multi Neutrosophic Tulpes (SVMNT):

$SVMNT = (T_1, T_2, \dots, T_{p_1};\ \ I_1, I_2, \dots, I_{r_1};\ \ F_1, F_2, \dots, F_{s_1})$ of (p_1, r_1, s_1)-form, and

$SVMNT' = (T_1', T_2', \dots, T_{p_2}'; I_1', I_2', \dots, I_{r_2}';\ \ F_1', F_2', \dots, F_{s_2}'\ \ )$ of (p_2, r_2, s_2)-form.

We make the classical averages of truth (T_a), indeterminancies (I_a) and falsehood (F_a), respectively for:

$SVMNT = \ \ (T_a, I_a, F_a)$

and the averages of truths (T_a'), indeterminancies (I_a'), and falsehood (F_a') respectively for:

$SVMNT' = (T'_a, I'_a, F'_a)$.

And then we apply the Score (S), Accuracy (A), and Certainty (C) Functions, as for the single valued neutrosophic set:

1. *Compute the Score Function* (average of positiveness)

$$S(T_a, I_a, F_a) = \frac{(T_a + (1 - I_a) + (1 - F_a)}{3}$$

$$S(T'_a, I'_a, F'_a) = \frac{T'_a + 1 - I'_a + \quad F'_a)}{3}$$

(i) if $S(T_a, I_a, F_a) > S(T'_a, I'_a, F'_a)$, then $SVMNT > SVMNT'$;

(ii) if $S(T_a, I_a, F_a) < S(T'_a, I'_a, F'_a)$, then $SVMNT < SVMNT'$;

(iii) and if $S(T_a, I_a, F_a) = S(T'_a, I'_a, F'_a)$,
then go to the second step.

2. *Compute the Accuracy Function* (difference between the truth and falsehood)

$$A(T_a, I_a, F_a) = T_a - \quad F_a$$

$$A(T'_a, I'_a, F'_a) = T'_a - F'_a$$

(i) if $A(T_a, I_a, F_a) > A(T'_a, I'_a, F'_a)$, then $SVMNT > SVMNT'$;

(ii) if $A(T_a, I_a, F_a) < A(T'_a, I'_a, F'_a)$, then $SVMNT < SVMNT'$;

(iii) and if $A(T_a, I_a, F_a) = A(T'_a, I'_a, F'_a)$,
then go to the third step.

3. *Compute the Certainty Function* (truth)

$$C(T_a, I_a, F_a) = T_a$$

$$C(T'_a, I'_a, F'_a) = T'_a$$

(i) if $C(T_a, I_a, F_a) > C(T'_a, I'_a, F'_a)$, then $SVMNT > SVMNT'$;

(ii) if $C(T_a, I_a, F_a) < C(T'_a, I'_a, F'_a)$, then $SVMNT < SVMNT'$;

(iii) if $C(T_a, I_a, F_a) = C(T'_a, I'_a, F'_a)$, then $SVMNT$ and $SVMNT'$ are multi-neutrosophically equal, i.e. $T_a = T'_a$, $I_a = I'_a$, $F_a = F'_a$, or their corresponding truth, indeterminancy, and falsehood averages are equal.

10. Example 1

Example where all sources providing information have equal weights.

Assume the student George is evaluated by several professors from his university with respect to his skills in science:

George$(\{0.8, 0.9, 0.3\}, \{0.2\}, \{0.6, 0.7\})$

While the student John is evaluated with respect to the same scientific skills by some of the previous professors and by other professors from the same university:

John$(\{0.7, 1.0, 0.6, 0.5\}, \{01., 0.4\}, \{0.2, 0.8, 0.7\})$

Which student does better than the others?

Let's compute the averages.

John$\left(\frac{0.7+1.0+0.6+0.5}{4}, \frac{0.1+0.4}{2}, \frac{0.2+0.8+0.7}{3}\right) \simeq$ John(0.70, 0.25, 0.57).

George$\left(\frac{0.8+0.9+0.3}{3}, \frac{0.2}{1}, \frac{0.6+0.7}{2}\right) \simeq$ George(0.67, 0.20, 0.65).

The Score Function:
S(George) = $\frac{0.67+(1-0.20)\_(1-0.65)}{3} \simeq 0.61$.
S(John) = $\frac{0.70+(1-0.25)\_(1-0.57)}{3} \simeq 0.63$.
John has better scientific skills than George, since S(John) $\simeq 0.63 > 0.61 \approx$S(George).

This may be explained from the fact that if more or less sources evaluate the same element x of a given set, we make the average of evaluations.

In cases some sources have a greater weight in evaluation than others, one uses the *weighted averages*, indexed as T_{wa}, I_{ua}, F_{va} and $T'_{w'a'}$ $I'_{u'a'}$ $F'_{v'a'}$ respectively.

Because the sources may be independent or partially independent, the sum of weights should not necessarily be equal to 1. As such, one has:

$$T_{wa} = \frac{w_1 T_1 + w_2 T_2 + \cdots + w_{p_1} T_{p_1}}{w_1 + w_2 + \cdots + w_{p_1}}$$

where $w_1,\ w_2,\ \ldots,\ w_{p_1} \in [0,1]$, while the sum $w_1 + w_2 + \cdots + w_{p_1}$ may be < 1, or $= 1$, or > 1;

$$I_{ua} = \frac{u_1 I_1 + u_2 I_2 + \cdots + u_{r_1} I_{r_1}}{u_1 + u_2 + \cdots + u_{r_1}}$$

where $u_1,\ u_2,\ \ldots,\ u_{r_1} \in [0,1]$, while the sum $u_1 + u_2 + \cdots + u_{r_1}$ may be < 1, or $= 1$, or > 1;

$$F_{va} = \frac{v_1 F_1 + v_2 F_2 + \cdots + v_{s_1} F_{s_1}}{v_1 + v_2 + \cdots + v_{s_1}}$$

where $v_1,\ v_2,\ \ldots,\ v_{s_1} \in [0,1]$, while the sum $v_1 + v_2 + \cdots + v_s$ may be < 1, or $= 1$, or > 1.

Similarly,

$$T'_{w'a} = \frac{w'_1 T'_1 + w'_2 T'_2 + \cdots + w'_{p_2} T'_{p_2}}{w'_1 + w'_2 + \cdots + w'_{p_2}}$$

where $w'_1,\ w'_2,\ \ldots,\ w'_{p_2} \in [0,1]$, while the sum $w'_1 + w'_2 + \cdots + w'_{p_1}$ may be < 1, or $= 1$, or > 1;

$$I'_{u'a} = \frac{u'_1 I'_1 + u'_2 I'_2 + \cdots + u'_{r_2} I'_{r_2}}{u_1 + u_2 + \cdots + u_{r_1}}$$

where $u'_1,\ u'_2,\ \ldots,\ u'_{r_2} \in [0,1]$, while the sum $u'_1 + u'_2 + \cdots + u'_{r_1}$ may be < 1, or $= 1$, or > 1;

$$F'_{v'a} = \frac{v'_1 F'_1 + v'_2 F'_2 + \cdots + v'_{s_2} F'_{s_2}}{v'_1 + v'_2 + \cdots + v'_{s_2}}$$

where $v'_1,\ v'_2,\ \ldots,\ v'_{s_2} \in [0,1]$, while the sum $v'_1 + v'_2 + \cdots + v'_s$ may be < 1, or $= 1$, or > 1.

And, similarly, one applies the Score, Accuracy, and Certainty Functions on these weighted averages to rank them.

$S(T_{wa}, I_{ua}, F_{va}) = \frac{T_{wa}+(1-I_{ua})+(1-F_{va})}{3}$
$A(T_{wa}, I_{ua}, F_{va}) = T_{wa} - F_{va}$
$C(T_{wa}, I_{ua}, F_{va}) = T_{wa}$

$S(T'_{w'a}, I'_{u'a}, F'_{v'a}) = \frac{T'_{w'a}+(1-I'_{u'a})+(1-F'_{v'a})}{3}$
$A(T'_{w'a}, I'_{u'a}, F'_{v'a}) = T'_{w'a} - F'_{v'a}$
$C(T'_{w'a}, I'_{u'a}, F'_{v'a}) = T'_{w'a}$

11. Example 2

Let's retake the *Example 1*:

George$(\{0.8, 0.9, 0.3\}, \{0.2\}, \{0.6, 0.7\})$,

and John$(\{0.7, 1.0, 0.6, 0.5\}, \{0.1, 0.4\}, \{0.2, 0.8, 0.7\})$,

and assume the six evaluators of George have the following corresponding weights respectively:

0.6, 0.7, 0.4; 0.3; 0.8, 0.7;

while the nine evaluators of John have the following corresponding weights respectively:

0.7, 0.2, 0.5, 0.1; 0.8, 0.3; 0.9, 0.4, 0.6.

Let's compute the weighted averages.

For George:

$$T_{wa} = \frac{0.8 \cdot (0.6) + 0.9 \cdot (0.7) + 0.3 \cdot (0.4)}{0.6 + 0.7 + 0.4} \simeq 0.72$$

$$I_{ua} = \frac{0.2 \cdot (0.3)}{0.3} = 0.20$$

$$F_{va} = \frac{0.6 \cdot (0.8) + 0.7 \cdot (0.7)}{0.8 + 0.7} \simeq 0.65.$$

We got George (0.72, 0.20, 0.65).

For John:

$$T'_{w\prime a} = \frac{0.7 \cdot (0.7) + 1.0 \cdot (0.2) + 0.6 \cdot (0.5) + 0.5 \cdot (0.1)}{0.7 + 0.2 + 0.5 + 0.1} \simeq 0.69$$

$$I'_{u\prime a} = \frac{0.1 \cdot (0.8) + 0.4 \cdot (0.3)}{0.8 + 0.3} \simeq 0.18$$

$$F'_{v\prime a} = \frac{0.2 \cdot (0.9) + 0.8 \cdot (0.4) + 0.7 \cdot (0.6)}{0.9 + 0.4 + 0.6} \simeq 0.48$$

We got John (0.69, 0.18, 0.48).

Compute the score functions in order to rank them.

S(George) = S(0.72, 0.20, 0.65) = $\frac{0.72+(1-0.20)+(1-0.65}{3} \simeq 0.62$.

S(John) = S(0.69, 0.20, 0.65) = $\frac{0.69+(1-0.20)+(1-0.65)}{3} \simeq 0.61$.

Therefore, now George is better, because S(George) = 0.62 $>$ 0.61 = S(John).

12. Isomorphism between Subset Refined Neutrosophic Set (*SRNS*) and Subset MultiNeutrosophic Set (*SMNS*)

The Subset Refined Neutrosophic Set was first introduced by Smarandache [4} in 2013.

12.1. Definition of Subset Refined Neutrosophic Set (SRNS)

Let $\mathcal{U}$ be a universe of discourse, and a set $R \subset \mathcal{U}$.

Then a Subset Refined Neutrosophic R is defined as follows:

$R = \{x, x(T, I, F), x \in \mathcal{U}\}$,

where T is refined/split into p sub-truths,

I is refined/split into r sub-indeterminacies,

$I = \langle I_1, I_2, \dots, I_r \rangle, I_k \subseteq [0, 1],\ 1 \le k \le r,$

and F is refined/split into s sub-falsehoods,

$F = \langle F_1, F_2, \dots, F_s \rangle, F_l \subseteq [0, 1],\ 1 \le l \le s,$

where $p, r, s \ge 0$ are integers, and $p + r + s = n \ge 2$, and at least one of p, r, s is ≥ 2 in order to ensure the existence of refinement (splitting).

Similarly, in particular cases, $p = r = s$, meaning that each component *T, I, F* is refined/split into the same member of sub-components.

The isomorphism is obvious:

$$\varphi: SMNS \to SRNS$$
$$\varphi(T_j) = T_j,\ 1 \le j \le p,$$
$$\varphi(I_k) = I_k,\ 1 \le k \le r,$$
$$\varphi(F_l) = F_l,\ 1 \le l \le s.$$

But while $T_j,\ I_k,\ F_l$ from SMNS are duplicates (or multi-truth, multi-indeterminacy, multi-falsehood respectively), the corresponding $T_j,\ I_k,\ F_l$ from SRNS are parts (or sub-truth, sub-indeterminacy, sub-falsehood respectively).

13. Operators on Multi (and Refined) Neutrosophic Sets/Logic

i. The case when the neutrosophic tuples have the same (p, r, s)-format.

Let $\vee_N,\ \wedge_N,\ \neg_N,\ \to_N,\ \leftrightarrow_N$ be the neutrosophic union, intersection, complement (negation), inclusion (implication), equality (equivalence) respectively.

While $\vee_F,\ \wedge_F,\ \neg_F,\ \to_F,\ \leftrightarrow_F$ the fuzzy operators respectively, where $\vee_F$ and $\wedge_F$ are t-conorm and t-norm respectively, afterwards fuzzy negation, fuzzy implication, and fuzzy equivalence respectively.

Also, by notation, one considers:

$(T_1, T_2, \dots, T_p;\ \ I_1, I_2, \dots, I_r;\ \ F_1, F_2, \dots, F_s) \equiv (T_j, 1 \le j \le p;\ \ I_k, 1 \le k \le r; F_l, 1 \le l \le s\ \ )$.

The operations will be a straightforward extension from the (1, 1, 1)-format (*T, I, F*)
to the (p, r, s)-format.

Multi/Refined Neutrosophic Union

$$(T_1, T_2, \dots, T_{p_1};\ \ I_1, I_2, \dots, I_{r_1};\ \ F_1, F_2, \dots, F_s) \vee_N (T_1', T_2', \dots, T_p'; I_1', I_2', \dots, I_r';\ \ F_1', F_2', \dots, F_s')$$
$$= (T_1 \vee_F T_1', T_2 \vee_F T_2'\ \ , \dots, T_p \vee_F T_p';\ \ I_1 \wedge_F I_2', \dots, I_r \wedge_F I_r'; F_1 \wedge_F F_1', F_2 \wedge_F F_2', \dots, F_s \wedge_F F_s')$$

Shortly, we may write:

$$(T_j, 1 \le j \le p; I_k, 1 \le k \le r; F_l, 1 \le l \le s) \vee_N (T'_j, 1 \le j \le p; I'_k, 1 \le k \le r; F'_l, 1 \le l \le s)$$
$$= (T_j \vee_F T_j', 1 \le j \le p; I_k \wedge_F I'_k, 1 \le k \le r; F_l \wedge_F F_l', 1 \le l \le s)$$

Multi/Refined Neutrosophic Intersection

$$(T_j, 1 \le j \le p; I_k, 1 \le k \le r; F_l, 1 \le l \le s) \wedge_N (T'_j, 1 \le j \le p; I'_k, 1 \le k \le r; F'_l, 1 \le l \le s)$$
$$= (T_j \wedge_F T'_j, 1 \le j \le p; I_k \vee_F I'_k, 1 \le k \le r; F_l \vee_F F'_l, 1 \le l \le s)$$

Multi/Refined Neutrosophic Negation

$$\neg_N(T_j, 1 \le j \le p; 1 \le k \le r; F_l, 1 \le l \le s) = (F_l, 1 \le l \le s; 1 - I_k, 1 \le k \le r; T_j, 1 \le j \le p)$$

Multi/Refined Neutrosophic Implication and Equivalence

Let $A = (T_j, 1 \le j \le p; I_k, 1 \le k \le r; F_l, 1 \le l \le s)$
and $A' = (T'_j, 1 \le j \le p; I'_k, 1 \le k \le r; F'_l, 1 \le l \le s)$.
Then $A \to_N A'$ means $(\neg_N A) \vee_N A'$
and $A \leftrightarrow_N A'$ means $[[A \to_N A']\text{and}[A' \to_N A]]$.

ii. The case when the neutrosophic tuples have different (p, r, s)-formats.

Let $B_1 = (T_j, 1 \le j \le p_1;\ \ I_k, 1 \le k \le r_1;\ \ F_l, 1 \le l \le s_1)$ of (p_1, r_1, s_1)-format,
and $B_2 = (T'_j, 1 \le j \le p_2;\ \ I'_k, 1 \le k \le r_2;\ \ F'_l, 1 \le l \le s_2)$ of (p_2, r_2, s_2)-format.
We compute the weight average of each neutrosophic component of both tuples, and we get:
$B_1 = (T_{wa}, I_{ua}, F_{va})$,
and $B_2 = (T'_{w'a}, I'_{u'a}, F'_{v'a})$,

which have the (1, 1, 1)-form, let's simplify their notation under the form:

$B_1 = (T, I, F)$

and $B_2 = (T', I', F')$.

and one applies the well-known and most used neutrosophic operators:

$$(T, I, F) \vee_N (T', I', F') = (T \vee_F T', I \wedge_F I', F \wedge_F F')$$
$$(T, I, F) \wedge_N (T', I', F') = (T \wedge_F T', I \vee_F I', F \vee_F F')$$
$$\neg(T, I, F) \vee_N (F, 1 - I, T)$$

$(T, I, F) \rightarrow_N (T', I', F')$ is the same as $(F, 1 - I, T) \vee_N (T', I', F')$, or $(F \vee_F T', (1 - I) \wedge_F I', T \wedge_F F')$

$(T, I, F) \leftrightarrow_N (T', I', F')$ is the same as $(T, I, F) \rightarrow_N (T', I', F')$ and $(T', I', F') \rightarrow_N (T, I, F)$

or $(F \vee_F T', (1 - I) \wedge_F I', T \wedge_F F')$ neutrosophic and $(F' \vee_F T, (1 - I_1') \wedge_F I, T' \wedge_F F)$,

or $([(F \vee_F T') \wedge_F (F' \vee_F T)], [(1 - I) \wedge_F I'] \vee_F [(1 - I_1') \wedge_F I], [(T \wedge_F F') \wedge_F (T' \wedge_F F)])$.

13.1. Weight Averaging and Neutrosophic Operators

The (weight) averaging and the neutrosophic operators for (*p, r, s*)-tuples, in general, do not commute.

13.2. Counter-Example

Let's consider the (2,3,2)-tuples:

A = $(\{0.2, 0.3\}, \{0.1, 0.4, 0.5\}, \{0.6, 0.9\})$

and B = $(\{0.8, 0.4\}, \{0.6, 0.0, 0.3\}, \{0.5, 0.6\})$

i. Union, then Averaging Union:

$A \vee_N B =$

$= \left(\left\{max\{0.2, 0.8\}, max\{0.3, 0.4\}, \left\{min\{0.1, 0.6\}\right\}, min\{0.4, 0.0\}, min\{0.5, 0.3\}, \left\{min\{0.6, 0.5\}\right\}, min\{0.9, 0.6\}\right\}\right)$

$= (\{0.8, 0.4\}, \{0.1, 0.0, 0.3\}, \{0.5, 0.6\})$

Averaging:

$$A \vee_N B = \left(\frac{0.8 + 0.4}{2}, \frac{0.1 + 0.0 + 0.3}{3}, \frac{0.5 + 0.6}{2}\right) \simeq (0.60, 0.13, 0.55)$$

ii. Reversely: Averaging, then Union.

Averaging:

$$A = \left(\frac{0.2 + 0.3}{2}, \frac{0.1 + 0.4 + 0.5}{3}, \frac{0.6 + 0.9}{2}\right) \simeq (0.25, 0.33, 0.75)$$
$$B = \left(\frac{0.8 + 0.4}{2}, \frac{0.6 + 0.0 + 0.3}{3}, \frac{0.5 + 0.6}{2}\right) \simeq (0.60, 0.30, 0.55)$$

Union:

$$A \vee_N B = (max\{0.25, 0.60\}, min\{0.33, 0.30\}, min\{0.75, 0.53\})$$
$$= (0.60, 0.30, 0.55) \neq$$

$(0.60, 0.13, 0.55)$.

Conclusion: The MultiNeutrosophic Set was introduced now for the first time. It is a neutrosophic set whose elements' degrees of truth / indeterminacy / falsehood are evaluated by many sources to get a better accurate result. The ranking of the n-tuples of the form (p, r, s) and their operators were also built on.

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Multi-Polar Interval-Valued Neutrosophic Hypersoft Set with Multi-criteria decision making of Cost-Effective Hydrogen Generation Technology Evaluation

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Abstract: The complex process of decision-making is addressed in this study, especially when dealing with diverse factors and input from several specialists. In the context of m-polar interval-valued neutrosophic hypersoft sets (m-PIVNHSSs), the paper proposes innovative adaptations of the correlation coefficient (CC) and weighted correlation coefficient (WCC), drawing on correlation analysis in statistics and engineering. The goal is to improve decision-making processes in scenarios with complicated features and input from several specialists. Through defined theorems and claims, the study offers a solid mathematical framework and presents methods based on CC and WCC to address decision-making complexity. These strategies show promise for enhancing decision accuracy in circumstances involving a wide range of features and expert inputs. AHP, TOPSIS, and other strategies that are now used might also be extended, according to the research. AHP, TOPSIS, and VIKOR are three possible methodologies that might be used to the m-PIVNHSSs environment, according to the research, opening opportunities for additional breakthroughs in the decision-making sector.

Keywords: Aggregate operators, Correlation Coefficient (CC), Neutrosophic Hypersoft Sets (NHSSs), Weighted Correlation Coefficients (WCC), Multi-Polar Neutrosophic Hypersoft Sets (m-PNHSSs).

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1. Introduction

The importance of hydrogen is due to its capacity to transport clean energy and serve as a flexible remedy for pressing global problems. Hydrogen is a clean fuel that enables emissions-free energy production in fuel cells, making it essential for moving away from fossil fuels and reducing global warming. Its capacity to store energy helps manage the oscillations of renewable energy, and its potential to decarbonize industrial sectors like steel and transportation demonstrates the breadth of its effects on emissions reduction. Hydrogen is also a major facilitator of a sustainable and low-carbon future since it drives technical innovation, encourages international cooperation, and spurs economic growth. Numerous research studies have investigated the use of hydrogen in Multi-Criteria Decision Making (MCDM) and its relevance in ambiguous situations. The adaptability of hydrogen as an energy carrier is especially beneficial in ambiguous situations. By storing surplus energy and releasing it when needed, it can help with renewable energy supply variations and provide grid stability in ambiguous energy situations [1]. Energy security in areas susceptible to supply outages can be improved by hydrogen's capacity for decentralized generation and delivery [2].

Hydrogen technology evaluation for MCDM sometimes entails considering several factors, including price, effectiveness, environmental impact, and scalability. This was demonstrated in research by [3], where a fuzzy MCDM technique was used to evaluate several hydrogen generation technologies considering economic, environmental, and technological issues. Furthermore, Qie, X. et al. [4] used an MCDM framework to assess hydrogen storage systems while taking economic, efficient, and safety considerations into account.

In MCDM applications, the flexibility of hydrogen to various situations is further demonstrated. Fayyazi et al.'s [5] analysis of its influence on transportation choices, for instance, considers the adoption of hydrogen fuel cell cars under ambiguous market conditions. Lu, Z., & Li, Y. applied fuzzy in MCDM approaches are used to evaluate various hydrogen generation processes while taking economic and environmental aspects into account.

A new theory was urgently needed to address inconsistencies. To deal with uncertain and inconsistent environments, Smarandache developed a new idea in 1998 [9]. This theory is referred to neutrosophic set (NS) with the addition of indeterminacy value along with membership and non-membership values (T, I, F) (all these values are independent of each other). Based on the numbers (T, I, and F) assigned by the decision-maker (DM) in the form of neutrosophic numbers, this concept of NS was further expanded. For instance, the single-valued neutrosophic set (SVNS) [10], the multi-valued neutrosophic set (MVNS) [11], the interval-valued neutrosophic set (IVNS) [12], and the multi-valued interval neutrosophic set (MVINS) [13]. The idea behind these statistics can be immediately applied to difficulties referred to as multi-criteria decision-making in real-world situations (MCDM). Numerous scholars have proposed various strategies to address MCDM issues using neutrosophic set based algorithms TOPSIS [14], MULTIMOORA [15], AHP [16], SWOT [17], and many more [18].

Numerous scholars have provided numerous uses for neutrosophic sets and their hybrids while taking into consideration MCDM approaches in daily life issues as an application [19-22]. Using mathematical methods, real-world issues such as human resource selection, gadget selection, shortest path selection, robot selection, security considerations, medical equipment selection, and environmental safety measures can be solved. To address ambiguities and get around the difficulties in the current set architectures, Maji suggested the idea of a soft set (SS) [23]. The SS theory was expanded by Cağman et al. [24] to include the features of the fuzzy soft set (FSS). Maji et al. [25] developed the idea of an intuitionistic fuzzy soft set (IFSS) and its attributes to address the issues with uncertainty. Like, Maji [26] extended the idea of neutrosophic sets by combining them to the soft set and presented the theory of neutrosophic soft sets (NSS) to overcome indeterminacy. Interval-

Valued Neutrosophic Soft Set (IVNSS) was introduced by Deli [27] with several fundamental concepts, operations, and decision-making techniques.

Hypersoft set (HSS) is a new set structure that Smarandache [28] proposed in 2018. In essence, this set is the mapping from the product of attributes (which are further divided) to the power set of the universal set and desire set of attributes. The concepts of fuzzy hypersoft sets, intuitionistic hypersoft sets, and neutrosophic hypersoft sets were also put out [28] to address truthiness, uncertainty, and indeterminacy. The definition of the neutrosophic hypersoft set (NHSSs) [29], aggregate operators, similarity measures, distance measures, and the concepts of single-valued neutrosophic hypersoft sets (SVNHSSs), multi-valued neutrosophic hypersoft sets (m-PNHSSs) [30], interval-valued neutrosophic hypersoft sets (IVNHSSs), and multi-valued interval neutrosophic hypersoft sets (m-PVINHSSs) were proposed by [31], along with matrix notations and using these definitions the applications, the algorithms with case studies has been presented by [32-33]. All these situations demonstrate how well hydrogen works in ambiguous situations and how well it works with MCDM [36] techniques for making decisions. Its adaptability and ability to consider a range of criteria and aspects highlight its value as a dynamic solution in both ambiguous situations and difficult decision-making processes. Novel approaches have been demonstrated by recent studies that have advanced a variety of sectors. Paul, Jana, and Pal [37] extended decision-making utilizing Pythagorean fuzzy Hamacher aggregation operators, Du, Wang, and Lu [38] maximized wireless power transmission with an improved approach, and Haq and Saqlain used machine learning for attendance tracking in a pandemic [39]. Convolutional neural networks were employed by Zulqarnain and Saqlain [40] to evaluate text readability in higher education, while Saqlain et al. [41] presented a multi-polar interval-valued neutrosophic hypersoft set for uncertainty and decision-making. These projects demonstrate a dedication to creativity and cross-domain problem-solving [42-46].

This paper makes significant contributions to the field of decision-making by addressing the limitations of existing approaches in dealing with m-PIVNHSs. By introducing the m-PIVNHSs model, this research offers a novel solution to the challenges posed by the abstract and context-dependent nature of language. The implementation of the proposed aggregate operators, correlation coefficients (CC) of a practical tool for solving decision-making issues and improving the overall understanding and application of m-PIVNHSs knowledge. This contribution has the potential to

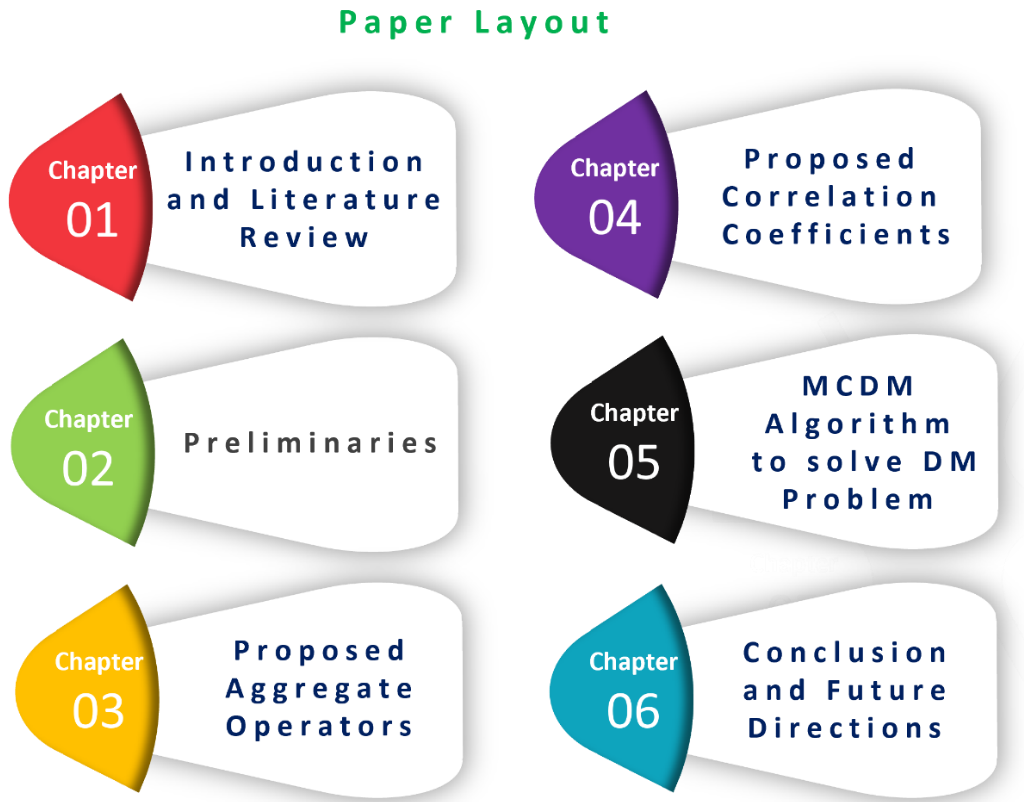

Figure 1. *Layout of the paper*

benefit various fields that rely on language-based decision-making, such as natural language processing, sentiment analysis, and artificial intelligence, among others. The following shows that, how the work has been organized: The fundamental ideas of m-PIVNHSs are broken down in detail in section 2. In section 3, we present a definition, notions, and examples of m-PIVNHSs with basic properties and operations. The aggregate operators, and correlation coefficients (CC) of m-PIVNHSs have been presented in section 4. In part 5, an MCDM framework is described for the "m-PIVNHSs algorithm to solve MCDM problem" with a case study to demonstrate the benefits of the proposed algorithm. The findings of the study have been summarized, along with their significance, in section 6, and concluded with future directions. The layout of the paper is also presented in figure 1.

2. Preliminary section

In this section, we go through some basic definitions that support the construction of the framework of this paper: hypersoft set (HSS), neutrosophic hypersoft set (NHSSs), m-polar neutrosophic hypersoft set (m-PNHSSs), and m-polar interval-valued neutrosophic hypersoft set (m-PIVNHSSs).

Definition 2.1: Hypersoft Set [28]

Assume that universal and power set of universal set is given that μ *and* $P(\mu)$. Considering $(i^1, i^2, i^3, \ldots, i^n)$ when $n \geq 1$, and suppose n be a well-defined attributives, whose corresponded attributive elements are sequentially, the set $(£^1, £^2, £^3, \ldots, £^n)$ with $£^i \cap £^j = \emptyset$, where $i \neq j$ and $i, j \epsilon \{1,2,3 \ldots n\}$, then $(\xi, £)$ is called a hypersoft set;

$$\xi: (£ = £^1 \times £^2 \times £^3 \times \ldots \times £^n) \rightarrow P(\mu) \qquad (1)$$

Definition 2.2: Single-Valued Neutrosophic Hypersoft Set [29]

In equation (1), if we assign the values to each attribute in the form of truthiness, indeterminacy, and falseness $< \mathrm{t}, \mathrm{i}, \mathrm{f} >$ where $\mathrm{t}, \mathrm{i}, \mathrm{f}: \mu \rightarrow [0,1]$ also $0 \leq \mathrm{t}(\xi(\varkappa)) + \mathrm{i}(\xi(\varkappa)) + \mathrm{f}(\xi(\varkappa)) \leq 3$. then the pair then $(\xi, £)$ is called a single-valued neutrosophic hypersoft set.

Definition 2.3: m-Polar Neutrosophic Hypersoft Set [30]

In equation (1) if we assign the values to each attribute in the form of

$$\boldsymbol{\xi}: ((£ = £^{\mathbf{1}} \times £^2 \times £^{\mathbf{3}} \times \ldots \times £^{\boldsymbol{n}}) \rightarrow \boldsymbol{P}(\boldsymbol{\mu})) = \left\{ \begin{matrix} < \boldsymbol{\varkappa}, . \boldsymbol{T}^{\boldsymbol{i}}(\boldsymbol{\xi}(\boldsymbol{\varkappa})) + \boldsymbol{I}^{\boldsymbol{j}}(\boldsymbol{\xi}(\boldsymbol{\varkappa})) + \boldsymbol{F}^{\boldsymbol{k}}(\boldsymbol{\xi}(\boldsymbol{\varkappa})) >. \boldsymbol{\varkappa} \in \boldsymbol{\mu}, \\ \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}. = \mathbf{1}, \mathbf{2}, \mathbf{3}, \ldots, \boldsymbol{n} \end{matrix} \right\} \text{ Also}$$

$$\mathbf{0} \leq \sum_{\boldsymbol{i}=\mathbf{1}}^{\boldsymbol{a}} \boldsymbol{T}^{\boldsymbol{i}}(\boldsymbol{\xi}(\boldsymbol{\varkappa})) \leq \mathbf{1}, \qquad \mathbf{0} \leq \sum_{\boldsymbol{j}=\mathbf{1}}^{\boldsymbol{b}} \boldsymbol{I}^{\boldsymbol{j}}(\boldsymbol{\xi}(\boldsymbol{\varkappa})) \leq \mathbf{1}, \qquad \mathbf{0} \leq \sum_{\boldsymbol{k}=\mathbf{1}}^{\boldsymbol{c}} \boldsymbol{F}^{\boldsymbol{k}}(\boldsymbol{\xi}(\boldsymbol{\varkappa})) \leq \mathbf{1}$$

Where $\boldsymbol{T}^{\boldsymbol{i}}(\boldsymbol{\xi}(\boldsymbol{\varkappa})), \boldsymbol{I}^{\boldsymbol{j}}(\boldsymbol{\xi}(\boldsymbol{\varkappa})), \boldsymbol{F}^{\boldsymbol{k}}(\boldsymbol{\xi}(\boldsymbol{\varkappa})) \subseteq [\mathbf{0}, \mathbf{1}]$ are the fuzzy numbers and

$$\mathbf{0} \leq \textstyle\sum_{\boldsymbol{i}=\mathbf{1}}^{\boldsymbol{a}} \boldsymbol{T}^{\boldsymbol{i}}(\boldsymbol{\xi}(\boldsymbol{\varkappa})) + \sum_{\boldsymbol{j}=\mathbf{1}}^{\boldsymbol{b}} \boldsymbol{I}^{\boldsymbol{j}}(\boldsymbol{\xi}(\boldsymbol{\varkappa})) + \sum_{\boldsymbol{k}=\mathbf{1}}^{\boldsymbol{c}} \boldsymbol{F}^{\boldsymbol{k}}(\boldsymbol{\xi}(\boldsymbol{\varkappa})) \leq \mathbf{3} \qquad (2)$$

then the pair then $(\xi, £)$ is called a m-Polar neutrosophic hypersoft set (m-PNHSSs).

Definition 2.4: m-Polar Interval-Valued Neutrosophic Hypersoft Set [31]

In equation (2) if we assign the values to each attribute in the form of

$$T^i(\xi(\varkappa)) = \left[\left(T^i(\xi(\varkappa))\right)^-, \left(T^i(\xi(\varkappa))\right)^+\right] \subseteq [0,1]$$

$$I^j(\xi(\varkappa)) = \left[\left(I^j(\xi(\varkappa))\right)^-, (I^j(\xi(\varkappa))^+\right] \subseteq [0,1]$$

$$F^k(\xi(\varkappa)) = \left[\left(F^k(\xi(\varkappa))\right)^-, \left(F^k(\xi(\varkappa))\right)^+\right] \subseteq [0,1]$$

Also

$$0 \leq \sum_{i=1}^{a} Sup \ \{T^i(\xi(\varkappa)) \leq 1, \qquad 0 \leq \sum_{j=1}^{b} Sup\{I^j(\xi(\varkappa))\} \leq 1, \qquad 0 \leq \sum_{k=1}^{c} \{F^k(\xi(\varkappa))\} \leq 1$$

And,

$$\mathbf{0} \leq \sum_{\boldsymbol{i}=\mathbf{1}}^{\boldsymbol{a}} \boldsymbol{T}^{\boldsymbol{i}}(\boldsymbol{\xi}(\boldsymbol{\varkappa})) + \sum_{\boldsymbol{j}=\mathbf{1}}^{\boldsymbol{b}} \boldsymbol{I}^{\boldsymbol{j}}(\boldsymbol{\xi}(\boldsymbol{\varkappa})) + \sum_{\boldsymbol{k}=\mathbf{1}}^{\boldsymbol{c}} \boldsymbol{F}^{\boldsymbol{k}}(\boldsymbol{\xi}(\boldsymbol{\varkappa})) \leq \mathbf{3} \qquad (\mathbf{3})$$

then the pair then $(\xi, £)$ is called a m-polar interval-valued neutrosophic hypersoft set (m-PIVNHSSs).

3. Calculations

In this section, we propose informational energy along with some necessary theorems and propositions. Informational energy and correlation coefficients are integral components of effective decision-making. Informational energy signifies the value and significance of available information, influencing decision quality, risk assessment, and resource allocation. Correlation coefficients facilitate the identification of relationships, predictive power, risk management, and decision optimization by quantifying the strength between variables. By leveraging high-energy information and understanding correlations, decision-makers can decide more precise and accurate.

Definition 3.1 Informational Energy of m-PIVNHSSs

Consider (ψ, α) and $((\psi, \beta))$ be two m-polar IVNHSSs; $(\psi, \alpha) =$

$$\{(v_i, [t_{\psi(d^-_k)}(v_i)^{i-}, t_{\psi(d^-_k)}(v_i)^{i+}], [i_{\psi(d^-_k)}(v_i)^{j-}, i_{\psi(d^-_k)}(v_i)^{j+}], ([f_{\psi(d^-_k)}(v_i)^{k-}, f_{\psi(d^-_k)}(v_i)^{k+}]) \mid v_i \in u\} \quad and\ (\phi, \beta)$$
$$= \{([v_i, t_{\phi(d^-_k)}(v_i)^{i-}, v_i, t_{\phi(d^-_k)}(v_i)^{i+}], [i_{\phi(d^-_k)}(v_i)^{j-}, i_{\phi(d^-_k)}(v_i)^{j+}], [f^-{}_{\phi(d^-_k)}(v_i)^{k-}, f^-{}_{\phi(d^-_k)}(v_i)^{k+}]) \mid v_i \in u\}$$

Then, their informational energies can be defined as;

$$S_{m-PIVNHSSs}(\psi, \alpha)$$
$$= \sum_{k=1}^{-}\sum_{i=1}^{+}(\sum_{i=1}^{p}\left(t_{\psi(d^-_k)t}^{i-}(v_i)\right)^2 + \sum_{i=1}^{p}\left(t_{\psi(d^-_k)t}^{i+}(v_i)\right)^2 + \sum_{j=1}^{q}\left(i_{p(a_k)j}^{j-}(v_i)\right)^2$$
$$+ \sum_{j=1}^{q}\left(i_{p(a_k)j}^{j+}(v_i)\right)^2 + \sum_{k=1}^{r}\left(\left(f_{\psi}^{k-}(d^-_k)k(v_i)\right)^2\right) + \sum_{k=1}^{r}\left(\left(f_{\psi}^{k+}(d^-_k)k(v_i)\right)^2\right)$$

$$S_{m-PIVNHSSs}(\phi, \beta)$$
$$= \sum_{k=1}^{-}\sum_{i=1}^{+}(\sum_{i=1}^{p}\left(t_{\phi(d^-_k)i}^{i-}(v_i)\right)^2 + \sum_{i=1}^{p}\left(t_{\phi(d^-_k)i}^{i+}(v_i)\right)^2 + \sum_{j=1}^{q}\left(i_{\phi(d^-_k)j}^{j-}(v_i)\right)^2$$
$$+ \sum_{j=1}^{q}\left(i_{\phi(d^-_k)j}^{j+}(v_i)\right)^2 + \sum_{k=1}^{r}\left(\tilde{f}_{\phi(d^-_k)k}^{k-}(v_i)\right)^2 + \sum_{k=1}^{r}\left(\tilde{f}_{\phi(d^-_k)k}^{k+}(v_i)\right)^2)$$

Definition 3.2 Correlation of two m-PIVNHSSs

Consider $(\psi, \ddot{\alpha})$ and $((\psi, \dddot{\beta}))$ be two m-PIVNHSSs; $(\psi, \ddot{\alpha}) =$

$$\{(v_i, [t_{\psi(d^-_k)}(v_i)^{i-}, t_{\psi(d^-_k)}(v_i)^{i+}], [i_{\psi(d^-_k)}(v_i)^{j-}, i_{\psi(d^-_k)}(v_i)^{j+}], ([f_{\psi(d^-_k)}(v_i)^{k-}, f_{\psi(d^-_k)}(v_i)^{k+}]) \mid v_i \in u\} \quad and\ (\phi, \ddot{\beta})$$
$$= \{([v_i, t_{\phi(d^-_k)}(v_i)^{i-}, v_i, t_{\phi(d^-_k)}(v_i)^{i+}], [i_{\phi(d^-_k)}(v_i)^{j-}, i_{\phi(d^-_k)}(v_i)^{j+}], [f^-{}_{\phi(d^-_k)}(v_i)^{k-}, f^-{}_{\phi(d^-_k)}(v_i)^{k+}]) \mid v_i \in u\}$$

Then, their correlation can be defined as;

$$\hat{C}_{m-PIVNHSSs}((\psi,\ddot{\alpha}),(\psi,\ddot{\beta}))$$
$$= \sum_{k=1}^{-}\sum_{i=1}^{+}(\sum_{i=1}^{p}\left(t_{\psi(d^{-}_{k})}(v_i)^{i-} * v_i, t_{\phi(d^{-}_{k})}(v_i)^{i-}\right)$$
$$+ (\sum_{i=1}^{p}\left(t_{\psi(d^{-}_{k})}(v_i)^{i+} * v_i, t_{\phi(d^{-}_{k})}(v_i)^{i+}\right) + \sum_{j=1}^{q}\left(i_{\psi(d^{-}_{k})}(v_i)^{j-} * i_{\phi(d^{-}_{k})}(v_i)^{j-}\right)$$
$$+ \sum_{j=1}^{q}\left(i_{\psi(d^{-}_{k})}(v_i)^{j+} * i_{\phi(d^{-}_{k})}(v_i)^{j+}\right) + \sum_{k=1}^{r}\left(f_{\psi(d^{-}_{k})}(v_i)^{k-} * f^{-}{}_{\phi(d^{-}_{k})}(v_i)^{k-}\right)$$
$$+ \sum_{k=1}^{r}\left(f_{\psi(d^{-}_{k})}(v_i)^{k+} * f^{-}{}_{\phi(d^{-}_{k})}(v_i)^{k+}\right)$$

And using equation (4) one can calculate the correlation coefficient.

$$C_{\ddot{\beta}}^{\ddot{\alpha}} = \frac{c_{m-PIVNHSSs}\left((\psi,\ddot{\alpha}),(\phi,\ddot{\beta})\right)}{\sqrt{S_{m-PIVNHSSs}((\psi,\ddot{\alpha})} * \sqrt{S_{m-PIVNHSSs}((\phi,\ddot{\beta})}} \qquad (4)$$

Example 3.3

$$\psi_{\ddot{\alpha}} = \left(\begin{array}{l} e_1,\left\{\begin{array}{l}(u_1,([0.4,0.9],[0.3,0.4],[0.3,0.3]),([0.4,0.4],[0.4,0.3],[0.3,0.5]),([0.5,0.6],[0.6,0.3],[0.7,2])) \\ (u_2,([0.4,0.5],[0.1,0.3],[0.7,0.4]),([0.8,0.2],[0.3,0.5],[0.7,0.4]),([0.7,2],[0.4,0.6],[0.5,0.3]))\end{array}\right\} \\ e_2,\left\{\begin{array}{l}(u_1,([0.4,0.5],[0.3,0.7],[0.4,0.3]),([0.1,0.3],[0.2,0.9],[0.2,04]),([0.4,0.8],[0.2,0.6],[0.7,0.2])) \\ (u_2,([0.3,0.6],[0.1,0.5],[0.6,0.5]),([0.4,0.6],[0.2,0.7],[0.3,0.3]),([0.5,0.8],[0.3,0.6],[0.3,0.4]))\end{array}\right\}\end{array}\right)$$

$$\psi_{\ddot{\beta}} = \left(\begin{array}{l} e_1,\left\{\begin{array}{l}(u_1,([0.6,0.1],[0.4,0.9],[0.1,0.5]),([0.5,0.6],[0.2,0.7],[0.3,0.1]),([0.9,0.3],[0.5,0.4],[0.2,0.4])) \\ (u_2,([0.4,0.5],[0.3,0.8],[0.3,0.1]),([0.5,0.7],[0.1,0.4],[0.3,0.6]),([0.4,0.9],[0.2,0.4],[0.3,0.5]))\end{array}\right\} \\ e_2,\left\{\begin{array}{l}(u_1,([0.1,0.9],[0.5,0.4],[0.3,0.4]),([0.2,0.7],[0.7,0.3],[0.5,0.1]),([0.4,0.8],[0.3,0.5],[0.7,0.2])) \\ (u_2,([0.3,0.4],[0.4,0.6],[0.3,0.8]),([0.5,0.2],[0.4,0.5],[0.5,0.3]),([0.6,0.9],[0.1,0.4],[0.3,0.5]))\end{array}\right\}\end{array}\right)$$

Proposition 3.4 Consider two m-PIVNHSSs;

$(\psi,\ddot{\alpha}) =$
$\{(v_i,\left[t_{\psi(d^{-}_{k})}(v_i)^{i-}, t_{\psi(d^{-}_{k})}(v_i)^{i+}\right],\left[i_{\psi(d^{-}_{k})}(v_i)^{j-}, i_{\psi(d^{-}_{k})}(v_i)^{j+}\right],\left(\left[f_{\psi(d^{-}_{k})}(v_i)^{k-}, f_{\psi(d^{-}_{k})}(v_i)^{k+}\right]\right) \mid v_i \in u\}$

and

$(\phi,\ddot{\beta}) =$
$\{(\left[v_i, t_{\phi(d^{-}_{k})}(v_i)^{i-}, v_i, t_{\phi(d^{-}_{k})}(v_i)^{i+}\right],\left[i_{\phi(d^{-}_{k})}(v_i)^{j-}, i_{\phi(d^{-}_{k})}(v_i)^{j+}\right],\left[f^{-}{}_{\phi(d^{-}_{k})}(v_i)^{k-}, f^{-}{}_{\phi(d^{-}_{k})}(v_i)^{k+}\right]) \mid v_i \in u\}$

and $C_{m-PIVNHSSs}\left((\psi,\dddot{A}),(\phi,\dddot{\beta})\right)$ correlation between them.

It satisfies the following properties:

1. $\hat{C}_{m-p_{IVNHSSs}}(\psi,\ddot{\alpha}),(\psi,\ddot{\alpha}) = \delta_{m-\ PIVNHSSs}\,(\psi,\ddot{\alpha})$
2. $C_{m-\ PIVNHSSs}\,(\phi,\ddot{\beta}),(\phi,\ddot{\beta}) = \delta_{m-\ PIVNHSSs}\,(\phi,\ddot{\beta})''$

Theorem 3.5 Let $(\psi,\alpha) =$

$\{(v_i,\left[t_{\psi(d^{-}_{k})}(v_i)^{i-}, t_{\psi(d^{-}_{k})}(v_i)^{i+}\right],\left[i_{\psi(d^{-}_{k})}(v_i)^{j-}, i_{\psi(d^{-}_{k})}(v_i)^{j+}\right],\left(\left[f_{\psi(d^{-}_{k})}(v_i)^{k-}, f_{\psi(d^{-}_{k})}(v_i)^{k+}\right]\right) \mid v_i \in u\}$ *and* $(\phi,\beta) =$
$\{(\left[v_i, t_{\phi(d^{-}_{k})}(v_i)^{i-}, v_i, t_{\phi(d^{-}_{k})}(v_i)^{i+}\right],\left[i_{\phi(d^{-}_{k})}(v_i)^{j-}, i_{\phi(d^{-}_{k})}(v_i)^{j+}\right],\left[f^{-}{}_{\phi(d^{-}_{k})}(v_i)^{k-}, f^{-}{}_{\phi(d^{-}_{k})}(v_i)^{k+}\right]) \mid v_i \in u\}$ be two m-PIVNHSSs, the following characteristics are satisfied by CC between them:

1. $0 \le \delta_{m-PIVNHSSs}((\psi,\alpha),(\phi,\beta)) \le 1$
2. $\delta_{m-PIVNHSSs}((\psi,\alpha),(\phi,\beta))$=$\delta_{m-PIVNHSSs}((\psi,\alpha),(\phi,\beta))$ *iff* $((\psi,\alpha) = (\phi,\beta))$

3. $T_{\psi(d_k)}(v_i)^i = T_{\phi(d_k)}(v_i)^i, I_{\psi(d_k)}(v_i)^j = I_{\phi(d_k)}(v_i)^j$ and $\hat{C}_{\psi(d_k)}(v_i)^k = \hat{C}_{\phi(d_k)}(v_i)^k$ then $\delta_{m-p_{IVNHSSs}}((\psi,\alpha),(\phi,\beta)) = 1$

4. Notion of Weighted Correlation Coefficients (WCC) under m-PIVNHSSs

When experts assign different weights to each option, the choice may be different. Therefore, it is essential to map the expert weights before putting together a conclusion. Assume that the experts' relative weights may be stated as $\Omega = \{\Omega_1, \Omega_2, \Omega_3, \dots, \Omega_m\}^T$, where $\Omega_k > 0, \sum \Omega_{k}{}_{k=1}^{m} = 1$. Assume the weights for the sub-attributes to be as follows. $= \{\gamma_1, \gamma_2, \gamma_3, \dots, \gamma_n\}^T$, where $\gamma_i > 0, \sum \gamma_{i}{}_{i=1}^{n} = 1$

Definition 4.1 Weighted correlation coefficient (WCC)

Let,

$$(\psi, \ddot{\alpha}) = \{(v_i, [t_{\psi(d^-_k)}(v_i)^{i-}, t_{\psi(d^-_k)}(v_i)^{i+}], [i_{\psi(d^-_k)}(v_i)^{j-}, i_{\psi(d^-_k)}(v_i)^{j+}], ([f_{\psi(d^-_k)}(v_i)^{k-}, f_{\psi(d^-_k)}(v_i)^{k+}]) \mid v_i \in u\} \quad and$$

$$(\phi, \ddot{\beta}) = \{([v_i, t_{\phi(d^-_k)}(v_i)^{i-}, v_i, t_{\phi(d^-_k)}(v_i)^{i+}], [i_{\phi(d^-_k)}(v_i)^{j-}, i_{\phi(d^-_k)}(v_i)^{j+}], [f^-{}_{\phi(d^-_k)}(v_i)^{k-}, f^-{}_{\phi(d^-_k)}(v_i)^{k+}]) \mid v_i \in u\}$$

be two m-PIVNHSSs, then WCC can be presented as;

$$\rho_{m-PIVNHSSs}((\psi, \dddot{\alpha}), (\phi, \dddot{\beta})) = \frac{\hat{C}_{m-PIVNHSSs}((\psi,\dddot{\alpha})),(\phi,\dddot{\beta})))}{max\big((S_{m-PIVNHSSs}(\psi,\alpha'')),(S_{m-PIVNHSSs}(\phi,\dddot{\beta})))\big)} \tag{5}$$

Theorem 4.2 Let $(\psi, \ddot{\alpha}) = \{(v_i, [t_{\psi(d^-_k)}(v_i)^{i-}, t_{\psi(d^-_k)}(v_i)^{i+}], [i_{\psi(d^-_k)}(v_i)^{j-}, i_{\psi(d^-_k)}(v_i)^{j+}], ([f_{\psi(d^-_k)}(v_i)^{k-}, f_{\psi(d^-_k)}(v_i)^{k+}]) \mid v_i \in u\}$ $and\ (\phi, \beta) = \{([v_i, t_{\phi(d^-_k)}(v_i)^{i-}, v_i, t_{\phi(d^-_k)}(v_i)^{i+}], [i_{\phi(d^-_k)}(v_i)^{j-}, i_{\phi(d^-_k)}(v_i)^{j+}], [f^-{}_{\phi(d^-_k)}(v_i)^{k-}, f^-{}_{\phi(d^-_k)}(v_i)^{k+}]) \mid v_i \in u\}$ the WCC between them meets the following qualities:

1. $0 \le \rho_{m-PIVNHSSs}\big((\psi, \ddot{\alpha})(\phi, \ddot{\beta})\big) \le 1$
2. $\rho_{m-PIVNHSSs}((\psi, \ddot{\alpha})(\phi, \ddot{\beta})) = \rho_{m-PIVNHSSs}((\phi, \ddot{\beta}), (\psi, \ddot{\alpha}))$ iff $(\psi, \ddot{\alpha}) = (\phi, \ddot{\beta})$
3. $T_{\psi(d_k)}(v_i)^i = T_{\phi(d_k)}(v_i), I_{\psi(d_k)}(v_i)^j = I_{\phi(d_k)}(v_i)^j$ and $\hat{C}_{\psi(d_k)}(v_i)^k = \hat{C}_{\phi(d_k)}(v_i)^k$ then $\rho_{m-PIVNHSSs}((\psi, \ddot{\alpha}), (\phi, \ddot{\beta})) = 1$

Definition 4.3 Properties of m-PIVNHSSs

Let $(\psi, \ddot{\alpha}) = \{(v_i, [t_{\psi(d^-_k)}(v_i)^{i-}, t_{\psi(d^-_k)}(v_i)^{i+}], [i_{\psi(d^-_k)}(v_i)^{j-}, i_{\psi(d^-_k)}(v_i)^{j+}], ([f_{\psi(d^-_k)}(v_i)^{k-}, f_{\psi(d^-_k)}(v_i)^{k+}]) \mid v_i \in u\}$

Consider, $\hat{J}_{d_k} = \langle T^i_{F(d_{ij})}, I_{F(d_{ij})}{}^j, \hat{C}_{F(d_{ij})}{}^k \rangle, \hat{J}_{d_{11}} = \langle T_{F(d_{11})})^i, I_{F(d_{11})}{}^j, \hat{C}_{F(d_{11})}{}^k \rangle$ and $\hat{J}_{d_{12}} = \langle T_{F(d_{11})})^i, I_{F(d_{12})}{}^j, \hat{C}_{F(d_{12})}{}^k \rangle$ be three m-PIVNHSSs and y be the positive real number, by algebraic norms, then;

1. $\hat{\jmath}_{d_{11}}{}^{i} \oplus \hat{\jmath}_{d_{12}}{}^{i} = \langle T_{F(d_{11})}{}^{i} + T_{F(d_{12})}{}^{i} - T_{F(d_{11})}{}^{i} T_{F(d_{12})}{}^{i}, \hat{\jmath}_{F(d_{11})}{}^{j} \hat{\jmath}_{F(d_{12})}{}^{j}, \hat{C}_{F(d_{11})}{}^{k} \hat{C}_{F(d_{12})}{}^{k} \rangle$
2. $\hat{\jmath}_{d_{11}}{}^{i} \otimes \hat{\jmath}_{d_{12}}{}^{i} = \langle T_{F(d_{11})}{}^{i} T_{F(d_{12})}{}^{i}, \hat{\jmath}_{F(d_{11})}{}^{j} + \hat{\jmath}_{F(d_{12})}{}^{j} - \hat{\jmath}_{F(d_{11})}{}^{j} \hat{\jmath}_{F(d_{12})}{}^{j}, \hat{C}_{F(d_{11})}{}^{k} + \hat{C}_{F(d_{12})}{}^{k} - \hat{C}_{F(d_{11})}{}^{k} \hat{C}_{F(d_{12})}{}^{k} \rangle$
3. $y\hat{\jmath}_{d_k} = \langle 1 - \left(1 - T_{d_k}{}^{i}\right)^{y}, \hat{\jmath}_{d_k}{}^{j^{y}}, \hat{C}_{d_k}{}^{k^{y}} \rangle$
4. $\hat{\jmath}_{d_k}{}^{i^{y}} = \langle, 1 - \left(1 - \hat{\jmath}_{d_k}{}^{j}\right)^{y}, 1 - \left(1 - \hat{C}_{d_k}{}^{k}\right)^{y} \rangle$

5. MCDM Algorithm (MULTIMOORA).

The Multi-Objective Optimization by Ratio Analysis (MOORA) method was initially developed by Brauers et al. [34]. In 2010, Brauers [35] further enhanced the MOORA technique by introducing the full multiplicative form, resulting in a more efficient and powerful method known as MULTIMOORA. The MULTIMOORA method consists of three stages: the ratio system approach (RSA), the reference point approach (RPA), and the full multiplicative form (FMF). These stages are utilized to rank the alternatives under consideration. The theory of dominance is then applied to determine the final ranking and decision. According to this theory, the alternative with the highest presence at the top position across all three ranking lists is selected as the best-ranked alternative."

Step 1: Construction of decision matrix.
Step 2: RSA approach.
In this approach, the general standing of the alternative i can be measured as follows:

$$\mathcal{Y}_i = \mathcal{Y}_i^{+} - \mathcal{Y}_i^{-}$$

Where,

$$\mathcal{Y}_i^{+} = \sum_{j \in \Omega_{max}} \omega_i \, r_{ij}$$

$$\mathcal{Y}_i^{-} = \sum_{j \in \Omega_{min}} \omega_i \, r_{ij}$$

$$r_{ij} = \frac{x_{ij}}{\sum_{i=1} x_{ij}}$$

where $\mathcal{Y}_i$ stands for ith position of the alternative on the base of all criteria; $\mathcal{Y}_i^{+} and$ $\mathcal{Y}_i^{-}$ denotes the position of the ith alternative according to benefit and cost criteria respectively, r_{ij} represents the normalized ith alternative under jth criteria; x_{ij} denotes the ith alternative related to jth criterion; the sets of benefit criteria are denoted by max and min denotes the cost criteria where $i = 1,2,3, \dots m$ and $j = 1,2,3, \dots, n$. The associated alternatives are positioned depending on $\mathcal{Y}_i$ in descending order so the alternative having the largest value of $\mathcal{Y}_i$ is the best in this approach.

Step 3: RPA approach
Using this approach best alternative selection could be done as below:

$$\eth_i^{max} = \max_j(\omega_j \left| r_j^* - r_{ij} \right|)$$

Where $\eth_i^{max}$ denotes the extreme distance of the alternative i with respect to the reference point and r_j^* represents the coordinate j of the reference point as follows.

$$r_j^* = \begin{cases} \max_i \ r_{ij} & , \quad j \in \Omega_{max} \\ \min_i \ r_{ij} & , \quad j \in \Omega_{min} \end{cases}$$

The final ranking in this approach is done by using ascending order of $\eth_i^{max}$ and accordingly the lowest $\eth_i^{max}$ value is the best one.

Step 4: FMF approach

For this form the total efficacy of the alternative could be obtained as follows:

$$u_i = \frac{a_i}{b_i}$$

Where,

$$a_i = \prod_{j \in \Omega_{max}} \omega_i r_{ij}$$

$$b_i = \prod_{j \in \Omega_{min}} \omega_i r_{ij}$$

Here: u_i means the overall efficacy of the ith alternative, aa_i and b_i indicate the product of the weighted performance ratings of the benefit and cost criteria of the ith alternative respectively. Like RSA, the associated alternatives are graded in descending order based on the value of u_i and the best alternative is selected having maximum value of u_i.

Step 5: The final rank of alternatives established through the MULTIMOORA method.

After the calculating using MULTIMOORA method, three ranking lists are obtained for the alternatives under consideration. According to Brauers [34], dominace theory is used and the alternative having the first positions in all ordered rankings is the best-ranked alternative.

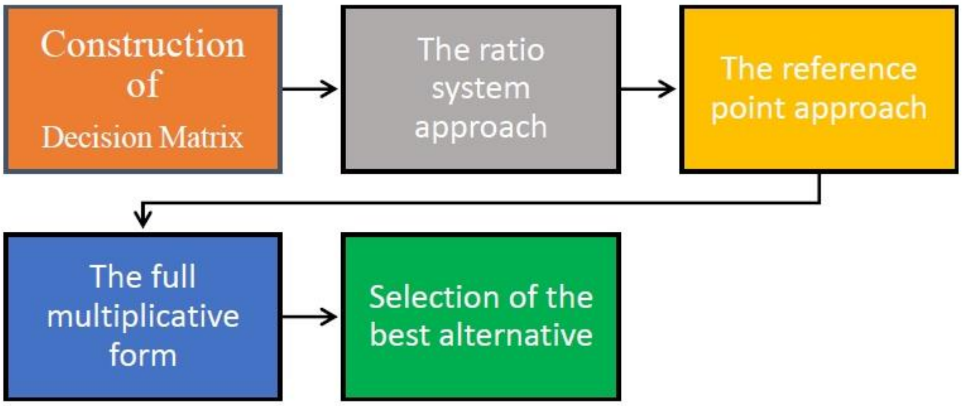

Figure 2. MCDM Algorithm (MULTIMOORA)

5.1 *Illustrative Example*

To determine the most efficient and affordable technology for hydrogen production, we conducted a case study. The process involved several steps, including selecting various alternatives, establishing a criteria system, and gathering relevant data. Within this study, we evaluated eight different hydrogen production technologies, focusing on their abstract descriptions. Drawing upon prior research in this field, we identified seven criteria that encompassed both cost and benefit aspects. The data used in this analysis was collected from diverse hydrogen production technologies available in table 1 [34].

Table 1. *Hydrogen production technologies statistics till 2013.*

Method		*CO2 E.E*	*EE*	*CC*	*FOC*	*VOC*	*FDC*	*EAC*
		C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	*SMR*	0.080	77.5	172.35	06.48	135.70	128.00	156.02
A_2	*CG*	0.076	55.8	511.48	25.81	37.550	33.190	104.40
A_3	*POX*	0.136	67.5	326.60	30.99	191.97	65.320	249.17
A_4	*BG*	0.020	42.5	262.06	16.71	69.420	44.030	107.16
A_5	*PV-EL*	0.040	31.2	388.32	16.71	250.66	246.31	298.53
A_6	*W-EL*	0.005	33.8	388.32	16.71	117.59	112.60	165.46
A_7	*H-EL*	0.010	52.0	388.32	16.71	92.840	87.970	140.71
A_8	*WS-CL*	0.012	21.0	857.46	131.67	12.820	11.540	213.29

The weights are calculated using the entropy method. $w1 = 0.2544$ $w2 = 0.0453$; $w3 = 0.0620$; $w4 = 0.2874$; $w5 = 0.1415$ $w6 = 0.1703$; $w7 = 0.0391$

Solution:

Step 1. *Construction of decision matrix and it is same as table 1.*

Step 2. RSA approach.

Applying the method we get,

$\mathcal{Y}_1 \;=\; -\; 0.0027401$

$\mathcal{Y}_2 \;=\; +\; 0.0032370$

$\mathcal{Y}_3 = \;\; 0.00719220$

$\mathcal{Y}_4 \;=\; -0.02882$

$\mathcal{Y}_5 = \;\; -0.099813$

$\mathcal{Y}_6 = \;\; -0.067408$

$\mathcal{Y}_7 = \;\; -0.051607$

$\mathcal{Y}_8 \;=\; -0.16064$

$$\mathcal{Y}_3 > \mathcal{Y}_2 > \mathcal{Y}_1 > \mathcal{Y}_4 > \mathcal{Y}_5 > \mathcal{Y}_6 > \mathcal{Y}_7 > \mathcal{Y}_8$$

Step 3. RPA approach.

Using this approach, the alternative orders are.

$ð_1^{max} = \;\; 0{:}\,0376$

$ð_2^{max} = \;\; 0{:}\,0403$

$ð_3^{max} = \;\; 0{:}\,0279$

$ð_4^{max} = \;\; 0{:}\,0779$

$ð_5^{max} = \;\; 0{:}\,0645$

$ð_6^{max} = \;\; 0{:}\,0879$

$ð_7^{max} = \;\; 0{:}\,0846$

$ð_8^{max} = \;\; 0{:}\,1375$

$$ð_3^{max} < ð_1^{max} < ð_2^{max} < ð_5^{max} < ð_4^{max} < ð_7^{max} < ð_6^{max} < ð_8^{max}$$

Step 4. FMF approach.

Using this approach, the total efficacy of the all the alternatives are obtained.

$u_1 \;=\; 8015519.718$

$u_2 \;=\; 9907856.161$

$u_3 \;=\; 1132088.766$

$u_4 \;=\; 2362483.732$

$u_5 \;=\; 40864.3973$

$u_6 \;=\; 47005.1677$

$u_7 \;=\; 273951.5255$

$u_8 \;=\; 277721.2461$

$$u_2 > u_i > u_4 > u_3 > u_8 > u_7 > u_6 > u_5$$

Step 5. Selection of best alternative.
The ranking of alternatives using all the approaches has been obtained.

Table 2. *Hydrogen production technologies ranking*

Method	Alternative Scores ranking
RSA	$\mathcal{Y}_3 > \mathcal{Y}_2 > \mathcal{Y}_1 > \mathcal{Y}_4 > \mathcal{Y}_5 > \mathcal{Y}_6 > \mathcal{Y}_7 > \mathcal{Y}_8$
RPA	$\mathfrak{d}_3^{max} < \mathfrak{d}_1^{max} < \mathfrak{d}_2^{max} < \mathfrak{d}_5^{max} < \mathfrak{d}_4^{max} < \mathfrak{d}_7^{max} < \mathfrak{d}_6^{max} < \mathfrak{d}_8^{max}$
FMF	$u_2 > u_i > u_4 > u_3 > u_8 > u_7 > u_6 > u_5$

According to [34] dominance theory is used and the alternative having the first positions in all ordered rankings is the best-ranked alternative. Table 2 shows that $\boldsymbol{A_3}$ is the best-ranked alternative. POX (Partial Oxidation) is recognized as another widely employed technique for hydrogen production from fossil fuels. This method involves the conversion of hydrocarbon-based fossil fuels, including natural gas, coal, and heavy oil, into hydrogen. Through the POX process, these fuels undergo partial oxidation, resulting in the production of hydrogen gas. POX is a well-established method utilized to harness the hydrogen potential inherent in fossil fuel resources.

5.2 *Result Discussion and Comparison*

We have been able to identify complicated linkages inside complex systems by using correlation coefficients. We have discovered possible connections that would have otherwise remained buried inside the complexity of the system by analyzing the interaction between several factors involved in hydrogen creation. This understanding is particularly useful since it provides a greater grasp of the fundamental mechanisms at work by illuminating how many elements interact and affect one another. Additionally, the MULTIMOORA a MCDM technique's inclusion of the multipolar analysis improves our capacity to negotiate this complexity.

We successfully combined the novel idea of multipolar analysis with correlation coefficients using the MULTIMOORA a MCDM method. The evaluation of efficient methods for producing hydrogen was the focus of our work. Our thorough investigation and implementation of these approaches produced insightful findings about the complex dynamics of the hydrogen generating environment. The development of a solid decision-making framework was made possible in large part by the identification of probable links and dependencies among various characteristics using the correlation coefficients. The multipolar method using MULTIMOORA was then used to provide a full evaluation of the many criteria associated with the hydrogen-generating systems. Using this method, we were able to weigh other important aspects in addition to cost efficiency. Consequently, we were able to rank and prioritize the various hydrogen generating processes efficiently, considering a wide range of factors. The effective use of correlation coefficients and the cutting-edge multipolar analysis using MULTIMOORA is an example of the power of this integrated strategy in tackling challenging real-world issues like the production of sustainable energy. Our findings indicate how this technique may be used in a variety of decision-making contexts, as well as providing contributions to the field of hydrogen generation.

6. Conclusion

The correlation coefficient (CC) and weighted correlation coefficient (WCC) for the m-polar interval-valued neutrosophic hypersoft set (m-PIVNHSSs) are presented in this paper, and their fundamental features are examined within the context of m-PIVNHSSs. This ground-breaking method has enormous promise for addressing difficult decision-making issues in a variety of fields, including education, healthcare, engineering, economics, and more. Additionally, the combination of the m-polar hypersoft set with other cutting-edge soft computing methods, such as bipolar fuzzy, Pythagorean set, and hybrid structures, holds the key to creating extraordinarily intelligent systems with improved machine intelligence (IQ). Such connections open new avenues for intelligent problem-solving and knowledge representation, offering interesting opportunities for applications in image processing, expert systems, and cognitive mapping.

Our study provides a thorough comparison of the recently suggested cost-effective hydrogen generating approaches versus current technologies by integrating correlation coefficients using the MULTIMOORA methodology. The correlation coefficients make possible trade-offs and synergies between factors visible, allowing for a more thorough review. The multipolar analysis then considers several factors, offering a comprehensive evaluation of each technique's performance in terms of economic viability, environmental effect, and technical maturity. This integrated methodology enables decision-makers to choose the most appropriate hydrogen generation technique while considering both novel solutions and tried-and-true methods, eventually directing sustainable energy choices and guiding future research paths.

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A Novel Classroom Teaching Evaluation Method for Assessing Learning Effectiveness Based on Machine Vision and Neutrosophic Sets

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Abstract: With the development of intelligent technology, machine vision is gradually applied to classroom teaching. Considering the uncertainty of students' class status, the application of neutrosophic sets provides a novel way for classroom evaluation. In this context, this study proposes a novel classroom teaching evaluation method based on machine vision and neutrosophic sets to better evaluate students' learning effectiveness. The main innovation of this study is to construct a temporal neutrosophic evaluation model that considers students' concentration. Specifically, machine vision technology first is used to detect students' status so as to construct temporal neutrosophic evaluation matrices on students' class status. Thereafter, this study proposes a novel time weight function considering students' concentration based on the Pearson correlation coefficient. Then, this study introduces evaluation based on distance from average solution to address multi-criteria decision-making issues. Finally, the validity and feasibility of the proposed evaluation model are illustrated through a case study and comparative analyses. The results indicate that the ranking of the proposed method is $1 \succ 3 \succ 4 \succ 2$, which is consistent with comparative analyses. The aforementioned study further validates the practical value and provides valuable insights for teaching evaluation methods.

Keywords: learning effectiveness evaluation; neutrosophic sets; machine vision; evaluation based on distance from average solution; multi-criteria decision-making

1. Introduction

With the continuous advancement of intelligent technology, colleges are progressively adopting intelligent classroom devices, incorporating cameras and sensors, for monitoring students' activities and emotional status in the classroom [1]. These intelligent devices not only capture some information such as students' facial expressions, movements, and postures, but also provide feedback on students' classroom engagement and emotional status [2]. In this context, machine vision technology has become crucial in enhancing education quality. Analyzing students' classroom behaviors, it accurately captures students' class learning status, concentration, participation, and even emotional states. It is obvious that machine vision provides educators with powerful tools to better comprehend students' needs, facilitating personalized adjustments in teaching methods. Despite remarkable progress in machine vision technology for classroom monitoring, challenges arise due to the diverse nature of students' participation, understanding of subjects, and learning patterns. To tackle this complexity, neutrosophic sets (NSs), a method for handling uncertainty, typically provide decision-makers in educational work with tools to make education decisions more discerningly [3].

To address the aforementioned challenges, teaching evaluation plays a crucial role as a key component in achieving high-quality education. The Global Education Monitoring Report emphasizes the necessity of establishing effective student assessment and monitoring mechanisms to track students' learning efficiency, thereby enhancing teaching quality [4]. This further underscores the vital role of teaching evaluation in achieving high-quality education, particularly in assessing students' learning effectiveness. However, traditional evaluation processes are often subjective and limited, lacking widely accepted methods [5]. In practical scenarios, teaching evaluation requires careful consideration of multidimensional factors, including subject characteristics, student diversity, and the allocation of teaching resources. Consequently, assessing teaching is treated as a multi-criteria decision-making (MCDM) problem, involving a comprehensive balance among diverse pivotal factors [6]. To ensure that the assessment of students' learning effectiveness is effective, equitable, and meaningful, it is essential to develop and adhere to a high-quality evaluation methodology, thereby promoting education quality.

Aiming at the above issues, this study proposes a quantification classroom teaching evaluation method that combines machine vision with NSs to better evaluate students' learning effectiveness. Specifically, the following summarizes the main contributions of this study.

First, this study uses machine vision technology to identify and process the data on students' class status. According to the identified data, this study constructs temporal neutrosophic evaluation matrices on students' class status.

Second, a novel weight calculation method considering students' concentration is proposed based on the Pearson correlation coefficient. It not only reflects students' class status over time, but also measures the correlation between time and students' concentration. Moreover, the proposed weight function is objective, avoiding subjective influences.

Third, a classical single-valued neutrosophic Dombi weighted arithmetic average (SVNDWAA) operator is introduced. In addition, a similarity function is presented to implement the evaluation problems based on distance from average solution (EDAS), facilitating a comprehensive assessment on students' learning effect.

The rest of this study is formed as follows. Section 2 introduces a series of literature on NSs, machine vision and MCDM. Section 3 introduces the related definitions of NSs, presents the process of machine vision recognition, and proposes a temporal neutrosophic evaluation model. Section 4 provides an example and comparative analysis to verify the practical value of the proposed model. Section 5 generalizes the work and future prospects. The research framework is shown in Figure 1.

2. Literature Review

Scholars have spent much effort exploring and investigating NSs, machine vision and MCDM for achieving high-quality education. This section reviews relevant previous research on assessing education quality, providing an overview of existing deficiencies that require attention.

To evaluate university courses, classroom learning effectiveness is usually an explicit indicator for assessing teaching quality. In this regard, some scholars have delved into the application of NSs in exploring the evaluation of classroom learning effectiveness [7]. For example, Tang et al. [8] established a compromise solution using single-valued neutrosophic measurement of alternatives for enhancing students' learning outcomes. Subsequently, Wu and Fang [9] constructed an innovation multilevel teaching quality evaluation framework in higher education, integrating the Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) with single-valued neutrosophic sets (SvNSs). At the same time, Mamites et al. [10] presented an analysis method on a neutrosophic decision-making trial and evaluation laboratory to study causal relationships affecting teaching quality in universities. Later, Rao and Xiao [11] proposed a novel generalized 2-tuple linguistic neutrosophic power Heronian mean operator applied in the MCDM algorithm, thereby better evaluating physical education quality. Then, Xie [12] presented a triangular fuzzy neutrosophic numbers grey relational analysis method, which expands the traditional classroom teaching mode and provides a novel insight for evaluating students' blended teaching effectiveness in colleges.

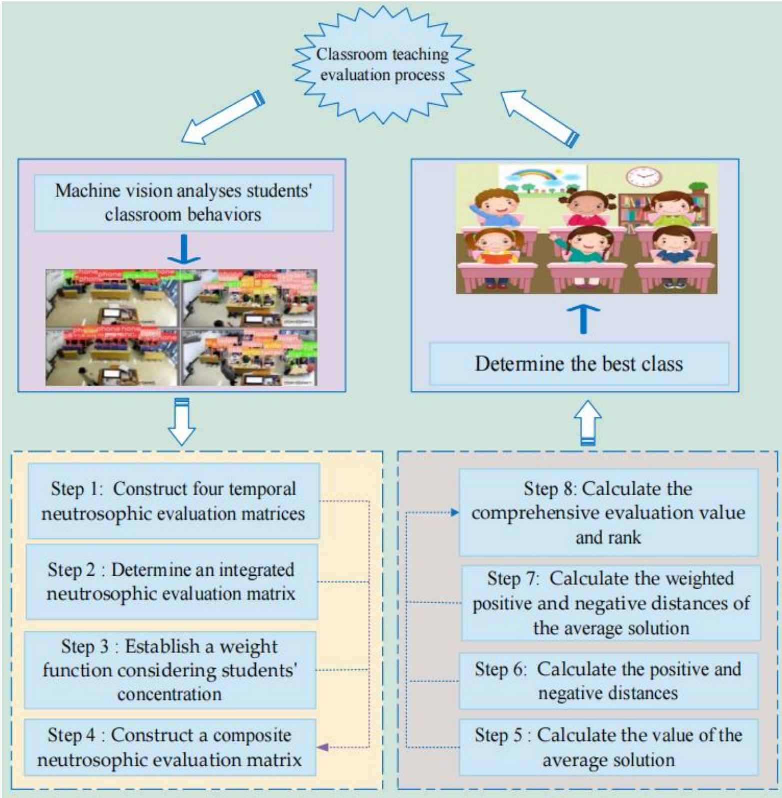

Figure 1. Research framework

According to evaluation on teaching quality in higher education, machine vision usually provides feedback on students' class learning behaviors. Currently, the application of machine vision technology in evaluating classroom teaching effectiveness has attracted widespread attention [13]. For example, Arashpour et al. [14] applied the YOLO algorithm in facial motion detection to predict students' engagement in the classroom, facilitating teachers in optimizing their instructional strategies. Subsequently, Shen et al. [15] delved into facial expression recognition to capture learners' emotional changes over time. On this basis, a domain-adaptive facial expression recognition method applied to the MOOC scenario was proposed to verify the effectiveness of students' learning engagement. Then, Pabba and Kumar [16] introduced a real-time system employing convolutional neural networks (CNN) for facial expression recognition related to students' status. At the same time, Liu [17] employed multi-task CNN and a quantitative evaluation method-class focus index to detect learners' facial features for determining students' status. Later, Gollapalli et al. [18] proposed a sustainable university field training framework and used machine vision technology to extract educational data to elucidate students' learning outcomes.

Meanwhile, the application of MCDM in education is gaining increasing attention. In this context, scholars are dedicated to exploring the practical application of MCDM models in classroom teaching to address its uncertainties effectively [19]. For example, Martin et al. [20] developed an MCDM method on Plithogenic contradictions, presenting a novel optimal decision-making method. Then, Priyadharshini and Irudayam [21] investigated a unique MCDM method using Plithogenic single-valued fuzzy sets, emphasizing the proposed method's effectiveness and practical adaptability to societal needs. At the same time, Abdel-Basset et al. [22] proposed a multi-stage approach integrating the application of the analytical network process method and TOPSIS to address information uncertainty within a hybrid technique. Additionally, Gamal et al. [23] presented a novel framework that integrated the α-discounting MCDM and the VlseKriterijumska Optimizacija I Kompromisno Resenje method, which was applied to address uncertain and fuzzy conditions under a neutrosophic environment. Later, Gamal et al. [24] extended a reliable MCDM approach based on the elimination effects of criteria and the combined compromise solution utilizing type-2 neutrosophic numbers for criteria assessment.

Based on the aforementioned, the introduced research shows some issues need to be settled in the teaching evaluation field. Firstly, the traditional evaluation methods mainly rely on subjective evaluation, leading to inconsistent evaluation criteria and difficulty in quantifying evaluation results. Secondly, although machine vision has been used in students' status assessment in recent studies, it has not yet been integrated with teaching evaluation methods. Thirdly, although NSs provide new ideas on students' learning effectiveness, they have not yet established a unified framework to solve various complexities and challenges in classroom teaching evaluation. Regarding the above issues, this study proposes a novel evaluation method based on NSs with machine vision, so as to better evaluate students' learning effectiveness.

3. Materials and Methods

For convenience, this section is segmented into several parts. The first part briefly introduces essential definitions regarding this study. The second part introduces the process of machine vision recognizing students' in-class status. The third part proposes a weight calculation method considering students' concentration. The fourth part proposes a temporal neutrosophic evaluation model considering students' concentration.

3.1. Preliminaries

Definition 1. [25] Let X be a set, and the elements of X are represented by x. If $\tilde{A} = \{< x, \alpha_1(x), \alpha_2(x),\ \alpha_3(x) > | x \in X\}$, $\tilde{A}$ is denoted as an SvNS, where $\alpha_1(x): X \to [0,1]$, $\alpha_2(x): X \to [0,1]$, $\alpha_3(x): X \to [0,1]$ depict the truth, indeterminacy and falsity membership degree, respectively. For simplicity, a single-valued neutrosophic number (SvNN) is expressed as the element $< x, \alpha_1(x), \alpha_2(x),\ \alpha_3(x) >$ in $\tilde{A}$.

Definition 2. [26] Presume that there exist two SvNNs, namely $\alpha_1 =< \alpha_{11}, \alpha_{21},\ \alpha_{31} >$ and $\alpha_2 =< \alpha_{12}, \alpha_{22},\ \alpha_{32} >$. Then, there are the following algorithms:

(1) $\alpha_1 \oplus \alpha_2 = \langle \alpha_{11} + \alpha_{12} - \alpha_{11} \cdot \alpha_{12}, \alpha_{21} \cdot \alpha_{22}, \alpha_{31} \cdot \alpha_{32} \rangle$;

(2) $\alpha_1 \otimes \alpha_2 = \langle \alpha_{11} \cdot \alpha_{12}, \alpha_{21} + \alpha_{22} - \alpha_{21} \cdot \alpha_{22}, \alpha_{31} + \alpha_{32} - \alpha_{31} \cdot \alpha_{32} \rangle$;

(3) $w\alpha_1 = (1-(1-\alpha_{11})^w, (\alpha_{21})^w, (\alpha_{31})^w)\ , w > 0.$

Definition 3. [27] Let $S(\alpha_1)$ be the cosine similarity of an SvNN $a_1 =< \alpha_{11}, \alpha_{21}, \alpha_{31} >$. Then $S(\alpha_1)$ is denoted as

$$S(\alpha_1) = \frac{\alpha_{11}}{\sqrt{(\alpha_{11})^2 + (\alpha_{21})^2 + (\alpha_{31})^2}}. \tag{1}$$

Definition 4. [28] Suppose F: $\mathbb{U}^q \to \mathbb{U}$ is a function of q. Then, an ordered weighted averaging (OWA) operator is as follows

$$F(a_1, a_2, \cdots, a_q) = \sum_{\sigma=1}^{q} w_\sigma b_\sigma, \tag{2}$$

where b_σ is the jth element of the descending sort in $\{a_1, a_2, \cdots, a_q\}$, and $w = \{w_1, w_2, \cdots,\ w_q\}^T$ is the weighted vector associated with F, satisfying $w_q \in [0,1]$ and $\sum_{\sigma=1}^{q} w_\sigma = 1$.

Definition 5. [29] Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be SvNNs. Let $\eta = (\eta_1, \eta_2, \cdots, \eta_n)$ be the weight vector of α_n with $\eta_n \geq 0$ and $\sum_{k=1}^{n} \eta_k = 1$. Then, an SVNDWAA operator is obtained as

$$SVNDWAA(\alpha_1,\alpha_2,\cdots,\alpha_n)=\sum_{k=1}^{n}\eta_k\alpha_k. \quad (3)$$

3.2. The process of machine vision recognition

In this study, the Yolov5 object detection model is employed to identify and classify in-class status of students, so as to objectively evaluate students' learning effectively in different classes and periods [30]. In the introduced process, the data on the students' class learning status is derived from classroom teaching videos, which are divided into image sequences. Here, 2000 images are selected as the datasets. The Labelimg software is then employed to detect the status of different students, which are categorized into 6 types: listen, write, distraction, talk, sleep, and phone. The specific identification process of Yolov5 is shown in Figure 2. And the effect of Yolov5 detection is illustrated in Figure 3.

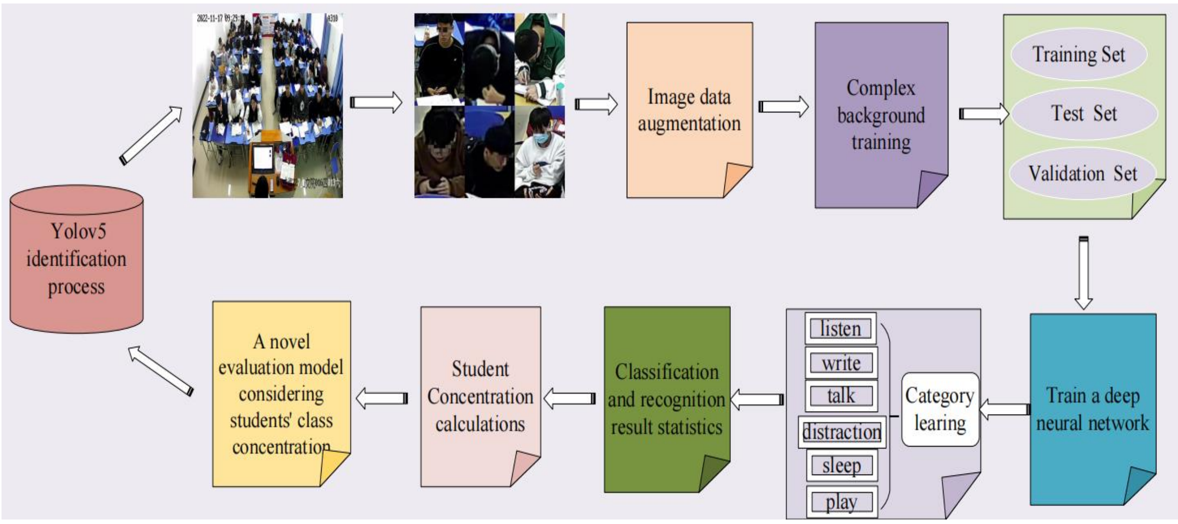

Figure 2. The identification process of Yolov5

For convenience, assume an SvNN exists, namely $\alpha_1 =< \alpha_{11},\alpha_{21}, \alpha_{31} >$. Then, the calculation method of α_{11},α_{21}, and α_{31} are as follows:

$$\alpha_{11}=\frac{A}{A+\sum_{p=1}^{3}B_p+\sum_{p=1}^{2}C_p},\alpha_{21}=\frac{\sum_{p=1}^{3}B_p}{A+\sum_{p=1}^{3}B_p+\sum_{q=1}^{2}C_q},\alpha_{31}=\frac{\sum_{q=1}^{2}C_q}{A+\sum_{p=1}^{3}B_p+\sum_{q=1}^{2}C_q}, \quad (4)$$

where α_{11} represents the truth-membership degree of 'listen' in machine vision recognition; α_{21} represents the indeterminacy-membership degree of 'distraction', 'write' and 'talk'; whereas α_{31} represents the falsity-membership degree of 'sleep' and 'phone'. A represents the count of students 'listen' in class, and $B_p(p = 1,2,3)$ represents the count of students 'distraction', 'write' and 'talk' in class. Whereas $C_q(q = 1,2)$ represents the count of students 'sleep' and 'phone' in class. Consequently, α_1 is obtained.

Figure 3. The effect chart of Yolov5 recognition

3.3. A calculation method of weight considering students' concentration

In classroom learning, students' listening effectiveness exhibits a transition from concentration to distraction as the class time extends [31]. In this context, this study proposes a novel time weight function $\omega(t)$ that considers students' concentration based on the Pearson correlation coefficient. The proposed $\omega(t)$ is a combination of decay and correlation. The decay factor signifies a gradual decline in students' concentration over time, reflecting that fatigue and distractions arise during extended study sessions. Conversely, the correlation factor evaluates the relationship between class duration and students' concentration, assessing variations in attention levels during different periods. In this study, the duration of a class is taken as $T-$minute, in which each 10-minute is divided into a period and each period is further subdivided into 2-minute as a time node. The following $\omega(t)$ is defined as

$$\omega(t)=\frac{\exp\left[\dfrac{\sum_{j=1}^{ti}(x_{ij}-\overline{x_i})\,|\,y_{ij}-\overline{y_i}\,|}{\sqrt{\sum_{j=1}^{ti}(x_{ij}-\overline{x_i})^2\sum_{j=1}^{ti}(y_{ij}-\overline{y_i})^2}}\cdot\left(w_s-\dfrac{w_s-w_e}{t_{\max}}\cdot t\right)\right]}{\sum_{j=1}^{n}\exp\left[\dfrac{\sum_{j=1}^{ti}(x_{ij}-\overline{x_i})\,|\,y_{ij}-\overline{y_i}\,|}{\sqrt{\sum_{j=1}^{ti}(x_{ij}-\overline{x_i})^2\sum_{j=1}^{ti}(y_{ij}-\overline{y_i})^2}}\cdot\left(w_s-\dfrac{w_s-w_e}{t_{\max}}\cdot t\right)\right]},\tag{5}$$

where $x_{ij}(i,j=1,2,\cdots,n)$ denotes the jth time node in the ith period, $\overline{x_i}$ represents the average time in the ith period, y_{ij} is on behalf of the number of students paying attention at the jth time node in the ith period, $\overline{y_i}$ means the average number of students listening attentively in the ith period, t denotes unit moments in a class divided into 10-minute periods, whereas t_{max} represents the maximum unit moment in class divided into 10-minute periods. In this study, w_s indicates the initial weight set to 1, while w_e represents the final weight set to 0.

Theorem 1: The $\omega(t)$ considering students' concentration satisfies the following properties.

(E1) For $t\in[0,T]$, $\omega(t)$ is a monotonically decreasing function.

(E2) When $t=0$, $\omega(t)$ has the maximum value.

(E3) When $t=a$, $\omega(t)$ has the minimum value.

Proof:

To prove the proposed properties, two fundamental functions are constructed as

$$f(t)=\exp\left[r\cdot(1-\frac{t}{T})\right],t\in[0,T],$$

$$F(t)=\frac{\exp\left[r\cdot(1-\frac{t}{T})\right]}{\sum_{t=0}^{T}\exp\left[r\cdot(1-\frac{t}{T})\right]},t\in[0,T].$$

Then, it gets

$$f'(t)=\frac{df(t)}{dt}=\left[\exp\left[r\cdot(1-\frac{t}{T})\right]\right]\cdot(-\frac{r}{T}),t\in[0,T], \tag{6}$$

where $r(r>0)$ represents the Pearson correlation coefficient.

Calculations indicate that $f'(t)$ is always less than 0. Then, it gets $f(t)$ is a monotonically decreasing function. Since Eq. (3) is a normalization of Eq. (2), $F(t)$ also monotonically decreases. Then, (E1) holds.☑

When $t=0$, $f(t)$ takes the maximum value $f_{max}(t)=e^{r}$. Then, (E2) holds.☑

When $t=T$, $f(t)$ takes the minimum value $f_{min}(t)=1$. Then, (E3) holds.☑

3.4. A temporal neutrosophic evaluation model considering students' concentration

To better evaluate students' learning effectiveness, this subsection proposes a novel temporal neutrosophic evaluation model that considers students' concentration. First, this study constructs four temporal neutrosophic evaluation matrices on students' class status over time and aggregates the proposed four matrices as one using a classical OWA operator. Second, a novel $\omega(t)$ considering students' concentration is proposed based on the Pearson correlation coefficient. Third, this study applies the EDAS method by introducing a classical SVNDWAA operator and a similarity function for comprehensive assessment and optimization of students' learning effectiveness. Specifically, the evaluation steps are as follows.

Step 1: Construct temporal neutrosophic evaluation matrices. This study focuses on ϕ classes, with l courses within a month as the research objects. By processing the data from students' class videos, a neutrosophic evaluation matrix $K_{l\times\phi i}$ is established for ϕ classes in the ith period by using Eq. (4). Taking the evaluation matrix $K_{l_1}^{\phi i}$ of a course within a month as an example, the following $K_{l_1}^{\phi i}$ is defined as

$$K_{l_1}^{\phi i}=\begin{bmatrix} <\alpha_{11}^{11},\alpha_{21}^{11},\alpha_{31}^{11}> & <\alpha_{11}^{12},\alpha_{21}^{12},\alpha_{31}^{12}> & \cdots & <\alpha_{11}^{1i},\alpha_{21}^{1i},\alpha_{31}^{1i}> \\ <\alpha_{11}^{21},\alpha_{21}^{21},\alpha_{31}^{21}> & <\alpha_{11}^{22},\alpha_{21}^{22},\alpha_{31}^{22}> & \cdots & <\alpha_{11}^{2i},\alpha_{21}^{2i},\alpha_{31}^{2i}> \\ \vdots & \vdots & \ddots & \vdots \\ <\alpha_{11}^{\phi 1},\alpha_{21}^{\phi 1},\alpha_{31}^{\phi 1}> & <\alpha_{11}^{\phi 2},\alpha_{21}^{\phi 2},\alpha_{31}^{\phi 2}> & \cdots & <\alpha_{11}^{\phi i},\alpha_{21}^{\phi i},\alpha_{31}^{\phi i}> \end{bmatrix},$$

where $\alpha_{11}^{\phi i},\alpha_{21}^{\phi i},\alpha_{31}^{\phi i}$ represent the truth-membership degree, indeterminacy-membership degree, and falsity-membership degree of class $\phi(\phi=1,2,\cdots,\varepsilon)$ in the ith period of the $K_{l_1}^{\phi i}$, respectively, whereas $i=1,2,\cdots,n$.

Step 2: Determine an integrated neutrosophic evaluation matrix. Introducing a classical OWA operator to integrate one-month courses from different classes into a course is to obtain a comprehensive evaluation matrix $L_{l\times\phi i}$, enhancing the depth and accuracy of students' performance assessment. Due to the same type of courses in each class, $w_\sigma = \frac{1}{l}(\sigma = 1,2,\cdots,l)$ is taken in this study. By using Eq. (2), the $L_{l\times\phi i}$ is defined as

$$L_l^{\phi i} = w_\sigma \times K_l^{\phi i}$$
$$= \begin{bmatrix} <\alpha_{12}^{11},\alpha_{22}^{11},\alpha_{32}^{11}> & <\alpha_{12}^{12},\alpha_{22}^{12},\alpha_{32}^{12}> & \cdots & <\alpha_{12}^{1i},\alpha_{22}^{1i},\alpha_{32}^{1i}> \\ <\alpha_{12}^{21},\alpha_{22}^{21},\alpha_{32}^{21}> & <\alpha_{12}^{22},\alpha_{22}^{22},\alpha_{32}^{22}> & \cdots & <\alpha_{12}^{2i},\alpha_{22}^{2i},\alpha_{32}^{2i}> \\ \vdots & \vdots & \ddots & \vdots \\ <\alpha_{12}^{\phi 1},\alpha_{22}^{\phi 1},\alpha_{32}^{\phi 1}> & <\alpha_{12}^{\phi 2},\alpha_{22}^{\phi 2},\alpha_{32}^{\phi 2}> & \cdots & <\alpha_{12}^{\phi i},\alpha_{22}^{\phi i},\alpha_{32}^{\phi i}> \end{bmatrix}.$$

Step 3: Establish a time weight function considering students' concentration. The overall effect of students' class status is evaluated through the $\omega(t)$ considering students' concentration constructed in Section 3.1. The proposed $\omega(t)$ not only reflects students' class status over time, but also measures the correlation between class time and students' concentration. Here, the $\omega(t)$ is shown in Eq. (5).

Step 4: Construct a composite neutrosophic evaluation matrix. By combining the proposed $\omega(t)$ with the $L_{l\times\phi i}$, a composite neutrosophic evaluation matrix $G_{l\times\phi i}$ is obtained. Integrating $L_{l\times\phi i}$ and $\omega(t)$, the $G_{l\times\phi i}$ is defined as

$$G_l^{\phi i} = \omega(t) \times L_l^{\phi i}$$
$$= \begin{bmatrix} <\alpha_{13}^{11},\alpha_{23}^{11},\alpha_{33}^{11}> & <\alpha_{13}^{12},\alpha_{23}^{12},\alpha_{33}^{12}> & \cdots & <\alpha_{13}^{1i},\alpha_{23}^{1i},\alpha_{33}^{1i}> \\ <\alpha_{13}^{21},\alpha_{23}^{21},\alpha_{33}^{21}> & <\alpha_{13}^{22},\alpha_{23}^{22},\alpha_{33}^{22}> & \cdots & <\alpha_{13}^{2i},\alpha_{23}^{2i},\alpha_{33}^{2i}> \\ \vdots & \vdots & \ddots & \vdots \\ <\alpha_{13}^{\phi 1},\alpha_{23}^{\phi 1},\alpha_{33}^{\phi 1}> & <\alpha_{13}^{\phi 2},\alpha_{23}^{\phi 2},\alpha_{33}^{\phi 2}> & \cdots & <\alpha_{13}^{\phi i},\alpha_{23}^{\phi i},\alpha_{33}^{\phi i}> \end{bmatrix}.$$

Step 5: Calculate the value of the class average solution. To calculate the average solution for each class, a classical SVNDWAA operator is introduced. It is used to convert the ranking results of all classes into a standard scoring scale for comparison and evaluation, effectively reducing errors in the final evaluation results. By using Eq. (3), the class average solution is obtained as

$$\overline{AF_\phi} = SVNDWAA(\alpha_1,\alpha_2,\cdots,\alpha_n) = \sum_{k=1}^{n}\eta_k\alpha_k, \tag{7}$$

where $\overline{AF_\phi}$ represents the class ϕ average solution, and η_k means the weight vector in α_k.

Step 6: Calculate the positive and negative distances. To evaluate the learning outcomes of each class compared to other classes, a similarity function is introduced. According to Eqs. (1) and (7), the positive distances $PDC_{\phi i}$ and negative distances $NDC_{\phi i}$ of class ϕ in the ith period are denoted as

$$PDC_{\phi i} = \frac{\max(0, S(\alpha_3^{\phi i}) - S(\overline{AF_\phi}))}{S(\overline{AF_\phi})}, \tag{8}$$

$$NDC_{\phi i} = \frac{\max(0, S(\overline{AF_\phi}) - S(\alpha_3^{\phi i}))}{S(\overline{AF_\phi})}, \tag{9}$$

where $S(\alpha_3^{\phi i})$ represents similarity calculation for each SvNN in $G_{l\times\phi i}$, whereas $S(\overline{AF_\phi})$ represents the similarity calculation of $\overline{AF_{\phi}}$.

Step 7: Calculate the weighted positive and negative distances. According to definition 2, the weighted $PDC_{\phi i}$ and $NDC_{\phi i}$ of class ϕ in the ith period are denoted as

$$SP_{\phi} = \sum_{i=1}^{n} \lambda_i PDC_{\phi i}, \tag{10}$$

$$SN_{\phi} = \sum_{i=1}^{n} \lambda_i NDC_{\phi i}, \tag{11}$$

where SP_{ϕ} and SN_{ϕ} indicate the weight sums of $PDC_{\phi i}$ and $NDC_{\phi i}$, respectively, whereas λ_i is treated as the weighted of the ith period.

Step 8: Calculate the comprehensive evaluation value. CS_{ϕ} is defined as

$$CS_{\phi} = \frac{1}{2}\left[\frac{SP_{\phi}}{\max SP} + (1 - \frac{SN_{\phi}}{\max SN})\right]. \tag{12}$$

Rank according to CS_{ϕ} and the highest CS_{ϕ} is the optimal one.

4. Results and Discussion

This section is mainly divided into three parts. 1) Give a practical example for the model used to illustrate the effectiveness. 2) Present a comparison of the proposed model with others to demonstrate the consistence. 3) Discuss the obtained results.

4.1. Case study application

In this subsection, the proposed model is applied in a case study. This study defines the duration of a class as 50 minutes, consisting of five periods, with each period further divided into 2-minute as time nodes. Besides, this study focuses on four Professional English courses in four classes over five periods within a month, where $i(i = 1,2,3,4,5)$ is considered the ith period. l_1, l_2, l_3 and l_4 correspond to the four Professional English courses. Moreover, $\phi(\phi = 1,2,3,4)$ represents Marine Engineering classes 221, 222, 223, and 224, respectively. For the above four classes, this study collects four videos of four Professional English courses from each class within a month, detects students' class status and processes the identification data. Based on the identified data, students' learning status in class is evaluated. The specific evaluation procedures are as follows.

Step 1: This study establishes four evaluation matrices ($K_{l_1}^{\phi i}$,$K_{l_2}^{\phi i}$,$K_{l_3}^{\phi i}$,$K_{l_4}^{\phi i}$) for four Professional English courses in four classes over five time periods within a month. By using Eq. (4), $K_{l_1}^{\phi i}$, $K_{l_2}^{\phi i}$, $K_{l_3}^{\phi i}$ *and* $K_{l_4}^{\phi i}$ are given as

$$K_{l_1}^{\phi i} = \begin{bmatrix} (0.87,0.06,0.07) & (0.85,0.07,0.08) & (0.88,0.05,0.07) & (0.78,0.13,0.09) & (0.87,0.12,0.01) \\ (0.94,0.06,0.00) & (0.98,0.02,0.00) & (0.98,0.02,0.00) & (0.94,0.06,0.00) & (0.94,0.06,0.00) \\ (0.94,0.06,0.00) & (0.93,0.07,0.00) & (0.84,0.08,0.08) & (0.83,0.10,0.07) & (0.85,0.08,0.07) \\ (0.39,0.50,0.11) & (0.47,0.37,0.16) & (0.40,0.41,0.19) & (0.35,0.52,0.13) & (0.34,0.52,0.14) \end{bmatrix},$$

$$K_{l_2}^{\phi i}=\begin{bmatrix}(0.80,0.19,0.01) & (0.87,0.13,0.01) & (0.87,0.13,0.01) & (0.80,0.18,0.01) & (0.71,0.28,0.01)\\(0.90,0.10,0.00) & (0.91,0.09,0.00) & (0.91,0.09,0.00) & (0.86,0.14,0.00) & (0.82,0.18,0.00)\\(0.89,0.06,0.05) & (0.90,0.05,0.05) & (0.91,0.05,0.04) & (0.95,0.05,0.00) & (0.96,0.04,0.00)\\(0.37,0.48,0.15) & (0.31,0.54,0.15) & (0.29,0.47,0.24) & (0.29,0.56,0.15) & (0.29,0.56,0.15)\end{bmatrix},$$

$$K_{l_3}^{\phi i}=\begin{bmatrix}(0.63,0.34,0.03) & (0.55,0.35,0.10) & (0.72,0.25,0.03) & (0.72,0.28,0.10) & (0.67,0.28,0.05)\\(0.85,0.15,0.00) & (0.78,0.22,0.00) & (0.75,0.25,0.00) & (0.75,0.25,0.00) & (0.73,0.27,0.00)\\(0.97,0.03,0.00) & (0.96,0.04,0.00) & (0.94,0.04,0.02) & (0.86,0.04,0.10) & (0.83,0.06,0.11)\\(0.32,0.54,0.14) & (0.30,0.53,0.17) & (0.23,0.53,0.24) & (0.25,0.57,0.18) & (0.29,0.52,0.19)\end{bmatrix},$$

$$K_{l_4}^{\phi i}=\begin{bmatrix}(0.62,0.32,0.06) & (0.70,0.24,0.06) & (0.68,0.22,0.10) & (0.60,0.20,0.20) & (0.51,0.27,0.22)\\(0.76,0.24,0.00) & (0.79,0.21,0.00) & (0.75,0.25,0.00) & (0.73,0.27,0.00) & (0.64,0.36,0.00)\\(0.84,0.05,0.11) & (0.79,0.04,0.17) & (0.83,0.07,0.10) & (0.85,0.07,0.08) & (0.87,0.06,0.07)\\(0.26,0.55,0.19) & (0.26,0.56,0.18) & (0.25,0.56,0.19) & (0.25,0.53,0.22) & (0.29,0.44,0.27)\end{bmatrix}.$$

Step 2: A classical OWA operator is introduced to integrate four courses into a course within a month. Since the four courses are the same type, the same weight value $w_\sigma = \frac{1}{4}(\sigma = 1,2,3,4)$ is taken throughout the study. By using Eq. (2), $L_{l\times\phi i}$ is obtained as

$$L_{l\times\phi i}=\begin{bmatrix}(0.75,0.24,0.04) & (0.77,0.20,0.06) & (0.81,0.17,0.05) & (0.74,0.20,0.10) & (0.72,0.24,0.08)\\(0.88,0.14,0.00) & (0.90,0.14,0.00) & (0.90,0.16,0.00) & (0.85,0.18,0.00) & (0.82,0.23,0.00)\\(0.92,0.05,0.04) & (0.91,0.05,0.06) & (0.89,0.06,0.06) & (0.88,0.07,0.06) & (0.89,0.06,0.06)\\(0.34,0.52,0.15) & (0.34,0.51,0.17) & (0.30,0.50,0.22) & (0.29,0.55,0.10) & (0.30,0.51,0.19)\end{bmatrix}.$$

Step 3: By using Eq. (5), the proposed $\omega(t)$ considering students' concentration is calculated. Specifically, the weight calculation results are shown in Table 1. The correlation between students' concentration and time is presented in Figure 4.

Table 1. Weight calculation results

time	10min	20min	30min	40min	50min
r	0.73	0.42	0.69	0.30	0.74
$\omega(t)$	0.25	0.22	0.19	0.18	0.16

Step 4: Integrating $L_{l\times\phi i}$ and $\omega(t)$, $G_{l\times\phi i}$ is given as

$$G_{l\times\phi i}=\begin{bmatrix}(0.29,0.70,0.45) & (0.28,0.70,0.54) & (0.27,0.71,0.57) & (0.22,0.75,0.63) & (0.18,0.80,0.67)\\(0.41,0.61,0.00) & (0.40,0.65,0.00) & (0.35,0.71,0.00) & (0.29,0.73.0.00) & (0.24,0.79,0.00)\\(0.47,0.47,0.45) & (0.41,0.52,0.54) & (0.34,0.59,0.59) & (0.32,0.62,0.60) & (0.30,0.64,0.64)\\(0.10,0.85,0.62) & (0.09,0.86,0.68) & (0.07,0.57,0.71) & (0.06,0.90,0.74) & (0.06,0.90,0.77)\end{bmatrix}.$$

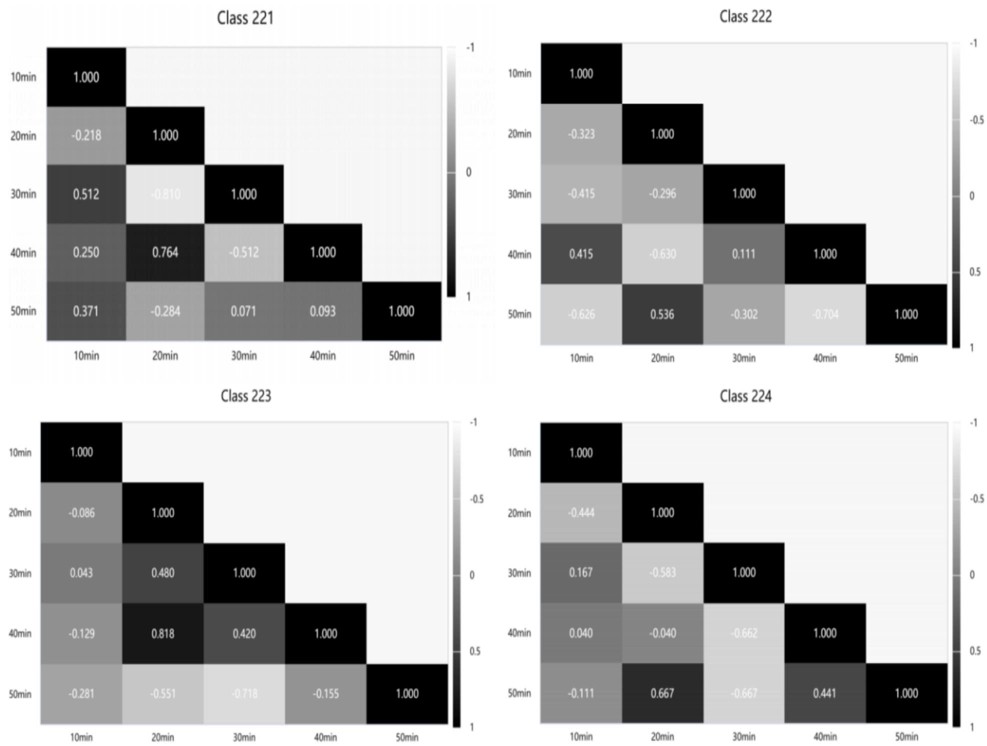

Figure 4. Correlation between students' concentration and time

Step 5: By using Eq. (7), $\overline{AF_1}$, $\overline{AF_2}$, $\overline{AF_3}$ and $\overline{AF_4}$ are obtained as

$$\overline{AF_1} = \frac{1}{5}\alpha_3^{11} + \frac{1}{5}\alpha_3^{12} + \frac{1}{5}\alpha_3^{13} + \frac{1}{5}\alpha_3^{14} + \frac{1}{5}\alpha_3^{15} = \langle 0.25, 0.73, 0.57 \rangle,$$

$$\overline{AF_2} = \frac{1}{5}\alpha_3^{21} + \frac{1}{5}\alpha_3^{22} + \frac{1}{5}\alpha_3^{23} + \frac{1}{5}\alpha_3^{24} + \frac{1}{5}\alpha_3^{25} = \langle 0.34, 0.70, 0.00 \rangle,$$

$$\overline{AF_3} = \frac{1}{5}\alpha_3^{31} + \frac{1}{5}\alpha_3^{32} + \frac{1}{5}\alpha_3^{33} + \frac{1}{5}\alpha_3^{34} + \frac{1}{5}\alpha_3^{35} = \langle 0.37, 0.57, 0.56 \rangle,$$

$$\overline{AF_4} = \frac{1}{5}\alpha_3^{41} + \frac{1}{5}\alpha_3^{42} + \frac{1}{5}\alpha_3^{43} + \frac{1}{5}\alpha_3^{44} + \frac{1}{5}\alpha_3^{45} = \langle 0.08, 0.82, 0.70 \rangle.$$

Step 6: By using Eqs. (8) and (9), $\mathrm{PDC}_{\phi i}$ and $\mathrm{NDC}_{\phi i}$ are obtained in Table 2 and Table 3.

Step 7: By using Eqs. (10) and (11), SP_ϕ and SN_ϕ are calculated. It gets

$$SP_1 = 0.10, SP_2 = 0.09, SP_3 = 0.11, SP_4 = 0.11,$$

$$SN_1 = 0.05, SN_2 = 0.11, SN_3 = 0.09, SN_4 = 0.12.$$

Step 8: By using Eq. (12), CS_ϕ is obtained. It gets

$$CS_1 = 0.75, CS_2 = 0.45, CS_3 = 0.63, CS_4 = 0.50.$$

Based on the above sorting results, the priority order of four classes is $1 \succ 3 \succ 4 \succ 2$, indicating that class 221 performs the best overall, while class 222 shows the worst performance. Specifically, the detailed analysis shown in Figure 5 presents the truth-membership degree for each class over time.

Table 2. Positive distances

ϕ	$PDC_{\phi 1}$	$PDC_{\phi 2}$	$PDC_{\phi 3}$	$PDC_{\phi 4}$	$PDC_{\phi 5}$
1	0.27	0.15	0.08	0.00	0.00
2	0.27	0.18	0.00	0.00	0.00
3	0.40	0.14	0.00	0.00	0.00
4	0.29	0.14	0.14	0.00	0.00

Table 3. Negative distances

ϕ	$NDC_{\phi 1}$	$NDC_{\phi 2}$	$NDC_{\phi 3}$	$NDC_{\phi 4}$	$NDC_{\phi 5}$
1	0.00	0.00	0.00	0.15	0.12
2	0.00	0.00	0.00	0.19	0.34
3	0.00	0.00	0.10	0.17	0.26
4	0.00	0.00	0.00	0.29	0.29

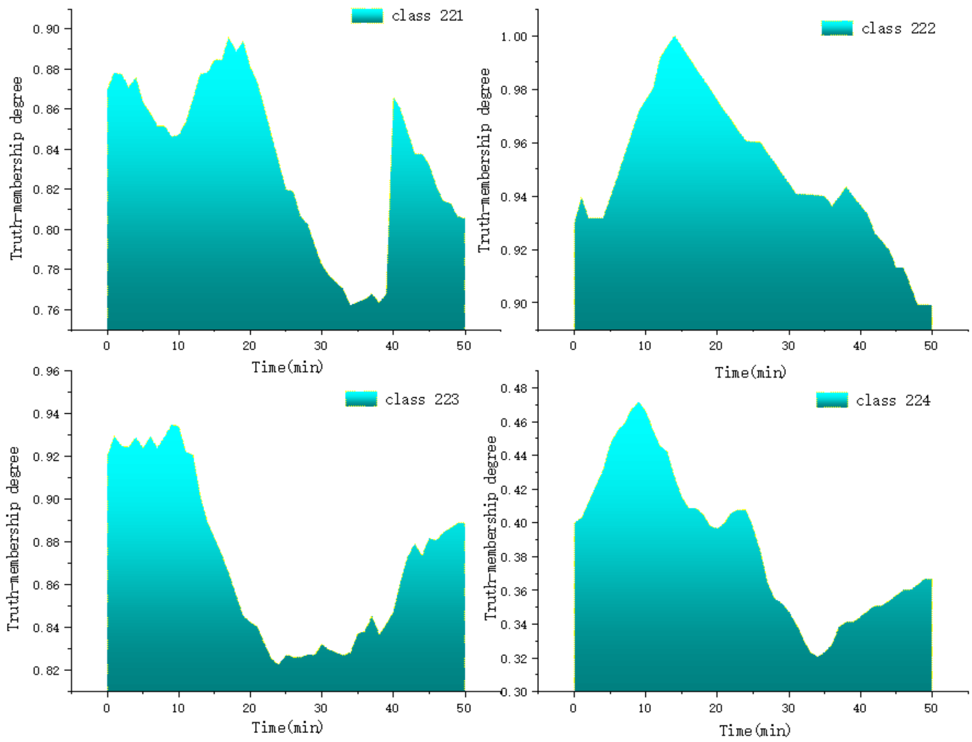

Figure 5. The truth-membership degree of the four classes over time

4.2. Comparative analysis

To further affirm the viability and practicability of the proposed method in assessing students' learning effectiveness, this study conducts comparative analysis with the traditional EDAS method, as well as the approaches introduced by Han et al. [32] and Biswas et al. [33]. Among them, the ranking results are presented in Table 4, which are consistent with those of existing methods. They all agree that class 221 is the optimal scheme. Based on this consistency, the proposed method is effective and reliable.

Table 4. Comparative analysis results

Method	Sorting results	Optimal scheme
The proposed method	$1 \succ 3 \succ 4 \succ 2$	1
Traditional EDAS method	$1 \succ 3 \succ 4 \succ 2$	1
Han et al.' method	$1 \succ 3 \succ 4 \succ 2$	1
Biswas et al.' method	$1 \succ 3 \succ 4 \succ 2$	1

4.3. Discussion

For convenience, a concise description of the experimental results is provided in this study. The details are as follows.

(1) This study utilizes machine vision technology to detect and analyze the videos on students' class state. Considering its uncertainty and diversity, this study proposes an SvNN calculation method to handle the obtained data. Compared to previous relevant research, the application of machine vision technology presents a more accurate and objective data analysis.

(2) This study proposes a novel classroom teaching evaluation method that combines machine vision technology with NSs. It is found that the ranking results are consistent with comparative analysis, indicating that the proposed method provides a novel idea and is suitable for solving the MCDM problem.

(3) This study proposes an objective weight function that considers students' concentration. Compared with other weight calculation methods (such as Analytic Hierarchy Process), the weight calculation method proposed in this study has reduced subjective challenges and a higher correlation performance on students' concentration.

As a result of the above, the proposed method presents a distinctive solution to address the subjectivity and inconsistency issues identified in previous research. This further expands the research depth in this field, providing a novel method for realizing high-quality education.

5. Conclusions and Prospects

To better evaluate students' learning effectiveness, this study proposes a novel classroom teaching evaluation method that combines NSs with machine vision technology, providing a reference for teaching quality evaluation. Specifically, the main contributions are summarized as follows.

First, this study identifies and classifies data on students' class status using Yolov5 detection mode, providing a novel idea to accurately and comprehensively evaluate students' class status. According to the obtained data, an SvNN calculation method is proposed, laying the foundation for constructing temporal neutrosophic evaluation matrices.

Second, this study proposes a novel time weight function on the basis of the Pearson correlation coefficient. The proposed weight function is a combination of decay and correlation that considers students' concentration. Moreover, the proposed weight function remains unaffected by subjective factors, enhancing the objectivity of the evaluation results.

Third, this study introduces a classical SVNDWAA operator to calculate class average solutions and utilizes a similarity function to implement the EDAS method. Besides, comparative analysis is given to verify the superiority of the model proposed, ensuring the accuracy and reliability of MCDM.

It is noteworthy that there is a relationship problem between samples and objects in the current accuracy of machine vision recognition. In some situations, the class status on the first three rows of students in the classroom is collected by the camera in this study, which fails to cover the learning status of the entire class. Therefore, future research should involve expanding the sample size to capture students' status more comprehensively, thus reducing potential sampling errors. Additionally, intuitionistic fuzzy sets can also be applied in this study. However, considering the specific context and requirements of this study, SvNSs are more suitable for effectively handling the uncertainties and fuzziness involved in the evaluation process, leading to more accurate assessment results.

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Properties of multiplication operation of neutrosophic fuzzy matrices

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Abstract: The article reveals the investigation of properties of multiplication operation of neutrosophic fuzzy matrices. We study commutative property, associative property, distributive property of them. We show with suitable example that neutrosophic fuzzy matrices do not obey commutative property with respect to multiplication operation. We prove that neutrosophic fuzzy matrices hold associative property property with respect to multiplication operation. We also prove that neutrosophic fuzzy matrices hold distributive property with respect to multiplication operation over addition. These results are further justified by providing suitable numerical examples. The important aspects of the article is that the investigation of commutative, associative and distributive properties of neutrosophic fuzzy matrices with respect to multiplication operation will fill up the gaps in the existing literature.

Keywords: *Neutrosophic Set; Fuzzy Matrix; Neutrosophic fuzzy Matrix; Properties of Neutrosophic Fuzzy Matrices.*

1. Introduction:

Classical methods often fail to deal real - life problems due to uncertainty. Thereafter, Zadeh [1] invented fuzzy set associating with membership value to resolve uncertainty. Sometimes, non - membership value is necessary to resolve uncertainty properly. In order to deal with such a situation, in 1986, Atanassov initiated the notion of intuitionistic fuzzy sets by associating truth and falsity-membership values. However, it fails to resolve indeterminate situation. Smarandache [2] talked about neutrosophic set after associating membership, non-membership and indeterminacy functions independently. It turned out to take decision for solving real life problem in complex situation. Pal et al. [3, 4] introduced minimal structures and continuity in neutrosophic topological spaces. Das and Das [5] investigated on neutrosophic separation axioms. Dhar [6] studied compactness and neutrosophic topological space via grills. Recently, Broumi et al. [7, 8] and Abdel-Basset et al. [9, 10, 11, 12] and some other authors [13, 14, 15, 16, 17, 18, 19, 20] have successfully applied neutrosophic sets to solve different problems.

Matrices have significant contribution in the field of science and technology. It is often seen that usual matrix theory cann't address all uncertainty. Thomas [21] invented fuzzy matrices.

Kandasamy and Smarandache [22, 23] referred neutrosophic relational maps and the classical algebraic structures converted to neutrosophic algebra after inserting the indeterminacy element I to it. The importance of matrices can be found in the theory of vector spaces. This concept has been generalized to neutrosophic matrices by Khaled et al. [24]. Addition and multiplication operations of square neutrosophic fuzzy matrices have been defined and investigated by Dhar et al. [25].

Gap in the literature:

Das et al. [26] studied on the subtraction operation and investigated algebraic properties of neutrosophic fuzzy matrices. However, they did not investigate on commutative, associative and distributive properties of them with respect to multiplication operation. In this article, we study commutative, associative and distributive properties of neutrosophic fuzzy matrices with respect to multiplication operation.

The innovative values of the article:

We have invented notion of multiplication operation of neutrosophic fuzzy matrices which is quite different from usual multiplication operation of other matrices of real or complex entries. We have discussed commutative property, associative property and distributive property of them with multiplication operation. We have also discussed suitable examples to justify the introduction of the notion.

We frame the paper in different sections. The next section procures few known definitions and results. We investigate few properties in section 3. Then conclusion appears.

2. **Preliminaries and Definitions:**

Necessary concepts and results have been procured in this section.

Definition 2.1. [2] The neutrosophic set η is the form η = {<x:$T_\eta(x)$, $I_\eta(x)$, $F_\eta(x)$>, $x \in U$}, where U is an universe set and the independent functions T, I, F : $U \rightarrow$]−0, 1+[referrer respectively degree of membership, indeterminacy and non-membership of $x \in U$ and $^-0 \leq T_\eta(x) + I_\eta(x) + F_\eta(x) \leq 3^+$.

It will be difficult to apply the interval]−0, 1+[in the applications of scientific and engineering problems. So we need to take [0, 1] in place of]−0, 1+[.

Definition 2.2. [25] The neutrosophic matrix is defined as $M_{m \times n}$= {(m_{ij}) : $m_{ij} \in K(I)$}. Here $K(I)$ denotes a neutrosophic field.

Definition 2.3. (One may refer to [26]) $A_{4\times3}=\begin{pmatrix} 5 & 0 & 2.1I \\ 3.5I & 3 & 5 \\ 7 & 4I & 0 \\ 8 & -5I & I \end{pmatrix}$

denotes a neutrosophic matrix involving the elements (entries) from the real and indeterminacy.

Definition 2.4. [22] Take P = [0, 1]∪I. The $p\times q$ matrices $C_{p\times q}$ = {(c_{ij}) : $c_{ij} \in$ [0, 1]∪I} is said to be fuzzy integral neutrosophic matrices. Evidently collection of fuzzy integral neutrosophic matrices contain collection of $p\times q$ matrices.

The fuzzy neutrosophic row and column matrices are the row vector $1\times q$ and column vector $p\times1$ respectively.

Definition 2.5. (One may refer [26]) Let$M_{4\times 3}$ = $\begin{pmatrix} 0.5 & 0 & 0.1I \\ I & 0.3 & 0.5 \\ 0.7 & 0.4I & 0 \\ 0.8 & 0.5I & I \end{pmatrix}$be a 4×3 integral fuzzy neutrosophic matrix.

Definition 2.6. [22] We denote N_s as fuzzy neutrosophic set where N_s = [0, 1]∪{bI :b∈ [0, 1]}. Then $M_{m\times n}$ = {(c_{ij}):c_{ij}∈ N_s, i = 1 to m and j = 1 to n} is defined as fuzzy neutrosophic matrices.

Example 2.7. (One may refer to [26]) Take N_s = [0, 1]∪{dI : d∈[0, 1]}as fuzzy neutrosophic set and

$$P = \begin{pmatrix} 0.5 & 0 & 0.1I \\ I & 0.3 & 0.5 \\ 0 & I & 0.01 \end{pmatrix}$$

is a fuzzy neutrosophic matrix of order 3× 3.

Definition 2.8. [21] A matrix with entries from unit fuzzy interval [0, 1] is said to be a fuzzy matrix and if the number of rows and column of that matrix are equal, then it is referred as fuzzy square matrix. An example is given below:

$$M = \begin{pmatrix} u & v \\ t & w \end{pmatrix}$$

Here u, v, t, w belong to [0, 1].

Definition 2.9. [22] The entries of a neutrosophic fuzzy matrix (in short, NFM) M are form $x + Iy$ (neutrosophic number). Here x, y are taken from [0, 1], I is an indeterminate where $I^n = I$, (n∈ ℕ). As for example

$$M= \begin{pmatrix} x_1 + Iy_1 & x_2 + Iy_2 \\ x_3 + Iy_3 & x_4 + Iy_4 \end{pmatrix}$$

is a neutrosophic fuzzy matrix.

Definition 2.10. [26] Let us take two matrices as below:

$$A = \begin{pmatrix} x_1 + Iy_1 & x_2 + Iy_2 \\ x_3 + Iy_3 & x_4 + Iy_4 \end{pmatrix} \text{ and } B = \begin{pmatrix} m_1 + In_1 & m_2 + In_2 \\ m_3 + In_3 & m_4 + In_4 \end{pmatrix}$$

The product of them as multiplication operation is as below

$$AB = \begin{pmatrix} x_1 + Iy_1 & x_2 + Iy_2 \\ x_3 + Iy_3 & x_4 + Iy_4 \end{pmatrix}\begin{pmatrix} m_1 + In_1 & m_2 + In_2 \\ m_3 + In_3 & m_4 + In_4 \end{pmatrix}$$

$$= \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} \text{ as defined in Definition 2.8 of [26].}$$

3. Main Results:

Here we investigate some properties of neutrosophic fuzzy matrices.

3.1. Proposition. Multiplication operation is not commutative in case of neutrosophic fuzzy matrices.

Proof. We consider $A = \begin{pmatrix} 0.1 + I0.2 & 0.2 + I0.4 \\ 0.3 + I0.5 & 0.4 + I0.6 \end{pmatrix}$, $B = \begin{pmatrix} 0.3 + I0.7 & 0.4 + I0.5 \\ 0.2 + I0.4 & 0.2 + I0.8 \end{pmatrix}$

$$AB = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

where, $C_{11} = \text{Max}\{\text{Min}(0.1, 0.3), \text{Min}(0.2, 0.2)\} + I\text{Max}\{\text{Min}(0.2, 0.7), \text{Min}(0.4, 0.4)\} = \text{Max}\{0.1, 0.2\} + I\text{Max}\{0.2, 0.4\} = 0.2 + I0.4$.

Similarly, one can show that

$C_{12} = 0.2 + I0.4,\ C_{21} = 0.3 + I0.5,\ C_{22} = 0.3 + I0.6.$

Thus, $AB = \begin{pmatrix} 0.2 + I0.4 & 0.2 + I0.4 \\ 0.3 + I0.5 & 0.3 + I0.6 \end{pmatrix}$……………(1)

Now,

$$BA = \begin{pmatrix} 0.3 + I0.7 & 0.4 + I0.5 \\ 0.2 + I0.4 & 0.2 + I0.8 \end{pmatrix}\begin{pmatrix} 0.1 + I0.2 & 0.2 + I0.4 \\ 0.3 + I0.5 & 0.4 + I0.6 \end{pmatrix}$$

$= \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}$ and

$$D_{11} = \text{Max}\{\text{Min}(0.3, 0.1), \text{Min}(0.4, 0.3)\} + I\text{Max}\{\text{Min}(0.7, 0.2), \text{Min}(0.5, 0.5)\}$$
$$= \text{Max}\{0.1, 0.3\} + I\text{Max}\{0.2, 0.5\} = 0.3 + I0.5$$

Similarly, one can show that

$D_{12} = 0.2 + I0.5,\ D_{21} = 0.2 + I0.5,\ D_{22} = 0.2 + I0.6.$

Thus, $BA = \begin{pmatrix} 0.3 + I0.5 & 0.2 + I0.5 \\ 0.2 + I0.5 & 0.2 + I0.6 \end{pmatrix}$………………(2)

From (1) and (2), it follows that $AB \neq BA$.

3.2. Proposition. Multiplication operation is associative in case of neutrosophic fuzzy matrices.

Proof. We consider $A = \begin{pmatrix} x_1 + Iy_1 & x_2 + Iy_2 \\ x_3 + Iy_3 & x_4 + Iy_4 \end{pmatrix}$, $B = \begin{pmatrix} c_1 + Id_1 & c_2 + Id_2 \\ c_3 + Id_3 & c_4 + Id_4 \end{pmatrix}$

$C = \begin{pmatrix} m_1 + In_1 & m_2 + In_2 \\ m_3 + In_3 & m_4 + In_4 \end{pmatrix}$

Now,

$$AB = \begin{pmatrix} x_1 + Iy_1 & x_2 + Iy_2 \\ x_3 + Iy_3 & x_4 + Iy_4 \end{pmatrix}\begin{pmatrix} c_1 + Id_1 & c_2 + Id_2 \\ c_3 + Id_3 & c_4 + Id_4 \end{pmatrix}$$
$$= \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix},$$

where, $M_{11} = \text{Max}\{\text{Min}(x_1, c_1), \text{Min}(x_2, c_3)\} + I\,\text{Max}\{\text{Min}(y_1, d_1), \text{Min}(y_2, d_3)\}$.

$M_{12} = \text{Max}\{\text{Min}(x_1, c_2), \text{Min}(x_2, c_4)\} + I\,\text{Max}\{\text{Min}(y_1, d_2), \text{Min}(y_2, d_4)\}$.

$M_{21} = \text{Max}\{\text{Min}(\mathrm{x}_3, \mathrm{c}_1), \text{Min}(\mathrm{x}_4, \mathrm{c}_3)\} + I\,\text{Max}\{\text{Min}(y_3, d_1), \text{Min}(y_4, d_3)\}$.

$M_{22} = \text{Max}\{\text{Min}(\mathrm{x}_3, \mathrm{c}_2), \text{Min}(x_4, c_4)\} + I\,\text{Max}\{\text{Min}(y_3, d_2), \text{Min}(y_4, d_4)\}$.

$\therefore AB = \begin{pmatrix} X_1 + IY_1 & X_2 + IY_2 \\ X_3 + IY_3 & X_4 + IY_4 \end{pmatrix}$,

where, $X_1 = \text{Max}\{\text{Min}(x_1, c_1), \text{Min}(x_2, c_3)\}$.

$X_2 = \text{Max}\{\text{Min}(x_1, c_2), \text{Min}(x_2, c_4)\}$.

$X_3 = \text{Max}\{\text{Min}(\mathrm{x}_3, \mathrm{c}_1), \text{Min}(\mathrm{x}_4, \mathrm{c}_3)\}$.

$X_4 = \text{Max}\{\text{Min}(\mathrm{x}_3, \mathrm{c}_2), \text{Min}(\mathrm{x}_4, \mathrm{c}_4)\}$.

$$Y_1 = \mathrm{Max}\{\mathrm{Min}(y_1, d_1), \mathrm{Min}(y_2, d_3)\}.$$
$$Y_2 = \mathrm{Max}\{\mathrm{Min}(y_1, d_2), \mathrm{Min}(y_2, d_4)\}.$$
$$Y_3 = \mathrm{Max}\{\mathrm{Min}(y_3, d_1), \mathrm{Min}(y_4, d_3)\}.$$
$$Y_4 = \mathrm{Max}\{\mathrm{Min}(y_3, d_2), \mathrm{Min}(y_4, d_4)\}.$$

$$\therefore (AB)C = \begin{pmatrix} X_1 + IY_1 & X_2 + IY_2 \\ X_3 + IY_3 & X_4 + IY_4 \end{pmatrix} \begin{pmatrix} m_1 + In_1 & m_2 + In_2 \\ m_3 + In_3 & m_4 + In_4 \end{pmatrix}$$
$$= \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix},$$

where, $N_{11} = \mathrm{Max}\{\mathrm{Min}(\mathrm{X}_1, \mathrm{m}_1), \mathrm{Min}(\mathrm{X}_2, \mathrm{m}_3)\} + I\,\mathrm{Max}\{\mathrm{Min}(Y_1, n_1), \mathrm{Min}(Y_2, n_3)\}$.

$$N_{12} = \mathrm{Max}\{\mathrm{Min}(X_1, m_2), \mathrm{Min}(\mathrm{X}_2, \mathrm{m}_4)\} + I\,\mathrm{Max}\{\mathrm{Min}(Y_1, n_2), \mathrm{Min}(Y_2, n_4)\}.$$
$$N_{21} = \mathrm{Max}\{\mathrm{Min}(\mathrm{X}_3, \mathrm{m}_1), \mathrm{Min}(\mathrm{X}_4, \mathrm{m}_3)\} + I\,\mathrm{Max}\{\mathrm{Min}(Y_3, n_1), \mathrm{Min}(Y_4, n_3)\}.$$
$$N_{22} = \mathrm{Max}\{\mathrm{Min}(\mathrm{X}_3, \mathrm{m}_2), \mathrm{Min}(\mathrm{X}_4, \mathrm{m}_4)\} + I\mathrm{Max}\{\mathrm{Min}(Y_3, n_2), \mathrm{Min}(Y_4, n_4)\}.$$

Again, $BC = \begin{pmatrix} c_1 + Id_1 & c_2 + Id_2 \\ c_3 + Id_3 & c_4 + Id_4 \end{pmatrix} \begin{pmatrix} m_1 + In_1 & m_2 + In_2 \\ m_3 + In_3 & m_4 + In_4 \end{pmatrix}$

$= \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}$, where

$$Q_{11} = \mathrm{Max}\{\mathrm{Min}(c_1, m_1), \mathrm{Min}(c_2, m_3)\} + I\,\mathrm{Max}\{\mathrm{Min}(d_1, n_1), \mathrm{Min}(d_2, n_3)\},$$
$$Q_{12} = \mathrm{Max}\{\mathrm{Min}(c_1, m_2), \mathrm{Min}(c_2, m_4)\} + I\,\mathrm{Max}\{\mathrm{Min}(d_1, n_2), \mathrm{Min}(d_2, n_4)\},$$
$$Q_{21} = \mathrm{Max}\{\mathrm{Min}(c_3, m_1), \mathrm{Min}(c_4, m_3)\} + I\mathrm{Max}\{\mathrm{Min}(d_3, n_1), \mathrm{Min}(d_4, n_3)\},$$
$$Q_{22} = \mathrm{Max}\{\mathrm{Min}(c_3, m_2), \mathrm{Min}(c_4, m_4)\} + I\,\mathrm{Max}\{\mathrm{Min}(d_3, n_2), \mathrm{Min}(d_4, n_4)\}.$$

$\therefore BC = \begin{pmatrix} G_1 + IH_1 & G_2 + IH_2 \\ G_3 + IH_3 & G_4 + IH_4 \end{pmatrix}$, where

$$G_1 = \mathrm{Max}\{\mathrm{Min}(\mathrm{c}_1, \mathrm{m}_1), \mathrm{Min}(\mathrm{c}_2, \mathrm{m}_3)\}.$$
$$G_2 = \mathrm{Max}\{\mathrm{Min}(\mathrm{c}_1, \mathrm{m}_2), \mathrm{Min}(\mathrm{c}_2, \mathrm{m}_4)\}.$$
$$G_3 = \mathrm{Max}\{\mathrm{Min}(\mathrm{c}_3, \mathrm{m}_1), \mathrm{Min}(\mathrm{c}_4, \mathrm{m}_3)\}.$$
$$G_4 = \mathrm{Max}\{Min(c_3, m_2), Min(c_4, m_4)\}.$$
$$H_1 = \mathrm{Max}\{\mathrm{Min}(\mathrm{d}_1, \mathrm{n}_1), \mathrm{Min}(\mathrm{d}_2, \mathrm{n}_3)\}.$$
$$H_2 = \mathrm{Max}\{\mathrm{Min}(d_1, n_2), \mathrm{Min}(d_2, n_4)\}.$$
$$H_3 = \mathrm{Max}\{\mathrm{Min}(d_3, n_1), \mathrm{Min}(d_4, n_3)\}.$$
$$H_4 = \mathrm{Max}\{\mathrm{Min}(d_3, n_2), \mathrm{Min}(d_4, n_4)\}.$$

$$\therefore A(BC) = \begin{pmatrix} x_1 + Iy_1 & x_2 + Iy_2 \\ x_3 + Iy_3 & x_4 + Iy_4 \end{pmatrix} \begin{pmatrix} G_1 + IH_1 & G_2 + IH_2 \\ G_3 + IH_3 & G_4 + IH_4 \end{pmatrix}$$

$= \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}$, where

$$R_{11} = \mathrm{Max}\{\mathrm{Min}(x_1, G_1), \mathrm{Min}(x_2, G_3)\} + I\,\mathrm{Max}\{\mathrm{Min}(y_1, H_1), \mathrm{Min}(y_2, H_3)\}.$$

$R_{12} = \text{Max}\{\text{Min}(x_1, G_2), \text{Min}(x_2, G_4)\} + I\,\text{Max}\{\text{Min}(y_1, H_2), \text{Min}(y_2, H_4)\}$.

$R_{21} = \text{Max}\{\text{Min}(x_3, G_1), \text{Min}(x_4, G_3)\} + I\,\text{Max}\{\text{Min}(y_3, H_1), \text{Min}(y_4, H_3)\}$.

$R_{22} = \text{Max}\{\text{Min}(\text{x}_3, \text{G}_2), \text{Min}(\text{x}_4, \text{G}_4)\} + I\,\text{Max}\{\text{Min}(y_3, H_2), \text{Min}(y_4, H_4)\}$.

In order to show $(AB)C = A(BC)$, we have to show $N_{11}= R_{11}$, $N_{12} = R_{12}$, $N_{21} = R_{21}$ and $N_{22} = R_{22}$.

In order to calculate N_{11}, we have to find X_1, X_2, Y_1 and Y_2.

Now, $X_1 = \text{Max}\{\text{Min}(x_1, c_1), \text{Min}(x_2, c_3)\}$

$= \text{Max}\{x_1,\ x_2\}$ (say)

$= x_1$ (say).

$X_2 = \text{Max}\{\text{Min}(x_1, c_2), \text{Min}(x_2, c_4)\} = \text{Max}\{x_1,\ x_2\}$ (say)

$= x_1$.

$Y_1 = \text{Max}\{Min(y_1, d_1), \text{Min}(y_2, d_3)\} = \text{Max}\{y_1,\ y_2\}$ [(say)

$= y_1$ (say).

$Y_2 = \text{Max}\{Min(y_1, d_2), Min(y_2, d_4)\} = \text{Max}\{y_1,\ y_2\}$ (say)

$\therefore N_{11} = \text{Max}\{\text{Min}(\text{X}_1, \text{m}_1), \text{Min}(\text{X}_2, \text{m}_3)\} + I\text{Max}\{\text{Min}(Y_1, n_1), \text{Min}(Y_2, n_3)\}$

$= \text{Max}\{\text{Min}(x_1, m_1),\ \text{Min}(x_1, m_3)\} + I\text{Max}\{\text{Min}(y_1,\ n_1), \text{Min}(y_1,\ n_3)\}$

$= \text{Max}\{x_1,\ x_1\} + I\text{Max}\{y_1,\ y_1\}$ [Assuming $\text{Min}(x_1, m_1) = x_1$, $\text{Min}(x_1,\ m_3) = x_1$,

$\text{Min}(y_1,\ n_1) = y_1$, $\text{Min}(y_1,\ n_3) = y_1$]

$= x_1 + Iy_1$

In order to calculate R_{11}, we have to find G_1, G_3, H_1 and H_3.

Now, $G_1 = \text{Max}\{\text{Min}(\text{c}_1, \text{m}_1), \text{Min}(\text{c}_2, \text{m}_3)\}$

$= \text{Max}\{m_1,\ m_3\}$ (say)

$= m_1$ (say).

$G_3 = \text{Max}\{\text{Min}(\text{c}_3, \text{m}_1), \text{Min}(\text{c}_4, \text{m}_3)\}$

$= \text{Max}\{m_1,\ m_3\}$ (say).

$H_1 = \text{Max}\{\text{Min}(\text{d}_1, \text{n}_1), \text{Min}(\text{d}_2, \text{n}_3)\} = \text{Max}\{\text{d}_1,\ d_2\}$ (say) $= d_1$ (say).

$H_3 = \text{Max}\{Min(d_3, n_1), Min(d_4, n_3)\} = \text{Max}\{d_3,\ d_4\}$ (say).

$= d_3$ (say).

$\therefore R_{11} = \text{Max}\{\text{Min}(x_1, G_1), \text{Min}(x_2, G_3)\} + I\text{Max}\{\text{Min}(y_1, H_1), \text{Min}(y_2, H_3)\}$

$= \text{Max}\{\text{Min}(x_1, m_1), \text{Min}(x_2, m_1)\} + I\text{Max}\{\text{Min}(y_1, d_1), \text{Min}(y_2, d_3)\}$

$= \text{Max}\{x_1,\ x_2\} + I\text{Max}\{y_1, y_2\}$(say)

$= x_1 + Iy_1$.

$\therefore N_{11} = R_{11}$. Assuming in all other cases, we can show that $N_{11} = R_{11}$.

Similarly we can show that $N_{12}= R_{12}$, $N_{21} = R_{21}$ and $N_{22} = R_{22}$.

Thus $(AB)C = A(BC)$.

This property is supported with a numerical example as shown below.

3.3. Numerical Example.

Let us consider

$$A = \begin{pmatrix} 0.3 + I0.2 & 0.4 + I0.5 \\ 0.4 + I0.1 & 0.5 + I0.6 \end{pmatrix}$$

$$B = \begin{pmatrix} 0.2 + I0.6 & 0.3 + I0.2 \\ 0.4 + I0.7 & 0.5 + I0.2 \end{pmatrix}$$

$$C = \begin{pmatrix} 0.8 + I0.3 & 0.3 + I0.2 \\ 0.5 + I0.6 & 0.4 + I0.7 \end{pmatrix}$$

$AB = \begin{pmatrix} 0.3 + I0.2 & 0.4 + I0.5 \\ 0.4 + I0.1 & 0.5 + I0.6 \end{pmatrix}\begin{pmatrix} 0.2 + I0.6 & 0.3 + I0.2 \\ 0.4 + I0.7 & 0.5 + I0.2 \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$ (say) and

C_{11} = Max{Min(0.3, 0.2), Min(0.4, 0.4)} + IMax{Min(0.2, 0.6), Min(0.5, 0.7)}

= Max{0.2, 0.4} + IMax{0.2, 0.5}

= 0.4 + I0.5.

Similarly, one can show that

$C_{12} = 0.4 + I0.2,\ C_{21} = 0.4 + I0.6,\ C_{22} = 0.5 + I0.2.$

$\therefore\ AB = \begin{pmatrix} 0.4 + I0.5 & 0.4 + I0.2 \\ 0.4 + I0.6 & 0.5 + I0.2 \end{pmatrix}$

$\therefore (AB)C = \begin{pmatrix} 0.4 + I0.5 & 0.4 + I0.2 \\ 0.4 + I0.6 & 0.5 + I0.2 \end{pmatrix}\begin{pmatrix} 0.8 + I0.3 & 0.3 + I0.2 \\ 0.5 + I0.6 & 0.4 + I0.7 \end{pmatrix} = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}$ (say) and

D_{11} = Max{Min(0.4, 0.8), Min(0.4, 0.5)} + IMax{Min(0.5, 0.3), Min(0.2, 0.6)}

= Max{0.4, 0.4} + IMax{0.3, 0.2}

= 0.4 + I0.3.

Similarly, one can show that

$D_{12} = 0.4 + I0.2,\ D_{21} = 0.5 + I0.3,\ D_{22} = 0.4 + I0.2.$

$\therefore (AB)C = \begin{pmatrix} 0.4 + I0.3 & 0.4 + I0.2 \\ 0.5 + I0.3 & 0.4 + I0.2 \end{pmatrix}$…………………(3)

$BC = \begin{pmatrix} 0.2 + I0.6 & 0.3 + I0.2 \\ 0.4 + I0.7 & 0.5 + I0.2 \end{pmatrix}\begin{pmatrix} 0.8 + I0.3 & 0.3 + I0.2 \\ 0.5 + I0.6 & 0.4 + I0.7 \end{pmatrix} = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix}$ (say) and

E_{11} = Max{Min(0.2, 0.8), Min(0.3, 0.5)} + IMax{Min(0.6, 0.3), Min(0.2, 0.6)}

= Max{0.2, 0.3} + IMax{0.3, 0.2}

= 0.3 + I0.3.

Similarly, one can show that

$E_{12} = 0.3 + I0.2,\ E_{21} = 0.5 + I0.3,\ E_{22} = 0.4 + I0.2.$

$\therefore\ BC = \begin{pmatrix} 0.3 + I0.3 & 0.3 + I0.2 \\ 0.5 + I0.3 & 0.4 + I0.2 \end{pmatrix}$

$\therefore\ A(BC) = \begin{pmatrix} 0.3 + I0.2 & 0.4 + I0.5 \\ 0.4 + I0.1 & 0.5 + I0.6 \end{pmatrix}\begin{pmatrix} 0.3 + I0.3 & 0.3 + I0.2 \\ 0.5 + I0.3 & 0.4 + I0.2 \end{pmatrix} = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}$ (say) and

F_{11} = Max{Min(0.3, 0.3), Min(0.4, 0.5)} + IMax{Min(0.2, 0.3), Min(0.5, 0.3)}

= Max{0.3, 0.4} + IMax{0.2, 0.3}

= 0.4 + I0.3.

Similarly, one can show that

$F_{12} = 0.4 + I0.2,\ F_{21} = 0.5 + I0.3,\ F_{22} = 0.4 + I0.2.$

$\therefore\ A(BC) = \begin{pmatrix} 0.4 + I0.3 & 0.4 + I0.2 \\ 0.5 + I0.3 & 0.4 + I0.2 \end{pmatrix}$…………(4)

From (3) and (4), it follows that $(AB)C = A(BC)$.

3.4. Proposition. Distributive property with respect to multiplication over addition holds in case of neutrosophic fuzzy matrices.

We consider $A = \begin{pmatrix} x_1 + Iy_1 & x_2 + Iy_2 \\ x_3 + Iy_3 & x_4 + Iy_4 \end{pmatrix}, B = \begin{pmatrix} c_1 + Id_1 & c_2 + Id_2 \\ c_3 + Id_3 & c_4 + Id_4 \end{pmatrix}$

$C = \begin{pmatrix} m_1 + In_1 & m_2 + In_2 \\ m_3 + In_3 & m_4 + In_4 \end{pmatrix}$

$\therefore D = B + C$

$= \begin{pmatrix} c_1 + Id_1 & c_2 + Id_2 \\ c_3 + Id_3 & c_4 + Id_4 \end{pmatrix} + \begin{pmatrix} m_1 + In_1 & m_2 + In_2 \\ m_3 + In_3 & m_4 + In_4 \end{pmatrix}$

where

$D_{11} = \text{Max}\{c_1, m_1\} + I\text{Max}\{d_1, n_1\} = x_{11} + Iy_{11}$(say),

$D_{12} = \text{Max}\{c_2, m_2\} + I\text{Max}\{d_2, n_2\} = x_{12} + Iy_{12}$(say),

$D_{21} = \text{Max}\{c_3, m_3\} + I\text{Max}\{d_3, n_3\} = x_{21} + Iy_{21}$(say),

$D_{22} = \text{Max}\{c_4, m_4\} + I\text{Max}\{d_4, n_4\} = x_{22} + Iy_{22}$(say).

$\therefore D = \begin{pmatrix} x_{11} + Iy_{11} & x_{12} + Iy_{12} \\ x_{21} + Iy_{21} & x_{22} + Iy_{22} \end{pmatrix}$

$\therefore AD = \begin{pmatrix} x_1 + Iy_1 & x_2 + Iy_2 \\ x_3 + Iy_3 & x_4 + Iy_4 \end{pmatrix} \begin{pmatrix} x_{11} + Iy_{11} & x_{12} + Iy_{12} \\ x_{21} + Iy_{21} & x_{22} + Iy_{22} \end{pmatrix} = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix}$,

where

$E_{11} = \text{Max}\{\text{Min}(x_1, x_{11}), \text{Min}(x_2, x_{21})\} + I\text{Max}\{\text{Min}(y_1, y_{11}), \text{Min}(y_2, y_{21})\}$,

$E_{12} = \text{Max}\{\text{Min}(x_1, x_{12}), \text{Min}(x_2, x_{22})\} + I\text{Max}\{\text{Min}(y_1, y_{12}), \text{Min}(y_2, y_{22})\}$,

$E_{21} = \text{Max}\{\text{Min}(x_3, x_{11}), \text{Min}(x_4, x_{21})\} + I\text{Max}\{\text{Min}(y_3, y_{11}), \text{Min}(y_4, y_{21})\}$,

$E_{22} = \text{Max}\{\text{Min}(x_3, x_{12}), \text{Min}(x_4, x_{22})\} + I\text{Max}\{\text{Min}(y_3, y_{12}), \text{Min}(y_4, y_{22})\}$.

In order to calculate E_{11}, we have to calculate x_{11}, x_{21}, y_{11} and y_{21}.

$x_{11} = \text{Max}\{c_1, m_1\} = c_1$ (say).

$y_{11} = \text{Max}\{d_1, n_1\} = d_1$ (say).

$x_{21} = \text{Max}\{c_3, m_3\} = c_3$ (say).

$y_{21} = \text{Max}\{d_3, n_3\} = c_3$ (say).

$\therefore E_{11} = \text{Max}\{\text{Min}(x_1, x_{11}), \text{Min}(x_2, x_{21})\} + I\text{Max}\{\text{Min}(y_1, y_{11}), \text{Min}(y_2, y_{21})\}$

$= \text{Max}\{\text{Min}(x_1, c_1), \text{Min}(x_2, c_3)\} + I\text{Max}\{\text{Min}(y_1, d_1), \text{Min}(y_2, d_3)\}$

$= \text{Max}\{c_1, c_3\} + I\text{Max}\{d_1, d_3\}$ [Assuming $\text{Min}(x_1, c_1\} = c_1, \text{Min}(x_2, c_3) = c_3$
$\text{Min}(y_1, d_1) = d_1$, $\text{Min}(y_2, d_3) = d_3$]

$= c_1 + Id_1$ (say).

Now $AB = \begin{pmatrix} x_1 + Iy_1 & x_2 + Iy_2 \\ x_3 + Iy_3 & x_4 + Iy_4 \end{pmatrix} \begin{pmatrix} c_1 + Id_1 & c_2 + Id_2 \\ c_3 + Id_3 & c_4 + Id_4 \end{pmatrix} = F$ (say), where,

F_{11} = Max{Min(x_1, c_1), Min(x_2, c_3)} + IMax{Min(y_1, d_1), Min(y_2, d_3)},

F_{12} = Max{Min(x_1, c_2), Min(x_2, c_4)} + IMax{Min(y_1, d_2), Min(y_2, d_4)},

F_{21} = Max{Min(x_3, c_1), Min(x_4, c_3)} + IMax{Min(y_3, d_1), Min(y_4, d_3)},

F_{22} = Max{Min(x_3, c_2), Min(x_4, c_4)} + IMax{Min(y_3, d_2), Min(y_4, d_4)}.

Now F_{11} = Max{Min(x_1, c_1), Min(x_2, c_3)} + IMax{Min(y_1, d_1), Min(y_2, d_3)}

= Max{c_1, c_3} + IMax{d_1, d_3}

= c_1 + Id_1 (say).

Now AC = $\begin{pmatrix} x_1 + Iy_1 & x_2 + Iy_2 \\ x_3 + Iy_3 & x_4 + Iy_4 \end{pmatrix}\begin{pmatrix} m_1 + In_1 & m_2 + In_2 \\ m_3 + In_3 & m_4 + In_4 \end{pmatrix}$ = G,

where,

G_{11} = Max{Min(x_1, m_1), Min(x_2, m_3)} + IMax{Min(y_1, n_1), Min(y_2, n_3)},

G_{12} = Max{Min(x_1, m_2), Min(x_2, m_4)} + IMax{Min(y_1, n_2), Min(y_2, n_4)},

G_{21} = Max{Min(x_3, m_1), Min(x_4, m_3)} + IMax{Min(y_3, n_1), Min(y_3, n_3)},

G_{22} = Max{Min(x_3, m_2), Min(x_4, m_4)} + IMax{Min(y_3, n_2), Min(y_4, n_4)}.

Now Max{c_1, m_1} = c_1 & Min(x_1, c_1) = $c_1$imply Min(x_1, m_1) = m_1.

Max{c_3, m_3} = c_3 & Min(x_2, c_3) = x_2 imply Min(x_2, m_3) = m_3.

Max{d_1, n_1} = d_1& Min(y_1, d_1) = y_1 imply Min(y_1, n_1) = n_1.

Max{d_3, n_3} = d_3& Min(y_2, d_3) = d_3 imply Min(y_2, n_3) = n_3.

So G_{11} = Max{Min(x_1, m_1), Min(x_2, m_3)} + IMax{Min(y_1, n_1), Min(y_2, n_3)}

= Max{m_1, m_3} + IMax{n_1, n_3}

= m_1 + In_1 (say).

$\therefore F_{11} + G_{11}$ = Max{c_1, m_1} + IMax{d_1, n_1}

= c_1 + Id_1[Since Max{c_1, m_1} = c_1 and Max{d_1, n_1} = d_1]

, $\therefore E_{11} = F_{11} + G_{11}$.

Similarly we can show that $E_{12} = F_{12}+ G_{12}$, $E_{21} = F_{21} + G_{21}$ and $E_{22} = F_{22} + G_{22}$.

$$\therefore A(B + C) = AB + AC.$$

We can also show that $A(B - C) = AB - AC$.

Thus $A(B \pm C) = AB \pm AC$.

This property is supported by numerical examples as given below.

3.5. Numerical Example. We consider the matrices

$$A = \begin{pmatrix} 0.2 + I0.1 & 0.3 + I0.5 \\ 0.4 + I0.3 & 0.5 + I0.7 \end{pmatrix}$$

$$B = \begin{pmatrix} 0.4 + I0.5 & 0.1 + I0.2 \\ 0.3 + I0.6 & 0.7 + I0.3 \end{pmatrix}$$

$$C = \begin{pmatrix} 0.7 + I0.2 & 0.2 + I0.3 \\ 0.6 + I0.5 & 0.4 + I0.8 \end{pmatrix}$$

$$\therefore D = B + C$$

$$= \begin{pmatrix} 0.4 + I0.5 & 0.1 + I0.2 \\ 0.3 + I0.6 & 0.7 + I0.3 \end{pmatrix} + \begin{pmatrix} 0.7 + I0.2 & 0.2 + I0.3 \\ 0.6 + I0.5 & 0.4 + I0.8 \end{pmatrix}$$

$$= \begin{pmatrix} 0.7 + I0.5 & 0.2 + I0.3 \\ 0.6 + I0.6 & 0.7 + I0.8 \end{pmatrix}$$

$$E = AD = A(B + C)$$

$$= \begin{pmatrix} 0.2 + I0.1 & 0.3 + I0.5 \\ 0.4 + I0.3 & 0.5 + I0.7 \end{pmatrix} \begin{pmatrix} 0.7 + I0.5 & 0.2 + I0.3 \\ 0.6 + I0.6 & 0.7 + I0.8 \end{pmatrix}$$

$$= \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix}$$

where, $E_{11} = \text{Max}\{\text{Min}(0.2, 0.3), \text{Min}(0.3, 0.6)\} + I\text{Max}\{\text{Min}(0.1, 0.5), \text{Min}(0.5, 0.6)\}$

$$= \text{Max}\{0.2, 0.3\} + I\text{Max}\{0.1, 0.5\} = 0.3 + I0.5.$$

Similarly, one can show that

$E_{12} = 0.3 + I0.5$, $E_{21} = 0.5 + I0.6$, $E_{22} = 0.5 + I0.7$.

$\therefore E = \begin{pmatrix} 0.3 + I0.5 & 0.3 + I0.5 \\ 0.5 + I0.6 & 0.5 + I0.7 \end{pmatrix}$ ………………………(5)

$$AB = \begin{pmatrix} 0.2 + I0.1 & 0.3 + I0.5 \\ 0.4 + I0.3 & 0.5 + I0.7 \end{pmatrix} \begin{pmatrix} 0.4 + I0.5 & 0.1 + I0.2 \\ 0.3 + I0.6 & 0.7 + I0.3 \end{pmatrix}$$

$= \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}$,

where, $F_{11} = \text{Max}\{\text{Min}(0.2, 0.4), \text{Min}(0.3, 0.3)\} + I\text{Max}\{\text{Min}(0.1, 0.5), \text{Min}(0.5, 0.6)\}$

$$= \text{Max}\{0.2, 0.3\} + I\text{Max}\{0.1, 0.5\} = 0.3 + I0.5$$

Similarly, one can show that

$F_{12} = 0.3 + I0.3$, $F_{21} = 0.4 + I0.6$, $F_{22} = 0.5 + I0.3$.

$$\therefore AB = \begin{pmatrix} 0.3 + I0.5 & 0.3 + I0.3 \\ 0.4 + I0.6 & 0.5 + I0.3 \end{pmatrix}$$

Now, $AC = \begin{pmatrix} 0.2 + I0.1 & 0.3 + I0.5 \\ 0.4 + I0.3 & 0.5 + I0.7 \end{pmatrix} \begin{pmatrix} 0.7 + I0.2 & 0.2 + I0.3 \\ 0.6 + I0.5 & 0.4 + I0.8 \end{pmatrix} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}$,

where, $G_{11} = \text{Max}\{\text{Min}(0.2, 0.7), \text{Min}(0.3, 0.6)\} + I\text{Max}\{\text{Min}(0.1, 0.2), \text{Min}(0.5, 0.5)\}$

$$= \text{Max}\{0.2, 0.3\} + I\text{Max}\{0.1, 0.5\} = 0.3 + I0.5.$$

Similarly, one can show that

$G_{12} = 0.3 + I0.5$, $G_{21} = 0.5 + I0.5$, $G_{22} = 0.4 + I0.7$.

$$\therefore AC = \begin{pmatrix} 0.3 + I0.5 & 0.3 + I0.5 \\ 0.5 + I0.5 & 0.4 + I0.7 \end{pmatrix}$$

$$\therefore AB + AC = \begin{pmatrix} 0.3 + I0.5 & 0.3 + I0.3 \\ 0.4 + I0.6 & 0.5 + I0.3 \end{pmatrix} + \begin{pmatrix} 0.3 + I0.5 & 0.3 + I0.5 \\ 0.5 + I0.5 & 0.4 + I0.7 \end{pmatrix}$$

$=$

$\begin{pmatrix} 0.3 + I0.5 & 0.3 + I0.5 \\ 0.5 + I0.6 & 0.5 + I0.7 \end{pmatrix}$ …………………..(6)

From (5) and (6) it follows that

$$A(B+C)=AB+AC$$

3.6. Numerical Example. Let us take $A=\begin{pmatrix}0.7+I0.3 & 0.2+I0.4\\ 0.5+I0.6 & 0.4+I0.3\end{pmatrix}$

$$B=\begin{pmatrix}0.1+I0.2 & 0.5+I0.3\\ 0.3+I0.4 & 0.7+I0.5\end{pmatrix}$$

$$C=\begin{pmatrix}0.5+I0.4 & 0.2+I0.1\\ 0.2+I0.3 & 0.3+I0.7\end{pmatrix}$$

$$B-C=\begin{pmatrix}0.1+I0.2 & 0.5+I0.3\\ 0.3+I0.4 & 0.7+I0.5\end{pmatrix}-\begin{pmatrix}0.5+I0.4 & 0.2+I0.1\\ 0.2+I0.3 & 0.3+I0.7\end{pmatrix}$$

$$=\begin{pmatrix}0.1+I0.2 & 0.2+I0.1\\ 0.2+I0.3 & 0.3+I0.5\end{pmatrix}$$

$$A(B-C)=\begin{pmatrix}0.7+I0.3 & 0.2+I0.4\\ 0.5+I0.6 & 0.4+I0.3\end{pmatrix}\begin{pmatrix}0.1+I0.2 & 0.2+I0.1\\ 0.2+I0.3 & 0.3+I0.5\end{pmatrix}$$

$$=\begin{pmatrix}D_{11} & D_{12}\\ D_{21} & D_{22}\end{pmatrix},$$

where, $D_{11}=\text{Max}\{\text{Min}(0.7,0.1),\text{Min}(0.2,0.2)\}+I\text{Max}\{\text{Min}(0.3,0.2),\text{Min}(0.4,0.3)\}$

$$=\text{Max}\{0.1,0.2\}+\ I\text{Max}\{0.2,0.3\}=0.2+I0.3.$$

Similarly, one can show that

$D_{12}=0.2+I0.4,\ D_{21}=0.2+I0.3,\ D_{22}=0.3+I0.3.$

$\therefore A(B-C)=\begin{pmatrix}0.2+I0.3 & 0.2+I0.4\\ 0.2+I0.3 & 0.3+I0.3\end{pmatrix}.$ …………………………(7)

Now,

$$AB=\begin{pmatrix}0.7+I0.3 & 0.2+I0.4\\ 0.5+I0.6 & 0.4+I0.3\end{pmatrix}\begin{pmatrix}0.1+I0.2 & 0.5+I0.3\\ 0.3+I0.4 & 0.7+I0.5\end{pmatrix}=\begin{pmatrix}E_{11} & E_{12}\\ E_{21} & E_{22}\end{pmatrix},$$

where, $E_{11}=\text{Max}\{\text{Min}(0.7,0.1),\text{Min}(0.2,0.3)\}+I\text{Max}\{\text{Min}(0.3,0.2),\text{Min}(0.4,0.4)\}$

$$=\text{Max}\{0.1,0.2\}+\ I\text{Max}\{0.2,0.4\}=0.2+I0.4.$$

Similarly, one can show that

$E_{12}=0.5+I0.4,\ E_{21}=0.3+I0.3,\ E_{22}=0.5+I0.3.$

$$\therefore AB=\begin{pmatrix}0.2+I0.4 & 0.5+I0.4\\ 0.3+I0.3 & 0.5+I0.3\end{pmatrix}$$

$$\therefore AC=\begin{pmatrix}0.7+I0.3 & 0.2+I0.4\\ 0.5+I0.6 & 0.4+I0.3\end{pmatrix}\begin{pmatrix}0.5+I0.4 & 0.2+I0.1\\ 0.2+I0.3 & 0.3+I0.7\end{pmatrix}$$

$$=\begin{pmatrix}F_{11} & F_{12}\\ F_{21} & F_{22}\end{pmatrix},$$

where, $F_{11}=\text{Max}\{\text{Min}(0.7,0.5),\text{Min}(0.2,0.2)\}+I\text{Max}\{\text{Min}(0.3,0.4),\text{Min}(0.4,0.3)\}$

$$=\text{Max}\{0.5,0.2\}+\ I\text{Max}\{0.3,0.3\}=0.5+I0.3.$$

Similarly, one can show that

$F_{12} = 0.2 + I0.4,\ F_{21} = 0.5 + I0.4,\ F_{22} = 0.3 + I0.3.$

$$\therefore AC = \begin{pmatrix} 0.5 + I0.3 & 0.2 + I0.4 \\ 0.5 + I0.4 & 0.3 + I0.3 \end{pmatrix}$$

$$\therefore AB - AC = \begin{pmatrix} 0.2 + I0.4 & 0.5 + I0.4 \\ 0.3 + I0.3 & 0.5 + I0.3 \end{pmatrix} - \begin{pmatrix} 0.5 + I0.3 & 0.2 + I0.4 \\ 0.5 + I0.4 & 0.3 + I0.3 \end{pmatrix}$$

$$= \begin{pmatrix} 0.2 + I0.3 & 0.2 + I0.4 \\ 0.3 + I0.3 & 0.3 + I0.3 \end{pmatrix} \text{ ……….(8)}$$

From (7) and (8) it follows that A(B – $C) = AB - AC$.

3.7. Identity element for multiplication.

We consider A $= \begin{pmatrix} x_1 + Iy_1 & x_2 + Iy_2 \\ x_3 + Iy_3 & x_4 + Iy_4 \end{pmatrix}, I_N = \begin{pmatrix} 1 + I1 & 0 + I0 \\ 0 + I0 & 1 + I1 \end{pmatrix}.$

Now A$I_N = \begin{pmatrix} x_1 + Iy_1 & x_2 + Iy_2 \\ x_3 + Iy_3 & x_4 + Iy_4 \end{pmatrix} \begin{pmatrix} 1 + I1 & 0 + I0 \\ 0 + I0 & 1 + I1 \end{pmatrix}$ = B (say),

where, B_{11} = Max{Min(x_1, 1), Min(x_2, 0)} + IMax{Min(y_1, 1), Min(y_2, 0)}

= Max{x_1, 0} + IMax{y_1, 0} = $x_1 + Iy_1$.

Similarly, one can show

B_{12} = Max{Min(x_1, 0), Min(x_2, 1)} + IMax{Min(y_1, 0), Min(y_2, 1)}

= $x_2 + Iy_2$.

B_{21} = Max{Min(x_3, 1), Min(x_4, 0)} + IMax{Min(y_3, 1), Min(y_4, 0)}

= $x_3 + Iy_3$.

B_{22} = Max{Min(x_3, 0), Min(x_4, 1)} + IMax{Min(y_3, 0), Min(y_4, 1)}

= $x_4 + Iy_4$.

∴AI_N = $\begin{pmatrix} x_1 + Iy_1 & x_2 + Iy_2 \\ x_3 + Iy_3 & x_4 + Iy_4 \end{pmatrix}$ = A …………..(9)

Again I_NA $= \begin{pmatrix} 1 + I1 & 0 + I0 \\ 0 + I0 & 1 + I1 \end{pmatrix} \begin{pmatrix} x_1 + Iy_1 & x_2 + Iy_2 \\ x_3 + Iy_3 & x_4 + Iy_4 \end{pmatrix}$ = C (say),

where, C_{11} = Max{Min(1, x_1), Min(0, x_3)} + IMax{Min(1, y_1), Min(0, y_3)}

= Max{x_1, 0} + IMax{y_1, 0} = $x_1 + Iy_1$.

C_{12} = Max{Min(1, x_2), Min(0, x_4)} + IMax{Min(1, y_2), Min(0, y_4)}

= $x_2 + Iy_2$.

C_{21} = Max{Min(0, x_1), Min(1, x_3)} + IMax{Min(0, y_1), Min(1, y_3)}

= $x_3 + Iy_3$.

C_{22} = Max{Min(0, x_2), Min(1, x_4)} + IMax{min(0, y_2), Min(1, y_4)}

= $x_4 + Iy_4$.

∴ I_NA = $\begin{pmatrix} x_1 + Iy_1 & x_2 + Iy_2 \\ x_3 + Iy_3 & x_4 + Iy_4 \end{pmatrix}$ = A …………..(10)

From (9) and (10) it follows that

$AI_N = I_N A = A$

Thus $I_N = \begin{pmatrix} 1+I1 & 0+I0 \\ 0+I0 & 1+I1 \end{pmatrix}$ is the identity multiplication for neutrosophic fuzzy matrices.

4. Conclusion:

This paper has provided the properties of multiplication operation of neutrosophic fuzzy matrices. We have shown that commutative property is not satisfied here. However, we have proved that associative property is satisfied. We have also proved that distributive property with respect to multiplication operation over addition of neutrosophic fuzzy matrices is satisfied. The results have further been examined with suitable numerical examples. In future, the authors will investigate on determinant, adjont, inverse and other relevant topics of neutrosophic fuzzy matrices.

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A novel approach for solving neutrosophic fractional transportation problem with non- linear discounting cost

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Abstract: Fractional transportation problem that includes source and destination may have fractional objective functions in real- world applications to maximize the profitability ratio like profit/ cost or profit/ time. We refere to such transportation problems as fractional transportation problem.The paper considers the interval- valued neutrosophic numbers and its aritemematic operations. This paper deals with fractional transportation problem having discounting cost in neutrosophic environment, where the supply, demand and transportation costs are uncertain. The problem is considered by introducing all the parameters as neutrosophic numbers. Using the benefits of the score function definition, the problem is transformed into the corresponding deterministic form which can be illustrated by any method. and hence by applying of least cost method with the help of Kuhn- Tucker' optimality conditions, the optimal solution is resulted. Our strategy is to assess the issue and can rank different sort of neutrosophic numbers. To claify the proposed technique, a numerical example is given to show the adequacy of the new model.

Keywords: Optmization, Optimization problems; Fractional programming, Transportation problem, Non-linear programming, Discounting cost, Pentagonal fuzzy neutrosophic numbers, Score function, Vogel's approximation method, Kuhn- Tucker optimality conditions, Optimal neutrosophic solution,Decision making

1. Introduction

Transportation problem is one of the oldest applications of linear programming problems. The basic transportation problem was originally developed by Hitchcock [1]. In a transportation problem, products have to be transported from a number of sources to a number of destinations. Decisions have to be taken according to the amount of products transported between each two locations to minimize total transportation cost [2]. Typically, only a variable cost proportional to the number of products transported is afforded. However, in many real-world problems, a fixed/setup cost is also afforded when the transportation amount is positive [3]. The transportation problem can be modeled as a standard linear programming problem, that can be solved by the simplex method. We can get an initial basic feasible solution for the transportation problem by using the North-West corner rule, Row Minima, Column Minima, Matrix Minima or the Vogel's Approximation Method. To get an optimal solution for the transportation problem, we use the MODI method (Modified Distribution Method). Transportation problem (TP) is a special type of linear programming (LP) problem; where the objective is to minimize the cost of distributing product from m sources or origins to n distributions and their capacities are $a_1, a_2, \dots, a_m$ and $b_1, b_2, \dots, b_m$, respectively. In other hand, there is a penalty c_{ij} connected with transportation a unit of product from source i to destination j. This penalty, perhaps cost or delivery time of safety of delivery, etc. A variable x_{ij} represents the unidentified quantity to be shipped from source i to destination j. Oheigeartaigh [4] developed an algorithm for fuzzy transportation problem (FTP) Chanas et al. [5] developed a parametric approach to solve single objective FTP. Thamaraiselvi and Santhi [6] studied FTP with hexagonal fuzzy numbers.

In Fractional problem (FP), decision problem arises to optimize the ratio subject to constraints. In real life decision conditions decision maker (DM) sometimes may face to evaluate ratio between inventory and sales, real cost and standard cost, output etc., with both denominator and numerator are linear. If only one ratio is considered as an objective function then under linear constraints, the problem is said to be linear fractional programming (LFP) problem. The Fractional programming problem, i. e., the maximization of a fraction of two functions subject to given conditions, arises in various decision making situations; for instance , fractional programming is used in the fields of traffic planning (Dantzig et al. [6]), network flows (Arisawa and Elmaghraby, [7]), and game theory (Isbell and Marlow, [8]). A review of various applications is given by Schaible, [9-11]. Tantawy [12-13] introduced two approaches to solve the LFP problem namely; a feasible direction approach and a duality approach. Odior [14] introduced an algebraic approach based on the duality concept and the partial fractions to solve the LFP problem. Gupta and Chakraborty [15] solved the LFP problem depending on the sign of the numerator under the assumption that the denominator is non -vanishing in the feasible region using the fuzzy programming approach. Stanojevic and Stancu- Minasian [16] proposed a method for solving fully fuzzified LFP problem. Buckley and Feuring (2000) studied fully fuzzified linear programming involving coefficients and decision variables as fuzzy quantities. Li and Chen [17] introduced a fuzzy LFP problem with fuzzy coefficients and present the concept of a fuzzy optimal solution. Pop and Stancu [18] studied LFP problem with all parameters and decision variables are triangular fuzzy numbers. Gomathi and Jayalakshmi [19] proposed an approach for solving linear fractional transportation problem. A nermous researchers studied fractional transportation (Veeramani et al. [20], Haque [21-24], Bas et al., [25], Akram et al., [26], El Sayed and Bakry [27], Khalifa et al., [28]).

In this paper, fractional transportation problem having discounting cost in neutrosophic environment is introduced. With the help of least cost method and the Kuhn- Tucker's optimality conditions, the optimal solution of the problem is resulted. The following are the study's main contributions and novelties:

1. Introducing suitable terminologies and measures that consider the properties of a possible optimal solution.
2. Presenting a parametric study by solving a parametric problem and determining the stability set of the first kind for collecting the most possible information about the possible optimal solutions in an uncertain situation
3. Interacting the analyst with the DM to assign a set of selected alternatives
4. Doing a multicriteria analysis by interacting with the DM for selecting one of the possible optimal as the satisfied optimal solution .

The rest of the paper is outlined as follows:

The following is how the paper is structured: Section 2 Presents some preliminaries and notation needed. Section 3, Formulates a neutrosophic fractional transportation problem with non- linear discounting cost. Section 4, proposes an algorithm combining with the least cost method and the Kuhn- Tucker's optimality conditions for solving the problem. Section 5, Introduces a numerical example for illustration. Section 6, Introduces discussion about the results. Section 7, introduces comparitive study with some existing relevant literature. Finally, some concluding remarks are reported.

2.Preliminaries

In This section, some of basic concepts and results related to neutrosophic set, single- valued trapezoidal neutrosophic numbers, and their arithmetic operations and its score function are recalled.

Definition1. (Atanason, [31]). A fuzzy set $\widetilde{B}$ is said to be an intuitionistic fuzzy set $\widetilde{B}^{IN}$of a non empty set X if $\widetilde{B}^{IN} = \{\langle x, \mu_{\widetilde{B}^{IN}}, \rho_{\widetilde{B}^{IN}}\rangle : x \in X\}$, where $\mu_{\widetilde{B}^{IN}}$, and $\rho_{\widetilde{B}^{IN}}$ are non-membership and membership functions such that $\mu_{\widetilde{B}^{IN}}$, $\rho_{\widetilde{B}^{IN}}: X \to [0,1]$ and $0 \le \mu_{\widetilde{B}^{IN}} + \rho_{\widetilde{B}^{IN}} \le 1$, for all $x \in X$.

Definition 2. (Atanason, [32]). An intuitionistic fuzzy set $\widetilde{B}^{IN}$of a $\mathbb{R}$ is named an Intuitionistic fuzzy number if the following conditions hold:

1. There exists $c \in \mathbb{R}: \mu_{\widetilde{B}^{IN}}(c) = 1$, and $\rho_{\widetilde{B}^{IN}}(c) = 0$,
2. $\mu_{\widetilde{B}^{IN}}: \mathbb{R} \to [0,1]$ is continuous function such that

   $0 \le \mu_{\widetilde{B}^{IN}} + \rho_{\widetilde{B}^{IN}} \le 1$, for all $x \in X$,
3. The membership and nonmembership functions of $\widetilde{B}^{IN}$ are

$$\mu_{\widetilde{B}^{IN}}(x) = \begin{cases} 0, & -\infty < x < r, \\ h(x), & r \le x \le s, \\ 1, & x = s, \\ l(x), & s \le x \le t, \\ 0, & t \le x < \infty, \end{cases}$$

$$\rho_{\widetilde{B}^{IN}}(x) = \begin{cases} 0, & -\infty < x < a, \\ f(x), & a \le x \le s, \\ 1, & x = s, \\ g(x), & s \le x \le b, \\ 0, & b \le x < \infty. \end{cases}$$

Where $f, g, h, l: \mathbb{R} \to [0,1]$, h and g are completely increasing functions, l and f are completely decreasing functions with the constraints $0 \le f(x) + h(x) \le 1$, and $0 \le l(x) + g(x) \le 1$.

Definition 3. (Jianqiang and Zhong, [33]). A trapezoidal intuitionistic fuzzy number is denoted by $\widetilde{B}^{IN} = (r,s,t,u),(p,s,t,q)$, where $p \le r \le s \le t \le u \le q$ with non-membership and membership functions are defined as

A trapezoidal intuitionistic fuzzy number is denoted by $\widetilde{B}^{IN} = (r,s,t,u),(p,s,t,q)$, where $p \le r \le s \le t \le u \le q$ with membership and nonmembership functions are defined as:

$$\mu_{\bar{B}^{INT}}(x)=\begin{cases}\frac{x-r}{s-r}, & r\le x<s,\\ 1, & s\le x\le t,\\ \frac{u-x}{u-t}, & t\le x\le u,\\ 0, & \text{otherwise,}\end{cases}\qquad \rho_{\bar{B}^{INT}}(x)=\begin{cases}\frac{s-x}{s-p}, & p\le x<s,\\ 0, & s\le x\le t,\\ \frac{x-t}{q-t}, & t\le x\le q,\\ 1, & \text{otherwise}\end{cases}$$

Definition 4. (Smarandache, [34]). A neutrosophic set $\overline{B}^N$ of non empty set X is defined as $\overline{B}^N=\{\langle x, I_{\overline{B}^N}(x), J_{\overline{B}^N}(x), V_{\overline{B}^N}(x)\rangle : x\in X, I_{\overline{B}^N}(x), J_{\overline{B}^N}(x), V_{\overline{B}^N}(x)\in\,]0_{-},1^{+}[\}$, where $I_{\overline{B}^N}(x), J_{\overline{B}^N}(x)$, and $V_{\overline{B}^N}(x)$ are an indeterminacy- membership function, truth membership function, and a falsity- membership function and there is no limit on the sum of $I_{\overline{B}^N}(x), J_{\overline{B}^N}(x)$, and $V_{\overline{B}^N}(x)$, so $0^-\le I_{\overline{B}^N}(x)+J_{\overline{B}^N}(x)+V_{\overline{B}^N}(x)\le 3^+$, and $]0_{-},1^{+}[$ is a nonstandard unit interval.

Definition 5. (Wang et al., [35]). A Single- valued neutrosophic set $\overline{B}^{SVN}$of a non empty set X is defined as $\overline{B}^{SVN}=\{\langle x, I_{\overline{B}^N}(x), J_{\overline{B}^N}(x), V_{\overline{B}^N}(x)\rangle : x\in X\}$, where $I_{\overline{B}^N}(x), J_{\overline{B}^N}(x)$, and $V_{\overline{B}^N}(x)\in[0,1]$ for each $x\in X$ and $0\le I_{\overline{B}^N}(x)+J_{\overline{B}^N}(x)+V_{\overline{B}^N}(x)\le 3$.

Definition 6. (Thamariselvi and Santhi, [36]). Let $\tau_{\tilde{q}}, \varphi_{\tilde{q}}, \omega_{\tilde{q}}\in[0,1]$ and $r,s,t,u\in\mathbb{R}$ such that $r\le s\le t\le u$. Then a single valued trapezoidal neutrosophic number, $\tilde{b}^N=\langle (r,s,t,u): \tau_{\tilde{q}}, \varphi_{\tilde{q}}, \omega_{\tilde{q}}\rangle$ is a special neutrosophic set on $\mathbb{R}$, whose truth-membership, indeterminacy- membership, and falsity- membership functions are

$$\mu_{\tilde{q}}{}^N(x)=\begin{cases}\tau_{\tilde{q}^N}\left(\frac{x-r}{s-r}\right), & r\le x<s,\\ \tau_{\tilde{b}}, & s\le x\le t,\\ \tau_{\tilde{q}^N}\left(\frac{u-x}{u-t}\right), & t\le x\le u,\\ 0, & \text{otherwise,}\end{cases}$$

$$\sigma_{\tilde{q}}{}^N(x)=\begin{cases}\frac{s-x+\omega_{\tilde{q}^N}(x-r)}{s-r}, & r\le x<s,\\ \omega_{\tilde{q}^N}, & s\le x\le t,\\ \frac{x-t+\omega_{\tilde{q}^N}(u-x)}{u-t}, & t\le x\le u,\\ 1, & \text{otherwise.}\end{cases}$$

Where $\tau_{\tilde{q}}, \varphi_{\tilde{q}}$, and $\omega_{\tilde{q}}$ indicate the maximum truth, minimum- indeterminacy, and minimum falsity membership degrees, respectively. A single- valued trapezoidal neutrosophic number $\tilde{q}^N=\langle (r,s,t,u): \tau_{\tilde{q}^N}, \varphi_{\tilde{q}^N}, \omega_{\tilde{q}^N}\rangle$ might express in ill- defined amount about q, which is roughly equal to $[s,t]$.

Definition 7. (Thamariselvi and Santhi, [36]). Let $\tilde{q}^N=\langle (r,s,t,u): \tau_{\tilde{q}^N}, \varphi_{\tilde{q}^N}, \omega_{\tilde{q}^N}\rangle$, and $\tilde{d}^N=\langle (r',s',t',u'): \tau_{\tilde{d}^N}, \varphi_{\tilde{d}^N}, \omega_{\tilde{d}^N}\rangle$ be two single- valued trapezoidal neutrosophic numbers and $v\neq 0$. The arithematic operations on $\tilde{q}^N$, and $\tilde{d}^N$are

1. $\tilde{q}^N\oplus\tilde{d}^N=\langle (r+r', s+s', t+t', u+u');\ \tau_{\tilde{q}^N}\wedge\tau_{\tilde{d}^N}, \varphi_{\tilde{q}^N}\vee\varphi_{\tilde{d}^N}, \omega_{\tilde{q}^N}\vee\omega_{\tilde{d}^N}\rangle$,
2. $\tilde{q}^N\ominus\tilde{d}^N=\langle (r-u', s-t', t-s', u'-r);\ \tau_{\tilde{q}^N}\wedge\tau_{\tilde{d}^N}, \varphi_{\tilde{q}^N}\vee\varphi_{\tilde{d}^N}, \omega_{\tilde{q}^N}\vee\omega_{\tilde{d}^N}\rangle$,
3. $\tilde{q}^N\otimes\tilde{d}^N=\begin{cases}\langle (rr', ss', tt', uu');\ \tau_{\tilde{q}^N}\wedge\tau_{\tilde{d}^N}, \varphi_{\tilde{q}^N}\vee\varphi_{\tilde{d}^N}, \omega_{\tilde{q}^N}\vee\omega_{\tilde{d}^N}\rangle, u,\ u'>0\\ \langle (ru', st', st', ru');\ \tau_{\tilde{q}^N}\wedge\tau_{\tilde{d}^N}, \varphi_{\tilde{q}^N}\vee\varphi_{\tilde{d}^N}, \omega_{\tilde{q}^N}\vee\omega_{\tilde{d}^N}\rangle, u<0,\ u'>0\\ \langle (uu', ss', tt', rr');\ \tau_{\tilde{q}^N}\wedge\tau_{\tilde{d}^N}, \varphi_{\tilde{q}^N}\vee\varphi_{\tilde{d}^N}, \omega_{\tilde{q}^N}\vee\omega_{\tilde{d}^N}\rangle, u<0,\ u'<0,\end{cases}$
4. $\tilde{q}^N\oslash\tilde{d}^N=\begin{cases}\langle (r/u', s/t', t/s', u/r');\ \tau_{\tilde{q}^N}\wedge\tau_{\tilde{d}^N}, \varphi_{\tilde{q}^N}\vee\varphi_{\tilde{d}^N}, \omega_{\tilde{q}^N}\vee\omega_{\tilde{d}^N}\rangle, u,\ u'>0\\ \langle (u/u', t/t', s/s', r/r');\ \tau_{\tilde{q}^N}\wedge\tau_{\tilde{d}^N}, \varphi_{\tilde{q}^N}\vee\varphi_{\tilde{d}^N}, \omega_{\tilde{q}^N}\vee\omega_{\tilde{d}^N}\rangle, u<0,\ u'>0\\ \langle (u/r', t/s', s/t', r/u');\ \tau_{\tilde{q}^N}\wedge\tau_{\tilde{d}^N}, \varphi_{\tilde{q}^N}\vee\varphi_{\tilde{d}^N}, \omega_{\tilde{q}^N}\vee\omega_{\tilde{d}^N}\rangle, u<0,\ u'<0,\end{cases}$

5. $k\tilde{d}^N = f(x) = \begin{cases} \langle (kr, ks, kt, k);\ \tau_{\tau_{\tilde{d}^N}},\ \varphi_{\tau_{\tilde{d}^N}}, \omega_{\tau_{\tilde{d}^N}}\ \rangle, k > 0, \\ \langle (ku, kt, ks, k\ \ r);\ \tau_{\tau_{\tilde{d}^N}},\ \varphi_{\tau_{\tilde{d}^N}}, \omega_{\tau_{\tilde{d}^N}}\ \rangle, k < 0, \end{cases}$

6. $\tilde{d}^{N^{-1}} = \langle (1/u', 1/t', 1/s', 1/r');\ \tau_{\tau_{\tilde{d}^N}},\ \varphi_{\tau_{\tilde{d}^N}}, \omega_{\tau_{\tilde{d}^N}}\ \rangle, \tilde{d}^N \neq 0.$

Definition 8. (Thamariselvi and Santhi, [37]). A two single- valued trapezoidal neutrosophic numbers$\tilde{b}$, and $\tilde{d}$ can be compared based on the score and accuracy functions as

1. Accuracy function $AC(\tilde{q}^N) = \left(\frac{1}{16}\right)[r + s + t + u] * [\mu_{\tilde{q}^N} + (1 - \rho_{\tilde{q}^N}(x) + \left(1 + \sigma_{\tilde{q}^N}(x)\right],$

2. Score function $SC(\tilde{q}^N) = \left(\frac{1}{16}\right)[r + s + t + u] * [\mu_{\tilde{q}^N} + (1 - \rho_{\tilde{q}^N}(x) + \left(1 - \sigma_{\tilde{q}^N}(x)\right].$

Definition 9. (Thamariselvi and Santhi, [37]). The order relations between $\tilde{b}^N$ and $\tilde{d}^N$ based on $SC(\tilde{q}^N)$ and $AC(\tilde{q}^N)$ are defined as

1. If $SC(\tilde{q}^N) < SC(\tilde{d}^N)$, then $\tilde{q}^N < \tilde{d}^N$
2. If $SC(\tilde{q}^N) = SC(\tilde{d}^N)$, then $\tilde{q}^N = \tilde{d}^N$,
3. If $AC(\tilde{q}^N) < AC(\tilde{d}^N)$, then $\tilde{q}^N < \tilde{d}^N$,
4. If $AC(\tilde{q}^N) > AC(\tilde{d}^N)$, then $\tilde{q}^N < \tilde{d}^N$,
5. If $AC(\tilde{q}^N) = AC(\tilde{d}^N)$, then $\tilde{q}^N = \tilde{d}^N$.

3. Problem statement and solution concepts

Consider the following general neutrosophic fractional transportation problem

(NFTP) $\max \tilde{F}^N(x) = \frac{\tilde{P}^N(x)}{\tilde{Q}^N(x)} = \frac{\sum_{i=1}^m \sum_{j=1}^n \tilde{p}_{ij}^N x_{ij}}{\sum_{i=1}^m \sum_{j=1}^n \tilde{q}_{ij}^N x_{ij}}$

Subject to

$$\sum_{j=1}^n x_{ij} = \tilde{a}_i^N, i = \overline{1, m},$$
$$\sum_{i=1}^m x_{ij} = \tilde{b}_j^N, j = \overline{1, n},$$
$$x_{ij} \geq 0;\ \forall i, j.$$

Where, $\tilde{p}_{ij}^N, \tilde{q}_{ij}^N, \tilde{a}_i^N$, and $\tilde{b}_j^N$, are neutrosophic numbers. Based on the score function introduced in Definition 8, the NFTP is converted into the following FTP as

(FTP) $\max F(x) = \frac{P(x)}{Q(x)} = \frac{\sum_{i=1}^m \sum_{j=1}^n p_{ij} x_{ij}}{\sum_{i=1}^m \sum_{j=1}^n q_{ij} x_{ij}}$

Subject to

$$\sum_{j=1}^n x_{ij} \leq a_i, i = \overline{1, m},$$
$$\sum_{i=1}^m x_{ij} \geq b_j, j = \overline{1, n},$$
$$x_{ij} \geq 0;\ \forall i, j.$$

It is supposed that $Q(x) > 0;\ \forall x = (x_{ij}) \in G$, where G is the feasible domain and $a_i > 0, b_j > 0$. Also, it is assumed that $\sum_{i=1}^m a_i \geq \sum_{j=1}^n b_j$.

Definition 10. (Bajalinov. [38]). A point $\bar{x} = \{\bar{x}_{ij}: i = \overline{1, m}; \overline{1, n}\}$ is said to be feasible solution to FTP if $\bar{x}$ satisfies the constraints in it.

Definition 11. A feasible point $\bar{x} = \{\bar{x}_{ij}: i = \overline{1, m}; \overline{1, n}\}$ is called an optimal solution to FTP if $F(\bar{x}) \geq F(x);\ \forall x$. The Lagrange function for the FTP can be formulated as

$$L(x, \zeta) = \frac{P(x)}{Q(x)} - \zeta_i\left(\sum_{j=1}^n x_{ij} - a_i\right) - \zeta_j\left(b_j - \sum_{i=1}^m x_{ij}\right) - \xi_{ij} x_{ij} = 0.$$

Where, ζ_i and ζ_j are Lagrange multipliers.

The optimal point $\overline{x}$ satisfies the Kuhn- Tucker's optimality conditions:

$$\frac{\partial F}{\partial x_{ij}} = \frac{\partial F(\overline{x})}{\partial x_{ij}} - (\zeta_i + \zeta_j) - \xi_{ij} = 0,$$

$$\zeta_i\left(\sum_{j=1}^{n} x_{ij} - a_i\right) = 0,$$

$$\zeta_j\left(b_j - \sum_{i=1}^{m} x_{ij}\right) = 0,$$

$$\sum_{j=1}^{n} x_{ij} \le a_i, i = \overline{1, m},$$

$$\sum_{i=1}^{m} x_{ij} \ge b_j, j = \overline{1, n},$$

$$\xi\,\hat{x}_{ij} = 0,$$

$$\zeta_i \ge 0 \text{ and } \zeta_j \ge 0 .$$

4. Solution Algorithm

In This section, a solution approach for solving NFTP is illustrated in the following steps:

Step1: Convert the NFTP into the corresponding crisp FTP based on the score function.

Step2: Consider the FTP $\left(\frac{p_{ij}}{q_{ij}}\right)$.

Step3: Search for the initial basic feasible solution of FTP using the least cost method.

Step4: Estimate the objective function value at $\overline{x}$ (i.e., $\frac{P(\overline{x})}{Q(\overline{x})}$). Add s_i and t_j to the R.H.S and the bottom of the TP Table 1

Step5: Add s_i and t_j to the R.H.S and the bottom of the TP tableau, respectively as

Table 1. Fractional transportation representation

$\frac{\partial F(\overline{x})}{\partial x_{ij}}$	…	…	$\frac{\partial F(\overline{x})}{\partial x_{1m}}$	a_1	s_1
…	…	…	…	…	…
…	$\frac{\partial F(\overline{x})}{\partial x_{ij}}$	…	…	a_i	s_i
$\frac{\partial F(\overline{x})}{\partial x_{n1}}$	…	…	$\frac{\partial F(\overline{x})}{\partial x_{nm}}$	a_n	s_n
b_1	…	…	b_m		
t_1	…	…	t_m		

Step 6: Calculate the values of s_i and t_j from the relation $\frac{\partial F}{\partial x_{B_{ij}}} = s_i + t_j$

$$\frac{\partial F}{\partial x_{B_{ij}}} = s_i + t_j \quad (1)$$

Step 7: If $M_{ij} = \frac{\partial F}{\partial x_{NB_{ij}}} - s_i - t_j \ge 0;\ \forall x_{ij}$ (2)

(non- basic variables), then $\overline{x}$ is Kuhn- Tucker point. Otherwise, go to step8 as x_{ij} (non- basic variables).

Step 8: Termination conditions:

(i). If all $\frac{\partial F}{\partial x_{NB_{ij}}} > 0 \Rightarrow$ the optimality and the uniqueness of the solution.

(ii). If all $\frac{\partial F}{\partial x_{NB_{ij}}} \ge 0$ with at least one $\frac{\partial F}{\partial x_{NB_{ij}}} = 0 \Rightarrow$ the optimality of the solution and the alternative solution exists.

(iii). If at least one $\frac{\partial F}{\partial x_{NB_{ij}}} < 0 \Rightarrow$ the solution is not optimal

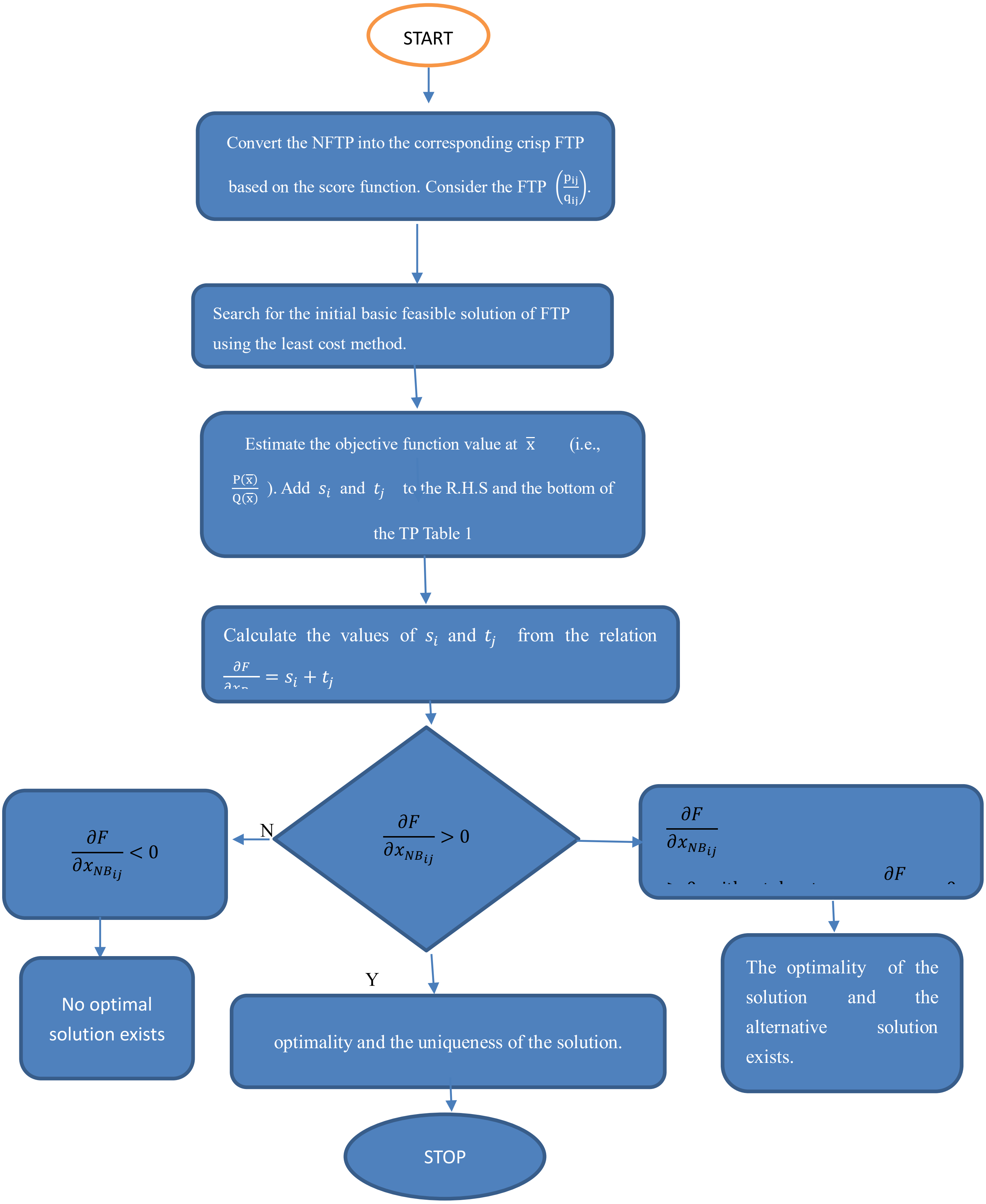

Fig. 1 The flow chart of the proposed solution procedure

5.Numerical example

Consider the following NFTP in which the objective function is the maximization of ratio of total profit given the total cost.

The following table illustrated the transportation company profit gained

Table 2. Input data of neutrosophic profit associated with shipment (in $)

	D_1	D_2	D_3	D_4
O_1	$\langle(14,17,21,28); 0.7,0.1$	$\langle(28,30,35,44); 0.9,0.2$	$\langle(10,16,18,20); 0.8,0.1$	$\langle(17,25,30,35); 0.7,0.2$
O_2	$\langle(18,20,22,25); 0.8,0.2$	$\langle(14,17,21,28); 0.7,0.1$	$\langle(31,35,40,45); 0.6,0.4$	$\langle(16,18,20,26); 0.8,0.2$
O_3	$\langle(18,21,23,26); 0.8,0.2$	$\langle(14,18,20,24); 0.6,0.4$	$\langle(25,30,35,40); 0.6,0.2$	$\langle(18,21,23,26); 0.8,0.2$

The cost of the shipping unit of the commodity from the supply to the demand is shown in the following table

Table 3. Input data of neutrosophic cost associated of shipment (in $)

	D_1	D_2	D_3	D_4
O_1	$\langle(25,30,35,40); 0.6,0.2$	$\langle(17,25,30,35); 0.7,0.2$	$\langle(28,30,35,44); 0.9,0.2$	$\langle(18,20,22,25); 0.8,0.2$
O_2	$\langle(18,20,22,25); 0.8,0.2$	$\langle(14,18,20,24); 0.6,0.4$	$\langle(23,27,30,35); 0.7,0.2$	$\langle(17,25,30,35); 0.7,0.2$
O_3	$\langle(23,28,30,34); 0.7,0.2$	$\langle(25,30,35,40); 0.6,0.2$	$\langle(14,17,21,28); 0.7,0.1$	$\langle(14,17,21,28); 0.7,0.1$

Supplies: $\tilde{a}_1^N = \langle(190,250,260,300); 0.7,0.2,0.1\rangle$, , $\tilde{a}_2^N = \langle(340,380,447,500); 0.7,0.2,0.1\rangle, \tilde{a}_3^N = \langle(283,300,350,400); 0.7,0.2,0.1\rangle$

Demands: $\tilde{b}_1^N = \langle(157,163,169,178); 0.7,0.1,0.2\rangle$, $\tilde{b}_2^N = \langle(340,380,446,500); 0.7,0.1,0.2\rangle$, $\tilde{b}_3^N = \langle(157,163,169,178); 0.7,0.1,0.2\rangle$, $\tilde{b}_4^N = \langle(190,250,260,300); 0.7,0.2,0.1\rangle$.

The discounting cost related to the commodity unit of purchased, transported and the discounted (%) resulted from some shipping policy is given in the following table

Table 4. Discount cost associated of shipment (%)

	D_1	D_2	D_3	D_4
O_1	0.02	0.03	0.05	0.02
O_2	0.03	0.01	0.005	0.02
O_3	0.014	0.04	0.013	0.04

In Table 2, 3: the profit and cost Based on the score function of the neutrosophic number are converted into:

Table 5. Input data of profit associated with shipment (in $)

	D_1	D_2	D_3	D_4	**Supply**
O_1	12	18	10	14	150

O_2	10	14	16	10	250
O_3	11	8	17	11	200
Demand	100	250	100	150	

Table 6. Input data of cost associated of shipment (in $)

	$\mathbf{D_1}$	$\mathbf{D_2}$	$\mathbf{D_3}$	$\mathbf{D_4}$	**Supply**
O_1	17	14	18	10	150
O_2	12	8	15	14	250
O_3	15	17	14	12	200
Demand	100	250	100	150	

From Table 5 and Table 6, the fractional transportation problem can be formulated as follows

$$\max F(x) = \frac{\begin{pmatrix} 12x_{11}+18x_{12}+10x_{13}+14x_{14} \\ +10x_{21}+14x_{22}+16x_{23}+10x_{24} \\ +11x_{31}+8x_{32}+17x_{33}+11x_{34} \end{pmatrix}}{\begin{pmatrix} 17x_{11}+14x_{12}+18x_{13}+10x_{14} \\ +12x_{21}+8x_{22}+15x_{23}+14x_{24} \\ +15x_{31}+17x_{32}+14x_{33}+12x_{34} \end{pmatrix}}$$

Subject to

$$\begin{aligned}
&x_{11} + x_{12} + x_{13} + x_{14} = 150, \\
&x_{21} + x_{22} + x_{23} + x_{24} = 250, \\
&x_{31} + x_{32} + x_{33} + x_{34} = 200, \\
&x_{11} + x_{21} + x_{31} = 100, \\
&x_{12} + x_{22} + x_{32} = 250, \\
&x_{13} + x_{23} + x_{33} = 100, \\
&x_{14} + x_{24} + x_{34} = 150, \\
&x_{ij} \geq 0, i = 1,2,3; j = 1,2,3,4.
\end{aligned}$$

Then, the cost function terms are:

$$\frac{p_{11}}{q_{11}}x_{11} = \frac{12}{17}x_{11} - d_{11}x_{11}^2 \Rightarrow \frac{p_{11}}{q_{11}}x_{11} = 0.706x_{11} - 0.02x_{11}^2,$$

$$\frac{p_{12}}{q_{12}}x_{12} = \frac{18}{14}x_{12} - d_{12}x_{12}^2 \Rightarrow \frac{p_{12}}{q_{12}}x_{12} = 1.286x_{12} - 0.03x_{12}^2,$$

$$\frac{p_{13}}{q_{13}}x_{13} = \frac{10}{18}x_{13} - d_{13}x_{13}^2 \Rightarrow \frac{p_{13}}{q_{13}}x_{13} = 0.556x_{13} - 0.05x_{13}^2,$$

$$\frac{p_{14}}{q_{14}}x_{14} = \frac{18}{14}x_{14} - d_{14}x_{14}^2 \Rightarrow \frac{p_{14}}{q_{14}}x_{14} = 1.4x_{14} - 0.02x_{14}^2,$$

$$\frac{p_{21}}{q_{21}}x_{21} = \frac{10}{12}x_{21} - d_{21}x_{21}^2 \Rightarrow \frac{p_{21}}{q_{21}}x_{21} = 0.833x_{21} - 0.03x_{21}^2,$$

$$\frac{p_{22}}{q_{22}}x_{22} = \frac{14}{8}x_{22} - d_{22}x_{22}^2 \Rightarrow \frac{p_{22}}{q_{22}}x_{22} = 1.75x_{22} - 0.01x_{22}^2,$$

$$\frac{p_{23}}{q_{23}}x_{23} = \frac{16}{15}x_{23} - d_{23}x_{23}^2 \Rightarrow \frac{p_{23}}{q_{23}}x_{23} = 1.067x_{23} - 0.005x_{23}^2,$$

$$\frac{p_{24}}{q_{24}}x_{24} = \frac{10}{14}x_{24} - d_{24}x_{24}^2 \Rightarrow \frac{p_{24}}{q_{24}}x_{24} = 0.714x_{24} - 0.02x_{24}^2,$$

$$\frac{p_{31}}{q_{31}}x_{31} = \frac{11}{15}x_{31} - d_{31}x_{31}^2 \Rightarrow \frac{p_{31}}{q_{31}}x_{31} = 0.733x_{31} - 0.014x_{31}^2,$$

$$\frac{p_{32}}{q_{32}}x_{32} = \frac{8}{17}x_{32} - d_{32}x_{32}^2 \Rightarrow \frac{p_{32}}{q_{32}}x_{32} = 0.4706x_{32} - 0.04x_{32}^2,$$

$$\frac{p_{33}}{q_{33}}x_{33} = \frac{17}{14}x_{33} - d_{33}x_{33}^2 \Rightarrow \frac{p_{33}}{q_{33}}x_{33} = 1.2143x_{33} - 0.013x_{33}^2,$$

$$\frac{p_{34}}{q_{34}}x_{34} = \frac{11}{12}x_{34} - d_{34}x_{34}^2 \Rightarrow \frac{p_{34}}{q_{34}}x_{34} = 0.917x_{34} - 0.04x_{34}^2,$$

Now, let us apply the Vogel's approximation method to determine the initial basic feasible solution for the transportation as

Table 7. Initial basic feasible solution tableau

	D_1		D_1		D_1		D_1		Supply	Raw penalty
O_1	12	**50**	18		10	**100**	14		150	4
	17		14		18		10			
O_2	10	**50**	14	**50**	16		10	**150**	250	4
	12		8		15		14			
O_3	11		8	**200**	17		11		200	2
	15		17		14		12			
Demand	100		250		100		150			
Column penalty	3		6		1		2			

Then, the initial basic feasible solution is

$\overline{x} = (\overline{x}_{11}, \overline{x}_{13}, \overline{x}_{21}, \overline{x}_{22}, \overline{x}_{24}, \overline{x}_{32}) = (50, 100, 50, 50, 150, 200)$, and $\overline{\mathrm{F}} = \frac{\mathrm{P}(\overline{\mathrm{x}})}{\mathrm{Q}(\overline{\mathrm{x}})} = 0.6448$

To improve the solution, let us apply the Kuhn- Tucker's optimality conditions as

$\mathrm{F}_{\mathrm{x}_{11}} = -1.294$, $\mathrm{F}_{\mathrm{x}_{12}} = 1.286$, $\mathrm{F}_{\mathrm{x}_{13}} = -9.444$, $\mathrm{F}_{\mathrm{x}_{14}} = 1.4$,
$\mathrm{F}_{\mathrm{x}_{21}} = -2.167$, $\mathrm{F}_{\mathrm{x}_{22}} = 0.75$, $\mathrm{F}_{\mathrm{x}_{23}} = 1.067$, $\mathrm{F}_{\mathrm{x}_{24}} = -5.286$,
$\mathrm{F}_{\mathrm{x}_{31}} = 0.733$, $\mathrm{F}_{\mathrm{x}_{32}} = -15.5294$, $\mathrm{F}_{\mathrm{x}_{33}} = 1.2143$, $\mathrm{F}_{\mathrm{x}_{34}} = 0.917$.

To determine the cost equation, let us use the equation (1)

$$\frac{\partial F}{\partial x_{B_{ij}}} = s_i + t_j$$

Since,

$F_{x_{11}} = s_1 + t_1 \Rightarrow s_1 + t_1 = -1.294$, $F_{x_{12}} = s_1 + t_2 \Rightarrow s_1 + t_2 = 1.286$,
$F_{x_{13}} = s_1 + t_3 \Rightarrow s_1 + t_3 = -9.444$, $F_{x_{14}} = s_1 + t_4 \Rightarrow s_1 + t_4 = 1.4$,
$F_{x_{21}} = s_2 + t_1 \Rightarrow s_2 + t_1 = -2.167$, $F_{x_{22}} = s_2 + t_2 \Rightarrow s_2 + t_2 = 0.75$,
$F_{x_{23}} = s_2 + t_3 \Rightarrow s_2 + t_3 = 1.067$, $F_{x_{24}} = s_2 + t_4 \Rightarrow s_2 + t_4 = -5.286$,
$F_{x_{31}} = s_3 + t_1 \Rightarrow s_3 + t_1 = 0.733$, $F_{x_{32}} = s_3 + t_2 \Rightarrow s_3 + t_2 = -15.5294$,
$F_{x_{33}} = s_3 + t_3 \Rightarrow s_3 + t_3 = 1.2143$, $F_{x_{34}} = s_3 + t_4 \Rightarrow s_3 + t_4 = 0.917$.
Put $s_1 = 0$, we have
$t_1 = -1.294, t_2 = 1.286, t_3 = -9.444$, $t_4 = 1.4, s_2 = -0.873, t_3 = 1.94, t_4 = -4.413$,
$s_3 = 2.027$,
Let us estimate the reduced cost from the equation (2) as

$$M_{12} = \frac{\partial F}{\partial x_{NB_{12}}} - s_1 - t_2 = -35,$$

$$M_{14} = \frac{\partial F}{\partial x_{NB_{14}}} - s_1 - t_4 = 5.81,$$

$$M_{23} = \frac{\partial F}{\partial x_{NB_{23}}} - s_2 - t_3 = 11.43,$$

$$M_{31} = \frac{\partial F}{\partial x_{NB_{14}}} - s_1 - t_4 = 19.18,$$

$$M_{33} = \frac{\partial F}{\partial x_{NB_{14}}} - s_1 - t_4 = 28.33,$$

$$M_{34} = \frac{\partial F}{\partial x_{NB_{14}}} - s_1 - t_4 = 22.99.$$

It is clear that $M_{12} = -35$, so x_{12} should be entered as basic variables. Then the next iteration resulted in the initial basic feasible solution

$\overline{x} = (\overline{x}_{12}, \overline{x}_{13}, \overline{x}_{21}, \overline{x}_{24}, \overline{x}_{32}) = (50, 100, 100, 100, 200)$, and $\overline{F} = \frac{P(\overline{x})}{Q(\overline{x})} = 0.5294$

By repeating the previous procedure, we obtain
$F_{x_{11}} = 0.706$, $F_{x_{12}} = -1.714$, $F_{x_{13}} = -9.444$, $F_{x_{14}} = 1.4$,
$F_{x_{21}} = -5.176$, $F_{x_{22}} = 1.75$, $F_{x_{23}} = 1.067$, $F_{x_{24}} = -3.286$,
$F_{x_{31}} = 0.733$, $F_{x_{32}} = -15.5294$, $F_{x_{33}} = 1.2143$, $F_{x_{34}} = 0.917$.
Let us use the equation (1), to determine the cost equation as
$F_{x_{11}} = s_1 + t_1 \Rightarrow s_1 + t_1 = 0.706$, $F_{x_{12}} = s_1 + t_2 \Rightarrow s_1 + t_2 = -1.714$,
$F_{x_{13}} = s_1 + t_3 \Rightarrow s_1 + t_3 = -9.444$, $F_{x_{14}} = s_1 + t_4 \Rightarrow s_1 + t_4 = 1.4$,
$F_{x_{21}} = s_2 + t_1 \Rightarrow s_2 + t_1 = -5.176$, $F_{x_{22}} = s_2 + t_2 \Rightarrow s_2 + t_2 = 1.75$,
$F_{x_{23}} = s_2 + t_3 \Rightarrow s_2 + t_3 = 1.067$, $F_{x_{24}} = s_2 + t_4 \Rightarrow s_2 + t_4 = -3.286$,
$F_{x_{31}} = s_3 + t_1 \Rightarrow s_3 + t_1 = 0.733$, $F_{x_{32}} = s_3 + t_2 \Rightarrow s_3 + t_2 = --15.5294$,
$F_{x_{33}} = s_3 + t_3 \Rightarrow s_3 + t_3 = 1.2143$, $F_{x_{34}} = s_3 + t_4 \Rightarrow s_3 + t_4 = 0.917$.
Set $s_1 = 0$, we get
$t_1 = 0.706, t_2 = -1.714, t_3 = -9.444, t_4 = 1.4, s_2 = -5.882, s_3 = 0.027$.
By applying the equation (2), let us estimate the reduced cost from as

$$M_{11} = \frac{\partial F}{\partial x_{NB_{12}}} - s_1 - t_1 = 23.2,$$

$$M_{14} = \frac{\partial F}{\partial x_{NB_{14}}} - s_1 - t_4 = 10.15,$$

$$M_{22} = \frac{\partial F}{\partial x_{NB_{23}}} - s_2 - t_2 = 4.0522,$$

$$M_{23} = \frac{\partial F}{\partial x_{NB_{14}}} - s_2 - t_3 = 7.1,$$

$$M_{31} = \frac{\partial F}{\partial x_{NB_{14}}} - s_3 - t_1 = 23.2,$$

$$M_{33} = \frac{\partial F}{\partial x_{NB_{14}}} - s_3 - t_3 = 24.5,$$

$$M_{34} = \frac{\partial F}{\partial x_{NB_{14}}} - s_3 - t_4 = 23.5.$$

Since al of $\frac{\partial F}{\partial x_{NB_{ij}}} > 0$ at $\bar{x} = (\bar{x}_{12}, \bar{x}_{13}, \bar{x}_{21}, \bar{x}_{24}, \bar{x}_{32}) = (50, 100, 100, 100, 200)$ this leads to the optimal solution with the optimum value equal to $\bar{F} = 0.5294$, and in neutrosophic is

$$\tilde{F}^N = \frac{\langle (8600, 10500, 11550, 14100); 0.6, 0.4, 0.7\rangle}{\langle (9850, 14750, 17200, 20150); 0.6, 0.4, 0.7\rangle} = \langle (0.4268, 0.61045, 0.78305, 1.4315); 0.6, 0.4, 0.7\rangle$$

6. Results and Discussions

It is clear that the neutrosophic optimum value is:
$\tilde{F}^N = \langle (0.4268, 0.61045, 0.78305, 1.4315); 0.6, 0.4, 0.7\rangle$ is better than the primary basic feasible solution, where it lies between 0.4268 *and* 1.4315. Also, as the optimum value lies between 0.61045 *and* 0.78305 , the overall acceptance level is60%. Also, the degrees of truthfulness and indeterminacy, respectively are given by:

$$\mu(x) = \begin{cases} 0.6\left(\frac{x-0.4268}{0.61045-0.4268}\right), & 0.4268 \le x < 0.61045, \\ 0.6, & 0.61045 \le x \le 0.78305, \\ 0.6\left(\frac{1.4315-x}{1.4315-0.78305}\right), & 0.78305 \le x \le 1.4315, \\ 0, & \text{otherwise,} \end{cases}$$

$$\rho(x) = \begin{cases} \frac{0.61045-x+0.4(x-0.4268)}{0.61045-0.4268}, & 0.4268 \le x < 0.61045, \\ 0.4, & 0.61045 \le x \le 0.78305, \\ \frac{x-0.78305+0.4(1.4315-x)}{1.4315-0.78305}, & 0.78305 \le x \le 1.4315, \\ 1, & \text{otherwise,} \end{cases}$$

$$\sigma(x) = \begin{cases} \frac{0.61045-x+0.7(x-0.4268)}{0.61045-0.4268}, & 0.4268 \le x < 0.61045, \\ 0.7, & 0.61045 \le x \le 0.78305, \\ \frac{x-0.78305+0.7(1.4315-x)}{1.4315-0.78305}, & 0.78305 \le x \le 1.4315, \\ 1, & \text{otherwise,} \end{cases}$$

Hence, the decision maker concludes that the optimum value range in between 0.4268 and 1.4315. On the other hand, the unit profit maximum is achieved with the supply 50 units from O_1 to D_2 with discount 3%, 100 units from O_1 to D_3 with discount 5%, 100 units from O_2 to D_1 with discount 3%, 100 units from O_2 to D_4 with discount 2%, and 200 units from O_3 to D_2 with discount 4%.

6.1 Advantages/Limitations of the proposed algorithm

The proposed algorithm's principal advantage is a novel combination of a parametric study, a multicriteria analysis, and the DM's vision. This combination uses the benefit of a parametric study that is used to scan the searching space smartly, the benefit of the multicriteria analysis that is used to rank the alternative solutions by employing the vision of the DM, and the benefit of involving the vision of the DM. Applying the proposed algorithm to real-life problems may encounter some limitations such as:

1- It does not take into account the complete parametric space, which has an endless number of possible scenarios. But, no other techniques can handle such situations where there are infinite scenarios.
2- It is impossible to assign a unified technique for assigning the interesting scenarios for the DM i.e. the approach does not involve a unified method; where the DM's vision and weights differ from one to another.
3- Many factors must be considered such as; (i) the possibility of formulating the problem as a FTP problem, (ii) the possibility of formulating the KKT conditions and solving it, and (iii) the capability of solving the PFTP problem's selected scenarios and finding their exact optimal solutions.

7. Comparitive Study

In this section, the proposed study is compared with some existing relevant literature to carve out the advantageous aspect of the proposed study. The Table 8 presents this comparison under certain parameters. It’s obvious that the result obtained by the proposed approach is less than the result by Gomathi and Jayalakshmi [19]

Table 8. Comparisons of different researcher's contributions

Author's name	Vogel's approximation method	Kuhn- Tucker optimality conditions	Optimal neutrosophic solution	Environment
Gomathi and Jayalakshmi [19]	√	√	×	crisp
Bas et al., [25],	×	×	√	Crisp
Akram et al., [26]	×	×	√	Fuzzy
Our proposed approach	√	√	√	Neutrosophic

8. Conclusions and future works

In this paper, the maximization fractional transportation problem has solved efficiently in neutrosophic environment. The method which has applied is can be used in all of road tax, discount cost and others. In addition, the analysis process in the proposed approach depends upon some proposed characteristics that consider the uncertainty in determining the optimal solution. Fundamental definitions for NINP problems, such as optimistic-optimal solutions, pessimistic-optimal solutions, satisfactory-optimal solutions, and feasibility-risk factors, were also introduced. Furthermore, the proposed approach involves the vision of the DM in the process of finding the optimal solution, and a utility function is used to rank the different alternatives so that the satisfactory optimal solution can be easily identified. Finally, an example is introduced to clarify the efficiency of the proposed approach. Finally, an example is introduced to clarify the efficiency of the proposed approach and compare the results obtained by one of the most prominent evolutionary algorithms, the genetic algorithm (GA), to validate the accuracy and reliability of simulation results. Future work may include the further extension of this study to other fuzzy- like structure (i. e., interval- valued fuzzy set, Neutrosophic set, Pythagorean fuzzy set, Spherical fuzzy set etc. with more discussion and suggestive comments.

Data Availibity

No data were used to support this study.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Evaluation of E-Commerce Sites using Novel Similarity Measure of Neutrosophic Hypersoft Sets

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Abstract: E-commerce has become a common method of making online purchases. Doorstep delivery, multiple options and variety of products to be bought, discounts and rewards are some of the advantages of online shopping. However, selection of the right shopping website is a challenge for online buyers across all markets. The multiple tangible and intangible factors involved in the e-commerce domain has made it a problem of MCDM. This paper proposes a new MCDM approach to develop a model for the necessary assessment of e-commerce websites. The notion of similarity measures for the single valued neutrosophic hypersoft sets is applied to get the best website on the bases of given criteria. Shopping websites are evaluated at three levels-below average, average and good. The attributes used to evaluate the websites are offers and deals made by the e-commerce sites to present and potential customers, qualitative assessment of the offered products and services, delivery timelines, payment safety and security concerns. Safety of personal data is the attribute which is judged as the most important factor in evaluation of shopping websites.

Keywords: *E-commerce, Similarity measure, Evaluation of shopping website, MCDM.*

1. Introduction

The e-commerce sector has gone under a remarkable evolutionary phase which gives significant impacts on entrepreneurs, bluechip/small-scaled manufacturing industries and ultimately the end-users/consumers. Adoption of internet and e-commerce has increased exponentially over the last few years. In 2022, sales from the e-commerce sector exceeded USD 5.7 trillion. The figure is projected for further increase in coming years. Global online retail sales is projected to exceed USD 7 Trillion by 2025 [1], [2]. Asia has the largest consumer base for the e-commerce market.India has the third largest position in terms of online shoppers base with 150 million in the year 2021. The E-commerce market in India is projected to achieve a growth of 350 billion USD in 2030 as compared to 46.2 billion USD in 2020 [3]. From fashion to groceries, household goods, electronic goods, and bill payments, e-commerce is taking over the traditional retail sector. Number of internet users in India is expected to increase to 320 million by the year 2025 [4]. Despite the robust growth, complete potential is not achieved by the sector. Different challenges like shopping cart abandonment, shifting to other shopping platforms, privacy, security concerns are deterring the growth [5]. Inconsistent, asymmetric and indeterminate information comprise the main limitations of e-commerce. The risks are prevalent not only in B2C but C2C or peer to peer market [6].With information inputs from multiple sources - social media, retail websites, advertisements, it becomes challenging for the consumer to select the best shopping website. Bindia and Daroch [7] outlined the factors affecting online shopping behavior. These include security, lack of trust,privacy policy, website, shopping experience, retailer brand, product information and financial risk. Consumers are skeptical about using online platforms because of cyber frauds, financial risks, hidden costs, long forms [8]. Further, assessment work has been done electric vehicle charge stations in order to enhance the green energy for smart cities [9]. In addition to this, an effective analysis has been done for photovoltaic power plants for the issues of green environment [10].

On the basis of different level survey data defined through a set of questions, various mathematical models have been formulated for obtaining the suitable assessment of the online shopping agencies under the framework of e-commerce services. Such assessments involve several criteria, e.g., "quality of product, cost, shipping services, safety of payments, etc."[11]. Therefore , different researchers have given due considerations on these criteria which are directly connected to access the qualitative aspects of e-commerce online shopping agencies.Since the customer's perceptions and experiences while using the features of e-commerce sectors comprise of precise and imprecise/incomplete both kind of values, therefore any possible judgement must incorporate the content of uncertainty in a considerable amount. For the sake of dealing such scenario, the concept of multi-criteria decision making problem under a certain fuzzified approach would be more useful and considerably better as the decision of buying/not buying is somehow dependent on customer's intuitions, common sense and past-experiences rather than on the crisp, precise and accurate information [12]. This paper presents a Multi Criteria Decision-making (*MCDM*) approach to select the shopping website which has the attributes required by the consumer.

Selection of online shopping platforms is affected by different criteria. Prior studies have identified several factors which affect consumer choice of a particular purchase platform. E-service quality is an important determinant of e-commerce. Different factors which categorized e-service quality are efficiency, fulfillment.compensation.privacy, in store experience contact, system availability, reliability, trust and quality of communication.[13], [14], [15] .Customer satisfaction results in repeat business and positive online reviews which are instrumental in increasing the brand equity [16] Website quality, content, conditions of product return, payment process, rapid response, transaction security issues are some other factors affecting satisfaction level of e commerce consumers [17]. Anushka et al. [18]. have identified ten factors with the help of a five-point fuzzy scale. The factors include fashion deals,product quality, fast shipping, prompt and regular customer care service, return policy, keeping track of the shipped products, refining search options, detailed product description, safe and secure payment options. The criteria are used to rank eight shopping websites Amazon, Flipkart, Myntra, BigBasket, Jabong, Ebay, Snapdeal and Paytm Mall. Safe payments and quality products are given maximum weightage according to the weight criteria matrix. Fuzzy TOPSIS and PROMETHEE are used to rank the websites based on the selected criteria. Amazon gets the highest rank as the most selected shopping website. Ilias O. Pappas[19] studies the impact of perceived trust, privacy and past experience on evaluation of online shopping sites. Fuzzy set analysis. The paper studies the impact of perceived trust, privacy, and past experience on customer purchase intention. The variables are related through a proposed research model, which is validated through fuzzy logic techniques. Ambiguity and uncertainty are the inherent characteristics of online shopping. Trust is an important factor for increased adoption and success of e-commerce sites. Also various extensions of fuzzy sets in terms of hypersoft sets have been done in the literature with applications in the renewable energy, robotic agrifarming [20], [21], [22], [23], [24]. Different factors have been identified in the prior studies to increase customer trust. These factors are existence, fulfillment, affiliation and company policy. Trust is an important factor for successful online transactions. Presence of trust enables the consumers to share their personal information with others. Presence of trust provides better experience even in the case of negative emotions [25]. Customer needs and security issues have gained attention in lieu of the exponential increase in the number of online customers in the last five years. Affective and cognitive factors such as trust and privacy affect consumer evaluation of online consumer sites [26]. The study highlights the importance of trust, user experience and privacy in defining the behavioral intention of online consumers. Another study by Aydın, Serhat & Kahraman, Cengiz [27] has identified ten criteria for assessing the shopping website. These include ease of use , product quality detail, security, privacy, customer relationship, accurate delivery, billing and safe payment gateways. A holistic approach is required to study the impact of these factors on online buyers' behavior. Researchers like Liang, R., Wang, J., & Zhang, H [28] have used *MCDM* techniques such as SVTN-DEMATEL module to show the relationship between different criteria, highlighting different priority areas.Prior studies on the topic use symmetric analysis tools based on regression like sequential equation modeling (SEM) or multiple regression analysis (MRA). Also, decision-making in neutrosophic topologies have been done along with the most influential sector of Industry 5.0 [29], [30], [31]. However, symmetric tests can be misleading in certain cases. Same technological factors may affect consumers differently. Thus a combination of factors needs to be studied to explain the complex consumer behavior. To address this gap fuzzy networks have been applied to provide a more comprehensive and exhaustive analysis.The paper attempts to propose a method for evaluation and selection of the shopping website that matches the attributes selected by the consumers. Multiple Criteria Decision-making (*MCDM*) approach employing similarity measures of is applied as the main method for data analysis. Fuzzy logic is a qualitative method to analyze complex system behaviors and patterns. It uses multi valued logic to develop effective reasoning and better decision making models [12]. The study uses fuzzy logic analysis to identify the configuration of perceived trust, privacy and user experience in influencing the levels of consumer perception in online buying. The tools are useful in explaining the complex relation among different research variables.Shopping websites are analyzed at three levels- below average, average and good. The attributes used to evaluate the websites are offers and deals offered by the ecommerce sites to their customers, qualitative assessment of the offered products and services, delivery timelines, payment safety and security concerns. Safety of personal data is the attribute which is judged as the most important factor in evaluation of shopping websites.

Novelty and Contribution of the present study

A novel type of similarity measure for single-valued neutrosophic sets has been prosposed for handling the formulated MCDM problem. The proposed notional description of similarity measures for the neutrosophic sets have been utilized in the evaluation of e-commerce sectors and are being integrated in decision-making methodology under a neutrosophic hypersoft setup.

In the present study, the prime focus of the work is to go for the assessment of e-commerce sites by utilizing the similarity measures for neutrosophic hypersoft sets:

- Introducing a new parametrized similarity for neutrosophic hypersoft sets.
- First, the proposed similarity measure has been proved mathematically on certain axioms for validation.
- Further, the proposed similarity measures have been successfully implemented in the evaluation for the e-commerce sites.

The paper's structure is: Section 2 presents fundamental definitions in connection with the proposed measure. Section 3 involves some binary operations and novel similarity measures of *SVNHSS* along with its proof of validation. Section 4 discusses the research methodology of the *MCDM* problem. Section 5 provides details of data analysis using the proposed similarity measure of *SVNHSS* in the E-commerce sector. Section 6 discusses the findings in the form of major conclusions, limitations and future scope.

2. Preliminaries

Definition 1. *[32]"Let X be the universal set and $P(X)$ be the power set of X. Consider $k^1, k^2, \dots, k^n$ for $n \geq 1$ be n well-defined attributes whose corresponding attribute values are respectively the sets $K^1, K^2, \dots, K^n$ with $K^i \cap K^j = \emptyset$, for $i \neq j$ and $i, j \in \{1,2,..,n\}$, then the pair $(\mathfrak{R}, K^1 \times K^2 \times \dots \times K^n)$ is said to be Hypersoft set over the set X, where $\mathfrak{R} : K^1 \times K^2 \times \dots \times K^n \to P(X)$."*

Definition 2. *[32]"Let $X = \{x_1, x_2, \dots, x_n\}$ be the universal set and $P(X)$ be the power set of X. Consider $K_1, K_2, \dots, K_m$ for $m \geq 1$ be m well-defined attributes whose corresponding attribute values are respectively the sets $K_1^a, K_2^b, \dots, K_m^z$ with the relation $K_1^a \times K_2^b \times \dots \times K_m^z = \Gamma$, where a, b, c, …, z = 1,2,..,n. Then the pair $(\mathfrak{R}, \Gamma)$ is said to be Neutrosophic Hypersoft set (NHSS) over X, where, $\mathfrak{R} : K_1^a \times K_2^b \times \dots \times K_m^z \to P(X)$ and $\mathfrak{R}(K_1^a \times K_2^b \times \dots \times K_m^z) = \{\langle x, T_{\mathfrak{R}(\xi)}(x), I_{\mathfrak{R}(\xi)}(x), F_{\mathfrak{R}(\xi)}(x)\rangle, x \in X, \xi \in \Gamma\}$; where T is the degree of truthness, I is the degree of indeterminacy and F is the degree of falsity such that T, I, F : V $\to (0^-, 1^+)$ and satisfies the constraint $0^- \leq T_{\mathfrak{R}(\xi)}(x) + I_{\mathfrak{R}(\xi)}(x) + F_{\mathfrak{R}(\xi)}(x)) \leq 3^+$.*

While dealing with applications of science and engineering, it becomes very difficult to handle situations under a neutrosophic environment. In order to deal with such situations notion of Single-Valued Neutrosophic HyperSoft sets ($SVNHSS$) is very useful and applicable."

Definition 3. *[33]"Let $X = \{x_1, x_2, \dots, x_n\}$ be the universal set and $P(X)$ be the power set of X. Consider $K_1, K_2, \dots, K_m$ for $m \geq 1$ be m well-defined attributes whose corresponding attribute values are respectively the sets $K_1^a, K_2^b, \dots, K_m^z$ with the relation $K_1^a \times K_2^b \times \dots \times K_m^z = \Gamma$, where a, b, c, …, z = 1,2,..,n. Then the pair $(\mathfrak{R}, \Gamma)$ is said to be a Single-Valued Neutrosophic Hypersoft set ($SVNHSS$) over X, where, $\mathfrak{R} : K_1^a \times K_2^b \times \dots \times K_m^z \to P(X)$ and $\mathfrak{R}(K_1^a \times K_2^b \times \dots \times K_m^z) = \{\langle x, T_{\mathfrak{R}(\xi)}(x), I_{\mathfrak{R}(\xi)}(x), F_{\mathfrak{R}(\xi)}(x)\rangle, x \in X, \xi \in \Gamma\}$; where T is the degree of truthness, I is the degree of indeterminacy and F is the degree of falsity such that T, I, F : V $\to [0, 1]$ and satisfies the constraint $0 \leq T_{\mathfrak{R}(\xi)}(x) + I_{\mathfrak{R}(\xi)}(x) + F_{\mathfrak{R}(\xi)}(x)) \leq 3$."*

Definition 4.*[30]"Consider A and B be two single-valued neutrosophic sets, then the axiomatic definition of similarity measure are as follows:*

i. $0 \leq \mathbb{S}(A,B) \leq 1$;

ii. $\mathbb{S}(A,A) = 1$.

iii. $\mathbb{S}(A,B) = \mathbb{S}(B,A) \;\; \forall \;\; B \in SVNS(X)$.

iv. *If* $A \subseteq B \subseteq C, \forall \;\; C \in SVNS(X)$, $\mathbb{S}(A,C) \leq \mathbb{S}(A,B)$ *and* $\mathbb{S}(A,C) \leq \mathbb{S}(B,C)$."

3. Binary Operations and Similarity Measure of Neutrosophic Hypersoft Sets

Some binary algebraic operations on $SVNHSSs$ have been presented where $SVNHSS(X)$ represents a collection of SVNHSSs over X. For $A, B \in SVNHSS(X)$, we present the operations as below:

- "**Union of** A **and**B":$A \cup B = \{x,\ T_{A\cup B}(\mathfrak{R}(\Gamma)),\ I_{A\cup B}(\mathfrak{R}(\Gamma)),\ F_{A\cup B}(\mathfrak{R}(\Gamma))|\ x \in X\ \}$

where,

$$T_{A\cup B}(\mathfrak{R}(\Gamma))(x) = max\{T_A(\mathfrak{R}(\Gamma))(x), \quad T_B(\mathfrak{R}(\Gamma))(x)\}\ , I_{A\cup B}(\mathfrak{R}(\Gamma))(x) = min\{I_A(\mathfrak{R}(\Gamma))(x), \quad I_B(\mathfrak{R}(\Gamma))(x)\}$$

and $F_{A\cup B}(\mathfrak{R}(\Gamma))(x) = min\{F_A(\mathfrak{R}(\Gamma))(x),\ F_B(\mathfrak{R}(\Gamma))(x)\}\forall\ \ x\ \ \in X$.

- "**Intersection of**A **and**B":$A \cap B = \{x,\ T_{A\cap B}(\mathfrak{R}(\Gamma)),\ I_{A\cap B}(\mathfrak{R}(\Gamma)),\ F_{A\cap B}(\mathfrak{R}(\Gamma))|\ x \in X\ \}$

where,

$$T_{A\cap B}(\mathfrak{R}(\Gamma))(x) = min\{T_A(\mathfrak{R}(\Gamma))(x), \quad T_B(\mathfrak{R}(\Gamma))(x)\}\ , I_{A\cap B}(\mathfrak{R}(\Gamma))(x) = max\{I_A(\mathfrak{R}(\Gamma))(x), \quad I_B(\mathfrak{R}(\Gamma))(x)\}$$

and $F_{A\cap B}(\mathfrak{R}(\Gamma))(x) = max\{F_A(\mathfrak{R}(\Gamma))(x),\ F_B(\mathfrak{R}(\Gamma))(x)\}\forall\ \ x\ \ \in X$.

- **Containment:**$A \subseteq B$ if and only if

$$T_A(\mathfrak{R}(\Gamma))(x) \leq T_B(\mathfrak{R}(\Gamma))(x),\ \ I_A(\mathfrak{R}(\Gamma))(x) \geq I_B(\mathfrak{R}(\Gamma))(x),\ \ F_A(\mathfrak{R}(\Gamma))(x) \geq F_B(\mathfrak{R}(\Gamma))(x)\forall\ \ x \in X.$$

- "**Complement:**

$T_{\bar{A}}(\mathfrak{R}(\Gamma))(x) = 1 - \ T_A(\mathfrak{R}(\Gamma))(x),\ I_{\bar{A}}(\mathfrak{R}(\Gamma))(x) = 1 - \ I_A(\mathfrak{R}(\Gamma))(x),\ F_{\bar{A}}(\mathfrak{R}(\Gamma))(x) = 1 - F_A(\mathfrak{R}(\Gamma))(x)$".

Further, we propose a new similarity measure for two $SVNHSSA$ and B,

$$\mathbb{S}(A,B) = \frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}\frac{1-\frac{1}{2}\left[\left(\min\left\{\left|T_{A(\xi_j^s)}(x^i)-T_{B(\xi_j^s)}(x^i)\right|,\left|I_{A(\xi_j^s)}(x^i)-I_{B(\xi_j^s)}(x^i)\right|\right\}+\left|F_{A(\xi_j^s)}(x^i)-F_{B(\xi_j^s)}(x^i)\right|\right)\right]}{1+\frac{1}{2}\left[\left(\max\left\{\left|T_{A(\xi_j^s)}(x^i)-T_{B(\xi_j^s)}(x^i)\right|,\left|I_{A(\xi_j^s)}(x^i)-I_{B(\xi_j^s)}(x^i)\right|\right\}+\left|F_{A(\xi_j^s)}(x^i)-F_{B(\xi_j^s)}(x^i)\right|\right)\right]} \tag{1}$$

where, $j = 1,2, \ldots, m; i = 1,2, \ldots, n; s = a, b, \ldots, z; z = 1,2, \ldots, n$ and $\xi_j^s \in \ K_1^a \times K_2^b \times \ldots \times \ K_m^z$.

Theorem 1.*The above-proposed similarity measure $\mathbb{S}(A,B)$ given in (1) is a valid similarity measure of $SVNHSSs$.*

Proof: Refer Section 2, we establish the axioms provided for checking the validity.

The axioms *(i)* and *(ii)* immidiately follows from the definition of the proposed measure.

(iii) Here, we assume that $A = B$.

Then, $T_{A(\xi_j^s)}(x^i) = T_{B(\xi_j^s)}(x^i)$, $I_{A(\xi_j^s)}(x^i) = I_{B(\xi_j^s)}(x^i)$, $F_{A(\xi_j^s)}(x^i) = F_{B(\xi_j^s)}(x^i)$.

$$\Longrightarrow \quad \mathbb{S}(A,B) = 1.$$

Conversely, let $\mathbb{S}(A,B) = 1$.

$$\Longrightarrow \frac{1-\frac{1}{2}\left[\left(\min\left\{\left|T_{A(\xi_j^s)}(x^i) - T_{B(\xi_j^s)}(x^i)\right|, \left|I_{A(\xi_j^s)}(x^i) - I_{B(\xi_j^s)}(x^i)\right|\right\} + \left|F_{A(\xi_j^s)}(x^i) - F_{B(\xi_j^s)}(x^i)\right|\right)\right]}{1+\frac{1}{2}\left[\left(\max\left\{\left|T_{A(\xi_j^s)}(x^i) - T_{B(\xi_j^s)}(x^i)\right|, \left|I_{A(\xi_j^s)}(x^i) - I_{B(\xi_j^s)}(x^i)\right|\right\} + \left|F_{A(\xi_j^s)}(x^i) - F_{B(\xi_j^s)}(x^i)\right|\right)\right]} = 1,$$

$$\Longrightarrow 1-\frac{1}{2}\left[\left(\min\left\{\left|T_{A(\xi_j^s)}(x^i) - T_{B(\xi_j^s)}(x^i)\right|, \left|I_{A(\xi_j^s)}(x^i) - I_{B(\xi_j^s)}(x^i)\right|\right\} + \left|F_{A(\xi_j^s)}(x^i) - F_{B(\xi_j^s)}(x^i)\right|\right)\right] = 1 +$$

$$\frac{1}{2}\left[\left(\max\left\{\left|T_{A(\xi_j^s)}(x^i) - T_{B(\xi_j^s)}(x^i)\right|, \left|I_{A(\xi_j^s)}(x^i) - I_{B(\xi_j^s)}(x^i)\right|\right\} + \left|F_{A(\xi_j^s)}(x^i) - F_{B(\xi_j^s)}(x^i)\right|\right)\right],$$

$$\Longrightarrow \frac{1}{2}\left[\left(\min\left\{\left|T_{A(\xi_j^s)}(x^i) - T_{B(\xi_j^s)}(x^i)\right|, \left|I_{A(\xi_j^s)}(x^i) - I_{B(\xi_j^s)}(x^i)\right|\right\} + \left|F_{A(\xi_j^s)}(x^i) - F_{B(\xi_j^s)}(x^i)\right|\right)\right] +$$

$$\frac{1}{2}\left[\left(\max\left\{\left|T_{A(\xi_j^s)}(x^i) - T_{B(\xi_j^s)}(x^i)\right|, \left|I_{A(\xi_j^s)}(x^i) - I_{B(\xi_j^s)}(x^i)\right|\right\} + \left|F_{A(\xi_j^s)}(x^i) - F_{B(\xi_j^s)}(x^i)\right|\right)\right]=0,$$

$$\Longrightarrow \left|T_{A(\xi_j^s)}(x^i) - T_{B(\xi_j^s)}(x^i)\right| = 0, \left|I_{A(\xi_j^s)}(x^i) - I_{B(\xi_j^s)}(x^i)\right| = 0, \text{ and } \left|F_{A(\xi_j^s)}(x^i) - F_{B(\xi_j^s)}(x^i)\right| = 0,$$

$$\Longrightarrow \quad A = B.$$

(iv) Let $A \subseteq B \subseteq C$,

$$\Longrightarrow \left|T_{A(\xi_j^s)}(x^i) - T_{B(\xi_j^s)}(x^i)\right| \le \left|T_{A(\xi_j^s)}(x^i) - T_{C(\xi_j^s)}(x^i)\right|, \ \left|I_{A(\xi_j^s)}(x^i) - I_{B(\xi_j^s)}(x^i)\right| \le \left|I_{A(\xi_j^s)}(x^i) - I_{C(\xi_j^s)}(x^i)\right|$$

and $\left|F_{A(\xi_j^s)}(x^i) - F_{B(\xi_j^s)}(x^i)\right| \le \left|F_{A(\xi_j^s)}(x^i) - F_{C(\xi_j^s)}(x^i)\right|$,

$$\Longrightarrow \min\left\{\left|T_{A(\xi_j^s)}(x^i) - T_{B(\xi_j^s)}(x^i)\right|, \left|I_{A(\xi_j^s)}(x^i) - I_{B(\xi_j^s)}(x^i)\right|\right\} + \left|F_{A(\xi_j^s)}(x^i) - F_{B(\xi_j^s)}(x^i)\right|$$

$$\le \min\left\{\left|T_{A(\xi_j^s)}(x^i) - T_{C(\xi_j^s)}(x^i)\right|, \left|I_{A(\xi_j^s)}(x^i) - I_{C(\xi_j^s)}(x^i)\right|\right\} + \left|F_{A(\xi_j^s)}(x^i) - F_{C(\xi_j^s)}(x^i)\right|$$

and $\max\left\{\left|T_{A(\xi_j^s)}(x^i) - T_{B(\xi_j^s)}(x^i)\right|, \left|I_{A(\xi_j^s)}(x^i) - I_{B(\xi_j^s)}(x^i)\right|\right\} + \left|F_{A(\xi_j^s)}(x^i) - F_{B(\xi_j^s)}(x^i)\right| \le \max\left\{\left|T_{A(\xi_j^s)}(x^i) -\right.\right.$

$\left.\left. T_{C(\xi_j^s)}(x^i)\right|, \left|I_{A(\xi_j^s)}(x^i) - I_{C(\xi_j^s)}(x^i)\right|\right\} + \left|F_{A(\xi_j^s)}(x^i) - F_{C(\xi_j^s)}(x^i)\right|$,

$$\Rightarrow 1-\frac{1}{2}[\min\left\{\left|T_{A\left(\xi_j^s\right)}(x^i)-T_{B\left(\xi_j^s\right)}(x^i)\right|,\left|I_{A\left(\xi_j^s\right)}(x^i)-I_{B\left(\xi_j^s\right)}(x^i)\right|\right\}+\left|F_{A\left(\xi_j^s\right)}(x^i)-F_{B\left(\xi_j^s\right)}(x^i)\right|]$$
$$\geq 1-\frac{1}{2}[\min\left\{\left|T_{A\left(\xi_j^s\right)}(x^i)-T_{C\left(\xi_j^s\right)}(x^i)\right|,\left|I_{A\left(\xi_j^s\right)}(x^i)-I_{C\left(\xi_j^s\right)}(x^i)\right|\right\}+\left|F_{A\left(\xi_j^s\right)}(x^i)-F_{C\left(\xi_j^s\right)}(x^i)\right|]$$

and $1+\frac{1}{2}[\max\left\{\left|T_{A\left(\xi_j^s\right)}(x^i)-T_{B\left(\xi_j^s\right)}(x^i)\right|,\left|I_{A\left(\xi_j^s\right)}(x^i)-I_{B\left(\xi_j^s\right)}(x^i)\right|\right\}+\left|F_{A\left(\xi_j^s\right)}(x^i)-F_{B\left(\xi_j^s\right)}(x^i)\right|] \leq 1+\frac{1}{2}[\max\left\{\left|T_{A\left(\xi_j^s\right)}(x^i)-T_{C\left(\xi_j^s\right)}(x^i)\right|,\left|I_{A\left(\xi_j^s\right)}(x^i)-I_{C\left(\xi_j^s\right)}(x^i)\right|\right\}+\left|F_{A\left(\xi_j^s\right)}(x^i)-F_{C\left(\xi_j^s\right)}(x^i)\right|]$

$\Rightarrow \mathbb{S}(A,C)\leq \mathbb{S}(A,B)$. On the similar lines, we can prove $\mathbb{S}(A,C)\leq \mathbb{S}(B,C)$.

Remark: Also, a tangent similar measure between the two $SVNHSSA$ and $B$is given by,

$$\mathbb{T}(A,B)=\frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}\left[1-tan\frac{\pi}{12}\left(\left|T_{A\left(\xi_j^s\right)}(x^i)-T_{B\left(\xi_j^s\right)}(x^i)\right|+\left|I_{A\left(\xi_j^s\right)}(x^i)-I_{B\left(\xi_j^s\right)}(x^i)\right|+\left|F_{A\left(\xi_j^s\right)}(x^i)-F_{B\left(\xi_j^s\right)}(x^i)\right|\right)\right]$$

where, $j=1,2,\ldots,m; i=1,2,\ldots,n; s=a,b,\ldots,z; z=1,2,\ldots,n$ and $\xi_j^s \in K_1^a \times K_2^b \times \ldots \times K_m^z$.
The validity of this trigonometric similarity measure can be done as above.

4. Implementation of the Proposed Similarity Measure in the MCDM Problem

The procedural phase wise computation involved in the proposed methodology has been elaborated through Figure 1. Let us assume that there are m alternatives $\{Y_1,Y_2,\ldots,Y_m\}$ and n attributes $K^1,K^2,\ldots,K^n$ and "whose corresponding attribute values are respectively the sets $K_1^a,K_2^b,\ldots,K_m^z$ with the relation $K_1^a\times K_2^b\times\ldots\times K_m^z=\Gamma$, where $a, b, c, \ldots, z=1,2,..,n$."The set of all possible $SVNHSSs$ are given by $(\mathfrak{R},\Gamma)$, where $\Gamma=K_1^a\times K_2^b\times\ldots\times K_m^z$. The objective of a decision-maker is to select the most appropriate choice among the available alternatives from the set of available ones satisfying the given attribute values. The decisions in the form of an information from all the decision-makers are tabulated in a matrix format, say, $H=[h_{ij}]_{m\times n}$ called single-valued neutrosophic hypersoft matrix where $h_{ij}=\left(T_{\mathfrak{R}(\xi)}(x)_{ij}, I_{\mathfrak{R}(\xi)}(x)_{ij}, F_{\mathfrak{R}(\xi)}(x)_{ij}\right)$. The procedure of the proposed methodology has been presented in Figure 1 below:

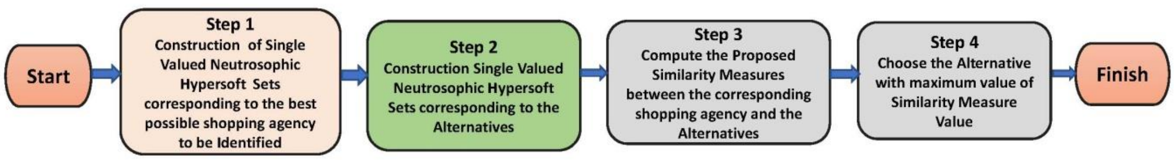

Figure 1: Methodology of MCDM

Step 1: Construction of a decision matrix on the basis of prescribed information in the form of SVNHSS.

Step 2: Further, we eliminate the heterogeneity present in the attributes & transform it in its homogenous form for the attribute. Next, the decision matrix $H=[h_{ij}]_{m\times n}$ has been changed to a revised decision matrix $H'=[h'_{ij}]_{m\times n}$ such that h'_{ij} is given by

$$h'_{ij} = \left(T_{\mathfrak{R}(\xi)}(x)_{ij}, I_{\mathfrak{R}(\xi)}(x)_{ij}, F_{\mathfrak{R}(\xi)}(x)_{ij}\right) = \begin{cases} \mathrm{h_{ij}} & ; \text{ for benefit criteria} \\ \mathrm{h_{ij}}^{c} & ; \text{ for cost criteria.} \end{cases}$$

Step 3: We evaluate the score from the proposed measure for the $Y_i's$ respectively with the sub-attributes on one to one basis.

Step 4: Finally, the necessary ranking of alternatives may be worked out on the basis of the score values obtained from the similarity measure.

5. Use of Proposed Similarity Measures in E-commerce sector.

On the basis of the methodology discussed above, we move on for the assessment of the best possible e-shopping agencies based on the similarity measure for $SVNHSSs$. Let there are four e-shopping agencies $\{Y_1, Y_2, Y_3, Y_4\}$. Suppose there are three stages of selectionfor assessing the shoping agencies as $below\ \ average(x^1)$, $average(x^2)$ and $good(x^3)$. The universal set $X = \{x^1, x^2, x^3\}$. Let $K = \{K^1 = captivating\ \ offers\ \ or\ \ deals,\ \ K^2 = qualitative\ \ assessment\ \ of\ \ items,\ K^3 = on\ \ time\ \ delivery,\ K^4 = paying\ \ safety\ \&\ security\ \}$ be a group of criterions which are categorized with sub-attributes:

$$K^1 = \text{“}captivating\ \ offers\ \ or\ \ deals = \{trendy,\quad elegant\}\text{”}$$

$$K^2 = \text{“}qualitative\ \ assessment\ \ of\ \ items$$
$$= \{test\ \ automation, defect\ \ rate, mean\ \ time\ \ to\ \ green\}\text{”}$$

$$K^3 = \text{“}on\ \ time\ \ delivery\text{”}$$

$$K^4 = \text{“}paying\ \ safety\ \ \&\ \ security$$
$$= \{online\ \ payment,\quad pay\ \ on\ \ delivery\ \ option,$$
$$assurance\ \ of\ \ safety\ \ of\ \ personal\ \ data\ \ of\ \ customers\ \}\text{”}$$

Now, let us define a relation $\mathfrak{R} : K_1^a \times K_2^b \times \ldots \times\ \ K_m^z \to P(X)$ defined as,
$\mathfrak{R}\ \ (K_1^a \times K_2^b \times \ldots \times\ \ K_m^z) = \{\xi = elegant, \zeta = test\ \ automation, \varrho = on\ \ time\ \ delivery, \varsigma = assurance\ \ of\ \ safety\ \ of\ \ personal\ \ data\ \ of\ \ customers\}$ is the most prominent choice of the sub-attributes for the assessment of online shopping agencies.

Step1: Let $(\mathfrak{R}, \Gamma)$ be a $SVNHSS(X)$ for best possible online shopping agency prepared with the help of experts in the field of e-commerce sector as given in Table 1.

Table 1. $SVNHSS(\mathfrak{R}, \Gamma)$ for best possible online shopping agency

$(\mathfrak{R}, \Gamma)$	K^1	K^2	K^3	K^4
x^1	$\xi(0.4,0.2,0.3)$	$\zeta(0.3,0.4,0.3)$	$\varrho(0.7,0.1,0.2)$	$\varsigma(0.4,0.2,0.3)$
x^2	$\xi(0.5,0.1,0.3)$	$\zeta(0.1,0.8,0.1)$	$\varrho(0.4,0.3,0.2)$	$\varsigma(0.5,0.2,0.3)$
x^3	$\xi(0.3,0.5,0.1)$	$\zeta(0.1,0.2,0.7)$	$\varrho(0.1,0.6,0.2)$	$\varsigma(0.5,0.4,0.1)$

The $SVNHSSs$ for the patients/subjects under observation are tabulated in Table 2-Table 5.

Table 2. $SVNHSS(\mathfrak{R}, \Gamma)$ for the shopping agency Y_1

$(\mathfrak{R}, \Gamma)$	K^1	K^2	K^3	K^4
x^1	$\xi(0.5,0.2,0.3)$	$\zeta(0.8,0.1,0.0)$	$\varrho(0.2,0.7,0.1)$	$\varsigma(0.9,0.1,0.0)$
x^2	$\xi(0.3,0.1,0.5)$	$\zeta(0.2,0.8,0.0)$	$\varrho(0.5,0.2,0.2)$	$\varsigma(0.6,0.1,0.2)$
x^3	$\xi(0.4,0.5,0.1)$	$\zeta(0.7,0.2,0.0)$	$\varrho(0.3,0.6,0.1)$	$\varsigma(0.4,0.5,0.1)$

Table 3. $SVNHSS(\mathfrak{R}, \Gamma)$ for the shopping agency Y_2

$(\mathfrak{N},\Gamma)$	K^1	K^2	K^3	K^4
x^1	$\xi(0.2,0.6,0.2)$	$\zeta(0.2,0.5,0.3)$	$\varrho(0.6,0.1,0.2)$	$\varsigma(0.7,0.2,0.1)$
x^2	$\xi(0.3,0.4,0.3)$	$\zeta(0.2,0.6,0.2)$	$\varrho(0.4,0.3,0.2)$	$\varsigma(0.5,0.2,0.3)$
x^3	$\xi(0.8,0.1,0.1)$	$\zeta(0.1,0.2,0.7)$	$\varrho(0.1,0.6,0.2)$	$\varsigma(0.8,0.1,0.1)$

Table 4. $SVNHSS(\mathfrak{N},\Gamma)$ for the shopping agency Y_3

$(\mathfrak{N},\Gamma)$	K^1	K^2	K^3	K^4
x^1	$\xi(0.3,0.4,0.3)$	$\zeta(0.2,0.6,0.1)$	$\varrho(0.3,0.6,0.0)$	$\varsigma(0.4,0.2,0.3)$
x^2	$\xi(0.5,0.1,0.3)$	$\zeta(0.1,0.8,0.1)$	$\varrho(0.4,0.3,0.2)$	$\varsigma(0.5,0.2,0.3)$
x^3	$\xi(0.5,0.5,0.0)$	$\zeta(0.2,0.0,0.8)$	$\varrho(0.4,0.5,0.1)$	$\varsigma(0.4,0.4,0.1)$

Table 5. $SVNHSS(\mathfrak{N},\Gamma)$ for the shopping agency Y_4

$(\mathfrak{N},\Gamma)$	K^1	K^2	K^3	K^4
x^1	$\xi(0.9,0.0,0.1)$	$\zeta(0.2,0.6,0.1)$	$\varrho(0.6,0.1,0.2)$	$\varsigma(0.5,0.2,0.2)$
x^2	$\xi(0.3,0.5,0.2)$	$\zeta(0.4,0.0,0.6)$	$\varrho(0.2,0.3,0.5)$	$\varsigma(0.7,0.2,0.1)$
x^3	$\xi(0.4,0.3,0.3)$	$\zeta(0.4,0.2,0.4)$	$\varrho(0.3,0.6,0.1)$	$\varsigma(0.0,0.1,0.9)$

Step 2: There is no requirement of excercising the normalization process as the given attributes are benefit type.

Step 3: Next, the proposed similarity measure has been utilized for computing the values of the similarity for various shopping agencies. In view of the proposed similarity measure (1), it is calculated that $\mathbb{S}(\mathfrak{N},Y_1) = 0.3457$ for theshopping agency Y_1, $\mathbb{I}_\gamma(\mathfrak{N},Y_2) = 0.6243$ for theshopping agency Y_2, $\mathbb{I}_\gamma(\mathfrak{N},Y_3) = 0.4892$ for theshopping agency Y_3 and $\mathbb{I}_\gamma(\mathfrak{N},Y_4) = 0.8657$ for theshopping agency Y_4.

Step 4: The maximum similarity measure is 0.8657 which is in reference with the shopping agency Y_1. Therefore, among all the four shopping agencies, Y_1 is the best possible shopping agency on the basis of the given criterions.

The following Table 6 breifly makes the indication in terms of benefits of utilizing the proposed notion and its analogous methodology in contrast with the available ones:

Table 6: Characteristic Comparitive Observations

Authors	**Information Measures**	**Truthiness**	**Indeterminacy**	**Falsity**	**Sub-Attributes**
Ohlan et al. [34]	"*Fuzzy Sets*"	Yes	No	No	No
Kadian et al. [35]	"*Intuitionistic Fuzzy Sets*"	Yes	No	Yes	No
Montes et al.[36]	"*Picture Fuzzy Sets*"	Yes	No	Yes	No
Proposed	"*Single-valued Neutrosophic Hypersoft Sets*"	Yes	Yes	Yes	Yes

6. Conclusion & Scope for Future Work

Evaluation method for selecting the best shopping website based on the needs of the customers is highly required in e-commerce sector. The characteristics defining a website quality can be both tangible and intangible. Making the evaluation process a multi-criteria decision making (MCDM) problem. The paper has employed fuzzy logic method of similarity measures of $SVNHSSs$ to assist the consumer in selection of the shopping website which is bets suited to their needs.The paper has added to the existing research on the topic by employing innovative data analysis method like fuzzy set analysis, which can be a better alternative to conventional methods based on variance [18]. The method can be used by the e –commerce vendors and consumers to evaluate the shopping platforms in the light of required attributes. E-comerce service providers can identify the main impact factors useful for framing customer centric strategies. However,selection of limited attributes is the major limitation in the present study. Including more attributes representing diverse areas and concerns of e-commerce can be undertaken in the future studies.

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Interval Valued Secondary k-Range Symmetric Neutrosophic Fuzzy Matrices

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Abstract. The characterization of interval valued (IV) secondary k- range symmetric (RS) Neutrosophic fuzzy matrices have been examined in this study with an example. It is discussed how IV s-k RS, s- RS, IV k- RS, and IV RS matrices relate to one another. We establish the necessary and sufficient criteria for IV s-k RS Neutrosophic fuzzy matrices. The existence of several generalized inverses of a matrix in IV Neutrosophic fuzzy matrices. It is also established what are the equivalent criteria for various g-inverses of an IV s – κ RS fuzzy matrix to be an IV s – κ RS. The generalized inverses of an IV s – κ RS P corresponding to the sets P{1, 2}, P{1, 2, 3} and P{1, 2, 4} are characterized.

Keywords: IV Neutrosophic Fuzzy matrix, IV RS Neutrosophic fuzzy matrix, s-k- RS IV Neutrosophic fuzzy matrix.

1. Introduction

Matrices are crucial in many fields of research in science and engineering. The traditional matrix theory is unable to address problems involving numerous kinds of uncertainties. Zadeh [1] first introduced fuzzy sets (FSs) in 1965. These are traditionally defined by their membership value or grade of membership. Assigning membership values to a fuzzy set can sometimes be challenging. Atanassov [2] introduced intuitionistic FSs to solve the problem of assigning non-membership values. Smarandache [3] introduced the concept of neutrosophic sets (NSs) to handle indeterminate information and deal with problems that involve imprecision, uncertainty, and inconsistency.

Fuzzy matrices are used to solve certain kinds of issues. Many researchers have since completed numerous works. Only membership values are addressed by fuzzy matrices. These matrices cannot handle values that are not membership. Khan, Shyamal, and Pal [4] have studied intuitionistic fuzzy matrices (IFMs) for the first time. Atanassov [5,6] has discussed IFS and Operations over IV IFS. Hashimoto [7] has studied Canonical form of a transitive matrix. Kim and Roush [8] have studied generalized fuzzy matrices. Lee [9] has studied Secondary Skew Symmetric, Secondary Orthogonal Matrices. Hill and Waters [10] have analyzed On k-Real and k-Hermitian matrices. Meenakshi [11] has focussed Fuzzy Matrix: Theory and Applications. Meenakshi and Jaya Shree [12] have studied On k-kernel symmetric matrices. Meenakshi and Krishanmoorthy [13] have characterized On Secondary k-Hermitian matrices. Meenakshi and Jaya Shree [14] have studied On K -range symmetric matrices. Jaya shree [15] has studied Secondary κ-Kernel Symmetric Fuzzy Matrices. Shyamal and Pal [16] Interval valued Fuzzy matrices. Meenakshi and Kalliraja [17] have studied Regular Interval valued Fuzzy matrices. But, practically it is difficult to measure the membership or non-membership value as a point. Anandhkumar [18,19] has studied Pseudo Similarity of NFM and On various Inverse of NFM. Anandhkumar,et.al [20] have studied Generalized Symmetric Neutrosophic Fuzzy Matrices. Anandhkumar,et.al [21] have discussed Reverse Sharp and Left-T Right-T Partial Ordering on Neutrosophic Fuzzy Matrices. Pal and Susanta Kha [22] have studied IV Intuitionistic Fuzzy Matrices. Vidhya and Irene Hepzibah [23] have discussed on Interval Valued Neutrosophic Fuzzy Matrices.

1.1 Research Gap

Jayashri [24] presented the concept of range and kernel-symmetry principles to fuzzy matrix. We have applied the range and principles to NFM in this context. We have examined some of the results and extended concepts to NFMs. We first present equivalent characterizations for a RS matrix. We then derive the equivalent conditions that NFMs must meet to show range symmetry. We also find equivalent conditions that allow various generalized inverses to have range symmetric.

Notations:

IVNFM = Interval valued Neutrosophic Fuzzy Matrix,

IV =Interval valued,

RS = Range Symmetric

$[P_\mu, P_\lambda, P_v]_L{}^T$ = Transpose of the IVNFM $[P_\mu, P_\lambda, P_v]_L$,

$[P_\mu, P_\lambda, P_v]_U{}^T$ = Transpose of the IVNFM $[P_\mu, P_\lambda, P_v]_U$,

$[P_\mu, P_\lambda, P_v]_L{}^+$ = Moore-Penrose inverse of IVNFM $[P_\mu, P_\lambda, P_v]_L$,

$[P_\mu, P_\lambda, P_v]_U{}^+$ = Moore-Penrose inverse of IVNFM $[P_\mu, P_\lambda, P_v]_U$,

$R\left([P_\mu, P_\lambda, P_v]_L\right)$ = Row space of $[P_\mu, P_\lambda, P_v]_L$

$R\left([P_\mu, P_\lambda, P_v]_U\right)$ = Row space of $[P_\mu, P_\lambda, P_v]_U$,

$C\left([P_\mu, P_\lambda, P_v]_L\right)$ = Column space of $[P_\mu, P_\lambda, P_v]_L$,

$C\left([P_\mu, P_\lambda, P_v]_U\right)$ = Column space of $[P_\mu, P_\lambda, P_v]_U$

2. Preliminaries and Definitions

2.1 Preliminary

If $\kappa(y)=(y_{k[1]}, y_{k[2]}, y_{k[3]},\ldots, y_{k[n]}) \in F_{n\times 1}$ for $y = y_1, y_2,..,y_n \in F_{[1\times n]}$, where K is involuntary, The corresponding Permutation matrix is satisfied using the conditions

(P.2.1) $KK^T = K^T K = I_n$, $K = K^T$, $K^2 = I$

By the definition of V, and $R(x) = Kx$

(P.2.2) $V = V^T$, $VV^T = V^T V = I_n$ and $V^2 = I$

(P.2.3) $R\left([P_\mu, P_\lambda, P_v]_L\right) = R\left([P_\mu, P_\lambda, P_v]_L\right)V$, $R\left([P_\mu, P_\lambda, P_v]_L\right) =$

$R\left([P_\mu, P_\lambda, P_v]_L\right)K$

$R\left([P_\mu, P_\lambda, P_v]_U\right) = R\left([P_\mu, P_\lambda, P_v]_U\right)V$, $R\left([P_\mu, P_\lambda, P_v]_U\right) =$

$R\left([P_\mu, P_\lambda, P_v]_U\right)K$

(P.2.4) $R\left([P_\mu, P_\lambda, P_v]_L\, V\right)^T = R\left(V[P_\mu, P_\lambda, P_v]_L^{\,T}\right)$, $R\left(V[P_\mu, P_\lambda, P_v]_L\right)^T =$

$R\left([P_\mu, P_\lambda, P_v]_L^{\,T} V\right)$

$R\left([P_\mu, P_\lambda, P_v]_U\, V\right)^T = R\left(V[P_\mu, P_\lambda, P_v]_U^{\,T}\right)$, $R\left(V[P_\mu, P_\lambda, P_v]_U\right)^T =$

$R\left([P_\mu, P_\lambda, P_v]_U^{\,T} V\right)$

Definition:2.1 IV Neutrosophic fuzzy matrix (IVNFM): An IV Neutrosophic fuzzy matrix P of order m×n is defined as $P = [x_{ij}, < p_{ij\mu}, p_{ij\lambda}\ p_{ijv} >]_{m\times n}$ where $p_{ij\mu}$, $p_{ij\lambda}$ and p_{ijv} are the subsets of [0,1] which are denoted by $p_{ij\mu} = [p_{ij\mu L}, p_{ij\mu U}]$, $p_{ij\lambda} = [p_{ij\lambda L}, p_{ij\lambda U}]$ and $p_{ijv} = [p_{ijvL}, p_{ijvU}]$ which maintaining the condition $0 \le p_{ij\mu U} + p_{ij\lambda U} + p_{ijvU} \le 3$, $0 \le p_{ij\mu L} + p_{ij\lambda L} + p_{ijvL} \le 3$, $0 \le p_{\mu L} \le p_{\mu U} \le 1$, $0 \le p_{\lambda L} \le p_{\lambda U} \le 1$, $0 \le p_{vL} \le p_{vU} \le 1$.

Example2.1 Consider an IV Neutrosophic Fuzzy Matrix

$$P = \begin{bmatrix} <[0,0],[1,1],[1,1]> & <[0.1,0.3],[0.2,0.4],[0.2,0.5]> \\ <[0.1,0.3],[0.2,0.4],[0.2,0.5]> & <[0,0],[1,1],[1,1]> \end{bmatrix}$$

Lower Limit NFM, $[P_\mu, P_\lambda, P_v]_L = \begin{bmatrix} <0,1,1> & <0.1,0.2,0,2> \\ <0.1,0.2,0.2> & <0,1,1> \end{bmatrix}$

Upper Limit NFM, $[P_\mu, P_\lambda, P_v]_U = \begin{bmatrix} <0,1,1> & <0.3,0.4,0.5> \\ <0.3,0.4,0.5> & <0,1,1> \end{bmatrix}$

and $Q = \begin{bmatrix} <[0,0],[1,1],[1,1]> & <[0.2,0.4],[0.3,0.5],[0.1,0.5]> \\ <[0.2,0.4],[0.3,0.5],[0.1,0.5]> & <[0,0],[1,1],[1,1]> \end{bmatrix}$

Then, $P+Q = \begin{bmatrix} <[0,0],[1,1],[1,1]> & <[0.2,0.4],[0.2,0.4],[0.1,0.5]> \\ <[0.2,0.4],[0.2,0.4],[0.1,0.5]> & <[0,0],[1,1],[1,1]> \end{bmatrix}$

$Q = \begin{bmatrix} <[0,0],[1,1],[1,1]> & <[0.1,0.3],[0.3,0.5],[0.2,0.5]> \\ <[0.1,0.3],[0.3,0.5],[0.2,0.5]> & <[0,0],[1,1],[1,1]> \end{bmatrix}$

$|P| = <[0,0],[1,1],[1,1]><[0,0],[1,1],[1,1]> + <[0.1,0.3],[0.2,0.4],[0.2,0.5]><[0.1,0.3],[0.2,0.4],[0.2,0.5]>$

$|P| = <[0,0],[1,1],[1,1]> + <[0.1,0.3],[0.2,0.4],[0.2,0.5]> = <[0.1,0.3],[0.2,0.4],[0.2,0.5]>$

Definition 2.2. For IV Neutrosophic fuzzy matrix P is RS fuzzy matrix iff $R\left([P_\mu, P_\lambda, P_v]_L\right) =$

$R\left([P_\mu, P_\lambda, P_v]_L{}^T\right)$ and $R\left([P_\mu, P_\lambda, P_v]_U\right) = R\left([P_\mu, P_\lambda, P_v]_U{}^T\right)$.

Lemma 2.1. For a matrix A belongs to F_n and a permutation fuzzy matrix P, R(A) = R(B) iff $R(PAQ^T) = R(PAQ^T)$.

Lemma2.2. For interval valued fuzzy matrix P=KP^{T}K iff KP=(KP)(KP)^T(KP), IV fuzzy matrix $\Leftrightarrow$ PK=(PK)(PK)^T(PK) IV fuzzy matrix.

3. Interval valued Secondary k-KS Neutrosophic fuzzy matrix

Definition 3.1. For a Neutrosophic fuzzy matrix $P = <[P_\mu, P_\lambda, P_v]_L, [P_\mu, P_\lambda, P_v]_U> \in \text{IVNFM}_{nn}$

is an IV s - symmetric fuzzy matrix iff $[P_\mu, P_\lambda, P_v]_L = V\left([P_\mu, P_\lambda, P_v]_L{}^T\right)V$ and $[P_\mu, P_\lambda, P_v]_U$

$= V\left([P_\mu, P_\lambda, P_v]_U{}^T\right)V$.

Definition 3.2 For a Neutrosophic fuzzy matrix P is an IV s- RS fuzzy matrix iff

$R\left([P_\mu, P_\lambda, P_v]_L\right) = R\left(V[P_\mu, P_\lambda, P_v]_L{}^T V\right)$, $R\left([P_\mu, P_\lambda, P_v]_U\right) = R\left(V[P_\mu, P_\lambda, P_v]_U{}^T V\right)$.

Definition 3.3. For a NFM $A = <[P_\mu, P_\lambda, P_v]_L, [P_\mu, P_\lambda, P_v]_U>$ is an IV s-k- RS fuzzy matrix iff

$R\left([P_\mu, P_\lambda, P_v]_L\right) = R\left(KV[P_\mu, P_\lambda, P_v]_L{}^T VK\right)$, $R\left([P_\mu, P_\lambda, P_v]_U\right) = R\left(KV[P_\mu, P_\lambda, P_v]_U{}^T VK\right)$.

Lemma 3.1. For a Neutrosophic fuzzy matrix P is an IV s- RS Neutrosophic fuzzy matrix $\Leftrightarrow$

$\mathrm{VA} =< V[P_\mu, P_\lambda, P_v]_L, \quad V[P_\mu, P_\lambda, P_v]_U >$ IV RS Neutrosophic fuzzy matrix

$\Leftrightarrow$ $\mathrm{AV} =< [P_\mu, P_\lambda, P_v]_L\, V\ ,[P_\mu, P_\lambda, P_v]_U\, V >$ is an IV RS Neutrosophic fuzzy matrix.

Proof. Let Neutrosophic fuzzy matrix P is s-RS fuzzy matrix

$\Leftrightarrow R\left([P_\mu, P_\lambda, P_v]_L\right) = R\left(V[P_\mu, P_\lambda, P_v]_L{}^T V\right)$ [Definition 3.2]

$\Leftrightarrow R\left([P_\mu, P_\lambda, P_v]_L\, V\right) = R\left([P_\mu, P_\lambda, P_v]_L V\right)^T$

$\Leftrightarrow [P_\mu, P_\lambda, P_v]_L\, V$ is RS. [By P.2.2]

$\Leftrightarrow R\left(V[P_\mu, P_\lambda, P_v]_L\, VV^T\right) = R\left(VV[P_\mu, P_\lambda, P_v]_L{}^T V\right)$

$\Leftrightarrow R\left(V[P_\mu, P_\lambda, P_v]_L\right) = R\left(V[P_\mu, P_\lambda, P_v]_L\right)^T$

$\Leftrightarrow V[P_\mu, P_\lambda, P_v]_L$ is RS.

Similar manner

$\Leftrightarrow R\left([P_\mu, P_\lambda, P_v]_U\right) = R\left(V[P_\mu, P_\lambda, P_v]_U{}^T V\right)$

$\Leftrightarrow R\left([P_\mu, P_\lambda, P_v]_U\, V\right) = R\left([P_\mu, P_\lambda, P_v]_U V\right)^T$

$\Leftrightarrow [P_\mu, P_\lambda, P_v]_U\, V$ is RS.

$\Leftrightarrow R\left(V[P_\mu, P_\lambda, P_v]_U\, VV^T\right) = R\left(VV[P_\mu, P_\lambda, P_v]_U{}^T V\right)$

$\Leftrightarrow R\left(V[P_\mu, P_\lambda, P_v]_U\right) = R\left(V[P_\mu, P_\lambda, P_v]_U\right)^T$

$\Leftrightarrow V[P_\mu, P_\lambda, P_v]_U$ is RS.

Therefore, $\mathrm{VP} =< V[P_\mu, P_\lambda, P_v]_L, V[P_\mu, P_\lambda, P_v]_U >$ is an IV symmetric.

Example 3.1 Let us consider IV NFM

$$P = \begin{bmatrix} <[0,0],[1,1],[1,1]> & <[0.1,0.3],[0.2,0.4],[0.2,0.5]> \\ <[0.1,0.3],[0.2,0.4],[0.2,0.5]> & <[0,0],[1,1],[1,1]> \end{bmatrix}$$

Lower Limit NFM, $[\mathrm{P}_{\mu}, \mathrm{P}_{\lambda}, \mathrm{P}_{v}]_{L} = \begin{bmatrix} <0,1,1> & <0.1,0.2,0,2> \\ <0.1,0.2,0.2> & <0,1,1> \end{bmatrix}$,

Upper Limit NFM, $[\mathrm{P}_{\mu}, \mathrm{P}_{\lambda}, \mathrm{P}_{v}]_{U} = \begin{bmatrix} <0,1,1> & <0.3,0.4,0.5> \\ <0.3,0.4,0.5> & <0,1,1> \end{bmatrix}$

$V = \begin{bmatrix} <0,0,0> & <1,1,0> \\ <1,1,0> & <0,0,0> \end{bmatrix}$, $\mathrm{K} = \begin{bmatrix} <1,1,0> & <0,0,0> \\ <0,0,0> & <1,1,0> \end{bmatrix}$,

$$KVP_L{}^T VK = \begin{bmatrix} <1,1,0> & <0,0,0> \\ <0,0,0> & <1,1,0> \end{bmatrix} \begin{bmatrix} <0,0,0> & <1,1,0> \\ <1,1,0> & <0,0,0> \end{bmatrix} \begin{bmatrix} <0,1,1> & <0.1,0.2,0,2> \\ <0.1,0.2,0.2> & <0,1,1> \end{bmatrix} \begin{bmatrix} <0,0,0> & <1,1,0> \\ <1,1,0> & <0,0,0> \end{bmatrix} \begin{bmatrix} <1,1,0> & <0,0,0> \\ <0,0,0> & <1,1,0> \end{bmatrix}$$

$$KVP_L{}^T VK = \begin{bmatrix} <0,1,0.2> & <0,0.2,0.2> \\ <0.1,0.2,0.2> & <0,1,0.2> \end{bmatrix}$$

$$KVP_L{}^T VK \neq P_L$$

Similarly, $KVP_U{}^T VK \neq P_U$

$$P_L = KP_L K$$

$$KP_L K = \begin{bmatrix} <1,1,0> & <0,0,0> \\ <0,0,0> & <1,1,0> \end{bmatrix} \begin{bmatrix} <0,1,1> & <0.1,0.2,0,2> \\ <0.1,0.2,0.2> & <0,1,1> \end{bmatrix} \begin{bmatrix} <1,1,0> & <0,0,0> \\ <0,0,0> & <1,1,0> \end{bmatrix}$$

$$KP_L K = \begin{bmatrix} <0,1,0.2> & <0.1,0.2,0.2> \\ <0.1,0.2,0.2> & <0.1,1,0.2> \end{bmatrix} \neq P_L$$

Similarly, $P_U \neq KP_U K$

N (P_L) = $N(KVP_L{}^T VK) = <0,0,0>$

Therefore P is symmetric NFM, range symmetric NFM, kernel symmetric, but not both κ –symmetric and s- κ - symmetric NFM.

Example 2.2. Let us consider IV NFM,

$$P=\begin{bmatrix} <[0.7,0.2],[0.3,0.4],[0.4,0.6]> & <[0.5,0.4],[0.3,0.3],[0.4,0.2]> \\ <[0.5,0.4],[0.3,0.3],[0.4,0.2]> & <[0.7,0.2],[0.3,0.4],[0.4,0.6]> \end{bmatrix}V$$

$$=\begin{bmatrix} <0,0,0> & <1,1,0> \\ <1,1,0> & <0,0,0> \end{bmatrix},\ K=\begin{bmatrix} <1,1,0> & <0,0,0> \\ <0,0,0> & <1,1,0> \end{bmatrix},$$

Lower Limit NFM, $P_L=\begin{bmatrix} <0.7,0.3,0.4> & <0.5,0.3,0.4> \\ <0.5,0.3,0.4> & <0.7,0.3,0.4> \end{bmatrix},$

Upper Limit NFM, $\mathrm{P}_U=\begin{bmatrix} <0.2,0.4,0.6> & <0.4,0.3,0.2> \\ <0.4,0.3,0.2> & <0.2,0.4,0.6> \end{bmatrix}$

$$KVP_L^{\ T}VK=\begin{bmatrix} <1,1,0> & <0,0,0> \\ <0,0,0> & <1,1,0> \end{bmatrix}\begin{bmatrix} <0,0,0> & <1,1,0> \\ <1,1,0> & <0,0,0> \end{bmatrix}\begin{bmatrix} <0.7,0.3,0.4> & <0.5,0.3,0.4> \\ <0.5,0.3,0.4> & <0.7,0.3,0.4> \end{bmatrix}$$
$$\begin{bmatrix} <0,0,0> & <1,1,0> \\ <1,1,0> & <0,0,0> \end{bmatrix}\begin{bmatrix} <1,1,0> & <0,0,0> \\ <0,0,0> & <1,1,0> \end{bmatrix}$$

$$KVP_L^{\ T}VK=\begin{bmatrix} <0.7,0.3,0.4> & <0.5,0.3,0.4> \\ <0.5,0.3,0.4> & <0.7,0.3,0.4> \end{bmatrix}=P_L$$

P is symmetric, RS, s-κ-symmetric and hence s- k- kernel symmetric.

Example 2.3. Let us consider IV NFM

Lower limit NFM, $[\mathrm{P}_\mu,\mathrm{P}_\lambda,\mathrm{P}_v]_L=\begin{bmatrix} <0,0,0> & <0,0,0> & <1,1,0> \\ <0.5,0.3,0.4> & <1,1,0> & <0,0,0> \\ <0.4,0.2,0.6> & <0.5,0.3,0.4> & <0,0,0> \end{bmatrix}$

$$K=\begin{bmatrix} <0,0,0> & <1,1,0> & <0,0,0> \\ <1,1,0> & <0,0,0> & <0,0,0> \\ <0,0,0> & <0,0,0> & <1,1,0> \end{bmatrix},V=\begin{bmatrix} <0,0,0> & <0,0,0> & <1,1,0> \\ <0,0,0> & <1,1,0> & <0,0,0> \\ <1,1,0> & <0,0,0> & <0,0,0> \end{bmatrix}$$

$$KV=\begin{bmatrix} <0,0,0> & <1,1,0> & <0,0,0> \\ <1,1,0> & <0,0,0> & <0,0,0> \\ <0,0,0> & <0,0,0> & <1,1,0> \end{bmatrix}\begin{bmatrix} <0,0,0> & <0,0,0> & <1,1,0> \\ <0,0,0> & <1,1,0> & <0,0,0> \\ <1,1,0> & <0,0,0> & <0,0,0> \end{bmatrix}$$

$$KV=\begin{bmatrix} <0,1,0> & <0,1,0> & <0,1,0> \\ <0,1,0> & <0,1,0> & <1,1,0> \\ <1,1,0> & <0,1,0> & <0,1,0> \end{bmatrix}$$

$$VK = \begin{bmatrix} <0,0,0> & <0,0,0> & <1,1,0> \\ <0,0,0> & <1,1,0> & <0,0,0> \\ <1,1,0> & <0,0,0> & <0,0,0> \end{bmatrix} \begin{bmatrix} <0,0,0> & <1,1,0> & <0,0,0> \\ <1,1,0> & <0,0,0> & <0,0,0> \\ <0,0,0> & <0,0,0> & <1,1,0> \end{bmatrix}$$

$$VK = \begin{bmatrix} <0,1,0> & <0,1,0> & <1,1,0> \\ <1,1,0> & <0,1,0> & <0,1,0> \\ <0,1,0> & <1,1,0> & <0,1,0> \end{bmatrix}$$

$$P_L{}^T VK = \begin{bmatrix} <0.5,0.8,0.4> & <0.4,0.8,0.6> & <0,0,0.4> \\ <0,0.7,0> & <0.5,0.7,0> & <0,0.7,0> \\ <0,0,0> & <0,0,0> & <1,0,0> \end{bmatrix}$$

$$KVP_L{}^T VK = \begin{bmatrix} <0,1,0> & <0,1,0> & <0,1,0> \\ <0,1,0> & <0,1,0> & <1,1,0> \\ <1,1,0> & <0,1,0> & <0,1,0> \end{bmatrix} \begin{bmatrix} <0.5,0.8,0.4> & <0.4,0.8,0.6> & <0,0,0.4> \\ <0,0.7,0> & <0.5,0.7,0> & <0,0.7,0> \\ <0,0,0> & <0,0,0> & <1,0,0> \end{bmatrix}$$

$$KVP_L{}^T VK = \begin{bmatrix} <0,0,0> & <0,0.2,0> & <0,0,0> \\ <0,0,0> & <0,0,0> & <1,0,0> \\ <0.5,0,0> & <0.4,0,0> & <0,0,0> \end{bmatrix} \neq P_L$$

$$P_L \neq KVP_L{}^T VK$$

Hence P is not s- k-symmetric and not RS. But s- k- kernel symmetric.

i.e) $N(P_L) = N(KVP_L{}^T VK) = <0,0,0>$

Theorem 3.1. The following conditions are equivalent for $P \in \mathrm{IVNFM}_n$

(i) $P = <[P_\mu, P_\lambda, P_v]_L, [P_\mu, P_\lambda, P_v]_U> \in \mathrm{IVNFM}_{nn}$ is an IV s – κ RS.

(ii) $\mathrm{KVP} = < KV[P_\mu, P_\lambda, P_v]_L, \mathrm{KV}[P_\mu, P_\lambda, P_v]_U >$ is an IV RS.

(iii) $\mathrm{PKV} = <[P_\mu, P_\lambda, P_v]_L KV, [P_\mu, P_\lambda, P_v]_U KV >$ is an IV RS.

(iv) $\mathrm{VP} = < V[P_\mu, P_\lambda, P_v]_L, \mathrm{V}[P_\mu, P_\lambda, P_v]_U >$ is an IV k- RS.

(v) $\mathrm{PK} = <[P_\mu, P_\lambda, P_v]_L \mathrm{K}, [P_\mu, P_\lambda, P_v]_U \mathrm{K} >$ is an IV s- RS.

(vi) P^T is an IV s-k RS.

(vii) $\mathrm{R}([P_\mu, P_\lambda, P_v]_L) = \left(\mathrm{R}[P_\mu, P_\lambda, P_v]_L{}^T \mathrm{VK}\right), \mathrm{R}([P_\mu, P_\lambda, P_v]_U) = \left(\mathrm{R}[P_\mu, P_\lambda, P_v]_U{}^T \mathrm{VK}\right)$

(viii) $\mathrm{R}([P_\mu, P_\lambda, P_v]_L{}^T) = \left(\mathrm{R}[P_\mu, P_\lambda, P_v]_L \mathrm{VK}\right), \mathrm{R}([P_\mu, P_\lambda, P_v]_U{}^T) = \left(\mathrm{R}[P_\mu, P_\lambda, P_v]_U \mathrm{VK}\right)$

(ix) $C(KV[P_\mu,P_\lambda,P_\nu]_L) = C\left(KV[P_\mu,P_\lambda,P_\nu]_L{}^T\right)^T, C(KV[P_\mu,P_\lambda,P_\nu]_U)$

$= C\left(KV[P_\mu,P_\lambda,P_\nu]_U{}^T\right)^T$

(x) $[P_\mu,P_\lambda,P_\nu]_L = VK[P_\mu,P_\lambda,P_\nu]_L{}^T\, VKH_1,\ [P_\mu,P_\lambda,P_\nu]_U$

$= VK[P_\mu,P_\lambda,P_\nu]_U{}^T\, VKH_1$ for $H_1 \in IVNFM$

(xi) $[P_\mu,P_\lambda,P_\nu]_L = H_1KV[P_\mu,P_\lambda,P_\nu]_L{}^T\, KV,\ [P_\mu,P_\lambda,P_\nu]_U$

$= H_1KV[P_\mu,P_\lambda,P_\nu]_U{}^T\, VK$ for $H_1 \in IVNFM$

(xii) $[P_\mu,P_\lambda,P_\nu]_L{}^T = KV[P_\mu,P_\lambda,P_\nu]_L\, VKH_1,\ [P_\mu,P_\lambda,P_\nu]_U{}^T$

$= KV[P_\mu,P_\lambda,P_\nu]_U\, VKH_1$ for $H_1 \in IVNFM$

(xiii) $[P_\mu,P_\lambda,P_\nu]_L{}^T = H_1KV[P_\mu,P_\lambda,P_\nu]_L\, KV,\ [P_\mu,P_\lambda,P_\nu]_U$

$= H_1KV[P_\mu,P_\lambda,P_\nu]_U\, VK$ for $H_1 \in IVNFM$

Proof: (i) iff (ii) iff (iv)

Let P is an IV s – κ RS

Let $[P_\mu,P_\lambda,P_\nu]_L$ is a s – κ RS.

$$\Leftrightarrow R([P_\mu,P_\lambda,P_\nu]_L) = R(KV[P_\mu,P_\lambda,P_\nu]_L{}^T VK), R([P_\mu,P_\lambda,P_\nu]_U) = R(KV[P_\mu,P_\lambda,P_\nu]_U{}^T VK),$$

(By Definition 3.3)

$$\Leftrightarrow R(KV[P_\mu,P_\lambda,P_\nu]_L) = R(KV[P_\mu,P_\lambda,P_\nu]_L)^T, R([P_\mu,P_\lambda,P_\nu]_U) = R(KV[P_\mu,P_\lambda,P_\nu]_U)^T$$

By (P.2.3)

$\Leftrightarrow$ $KVP = < KV[P_\mu,P_\lambda,P_\nu]_L, KV[P_\mu,P_\lambda,P_\nu]_U >$ is an IV RS

$\Leftrightarrow$ $VP = < V[P_\mu,P_\lambda,P_\nu]_L, V[P_\mu,P_\lambda,P_\nu]_U >$ is an IV κ- RS

As a conclusion (i) iff (ii) iff (iv) is true

(i) iff (ii) iff (v)

Let P is an IV s – κ RS

$\Leftrightarrow R(KV[P_\mu,P_\lambda,P_v]_L)=R(KV[P_\mu,P_\lambda,P_v]_L)^T, R(KV[P_\mu,P_\lambda,P_v]_U)=R(KV[P_\mu,P_\lambda,P_v]_U)^T,$

$\Leftrightarrow R(VK(KV[P_\mu,P_\lambda,P_v]_L))=R((VK)[P_\mu,P_\lambda,P_v]_L{}^T VK(VK)^T)$,

$R(VK(KV[P_\mu,P_\lambda,P_v]_U))=R((VK)[P_\mu,P_\lambda,P_v]_U{}^T VK(VK)^T)$

$\Leftrightarrow AKV=\left[[P_\mu,P_\lambda,P_v]_L KV,[P_\mu,P_\lambda,P_v]_U KV\right]$ is an IV RS

$\Leftrightarrow AK=\left[[P_\mu,P_\lambda,P_v]_L K,[P_\mu,P_\lambda,P_v]_U K\right]$ is an IV s- RS

As a conclusion (i) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (v) is true. (ii) $\Leftrightarrow$ (ix)

$KVA=\left[KV[P_\mu,P_\lambda,P_v]_L, KV[P_\mu,P_\lambda,P_v]_U\right]$ is an IV RS

$\Leftrightarrow R\left(KV[P_\mu,P_\lambda,P_v]_L\right)=R\left(\left(KV[P_\mu,P_\lambda,P_v]_L\right)^T\right), R\left(KV[P_\mu,P_\lambda,P_v]_U\right)$

$=R\left(\left(KV[P_\mu,P_\lambda,P_v]_U\right)^T\right)$

(ii) $\Leftrightarrow$ (ix) is true. (ii) $\Leftrightarrow$ (vii)

$KVP=\left[KV[P_\mu,P_\lambda,P_v]_L, KV[P_\mu,P_\lambda,P_v]_U\right]$ is an IV RS.

$\Leftrightarrow R\left(KV[P_\mu,P_\lambda,P_v]_L\right)=R\left(\left(KV[P_\mu,P_\lambda,P_v]_L\right)^T\right), R\left(KV[P_\mu,P_\lambda,P_v]_U\right)$

$=R\left(\left(KV[P_\mu,P_\lambda,P_v]_U\right)^T\right)$

$\Leftrightarrow R\left([P_\mu,P_\lambda,P_v]_L\right)=R\left([P_\mu,P_\lambda,P_v]_L{}^T VK\right), R\left([P_\mu,P_\lambda,P_v]_U\right)=R\left([P_\mu,P_\lambda,P_v]_U{}^T VK\right)$

As a conclusion (ii) $\Leftrightarrow$ (vii) is true. (iii) $\Leftrightarrow$ (viii)

$PVK=\left[[P_\mu,P_\lambda,P_v]_L VK,[P_\mu,P_\lambda,P_v]_U VK\right]$

$\Leftrightarrow R\left([P_\mu,P_\lambda,P_v]_L VK\right)=R\left(\left([P_\mu,P_\lambda,P_v]_L VK\right)^T\right), R\left([P_\mu,P_\lambda,P_v]_U VK\right)$

$=R\left(\left([P_\mu,P_\lambda,P_v]_U VK\right)^T\right)$

$\Leftrightarrow R\left([P_\mu,P_\lambda,P_v]_L VK\right)=R\left([P_\mu,P_\lambda,P_v]_L\right)^T, R\left([P_\mu,P_\lambda,P_v]_U VK\right)=R\left([P_\mu,P_\lambda,P_v]_U\right)^T$

As a conclusion (iii) $\Leftrightarrow$ (viii) is true. (i) $\Leftrightarrow$ (vi)

Let $A=<[P_\mu,P_\lambda,P_v]_L,[P_\mu,P_\lambda,P_v]_U>\in \text{IVNFM}_{nn}$ is an IV s – κ RS

$\Leftrightarrow R([P_\mu,P_\lambda,P_v]_L)=R(KV[P_\mu,P_\lambda,P_v]_L{}^T VK), R([P_\mu,P_\lambda,P_v]_U)=R(KV[P_\mu,P_\lambda,P_v]_U{}^T VK),$

(By Definition 3.3)

$\Leftrightarrow \quad (KVA)^T = (KV[P_\mu, P_\lambda, P_v]_L, KV[P_\mu, P_\lambda, P_v]_U)^T$ is an IV RS

$\Leftrightarrow \quad A^T VK = ([P_\mu, P_\lambda, P_v]_L VK, [P_\mu, P_\lambda, P_v]_U VK)$ is an IV RS

$\Leftrightarrow \quad P^T = \left([P_\mu, P_\lambda, P_v]_L{}^T, [P_\mu, P_\lambda, P_v]_U{}^T\right)$ is an IV s – κ RS

As a conclusion (i) $\Leftrightarrow$ (vi) is true

(i) $\Leftrightarrow$ (xii) $\Leftrightarrow$ (xi)

Let P is an IV s – κ RS

Consider $[P_\mu, P_\lambda, P_v]_L$ is a s – κ RS

$$\Leftrightarrow C([P_\mu, P_\lambda, P_v]_L{}^T) = C(\mathrm{KV}[P_\mu, P_\lambda, P_v]_L VK), \mathrm{C}([P_\mu, P_\lambda, P_v]_U{}^T) = C(\mathrm{KV}[P_\mu, P_\lambda, P_v]_U VK)$$

By (P.2.3)

$$\Leftrightarrow [P_\mu, P_\lambda, P_v]_L = H_1 \mathrm{KV}[P_\mu, P_\lambda, P_v]_L{}^T VK, [P_\mu, P_\lambda, P_v]_U{}^T = H_1 \mathrm{KV}[P_\mu, P_\lambda, P_v]_U VK$$

for $H_1 \in IVNFM$. As a result (i) $\Leftrightarrow$ (xii) $\Leftrightarrow$ (xi) true.

(ii) $\Leftrightarrow$ (xiii) $\Leftrightarrow$ (x)

$\Leftrightarrow AVK = \left[[P_\mu, P_\lambda, P_v]_L VK, [P_\mu, P_\lambda, P_v]_U VK\right]$ is an IV RS

$\Leftrightarrow AV = \left[[P_\mu, P_\lambda, P_v]_L V, [P_\mu, P_\lambda, P_v]_U V\right]$ is an IV κ- RS

$$\Leftrightarrow R(\mathrm{V}[P_\mu, P_\lambda, P_v]_L) = R(\mathrm{K}\left(\mathrm{V}[P_\mu, P_\lambda, P_v]_L\right)^T K),$$

$$\mathrm{R}(\mathrm{V}[P_\mu, P_\lambda, P_v]_U) = R(K\left(V[P_\mu, P_\lambda, P_v]_U\right)^T K),$$ [By Definition 3.3]

$$\Leftrightarrow R([P_\mu, P_\lambda, P_v]_L) = R\left([P_\mu, P_\lambda, P_v]_L\right)^T VK), \ \mathrm{R}([P_\mu, P_\lambda, P_v]_U) = R\left([P_\mu, P_\lambda, P_v]_U\right)^T VK),$$

$$\Leftrightarrow R([P_\mu, P_\lambda, P_v]_L{}^T) = R\left(\mathrm{KV}[P_\mu, P_\lambda, P_v]_L\right)^T K), \ R([P_\mu, P_\lambda, P_v]_U{}^T) = R\left(KV[P_\mu, P_\lambda, P_v]_U\right)$$

$$[P_\mu, P_\lambda, P_v]_L = VK[P_\mu, P_\lambda, P_v]_L{}^T \ \mathrm{VK}\, H_1, [P_\mu, P_\lambda, P_v]_U = VK[P_\mu, P_\lambda, P_v]_U \ \mathrm{VK} \ for\ H_1 \in IVNFM$$

As a conclusion (ii) $\Leftrightarrow$ (xiii) $\Leftrightarrow$ (x) is true

The above statement can be reduced to the equivalent requirement that a matrix be an IV s- RS for K = I in particular.

Corollary:3.1 The following statements are equivalent for $P \in \mathrm{IVNFM}_{nn}$

(i) $P = \langle [P_\mu, P_\lambda, P_v]_L, [P_\mu, P_\lambda, P_v]_U \rangle \in \mathrm{IVNFM}_{nn}$ is an IV s-RS.

(ii) $VP = < V[P_\mu, P_\lambda, P_v]_L, V[P_\mu, P_\lambda, P_v]_U >$ is an IV RS.

(iii) $PV = < [P_\mu, P_\lambda, P_v]_L\ V, [P_\mu, P_\lambda, P_v]_U\ V >$ is an IV RS.

(iv) $P^T = < [P_\mu, P_\lambda, P_v]_L{}^T, [P_\mu, P_\lambda, P_v]_U{}^T >$ is an IV s – RS.

(v) $R\left([P_\mu, P_\lambda, P_v]_L\right) = R\left([P_\mu, P_\lambda, P_v]_L{}^T V\right), R\left([P_\mu, P_\lambda, P_v]_U\right) = R\left([P_\mu, P_\lambda, P_v]_U{}^T V\right)$

(vi) $R\left([P_\mu, P_\lambda, P_v]_L{}^T\right) = R\left([P_\mu, P_\lambda, P_v]_L V\right), R\left([P_\mu, P_\lambda, P_v]_U{}^T\right) = R\left([P_\mu, P_\lambda, P_v]_U\ V\right)$

(vii) $C\left(KV[P_\mu, P_\lambda, P_v]_L\right) = C\left(V[P_\mu, P_\lambda, P_v]_L\right)^T, C\left(KV[P_\mu, P_\lambda, P_v]_U\right) = C\left(V[P_\mu, P_\lambda, P_v]_U\right)^T$

(viii) $[P_\mu, P_\lambda, P_v]_L = V[P_\mu, P_\lambda, P_v]_L{}^T\ VH_1,\ [P_\mu, P_\lambda, P_v]_U$

$= V[P_\mu, P_\lambda, P_v]_U{}^T\ VH_1 \text{ for } H_1 \in IVNFM$

(ix) $[P_\mu, P_\lambda, P_v]_L = H_1V[P_\mu, P_\lambda, P_v]_L{}^T\ V,\ [P_\mu, P_\lambda, P_v]_U$

$= H_1V[P_\mu, P_\lambda, P_v]_U{}^T\ V \text{ for } H_1 \in IVNFM$

(x) $[P_\mu, P_\lambda, P_v]_L{}^T = V[P_\mu, P_\lambda, P_v]_L\ VH_1,\ [P_\mu, P_\lambda, P_v]_U{}^T$

$= V[P_\mu, P_\lambda, P_v]_U\ VH_1 \text{ for } H_1 \in IVNFM$

(xi) $[P_\mu, P_\lambda, P_v]_L{}^T = H_1V[P_\mu, P_\lambda, P_v]_L\ V,\ [P_\mu, P_\lambda, P_v]_U$

$= H_1V[P_\mu, P_\lambda, P_v]_U\ V \text{ for } H_1 \in IVNFM$

Theorem 3.2. For $P = < [P_\mu, P_\lambda, P_v]_L, [P_\mu, P_\lambda, P_v]_U > \in IVNFM_{nn}$ then any two of the conditions below imply the other

(i) P is an IV κ- RS.

(ii) P is an IV s-κ- RS.

(iii) $R\left([P_\mu, P_\lambda, P_v]_L\right)^T = R\left(VK[P_\mu, P_\lambda, P_v]_L\right)^T, R\left([P_\mu, P_\lambda, P_v]_U\right)^T = R\left(VK[P_\mu, P_\lambda, P_v]_U{}^T\right)$

Proof: (i) and (ii) implies (iii)

Let P is an IV s – κ RS

$$\Rightarrow R\left([P_\mu,P_\lambda,P_v]_L\right)=R\left([P_\mu,P_\lambda,P_v]_L{}^T VK\right),R\left([P_\mu,P_\lambda,P_v]_U\right)=R\left([P_\mu,P_\lambda,P_v]_U{}^T VK\right)$$

[By Theorem 3.1]

$$\Rightarrow R\left(K[P_\mu,P_\lambda,P_v]_L K\right)=R\left(K[P_\mu,P_\lambda,P_v]_L{}^T K\right),R\left(K[P_\mu,P_\lambda,P_v]_U K\right)$$

$$=R\left(K[P_\mu,P_\lambda,P_v]_U{}^T K\right)$$ [By Lemma 2.2]

$$\Rightarrow R[P_\mu,P_\lambda,P_v]_L{}^T=R\left(\left(VK[P_\mu,P_\lambda,P_v]_L\right)^T\right),R\left([P_\mu,P_\lambda,P_v]_U\right)^T=R\left(\left(VK[P_\mu,P_\lambda,P_v]_U\right)^T\right)$$

(i) & (ii) implies (iii) is true

(i)& (iii) implies (ii)

P is an IV κ- RS

$$\Rightarrow R\left(K[P_\mu,P_\lambda,P_v]_L K\right)=R\left(\left([P_\mu,P_\lambda,P_v]_L\right)^T\right),R\left(K[P_\mu,P_\lambda,P_v]_U K\right)=R\left(\left([P_\mu,P_\lambda,P_v]_U\right)^T\right)$$

Therefore, (i) & (iii)

$$\Rightarrow R\left([P_\mu,P_\lambda,P_v]_L\right)=R\left([P_\mu,P_\lambda,P_v]_L{}^T VK\right),R\left([P_\mu,P_\lambda,P_v]_U\right)=R\left([P_\mu,P_\lambda,P_v]_U{}^T VK\right)$$

$$\Rightarrow R\left([P_\mu,P_\lambda,P_v]_L\right)=R\left(\left(KV[P_\mu,P_\lambda,P_v]_L\right)^T\right),R\left([P_\mu,P_\lambda,P_v]_U\right)=R\left(\left(KV[P_\mu,P_\lambda,P_v]_U\right)^T\right)$$

P is an IV s-k-RS (By Theorem 3.1)

$\Rightarrow$ (ii) is true

(ii) & (iii) implies (i)

P is an IV s- κ- RS

$$\Rightarrow R\left(K[P_\mu,P_\lambda,P_v]_L K\right)=R\left(K[P_\mu,P_\lambda,P_v]_L{}^T K\right),R\left(K[P_\mu,P_\lambda,P_v]_U K\right)=R\left(K[P_\mu,P_\lambda,P_v]_U{}^T K\right)$$ There fore,(ii) and (iii)

$$\Rightarrow R\left([P_\mu,P_\lambda,P_v]_L\right)=R\left(K[P_\mu,P_\lambda,P_v]_L{}^T K\right),R\left([P_\mu,P_\lambda,P_v]_U\right)=R\left(K[P_\mu,P_\lambda,P_v]_U{}^T K\right)$$

$P=<[P_{\mu L},P_{\lambda L},P_{vL}],[P_\mu,P_\lambda,P_v]_U>\in \mathrm{IVNFM}_{nn}$ is an IV κ- RS .

Therefore, (i) is true, Hence the theorem.

4. IV s – κ RS regular Neutrosophic fuzzy matrices

In this section, it was discovered that there are various generalized inverses of matrices in IVNFM. The comparable standards for different g-inverses of an IV s-k RS Neutrosophic fuzzy matrix to be IV s-k RS are also established. The generalized inverses of an IV s – κ RS P corresponding to the sets P{1, 2}, P{1, 2, 3} and P{1, 2, 4} are characterized.

Theorem 4.1: Let $P\in \mathrm{IVNFM}_{nn}$, Z belongs to P{1,2} and PW, ZW are an IV s- κ- RS. Then P is an IV s- κ - RS iff $W=<[W_\mu,W_\lambda,W_v]_L,[W_\mu,W_\lambda,W_v]_U>$ is an IV s- κ – RS.

Proof: Let $P = <[P_\mu, P_\lambda, P_v]_L, [P_\mu, P_\lambda, P_v]_U> \in \mathrm{IVNFM}_{nn}$

$$R\left(KV[P_\mu, P_\lambda, P_v]_L\right) = R\left(KV[P_\mu, P_\lambda, P_v]_L W[P_{\mu L}, P_{\lambda L}, P_{vL}]\right) \subseteq R\left(W[P_\mu, P_\lambda, P_v]_L\right)$$

$$= R\left(WVV[P_\mu, P_\lambda, P_v]_L\right) \subseteq R\left(WVKKV[P_\mu, P_\lambda, P_v]_L\right) \subseteq R\left(KV[P_\mu, P_\lambda, P_v]_L\right)$$

Hence, $\mathrm{R}\left(KV[P_\mu, P_\lambda, P_v]_L\right) = R\left(W[P_\mu, P_\lambda, P_v]_L\right)$

$$= \mathrm{R}\left(KV\left(W[P_\mu, P_\lambda, P_v]_L\right)^T VK\right) \qquad [\text{WP is IV s- κ-RS}]$$

$$= \mathrm{R}\left([P_\mu, P_\lambda, P_v]_L{}^T [W_\mu, W_\lambda, W_v]_L{}^T VK\right)$$

$$= \mathrm{R}\left([W_\mu, W_\lambda, W_v]_L{}^T VK\right) = \mathrm{R}\left(\left(KV[W_\mu, W_\lambda, W_v]_L\right)^T\right)$$

$$R\left(\left(KV[P_\mu, P_\lambda, P_v]_L\right)^T\right) = R\left([P_\mu, P_\lambda, P_v]_L{}^T VK\right)$$

$$= \mathrm{R}\left(KV[P_\mu, P_\lambda, P_v]_L [W_\mu, W_\lambda, W_v]_L\right) \qquad [\text{ VP is s- κ} - \text{IV RS}]$$

$$= \mathrm{R}\left(KV[W_\mu, W_\lambda, W_v]_L\right)$$

Similarly,

Hence, $\mathrm{R}\left(KV[W_\mu, W_\lambda, W_v]_U\right) = R\left(\left(KV[P_\mu, P_\lambda, P_v]_U\right)^T\right)$ (KVW is an IV RS)

$$\Leftrightarrow R\left(KV[P_\mu, P_\lambda, P_v]_L\right) = R\left(\left(KV[P_{\mu L}, P_{\lambda L}, P_{vL}]\right)^T\right),\ R\left(KV[P_\mu, P_\lambda, P_v]_U\right) = R\left(\left(KV[P_\mu, P_\lambda, P_v]_U\right)^T\right)$$

$$\Leftrightarrow R\left(KV[W_\mu, W_\lambda, W_v]_L\right) = R\left(\left(KV[W_\mu, W_\lambda, W_v]_L\right)^T\right),$$

$$R\left(KV[W_\mu, W_\lambda, W_v]_U\right) = R\left(\left(KV[W_\mu, W_\lambda, W_v]_U\right)^T\right)$$

$\Leftrightarrow KVX = \left[KV[W_\mu, W_\lambda, W_v]_L, KV[W_\mu, W_\lambda, W_v]_U\right]$ is an IV RS

$W = <[W_\mu, W_\lambda, W_v]_L, [W_\mu, W_\lambda, W_v]_U>$ is an IV RS.

Theorem 4.2:, Let $P \in \mathrm{IVNFM}_{nn}$ $W = <[W_\mu, W_\lambda, W_v]_L, [W_\mu, W_\lambda, W_v]_U> \in$P{1,2,3}, $\mathrm{R}(\mathrm{KV}[P_\mu, P_\lambda, P_v]_L) = R\left(\mathrm{KV}[X_{\mu L}, X_{\lambda L}, X_{vL}]\right)^T$, $\mathrm{R}(\mathrm{KV}[P_\mu, P_\lambda, P_v]_U) = R\left(\mathrm{KV}[Z_{\mu U}, Z_{\lambda U}, Z_{vU}]\right)^T$.Then

$P = <[P_\mu, P_\lambda, P_v]_L, [P_\mu, P_\lambda, P_v]_U> \in \mathrm{IVNFM}_{nn}$ is IV s-κ- RS $\Leftrightarrow W = <[W_\mu, W_\lambda, W_v]_L, [W_\mu, W_\lambda, W_v]_U>$ is IV s- κ – RS.

Proof: Given P{1,2,3}, Hence ,

$[\mathrm{P}_\mu, \mathrm{P}_\lambda, \mathrm{P}_v]_L[W_\mu, W_\lambda, W_v]_L[\mathrm{P}_\mu, \mathrm{P}_\lambda, \mathrm{P}_v]_L = [\mathrm{P}_\mu, \mathrm{P}_\lambda, \mathrm{P}_v]_L,$

$[W_\mu, W_\lambda, W_v]_L[\mathrm{P}_{\mu L}, \mathrm{P}_{\lambda L}, \mathrm{P}_{vL}][W_\mu, W_\lambda, W_v]_L = [W_\mu, W_\lambda, W_v]_L,$

$\left([\mathrm{P}_\mu, \mathrm{P}_\lambda, \mathrm{P}_v]_L[W_\mu, W_\lambda, W_v]_L\right)^T = [\mathrm{P}_\mu, \mathrm{P}_\lambda, \mathrm{P}_v]_L[W_\mu, W_\lambda, W_v]_L$

Consider, $\mathrm{R}\left(\left(KV[\mathrm{P}_\mu, \mathrm{P}_\lambda, \mathrm{P}_v]_L\right)^T\right) = \mathrm{R}\left([W_\mu, W_\lambda, W_v]_L{}^T[\mathrm{P}_\mu, \mathrm{P}_\lambda, \mathrm{P}_v]_L{}^T VK\right)\left[\text{By using P} = \text{PW}P\right]$

$= \mathrm{R}\left(KV\left([\mathrm{P}_\mu, \mathrm{P}_\lambda, \mathrm{P}_v]_L[W_\mu, W_\lambda, W_v]_L\right)^T\right)$

$= \mathrm{R}\left(\left([\mathrm{P}_\mu, \mathrm{P}_\lambda, \mathrm{P}_v]_L[W_\mu, W_\lambda, W_v]_L\right)^T\right)$ $\left[By\ P._{2.3}\right]$

$= \mathrm{R}\left([\mathrm{P}_\mu, \mathrm{P}_\lambda, \mathrm{P}_v]_L[W_\mu, W_\lambda, W_v]_L\right)$

$= \mathrm{R}\left([W_\mu, W_\lambda, W_v]_L\right)$

$\left[\text{By using } [W_\mu, W_\lambda, W_v]_L = [W_\mu, W_\lambda, W_v]_L[\mathrm{P}_\mu, \mathrm{P}_\lambda, \mathrm{P}_v]_L[W_\mu, W_\lambda, W_v]_L\right]$

$= \mathrm{R}\left(KV[W_\mu, W_\lambda, W_v]_L\right)$ $\left[By\ P_{2.3}\right]$

Similarly, we can consider, $\mathrm{R}\left(\left(KV[\mathrm{P}_\mu, \mathrm{P}_\lambda, \mathrm{P}_v]_U\right)^T\right) = \mathrm{R}\left([W_\mu, W_\lambda, W_v]_U{}^T[\mathrm{P}_\mu, \mathrm{P}_\lambda, \mathrm{P}_v]_U{}^T VK\right)$

$= \mathrm{R}\left(KV\left([\mathrm{P}_\mu, \mathrm{P}_\lambda, \mathrm{P}_v]_U[W_\mu, W_\lambda, W_v]_U\right)^T\right)$

$= \mathrm{R}\left(\left([\mathrm{P}_\mu, \mathrm{P}_\lambda, \mathrm{P}_v]_U[W_\mu, W_\lambda, W_v]_U\right)^T\right)$ $\left[By\ P._{2.3}\right]$

$= \mathrm{R}\left([\mathrm{P}_\mu, \mathrm{P}_\lambda, \mathrm{P}_v]_U[W_\mu, W_\lambda, W_v]_U\right)$ $\left[(PW)^T = \text{PW}\right]$

$= \mathrm{R}\left([W_\mu, W_\lambda, W_v]_U\right)$ $\left[\text{By using W} = \text{W}PW\right]$

$= \mathrm{R}\left(KV[W_\mu, W_\lambda, W_v]_U\right)$ $\left[\text{By } P_{2.3}\right]$

If KVA is an IV RS

$\Leftrightarrow R\left(KV[\mathrm{P}_\mu, \mathrm{P}_\lambda, \mathrm{P}_v]_L\right) = \mathrm{R}\left(\left(KV[\mathrm{P}_\mu, \mathrm{P}_\lambda, \mathrm{P}_v]_L\right)^T\right),$

$R\left(KV[\mathrm{P}_\mu, \mathrm{P}_\lambda, \mathrm{P}_v]_U\right) = \mathrm{R}\left(\left(KV[\mathrm{P}_\mu, \mathrm{P}_\lambda, \mathrm{P}_v]_U\right)^T\right)$

$\Leftrightarrow R\left(KV[W_\mu, W_\lambda, W_v]_L\right) = \mathrm{R}\left(\left(KV[W_\mu, W_\lambda, W_v]_L\right)^T\right),$

$KVX = [KV[W_\mu, W_\lambda, W_v]_L, KV[W_\mu, W_\lambda, W_v]_U]$ is an IV RS.

$W =< [W_\mu, W_\lambda, W_v]_L, [W_\mu, W_\lambda, W_v]_U >$ is an IV s-k RS.

Theorem 4.3: Let $P \in \mathrm{IVNFM}_{nn}$, Z ∈ A {1, 2, 4}, $\mathrm{R}(\mathrm{KV}[\mathrm{P}_\mu, \mathrm{P}_\lambda, \mathrm{P}_v]_L)^T = R\left(\mathrm{KV}[W_\mu, W_\lambda, W_v]_L\right)$,

$\mathrm{R}(\mathrm{KV}[\mathrm{P}_\mu, \mathrm{P}_\lambda, \mathrm{P}_v]_U)^T = R\left(\mathrm{KV}[W_\mu, W_\lambda, W_v]_U\right)$. Then KVP is an IV s- κ-Ks iff

$W =< [W_\mu, W_\lambda, W_v]_L , [W_\mu, W_\lambda, W_v]_U >$ is an IV s- κ- RS.

Proof: Given, P {1, 2, 4}, Hence $[\mathrm{P}_\mu, \mathrm{P}_\lambda, \mathrm{P}_v]_L [W_\mu, W_\lambda, W_v]_L [\mathrm{P}_\mu, \mathrm{P}_\lambda, \mathrm{P}_v]_L = [\mathrm{P}_\mu, \mathrm{P}_\lambda, \mathrm{P}_v]_L$,

$[W_\mu, W_\lambda, W_v]_L [\mathrm{P}_\mu, \mathrm{P}_\lambda, \mathrm{P}_v]_L [W_\mu, W_\lambda, W_v]_L = [W_\mu, W_\lambda, W_v]_L$,

$\left([W_\mu, W_\lambda, W_v]_L [\mathrm{P}_\mu, \mathrm{P}_\lambda, \mathrm{P}_v]_L\right)^T = [W_\mu, W_\lambda, W_v]_L [\mathrm{P}_\mu, \mathrm{P}_\lambda, \mathrm{P}_v]_L$

Consider, $\mathrm{R}\left(\left(KV[\mathrm{P}_\mu, \mathrm{P}_\lambda, \mathrm{P}_v]_L\right)^T\right) = \mathrm{R}\left([W_\mu, W_\lambda, W_v]_L{}^T [\mathrm{P}_\mu, \mathrm{P}_\lambda, \mathrm{P}_v]_L{}^T VK\right)$ $\left[\text{By using P} = \text{PW}P\right]$

$= R\left(KV\left([\mathrm{P}_\mu, \mathrm{P}_\lambda, \mathrm{P}_v]_L [W_\mu, W_\lambda, W_v]_L\right)^T\right)$

$= R\left(\left([\mathrm{P}_\mu, \mathrm{P}_\lambda, \mathrm{P}_v]_L [W_\mu, W_\lambda, W_v]_L\right)^T\right)$ $\left[By\ P_{2.3}\right]$

$= R\left([\mathrm{P}_\mu, \mathrm{P}_\lambda, \mathrm{P}_v]_L [W_\mu, W_\lambda, W_v]_L\right) = R\left([W_\mu, W_\lambda, W_v]_L\right)$

$= R\left(\mathrm{KV}[W_\mu, W_\lambda, W_v]_L\right)$ $\left[By\ P_{2.3}\right]$

$R\left(\left(KV[\mathrm{P}_\mu, \mathrm{P}_\lambda, \mathrm{P}_v]_U\right)^T\right) = \mathrm{R}\left([W_\mu, W_\lambda, W_v]_U{}^T [\mathrm{P}_\mu, \mathrm{P}_\lambda, \mathrm{P}_v]_U{}^T VK\right)$ $\left[\text{By using P} = \text{PW}P\right]$

$= R\left(KV\left([\mathrm{P}_\mu, \mathrm{P}_\lambda, \mathrm{P}_v]_U [W_\mu, W_\lambda, W_v]_U\right)^T\right)$

$= R\left(\left([\mathrm{P}_\mu, \mathrm{P}_\lambda, \mathrm{P}_v]_U [W_\mu, W_\lambda, W_v]_U\right)^T\right)$ $\left[By\ P_{2.3}\right]$

$= R\left([\mathrm{P}_\mu, \mathrm{P}_\lambda, \mathrm{P}_v]_U [W_\mu, W_\lambda, W_v]_U\right)$ $\left[\ (\mathrm{P}W)^T = PW\right]$

$= N\left([\mathrm{X}_{\mu U}, \mathrm{X}_{\lambda U}, \mathrm{X}_{vU}]\right) = R\left(\mathrm{KV}[W_\mu, W_\lambda, W_v]_U\right)$ $\left[By\ P_{2.3}\right]$

If KVP is an IV RS

$\Leftrightarrow R\left(KV[\mathrm{P}_\mu, \mathrm{P}_\lambda, \mathrm{P}_v]_L\right) = \mathrm{R}\left(\left(KV[\mathrm{P}_\mu, \mathrm{P}_\lambda, \mathrm{P}_v]_L\right)^T\right)$,

$R\left(KV[\mathrm{P}_{\mu U}, \mathrm{P}_{\lambda U}, \mathrm{P}_{vU}]\right) = \mathrm{R}\left(\left(KV[\mathrm{P}_{\mu U}, \mathrm{P}_{\lambda U}, \mathrm{P}_{vU}]\right)^T\right)$

$\Leftrightarrow R\left(KV[W_\mu, W_\lambda, W_v]_L\right) = \mathrm{R}\left(\left(KV[W_\mu, W_\lambda, W_v]_L\right)^T\right)$,

$KVX = [KV[W_{\mu}, W_{\lambda}, W_{v}]_{L}, KV[W_{\mu}, W_{\lambda}, W_{v}]_{U}]$ is an IV RS.

$W = < [W_{\mu}, W_{\lambda}, W_{v}]_{L}, [W_{\mu}, W_{\lambda}, W_{v}]_{U} >$ is an IV s-k RS.

The aforementioned Theorems reduce to comparable criteria, in particular for K = I, for different g-inverses of interval valued s- RS to be IV secondary RS.

Corollary 4.1: For $P \in \text{IVNFM}_{nn}$, Z∈ P {1, 2} and $PW = < [\mathrm{P}_{\mu}, \mathrm{P}_{\lambda}, \mathrm{P}_{v}]_{L}[W_{\mu}, W_{\lambda}, W_{v}]_{L}$ $, [\mathrm{P}_{\mu}, \mathrm{P}_{\lambda}, \mathrm{P}_{v}]_{U}[W_{\mu}, W_{\lambda}, W_{v}]_{U} >, WP = < [W_{\mu}, W_{\lambda}, W_{v}]_{L}[\mathrm{P}_{\mu}, \mathrm{P}_{\lambda}, \mathrm{P}_{v}]_{L}, [W_{\mu}, W_{\lambda}, W_{v}]_{U}[\mathrm{P}_{\mu}, \mathrm{P}_{\lambda}, \mathrm{P}_{v}]_{U} >,$ are is an IV s- RS. Then P is an IV s- RS iff $W = < [W_{\mu}, W_{\lambda}, W_{v}]_{L}, [W_{\mu}, W_{\lambda}, W_{v}]_{U} >$ is an IV s- RS.

Corollary 4.2: For $P \in \text{IVNFM}_{nn}$, W∈ P {1, 2, 3}, $\mathrm{R}(\mathrm{KV}[\mathrm{P}_{\mu}, \mathrm{P}_{\lambda}, \mathrm{P}_{v}]_{L}) = R\left(\mathrm{V}[W_{\mu}, W_{\lambda}, W_{v}]_{L}\right)^{T}$ $, \mathrm{R}(\mathrm{KV}[\mathrm{P}_{\mu U}, \mathrm{P}_{\lambda U}, \mathrm{P}_{vU}]) = R\left(\mathrm{V}[W_{\mu}, W_{\lambda}, W_{v}]_{U}\right)^{T}$. Then P is an IV s- RS iff $W = < [W_{\mu}, W_{\lambda}, W_{v}]_{L}$ $, [W_{\mu}, W_{\lambda}, W_{v}]_{U} >$ is an IV s- RS.

Corollary 4.3: For $P \in \text{IVNFM}_{nn}$, W∈P, $\mathrm{R}(\mathrm{V}[\mathrm{P}_{\mu}, \mathrm{P}_{\lambda}, \mathrm{P}_{v}]_{L})^{T} = R\left(\mathrm{V}[W_{\mu}, W_{\lambda}, W_{v}]_{L}\right)^{T}, \mathrm{R}(\mathrm{V}[\mathrm{P}_{\mu}, \mathrm{P}_{\lambda}, \mathrm{P}_{v}]_{U})^{T}$, $, = R\left(\mathrm{V}[W_{\mu}, W_{\lambda}, W_{v}]_{U}\right)$. Then P is an IV s- RS iff W is an IV s-RS.

5. Conclusion:

We present equivalent characterizations of an IV k- RS, IV RS, IV s- RS, IV s-k RS NFM. Also, we give the example of s-k-symmetric fuzzy matrix is s-k- RS Neutrosophic fuzzy matrix the opposite isn't always true. We discuss various g-inverse associated with a regular matrices and obtain characterization of set of all inverses. Equivalent conditions for various g-inverses of an Interval Valued s-k-range Symmetric and s-range Symmetric NFMs are determined. In future, we shall prove some related properties of Interval Valued Secondary k-range Symmetric Neutrosophic Fuzzy Matrices.

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An Overview of Neutrosophic Theory in Medicine and Healthcare

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Abstract

Neutrosophic ranking studies are an important part of medicine that determines the ranks of tests, risk factors, attributes, and medical suppliers. Neutrosophic clustering in healthcare can split data into groups (called clusters) to determine usage patterns for purposes, in which objects within the same cluster have similar properties and objects of different clusters have different properties. Like neutrosophic clustering, neutrosophic classification studies are also a data mining technique. Neutrosophic pattern recognition is a machine learning process to decipher the underlying patterns in the concerned subjects. Neutrosophic time series analysis tries to find patterns and rules depending on time, neutrosophic recognition of medical images belongs to this type of study. This article comes as an attempt to review and shed light on almost all studies and subjects that used neutrosophic studies and algorithms related to medicine and healthcare for dozens of articles and authors.

Keywords: Neutrosophic logic, medical diagnosis, healthcare, neutrosophic sets, neutrosophic soft sets and lung disease diagnosis.

1. Introduction

Indeterminacy stems from real-world problems. It is well known that between the two colours white and black, there are unlimited grey colour gradients. It is the same as the infinite decimal numbers between zero and one. Also, between absolute truthiness and absolute falseness, there are many situations and logistic phrases can hold a percentage of truth with a percentage of false, as well as, when the concepts get mixed up, we will see the situation that should be true may be regarded as absolutely false and vice versa. The solver won't be able to meet all the requirements unless the solving tools are flexible and soft and cover the incompleteness, inconsistency, and indeterminate data to analyze them fairly without neglecting any part of the data.

In medicine, it is not applicable to describe medical concepts and relationships precisely. Consider the phrase (If the back pain is severe and the patient is old, then apply acupuncture to a certain point for a long time), aiming to program the above statement, we need to reformulate it as an If-Then loop regarded as a model in a computer system, so we need a mathematical interpretation for the following linguistic terms:

'' serve'', ''old'', ''certain point'', and ''long-time'', are all linguistic words that are vague and contain indeterminate boundaries. This is why the information on healthcare should be interpreted by neutrosophic methodologies.

Imperfect knowledge is unavoidable in medicine and the nature of medical data causes many uncertainties in medical decision-making, arising from a number of areas.

such as an incomplete understanding of biological mechanisms, imprecise test measurements, uncertainty about normal ranges for test results, the simultaneous presence of more than one condition, and missing information occurring in a large percentage of cases.

The Romanian scholar Florentin Smarandache set up the neutrosophic sets, neutrosophic numbers, neutrosophic theory, neutrosophic logic, and neutrosophic probability in 1995 [1], he also presented hundreds of new unfathomable concepts, and theorems in neutrosophic calculus, neutrosophic probability and statistics, neutrosophic number theory, neutrosophic graph theory, neutrosophic geometry and so on, especially, in decision- making.

Maria et al. [2] used the concepts of single-valued neutrosophic sets (SVS) with a score function S of a single-valued neutrosophic function based on the truth-membership degree, indeterminacy-membership degree, and falsity membership degree, with neutrosophic statistic tool (i.e. neutrosophic frequency distribution) to development the assertive communication competencies that allow nursing professionals to keep a good relationship with their therapeutic team and the patient and avoid professional burnout.

Medicine is one of the fastest-growing fields when compared to other computer-aided technology. This fast growth, together with the vague nature the medicine, brings the need for different strategies and creative technologies such as neutrosophic logic or its combination with other artificial intelligence techniques.

All of us suffered from COVID-19, its propagation and virulence, which has constituted the second global pandemic of the XXI century, reasons why have generated social distancing as a preventive measure, Marylin et al [3] point out the uncertainties in discursive analysis using a qualitative research approach in line with the Smarandache proposal. Datamining tool orange was adapted to the neutrosophic environment, improves the social and emotional facilitated by parents strengthens the development of adaptive behaviour skills, and generates active and coherent of the special educational needs of students, strengthening their inclusive education.

Because of its ability to extend the classical Boolean logic (two-valued logic) of the computer applications, healthcare computer-aided applications employ neutrosophic logic to handle the semantics of the related domain. It compares, constraints, extends, and particularizes concepts as humans do in reasoning; it connects symbols and concepts.

This article is arranged as follows: the upcoming section (section two) has been dedicated to demonstrating neutrosophic logic, while section three is regarded as the main part entitled ʻʻ Neutrosophic Logic in Medical Domainʼʼ covering the common neutrosophic applications and techniques in medicine. As well as the uncertainty of medicine will be explained in detail in this section. The ending section will be the conclusion section.

2. Preliminaries of Neutrosophic Theory

As previously mentioned, the neutrosophic set has been presented in its current seemliness by Florentin Smarandache in 1995 [1], Huda et al [4] gave the differences between fuzzy logic and neutrosophic logic in the application of linear programming as follows:

In Fuzzy Linear Programming Problems (FLP), the optimal solution depends on a limited number of constraints, therefore, much of the information that should be collected and have a good impact on the solution is absent, this is exactly what Neutrosophic Linear Programming (NLP) provides.

Given the power of LP, one could have expected even more applications. This might be because LP requires many well-defined and precise data which involves high information costs. In real-world applications certainty, reliability, and precision of data are often illusory. Being able to deal with vague and imprecise data may greatly contribute to the diffusion and application of LP. Neutrosophic Linear Programming problems can reformulate the soft linear programming problems through three membership functions which are truth membership function, indeterminacy membership function, and falsity membership functions, while fuzzy linear programming deals with just one membership function.

We won't be unfair if we say that neutrosophic logic can be defined as the efforts of simulation of the human thinking model, which uses linguistic variables and concepts, into computer applications. In this way, digital computers can easily be able to handle linguistic variables and their degrees of membership, non-membership, and indeterminate membership rather than fuzzy or crisp systems.

2.1 Neutrosophic Sets

The neutrosophic set has a meaning that differs from the fuzzy set or intuitionistic fuzzy set, where the fuzzy set has members belonging to it partially, while the intuitionistic set has members partially belonging to it side by side with partially un-belonging to it. wherein any member can find a well belonging definition to any set in the perspective of neutrosophic theory, since any element either belongs partially to its truth membership function or belongs partially to its falsity membership function, there is another chance to belonging to an indeterminate membership function, the following examples can clearly determine the global comprehensive vision of neutrosophic thought:

Ex.1 [5]

Suppose 5 professors conduct PhD dissertations in neutrosophic statistics. Each professor has a number of graduate students, but some students are undecided about whether to pursue their dissertations in classical or neutrosophic statistics. The professors represent the clusters. One randomly selects 2 professors to interview their students about research in neutrosophic statistics. But, because some students are undecided (indeterminate) with respect to their research topic, we have a neutrosophic cluster sampling.

Ex.2 [5]

For example: tossing a coin on an irregular surface which has cracks, the coin can fall inside a crack on its edge, and thus one gets neither head, nor tail, but indeterminacy.

Ex. 3 [6]

A cloud is a neutrosophic set, because its borders are ambiguous, and each element (water drop) belongs with a neutrosophic probability to the set. (e.g. there are a kind of separated water drops, around a compact mass of water drops, that we don't know how to consider them: in or out of the cloud).

2.2 Why the Neutrosophic Set is an Essential Tool in Medical Diagnosis?

Neutrosophic sets suit the requirements of medical data representation. It is very rare that a doctor tends to diagnose/ judge the disease in definite environments. Imprecision could be due to the lack of confidence on the part of patients in reporting symptoms, or imperfection leads to doubt about the value of a variable, a decision to be taken or a conclusion to be drawn for the actual symptom. Multiple factors could lead to uncertainty like incomplete knowledge (ignorance of the patient, limited view of a system because of its complexity), stochasticity (the case of intrinsic imperfection where a typical and single value does not exist), or acquisition errors (intrinsically imperfect lab observations, the quantitative errors in measures). So, the neutrosophic technique would have indeterminate features and behaviors associated, and there would always be unanticipated happening conditions which are uncontrollable - we mean the indeterminacy plays a role as well [7].

2.3. Mathematical Representation of Neutrosophic Set

Definition2.3.1 [8]

Let X be a space of points (objects), with a generic element in X denoted by x. A neutrosophic set A in X is characterized by a truth-membership function T_A , an indeterminacy-membership function I_A and a falsity-membership function F_A. $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or non-standard subsets of $]0-\ ,1+\ [$. That is:

$T_A: X \to]\,^{-}0, 1^{+}[$

$I_A: X \to]\,^{-}0, 1^{+}[$

$F_A: X \to]\,^{-}0, 1^{+}[$

Definition 2.3.2[8]

(Single Valued Neutrosophic Set). Let X be a space of points (objects), with a generic element in X denoted by x. A single valued neutrosophic set ($SVNS$) A in X is characterized by truth-membership function T_A , indeterminacy-membership function I_A and falsity-membership function F_A. For each point x in X, $T_A(x),\ I_A(x), F_A(x) \in\ [0,1]$.

2.4 Neutrosophic's Analytical Comparison to Other Logics [9]

Neutrosophic logic is a far better representation of real-world data/executions because of the following reasons:

a. Fuzzy logic though ensures multiple belongingness of a particular element to multiple classes with a varied degree but capturing of neutralities due to indeterminacy is missing, it is further limited by the fact that membership and non-membership value of an element to a particular class should sum up to 1.

b. Similarly other allied logics like Lukasiewicz logic considered three values $(1, 1/2, 0)$, Post considered m values, etc., but all are handicapped with the constraint that values can vary between 0 and 1 only.

c. Intuitionistic fuzzy logic though deals with indeterminacy parameter related to a particular element, but this fact is still constrained with the condition that, for any element x , indeterminacy value $(x) = 1 - [\text{membership value}(x) + \text{non} - \text{membership value}(x)]$. There is no provision of distinguishing between relative and absolute truth/indeterminacy/falsity.

d. In a rough set, an element x on the boundary line cannot be classified as a member of a particular class nor of its complement with certainty; but can be very well described by neutrosophic logic, such that $x\ (T, I, F)$ where T, I, F are standard or non-standard subsets of the nonstandard interval $]\ {}^{-}0, 1^{+}[$.

3. Neutrosophic Logic in Medical Domain

As shown below figure 1 displays the conceptual membership functions of blood sugar=normal, the neutrosophic logic would help in explicitly listing out three important components of the input values captured: truthiness, indeterminacy, and falsity.

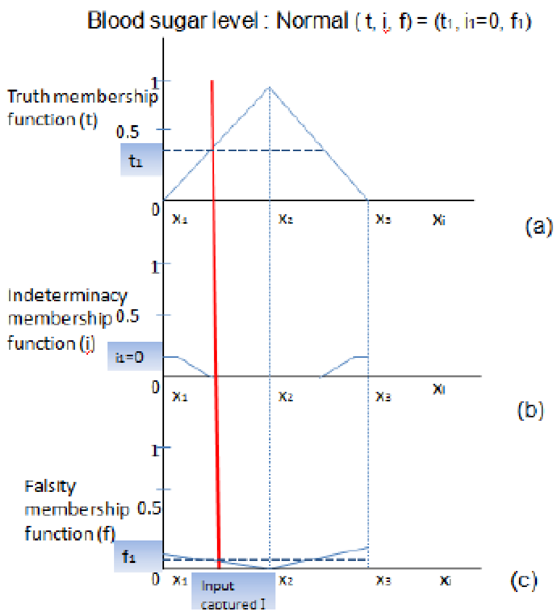

Figure 1: Neutrosophication of the input captured

Again, Figure 1 shows the conceptual membership functions of blood sugar level=normal; designed for capturing the truthiness, indeterminacy and falsity of the input record. The designing of these three membership functions would vary considerably for different parameters. Here in this figure, the captured input (I) is mapped onto the three membership functions, it is assumed that indeterminacy related to deciding whether the blood sugar is normal is high on the tapering ends of the truth membership functions designed and falsity membership function corresponds to the lab equipment error or degradation of equipment noticed which can give erroneous results. So as per Figure 1, the three-component values generated after neutrosophication of captured input I is $I(t_1, i_1 = 0, f_1)$. Figure 2 discusses the neutrosophication process applied to multiple parameters simultaneously. As it is very common in the medical domain to infer D, analysis of multiple parameters is required. Here in this figure, it is assumed that analysis of 4 input parameters is required to infer D. As clearly shown input parameter (I_1) lies in the indeterminacy zone (which could be possible due to lack of information or early onset of the disease during which input I_1 cannot be captured); input parameters (I_2 and I_3) lies in the truth zone indicating favor for D and the corresponding mapping shows I_3 favors strongly for D in comparison to I_2 as $t_1 > t_2$; input parameter I_4 contradicts the possibility of D by f_1 value.

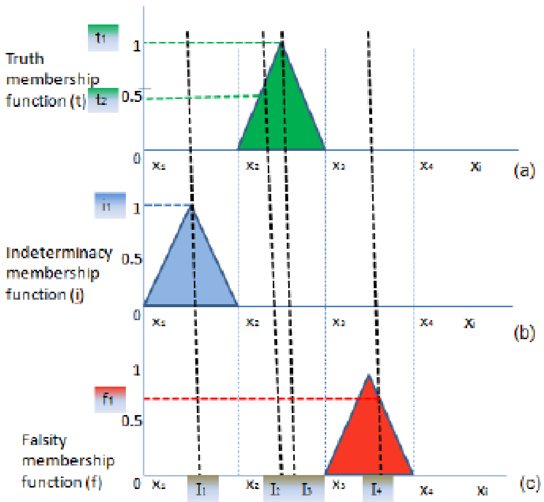

Figure 2: Neutrosophication involving multiple parameters

Fuzzy logic designs only the truth membership functions that give a description of the degree of membership value to a particular class. Contrary to fuzzy logic in which there is no provision of capturing indeterminacy corresponding to non-availability of information, or falsity functions to record the imprecision or degradation of the equipment with which input is captured, neutrosophic logic is a better representation of the medical data as it gives the clear insight of the truthiness, indeterminacy and falsity associated with the input captured. This should indeed be of considerable interest to the medical artificial intelligence community, because, as indicated above, medicine is essentially a continuous domain where the captured inputs could have uncertainty, indeterminacy and sometimes falsity associated [9].

The incorporation of neutrosophic logic in the medical models would retain the continuous behavior as displayed by the fuzzy logic. The medical domain is the field where there is indeterminacy, unknown, hidden parameters, imprecision, the high conflict between sources of information, and non-exhaustive or non-exclusive elements of the frame of discernment so neutrosophic could be applied. Similar to fuzzy systems, depending on the design of the rule base of the neutrosophic medical model, the output of such a system can be a continuous function (Sugeno model [10]) or it can be a single value output (Mamdani and Tsukamoto model [11], [12]). Generally, the continuous output function would be a better estimation of the modelled medical relationship than its underlying discrete specification. As suggested neutrosophic medical systems can be utilized for neutrosophic scores; continuous truth/indeterminate/falsity versions of conventional score schemes. The approach of incorporation of neutrosophic sets in the medical domain would lead to tabular or rule-based mapping from input to output variables effectively implementing a continuous control law. Neutrosophic qualitative simulation and, more generally, neutrosophic model-based diagnosis are promising candidates for future research. The proposed neutrosophic logic theory is not a substitute for existing fuzzy medical models, but an extension and enhancement of the classical AI approach. Due to the inherent advantages of neutrosophic sets, such systems would address medical problems more adequately as discussed in previous sections. What makes the inclusion of neutrosophic logic in the medical domain a powerful tool is its desirable properties of allowing continuity, gradation of reality, capturing of truthiness, indeterminacy, and falsity.

3.1 Miscellaneous Neutrosophic Works in Medicine

If we focused on the work of G. Shahzadi et al. [13] in medical diagnosis, we would find that they adapted and normalized the Hamming distance and Euclidean distance to be appropriate for medical diagnosis via distances between neutrosophic sets. They aimed to find an accurate diagnosis for three patients and gave the relation between neutrosophic sets for all symptoms of the $i-th$ patient from the $k-th$ diagnosis. the symptoms of the three patients were Temperature, Insulin, Blood pressure, Blood plates, Cough and finally, they were diagnosed with Diabetes, Dengue, and Tuberculosis. The readers can return to their article to see two algorithms with two different techniques, those two algorithms with fourteen tables of data enable the authors to identify that the first patient suffers from Dengue, the second patient suffers from Diabetes, and the last patient suffers from Tuberculosis. There are many other fields of the application area of neutrosophic logic in medicine, but not limited to, are as follows:

1. Managing malaria disease.
2. HIV infection cell determination.
3. Anaesthesia monitoring.
4. Image segmentation for tumours.
5. Lymph disease.
6. Monitoring and control in intensive care units.
7. Lung disease diagnosis.
8. Cancer risk prognosis.

The computer-based tools for medical decision-making help medical staff diagnose disease. One of these computer-based tools that ease medical decision-making is the neutrosophic expert system which has proven to be useful in the quantitative analysis and qualitative evaluation of medical data, by achieving the correctness of results.

The following Algorithm was used in the upcoming case study as an example published by M.N. Jafar et al [14],

Algorithm:

In the forthcoming steps, the authors describe a process used for medical diagnosis by neutrosophic soft sets (NSS), at the hypothesis that P° is a set of patients, $\check{S}$ is the set of symptoms, $D^{\sim}$ is the set of diseases.

The set of diseases related to their symptoms is obtained from the symptom-disease relation R_1.

The patient symptoms set has obtained the relation of the symptoms Q_1.

Evaluate their corresponding complement matrices $\widehat{R_2}$ and Q_2.

The relation between the patient symptoms and the disease matrices is T_1.

Compute relation T_2 called patient non-symptoms non-disease matrices.

Evaluate $\check{S}_{T_1}$ and $\check{S}_{T_2}$ neutrosophic soft sets by using the definition of (evaluation of neutrosophic soft sets).

Compute $\check{S}_k$, i.e. the higher value of the possibility of the patient suffering from that disease.

Using the above algorithm in the following case study:

Assume that the three patients P_1, P_2, P_3 in the hospital with symptoms of headache, temperature and severe pain are represented by c_1, c_2, c_3. Now consider $P^{\circ} = \{P_1, P_2, P_3\}$ represents the patients and $\check{S} = \{c_1, c_2, c_3.\}$ Shows the symptoms and $D^{\sim} = \{d_1^{\sim}, d_2^{\sim}, d_3^{\sim}\}$ shows the diseases: fever, typhoid, and malaria.

Solving the above case study using the mentioned algorithm, the authors conclude that there are the following possibilities for the patients suffering from the disease:

The patient P_1 has 0.85 possible suffering from fever, 0.9 suffering from typhoid, and 0.5 suffering from malaria. The patient P_2 has 0.9 possible suffering from fever, 0.8 possible suffering from typhoid, and 1 possible suffering from malaria which means the patient P_2 guaranteed suffering from malaria. The patient P_3 has 0.8 possible suffering from fever, 0.3 suffering from typhoid, and 0.7 suffering from malaria.

4. Conclusion

This article reviews the applications of the uncertainty mathematical tools/ neutrosophic logic in the medical domain. Dozens of papers and authors are interested in medical diagnosis, decision-making, pattern recognition studies, and performance comparison studies. The neutrosophic nature enables the computer programs' algorithms in medicine fields and healthcare to be flexible by taking into consideration all possible values including the indeterminate ones, and by making the process more robust when compared to traditional techniques with the feature of taking indeterminate data into consideration, and by efficiency by using more available data.

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Time for a New Player in Business Analytics: An MCDM Scheme Based on One-Dimensional Uncertain Linguistic Interval-Valued Neutrosophic Fuzzy Data

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Abstract: Pooling the opinions of the decision makers about premier alternative from a list of given choices is the most crucial phenomenon theory of decision making. The MSM operator is an effective approach and can identify the collective connection among various input viewpoints. So, tacking the fully benefit of MSM operator, this paper presents a new novel in depth investigation of MSM in context of Neutrosophic fuzzy variables. Initially we developed the structure of One-dimension uncertain linguistic interval valued neutrosophic fuzzy variables, deliberate their fundamental properties along with meaningful laws. Moreover, we encompass the MSM operator to the extent of one dimensional uncertain-linguistic Neutrosophic fuzzy variables and introduced some novel aggregations operators including one-dimensional uncertain linguistic interval-valued Neutrosophic fuzzy, weighted fuzzy and ordered weighted fuzzy McLaurin symmetric mean operator. The practical utilization of the proposed aggregation operators in business analytic are also included in the paper. In last comparison is made between current vs old studies which proved solidity and reliability of the new operator's vs existing operators in the literature.

Keywords: Linguistic numbers; Fuzzy variables; MSM; Aggregation operators; MCDM; Business analytic; Interval Valued fuzzy Data

1. Introduction

There exist imprecise, uncertain, and vague situations in real life, like in engineering, economics, trade, life, and social sciences. The techniques of classical mathematics are not fruitful to meet this problem. Therefore, a number of models, mainly including the theory of probability, fuzzy set [1] and its interval-valued extension [2], soft set (SS) [3], intuitionistic-fuzzy set [4], Pythagorean fuzzy-Set [5], linguistic term sets [6] and neutrosophic sets [7] with some of its modifications and extensions [8, 9,10] etc. have been presented to take in hand such situations.

In decision making strategies, indeterminacy is a substantial factor existent in theory of decision. In review forms, consider three assortments for sexual category: (1) Male (2) Female (3) Other. Therefore, the imprecision and uncertainty with indeterminacy cannot be described merely with the assistance intuitionistic and/or soft sets. Thus, Smarandache [7] coined the perception of neutrosophic sets (NSs). Beyond, NS comprises truth, false and indeterminacy membership functions. Currently, the research on the theory of NSs has been established vibrantly [11]. Later, NSs were hybridized with SSs to initiate neutrosophic soft set (NSS) [12]. Subsequently, several researchers have worked on this idea [12-16]. Wang et al. [17] commenced the conception of an IVN set and later Zhang et al. [18] applied this notion in theory of multiple attributes. Unlike IFS, centralized calculations are more adaptable as they have fewer limitations. Utilizing a neutrosophic set ensemble can effectively analyze uncertainty in datasets, particularly those of a significant scale. In addition, the concept behind this collection has developed and united with SSs to generate a new kind of set known as interval-valued neutrosophic set. Here are [19-25] also some significant characteristics of these sets when subjected to various algebraic operations.

For the sake of tracking down the finest alternative while solving MCDM issues and other related complications, one of the critical phases is aggregation whereby the views of the decision experts are required to be aggregated by using suitable technique of aggregation.

A straightforward way to understand this task is to see it as a MCDM issue, where the selection of suitable aggregation operators is crucial. The role of aggregation operators is like an appliance to pool the estimations of several decision makers into a collective one. In the decision-making area, there are many aggregation operators e.g., the arithmetic and geometric weighted average [26, 27]. Various strategies can be used in literature to combine numbers using different methodologies (Aggregation operators) in research areas. A method called intuitionistic fuzzy soft aggregation operator, developed by Garg and Arora [26], was created to combine information for decision making. Ye [27] introduced two innovative methods for combining data using trapezoidal intuitionistic fuzzy numbers. Awang et al. [28] and his colleagues proposed using the SVNWA operator as a method for combining information. Later, the notion of INWA is more explored and expanded by Huang et al. [29]. Zhang et al. [30] discussed the (INNWA) and Geometric-averaging operators. The primary emphasis of this paper is on resolving MCDM problems with heterogeneous priority levels for criteria, and it suggests employing neutrosophic uncertain linguistic factors to simplify the computation. The two eye catching operators included IVNWAA and IVNWGA was developed by Hussain et al. [31]. Aiwu et al., [32] put forward the IVNSGWA, a generalized weighted aggregation operator for neutrosophic aggregates incorporating interval values. Peng and Wang [33] introduced multiple aggregation operators for combining multi-valued neutral environments. Recently, Hamid et al. [34] presented multistage decision analysis in the framework of q-rung m-polar fuzzy setting. In [35], Naeem et al. employed generalized aggregation operators for medical diagnosis in Pythagorean fuzzy soft environment. Riaz et al. [36] made use of weighted aggregation operators for investment strategy using multipolar Pythagorean numbers. Ye [37] originated the idea of a (TNNWAA) and (TNNWGA) operator in the framework of trapezoidal Neutrosophic-number. Furthermore, a modified "TOPSIS" approach is introduced for handling trapezoidal opacification neutrosophic data. Jana and Pal [38] studied (SVNSWA) and (SVNSWGA) using single-valued neutrosophic soft environment.

Neutrosophic Frameworks with Applications can be explained in simpler terms as the use of a specific approach to analyze and solve problems. This approach considers three different perspectives - true, false, and indeterminate - and considers their interplay to come up with practical solutions. These frameworks can be applied in various fields and have practical uses in problem-solving and decision-making processes. Abdel et al. [39] discussed a strategy for managing rural water administration using a Coordinates Neutrosophic Territorial Administration Positioning Strategy. Hafeez et al. [40]. Developed the Neutrosophic MCDM Model to rank and evaluate different ways of managing healthcare waste to achieve better effectiveness and sustainability. Karak et al. [41] created a plan on how to solve transportation problems in a unique situation.

The MSM operator has gained the attention of the researchers working on decision making techniques from the past few years. Wang et al. [42] suggested single-Valued neutrosophic linguistic MSM aggregation operator by making use of operational laws of the underlying set. Wu et al. [43] presented some practical utilities of single valued neutrosophic linguistic set (SVNLS). In current, some generalized aggregation-operators have been efficaciously explored by many researchers and scholar across the globe. Some prominent operators are studied in [44-47]. Even though, the aggregation operators presented so far have far-flung applications in decision making, but sometimes there exist complex fuzzy information where these operators fail to work. According to the best knowledge of authors, no worth mentioning work has been done on the utility of one-dimensional uncertain linguistic interval-valued neutrosophic fuzzy variables.

To fill the gaps followings study is conducted and for smooth understanding of the notions presented: The basic ideas in Section-2 can be viewed as an introductory overview. In the very next section, operational laws, modified operational laws, expectancy, and some basic attributes of ODULIVNF variables are studied. Along with some novel operators meant for aggregation in this section have also been presented there. These operators include un-weighted, weighted, and ordered weighted one-dimensional uncertain linguistic interval-valued neutrosophic fuzzy MSM operator. An MCDM via proposed scheme as well as an algorithm is added in the section four. The same section also presents an MCDM model based upon one-dimension uncertain linguistic interval-valued neutrosophic fuzzy variables i.e., practical utility of the proposed aggregation operators in business analytics. Comparison study is of the current technique with some existing methodologies is also made part of the same section. Results and discussion are presented therein too. Finally, Section-5 concludes the article.

Motivation:

The MSM operator has gained attention of the researchers working on decision making techniques from last decade because of its valuable performance. In recent years, there have been new methods developed for integrating data that have proven to be effective. Amongst all, MSM operator is a prominent because this operator is empowered with the attribute of catching the conjoint association amongst the multiple input opinions. Even though, the aggregation operators based on MSM have far-flung applications in decision-making, but sometimes there exist complex fuzzy information where these operators fail to work. According to the best knowledge of authors, no worth mentioning work has been done on the utility of one-dimensional uncertain linguistic interval-valued neutrosophic fuzzy variables. To reduce such issues in the way of fuzzy theory and to fill this gap the following research is conducted.

Novelty:

The primary aim of this article is to meet this end by initiating the notion of one-dimensional uncertain linguistic interval-valued neutrosophic fuzzy variables. Accompanied by their various necessary characteristics. This model is equipped with the ability to take in hand uncertain statistics and moderate the complexity prevailing in the data. Operational laws modified operational laws, expectancy, and some basic attributes of these variables leaded us towards some novel MSM operators for aggregation. These operators include un-weighted, weighted, and ordered weighted one-dimensional uncertain linguistic interval-valued neutrosophic fuzzy MSM operators. The practical utility of the current aggregation operators is an influential in business analytics and other trade Markets.

2. Basic and Fundamental Concepts

From now onward, X will represent a non-void universe, unless stated otherwise.

Definition 2.1 [1] A collection of the form $T=\{(s,\mu_T(s)):s\in X\}$ with the map $\mu_T:T\to[0,1]$ declared the degree of belongness of elements to the set is called fuzzy set.

Example: Consider the reference set of students Y = {g1, g2, g3, g4}. Choose B = {(g1, 0.9) (g2, O.4) (g3, 0.8) (g4, 1)}. The set B indicates the degree of smartness. i.e., g1 is 0.9 and so on.

Definition 2.2 [4] By an IFS, we mean an assemblage of the form $\tilde{M}=\{(\tilde{x},\tilde{\mu}_{\tilde{M}}(\tilde{x}),\tilde{\nu}_{\tilde{M}}(\tilde{x})):\tilde{x}\in\tilde{X}\}$. The maps $\tilde{\nu}_{\tilde{M}},\tilde{\mu}_{\tilde{M}}:\tilde{M}\to[0,1]$ with sum equal to 1 and acknowledged belongness and non-belongness to a set.

Example: Consider B to be an IFS with νB(y) = 0.2 and μB(y) = 0.5 then, μB(x) = 0.3, πB(x) = 0.7, and ∂B(y) = 0.18. μB(y) = 0.5 and νB(y) = 0.2 show the belongness and not belongness of object to IFS respectively.

Definition 2.3 [7] A family of the form $\tilde{M}=\{(\tilde{x},\tilde{\eta}_{\tilde{M}}(\tilde{x}),\nu_{\tilde{M}}(\tilde{x}),\tilde{\mu}_{\tilde{M}}(\tilde{x})):\tilde{x}\in\tilde{X}\}$ is called a *neutrosophic set*. The maps $\tilde{\eta}_{\tilde{M}},\tilde{\nu}_{\tilde{M}},\tilde{\mu}_{\tilde{M}}:\tilde{T}\to\left]0^-,1^+\right[$ along with the constraint that their sum lies in $\left]0^-,3^+\right[$, and acknowledged value of truth, indeterminacy, and value of falsity in a set.

Example: Consider the triplet (0.3,0.1,0.2) ∈M, the degree of an in A is 0.3, 0.1, and 0.2 denotes the membership, indeterminacy, and non-membership respectively. Similarly, the element b (0.4,0.5,0.1) ∈M and the element C (0.1,0.2,0.7) ∈M with components sum equal to 1.

Definition 2.4 [17] An object of the form

$$\tilde{M}=\left\{\left(\tilde{x},\mu_{\tilde{M}}(\tilde{x})=\left[\mu^{L}_{\tilde{M}}(\tilde{x}),\mu^{U}_{\tilde{M}}(\tilde{x})\right],\eta_{\tilde{M}}(s)=\left[\eta^{L}_{\tilde{M}}(\tilde{x}),\eta^{U}_{\tilde{M}}(\tilde{x})\right],\nu_{\tilde{M}}(\tilde{x})=\left[\nu^{L}_{\tilde{M}}(\tilde{x}),\nu^{U}_{\tilde{M}}(\tilde{x})\right]\right):\tilde{x}\in\tilde{X}\right\}$$

is acknowledged as an interval-valued neutrosophic set. The maps $\eta_{\tilde{M}},\mu_{\tilde{M}},\nu_{\tilde{M}}:\tilde{M}\to\left]0^{-},1^{+}\right[$ along with the restriction known as "value of truth, indeterminacy and value of falsity, respectively".

Example: Consider M be an IVNS. The element m ((0.50-0.51), (0.10–0.15) ∪ [0.20–0.30], {0.20, 0.24, and 0.28}) ∈M. The membership is between 0.50-0.51 indeterminacy of m to M is between 0.10-0.15 or between 0.20-0.30 and non-membership is 0.20 or 0.24 or 0.28. limits don't exclude while actual approximation are not considered because of numerous sources.

Definition 2.5 [6] Assume that $S_{[0,h]}$ is a continuous linguistic-Term set (CLTs). A LIF set is defined as $\left\{\left(\hat{a},\hat{S}_{\hat{\alpha}}(\hat{a}),\hat{S}_{\hat{\beta}}(\hat{a})\right):\hat{a}\in\hat{M};\hat{S}_{\hat{\alpha}},\hat{S}_{\hat{\beta}}\in\hat{S}_{[0,r]}\right\}$ where $\hat{\alpha}+\hat{\beta}\in[0,r]$. $\hat{S}_{\alpha}$ and $\hat{S}_{\beta}$ are "Linguistic membership" and "non-membership values". The quantity $\hat{S}_{-\left(\hat{\alpha}+\hat{\beta}-r\right)}$ is acknowledged as degree of indeterminacy. The doublet $\left(\hat{S}_{\hat{\alpha}},\hat{S}_{\hat{\beta}}\right)$ is reckoned as linguistic intuitionistic fuzzy number (LIFN).

Example: Consider finite discrete ordered LT values by $\hat{S} = \{\hat{s}_0, \hat{s}_1, ..., \hat{s}_r\}$, where *r* is the even. E.g., for *r*=4, then the chosen linguistic term $\tilde{S}$ with following corresponding semantics is expressed as follows: " $\hat{S} = \begin{Bmatrix} \hat{s}_0, \\ \hat{s}_1, \\ \hat{s}_2, \\ \hat{s}_3 \end{Bmatrix} = \begin{Bmatrix} \hat{s}_0(Low), \\ \hat{s}_1(Slightly-low), \\ \hat{s}_2(Medium), \\ \hat{s}_3(Slightly-high) \end{Bmatrix}$

3. One-dimension uncertain linguistic Interval-valued neutrosophic fuzzy variables, operational laws, and their basic properties

Definition 3.1 $\widehat{S} = \left\langle \begin{matrix} [\widehat{s}_l, \widehat{s}_m], \\ [\widehat{s}_n, \widehat{s}_o], \\ [\widehat{s}_p, \widehat{s}_q] \end{matrix} \right\rangle$; $\widehat{S}$ is said to be one-dimension uncertain linguistic neutrosophic fuzzy variable with following condition. $\left\langle \widehat{s}_l, \widehat{s}_m, \widehat{s}_n, \widehat{s}_o, \widehat{s}_p, \widehat{s}_q \in \widehat{S} \right\rangle$ while $\langle l \le m \,; n \le o \text{ and } p \le q \rangle$ whereas $\left\langle \widehat{s}_l, \widehat{s}_n, \widehat{s}_p \right\rangle$ and $\left\langle \widehat{s}_m, \widehat{s}_o, \widehat{s}_q \right\rangle$ are the lower and upper limits.

Definition 3.2 If $\tilde{S}_1 = \left\langle \begin{matrix} \left[\tilde{s}_{l_1}, \tilde{s}_{m_1}\right], \\ \left[\tilde{s}_{n_1}, \tilde{s}_{o_1}\right], \\ \left[\tilde{s}_{p_1}, \tilde{s}_{q_1}\right] \end{matrix} \right\rangle$ and $\tilde{S}_2 = \left\langle \begin{matrix} \left[\tilde{s}_{l_2}, \tilde{s}_{m_2}\right], \\ \left[\tilde{s}_{n_2}, \tilde{s}_{o_2}\right], \\ \left[\tilde{s}_{p_2}, \tilde{s}_{q_2}\right] \end{matrix} \right\rangle$; Then operational rules for them are defined as fallows.

1:
$$\begin{aligned} \tilde{S}_1 \oplus \tilde{S}_2 &= \left(\left[\tilde{s}_{l_1}, \tilde{s}_{m_1}\right], \left[\tilde{s}_{n_1}, \tilde{s}_{o_1}\right], \left[\tilde{s}_{p_1}, \tilde{s}_{q_1}\right]\right) \oplus \left(\left[\tilde{s}_{l_2}, \tilde{s}_{m_2}\right], \left[\tilde{s}_{n_2}, \tilde{s}_{o_2}\right], \left[\tilde{s}_{p_2}, \tilde{s}_{q_2}\right]\right) \\ &= \left(\left[s_{l_1+l_2}, s_{m_1+m_2}\right], \left[s_{n_1+n_2}, s_{o_1+o_2}\right], \left[s_{p_1+p_2}, s_{q_1+q_2}\right]\right) \end{aligned} \tag{1}$$

2:
$$\begin{aligned} \tilde{S}_1 \otimes \tilde{S}_2 &= \left(\left[\tilde{s}_{l_1}, \tilde{s}_{m_1}\right], \left[\tilde{s}_{n_1}, \tilde{s}_{o_1}\right], \left[\tilde{s}_{p_1}, \tilde{s}_{q_1}\right]\right) \otimes \left(\left[\tilde{s}_{l_2}, \tilde{s}_{m_2}\right], \left[\tilde{s}_{n_2}, \tilde{s}_{o_2}\right], \left[\tilde{s}_{p_2}, \tilde{s}_{q_2}\right]\right) \\ &= \left(\left[s_{l_1 \times l_2}, s_{m_1 \times m_2}\right], \left[s_{n_1 \times n_2}, s_{o_1 \times o_2}\right], \left[s_{p_1 \times p_2}, s_{q_1 \times q_2}\right]\right) \end{aligned} \tag{2}$$

3: $\dfrac{\tilde{S}_1}{\tilde{S}_2} = \dfrac{\left(\left[\tilde{s}_{l_1}, \tilde{s}_{m_1}\right], \left[\tilde{s}_{n_1}, \tilde{s}_{o_1}\right], \left[\tilde{s}_{p_1}, \tilde{s}_{q_1}\right]\right)}{\left(\left[\tilde{s}_{l_2}, \tilde{s}_{m_2}\right], \left[\tilde{s}_{n_2}, \tilde{s}_{o_2}\right], \left[\tilde{s}_{p_2}, \tilde{s}_{q_2}\right]\right)} = \left(\left[\tilde{s}_{\frac{l_1}{l_2}}, \tilde{s}_{\frac{m_1}{m_2}}\right], \left[\tilde{s}_{\frac{n_1}{n_2}}, \tilde{s}_{\frac{o_1}{o_2}}\right], \left[\tilde{s}_{\frac{p_1}{p_2}}, \tilde{s}_{\frac{q_1}{q_2}}\right]\right)$ (3)

4: $k \otimes \tilde{S}_1 = k\tilde{S}_1 = \left[\tilde{s}_{l_1}, \tilde{s}_{m_1}\right], \left[\tilde{s}_{n_1}, \tilde{s}_{o_1}\right], \left[\tilde{s}_{p_1}, \tilde{s}_{q_1}\right] = \left(\left[\tilde{s}_{kl_1}, \tilde{s}_{km_1}\right], \left[\tilde{s}_{kn_1}, \tilde{s}_{ko_1}\right], \left[\tilde{s}_{kp_1}, \tilde{s}_{kq_1}\right]\right);\ \mathrm{k} \geq 0$ (4)

5: $\tilde{S}_1^k = \left(\left[\tilde{s}_{l_1}, \tilde{s}_{m_1}\right], \left[\tilde{s}_{n_1}, \tilde{s}_{o_1}\right], \left[\tilde{s}_{p_1}, \tilde{s}_{q_1}\right]\right)^k = \left(\left[\tilde{s}_{l_1^k}, \tilde{s}_{m_1^k}\right], \left[\tilde{s}_{n_1^k}, \tilde{s}_{o_1^k}\right], \left[\tilde{s}_{p_1^k}, \tilde{s}_{q_1^k}\right]\right);\ \mathrm{k} \geq 0$ (5)

Definition 3.3 Consider $\tilde{S} = \left\langle \begin{matrix} \left[\tilde{s}_l, \tilde{s}_m\right], \\ \left[\tilde{s}_n, \tilde{s}_o\right], \\ \left[\tilde{s}_p, \tilde{s}_q\right] \end{matrix} \right\rangle$; expectation of $\tilde{S}$ is stated as by eq#6.

$$E\left(\tilde{S}_1\right) = \frac{l+m}{6(\alpha-1)} \times \frac{n+o}{6(\beta-1)} \times \frac{p+q}{6(\gamma-1)} \tag{6}$$

Definition 3.4 Consider

$$\tilde{S}_1 = \left\langle \begin{matrix} \left[\tilde{s}_{l_1}, \tilde{s}_{m_1}\right], \\ \left[\tilde{s}_{n_1}, \tilde{s}_{o_1}\right], \\ \left[\tilde{s}_{p_1}, \tilde{s}_{q_1}\right] \end{matrix} \right\rangle \text{ and } \tilde{S}_2 = \left\langle \begin{matrix} \left[\tilde{s}_{l_2}, \tilde{s}_{m_2}\right], \\ \left[\tilde{s}_{n_2}, \tilde{s}_{o_2}\right], \\ \left[\tilde{s}_{p_2}, \tilde{s}_{q_2}\right] \end{matrix} \right\rangle;$$

as any two ODULNFVs. Then $\tilde{E}\left(\tilde{s}_1\right) \leq \tilde{E}\left(\tilde{s}_2\right)$ indicates expectancy of first number is lesser than expectancy of second number that is $\left\langle \tilde{S}_1 \leq \tilde{s}_2 \right\rangle$ or vice versa.

Definition 3.5 Let $\widehat{s}_1 = \left\langle \begin{matrix} \left[\tilde{s}_{l_1}, \tilde{s}_{m_1}\right], \\ \left[\tilde{s}_{n_1}, \tilde{s}_{o_1}\right], \\ \left[\tilde{s}_{p_1}, \tilde{s}_{q_1}\right] \end{matrix} \right\rangle$ and $\widehat{s}_2 = \left\langle \begin{matrix} \left[\tilde{s}_{l_2}, \tilde{s}_{m_2}\right], \\ \left[\tilde{s}_{n_2}, \tilde{s}_{o_2}\right], \\ \left[\tilde{s}_{p_2}, \tilde{s}_{q_2}\right] \end{matrix} \right\rangle$ be any two ONDLNFVs. Then, for three scalars $\left\langle \varphi, \dot{\varphi}\, and\, \ddot{\varphi} > 0 \right\rangle$ the following results hold: noted that $\langle\ \rangle$ is just a notation.

1) $\left\langle \widehat{s}_{\widehat{p}} \oplus \widehat{s}_{\widehat{q}} \right\rangle = \left\langle \widehat{s}_{\widehat{q}} \oplus \widehat{s}_{\widehat{p}} \right\rangle$ (7)

2) $\left\langle \widehat{s}_{\widehat{p}} \otimes \widehat{s}_{\widehat{q}} \right\rangle = \left\langle \widehat{s}_{\widehat{q}} \otimes \widehat{s}_{\widehat{p}} \right\rangle$ (8)

3) $\left\langle \varphi\left(\widehat{s}_{\widehat{p}} \oplus \widehat{s}_{\widehat{q}}\right) \right\rangle = \left\langle \varphi\widehat{s}_{\widehat{q}} \oplus \varphi\widehat{s}_{\widehat{p}} \right\rangle$ (9)

4) $\left\langle \varphi\widehat{s}_{\widehat{p}} \oplus \dot{\varphi}\widehat{s}_{\widehat{p}} \right\rangle = \left\langle \left(\varphi + \dot{\varphi}\right)\widehat{s}_{\widehat{p}} \right\rangle$ (10)

5) $\left\langle \left(\hat{s}_{\tilde{p}}\right)^{\varphi} \otimes \left(\hat{s}_{\tilde{p}}\right)^{\varphi} \right\rangle = \left\langle \left(\hat{s}_{\tilde{p}} \otimes \hat{s}_{\tilde{p}}\right)^{\varphi} \right\rangle$ (11)

6) $\left\langle \hat{s}_{\tilde{p}}^{\varphi} \otimes \hat{s}_{\tilde{p}}^{\dot{\varphi}} \right\rangle = \left\langle \hat{s}_{\tilde{p}}^{\varphi+\dot{\varphi}} \right\rangle$ (12)

3.1 One-dimension uncertain linguistic interval-valued neutrosophic fuzzy Maclaurin symmetric mean operators.

The MSM was originally developed by Maclaurin. MSM supports decision making by merging and evaluating information about various alternatives and their associations. MSM helps make decisions by combining information and thus widely used as a most beneficial trick to capture the "interrelationship among the multi-input values". By taking the fully command over MSM operator with utilizing the concept of ODULNFVs with its meaningful properties, now in this section we built up some new novel aggregation operators i.e., ODULIVNFMSM, ODULIVNWFMSM and ODULIVNOWFMSM.

Definition 3.1.1 consider $\tilde{S}_r = \left\langle \begin{array}{c} \left[s_{\tilde{a}_i}, s_{\tilde{b}_i} \right], \\ \left[s_{\tilde{c}_i}, s_{\tilde{d}_i} \right], \\ \left[s_{\tilde{e}_i}, s_{\tilde{f}_i} \right] \end{array} \right\rangle$ such that $\mathrm{r} = [1:1:n]$; be a non-empty collection of ODULNF variable. The w be a vector of

$\tilde{S}_r; \mathrm{r} = [1:1:n]$ with $w_t \in [0,1], t = 1,2,...,n;$ and $\sum_{t=1}^{n} w_t = 1.$ Then the defining relation below

$$\mathrm{ODULIVNWFMSM}^{(k)}\left(\tilde{S}_1, \tilde{S}_2, \tilde{S}_3, ..., \tilde{S}_n\right) = \left(\frac{\otimes_{1 \le t_1 < ... < t_k \le n} \left(\oplus_{r=1}^{k} \left(W_{t_r} \otimes \tilde{S}_{t_r} \right) \right)}{C_n^k} \right)^{\frac{1}{k}}; \tag{13}$$

Is called to be a WODULNFMSM operator. Where k tuples combination of {1…, n} of t is $\left(t_1, t_2, t_3, ..., t_k\right)$ and C_n^k denotes the binomial coefficient.

Theorem 3.1.2. Consider $\tilde{S}_r = \left(\left[s_{\tilde{a}_r}, s_{\tilde{b}_r}\right], \left[s_{\tilde{c}_r}, s_{\tilde{d}_r}\right], \left[s_{\tilde{e}_r}, s_{\tilde{f}_r}\right]\right); \mathrm{r} = [1:1:n]$; be a non-void assemblage of ODULNF variables. "w" is representing the weights of $\tilde{S}_r; \mathrm{r} = [1:1:n]$ with $w_t \in [0,1],\ \ t = 1, 2, ..., n;$ and $\sum_{t=1}^{n} w_t = 1$. Then the aggregated value is also a ODULNF variable and can be obtained as follow.

$$\mathrm{ODULIVNFMSM}^{(k)}\left(\tilde{S}_1, \tilde{S}_2, \tilde{S}_3, ..., \tilde{S}_n\right) =$$

$$
= \left(\begin{array}{l}
\left\langle \tilde{S}\left(\frac{\sum_{1\le t_1<\dots<t_k\le n}\prod_{r=1}^{k} w_{t_r}\tilde{a}_{t_r}}{C_n^k}\right)^{\left(\frac{1}{k}\right)}, \tilde{S}\left(\frac{\sum_{1\le j_1<\dots<j_k\le n}\prod_{i=1}^{k} w_{t_r}\tilde{b}_{t_r}}{C_n^k}\right)^{\left(\frac{1}{k}\right)} \right\rangle, \\
\left\langle \tilde{S}\left(\frac{1}{(\hat{\beta}-1)^{-1}}\left(1-\left[\prod\left(1\le t_1\le t_2\le\dots\le t_k\le n\right)\left(1-\frac{\prod_{r=1}^{n}\frac{1}{(\hat{\beta}-1)^{-1}}\left[1-\left(1-\frac{c_{t_r}}{\hat{\beta}-1}\right)^{w_{t_r}}\right]}{\left(\frac{1}{(\hat{\beta}-1)^{-k}}\right)}\right)\right]^{\frac{1}{C_n^k}}\right)\right)^{\left(\frac{1}{k}\right)}, \tilde{S}\left(\frac{1}{(\hat{\beta}-1)^{-1}}\left(1-\left[\prod\left(1\le t_1\le t_2\le\dots\le t_k\le n\right)\left(1-\frac{\prod_{r=1}^{n}\frac{1}{(\hat{\beta}-1)^{-1}}\left[1-\left(1-\frac{d_{t_r}}{\hat{\beta}-1}\right)^{w_{t_r}}\right]}{\left(\frac{1}{(\hat{\beta}-1)^{-k}}\right)}\right)\right]^{\frac{1}{C_n^k}}\right)\right)^{\left(\frac{1}{k}\right)} \right\rangle, \\
\left\langle \tilde{S}\left(\frac{1}{(\hat{\beta}-1)^{-1}}\left(1-\left[\prod\left(1\le t_1\le t_2\le\dots\le t_k\le n\right)\left(1-\frac{\prod_{r=1}^{n}\frac{1}{(\hat{\beta}-1)^{-1}}\left[1-\left(1-\frac{e_{t_r}}{\hat{\beta}-1}\right)^{w_{t_r}}\right]}{\left(\frac{1}{(\hat{\beta}-1)^{-k}}\right)}\right)\right]^{\frac{1}{C_n^k}}\right)\right)^{\left(\frac{1}{k}\right)}, \tilde{S}\left(\frac{1}{(\hat{\beta}-1)^{-1}}\left(1-\left[\prod\left(1\le t_1\le t_2\le\dots\le t_k\le n\right)\left(1-\frac{\prod_{r=1}^{n}\frac{1}{(\hat{\beta}-1)^{-1}}\left[1-\left(1-\frac{f_{t_r}}{\hat{\beta}-1}\right)^{w_{t_r}}\right]}{\left(\frac{1}{(\hat{\beta}-1)^{-k}}\right)}\right)\right]^{\frac{1}{C_n^k}}\right)\right)^{\left(\frac{1}{k}\right)} \right\rangle
\end{array} \right) \tag{14}
$$

Proof:

In accordance with the operational rules of ODULIVNF variable

$$\oplus_{r=1}^{k}\tilde{S}_{t_r}=\left(\begin{array}{c}\left\langle \tilde{S}_{\prod_{r=1}^{n}a_{t_r}},\tilde{S}_{\prod_{r=1}^{n}b_{t_r}}\right\rangle,\\ \left\langle \dfrac{\tilde{S}_{\prod_{r=1}^{n}c_{t_r}}}{\left(\dfrac{1}{\left(\widehat{\beta}-1\right)^{1-k}}\right)},\dfrac{\tilde{S}_{\prod_{r=1}^{n}d_{t_r}}}{\left(\dfrac{1}{\left(\widehat{\beta}-1\right)^{1-k}}\right)}\right\rangle,\\ \left\langle \dfrac{\tilde{S}_{\prod_{r=1}^{n}d_{t_r}}}{\left(\dfrac{1}{\left(\widehat{\beta}-1\right)^{1-k}}\right)},\dfrac{\tilde{S}_{\prod_{t=1}^{n}e_{t_r}}}{\left(\dfrac{1}{\left(\widehat{\beta}-1\right)^{1-k}}\right)}\right\rangle\end{array}\right)$$

$$\oplus_{r=1}^{k}w_{t_r}\tilde{S}_{t_r}=\left(\begin{array}{c}\left\langle \tilde{S}_{\left(\prod_{r=1}^{n}w_{t_r}a_{t_r}\right)},\tilde{S}_{\left(\prod_{r=1}^{n}w_{t_r}b_{t_r}\right)}\right\rangle,\\ \left\langle \dfrac{\tilde{S}_{\left(\prod_{r=1}^{n}w_{t_r}c_{t_r}\right)}}{\left(\dfrac{1}{\left(\widehat{\beta}-1\right)^{1-k}}\right)},\dfrac{\tilde{S}_{\left(\prod_{r=1}^{n}w_{t_r}d_{t_r}\right)}}{\left(\dfrac{1}{\left(\widehat{\beta}-1\right)^{1-k}}\right)}\right\rangle,\\ \left\langle \dfrac{\tilde{S}_{\left(\prod_{r=1}^{n}w_{t_r}d_{t_r}\right)}}{\left(\dfrac{1}{\left(\widehat{\beta}-1\right)^{1-k}}\right)},\dfrac{\tilde{S}_{\left(\prod_{r=1}^{n}w_{t_r}e_{t_r}\right)}}{\left(\dfrac{1}{\left(\widehat{\beta}-1\right)^{1-k}}\right)}\right\rangle\end{array}\right)$$

And

$$\bigoplus_{\left(1\le t_1\le t_2\le\ldots\le t_n\right)}\left(\oplus_{r=1}^{k} w_{t_r}\tilde{S}_{t_r}\right)=\left(\begin{array}{c}\left\langle \tilde{S}_{\sum_{\left(1\le t_1\le t_2\le\ldots\le t_n\right)}\prod_{r=1}^{n} w_{t_r}a_{t_r}},\tilde{S}_{\sum_{\left(1\le t_1\le t_2\le\ldots\le t_n\right)}\prod_{r=1}^{n} w_{t_r}b_{t_r}}\right\rangle,\\ \left\langle \tilde{S}_{\frac{\frac{1}{\left(\hat{\beta}-1\right)^{-1}}\left(1-\prod_{\left(1\le t_1\le t_2\le\ldots\le t_n\right)}\left(1-\frac{\prod_{r=1}^{n} w_{t_r}c_{t_r}}{\left(\hat{\beta}-1\right)}\right)\right)}{\left(\frac{1}{\left(\hat{\beta}-1\right)^{1-k}}\right)}},\tilde{S}_{\frac{\frac{1}{\left(\hat{\beta}-1\right)^{-1}}\left(1-\prod_{\left(1\le t_1\le t_2\le\ldots\le t_n\right)}\left(1-\frac{\prod_{r=1}^{n} w_{t_r}d_{t_r}}{\left(\hat{\beta}-1\right)}\right)\right)}{\left(\frac{1}{\left(\hat{\beta}-1\right)^{1-k}}\right)}}\right\rangle,\\ \left\langle \tilde{S}_{\frac{\frac{1}{\left(\hat{\beta}-1\right)^{-1}}\left(1-\prod_{1\le t_1\le t_2\le\ldots\le t_n}\left(1-\frac{\prod_{r=1}^{n} w_{t_r}e_{t_r}}{\left(\hat{\beta}-1\right)}\right)\right)}{\left(\frac{1}{\left(\hat{\beta}-1\right)^{1-k}}\right)}},\tilde{S}_{\frac{\frac{1}{\left(\hat{\beta}-1\right)^{-1}}\left(1-\prod_{1\le t_1\le t_2\le\ldots\le t_n}\left(1-\frac{\prod_{t=1}^{n} w_{t_r}f_{t_r}}{\left(\hat{\beta}-1\right)}\right)\right)}{\left(\frac{1}{\left(\hat{\beta}-1\right)^{1-k}}\right)}}\right\rangle\end{array}\right)$$

Then we obtain $\frac{1}{C_n^k}\left(\bigoplus_{\left(1\le t_1\le t_2\le\ldots\le t_n\right)}\left(\oplus_{r=1}^{k} w_{t_r}\tilde{S}_{t_r}\right)\right)=\left(\frac{\bigoplus_{\left(1\le t_1\le t_2\le\ldots\le t_n\right)}\left(\oplus_{r=1}^{k} w_{t_r}\tilde{S}_{t_r}\right)}{C_n^k}\right)^{\frac{1}{k}}$; Therefore

$$\text{ODULIVNFMSM}^{(k)}\left(\tilde{S}_1,\tilde{S}_2,\tilde{S}_3,\ldots,\tilde{S}_n\right)=\left(\frac{\bigoplus_{\left(1\le t_1\le t_2\le\ldots\le t_{k\le} n\right)}\left(\oplus_{r=1}^{k} w_{t_r}\tilde{S}_{t_r}\right)}{C_n^k}\right)^{\frac{1}{k}}=$$

$$
= \left(
\begin{array}{c}
\left\langle \tilde{s}_{\left(\frac{\sum_{(1\le t_1\le t_2\le\ldots\le t_k\le n)} \Pi_{r=1}^{k} w_{t_r}\tilde{a}_{t_r}}{C_n^k}\right)^{\left(\frac{1}{k}\right)}}, \tilde{s}_{\left(\frac{\sum_{(1\le t_1\le t_2\le\ldots\le t_k\le n)} \Pi_{r=1}^{k} w_{t_r}\tilde{b}_{t_r}}{C_n^k}\right)^{\left(\frac{1}{k}\right)}} \right\rangle \\
\left\langle \tilde{s}_{\frac{1}{(\hat{\beta}-1)^{-1}}\left(1-\left[\Pi_{(1\le t_1\le t_2\le\ldots\le t_k\le n)}\left(1-\frac{\Pi_{r=1}^{n}\frac{1}{(\hat{\beta}-1)^{-1}}\left[1-\left(1-\frac{c_{t_r}}{\hat{\beta}-1}\right)^{w_{t_r}}\right]}{\left(\frac{1}{(\hat{\beta}-1)^{-k}}\right)}\right)\right]^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}}, \tilde{s}_{\frac{1}{(\hat{\beta}-1)^{-1}}\left(1-\left[\Pi_{(1\le t_1\le t_2\le\ldots\le t_k\le n)}\left(1-\frac{\Pi_{r=1}^{n}\frac{1}{(\hat{\beta}-1)^{-1}}\left[1-\left(1-\frac{d_{t_r}}{\hat{\beta}-1}\right)^{w_{t_r}}\right]}{\left(\frac{1}{(\hat{\beta}-1)^{-k}}\right)}\right)\right]^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}} \right\rangle, \\
\left\langle \tilde{s}_{\frac{1}{(\hat{\beta}-1)^{-1}}\left(1-\left[\Pi_{(1\le t_1\le t_2\le\ldots\le t_k\le n)}\left(1-\frac{\Pi_{r=1}^{n}\frac{1}{(\hat{\beta}-1)^{-1}}\left[1-\left(1-\frac{e_{t_r}}{\hat{\beta}-1}\right)^{w_{t_r}}\right]}{\left(\frac{1}{(\hat{\beta}-1)^{-k}}\right)}\right)\right]^{\left(\frac{1}{C_n^k}\right)}\right)^{\frac{1}{k}}}, \tilde{s}_{\frac{1}{(\hat{\beta}-1)^{-1}}\left(1-\left[\Pi_{(1\le t_1\le t_2\le\ldots\le t_k\le n)}\left(1-\frac{\Pi_{r=1}^{n}\frac{1}{(\hat{\beta}-1)^{-1}}\left[1-\left(1-\frac{e_{t_r}}{\hat{\beta}-1}\right)^{w_{t_r}}\right]}{\left(\frac{1}{(\hat{\beta}-1)^{-k}}\right)}\right)\right]^{\left(\frac{1}{C_n^k}\right)}\right)^{\frac{1}{k}}} \right\rangle,
\end{array}
\right)
$$

Theorem 3.1.3. Idempotency

Consider $\breve{S}_r = \left\langle \begin{matrix} \left[\breve{s}_{\tilde{a}_r}, \breve{s}_{\tilde{b}_r}\right], \\ \left[\breve{s}_{\tilde{c}_r}, \breve{s}_{\tilde{d}_r}\right], \\ \left[\breve{s}_{\tilde{e}_r}, \breve{s}_{\tilde{f}_r}\right] \end{matrix} \right\rangle$ such that $\langle \mathrm{r} = 1, 2, 3, ..., n \rangle$; be a collection of ODULIVNF variable

then $ODULIVNFMSM^{(k)}\left(\breve{s}, \breve{s}, \breve{s}, ..., \breve{s}\right) = \breve{S}$.

Proof:

Since must write $ODULIVNFMSM^{(k)}\left(\breve{s}, \breve{s}, \breve{s}, ..., \breve{s}\right) = \breve{S}$

$$= \left[\begin{matrix} \left\langle \breve{s}_{\left(\frac{\sum_{(1\le t_1<...<t_k\le n)} \Pi_{i=1}^{k} \tilde{a}_{t_r}}{C_n^k}\right)^{\frac{1}{k}}}, \breve{s}_{\left(\frac{\sum_{(1\le t_1<...<t_k\le n)} \Pi_{i=1}^{k} \tilde{b}_{t_r}}{C_n^k}\right)^{\frac{1}{k}}} \right\rangle, \\ \left\langle \breve{s}_{\frac{1}{(\hat{\beta}-1)^{-1}}\left(1-\left[\Pi_{\left(1\le t_1<...<t_k\le n\right)}\left(1-\frac{\Pi_{i=1}^{n}\tilde{c}_{t_r}}{\frac{1}{(\hat{\beta}-1)^{-k}}}\right)\right]^{\left(\frac{1}{C_n^k}\right)}\right)^{\frac{1}{k}}}, \breve{s}_{\frac{1}{(\hat{\beta}-1)^{-1}}\left(1-\left[\Pi_{\left(1\le t_1<...<t_k\le n\right)}\left(1-\frac{\Pi_{i=1}^{n}\tilde{d}_{t_r}}{\frac{1}{(\hat{\beta}-1)^{-k}}}\right)\right]^{\left(\frac{1}{C_n^k}\right)}\right)^{\frac{1}{k}}} \right\rangle, \\ \left\langle \breve{s}_{\frac{1}{(\hat{\beta}-1)^{-1}}\left(1-\left[\Pi_{\left(1\le t_1<...<t_k\le n\right)}\left(1-\frac{\Pi_{i=1}^{n}\tilde{e}_{t_r}}{\frac{1}{(\hat{\beta}-1)^{-k}}}\right)\right]^{\left(\frac{1}{C_n^k}\right)}\right)^{\frac{1}{k}}}, \breve{s}_{\frac{1}{(q-1)^{-1}}\left(1-\left[\Pi_{\left(1\le t_1<...<t_k\le n\right)}\left(1-\frac{\Pi_{i=1}^{n}\tilde{f}_{t_r}}{\frac{1}{(\hat{\beta}-1)^{-k}}}\right)\right]^{\left(\frac{1}{C_n^k}\right)}\right)^{\frac{1}{k}}} \right\rangle \end{matrix} \right] \quad (16)$$

$$
=\left[\begin{array}{l}
\left\langle \breve{S}_{\left(\frac{C_n^k \tilde{a}^k}{C_n^k}\right)^{\frac{1}{k}}}, \breve{S}_{\left(\frac{C_n^k \tilde{b}^k}{C_n^k}\right)^{\frac{1}{k}}} \right\rangle, \\
\left\langle \breve{S}_{\frac{1}{\left(\hat{\beta}-1\right)^{-1}}\left(1-\left[\left(1-\frac{\tilde{c}^k}{\frac{1}{(\hat{\beta}-1)^{-k}}}\right)^{\frac{1}{C_n^k}}\right]^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}}, \breve{S}_{\frac{1}{\left(\hat{\beta}-1\right)^{-1}}\left(1-\left[\left(1-\frac{\tilde{d}^k}{\frac{1}{(\hat{\beta}-1)^{-k}}}\right)^{\frac{1}{C_n^k}}\right]^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}} \right\rangle, \\
\left\langle \breve{S}_{\frac{1}{\left(\hat{\beta}-1\right)^{-1}}\left(1-\left[\left(1-\frac{\tilde{e}^k}{\frac{1}{(\hat{\beta}-1)^{-k}}}\right)^{\frac{1}{C_n^k}}\right]^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}}, \breve{S}_{\frac{1}{\left(q-1\right)^{-1}}\left(1-\left[\left(1-\frac{\tilde{f}^k}{\frac{1}{(\hat{\beta}-1)^{-k}}}\right)^{\frac{1}{C_n^k}}\right]^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}} \right\rangle
\end{array}\right]
$$

$$= \left(\begin{array}{l} \left\langle \breve{S}_{\left(\tilde{a}^{k}\right)^{\frac{1}{k}}}, \breve{S}_{\left(\tilde{b}^{k}\right)^{\frac{1}{k}}} \right\rangle, \\ \left\langle \breve{S}_{\frac{1}{\left(\widehat{\beta}-1\right)^{-1}}\left(1-\left[\left(1-\frac{\tilde{c}^{k}}{\frac{1}{(\widehat{\beta}-1)^{-k}}}\right)\right]\right)^{\frac{1}{k}}}, \breve{S}_{\frac{1}{\left(\widehat{\beta}-1\right)^{-1}}\left(1-\left[\left(1-\frac{\tilde{d}^{k}}{\frac{1}{(\widehat{\beta}-1)^{-k}}}\right)\right]\right)^{\frac{1}{k}}} \right\rangle, \\ \left[\breve{S}_{\frac{1}{\left(\widehat{\beta}-1\right)^{-1}}\left(1-\left[\left(1-\frac{\tilde{e}^{k}}{\frac{1}{(\widehat{\beta}-1)^{-k}}}\right)\right]\right)^{\frac{1}{k}}}, \breve{S}_{\frac{1}{\left(\widehat{\beta}-1\right)^{-1}}\left(1-\left[\left(1-\frac{\tilde{f}^{k}}{\frac{1}{(\widehat{\beta}-1)^{-k}}}\right)\right]\right)^{\frac{1}{k}}} \right] \end{array} \right)$$

$$= = \left\langle \begin{matrix} \left[s_{\tilde{a}}, s_{\tilde{b}} \right], \\ \left[s_{\tilde{c}}, s_{\tilde{d}} \right], \\ \left[s_{\tilde{e}}, s_{\tilde{f}} \right] \end{matrix} \right\rangle$$

Which is required proof.

Theorem 3.1.4: Commutativity

Let us consider $(\tilde{s}_1, \tilde{s}_2, ..., \tilde{s}_n)$ is any permutation of $(\hat{s}_1, \hat{s}_2, ..., \hat{s}_n)$

$$ODULIVNFMSM^{(k)}(\tilde{s}_1, \tilde{s}_2, ..., \tilde{s}_n) = ODULIVNFMSM^{(k)}(\hat{s}_1, \hat{s}_2, ..., \hat{s}_n)$$

Proof:

Since $ODULIVNFMSM^{(k)}(\hat{s}_1, \hat{s}_2, ..., \hat{s}_n) = \left(\frac{\sum_{(1 \le t_1 < ... < t_k \le n)} \Pi_{r=1}^{k} \tilde{s}_{t_r}}{C_n^k} \right)^{\frac{1}{k}} = \left(\frac{\sum_{(1 \le t_1 < ... < t_k \le n)} \Pi_{r=1}^{k} \hat{s}_{t_r}}{C_n^k} \right)^{\frac{1}{k}} =$

$ODULIVNFMSM^{(k)}(\hat{s}_1, \hat{s}_2, ..., \hat{s}_n)$. Which is required result.

Theorem 3.1.5: Monotonicity

If $\breve{a}_{t_r} \ge a_{t_r}, \breve{b}_{t_r} \ge b_{t_r}, \breve{c}_{t_r} \ge c_{t_r}, \breve{d}_{t_r} \ge d_{t_r}, \breve{e}_{t_r} \ge e_{t_r}$ and $\breve{f}_{t_r} \ge f_{t_r}$. for all r,

$$ODULIVNFMSM^{(k)}(\breve{s}_1, \breve{s}_2, \breve{s}_3, ..., \breve{s}_n) = ODULIVNFMSM^{(k)}(\hat{s}_1, \hat{s}_2, ..., \hat{s}_n)$$

Proof:

Since $\mathrm{ODULIVNFMSM}^{(k)}(\breve{s}_1, \breve{s}_2, \breve{s}_3, ..., \breve{s}_n) = \left(\frac{\sum_{(1 \le t_1 < ... < t_k \le n)} \Pi_{r=1}^{k} \breve{s}_{t_r}}{C_n^k} \right)^{\frac{1}{k}}$

$$= \left[\begin{array}{c} \left\langle \tilde{s}_{\left(\frac{\sum_{(1\le t_1<\ldots<t_k\le n)}\Pi_{r=1}^{k}\breve{a}}{C_n^k}\right)^{\frac{1}{k}}}, \tilde{s}_{\left(\frac{\sum_{(1\le t_1<\ldots<t_k\le n)}\Pi_{r=1}^{k}\breve{b}}{C_n^k}\right)^{\frac{1}{k}}} \right\rangle \\ \left\langle \tilde{s}_{\left(\frac{1}{(\hat{\beta}-1)^{-1}}\left(1-\left[\Pi_{(1\le t_1<\ldots<t_k\le n)}\left(1-\frac{\Pi_{r=1}^{n}\breve{c}}{\left(\frac{1}{(\hat{\beta}-1)^{-k}}\right)}\right)\right]^{\frac{1}{C_n^k}}\right)\right)^{\frac{1}{k}}}, \tilde{s}_{\left(\frac{1}{(\hat{\beta}-1)^{-1}}\left(1-\left[\Pi_{(1\le t_1<\ldots<t_k\le n)}\left(1-\frac{\Pi_{r=1}^{n}\breve{d}}{\left(\frac{1}{(\hat{\beta}-1)^{-k}}\right)}\right)\right]^{\frac{1}{C_n^k}}\right)\right)^{\frac{1}{k}}} \right\rangle, \\ \left\langle \tilde{s}_{\left(\frac{1}{(\hat{\beta}-1)^{-1}}\left(1-\left[\Pi_{(1\le t_1<\ldots<t_k\le n)}\left(1-\frac{\Pi_{r=1}^{n}\breve{e}}{\left(\frac{1}{(\hat{\beta}-1)^{-k}}\right)}\right)\right]^{\frac{1}{C_n^k}}\right)\right)^{\frac{1}{k}}}, \tilde{s}_{\left(\frac{1}{(\hat{\beta}-1)^{-1}}\left(1-\left[\Pi_{(1\le t_1<\ldots<t_k\le n)}\left(1-\frac{\Pi_{r=1}^{n}\breve{f}}{\left(\frac{1}{(\hat{\beta}-1)^{-k}}\right)}\right)\right]^{\frac{1}{C_n^k}}\right)\right)^{\frac{1}{k}}} \right\rangle \end{array} \right] \quad (17)$$

If $\breve{a}_{t_r} \geq a_{t_r}, \breve{b}_{t_r} \geq b_{t_r}, \breve{c}_{t_r} \geq c_{t_r}, \breve{d}_{t_r} \geq d_{t_r}, \breve{e}_{t_r} \geq e_{t_r}$ and $\breve{f}_{t_r} \geq f_{t_r}$.

Since $\left\langle \breve{a}_{t_r} \geq a_{t_r} \right\rangle$, then

$$\left(\frac{1}{C_n^k}\left(\sum_{(1\leq t_1<\ldots<t_k\leq n)}\Pi_{r=1}^{k}\breve{a}_{t_r}\right)\right)^{\frac{1}{k}} \geq \left(\frac{1}{C_n^k}\left(\sum_{(1\leq t_1<\ldots<t_k\leq n)}\Pi_{r=1}^{k}a_{t_r}\right)\right)^{\frac{1}{k}}$$

Same for $\left\langle \breve{b}_{t_r} \geq b_{t_r} \right\rangle$ we must write,

$$\left(\frac{1}{C_n^k}\sum_{1\leq t_1<\ldots<t_k\leq n}\Pi_{r=1}^{k}\breve{b}_{t_r}\right)^{\frac{1}{k}} \geq \left(\frac{1}{C_n^k}\sum_{1\leq t_1<\ldots<t_k\leq n}\Pi_{r=1}^{k}b_{t_r}\right)^{\frac{1}{k}}$$

Since $\left\langle \breve{c}_{t_r} \geq c_{t_r} \right\rangle$, then $\left\langle \frac{\left(\Pi_{r=1}^{n}\breve{c}_{t_r}\right)}{\left(\frac{1}{(\hat{\beta}-1)^{-k}}\right)} \right\rangle \geq \left\langle \frac{\left(\Pi_{r=1}^{n}c_{t_r}\right)}{\left(\frac{1}{(\hat{\beta}-1)^{-k}}\right)} \right\rangle$ and

$$\left[\Pi_{(1\leq t_1<\ldots<t_k\leq n)}\left(1-\frac{\Pi_{r=1}^{n}\breve{c}}{\left(\frac{1}{(\hat{\beta}-1)^{-k}}\right)}\right)\right]^{\frac{1}{C_n^k}} \leq \left[\Pi_{(1\leq t_1<\ldots<t_k\leq n)}\left(1-\frac{\Pi_{r=1}^{n}c}{\left(\frac{1}{(\hat{\beta}-1)^{-k}}\right)}\right)\right]^{\frac{1}{C_n^k}}$$

$$\Rightarrow 1-\left[\Pi_{(1\leq t_1<\ldots<t_k\leq n)}\left(1-\frac{\Pi_{r=1}^{n}\breve{c}}{\left(\frac{1}{(\hat{\beta}-1)^{-k}}\right)}\right)\right]^{\frac{1}{C_n^k}} \geq 1-\left[\Pi_{(1\leq t_1<\ldots<t_k\leq n)}\left(1-\frac{\Pi_{r=1}^{n}c}{\left(\frac{1}{(\hat{\beta}-1)^{-k}}\right)}\right)\right]^{\frac{1}{C_n^k}}$$

Then

$$\left\langle \left(\frac{1}{(\hat{\beta}-1)^{-1}}\right)\left(1-\left[\Pi_{(1\leq t_1<\ldots<t_k\leq n)}\left(1-\frac{\Pi_{r=1}^{n}\breve{c}}{\left(\frac{1}{(\hat{\beta}-1)^{-k}}\right)}\right)\right]^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}} \right\rangle \geq \left\langle \left(\frac{1}{(\hat{\beta}-1)^{-1}}\right)\left(1-\left[\Pi_{(1\leq t_1<\ldots<t_k\leq n)}\left(1-\frac{\Pi_{r=1}^{n}c}{\left(\frac{1}{(\hat{\beta}-1)^{-k}}\right)}\right)\right]^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}} \right\rangle$$

Similarly, for $\breve{d}_{t_r} \geq d_{j_i}, \breve{e}_{t_r} \geq e_{t_r}$ and $\breve{f}_{t_r} \geq f_{t_r}$

$$\left\langle \left(\frac{1}{(\hat{\beta}-1)^{-1}}\right)\left(1-\left[\Pi_{(1\leq t_1<\ldots<t_k\leq n)}\left(1-\frac{\Pi_{r=1}^{n}\breve{d}}{\left(\frac{1}{(\hat{\beta}-1)^{-k}}\right)}\right)\right]^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}} \right\rangle \geq \left\langle \left(\frac{1}{(\hat{\beta}-1)^{-1}}\right)\left(1-\left[\Pi_{(1\leq t_1<\ldots<t_k\leq n)}\left(1-\frac{\Pi_{r=1}^{n}d}{\left(\frac{1}{(\hat{\beta}-1)^{-k}}\right)}\right)\right]^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}} \right\rangle$$

$$\left\langle \left(\frac{1}{(\hat{\beta}-1)^{-1}}\right)\left(1-\left[\Pi_{(1\le t_1<\ldots<t_k\le n)}\left(1-\frac{\Pi_{r=1}^{n}\breve{e}}{\left(\frac{1}{(\hat{\beta}-1)^{-k}}\right)}\right)\right]^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}\right\rangle \ge \left\langle \left(\frac{1}{(\hat{\beta}-1)^{-1}}\right)\left(1-\left[\Pi_{(1\le t_1<\ldots<t_k\le n)}\left(1-\frac{\Pi_{r=1}^{n}e}{\left(\frac{1}{(\hat{\beta}-1)^{-k}}\right)}\right)\right]^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}\right\rangle$$

$$\left\langle \left(\frac{1}{(\hat{\beta}-1)^{-1}}\right)\left(1-\left[\Pi_{(1\le t_1<\ldots<t_k\le n)}\left(1-\frac{\Pi_{r=1}^{n}\breve{f}}{\left(\frac{1}{(\hat{\beta}-1)^{-k}}\right)}\right)\right]^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}\right\rangle \ge \left\langle \left(\frac{1}{(\hat{\beta}-1)^{-1}}\right)\left(1-\left[\Pi_{(1\le t_1<\ldots<t_k\le n)}\left(1-\frac{\Pi_{r=1}^{n}f}{\left(\frac{1}{(\hat{\beta}-1)^{-k}}\right)}\right)\right]^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}\right\rangle$$

Using # 3.3, we have

$$E\left(\tilde{s}_r\right)=\left\langle \frac{\left(\breve{a}_r+\breve{b}_r\right)}{6(p-1)}\times\frac{\left(\tilde{c}_r+\breve{d}_r\right)}{6(q-1)}\times\frac{\left(\breve{e}_r+\breve{f}_r\right)}{6(r-1)}\right\rangle \tag{18}$$

And

$$E\left(\dot{s}_r\right)=\left\langle \frac{\left(a_r+b_r\right)}{6(p-1)}\times\frac{\left(c_r+d_r\right)}{6(q-1)}\times\frac{\left(e_r+f_r\right)}{6(r-1)}\right\rangle \tag{19}$$

And then by definition 3.4 we can write $E\left(\tilde{s}_r\right)\ge E\left(\dot{s}_r\right)$. So, finally we have

$$ODULIVNFMSM^{(k)}\left(\breve{s}_1,\breve{s}_2,\breve{s}_3,\ldots,\breve{s}_n\right)=ODULIVNFMSM^{(k)}\left(\hat{s}_1,\hat{s}_2,\ldots,\hat{s}_n\right).$$

Which is the required result.

Theorem 3.1.6: Boundedness

If $\begin{bmatrix}\left\langle \breve{s}^- = \min\left(\breve{s}_1^-,\breve{s}_2^-,\ldots,\breve{s}_n^-\right)\right\rangle \\ \left\langle \breve{s}^+ = \max\left(\breve{s}_1^+,\breve{s}_2^+,\ldots,\breve{s}_n^+\right)\right\rangle\end{bmatrix}$, then $\breve{S}^- \le ODULNFMSM^{(k)}\left(\breve{s}_1,\breve{s}_2,\ldots,\breve{s}_n\right)\le \breve{S}^+$

Proof:

Suppose $\begin{bmatrix}\left\langle \breve{s}^- = \min\left(\breve{s}_1^-,\breve{s}_2^-,\ldots,\breve{s}_n^-\right)\right\rangle \\ \left\langle \breve{s}^+ = \max\left(\breve{s}_1^+,\breve{s}_2^+,\ldots,\breve{s}_n^+\right)\right\rangle\end{bmatrix}$. According to the above theorem, we have

$ODULIVNFMSM^{(k)}\left(\breve{s}^-,\breve{s}^-,\ldots,\breve{s}^-\right)\le ODULIVNFMSM^{(k)}\left(\breve{s},\breve{s},\ldots,\breve{s}\right)\le ODULIVNFMSM^{(k)}\left(\breve{S}^+,\breve{S}^+,\ldots,\breve{S}^+\right)$ Now, according to theorem 3.1.3 we have,

Thus,

$$s^- \le ODULNFMSM^{(k)}\left(\tilde{s}_1,\tilde{s}_2,\tilde{s}_3,\ldots,\tilde{s}_n\right)\le s^+.$$

Which is required result.

3.2 Some special k-based feature of ODULNFMSM operator

(1): By taking k=1, the ODULIVNFMSM operator took the form of ODULIVNF arithmetic operator.

$$ODULIVNFMSM^{(1)}\left(\tilde{s}_1,\tilde{s}_2,\tilde{s}_3,\ldots,\tilde{s}_n\right)=$$

$$\left[\begin{array}{l}\left\langle \tilde{S}_{\frac{\sum_{i=1}^{n}\tilde{a}_i}{n}},\tilde{S}_{\frac{\sum_{i=1}^{n}\tilde{b}_i}{n}}\right\rangle, \\ \left\langle \tilde{S}_{\left(\frac{1}{(\hat{\beta}-1)^{-1}}\right)\left(1-\left[\Pi_{j=1}^{n}\left(1-\frac{\tilde{c}_i}{\left(\frac{1}{(\hat{\beta}-1)^{-k}}\right)}\right)\right]\right)^{1/n}},\tilde{S}_{\left(\frac{1}{(\hat{\beta}-1)^{-1}}\right)\left(1-\left[\Pi_{j=1}^{n}\left(1-\frac{\breve{d}_i}{\left(\frac{1}{(\hat{\beta}-1)^{-k}}\right)}\right)\right]\right)^{1/n}}\right\rangle, \\ \left\langle \tilde{S}_{\left(\frac{1}{(\hat{\beta}-1)^{-1}}\right)\left(1-\left[\Pi_{j=1}^{n}\left(1-\frac{\breve{e}_i}{\left(\frac{1}{(\hat{\beta}-1)^{-k}}\right)}\right)\right]\right)^{1/n}},\tilde{S}_{\left(\frac{1}{(\hat{\beta}-1)^{-1}}\right)\left(1-\left[\Pi_{j=1}^{n}\left(1-\frac{\breve{f}_i}{\left(\frac{1}{(\hat{\beta}-1)^{-k}}\right)}\right)\right]\right)^{1/n}}\right\rangle\end{array}\right] \tag{20}$$

(2): By taking k=2, the ODULNFMSM operator took the form of ODULIVNFB operator.

$$\mathrm{ODULIVNFMSM}^{(1)}\left(\tilde{S}_1,\tilde{S}_2,\tilde{S}_3,\ldots,\tilde{S}_n\right)=$$

$$\left[\begin{array}{l}\left\langle \tilde{s}_{\left(\frac{\sum_{(1\le j_1<\ldots<j_k\le n)}\Pi_{i=1}^{k}\tilde{a}_{j_i}}{C_n^2}\right)^{\frac{1}{2}}},\tilde{s}_{\left(\frac{\sum_{(1\le j_1<\ldots<j_k\le n)}\Pi_{i=1}^{k}\tilde{b}_{j_i}}{C_n^2}\right)^{\frac{1}{2}}}\right\rangle, \\ \left\langle \tilde{s}_{\left(\frac{1}{(\hat{\beta}-1)^{-1}}\right)\left(1-\left[\Pi_{(1\le j_1<\ldots<j_k\le n)}\left(1-\frac{\Pi_{i=1}^{n}\tilde{c}_{j_i}}{\left(\frac{1}{(\hat{\beta}-1)^{-k}}\right)}\right)\right]^{\frac{1}{C_n^2}}\right)^{\frac{1}{2}}},\tilde{s}_{\left(\frac{1}{(\hat{\beta}-1)^{-1}}\right)\left(1-\left[\Pi_{(1\le j_1<\ldots<j_k\le n)}\left(1-\frac{\Pi_{i=1}^{n}\tilde{d}_{j_i}}{\left(\frac{1}{(\hat{\beta}-1)^{-k}}\right)}\right)\right]^{\frac{1}{C_n^2}}\right)^{\frac{1}{2}}}\right\rangle, \\ \left\langle \tilde{s}_{\left(\frac{1}{(\hat{\beta}-1)^{-1}}\right)\left(1-\left[\Pi_{(1\le j_1<\ldots<j_k\le n)}\left(1-\frac{\Pi_{i=1}^{n}\tilde{e}_{j_i}}{\left(\frac{1}{(\hat{\beta}-1)^{-k}}\right)}\right)\right]^{\frac{1}{C_n^2}}\right)^{\frac{1}{2}}},\tilde{s}_{\left(\frac{1}{(\hat{\beta}-1)^{-1}}\right)\left(1-\left[\Pi_{(1\le j_1<\ldots<j_k\le n)}\left(1-\frac{\Pi_{i=1}^{n}\tilde{f}_{j_i}}{\left(\frac{1}{(\hat{\beta}-1)^{-k}}\right)}\right)\right]^{\frac{1}{C_n^2}}\right)^{\frac{1}{2}}}\right\rangle\end{array}\right] \tag{21}$$

$$=\mathrm{ODULIVNFMSM}^{(2)}\left(\tilde{s}_1,\tilde{s}_2,\tilde{s}_3,\ldots,\tilde{s}_n\right).$$

(3): By taking k=n, the ODULNFMSM operator took the form of ODULNFM operator (p=1, q=1).

$$\text{ODULIVNFMSM}^{(n)}\left(\tilde{S}_1,\tilde{S}_2,\tilde{S}_3,\ldots,\tilde{S}_n\right)=\begin{bmatrix}\left\langle \tilde{s}_{\left(\Pi_{i=1}^{k}\tilde{a}_{j_i}\right)^{\frac{1}{n}}},\tilde{s}_{\left(\Pi_{i=1}^{k}\tilde{b}_{j_i}\right)^{\frac{1}{n}}}\right\rangle,\\ \left\langle \tilde{s}_{\left(\Pi_{i=1}^{k}\tilde{c}_{j_i}\right)^{\frac{1}{n}}},\tilde{s}_{\left(\Pi_{i=1}^{k}\tilde{d}_{j_i}\right)^{\frac{1}{n}}}\right\rangle,\\ \left\langle \tilde{s}_{\left(\Pi_{i=1}^{k}\tilde{e}_{j_i}\right)^{\frac{1}{n}}},\tilde{s}_{\left(\Pi_{i=1}^{k}\tilde{f}_{j_i}\right)^{\frac{1}{n}}}\right\rangle\end{bmatrix}$$

3.3 One-dimension uncertain linguistic interval valued neutrosophic weighted fuzzy MSM aggregation operator (ODULIVNWFMSM)

Def: 3.3.1. Consider $\tilde{S}_r=\left\langle\begin{matrix}\left[s_{\tilde{a}_r},s_{\tilde{b}_r}\right],\\ \left[s_{\tilde{c}_r},s_{\tilde{d}_r}\right],\\ \left[s_{\tilde{e}_r},s_{\tilde{f}_r}\right]\end{matrix}\right\rangle$ such that $\text{r}=[1:1:n]$; be a non-empty collection of ODULIVNF variable and w denoting the weight values that is.

$\tilde{S}_r$; with $w_t\in[0,1], t=1,2,\ldots,n;$ and $\sum_{t=1}^{n}w_t=1.$

If $$\text{WODULNFMSM}^{(k)}\left(\tilde{S}_1,\tilde{S}_2,\tilde{S}_3,\ldots,\tilde{S}_n\right)=\left(\frac{\otimes_{1\le t_1<\ldots<t_k\le n}\left(\oplus_{r=1}^{k}\left(W_{t_r}\otimes\tilde{S}_{t_r}\right)\right)}{C_n^k}\right)^{\frac{1}{k}} \tag{22}$$

Then ODULIVNFMSM is said to be ODULIVNWFMSM operator. Where k tuples combination of 1,…,n of t is $\left(t_1,t_2,t_3,\ldots,t_k\right)$ and C_n^k denotes the binomial coefficient.

Theorem 3.3.2. Consider $\tilde{S}_r=\left\langle\begin{matrix}\left[s_{\tilde{a}_r},s_{\tilde{b}_r}\right],\\ \left[s_{\tilde{c}_r},s_{\tilde{d}_r}\right],\\ \left[s_{\tilde{e}_r},s_{\tilde{f}_r}\right]\end{matrix}\right\rangle$ such that $\text{r}=[1:1:n]$; be a non-empty collection of ODULIVNF variable and w denoting the Weight values of $\tilde{S}_r$; $w_t\in[0,1], t=1,2,\ldots,n;$ and $\sum_{t=1}^{n}w_t=1.$ Then ODULIVNFN can be obtained as fallows.

$$ODULIVNWFMSM^{(k)}\left(\tilde{S}_1,\tilde{S}_2,\tilde{S}_3,\ldots,\tilde{S}_n\right)=$$

$$
= \left\langle
\begin{array}{l}
\left\langle \tilde{S}\left(\frac{\Sigma \left(1 \le t_1 < \ldots < t_k \le n\right) \Pi_{r=1}^{k} w_{t_r} \tilde{a}_{t_r}}{C_n^k} \right)^{\frac{1}{k}}, \tilde{S}\left(\frac{\Sigma \left(1 \le j_1 < \ldots < j_k \le n\right) \Pi_{i=1}^{k} w_{t_r} \tilde{b}_{t_r}}{C_n^k} \right)^{\frac{1}{k}} \right\rangle, \\
\left\langle \tilde{S} \left(\left(\frac{1}{(\hat{\beta}-1)^{-1}}\right) \left[1 - \Pi \left(1 \le t_1 \le t_2 \le \ldots \le t_k \le n\right) \left(1 - \frac{\Pi_{r=1}^{n} \left(\frac{1}{(\hat{\beta}-1)^{-1}}\right) \left[\left(1 - \left(1 - \frac{c_{t_r}}{\left(\hat{\beta}-1\right)}\right)\right)^{w_{t_r}} \right]}{\left(\frac{1}{(\hat{\beta}-1)^{-k}}\right)} \right) \right]^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}}, \tilde{S} \left(\left(\frac{1}{(\hat{\beta}-1)^{-1}}\right) \left[1 - \Pi \left(1 \le t_1 \le t_2 \le \ldots \le t_k \le n\right) \left(1 - \frac{\Pi_{r=1}^{n} \left(\frac{1}{(\hat{\beta}-1)^{-1}}\right) \left[\left(1 - \left(1 - \frac{d_{t_r}}{\left(\hat{\beta}-1\right)}\right)\right)^{w_{t_r}} \right]}{\left(\frac{1}{(\hat{\beta}-1)^{-k}}\right)} \right) \right]^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}} \right\rangle, \\
\left\langle \tilde{S} \left(\left(\frac{1}{(\hat{\beta}-1)^{-1}}\right) \left[1 - \Pi \left(1 \le t_1 \le t_2 \le \ldots \le t_k \le n\right) \left(1 - \frac{\Pi_{r=1}^{n} \left(\frac{1}{(\hat{\beta}-1)^{-1}}\right) \left[\left(1 - \left(1 - \frac{e_{t_r}}{\left(\hat{\beta}-1\right)}\right)\right)^{w_{t_r}} \right]}{\left(\frac{1}{(\hat{\beta}-1)^{-k}}\right)} \right) \right]^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}}, \tilde{S} \left(\left(\frac{1}{(\hat{\beta}-1)^{-1}}\right) \left[1 - \Pi \left(1 \le t_1 \le t_2 \le \ldots \le t_k \le n\right) \left(1 - \frac{\Pi_{r=1}^{n} \left(\frac{1}{(\hat{\beta}-1)^{-1}}\right) \left[\left(1 - \left(1 - \frac{f_{t_r}}{\left(\hat{\beta}-1\right)}\right)\right)^{w_{t_r}} \right]}{\left(\frac{1}{(\hat{\beta}-1)^{-k}}\right)} \right) \right]^{\frac{1}{C_n^k}} \right)^{\frac{1}{k}} \right\rangle
\end{array}
\right\rangle \tag{23}
$$

Proof:

From the point of view of operational rules of ODULIVNF Variables, we must write

$$\oplus_{r=1}^{k}\left(\tilde{S}_{t_r}\right)=\left(\begin{array}{l}\left\langle \tilde{s}_{\prod_{r=1}^{n} a_{t_r}}, \tilde{s}_{\prod_{r=1}^{n} b_{t_r}} \right\rangle, \\ \left\langle \dfrac{\tilde{s}_{\prod_{r=1}^{n} c_{t_r}}}{\left(\dfrac{1}{\left(\widehat{\beta}-1\right)^{1-k}}\right)}, \dfrac{\tilde{s}_{\prod_{r=1}^{n} d_{t_r}}}{\left(\dfrac{1}{\left(\widehat{\beta}-1\right)^{1-k}}\right)} \right\rangle, \\ \left\langle \dfrac{\tilde{s}_{\prod_{r=1}^{n} e_{t_r}}}{\left(\dfrac{1}{\left(\widehat{\beta}-1\right)^{1-k}}\right)}, \dfrac{\tilde{s}_{\prod_{t=1}^{n} f_{t_r}}}{\left(\dfrac{1}{\left(\widehat{\beta}-1\right)^{1-k}}\right)} \right\rangle \end{array}\right)$$

$$\oplus_{r=1}^{k}\left(w_{t_r}\tilde{S}_{t_r}\right)=\left(\begin{array}{c}\left\langle \tilde{s}_{\prod_{r=1}^{n} w_{t_r} a_{t_r}}, \tilde{s}_{\prod_{r=1}^{n} w_{t_r} b_{t_r}} \right\rangle, \\ \left\langle \frac{\tilde{s}_{\prod_{r=1}^{n} w_{t_r} c_{t_r}}}{\left(\frac{1}{\left(\widehat{\beta}-1\right)^{1-k}}\right)}, \frac{\tilde{s}_{\prod_{r=1}^{n} w_{t_r} d_{t_r}}}{\left(\frac{1}{\left(\widehat{\beta}-1\right)^{1-k}}\right)} \right\rangle, \\ \left\langle \frac{\tilde{s}_{\prod_{r=1}^{n} w_{t_r} e_{t_r}}}{\left(\frac{1}{\left(\widehat{\beta}-1\right)^{1-k}}\right)}, \frac{\tilde{s}_{\prod_{r=1}^{n} w_{t_r} f_{t_r}}}{\left(\frac{1}{\left(\widehat{\beta}-1\right)^{1-k}}\right)} \right\rangle \end{array}\right)$$

And

$$\bigoplus_{\left(1\le t_1 \le t_2 \le \ldots \le t_n\right)}\left(\oplus_{r=1}^{k}\left(w_{t_r}\tilde{S}_{t_r}\right)\right)=$$

$$\left(\begin{array}{l}\left\langle \tilde{s}_{\sum\limits_{(1\le t_1\le t_2\le\ldots\le t_n)}\prod_{r=1}^{n} w_{t_r} a_{t_r}}, \tilde{s}_{\sum\limits_{(1\le t_1\le t_2\le\ldots\le t_n)}\prod_{r=1}^{n} w_{t_r} b_{t_r}} \right\rangle, \\ \left\langle \tilde{s}_{\frac{\frac{1}{(\hat{\beta}-1)^{-1}}\left(1-\prod\limits_{(1\le t_1\le t_2\le\ldots\le t_n)}\left(1-\frac{\prod_{r=1}^{n} w_{t_r} c_{t_r}}{\left(\frac{1}{(\hat{\beta}-1)^{-1}}\right)}\right)\right)}{\left(\frac{1}{(\hat{\beta}-1)^{1-k}}\right)}}, \tilde{s}_{\frac{\frac{1}{(\hat{\beta}-1)^{-1}}\left(1-\prod\limits_{(1\le t_1\le t_2\le\ldots\le t_n)}\left(1-\frac{\prod_{r=1}^{n} w_{t_r} d_{t_r}}{\left(\frac{1}{(\hat{\beta}-1)^{-1}}\right)}\right)\right)}{\left(\frac{1}{(\hat{\beta}-1)^{1-k}}\right)}} \right\rangle, \\ \left\langle \tilde{s}_{\frac{\frac{1}{(\hat{\beta}-1)^{-1}}\left(1-\prod\limits_{(1\le t_1\le t_2\le\ldots\le t_n)}\left(1-\frac{\prod_{r=1}^{n} w_{t_r} e_{t_r}}{\left(\frac{1}{(\hat{\beta}-1)^{-1}}\right)}\right)\right)}{\left(\frac{1}{(\hat{\beta}-1)^{1-k}}\right)}}, \tilde{s}_{\frac{\frac{1}{(\hat{\beta}-1)^{-1}}\left(1-\prod\limits_{(1\le t_1\le t_2\le\ldots\le t_n)}\left(1-\frac{\prod_{t=1}^{n} w_{t_r} f_{t_r}}{\left(\frac{1}{(\hat{\beta}-1)^{-1}}\right)}\right)\right)}{\left(\frac{1}{(\hat{\beta}-1)^{1-k}}\right)}} \right\rangle \end{array}\right)$$

Then we obtain

$$\frac{1}{C_n^k}\left(\bigoplus_{1\le t_1\le t_2\le\ldots\le t_n}\left(\oplus_{r=1}^{k}\left(w_{t_r}\tilde{S}_{t_r}\right)\right)\right)=\left(\frac{\bigoplus\limits_{1\le t_1\le t_2\le\ldots\le t_n}\left(\oplus_{r=1}^{k}\left(w_{t_r}\tilde{S}_{t_r}\right)\right)}{C_n^k}\right)^{\frac{1}{k}}$$

$$= \left(\begin{array}{c} \left\langle \tilde{s}_{\frac{\sum_{(1\le t_1\le t_2\le\ldots\le t_{k\le n})} \Pi_{r=1}^{n} w_{t_r} a_{t_r}}{C_n^k}}, \tilde{s}_{\frac{\sum_{(1\le t_1\le t_2\le\ldots\le t_{k\le n})} \Pi_{r=1}^{n} w_{t_r} b_{t_r}}{C_n^k}} \right\rangle, \\ \left\langle \tilde{s}_{\frac{\frac{1}{(\hat{\beta}-1)^{-1}}\left(1-\prod_{(1\le t_1\le t_2\le\ldots\le t_{k\le n})}\left[1-\frac{\Pi_{r=1}^{n} w_{t_r} c_{t_r}}{\left(\frac{1}{(\hat{\beta}-1)^{-1}}\right)}\right]^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}}{\left(\frac{1}{(\hat{\beta}-1)^{1-k}}\right)}}, \tilde{s}_{\frac{\frac{1}{(\hat{\beta}-1)^{-1}}\left(1-\prod_{(1\le t_1\le t_2\le\ldots\le t_{k\le n})}\left[1-\frac{\Pi_{r=1}^{n} w_{t_r} d_{t_r}}{\left(\frac{1}{(\hat{\beta}-1)^{-1}}\right)}\right]^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}}{\left(\frac{1}{(\hat{\beta}-1)^{1-k}}\right)}} \right\rangle, \\ \left\langle \tilde{s}_{\frac{\left(\frac{1}{(\hat{\beta}-1)^{-1}}\right)\left(1-\prod_{1\le t_1\le t_2\le\ldots\le t_{k\le n}}\left[1-\frac{\Pi_{r=1}^{n} w_{t_r} e_{t_r}}{\left(\frac{1}{(\hat{\beta}-1)^{-1}}\right)}\right]^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}}{\left(\frac{1}{(\hat{\beta}-1)^{1-k}}\right)}}, \tilde{s}_{\frac{\left(\frac{1}{(\hat{\beta}-1)^{-1}}\right)\left(1-\prod_{(1\le t_1\le t_2\le\ldots\le t_{k\le n})}\left[1-\frac{\Pi_{i=1}^{n} w_{t_r} e_{t_r}}{\left(\frac{1}{(\hat{\beta}-1)^{-1}}\right)}\right]^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}}{\left(\frac{1}{(\hat{\beta}-1)^{1-k}}\right)}} \right\rangle \end{array} \right)$$

Therefore

$$\text{ODULIVNWFMSM}^{(k)}\left(\tilde{S}_1, \tilde{S}_2, \tilde{S}_3, \ldots, \tilde{S}_n\right) = \left(\frac{\bigoplus_{(1\le t_1 \ldots \le t_{k\le} n)} \left(\oplus_{r=1}^{k}\left(w_{t_r}\tilde{S}_{t_r}\right)\right)}{C_n^k} \right)^{\frac{1}{k}}$$

$$
\left(\begin{array}{l}
\left\langle \tilde{s}_{\left(\frac{\sum_{(1\le t_1\le t_2\le\ldots\le t_{k\le} n)}\prod_{r=1}^{k} w_{t_r}\tilde{a}_{t_r}}{C_n^k}\right)^{\frac{1}{k}}}, \tilde{s}_{\left(\frac{\sum_{(1\le t_1\le t_2\le\ldots\le t_{k\le} n)}\prod_{r=1}^{k} w_{t_r}\tilde{b}_{t_r}}{C_n^k}\right)^{\frac{1}{k}}} \right\rangle, \\
= \left\langle \begin{array}{l}
\tilde{s}_{\frac{1}{(\hat{\beta}-1)^{-1}}\left(1-\left[\prod\left(1\le t_1\le t_2\le\ldots\le t_{k\le} n\right)\left(1-\frac{\prod_{r=1}^{n}\left(\frac{1}{(\hat{\beta}-1)^{-1}}\right)\left[1-\left(1-\frac{c_{t_r}}{\left(\frac{1}{(\hat{\beta}-1)^{-1}}\right)}\right)^{w_{t_r}}\right]}{\left(\frac{1}{(\hat{\beta}-1)^{-k}}\right)}\right)\right]^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}}, \tilde{s}_{\frac{1}{(q-1)^{-1}}\left(1-\left[\prod\left(1\le t_1\le t_2\le\ldots\le t_{k\le} n\right)\left(1-\frac{\prod_{r=1}^{n}\left(\frac{1}{(\hat{\beta}-1)^{-1}}\right)\left[1-\left(1-\frac{d_{t_r}}{\left(\frac{1}{(\hat{\beta}-1)^{-1}}\right)}\right)^{w_{t_r}}\right]}{\left(\frac{1}{(\hat{\beta}-1)^{-k}}\right)}\right)\right]^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}} \\
\tilde{s}_{\frac{1}{(\hat{\beta}-1)^{-1}}\left(1-\left[\prod\left(1\le t_1\le t_2\le\ldots\le t_{k\le} n\right)\left(1-\frac{\prod_{r=1}^{n}\frac{1}{(\hat{\beta}-1)^{-1}}\left[1-\left(1-\frac{e_{t_r}}{\left(\frac{1}{(\hat{\beta}-1)^{-1}}\right)}\right)^{w_{t_r}}\right]}{\left(\frac{1}{(\hat{\beta}-1)^{-k}}\right)}\right)\right]^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}}, \tilde{s}_{\frac{1}{(\hat{\beta}-1)^{-1}}\left(1-\left[\prod\left(1\le t_1\le t_2\le\ldots\le t_{k\le} n\right)\left(1-\frac{\prod_{r=1}^{n}\frac{1}{(\hat{\beta}-1)^{-1}}\left[1-\left(1-\frac{f_{t_r}}{\left(\frac{1}{(\hat{\beta}-1)^{-1}}\right)}\right)^{w_{t_r}}\right]}{\left(\frac{1}{(\hat{\beta}-1)^{-k}}\right)}\right)\right]^{\frac{1}{C_n^k}}\right)^{\frac{1}{k}}}
\end{array}\right\rangle
\end{array}\right)
$$

The other desirable properties may be easily proved on same lines as stated above.

4. MCDM via OULIVNWFMSM operator

For tactful evaluation, consider a finite set A of alternatives D be a set of decision makers values and λ represent a weight vector made by DMs. D_μ obtained by $\lambda_\mu \in [0,1]$ $for\,\mu$=1,2,...,p; while weight sum is equal to 1 that is $\sum_{\mu=1}^{p} \lambda_\mu = 1$. The attributes set is C with $\omega = \left(\omega_{1,}\omega_2, ..., \omega_p\right)^T$. The $\omega_j \in [0,1]$ $and \sum_{j=1}^{n} \omega_j = 1 with\ j = 1, 2, ..., n.$ Thus $S^\mu = \left[s_{i_j}^\mu\right]_{m\times n}; \mu = 1, 2, ..., p$

$S^\mu = \left[s_{i_j}^\mu\right]_{m\times n}; \mu = 1, 2, ..., p$ is a decision matrix and looks $s_{i_j}^\mu = \left(\left[s_{a_{i_j}^\mu}, s_{b_{i_j}^\mu}\right], \left[s_{c_{i_j}^\mu}, s_{d_{i_j}^\mu}\right], \left[s_{e_{i_j}^\mu}, s_{f_{i_j}^\mu}\right]\right)$ gives the evaluation value alternatives because of attributes values. The process can be shortly demonstrated by algorithm with flow chart.

4.1 ALGORITHM

The following are the necessary and sufficient steps for evaluating numerical data.

Step-1: Initially we calculate the ODULIVN fuzzy decision-matrix".

Step-2: The Normalization of attributes will be sort out if required e.g., if they are different types i.e., benefit or cost .

Step-3: We aggregate the fuzzy informative data of each decision maker by proposed operator.

Step-4: In third step we calculate the expectancy of each ODULIVNF variable using $E\left(\tilde{S}_1\right) = \left(\frac{l+m}{6(\alpha-1)} \times \frac{n+o}{6(\beta-1)} \times \frac{p+q}{6(\gamma-1)}\right)$

Step-5: In last step we rank the chosen alternatives by adopting the expectancy criteria stated in definition 3.4.

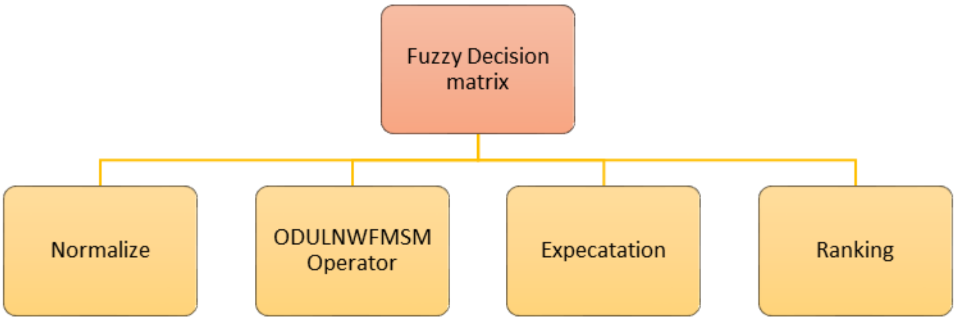


Figure-1: Flow chart of proposed mechanism

4.2 Practical utility of the proposed aggregation operators in business analytics

Business analytics (BA) is an area that drives concrete, data-driven modifications in a business. Indeed, it is a tool a business group requires to take accurate decisions that are probable to influence the whole organization for they assist in improving lucrativeness. BA is a real-world utility of arithmetical analysis that emphasizes on providing practicable commendations. BA specialists engage in how to use the perceptions they derive from the data. Their objective is to draw concrete conclusions pertaining to a business by finding answers to specific queries about why things happened (past analysis of the happening), what will happen (forecasting) and what should be done (recommending necessary measures to be taken essentially). The experts of BA pool the fields of administration and business accompanied by the techniques of information technology that are successfully employed in this field. The business feature involves a preeminent knowledge of the business in addition to the practical inhibitions that subsist. The analytical part comprises a clear and flawless perception of the data handling using information technology techniques whose combination certainly bridges the gap amongst administration and technology.

Exploring the existing data, the business analytics give valuable suggestions to tackle hindrances and improve businesses. Many take-away restaurants and fast-food companies around the globe have been successfully implementing BA to enhance their business that leads to reasonable increase in profits and expansion of their business. By keeping an eye on how engaged the drive-thru is, these businesses can boost their effectualness in the course of prime times of their business. When the que becomes over-crowded, the digital order boards change. They start highlighting those products which can be readied and offered expeditiously. When there is less traffic, employees can suggest items that are more expensive (having higher margins) and take more time in preparation. Other sorts of BA applications perform more than merely responding to the prevailing situation. These methodologies give a helping hand to businesses anticipate which customers are least probable to come again. In such a case, they can then focus on promotions and advertisement to such customers to lift the rate of retention.

4.3 Example:

Assume that a fast-food restaurant wants to expand its business by increasing its customers. The restaurant works from 2:00 PM to 2:00 AM. The restaurant hires the services of BA to flourish its business and make it more lucrative. For this purpose, the restaurant assigns the task of deciding at which time slots the restaurant should offer which package of fast food to gain and retain maximum customers. The time slots available are in table-1 and available packages are shown in table-2. noted that slots will represent the alternatives and packages will show attributes values in numerical computation.

Slots	Time distribution	
	startup	end up
S-1	2-PM	4-PM
S-2	4-PM	6-PM
S-3	6- PM	6-PM
S-4	6-PM	8-PM
S-5	8-PM	10-PM
S-6	10-PM	12-PM

Table-1

Suppose that there are six packages available.

P1	Package-1
P2	Package-2
P3	Package-1
P4	Package-1
P5	Package-5
P6	Package-6

Table-2

The evaluation steps by the proposed method

Alternatives /Attributes	C_1	C_2	C_3	C_4	C_5	C_6
$\tilde{A}_1$	$\begin{pmatrix}{[s_1,s_3]}\\ {[s_3,s_5]}\\ {[s_2,s_4]}\end{pmatrix}$	$\begin{pmatrix}{[s_2,s_4]}\\ {[s_1,s_4]}\\ {[s_2,s_3]}\end{pmatrix}$	$\begin{pmatrix}{[s_5,s_6]}\\ {[s_1,s_2]}\\ {[s_3,s_4]}\end{pmatrix}$	$\begin{pmatrix}{[s_2,s_3]}\\ {[s_1,s_5]}\\ {[s_2,s_3]}\end{pmatrix}$	$\begin{pmatrix}{[s_1,s_2]}\\ {[s_4,s_5]}\\ {[s_3,s_4]}\end{pmatrix}$	$\begin{pmatrix}{[s_4,s_6]}\\ {[s_2,s_5]}\\ {[s_1,s_6]}\end{pmatrix}$
$\tilde{A}_2$	$\begin{pmatrix}{[s_1,s_2]}\\ {[s_4,s_5]}\\ {[s_3,s_4]}\end{pmatrix}$	$\begin{pmatrix}{[s_4,s_6]}\\ {[s_2,s_5]}\\ {[s_1,s_6]}\end{pmatrix}$	$\begin{pmatrix}{[s_5,s_6]}\\ {[s_1,s_2]}\\ {[s_3,s_4]}\end{pmatrix}$	$\begin{pmatrix}{[s_5,s_6]}\\ {[s_1,s_2]}\\ {[s_3,s_4]}\end{pmatrix}$	$\begin{pmatrix}{[s_2,s_4]}\\ {[s_1,s_4]}\\ {[s_2,s_3]}\end{pmatrix}$	$\begin{pmatrix}{[s_1,s_3]}\\ {[s_3,s_5]}\\ {[s_2,s_4]}\end{pmatrix}$
$\tilde{A}_3$	$\begin{pmatrix}{[s_4,s_6]}\\ {[s_2,s_5]}\\ {[s_1,s_6]}\end{pmatrix}$	$\begin{pmatrix}{[s_1,s_2]}\\ {[s_4,s_5]}\\ {[s_3,s_4]}\end{pmatrix}$	$\begin{pmatrix}{[s_5,s_6]}\\ {[s_1,s_2]}\\ {[s_3,s_4]}\end{pmatrix}$	$\begin{pmatrix}{[s_2,s_4]}\\ {[s_1,s_4]}\\ {[s_2,s_3]}\end{pmatrix}$	$\begin{pmatrix}{[s_1,s_3]}\\ {[s_3,s_5]}\\ {[s_2,s_4]}\end{pmatrix}$	$\begin{pmatrix}{[s_4,s_6]}\\ {[s_2,s_5]}\\ {[s_1,s_6]}\end{pmatrix}$
$\tilde{A}_4$	$\begin{pmatrix}{[s_1,s_3]}\\ {[s_3,s_5]}\\ {[s_2,s_4]}\end{pmatrix}$	$\begin{pmatrix}{[s_4,s_6]}\\ {[s_2,s_5]}\\ {[s_1,s_6]}\end{pmatrix}$	$\begin{pmatrix}{[s_4,s_6]}\\ {[s_2,s_5]}\\ {[s_1,s_6]}\end{pmatrix}$	$\begin{pmatrix}{[s_1,s_3]}\\ {[s_3,s_5]}\\ {[s_2,s_4]}\end{pmatrix}$	$\begin{pmatrix}{[s_1,s_2]}\\ {[s_4,s_5]}\\ {[s_3,s_4]}\end{pmatrix}$	$\begin{pmatrix}{[s_5,s_6]}\\ {[s_1,s_2]}\\ {[s_3,s_4]}\end{pmatrix}$
$\tilde{A}_5$	$\begin{pmatrix}{[s_1,s_2]}\\ {[s_4,s_5]}\\ {[s_3,s_4]}\end{pmatrix}$	$\begin{pmatrix}{[s_1,s_3]}\\ {[s_3,s_5]}\\ {[s_2,s_4]}\end{pmatrix}$	$\begin{pmatrix}{[s_5,s_6]}\\ {[s_1,s_2]}\\ {[s_3,s_4]}\end{pmatrix}$	$\begin{pmatrix}{[s_4,s_6]}\\ {[s_2,s_5]}\\ {[s_1,s_6]}\end{pmatrix}$	$\begin{pmatrix}{[s_5,s_6]}\\ {[s_1,s_2]}\\ {[s_3,s_4]}\end{pmatrix}$	$\begin{pmatrix}{[s_1,s_3]}\\ {[s_3,s_5]}\\ {[s_2,s_4]}\end{pmatrix}$
$\tilde{A}_6$	$\begin{pmatrix}{[s_1,s_3]}\\ {[s_3,s_5]}\\ {[s_2,s_4]}\end{pmatrix}$	$\begin{pmatrix}{[s_5,s_6]}\\ {[s_1,s_2]}\\ {[s_3,s_4]}\end{pmatrix}$	$\begin{pmatrix}{[s_2,s_4]}\\ {[s_1,s_4]}\\ {[s_2,s_3]}\end{pmatrix}$	$\begin{pmatrix}{[s_1,s_3]}\\ {[s_3,s_5]}\\ {[s_2,s_4]}\end{pmatrix}$	$\begin{pmatrix}{[s_4,s_6]}\\ {[s_2,s_5]}\\ {[s_1,s_6]}\end{pmatrix}$	$\begin{pmatrix}{[s_5,s_6]}\\ {[s_1,s_2]}\\ {[s_3,s_4]}\end{pmatrix}$

Table-3

$\begin{pmatrix}{[s_{0.3210},s_{0.4203}]}\\ {[s_{0.3213},s_{1.4312}]}\\ {[s_{0.2642},s_{1.0204}]}\end{pmatrix}$	$\begin{pmatrix}{[s_{1.0231},s_{1.4323}]}\\ {[s_{0.2393},s_{1.1125}]}\\ {[s_{0.3542},s_{1.5304}]}\end{pmatrix}$	$\begin{pmatrix}{[s_{1.2791},s_{1.9433}]}\\ {[s_{0.9823},s_{1.3625}]}\\ {[s_{0.3352},s_{0.7634}]}\end{pmatrix}$	$\begin{pmatrix}{[s_{0.1013},s_{0.4002}]}\\ {[s_{0.4913},s_{1.0391}]}\\ {[s_{1.2622},s_{1.9214}]}\end{pmatrix}$	$\begin{pmatrix}{[s_{0.0231},s_{0.1322}]}\\ {[s_{0.2991},s_{0.6021}]}\\ {[s_{0.8572},s_{1.5201}]}\end{pmatrix}$	$\begin{pmatrix}{[s_{0.2701},s_{0.3433}]}\\ {[s_{1.3121},s_{1.5625}]}\\ {[s_{0.6752},s_{0.8694}]}\end{pmatrix}$
$\begin{pmatrix}{[s_{1.9210},s_{1.9243}]}\\ {[s_{0.9243},s_{0.9742}]}\\ {[s_{1.2132},s_{1.5204}]}\end{pmatrix}$	$\begin{pmatrix}{[s_{0.0311},s_{0.5353}]}\\ {[s_{0.0333},s_{0.1721}]}\\ {[s_{0.2507},s_{0.6314}]}\end{pmatrix}$	$\begin{pmatrix}{[s_{1.0071},s_{1.0233}]}\\ {[s_{0.7403},s_{1.3155}]}\\ {[s_{0.2132},s_{0.6534}]}\end{pmatrix}$	$\begin{pmatrix}{[s_{1.0248},s_{1.9463}]}\\ {[s_{0.2213},s_{1.2071}]}\\ {[s_{0.2034},s_{1.1264}]}\end{pmatrix}$	$\begin{pmatrix}{[s_{0.0301},s_{0.5823}]}\\ {[s_{0.5631},s_{0.6741}]}\\ {[s_{0.5436},s_{1.0014}]}\end{pmatrix}$	$\begin{pmatrix}{[s_{1.4829},s_{1.8213}]}\\ {[s_{0.7403},s_{1.3895}]}\\ {[s_{0.2082},s_{1.6545}]}\end{pmatrix}$
$\begin{pmatrix}{[s_{1.5213},s_{1.6243}]}\\ {[s_{0.3219},s_{0.4710}]}\\ {[s_{0.0142},s_{0.1294}]}\end{pmatrix}$	$\begin{pmatrix}{[s_{0.0235},s_{1.0393}]}\\ {[s_{1.0303},s_{1.1985}]}\\ {[s_{0.5102},s_{0.5397}]}\end{pmatrix}$	$\begin{pmatrix}{[s_{0.6531},s_{1.3106}]}\\ {[s_{0.6433},s_{0.9891}]}\\ {[s_{0.2317},s_{0.7994}]}\end{pmatrix}$	$\begin{pmatrix}{[s_{0.0413},s_{1.0244}]}\\ {[s_{0.3419},s_{0.7360}]}\\ {[s_{0.0102},s_{0.5936}]}\end{pmatrix}$	$\begin{pmatrix}{[s_{0.5615},s_{0.7390}]}\\ {[s_{0.3751},s_{0.5721}]}\\ {[s_{1.5462},s_{1.5017}]}\end{pmatrix}$	$\begin{pmatrix}{[s_{0.5329},s_{1.0986}]}\\ {[s_{0.2671},s_{1.3891}]}\\ {[s_{1.4317},s_{1.0207}]}\end{pmatrix}$

Table-2

$\begin{pmatrix}{[s_{0.3241},s_{0.5378}]}\\ {[s_{0.1090},s_{1.4218}]}\\ {[s_{0.2902},s_{1.3284}]}\end{pmatrix}$	$\begin{pmatrix}{[s_{0.0192},s_{1.3109}]}\\ {[s_{1.1343},s_{1.9340}]}\\ {[s_{0.2132},s_{1.5004}]}\end{pmatrix}$	$\begin{pmatrix}{[s_{0.3081},s_{0.6597}]}\\ {[s_{0.3219},s_{1.8710}]}\\ {[s_{0.0142},s_{0.1004}]}\end{pmatrix}$	$\begin{pmatrix}{[s_{0.5219},s_{1.8243}]}\\ {[s_{0.3219},s_{0.8719}]}\\ {[s_{0.4173},s_{1.6494}]}\end{pmatrix}$	$\begin{pmatrix}{[s_{1.0815},s_{1.4203}]}\\ {[s_{0.2381},s_{0.4710}]}\\ {[s_{1.0102},s_{1.1291}]}\end{pmatrix}$	$\begin{pmatrix}{[s_{0.8213},s_{1.3462}]}\\ {[s_{0.1210},s_{0.6745}]}\\ {[s_{1.4523},s_{1.5274}]}\end{pmatrix}$

Table -4

$\tilde{A}_i; i=1,2,\ldots,6$	$\tilde{A}_2$	$\tilde{A}_2$	$\tilde{A}_3$	$\tilde{A}_4$	$\tilde{A}_5$	$\tilde{A}_6$

Expected value	0.1161	0.1149	0.1151	0.1156	0.1152	0.1159

Table-5

Ranking based on proposed scheme

$$A_1 \succ A_6 \succ A_4 \succ A_5 \succ A_3 \succ A_2$$

Table-6

4.4 Comparative study based on proposed-model vs. existing-models.

A comparative study is presented for validation, feasibility, and effectiveness of the propound operator.

Modeling tool	2DULWBOWA Operator	W2DULMSM Operator	FLIOWA Operator
Alternatives with expected values	A_1 0.8231 A_2 0.9001 A_3 0.3443 A_4 0.2367 A_5 0.7002 A_6 0.7691	A_1 0.6199 A_2 0.3183 A_3 0.7115 A_4 0.5146 A_5 0.9201 A_6 0.2117	A_1 0.5359 A_2 0.7738 A_3 0.4932 A_4 0.8823 A_5 0.7032 A_6 0.6786
Ranking	$A_2>A_1>A_6>A_5>A_3>A_4$	$A_5>A_3>A_1>A_4>A_2>A_6>$	$A_4>A_2>A_5>A_6>A_1>A_3$

Table-7

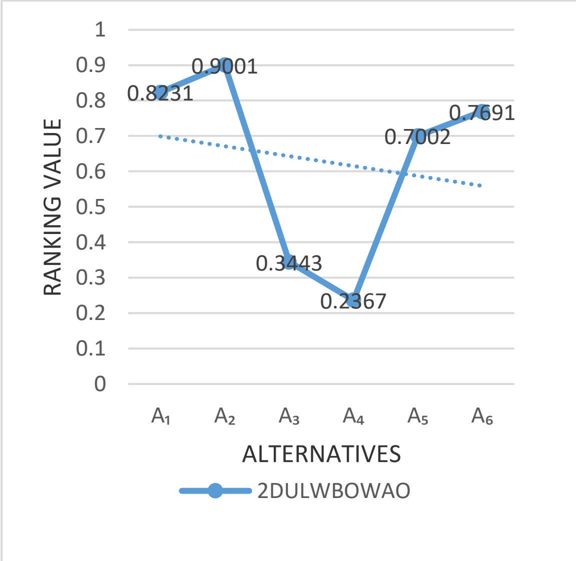


Figure 2: MCDM based on 2DULWBOWA operator.

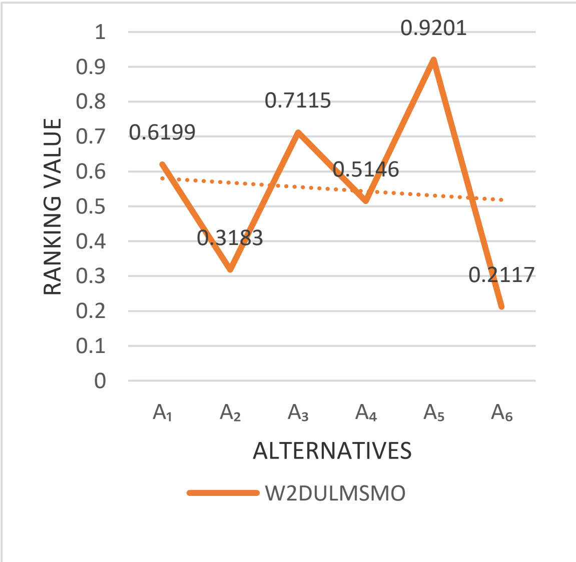


Figure-3: MCDM based on W2DULMSM operator.

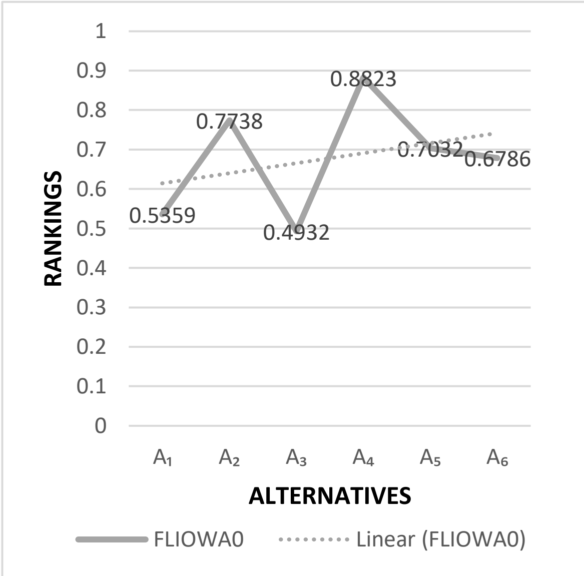


Figure-4: MCDM based on FLIOWA operator.

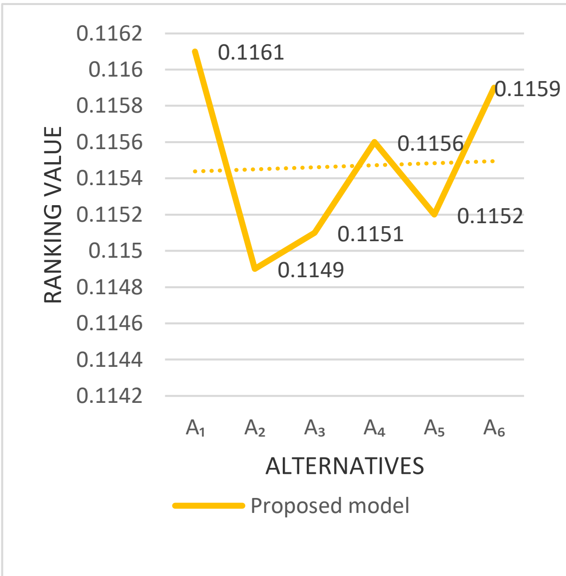
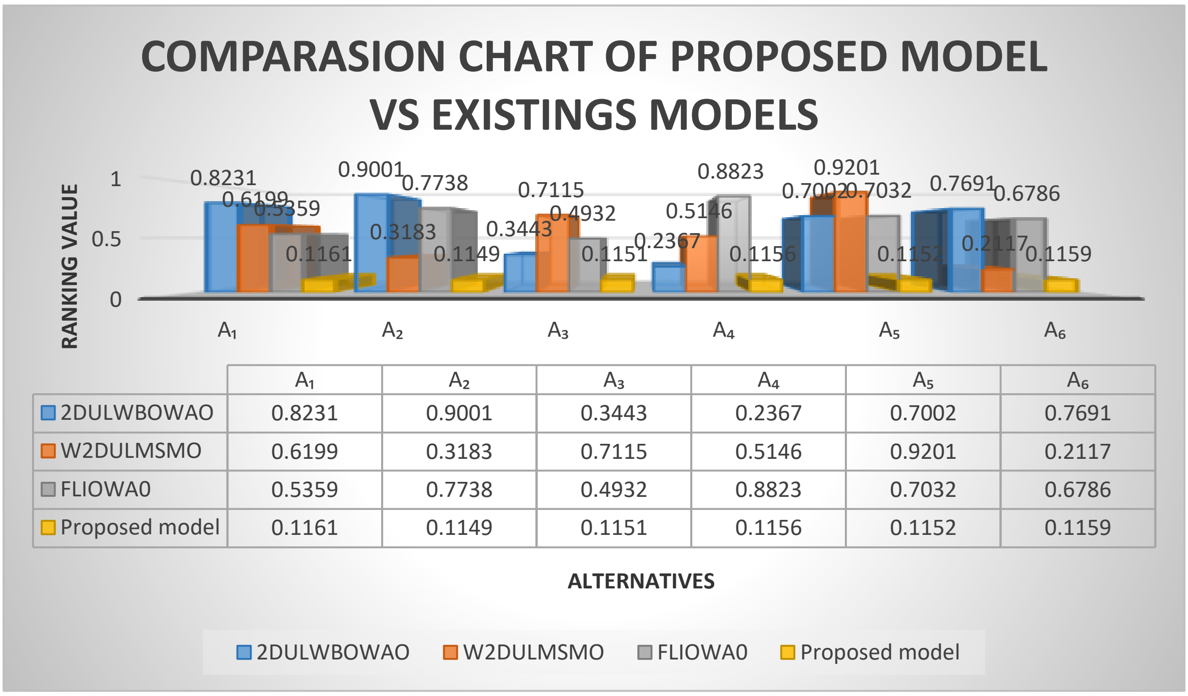


Figure-5: MCDM based on proposed scheme.



	A_1	A_2	A_3	A_4	A_5	A_6
2DULWBOWAO	0.8231	0.9001	0.3443	0.2367	0.7002	0.7691
W2DULMSMO	0.6199	0.3183	0.7115	0.5146	0.9201	0.2117
FLIOWA0	0.5359	0.7738	0.4932	0.8823	0.7032	0.6786
Proposed model	0.1161	0.1149	0.1151	0.1156	0.1152	0.1159

Figure-6: Comparatively Study based on proposed model vs. existing models.

4.5 Result and Discussion

No doubt, there exist hundreds of aggregations tools which play a vital role and to be considered very useful in theory of multiple decision. The proposed operator would be very beneficial and fruitful in the same area of interest because of its extra features and capabilities. In numerically analysis we examined that the results obtained by proposed MAGDM scheme are more flexible, reliable, and valuable as compared to all the MCDM strategies and tools which are already designed for the same set of modeling and plotting the data. To evaluate the new method's effectiveness, we experimented with it in comparison to three previously developed methods on a specific example. These methods are the 2DULWBOWAO method by Liu and Shi [50], the W2DULMSMO method by Chu and Liu [51], and the FLIOWAO method by Liu et al. [52]. To make things easy, we assigned values to certain variables: k is set as 1 or 2, while p and q are both set as 1. Using these values, we determined the final rankings for the options. The outcomes of these methods are displayed in Figure-6 together with Table-7. From Table-7, we can tell that the options we chose have a significant difference in ranking. They are spread out and separated by large margins, and not very correct and seem unsuitable. However, the values obtained by the proposed method are very close and more correct and suitable. So, these values are like human thinking as well. The method advocated in this research proves to be productive. Which notify that proposed model has the high level of accuracy and validity. To comprehend the advantages of the proposed technique, these points will provide further explanation.

1. These two methods share the same operational guidelines as suggested by Liu and Shi [50], although they exhibit enhanced accuracy. While the new method considers the relationships between various elements, the old method overlooks such connections. The suggested method implies that the value of k can reflect an individual's willingness to take risks.
2. In comparison to the W2DULMSMO method proposed by Liu et al. [52], these two methods could incorporate the relationship between input data/chosen values. However, the suggested approach can think about how all the input arguments are connected to each other, whereas the approach suggested by Chu and Liu [51] can only think about how two input arguments are connected to each other. Furthermore, our suggested method uses the latest operational rules with precise actions, whereas Chu and Liu's method [51] only use the old-fashioned

operational-rule. Clearly, the suggested technique is more adaptable and inclusive in solving the multi-input data problems compared to the method proposed by Chu and Liu [51].

3. These two approaches share the same rules as the FLIOWAO method introduced by Liu et al. [52], and they both account for the correlation between input arguments. Nevertheless, the recently developed technique enables an analysis of the relationships among multiple arguments. The method proposed by Liu et al. [52] is designed to specifically address the relationship between two arguments, which serves as a specific case within the overall application of the innovative approach.

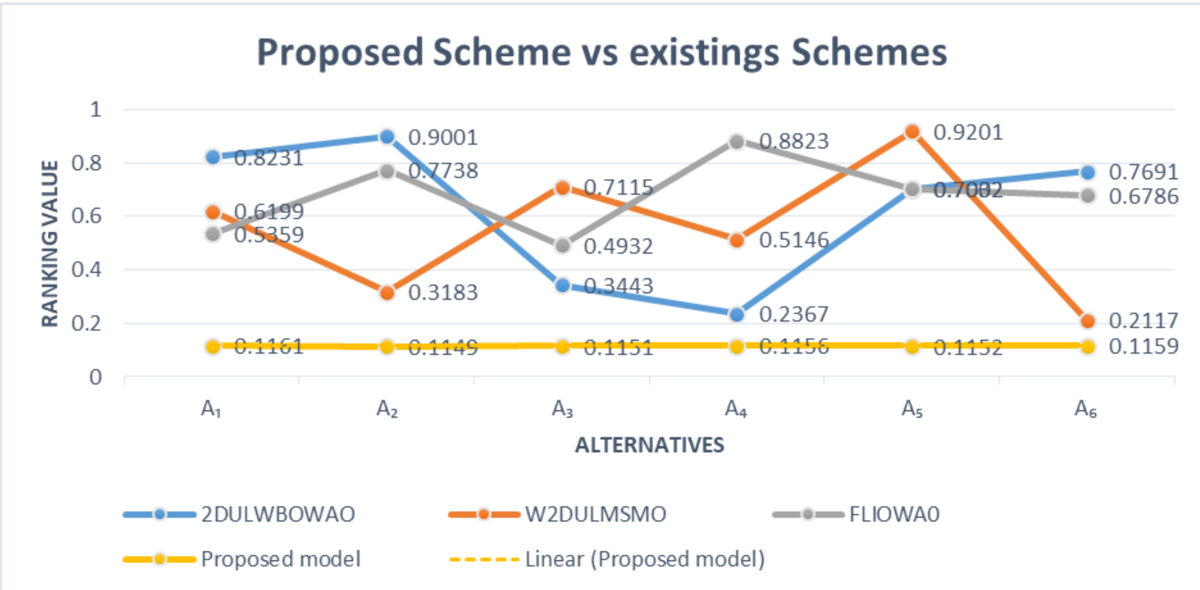


Figure-7: Numerical Data for Proposed model vs. existing schemes

5. Conclusions

The complexities of considering multiple criteria make decision-making an interesting and crucial component of modern group dynamics. Descriptive terms (linguistic variables) make it easier for decision makers to display assessment values. Decision makers utilize linguistic variables to incorporate their personal judgments regarding the probabilities of specific outcomes. Well-formulated and clearly defined linguistic variables can eliminate any potential gaps in information and enhance the accuracy of the process. In current research paper, we stated the idea of one-dimensional-uncertain-linguistic-neutrosophic-fuzzy variable. We extracted their modified operational laws, basic properties and stated the expectation of these variables. Moreover, For the sake of tracking down the finest alternative while solving multi-input data values (MCDM) problems and other utilities, one of the critical phases is aggregation whereby the suggestion of the decision expert is required to be calculated by using suitable multi-input techniques/operators. We developed some new novel aggregation operators to overcome the power of rapidly growing of convolution, complication, and the vagueness of the socioeconomic environment time to time. Moreover, one dimensional uncertain linguistic interval valued neutrosophic fuzzy information was successfully aggregated by proposed aggregation-operators. Furthermore, the use of a particular numerical example illustrates the strength and validity of the exhibit approach in facilitating group decision-making. Graphically representation for evaluated data based on proposed scheme is also added for easy understanding. Conclusively, comparative study based on proposed model vs. existing-models was managed by the help of decision-experts.

Future work

In the future, using our identified ideas, the research work is expected to be significantly expanded, some beneficial methodological extensions and other useful applications. The current tool can be considered very effective and powerful in a broad field of decision-making methods, such as Average probability, data information and moving data. Implementing the suggested method in combination with other decision-making methods can enhance the functionality of various financial models.

Conflicts of Interest

The authors announce that no conflicts of interest exist.

Author Contributions

Each author participated equally to the writing and editing of the paper. The paper was examined by each author.

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Data availability

This article has all the data that were created or evaluated during this investigation.

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Novel Neutrosophic Objects Within Neutrosophic Topology $(N(X), \tau)$

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Abstract

This essay intends to introduce and study many new neutrosophic objects within neutrosophic topologies $(N(X), \tau)$, such as the neutrosophic point, the neutrosophic quasi-concomitant, the neutrosophic quasi neighborhood, the neutrosophic ideals, the neutrosophic local function, the neutrosophic closure operator, the generated neutrosophic topology, and quite a few of theorems, corollaries, examples related the above- mentioned concepts.

Keywords: Neutrosophic Topology $(N(X), \pi)$; Neutrosophic Point; Neutrosophic Quasi-Concomitant; Neutrosophic Quasi Neighborhood; Neutrosophic Ideals; Neutrosophic Local Function.

1. Introduction

The reformulations of all scientific fields in the perspective of the existence of indeterminacy were by the paradigm shift man F. Smarandache [1-3]. As of 1995 so far, he redefined almost all branches of knowledge, setting up the neutrosophic theory through the implication of the indeterminacy part into all components, elements, operations, thoughts, opinions,etc. [10-13].

The topological space took its share of that evolution, the eminent scientist who led changing the topological spaces into the frame of neutrosophic theory was A.A. Salama [4-6], with the collaboration of F. Smarandache, later dozens of mathematicians who were interested in the topological field joined them [7-9]. This paper comes as a point in the sea of this scientific promotion, it is not the first nor the last in the field of neutrosophic topological spaces. This article is organized by dedicating section two to new neutrosophic notions which are presented in this paper for the first time, Section three goes for demonstrates the neutrosophic ideal, neutrosophic local function, and generated neutrosophic topology, while section four encloses the basic structure of generated neutrosophic topology, eventually section five contains the conclusions and recommendations.

2. New Notions in Neutrosophic Topological Spaces $(\boldsymbol{N(X)}, \boldsymbol{\tau})$

2.1 Definition (Neutrosophic point):

Suppose $(N(X), \tau)$ to be neutrosophic topological space, a neutrosophic point $x \in N(X)$ is denoted by $x_{\langle \lambda_1, \lambda_2, \lambda_3 \rangle}$, where $0^- \leq \lambda_1, \lambda_2, \lambda_3 \leq 1^+$. A neutrosophic point $x_{\langle \lambda_1, \lambda_2, \lambda_3 \rangle}$ is belonging to a neutrosophic set $A \subseteq N(X)$ iff $\lambda_1 \leq A(x_{\lambda_1}), \lambda_2 \leq A(x_{\lambda_2}), \lambda_3 \leq A(x_{\lambda_3})$ and symbolized by $x_{\langle \lambda_1, \lambda_2, \lambda_3 \rangle} \in_N A$.

2.2 Definition (Neutrosophic Quasi-concomitant):

A neutrosophic set $A \in N(X)$ is said to be neutrosophic quasi-concomitant to another neutrosophic set B if there exists a neutrosophic point $x_{\langle \lambda_1, \lambda_2, \lambda_3 \rangle} \in_N N(X)$ such that the following conditions are hold together:

1- $A(x_{\lambda_1}) + B(x_{\lambda_1}) > 1^+_{x_{\lambda_1}}$,

2- $A(x_{\lambda_2}) + B(x_{\lambda_2}) > 1^+_{x_{\lambda_2}}$,

3- $A(x_{\lambda_3}) + B(x_{\lambda_3}) > 1^+_{x_{\lambda_3}}$,

And it is symbolized by $A\ qc\ B$, for any two neutrosophic sets $A \& B$, $A\ qc\ B \Leftrightarrow B\ qc\ A$.

2.3 Definition (Neutrosophic quasi neighborhood):

A neutrosophic set A in a neutrosophic topological space $(N(X), \tau)$ is called a neutrosophic quasi neighborhood of a neutrosophic point $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}$ iff there exist a neutrosophic open set $\mu \subseteq A$ such that $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}\, qn\, \mu$. The set of all neutrosophic quasi neighborhood of $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}\ in\ (N(X), \tau)$ is symbolized by $NQN(x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle})$.

2.4 Definition (Accumulation neutrosophic point):

A neutrosophic point $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}$ is called an accumulation neutrosophic point of a neutrosophic set A in the neutrosophic topological space $(N(X), \tau)$ iff the following condition holds:

1- Any neutrosophic quasi neighborhood of $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}$ is neutrosophic quasi concomitant,

2- If $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle} \in_N A$, any quasi neighborhood of $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}$ and A are quasi concomitant at some neutrosophic point $y_{\langle t_1,t_2,t_3\rangle}$ such that $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle} \neq y_{\langle t_1,t_2,t_3\rangle}$.

Note *

In the above definition of accumulation neutrosophic point, if only the condition 1 holds, then $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}$ is called an adherence neutrosophic point of A.

Note **

It is obvious that any neutrosophic point $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}$ is belonging to the closure of a neutrosophic set A (i.e. $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle} \in_N Ncl(A)$) iff for every quasi neighborhood B of $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}$, $B\ qc\ A$.

3. Neutrosophic Ideal, Neutrosophic Local Function and Generated Neutrosophic Topology

3.1 Definition (Neutrosophic Ideal):

Suppose that $\pi, v \in N(X)$, A nonempty collection of neutrosophic sets I of $N(X)$ is called ideal on $N(X)$ if and only if

1- $\pi \in_N I$ and $v \subseteq \pi \Longrightarrow v \in I$ [heredity],

2- $\pi \in_N I$ and $v \in_N I \Longrightarrow \pi \cup v \in_N I$ [finite additivity].

3.2 Definition (Neutrosophic Local Function):

Let $(N(X),\tau)$ be a neutrosophic topological space and I be neutrosophic ideal on $N(X)$. Let A be any neutrosophic set of $N(X)$. Then the neutrosophic local function $A^*(I,\tau)$ of A is the union of all neutrosophic points $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}$, such that if $\mu \in_N NQN(x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle})$ and $I_1 \in I$ then there is at least one $y_{\langle t_1,t_2,t_3\rangle} \in_N N(X)$ for which $\mu(y_{t_1}) + A(y_{t_1}) - 1^+_{y_{t_1}} > I_1(y_{t_1})$, $\mu(y_{t_2}) + A(y_{t_2}) - 1^+_{y_{t_2}} > I_1(y_{t_2})$, $\mu(y_{t_3}) + A(y_{t_3}) - 1^+_{y_3} > I_1(y_{t_3})$. therefore, any $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle} \notin_N A^*(I,\tau)$ [i.e. implies to $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}$ is not contained in the neutrosophic set A, i.e. $\lambda_1 > A(x_{\lambda_1})$, $\lambda_2 > A(x_{\lambda_2})$, $\lambda_3 > A(x_{\lambda_3})$] implies there is at least one $\mu \in_N NQN(x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle})$ such that for every $y_{\langle t_1,t_2,t_3\rangle} \in_N N(X)$, $\mu(y_{t_1}) + A(y_{t_1}) - 1^+_{y_{t_1}} \le I_1(y_{t_1})$, $\mu(y_{t_2}) + A(y_{t_2}) - 1^+_{y_{t_2}} \le I_1(y_{t_2})$, $\mu(y_{t_3}) + A(y_{t_3}) - 1^+_{y_3} \le I_1(y_{t_3})$ for some $I_1 \in I$. The symbols A^* *or* $A^*(I)$ will be written as abbreviate instead of $A^*(I,\tau)$.

3.3 Note:

The neutrosophic empty set and the neutrosophic universal set on $N(X)$ is denoted by $0_{x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}}$ and $1_{x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}}$, respectively.

3.4 Example

The simplest neutrosophic ideals on $N(X)$ are $\{0_{x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}}\}$ and $H(N(X))$, the set of all neutrosophic sets of $N(X)$ (hereafter, if necessary, $H(N(X))$ will carry the same meaning). Obviously, $I = \{0_{x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}}\} \Leftrightarrow A^*(I,\tau) = cl(A)$, for any neutrosophic set A of $N(X)$ and $I = H(N(X)) \Leftrightarrow A^*(I,\tau) = 0_{x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}}$.

3.5 Theorem

Let $(N(X),\tau)$ be a neutrosophic topological space and I_1, I_2 are two neutrosophic ideals on $N(X)$. Then for any neutrosophic sets A, B *of* $N(X)$, the following mathematical phrases are true:

1- $A \subseteq B \Longrightarrow A^*(I_1,\tau) \subseteq B^*(I_1,\tau)$.

2- $I_1 \subseteq I_2 \Longrightarrow A^*(I_2,\tau) \subseteq A^*(I_1,\tau)$.

3- $A^* = cl(A^*) \subseteq cl(A)$.

4- $(A^*)^* \subseteq A^*$.

5- $(A \cup B)^* = A^* \cup B^*$.

6- $I_1 \in I \Longrightarrow (A \cup I_1)^* = A^*$.

Proof.

1- Since $A \subseteq B$ this implies that $A_{x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}} \le B_{x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}}$ for every $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle} \in_N N(X)$, therefore and by Definition 3.2, $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle} \in_N A^*$implies that $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle} \in_N B^*$, which complete the proof of (1).

2- Suppose $I_1 \subseteq I_2 \Longrightarrow A^*(I_2) \subseteq A^*(I_1)$,as there may be other neutrosophic sets which belong to I_2 so that for a neutrosophic point $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}$, $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle} \in_N A^*(I_1)$ but $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}$ may not be contained in $A^*(I_2)$.

3- Since the empty neutrosophic set $\left\{0_{x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}}\right\}$ is contained in any neutrosophic ideal I_1 on $N(X)$ (i.e. $\left\{0_{x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}}\right\} \subseteq I_1$),[therefore by (2) and because of the reality that the simplest neutrosophic ideal on $N(X)$ is $\left\{0_{x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}}\right\}$. Obviously, $A^*\left(\left\{0_{x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}}\right\}\right) = cl(A)$, for any neutrosophic set A of $N(X)$], $A^*(I_1) \subseteq A^*\left(\left\{0_{x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}}\right\}\right) = cl(A)$, for any neutrosophic set A of $N(X)$. Suppose, $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle} \in_N cl(A^*)$.So there is at least one $y_{\langle t_1,t_2,t_3\rangle} \in_N N(X)$ such that $\mu(y_{t_1}) + A^*(y_{t_1}) > 1^+_{y_{t_1}}$, $\mu(y_{t_2}) + A^*(y_{t_2}) > 1^+_{y_{t_2}}$, $\mu(y_{t_3}) + A^*(y_{t_3}) > 1^+_{y_3}$ for each neutrosophic quasi neighborhood μ of $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}$. Hence $A^*(y_{\langle t_1,t_2,t_3\rangle}) \neq 0_{y_{\langle t_1,t_2,t_3\rangle}}$. Let $f_{\langle f_1,f_2,f_3\rangle} = A^*(y_{\langle t_1,t_2,t_3\rangle})$. Clearly, $y_f \in_N A^*$ and $f_{f_1} + \mu(y_{t_1}) > 1^+_{y_{t_1}}$, $f_{f_2} + \mu(y_{t_2}) > 1^+_{y_{t_2}}$, $f_{f_3} + \mu(y_{t_3}) > 1^+_{y_3}$, so that μ is also neutrosophic quasi neighborhood of y_f.

Now $y_f \in_N A^*$ implies there is at least one $x'_{\langle\lambda_1{}',\lambda_2{}',\lambda_3{}'\rangle} \in_N N(X)$ such that $v\left(x'_{\lambda_1{}'}\right) + A\left(x'_{\lambda_1{}'}\right) - 1^+_{x'_{\lambda_1{}'}} > I_1(x'_{\lambda_1{}'})$, $v\left(x'_{\lambda_2{}'}\right) + A\left(x'_{\lambda_2{}'}\right) - 1^+_{x'_{\lambda_2{}'}} > I_1(x'_{\lambda_2{}'})$, $v\left(x'_{\lambda_3{}'}\right) + A\left(x'_{\lambda_3{}'}\right) - 1^+_{x'_{\lambda_3{}'}} > I_1(x'_{\lambda_3{}'})$, for each $v \in NQN(x'_{\langle\lambda_1{}',\lambda_2{}',\lambda_3{}'\rangle})$ and $I_1 \in I$. This is also true for μ.

So there is at least one $x''_{\langle\lambda_1{}'',\lambda_2{}'',\lambda_3{}''\rangle} \in_N N(X)$ such that $\mu\left(x''_{\lambda_1{}''}\right) + A\left(x''_{\lambda_1{}''}\right) - 1^+_{x''_{\lambda_1{}''}} > I_1(x''_{\lambda_1{}''})$, $\mu\left(x''_{\lambda_2{}''}\right) + A\left(x''_{\lambda_2{}''}\right) - 1^+_{x''_{\lambda_2{}''}} > I_1(x''_{\lambda_2{}''})$, $v\left(x''_{\lambda_3{}''}\right) + A\left(x''_{\lambda_3{}''}\right) - 1^+_{x''_{\lambda_3{}''}} >$

$I_1(x''_{\lambda_3}'')$, for each $I_1 \in I$, and. Since μ is an arbitrary neutrosophic quasi neighborhood of $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}$, therefore $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle} \in_N A^*$. Hence, $A^* \subseteq cl(A^*) \subseteq cl(A)$.

4- By (3), we have $A^{**} = cl((A^*)^*) \subseteq cl(A^*) = A^*$.

5- Suppose, $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle} \notin_N A^* \cup B^*$, i.e. $\lambda_1 > (A^* \cup B^*)(x_{\lambda_1}) = max\{A^*(x_{\lambda_1}), B^*(x_{\lambda_1})\}$, $\lambda_2 > (A^* \cup B^*)(x_{\lambda_2}) = max\{A^*(x_{\lambda_2}), B^*(x_{\lambda_2})\}$, $\lambda_3 > (A^* \cup B^*)(x_{\lambda_3}) = max\{A^*(x_{\lambda_3}), B^*(x_{\lambda_3})\}$. So $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}$ is not contained in both A^* and B^*. This implies there is at least one neutrosophic quasi neighborhood μ_1 of $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}$ such that for every $y_{\langle t_1,t_2,t_3\rangle} \in_N N(X)$, $\mu_1(y_{t_1}) + A(y_{t_1}) - 1^+_{y_{t_1}} > I_2(y_{t_1})$, $\mu_1(y_{t_2}) + A(y_{t_2}) - 1^+_{y_{t_2}} > I_2(y_{t_2})$, $\mu_1(y_{t_3}) + A(y_{t_3}) - 1^+_{y_{t_3}} > I_2(y_{t_3})$, for some $I_2 \in I$ and similarly there is at least one neutrosophic quasi neighborhood μ_2 of $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}$ such that for every $y_{\langle t_1,t_2,t_3\rangle} \in_N N(X)$, $\mu_2(y_{t_1}) + B(y_{t_1}) - 1^+_{y_{t_1}} > I_3(y_{t_1})$, $\mu_2(y_{t_2}) + B(y_{t_2}) - 1^+_{y_{t_2}} > I_3(y_{t_2})$, $\mu_2(y_{t_3}) + B(y_{t_3}) - 1^+_{y_{t_3}} > I_3(y_{t_3})$, for some $I_3 \in I$. Let $\mu = \mu_1 \cap \mu_2$. So μ is also a neutrosophic quasi neighborhood of $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}$ and $\mu(y_{t_1}) + (A \cup B)(y_{t_1}) - 1^+_{y_{t_1}} \leq (I_2 \cup I_3)(y_{t_1})$, $\mu(y_{t_2}) + (A \cup B)(y_{t_2}) - 1^+_{y_{t_2}} \leq (I_2 \cup I_3)(y_{t_2})$, $\mu(y_{t_3}) + A(y_{t_3}) - 1^+_{y_3} \leq I_1(y_{t_3})$, for every $y_{\langle t_1,t_2,t_3\rangle} \in_N N(X)$. Therefore, by finite additivity of neutrosophic ideal, as $I_2 \cup I_3 \in I$, $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle} \notin_N (A \cup B)^*$. Hence $(A \cup B)^* \subseteq A^* \cup B^*$. Clearly, both A and $B \subseteq A \cup B$ implies $A^* \cup B^* \subseteq (A \cup B)^*$ and this complete the prove of (5).

6- It is obvious that $I_1 \in I$ implies $I^* = 0_x$ so that $(A \cup I)^* = A^* \cup I^* = A^*$.

3.6 Definition

A neutrosophic closure operator $\psi: H(N(X)) \longrightarrow H(N(X))$ is defined by

1- $\psi\left(0_{x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}}\right) = 0_{x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}}$.

2- $A \in H(N(X)) \Longrightarrow A \subseteq \psi(A)$.

3- $A, B \in H(N(X)) \Longrightarrow \psi(A \cup B) = \psi(A) \cup \psi(B)$.

4- $A \in H(N(X)) \Longrightarrow \psi(\psi(A)) = \psi(A)$.

Obviously, $\{A: \psi(A) = A\}$ constitutes as a collection of neutrosophic closed sets for a neutrosophic topology on $N(X)$.

3.7 Theorem

Let $D: H(N(X)) \longrightarrow H(N(X))$ be a function such that:

1- $D\left(0_{x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}}\right) = 0_{x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}}$.

2- $D(A \cup B) = D(A) \cup D(B)$.

3- $D(D(A)) \subseteq D(A)$

Where A, B are any neutrosophic sets of $N(X)$. Then $\psi: H(N(X)) \longrightarrow H(N(X))$ defined by $\psi(A) = A \cup D(A)$ is a neutrosophic closure operator. Clearly, D does not necessarily coincide with neutrosophic derived set operator in the generated neutrosophic operator.

3.8 Theorem

$*: H(N(X)) \longrightarrow H(N(X))$ satisfies all the required condition for D.

Proof.

Since $0^*_{x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}} = 0_{x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}}$, $(A \cup B)^* = A^* \cup B^*$ and $(A^*)^* \subseteq A^*$, the proof is complete.

Let $(N(X), \tau)$ be a neutrosophic topological space and I be a neutrosophic ideal on $N(X)$. Let us define $cl^*(A) = A \cup A^*$ for any neutrosophic set A of $N(X)$. Clearly, $cl^*$is a neutrosophic closure operator. Let $\tau^*(I)$ be the neutrosophic topology generated by cl^*, i.e., $\tau^*(I) = \{A: cl^*(A^c) = A^c\}$, A^c will denote the complement of A.

Now, let $I = \left\{0_{x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}}\right\} \Longrightarrow cl^*(A) = A \cup A^* = A \cup cl(A) = cl(A)$, for every $A \in H(N(X))$. So, $\tau^*\left(\left\{0_{x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}}\right\}\right) = \tau$. Again, let $I = H\big(N(X)\big) \Longrightarrow cl^*(A) = A$, because $A^* = 0_{x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}}$, for every $A \in H(N(X))$. So, $\tau^*(H(N(X))$ is the neutrosophic discrete topology on $N(X)$. Since $\left\{0_{x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}}\right\}$ and $H(N(X))$ are two extreme neutrosophic ideals on $N(X)$, therefore for any neutrosophic ideal I on $N(X)$ we have $\left\{0_{x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}}\right\} \subseteq I \subseteq H(N(X))$. So we can conclude by theorem 3.5 (2), $\tau^*\left(\left\{0_{x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}}\right\}\right) \subseteq \tau^*\big(H(N(X))\big)$, i.e. $\tau \subseteq \tau^*(I) \subseteq$ neutrosophic discrete topology, for any neutrosophic ideal I on X. In particular, we have, for any two neutrosophic ideals I_1 and I_2 on $N(X)$, $I_1 \subseteq I_2 \Longrightarrow \tau^*(I_1) \subseteq \tau^*(I_2)$.

3.9 Theorem

Let τ_1, τ_2 be two neutrosophic topologies on $N(X)$. Then for any neutrosophic ideal I on $N(X)$, $\tau_1 \subseteq \tau_2$ implies that:

1- $A^*(\tau_2, I) \subseteq A^*(\tau_1, I)$, for every $A \in H(N(X))$.

2- $\tau_1^* \subseteq \tau_2^*$.

Proof.

Since every τ_1 neutrosophic quasi neighborhood of any neutrosophic point $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}$ is also a τ_2 neutrosophic quasi neighborhood of $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}$. Therefore, $A^*(\tau_2, I) \subseteq A^*(\tau_1, I)$ as there may be other τ_2 neutrosophic quasi neighborhood of $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}$ where the condition for $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle} \in_N A^*(\tau_2, I)$ may not hold true, although $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle} \in_N A^*(\tau_1, I)$. Clearly, $\tau_1^*(I) \subseteq \tau_2^*(I)$ as $A^*(\tau_2, I) \subseteq A^*(\tau_1, I)$.

3.10 Theorem

Suppose $A^{D^*}$is the neutrosophic derived set of A in τ^*neutrosophic topology, then, $A^{D^*} \subseteq A^D$ and $A^{D^*} \subseteq A^*$for all neutrosophic set A of $N(X)$.

Proof.

Since $\tau \subseteq \tau^*$. Therefore, $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle} \in_N A^{D^*}$ implies that every neutrosophic quasi neighborhood of $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}$ in neutrosophic topology τ^* is neutrosophic quasi concomitant with $A \Longrightarrow x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle} \in_N A^D$, so that $A^{D^*} \subseteq A^*$.

Again, for any neutrosophic point $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle} \in_N A^{D^*}$ implies $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle} \in_N cl^*(A) = A \cup A^*$, i.e. $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle} \in_N A$ or $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle} \in_N A^*$ or both. Now, if $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle} \in_N A$, then for any neutrosophic quasi neighborhood μ of $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}$ in neutrosophic topology τ^* , there exists $y_{\langle t_1,t_2,t_3\rangle} \in_N N(X)$ such that $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle} \neq y_{\langle t_1,t_2,t_3\rangle}$ and $A(y_{t_1}) + \mu(y_{t_1}) > 1^+_{y_{t_1}}$, $A(y_{t_2}) + \mu(y_{t_2}) > 1^+_{y_{t_2}}$, $A(y_{t_3}) + \mu(y_{t_3}) > 1^+_{y_3}$, this implies $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}$ is accumulation neutrosophic point of the neutrosophic set A' such that

$$A'(y_{\langle t_1,t_2,t_3\rangle}) = \begin{cases} A(y_{\langle t_1,t_2,t_3\rangle}) & if \quad y_{t_1} \neq x_{\lambda_1}, y_{t_2} \neq x_{\lambda_2}, y_{t_3} \neq x_{\lambda_3} \\ t_1 & if \quad y_{t_1} = x_{\lambda_1} \ and \ t_1 < \lambda_1 \\ t_2 & if \quad y_{t_2} = x_{\lambda_2} \ and \ t_2 < \lambda_2 \\ t_3 & if \quad y_{t_3} = x_{\lambda_3} \ and \ t_3 < \lambda_3 \end{cases}$$

Obviously, $A' \subseteq A$, so that $(A')^* \subseteq A^*$ also $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle} \notin_N A'$. Hence, $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle} \in_N (A')^*$, because $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle} \in_N cl^*(A') = A' \cup (A')^*$. So, $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle} \in_N A^*$, therefore, $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle} \in_N A^{D^*} \Longrightarrow x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle} \in_N A^* \Longrightarrow A^{D^*} \subseteq A^*$.

3.11 Definition

A neutrosophic set μ of $N(X)$ is called neutrosophic closed and discrete if and only if $\mu^D = 0_{x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}}$.

3.12 Theorem

Let $(N(X), \tau)$ be a neutrosophic topology space with I a fuzzy ideal on $N(X)$. Then,

1- $I_1 \in I$ is closed and discrete in $(N(X),\ \tau^*)$.

2- $A^* = cl(A - I_1)$ for every $I_1 \in I$ and for any neutrosophic set A of $N(X)$, where $A - I_1$ is the neutrosophic set defined by $(A - I_1)(x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}) = max\left\{A(x_{\lambda_1}) - I_1(x_{\lambda_1}), A(x_{\lambda_2}) - I_1(x_{\lambda_2}), A(x_{\lambda_3}) - I_1(x_{\lambda_3}), 0_{x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}}\right\}$, for every $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle} \in_N N(X)$.

Proof.

1) $I_1 \in I \Longrightarrow I_1^* = 0_{x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}}$, therefore by theorem 3.10, $I_1^{D^*} = 0_{x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}}$.

2) The proof is clear from the definition of neutrosophic local function and the closure of a neutrosophic set.

The above theorem characterizes a useful fact about the construction of different neutrosophic ideals in relation with the original neutrosophic topology and the generated neutrosophic topology. The following examples show some cases where the two neutrosophic topologies τ and τ^* on $N(X)$ are equal.

3.12 Example:

1- If I_1 be a neutrosophic ideal on $N(X)$ such that $A^D \subseteq cl(A - I_1)$ for every $I_1 \in I$ and for any neutrosophic set A of $N(X)$, then it is clear that $A^D \subseteq A^*$ so that $cl(A) = cl^*(A)$. Therefore, $\tau = \tau^*$.

2- Again, if I_1 be such that $A^D = (A - I_1)^D$ for every $I_1 \in I$, then it is obvious that $\tau = \tau^*$.

3- Also, $A^D = A^*$, for a neutrosophic ideal I on $N(X)$ implies $\tau = \tau^*$.

4. Basic Structure of Generated Neutrosophic Topology

Let $(N(X), \tau)$ be a neutrosophic topological space and I is a neutrosophic ideal on $N(X)$. Let A be a neutrosophic quasi neighborhood of a neutrosophic point $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}$ in the neutrosophic topology τ^*. Therefore, there exists $\mu \in \tau^*$ such that, $\lambda_1 + \mu(x_{\lambda_1}) > 1^+_{x_{\lambda_1}}$, $\lambda_2 + \mu(x_{\lambda_2}) > 1^+_{x_{\lambda_2}}$, $\lambda_3 + \mu(x_{\lambda_3}) > 1^+_{x_{\lambda_3}}$. And $\mu \subseteq A$. Now, $\mu \in \tau^* \Leftrightarrow \mu^c$ is closed in $\tau^* \Leftrightarrow cl^*(\mu^c) = \mu^c \Leftrightarrow (\mu^c)^* \subseteq \mu^c \Leftrightarrow \mu \subseteq \{(\mu^c)^*\}^c$. Therefore, $\lambda_1 + \mu(x_{\lambda_1}) > 1^+_{x_{\lambda_1}}$, $\lambda_2 + \mu(x_{\lambda_2}) > 1^+_{x_{\lambda_2}}$, $\lambda_3 + \mu\left(x_{\lambda_3}\right) > 1^+_{x_{\lambda_3}} \Rightarrow \lambda_1 + \{(\mu^c)^*\}^c(x_{\lambda_1}) > 1^+_{x_{\lambda_1}}$, $\lambda_2 + \{(\mu^c)^*\}^c(x_{\lambda_2}) > 1^+_{x_{\lambda_2}}$, $\lambda_3 + \{(\mu^c)^*\}^c(x_{\lambda_3}) > 1^+_{x_{\lambda_3}} \Rightarrow \lambda_1 > (\mu^c)^*(x_{\lambda_1})$, $\lambda_2 > (\mu^c)^*(x_{\lambda_2})$, $\lambda_3 > (\mu^c)^*(x_{\lambda_3}) \Rightarrow x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle} \notin_N (\mu^c)^*$. this implies there exists at least one neutrosophic quasi neighborhood v_1 of $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}$ in τ such that for every $y_{\langle t_1,t_2,t_3\rangle} \in_N N(X)$, $v_1\left(y_{t_1}\right) + \mu^c\left(y_{t_1}\right) - 1^+_{y_{t_1}} > I_1(y_{t_1})$, $v_1\left(y_{t_2}\right) + \mu^c\left(y_{t_2}\right) - 1^+_{y_{t_2}} > I_1(y_{t_2})$, $v_1\left(y_{t_3}\right) + \mu^c\left(y_{t_3}\right) - 1^+_{y_3} > I_1(y_{t_3})$, for some $I_1 \in I$, i.e. $v_1\left(y_{t_1}\right) - I_1\left(y_{t_1}\right) \leq \mu(y_{t_1})$, $v_1\left(y_{t_2}\right) - I_1\left(y_{t_2}\right) \leq \mu(y_{t_2})$, $v_1\left(y_{t_3}\right) - I_1\left(y_{t_3}\right) \leq \mu(y_{t_3})$, for every $y_{\langle t_1,t_2,t_3\rangle} \in_N N(X)$. Therefore, as v_1 is a neutrosophic quasi neighborhood of $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}$ in τ, there is $v \in \tau$ such that $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}$ is a neutrosophic quasi concomitant to $v \subseteq v_1$, and by heredity property of neutrosophic ideal we have $I_2 \in I$ for which $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}\, qc\, (v - I_2) \subseteq \mu$, we have $(v - I_2)\left(y_{\langle t_1,t_2,t_3\rangle}\right) = max\left\{v\left(y_{\langle t_1,t_2,t_3\rangle}\right) - I_2\left(y_{\langle t_1,t_2,t_3\rangle}\right), 0_{x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}}\right\}$, for every $y_{\langle t_1,t_2,t_3\rangle} \in_N N(X)$. Here for $\mu \in \tau^*$, we have $v \in \tau$ and $I_2 \in I$ such that, $(v - I_2) \subseteq \mu$. Let us denote $\beta(I, \tau) = \{v - I_2 : v \in_N \tau, I_2 \in I\}$.

4.1 Theorem

β forms a basis for the generated neutrosophic topology τ^* of the neutrosophic ideal I on $N(X)$.

Proof. Straight forward.

The following example is very important for the further results that justifies the above construction.

4.2 Example:

Let W be the neutrosophic indiscrete topology on $N(X)$, i.e. $W = \left\{0_{x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}}, 1_{x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}}\right\}$. So $1_{x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}}$ is the only neutrosophic quasi neighborhood of every neutrosophic point $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}$. Now, let

$x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle} \in_N A^*$for a neutrosophic set $A \Leftrightarrow$ for each $I_1 \in I$, there is at least one $y_{\langle t_1,t_2,t_3\rangle} \in_N N(X)$ such that $1^+_{y_{t_1}} + A(y_{t_1}) - 1^+_{y_{t_1}} > I_1\left(1^+_{y_{t_1}}\right), 1^+_{y_{t_2}} + A(y_{t_2}) - 1^+_{y_2} > I_1\left(1^+_{y_{t_2}}\right), 1^+_{y_{t_3}} + A(y_{t_3}) - 1^+_{y_{t_3}} > I_1(1^+_{y_{t_3}})$, this implies for each $I_1 \in I$, $A(y_{\langle t_1,t_2,t_3\rangle}) > I_1(y_{\langle t_1,t_2,t_3\rangle})$ for at least one $y_{\langle t_1,t_2,t_3\rangle} \in_N N(X)$. So $A \notin I$. Therefore, $A^* = 1_{x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}}$ if $A \notin I$ and $A^* = 0_{x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}}$ if $A \in I$. This implies that we have, $cl^*(A) = A \cup A^* = 1_{x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}}$, if $A \notin I$ and $cl^*(A) = A$, if $A \in I$, for any neutrosophic set A of $N(X)$. Hence $W^* = \{\mu: \mu^c \in I\}$. Let $\tau \vee W^*(I)$ be the supremum neutrosophic topology of τ and $W^*(I)$, i.e. the smallest neutrosophic topology generated by $\tau \cup W^*(I)$. Then we have the following theorem:

4.3 Theorem

$\tau^*(I) = \tau \vee W^*(I)$

Proof.

Follows from the fact that β forms a basis for τ^*.

4.4 Corollary

For any two neutrosophic ideals I_1 and I_2 on $N(X)$, $I_1 \vee I_2 = \{I_1 \cup I_2 : J_1 \in I_1, J_2 \in I_2\}$ and $I_1 \cap I_2$ are neutrosophic ideals on $N(X)$.

Proof.

It is clear and straight forward.

4.5 Theorem

Let $(N(X), \tau)$ be a neutrosophic topological space and I_1, I_2 be two neutrosophic ideals on $N(X)$. Then, for any neutrosophic set A of $N(X)$,

1- $A^*(I_1 \cap I_2) = A^*(I_1) \cup A^*(I_2)$

2- $A^*(I_1 \vee I_2, \tau) = A^*(I_1, \tau^*(I_2)) \cap A^*(I_2, \tau^*(I_1))$

Proof.

1- Let $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle} \notin_N A^*(I_1) \cup A^*(I_2)$. Then, $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle} \notin_N$ both $A^*(I_1)$ and $A^*(I_2)$. Now $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle} \notin_N A^*(I_1)$ implies there is at least one quasi neighborhood μ of $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}$ (in τ) such that for every $y_{\langle t_1,t_2,t_3\rangle} \in_N N(X)$, we have $\mu(y_{t_1}) + A(y_{t_1}) - 1^+_{y_{t_1}} \le I(y_{t_1})$, $\mu(y_{t_2}) + A(y_{t_2}) - 1^+_{y_{t_2}} > I(y_{t_2})$, $\mu(y_{t_3}) + A(y_{t_3}) - 1^+_{y_3} > I(y_{t_3})$, for some $I \in I_1$.

Again, $x_{\langle \lambda_1,\lambda_2,\lambda_3\rangle} \notin_N A^*(I_2)$ implies there is at least one quasi neighborhood v of $x_{\langle \lambda_1,\lambda_2,\lambda_3\rangle}$ (in τ) such that for every $y_{\langle t_1,t_2,t_3\rangle} \in_N N(X)$, we have $v(y_{t_1}) + A(y_{t_1}) - 1^+_{y_{t_1}} \le J(y_{t_1})$, $v(y_{t_2}) + A(y_{t_2}) - 1^+_{y_{t_2}} > J(y_{t_2})$, $v(y_{t_3}) + A(y_{t_3}) - 1^+_{y_3} > J(y_{t_3})$, for some $J \in I_2$ .therefore, we have $(\mu \cap v)(y_{t_1}) + A(y_{t_1}) - 1^+_{y_{t_1}} \le (I \cap J)(y_{t_1})$, $(\mu \cap v)(y_{t_2}) + A(y_{t_2}) - 1^+_{y_{t_2}} > (I \cap J)(y_{t_2})$, $(\mu \cap v)(y_{t_3}) + A(y_{t_3}) - 1^+_{y_3} > (I \cap J)(y_{t_3})$, for every $y_{\langle t_1,t_2,t_3\rangle} \in_N N(X)$. Since $(\mu \cap v)$ is also a quasi-neighborhood of $x_{\langle \lambda_1,\lambda_2,\lambda_3\rangle}$ (in τ) and $I \cap J \in I_1 \cap I_2$, therefore $x_{\langle \lambda_1,\lambda_2,\lambda_3\rangle} \notin_N A^*(I_1 \cap I_2)$, so that $A^*(I_1 \cap I_2) \subseteq A^*(I_1) \cap A^*(I_2)$. Also, $I_1 \cap I_2$ is included in both I_1 and I_2, so by theorem (3.4/2), reverse inclusion is obvious, which completes the proof of (1).

2- Since $x_{\langle \lambda_1,\lambda_2,\lambda_3\rangle} \notin_N A^*(I_1 \vee I_2, \tau)$ implies there is at least one quasi-neighborhood μ of $x_{\langle \lambda_1,\lambda_2,\lambda_3\rangle}$ $(in\ \tau)$ such that, for every $y_{\langle t_1,t_2,t_3\rangle} \in_N N(X)$, $\mu(y_{t_1}) + A(y_{t_1}) - 1^+_{y_{t_1}} \le I'(y_{t_1})$, $\mu(y_{t_2}) + A(y_{t_2}) - 1^+_{y_{t_2}} > I'(y_{t_2})$, $\mu(y_{t_3}) + A(y_{t_3}) - 1^+_{y_3} > I'(y_{t_3})$, for some $I' \in I_1 \vee I_2$. Therefore, by heredity of neutrosophic ideals and considering the structure of neutrosophic open sets in generated neutrosophic topology, we can find v or v', the quasi-neighborhood of the neutrosophic point $x_{\langle \lambda_1,\lambda_2,\lambda_3\rangle}$ in $\tau^*(I_1)$ or $\tau^*(I_2)$ respectively, such that for every $y_{\langle t_1,t_2,t_3\rangle} \in_N N(X)$, $v(y_{t_1}) + A(y_{t_1}) - 1^+_{y_{t_1}} \le J(y_{t_1})$, $v(y_{t_2}) + A(y_{t_2}) - 1^+_{y_{t_2}} > J(y_{t_2})$, $v(y_{t_3}) + A(y_{t_3}) - 1^+_{y_3} > J(y_{t_3})$, OR $v'(y_{t_1}) + A(y_{t_1}) - 1^+_{y_{t_1}} \le I(y_{t_1})$, $v'(y_{t_2}) + A(y_{t_2}) - 1^+_{y_{t_2}} > I(y_{t_2})$, $v'(y_{t_3}) + A(y_{t_3}) - 1^+_{y_3} > I(y_{t_3})$, for some $I \in I_1$ or $J \in I_2$. This implies $x_{\langle \lambda_1,\lambda_2,\lambda_3\rangle} \notin_N A^*(I_2, \tau^*(I_1))$ or $x_{\langle \lambda_1,\lambda_2,\lambda_3\rangle} \notin_N A^*(I_1, \tau^*(I_2))$, thus we have, $A^*(I_2, \tau^*(I_1)) \cap A^*(I_1, \tau^*(I_2)) \subseteq A^*(I_1 \vee I_2, \tau)$. Conversely, let $x_{\langle \lambda_1,\lambda_2,\lambda_3\rangle} \notin_N A^*(I_1, \tau^*(I_2))$. This implies there is at least one quasi-neighborhood μ in $\tau^*(I_2)$ of $x_{\langle \lambda_1,\lambda_2,\lambda_3\rangle}$ such that for every $y_{\langle t_1,t_2,t_3\rangle} \in_N N(X)$, $\mu(y_{t_1}) + A(y_{t_1}) - 1^+_{y_{t_1}} \le I_3(y_{t_1})$, $\mu(y_{t_2}) + A(y_{t_2}) - 1^+_{y_{t_2}} > I_3(y_{t_2})$, $\mu(y_{t_3}) + A(y_{t_3}) - 1^+_{y_3} > I_3(y_{t_3})$, for some $I_3 \in I_1$.Since μ is a $\tau^*(I_2)$ quasi-neighborhood of $x_{\langle \lambda_1,\lambda_2,\lambda_3\rangle}$, by heredity of neutrosophic ideals we have a quasi-neighborhood v of $x_{\langle \lambda_1,\lambda_2,\lambda_3\rangle}$ $(in\ \tau)$ such that for every $y_{\langle t_1,t_2,t_3\rangle} \in_N N(X)$, $v(y_{t_1}) + A(y_{t_1}) - 1^+_{y_{t_1}} \le (I \cup J)(y_{t_1})$, $v(y_{t_2}) + A(y_{t_2}) - 1^+_{y_{t_2}} > (I \cup J)(y_{t_2})$, $v(y_{t_3}) + A(y_{t_3}) - 1^+_{y_3} > (I \cup J)(y_{t_3})$, for some $I \in I_1$ and $J \in I_2$, i.e. $x_{\langle \lambda_1,\lambda_2,\lambda_3\rangle} \notin_N A^*(I_1 \vee I_2, \tau)$.

Thus, $A^*(I_1 \vee I_2, \tau) \subseteq A^*(I_1, \tau^*(I_2))$. Similarly, $A^*(I_1 \vee I_2, \tau) \subseteq A^*(I_2, \tau^*(I_1))$ and this completes the proof.

An important result follows from the above theorem that $\tau^*(I_1)$ and $[\tau^*(I_1)]^*$ [in short τ^{**}] are equal for any neutrosophic ideal on $N(X)$.

4.6 Corollary

Let $(N(X), \tau)$ be a neutrosophic topological space and I_1 be a neutrosophic ideal on $N(X)$. Then, $A^*(I_1, \tau) = A^*(I_1,\ \tau^*)$ and $\tau^*(I_1) = [\tau^*(I_1)]^*(I_1)$.

Proof.

By putting $I_1 = I_2$ in theorem (4.5/2) we have the required result.

4.7 Corollary

Let $(N(X), \tau)$ be a neutrosophic topological space and I_1, I_2 be two neutrosophic ideals on $N(X)$. Then,

i) $\tau^*(I_1 \vee I_2) = [\tau^*(I_2)]^*(I_1) = [\tau^*(I_1)]^*(I_2)$,

ii) $\tau^*(I_1 \vee I_2) = \tau^*(I_1) \vee \tau^*(I_2)$,

iii) $\tau^*(I_1 \cap I_2) = \tau^*(I_1) \cap \tau^*(I_2)$.

Proof.

(i) By theorem (4.5/2) the result follows.

(ii) By (i), we have, $\tau^*(I_1 \vee I_2) = [\tau^*(I_2)]^*(I_1) = [\tau^*(I_1)]^*(I_2)$ [by theorem 4.5/2). Since, $\tau \subseteq \tau^*$ for any neutrosophic ideal on $N(X)$. Therefore, $\tau^*(I_1 \vee I_2) = (\tau \vee \tau^*)(I_2) \vee \tau^*(I_1) = \tau^*(I_1) \vee \tau^*(I_2)$.

(iii) Since $I_1 \cap I_2$ is included in both I_1 and I_2, $\tau^*(I_1 \cap I_2)$ is included in both $\tau^*(I_1)$ and $\tau^*(I_2)$. Now μ is a neutrosophic open set in $\tau^*(I_1) \cap \tau^*(I_2)$, implies μ^c is neutrosophic closed set in both $\tau^*(I_1)$ and $\tau^*(I_2)$. That means $(\mu^c)^*(I_1) \subseteq \mu^c$ and $(\mu^c)^*(I_2) \subseteq \mu^c$. So, $(\mu^c)^*(I_1) \cup (\mu^c)^*(I_2) \subseteq \mu^c$. Therefore, by theorem (4.5/1), $(\mu^c)^*(I_1 \cap I_2) \subseteq \mu^c$. Hence, $\mu \in \tau^*(I_1 \cap I_2)$. This completes the proof.

5. Conclusion:

This work contains new insight into defining many mathematical notions from corners that have not been addressed before, such as neutrosophic topological space, neutrosophic ideal, neutrosophic quasi neighborhood, and neutrosophic point $x_{\langle\lambda_1,\lambda_2,\lambda_3\rangle}$ in the neutrosophic topology τ^*. As well as, many theorems and corollaries, some examples that have support the theoretical concepts.

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On neutrosophic $n-$normed linear spaces

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Abstract. In this current paper, our objective is to establish neutrosophic $n-$normed linear space(briefly called $N-n-NLS$) and introduce the convergence structure within these spaces. We also define Cauchy sequences, completeness in $N-n-NLS$ and obtained the relations between these notions.

Keywords: Neutrosophic normed linear spaces, $n-$normed space, convergence and Cauchy sequences.

1. Introduction

Nearly about 60 years ago, Zadeh [42] introduced fuzzy sets as a generalization of crisp set to address those problems which can't be modeled in the framework of crisp sets. These sets have vast applicability in many areas of science and engineering especially, in control engineering, decision making theory [40], fuzzy physics [28], artificial intelligence and robotics [41]. Katsaras [24] first observed that there are some situations in which the precise value of the norm of a vector can't be determined and therefore the concept of a fuzzy norm seems more appropriate as compared to the crisp norm. In view of this, he introduced the concept of fuzzy normed linear spaces and proved that any two fuzzy norms are equivalent. For further developments on these spaces, we recommend to the reader [24], [27-28],[30], [32], etc. Sadati and Park [34] generalized fuzzy normed linear space, called intutionistic fuzzy normed space while studying fuzzy topological spaces. Later, Karakus et al. [20] studied generalized convergence called statistical convergence in intutionistic fuzzy normed space. However, Srinivasan Vijayabalaji et al. [40] defined the intutionistic fuzzy n-normed linear space (IF-n-NLS) and introduced the notions of convergence and Cauchy sequence. Subsequently, these spaces have been developed in [4, 15, 18, 20] etc.

Recently, Kirişci and Şimşek [22] introduce a more generalized form of fuzzy normed space called neutrosophic normed linear space and studied statistical convergence in these spaces. After their pioneer work, many papers have been appeared on $NNLS$ and linked with summability theory. For an extensive view, we refer [21,23, 29-32, 37, 38] etc. In present work, we are motivated by the works of [40] and [22] to define a $N-n-NLS$ and develop some fundamental notions of convergence and Cauchy sequences in these spaces.

2. **Preliminaries and background**

This section starts with some basic definitions, results and terminology on neutrosophic normed linear spaces and $n-$norm [22].

Definition 2.1 "A binary operation $\circ : \Im \times \Im \to \Im$, where $\Im = [0,\ 1]$ is continuous $t-$norm, for each τ_1, τ_2, τ_3 and $\tau_4 \in [0, 1]$ if the below conditions are satisfied:
(i) $\circ$ is continuous, commutative and associative;
(ii) $\tau_1 \circ 1 = \tau_1$
(iii) $\tau_1 \circ \tau_2 \le \tau_3 \circ \tau_4$ whenever $\tau_1 \le \tau_3$ and $\tau_2 \le \tau_4$."

Definition 2.2 "A binary operation $\diamond : \Im \times \Im \to \Im$, where $\Im = [0,\ 1]$ is continuous $t-$conorm, for each τ_1, τ_2, τ_3 and $\tau_4 \in [0, 1]$ if the below conditions are satisfied:
(i) $\diamond$ is continuous, commutative and associative;
(ii) $\tau_1 \diamond 0 = \tau_1$
(iii) $\tau_1 \diamond \tau_2 \le \tau_3 \diamond \tau_4$ whenever $\tau_1 \le \tau_3$ and $\tau_2 \le \tau_4$.

Kirişci and Şimşek [22] used Definition 2.1 and Definition 2.2 to define neutrosophic normed linear space as follows."

Definition 2.3 [22]"Let U be a linear space over F and $\circ$, $\diamond$ respectively denotes t-norm and t-conorm, let G, B, Y are function from $(\varrho, \lambda_1) \in U \times (0, \infty)$ to $[0, 1]$. A six tuple $(U, G, B, Y, \circ, \diamond)$ is called a neutrosophic normed linear space, if the below properties are satisfied:

For every $\varrho, v \in U$, $\lambda_1, \lambda_2 > 0$ and scaler $\alpha \neq 0$, we have
(i) $0 \le G(\varrho,\ \lambda_1) \le 1, 0 \le\ B(\varrho,\ \lambda_1) \le 1, 0 \le Y(\varrho,\ \lambda_1) \le 1$;
(ii) $G(\varrho,\ \lambda_1) +\ B(\varrho,\ \lambda_1) + Y(\varrho,\ \lambda_1) \le 3$;
(iii) $G(\varrho,\ \lambda_1) = 1,\ B(\varrho,\ \lambda_1) = 0$ and $Y(\varrho,\ \lambda_1) = 0$ if and only if $\varrho = 0$;
(iv) $G(\alpha\varrho,\ \lambda_1) = G\left(\varrho,\ \frac{\lambda_1}{|\alpha|}\right),\ B(\alpha\varrho,\ \lambda_1) = B\left(\varrho,\ \frac{\lambda_1}{|\alpha|}\right)$ and $Y(\alpha\varrho,\ \lambda_1) = Y\left(\varrho,\ \frac{\lambda_1}{|\alpha|}\right)$;
(v) $G(\varrho,\ \lambda_1) \circ G(v,\ \lambda_2) \le G(\varrho + v,\ \lambda_1 + \lambda_2),\ B(\varrho,\ \lambda_1) \diamond B(v,\ \lambda_2) \ge B(\varrho + v,\ \lambda_1 + \lambda_2)$ and $Y(\varrho,\ \lambda_1) \diamond Y(v,\ \lambda_2) \ge Y(\varrho + v,\ \lambda_1 + \lambda_2)$;
(vi) $G(\varrho,\ .)$ is a non-decreasing continuous function;

(vii) $B(\varrho,\ .)$ and $Y(\varrho,\ .)$ are non-increasing continuous function;

(viii) $\lim_{\lambda_1\to\infty} G(\varrho,\ \lambda_1) = 1$, $\lim_{\lambda_1\to\infty} B(\varrho,\ \lambda_1) = 0$ and $\lim_{\lambda_1\to\infty} Y(\varrho,\ \lambda_1) = 0$;

(ix) If $\lambda_1 \leq 0$, then $G(\varrho,\ \lambda_1) = 0, \quad B(\varrho,\ \lambda_1) = 1$ and $Y(\varrho,\ \lambda_1) = 1$.

We call $N(G, B, Y)$ as the neutrosophic norm and $(U, G, B, Y, \circ, \diamond)$ the neutrosophic normed linear space. For some examples on these spaces, we refer [22]."

Finally we recall the concept of $n-$norm as given in [16].

Definition 2.4 [16] "Let U be a real space of dimension $m \geq n$ (m is finite or infinite, $n \in \mathfrak{n}$) the real valued function $||.||_{\mathfrak{n}}$ on $U \times U \times ... \times U = U^{\mathfrak{n}}$ is called $n-$norm on U if and only if it satisfying the below axioms:

(i): $||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}}||_{\mathfrak{n}} = 0$ iff $\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}} \in U$ are linearly dependent;

(ii): $||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}}||_{\mathfrak{n}}$ remains invariant for $1 \leq i \leq n$;

(iii): $||\varrho_1, \varrho_2, \ldots, \alpha\varrho_{\mathfrak{n}}||_{\mathfrak{n}} = |\alpha|||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}}||_{\mathfrak{n}}$ for any $\alpha \in \mathbb{R}$;

(iv): $||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, v + w||_{\mathfrak{n}} \leq ||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, v||_{\mathfrak{n}} + ||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, w||_{\mathfrak{n}}$.

The pair $(U, ||.||_{\mathfrak{n}})$ is known as $n-$normed linear space."

3. N-n-NLS

We, now turn towards our main results. We start with the following definition of neutrosophic $-n-$ normed space.

Definition 3.1 Let U be a linear space over F and $\circ$, $\diamond$ respectively denotes t-norm and t-conorm, let G_0, B_0, Y_0 are function from $U^{\mathfrak{n}} \times (0, \infty)$ to $[0, 1]$. A six tuple $(U, G_0, B_0, Y_0, \circ, \diamond)$ is called a neutrosophic $n-$normed linear space, $(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) \in U^{\mathfrak{n}} \times (0, \infty) \to [0, 1]$, if the below properties are satisfied:

(i)$G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) + B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) + Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) \leq 3$;

(ii) $G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) > 0$;

(iii) $G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) = 1$, iff ϱ_i are dependent for $1 \leq i \leq \mathfrak{n}$;

(iv) $G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1)$ remains invariant, ϱ_i for $1 \leq i \leq \mathfrak{n}$;

(v) $G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \alpha\varrho_{\mathfrak{n}}, \lambda_1) = G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \frac{\lambda_1}{|\alpha|})$ for $\alpha \neq 0,\ \alpha \in F$;

(vi) $G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) \circ G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho'_{\mathfrak{n}}, \lambda_2)$
$\geq G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}} + \varrho'_{\mathfrak{n}}, \lambda_1 + \lambda_2)$;

(vii) $G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1)$ is non-decreasing continuous in λ_1

(viii) $\lim\limits_{\lambda_1\to\infty} G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) = 1$ and $\lim\limits_{\lambda_1\to 0} G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) = 0$;

(ix) $B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) > 0$;

(x) $B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) = 1$ iff ϱ_i are dependent for $1 \leq i \leq \mathfrak{n}$

(xi) $B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1)$ remains invariant, ϱ_i for $1 \leq i \leq \mathfrak{n}$;

(xii) $B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \alpha\varrho_{\mathfrak{n}}, \lambda_1) = B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \frac{\lambda_1}{|\alpha|})$ for $\alpha \neq 0,\ \alpha \in F$;

(xiii) $B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) \diamond B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}, \lambda_2)$
$\geq B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}} + \varrho_{\mathfrak{n}}^{'}, \lambda_1 + \lambda_2)$;
(xiv) $B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1)$ is non-increasing continuous in λ_1;
(xv) $\lim_{\lambda_1 \to \infty} B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) = 0$ and $\lim_{\lambda_1 \to 0} B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) = 1$;
(xvi) $Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) > 0$;
(xvii) $Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) = 1$ iff ϱ_i are dependent for $1 \leq i \leq \mathfrak{n}$
(xviii) $Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1)$ remains invariant, ϱ_i for $1 \leq i \leq \mathfrak{n}$;
(xix) $Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \alpha\varrho_{\mathfrak{n}}, \lambda_1) = Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \frac{\lambda_1}{|\alpha|})$ for $\alpha \neq 0,\ \alpha \in F$;
(xx) $Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) \diamond Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}, \lambda_2)$
$\geq Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}} + \varrho_{\mathfrak{n}}^{'}, \lambda_1 + \lambda_2)$;
(xxi) $Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1)$ is non-increasing continuous in λ_1;
(xxii) $\lim_{\lambda_1 \to \infty} Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) = 0$ and $\lim_{\lambda_1 \to 0} Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) = 1$.
For simplicity, we shall denote the neutrosophic $n-$norm by $N_{\mathfrak{n}}(G_0, B_0, Y_0)$.

Example 3.1 Let $(U, \|.\|_{\mathfrak{n}})$ be an $n-$normed space. For $\tau_1, \tau_2 \in [0, 1]$, define, $t-$norm, $t-$conorm by $\tau_1 \circ \tau_2 = \min\{\tau_1, \tau_2\}$ and $\tau_1 \diamond \tau_2 = \max\{\tau_1, \tau_2\}$ and fuzzy sets G_0, B_0, Y_0 on $U^{\mathfrak{n}} \times (0, \infty)$ by

$$G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) = \frac{\lambda_1}{\lambda_1 + \|\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}\|_{\mathfrak{n}}} \text{ and}$$

$$B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) = \frac{\|\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}\|_{\mathfrak{n}}}{\lambda_1 + \|\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}\|_{\mathfrak{n}}},$$

$$Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) = \frac{\|\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}\|_{\mathfrak{n}}}{\lambda_1},$$

then $(U, G_0, B_0, Y_0, \circ, \diamond)$ is a $N - n - NLS$.

Proof (i) and (ii) directly follows from definition of G_0, B_0, Y_0, i.e.,

$$(i)\ G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) + B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) + Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) \leq 3;$$

$$(ii)\ G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) > 0;$$

$$\begin{aligned}(iii)\ G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) = 1 &\Leftrightarrow \frac{\lambda_1}{\lambda_1 + \|\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}\|_{\mathfrak{n}}} = 1 \\ &\Leftrightarrow ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}} = 0 \\ &\Leftrightarrow \varrho_i \text{ are linearly dependent for } 1 \leq i \leq \mathfrak{n}.\end{aligned}$$

$$\begin{aligned}(iv)\ G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) = \frac{\lambda_1}{\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}}} &= \frac{\lambda_1}{\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}}} \\ &= \frac{\lambda_1}{\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}}, \varrho_{\mathfrak{n}-1}||_{\mathfrak{n}}} = \cdots\end{aligned}$$

$$(v)\ G_0\left(\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n, \frac{\lambda_1}{|\alpha|}\right) = \frac{\frac{\lambda_1}{|\alpha|}}{\frac{\lambda_1}{|\alpha|} + ||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n||_n} = \frac{\lambda_1}{\lambda_1 + |\alpha|||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n||_n}$$
$$= \frac{\lambda_1}{\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \alpha\varrho_n||_n} = G_0(\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \alpha\varrho_n, \lambda_1).$$

(vi) In general, let's suppose that,

$$\begin{aligned}
&G_0(\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n^{'}, \lambda_1) \leq G_0(\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n, \lambda_2)\\
&\Rightarrow \frac{\lambda_1}{\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n^{'}||_n} \leq \frac{\lambda_2}{\lambda_2 + ||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n||_n}\\
&\Rightarrow \lambda_1(\lambda_2 + ||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n||_n) \leq \lambda_2(\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n^{'}||_n)\\
&\Rightarrow \lambda_1||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n||_n \leq \lambda_2||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n^{'}||_n\\
&\Rightarrow ||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n||_n \leq \left(\frac{\lambda_2}{\lambda_1}\right)||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n^{'}||_n.
\end{aligned}$$

$$\begin{aligned}
\therefore\quad &||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n||_n + ||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n^{'}||_n\\
&\leq \left(\frac{\lambda_2}{\lambda_1}\right)||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n^{'}||_n + ||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n^{'}||_n\\
&\leq \left(\frac{\lambda_2}{\lambda_1} + 1\right)||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n^{'}||_n = \left(\frac{\lambda_2 + \lambda_1}{\lambda_1}\right)||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n^{'}||_n.
\end{aligned}$$

Now,

$$\begin{aligned}
&||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n + \varrho_n^{'}||_n \leq ||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n||_n + ||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n^{'}||_n\\
&\qquad \leq \left(\frac{\lambda_2 + \lambda_1}{\lambda_1}\right)||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n^{'}||_n\\
&\Rightarrow \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n + \varrho_n^{'}||_n}{\lambda_2 + \lambda_1} \leq \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n^{'}||_n}{\lambda_1}\\
&\Rightarrow 1 + \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n + \varrho_n^{'}||_n}{\lambda_2 + \lambda_1} \leq 1 + \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n^{'}||_n}{\lambda_1}\\
&\Rightarrow \frac{\lambda_2 + \lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n + \varrho_n^{'}||_n}{\lambda_2 + \lambda_1} \leq \frac{\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n^{'}||_n}{\lambda_1}\\
&\Rightarrow \frac{\lambda_2 + \lambda_1}{\lambda_2 + \lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n + \varrho_n^{'}||_n} \geq \frac{\lambda_1}{\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n^{'}||_n}\\
&\Rightarrow G_0(\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n + \varrho_n^{'}, \lambda_2 + \lambda_1)\\
&\qquad \geq \min\{G_0(\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n, \lambda_2), G_0(\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n^{'}, \lambda_1)\}\\
&\qquad = G_0(\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n, \lambda_2) \circ G_0(\varrho_1, \varrho_2, \ldots, \varrho_{n-1}, \varrho_n^{'}, \lambda_1).
\end{aligned}$$

(vii) Clearly $G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1)$ is non-decreasing continuous in λ_1.

(viii) $B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) > 0$.

$$(ix)\ B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) = 0 \Rightarrow \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}}}{\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}}} = 0$$
$$\Rightarrow ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}} = 0$$

$\Rightarrow \varrho_i$ are linearly dependent for $1 \le i \le \mathfrak{n}$.

$$(x)\ B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) = \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}}}{\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}}} = \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}}}{\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}}} = \cdots$$

$$(xi)\ B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \alpha\varrho_{\mathfrak{n}}, \lambda_1) = \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \alpha\varrho_{\mathfrak{n}}||_{\mathfrak{n}}}{\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \alpha\varrho_{\mathfrak{n}}||_{\mathfrak{n}}} = \frac{|\alpha|||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}}}{\lambda_1 + |\alpha|||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}}}$$
$$= \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}}}{\frac{\lambda_1}{|\alpha|} + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}}} = B_0\left(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \frac{\lambda_1}{|\alpha|}\right).$$

(xii) In general, let's suppose that,

$$B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_2) \le B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}, \lambda_1)$$
$$\Rightarrow \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}}}{\lambda_2 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}}} \le \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}}}{\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}}}$$
$$\Rightarrow ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}}(\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}})$$
$$\le ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}}(\lambda_2 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}})$$
$$\Rightarrow \lambda_1||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}} \le \lambda_2||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}}.$$
$$\Rightarrow \lambda_1||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}} - \lambda_2||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}} \le 0.$$

Now,

$$\frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}} + \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}}}{\lambda_2 + \lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}} + \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}}} - \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}}}{\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}}}$$
$$\le \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}} + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}}}{\lambda_2 + \lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}} + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}}} - \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}}}{\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}}}$$
$$= \frac{\lambda_1||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}} - \lambda_2||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}}}{(\lambda_2 + \lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}} + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}})(\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}})} \le 0.$$

This implies that, $$\frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}} + \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}}}{\lambda_2 + \lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}} + \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}}} \le \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}}}{\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}}}.$$

Similarly, $$\frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}} + \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}}}{\lambda_2 + \lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}} + \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}}} \le \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}}}{\lambda_2 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}}}.$$

This implies that,

$$B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}} + \varrho_{\mathfrak{n}}^{'}, \lambda_2 + \lambda_1)$$
$$\le \max\{B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_2), B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}, \lambda_1)\}.$$
$$B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_2) \diamond B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}, \lambda_1).$$

(xiii) Clearly $B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1)$ is non-increasing continuous in λ_1.

(xiv) $Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) > 0$.

$$(xv)\ Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) = 0 \Rightarrow \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}}}{\lambda_1} = 0$$
$$\Rightarrow ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}} = 0$$
$$\Rightarrow \varrho_i \text{ are linearly dependent for } 1 \leq i \leq \mathfrak{n}.$$

$$(xvi)\ Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1) = \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}}}{\lambda_1} = \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}}, \varrho_{\mathfrak{n}-1}||_{\mathfrak{n}}}{\lambda_1}$$
$$= Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}}, \varrho_{\mathfrak{n}-1}, \lambda_1) = \cdots .$$

$$(xvii)\ Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \alpha\varrho_{\mathfrak{n}}, \lambda_1) = \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \alpha\varrho_{\mathfrak{n}}||_{\mathfrak{n}}}{\lambda_1} = \frac{|\alpha|||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}}}{\lambda_1}$$
$$= \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}}}{\frac{\lambda_1}{|\alpha|}} = Y_0\left(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \frac{\lambda_1}{|\alpha|}\right).$$

$(xviii)$In general, let's suppose that,

$$Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_2) \leq Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}, \lambda_1)$$
$$\Rightarrow \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}}}{\lambda_2} \leq \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}}}{\lambda_1}$$
$$\Rightarrow \lambda_1||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}} \leq \lambda_2||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}}$$
$$\Rightarrow \lambda_1||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}} - \lambda_2||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}} \leq 0.$$

Now,

$$\frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}} + \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}}}{\lambda_2 + \lambda_1} - \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}}}{\lambda_1}$$
$$\leq \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}} + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}}}{\lambda_2 + \lambda_1} - \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}}}{\lambda_1}$$
$$= \frac{\lambda_1||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}} - \lambda_2||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}}}{\lambda_1(\lambda_2 + \lambda_1)} \leq 0.$$

This implies that, $$\frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}} + \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}}}{\lambda_2 + \lambda_1} \leq \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}}}{\lambda_1}.$$

Similarly, $$\frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}} + \varrho_{\mathfrak{n}}^{'}||_{\mathfrak{n}}}{\lambda_2 + \lambda_1} \leq \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}||_{\mathfrak{n}}}{\lambda_2}$$

Therefore, $$Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}} + \varrho_{\mathfrak{n}}^{'}, \lambda_2 + \lambda_1)$$
$$\leq \max\{Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_2), Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}, \lambda_1)\}.$$
$$Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_2) \diamond Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}^{'}, \lambda_1).$$

(xix) Clearly $Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \varrho_{\mathfrak{n}}, \lambda_1)$ is non-increasing continuous in λ_1.

Thus, $N_{\mathfrak{n}}(G_0, B_0, Y_0)$ satisfy all conditions of a neutrosophic $n-$norm and the space $(U, G_0, B_0, Y_0, \circ, \diamond)$ is a $N-n-NLS$ becomes a neutrosophic $n-$normed linear space $\square$

Definition 3.2 A sequence $v = (v_k)$ in a $N - n - NLS$ $(U, G_0, B_0, Y_0, \circ, \diamond)$ is said to be convergent to v_0 w.r.t. the norm $N_{\mathfrak{n}}(G_0, B_0, Y_0)$ if for $\epsilon > 0$, $\lambda_1 > 0$ and $\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1} \in U$, $\exists\ \mathfrak{n}_0 \in \mathbb{N}$ s.t. $G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) > 1 - \epsilon$ and $B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) < \epsilon$, $Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) < \epsilon$, $\forall\ k \geq \mathfrak{n}_0$. In this case, we write $N_{\mathfrak{n}}(G_0, B_0, Y_0) - \lim_{k\to\infty} v_k$. $\square$

Example 3.2 Consider the neutrosophic $n-$normed linear space as given in example 3.1. Define a sequence $v = (v_k)$ by $v_k = \frac{1}{k}$, then for each $\epsilon > 0$, $\lambda_1 > 0$ and $\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1} \in U$ we have

$$
\begin{aligned}
G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k, \lambda_1) &= G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \frac{1}{k}, \lambda_1) \\
&= \frac{\lambda_1}{\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \frac{1}{k}||_{\mathfrak{n}}} \to 1 \text{ as } k \to \infty \text{ and} \\
B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k, \lambda_1) &= B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \frac{1}{k}, \lambda_1) \\
&= \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \frac{1}{k}||_{\mathfrak{n}}}{\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \frac{1}{k}||_{\mathfrak{n}}} \to 0 \text{ as } k \to \infty, \\
Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k, \lambda_1) &= Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \frac{1}{k}, \lambda_1) \\
&= \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \frac{1}{k}||_{\mathfrak{n}}}{\lambda_1} \to 0 \text{ as } k \to \infty,
\end{aligned}
$$

this shows that the sequence $v_k = (\frac{1}{k})$ is convergent to θ where θ denotes the zero element in U w.r.t. $N_{\mathfrak{n}}(G_0, B_0, Y_0)$.

Theorem 3.1 For any sequence $v = (v_k)$, in a $N - n - NLS$ $(U, G_0, B_0, Y_0, \circ, \diamond)$ with $N_{\mathfrak{n}}(G, B, Y) - \lim_k v_k = v_0$, v_0 is unique.

Proof Let, if possible $N_{\mathfrak{n}}(G_0, B_0, Y_0) - \lim_k v_k = v_0$ and $N_{\mathfrak{n}}(G_0, B_0, Y_0) - \lim_k v_k = w_0$. Let $\epsilon > 0$ and $\lambda_1 > 0$ be given. Choose $\epsilon_1 > 0$ s.t.

$$(1 - \epsilon_1) \circ (1 - \epsilon_1) > 1 - \epsilon \text{ and } \epsilon_1 \diamond \epsilon_1 < \epsilon. \tag{1}$$

Since $N_{\mathfrak{n}}(G_0, B_0, Y_0) - \lim_k v_k = v_0$ so there exists $\mathfrak{n}_1 \in \mathbb{N}$ s.t.$\forall\ k \geq \mathfrak{n}_1$

$$G\left(v_k - v_0, \frac{\lambda_1}{2}\right) > 1 - \epsilon_1 \text{ and } B\left(v_k - v_0, \frac{\lambda_1}{2}\right) < \epsilon_1, Y\left(v_k - v_0, \frac{\lambda_1}{2}\right) < \epsilon_1.$$

Further, $N_{\mathfrak{n}}(G_0, B_0, Y_0) - \lim_k v_k = w_0$, will give another $\mathfrak{n}_2 \in \mathbb{N}$ s.t. $\forall k \geq \mathfrak{n}_2$

$$G\left(v_k - w_0, \frac{\lambda_1}{2}\right) > 1 - \epsilon_1 \text{ and } B\left(v_k - w_0, \frac{\lambda_1}{2}\right) < \epsilon_1, Y\left(v_k - w_0, \frac{\lambda_1}{2}\right) < \epsilon_1.$$

Let, $\mathfrak{n}_0 = \max\{\mathfrak{n}_1, \mathfrak{n}_2\}$, then for all $k \geq \mathfrak{n}_0$

$$\begin{aligned} G(v_0 - w_0, \lambda_1) &= G\bigg(v_0 - v_k + v_k - w_0, \frac{\lambda_1}{2} + \frac{\lambda_1}{2}\bigg) \\ &\geq G\bigg(v_k - v_0, \frac{\lambda_1}{2}\bigg) \circ G\bigg(v_k - w_0, \frac{\lambda_1}{2}\bigg) \\ &> (1-\epsilon_1) \circ (1-\epsilon_1) > 1-\epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary so $G(v_0 - w_0, \lambda_1) = 1$ and therefore $v_0 - w_0 = 0$ i.e., $v_0 = w_0$.
Now,

$$\begin{aligned} B(v_0 - w_0, \lambda_1) &= B\bigg(v_0 - v_k + v_k - w_0, \frac{\lambda_1}{2} + \frac{\lambda_1}{2}\bigg) \\ &\leq B\bigg(v_k - v_0, \frac{\lambda_1}{2}\bigg) \diamond B\bigg(v_k - w_0, \frac{\lambda_1}{2}\bigg) \\ &< \epsilon_1 \diamond \epsilon_1 < \epsilon, \\ Y(v_0 - w_0, \lambda_1) &= Y\bigg(v_0 - v_k + v_k - w_0, \frac{\lambda_1}{2} + \frac{\lambda_1}{2}\bigg) \\ &\leq Y\bigg(v_k - v_0, \frac{\lambda_1}{2}\bigg) \diamond Y\bigg(v_k - w_0, \frac{\lambda_1}{2}\bigg) \\ &< \epsilon_1 \diamond \epsilon_1 < \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary so we have $B(v_0 - w_0, \lambda_1) = Y(v_0 - w_0, \lambda_1) = 0$ which gives $v_0 - w_0 = 0$ i.e., $v_0 = w_0$. Hence, in all cases v_0 is uniquely determined.□

Theorem 3.2 If $v = (v_k)$ and $w = (w_k)$ be any two sequences in a $N-n-NLS$ $(U, G_0, B_0, Y_0, \circ, \diamond)$ s.t. $\lim v_k = v_0$ and $\lim w_k = w_0$ then,

(i) $\lim_k \alpha v_k = \alpha v_0,\ \alpha \neq 0$ for any scalar.

(ii) $\lim_k (v_k + w_k) = v_0 + w_0$.

Proof. The proof of the theorem follow parallel lines of the proof of theorem 3.1, so omitted.
□

Theorem 3.3 A sequence $v = (v_k)$ in a $N-n-NLS$ $(U, G_0, B_0, Y_0, \circ, \diamond)$ is convergent to v_0 w.r.t. $N_\mathfrak{n}(G_0, B_0, Y_0)$, if and only if, $G(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) \to 1$ and $B(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) \to 0$, $Y(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) \to 0$ as $\mathfrak{n} \to \infty$, where $\lambda_1 > 0$.

Proof Let $v = (v_k)$ converges to v_0 w.r.t. $N_\mathfrak{n}(G_0, B_0, Y_0)$. So, for $0 < \epsilon < 1$ and $\lambda_1 > 0$, $\exists\ \mathfrak{n}_0 \in \mathbb{N}$ s.t. $G(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) > 1 - \epsilon$ and $B(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) < \epsilon$, $Y(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) < \epsilon$, $\forall\ k \geq \mathfrak{n}_0$. This implies, for $\forall\ k \geq \mathfrak{n}_0$. $1 - G(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) < \epsilon$ and $B(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) < \epsilon$, $Y(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) < \epsilon$, which shows $G(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) \to 1$ and $B(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) \to 0$, $Y(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) \to 0$ as $k \to \infty$.

Conversely, suppose that for $\lambda_1 > 0$, $G(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) \to 1$ and $B(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) \to 0$, $Y(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) \to 0$ as $k \to \infty$. For $0 < \epsilon <$

1, $\exists\ \mathfrak{n}_0 \in \mathbb{N}$ s.t. $1 - G(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) < \epsilon$ and $B(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) < \epsilon$, $Y(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) < \epsilon$, which gives $G(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) > 1 - \epsilon$ and $B(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) < \epsilon$, $Y(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) < \epsilon$. Hence, (v_k) converges to v_0 w.r.t. $N_{\mathfrak{n}}(G, B, Y)$. $\square$

Definition 3.3 A sequence $v = (v_k)$ in a $N-n-NLS$ $(U, G_0, B_0, Y_0, \circ, \diamond)$ is said to be Cauchy w.r.t. $N_{\mathfrak{n}}(G_0, B_0, Y_0)$ if for $\epsilon > 0$ and $\lambda_1 > 0$, $\exists\ \mathfrak{n}_0 \in \mathbb{N}$ s.t. $G(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_p, \lambda_1) > 1-\epsilon$ and $B(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_p, \lambda_1) < \epsilon$, $Y(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_p, \lambda_1) < \epsilon$, $\forall\ k, p \geq \mathfrak{n}_0$. $\square$

Example 3.3 Let us consider the space $U = (0, 1]$. If we define G_0, B_0 and Y_0 by $G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k; \lambda_1) = \frac{\lambda_1}{\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k||_{\mathfrak{n}}}$, $B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k; \lambda_1) = \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k||_{\mathfrak{n}}}{\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k||_{\mathfrak{n}}}$, $Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k; \lambda_1) = \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k||_{\mathfrak{n}}}{\lambda_1}$ and the $t-$norm, $t-$conorm respectively as $\circ = \min$ and $\diamond = \max$, then $(U, G_0, B_0, Y_0, \circ, \diamond)$ becomes a $N - n - NLS$.

Defined a sequence $v = (v_k)$ where $v_k = \frac{1}{k}$ as in example 3.2 then for $\epsilon > 0$, $\lambda_1 > 0$ and $\varrho_1, \varrho_2, \ldots, \varrho_{n-1} \in U$ we have

$G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_p, \lambda_1) = G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \frac{1}{k} - \frac{1}{p}, \lambda_1) = \frac{\lambda_1}{\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \frac{1}{k} - \frac{1}{p}||_{\mathfrak{n}}} \to 1$ as $k, p \to \infty$,

$B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_p, \lambda_1) = B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \frac{1}{k} - \frac{1}{p}, \lambda_1) = \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \frac{1}{k} - \frac{1}{p}||_{\mathfrak{n}}}{\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \frac{1}{k} - \frac{1}{p}||_{\mathfrak{n}}} \to 0$,

$Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_p, \lambda_1) = Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \frac{1}{k} - \frac{1}{p}, \lambda_1) = \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, \frac{1}{k} - \frac{1}{p}||_{\mathfrak{n}}}{\lambda_1} \to 0$, as $k, p \to \infty$.

This shows that the sequence $v = (v_k)$ is Cauchy w.r.t. the neutrosophic $n-$norm $N_{\mathfrak{n}}(G_0, B_0, Y_0)$.

Theorem 3.4 Every convergent sequence in a $N - n - NLS$ $(U, G_0, B_0, Y_0, \circ, \diamond)$ is a Cauchy w.r.t. $N_{\mathfrak{n}}(G_0, B_0, Y_0)$.

Proof Let $v = (v_k)$ converges to v_0 w.r.t. $N_{\mathfrak{n}}(G_0, B_0, Y_0)$. For $0 < \epsilon < 1$ and $\lambda > 0$, choose $\epsilon_1 \in (0, 1)$ such that (1) is satisfied. Since $v_k \to v_0$ w.r.t. $N_{\mathfrak{n}}(G_0, B_0, Y_0)$, so $\exists$ $\mathfrak{n}_0 \in \mathbb{N}$ s.t. $G(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \frac{\lambda_1}{2}) > 1 - \epsilon$ and $B(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \frac{\lambda_1}{2}) < \epsilon$, $Y(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \frac{\lambda_1}{2}) < \epsilon$, $\forall\ k, p \geq \mathfrak{n}_0$.

Now,

$$G(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_p, \lambda_1) = G\Bigg(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0 + v_0 - v_p, \frac{\lambda_1}{2} + \frac{\lambda_1}{2}\Bigg)$$

$$\geq G\Bigg(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \frac{\lambda_1}{2}\Bigg) \circ G\Bigg(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_p - v_0, \frac{\lambda_1}{2}\Bigg)$$

$$> (1 - \epsilon_1) \circ (1 - \epsilon_1)$$

$> 1 - \epsilon$ and

$$\begin{aligned} B(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_p, \lambda_1) &= B\left(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0 + v_0 - v_p, \frac{\lambda_1}{2} + \frac{\lambda_1}{2}\right) \\ &\leq B\left(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \frac{\lambda_1}{2}\right) \diamond B\left(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_p - v_0, \frac{\lambda_1}{2}\right) \\ &< \epsilon_1 \diamond \epsilon_1 \\ &< \epsilon, \end{aligned}$$

$$\begin{aligned} Y(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_p, \lambda_1) &= Y\left(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0 + v_0 - v_p, \frac{\lambda_1}{2} + \frac{\lambda_1}{2}\right) \\ &\leq Y\left(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \frac{\lambda_1}{2}\right) \diamond Y\left(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_p - v_0, \frac{\lambda_1}{2}\right) \\ &< \epsilon_1 \diamond \epsilon_1 \\ &< \epsilon. \end{aligned}$$

This shows that (v_k) is a Cauchy sequence w.r.t. $N_{\mathfrak{n}}(G_0, B_0, Y_0)$. $\square$

Theorem 3.5 Consider the neutrosophic $n-$norm linear space as defined in Example 3.1. Let $v = (v_k)$ be any sequence in U, then

(i) (v_k) is Cauchy in $(U, ||.||_{\mathfrak{n}})$ iff $(v_{\mathfrak{n}})$ is Cauchy in $(U, G_0, B_0, Y_0, \circ, \diamond)$.

(ii) (v_k) is a convergent in $(U, ||.||_{\mathfrak{n}})$ iff (v_k) is convergent in $(U, G_0, B_0, Y_0, \circ, \diamond)$.

Proof (i) Let $v = (v_k)$ be a Cauchy in $(U, ||.||_{\mathfrak{n}})$, then

$\lim_{k,p\to\infty} ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_p||_{\mathfrak{n}} = 0.$

Now, for $\lambda_1 > 0$

$$\begin{aligned} &\lim_{k,p\to\infty} G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_p, \lambda_1) \\ &\qquad = \lim_{k,p\to\infty} \frac{\lambda_1}{\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_p||_{\mathfrak{n}}} = 1 \text{ and;} \\ &\lim_{k,p\to\infty} B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_p, \lambda_1) \\ &\qquad = \lim_{k,p\to\infty} \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_p||_{\mathfrak{n}}}{\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_p||_{\mathfrak{n}}} = 0, \\ &\lim_{k,p\to\infty} Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_p, \lambda_1) \\ &\qquad = \lim_{k,p\to\infty} \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_p||_{\mathfrak{n}}}{\lambda_1} = 0. \end{aligned}$$

This shows that, $G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_p, \lambda_1) \to 1$, $B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_p, \lambda_1) \to 0$ and $Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_p, \lambda_1) \to 0$, as $k, p \to \infty$. So for $0 < \epsilon < 1$ and $\lambda_1 > 0$, $\exists$ $\mathfrak{n}_0 \in \mathbb{N}$ s.t. $G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_p, \lambda_1) > 1 - \epsilon$ and $B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_p, \lambda_1) < \epsilon$, $Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_p, \lambda_1) < \epsilon$, and therefore (v_k) is Cauchy sequence in $(U, G_0, B_0, Y_0, \circ, \diamond)$.

Conversely, if (v_k) is Cauchy in $(U, G_0, B_0, Y_0, \circ, \diamond)$ then it is clearly (v_k) is Cauchy in $(U, ||.||_{\mathfrak{n}})$.

(ii) Let, $v = (v_k)$ be a convergent in $(U, ||.||_{\mathfrak{n}})$ and converges to v_0. Then $\lim_{k\to\infty} ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0|| = 0$.
Now,

$$\lim_{k\to\infty} G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) = \lim_{k\to\infty} \frac{\lambda_1}{\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0||_{\mathfrak{n}}} = 1 \text{ and;}$$

$$\lim_{k\to\infty} B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) = \lim_{k\to\infty} \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0||_{\mathfrak{n}}}{\lambda_1 + ||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0||_{\mathfrak{n}}} = 0,$$

$$\lim_{k\to\infty} Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) = \lim_{k\to\infty} \frac{||\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0||_{\mathfrak{n}}}{\lambda_1} = 0.$$

This shows that, $G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) \to 1$, $B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) \to 0$ and $Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) \to 0$, as $k \to \infty$. So for $\epsilon > 0$ and $\lambda_1 > 0$, $\exists\ \mathfrak{n}_0 \in \mathbb{N}$ s.t. $G_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) > 1 - \epsilon$ and $B_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) < \epsilon$, $Y_0(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) < \epsilon$, and therefore (v_k) is convergent sequence in $(U, G_0, B_0, Y_0, \circ, \diamond)$.

Converse part of the theorem can be obtained similarly and therefore omitted. □

Definition 3.4 A $N - n - NLS$ $(U, G_0, B_0, Y_0, \circ, \diamond)$ is said to be complete, iff, every Cauchy sequence is convergent in $(U, G_0, B_0, Y_0, \circ, \diamond)$.

Example 3.5 The sequence $v = (v_k) = \frac{1}{k}$ as in Example 3.4 is a Cauchy sequence that converges to 0 w.r.t $N_{\mathfrak{n}}(G_0, B_0, Y_0)$. But, $0 \notin (0, 1] = U$ and therefore the $N - n - NLS$ $(U, G_0, B_0, Y_0, \circ, \diamond)$ where $U = (0, 1]$ is not complete.

Theorem 3.6 If every Cauchy sequence in a $N - n - NLS$ $(U, G_0, B_0, Y_0, \circ, \diamond)$ has a convergent subsequence, then it is complete.

Proof Let $v = (v_k)$ be a Cauchy in a $N - n - NLS$ U and (v_{k_p}) be a subsequence of (v_k) that converges to v_0. We shell show that (v_k) converges to v_0. Let $\lambda_1 > 0$ and $\epsilon \in (0, 1)$. Choose $\epsilon_1 \in (0, 1)$ s.t. (1) is satisfied.

Since (v_k) is a Cauchy sequence, so $\exists\ \mathfrak{n}_0 \in \mathbb{N}$ s.t. $\forall\ k, p \geq \mathfrak{n}_0$
$G\left(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_p, \frac{\lambda_1}{2}\right) > 1 - \epsilon_1$ and
$B\left(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_p, \frac{\lambda_1}{2}\right) < \epsilon_1, Y\left(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_p, \frac{\lambda_1}{2}\right) < \epsilon_1.$

Since (v_{k_p}) converges to v_0, so $\exists\ i_p \in \mathfrak{n}$ with $i_p > \mathfrak{n}_0$ s.t.

$$G\left(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_{i_p} - v_0, \frac{\lambda_1}{2}\right) > 1 - \epsilon_1 \text{ and}$$
$$B\left(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_{i_p} - v_0, \frac{\lambda_1}{2}\right) < \epsilon_1, Y\left(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_{i_p} - v_0, \frac{\lambda_1}{2}\right) < \epsilon_1.$$

Now,

$G(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) = G\bigg(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_{i_p} + v_{i_p} - v_0, \frac{\lambda_1}{2} + \frac{\lambda_1}{2}\bigg)$
$\geq G\bigg(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_{i_p}, \frac{\lambda_1}{2}\bigg) \circ G\bigg(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_{i_0} - v_p, \frac{\lambda_1}{2}\bigg)$
$> (1 - \epsilon_1) \circ (1 - \epsilon_1) > 1 - \epsilon$ and;
$B(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) = B\bigg(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_{i_p} + v_{i_p} - v_0, \frac{\lambda_1}{2} + \frac{\lambda_1}{2}\bigg)$
$\leq B\bigg(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_{i_p}, \frac{\lambda_1}{2}\bigg) \diamond B\bigg(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_{i_p} - v_0, \frac{\lambda_1}{2}\bigg) < \epsilon_1 \diamond \epsilon_1 < \epsilon;$
$Y(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_0, \lambda_1) = Y\bigg(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_{i_p} + v_{i_p} - v_0, \frac{\lambda_1}{2} + \frac{\lambda_1}{2}\bigg)$
$\leq Y\bigg(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_k - v_{i_p}, \frac{\lambda_1}{2}\bigg) \diamond Y\bigg(\varrho_1, \varrho_2, \ldots, \varrho_{\mathfrak{n}-1}, v_{i_p} - v_0, \frac{\lambda_1}{2}\bigg) < \epsilon_1 \diamond \epsilon_1 < \epsilon.$
This shows that $v_k \to v_0$ in U w.r.t. $N_{\mathfrak{n}}(G_0, B_0, Y_0)$ and therefore U is complete. □

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Solving Neutrosophic Zero-Sum Two-Person Matrix Game using Mellin's Transform

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Abstract. This paper addresses the research gap in neutrosophic game theory, specifically the resolution of zero-sum two-person matrix games characterized by single-valued neutrosophic triangular numbers. We introduce a novel de-neutrosophication method leveraging Mellin's transform to obtain crisp value indices, thereby translating neutrosophic linear programming problems into their crisp counterparts. The effectiveness and precision of our approach are demonstrated through a real-world telecom sector case study, showcasing its potential for yielding more accurate and dependable solutions.

Keywords: Neutrosophic Triangular Numbers, Neutrosophic Triangular Matrix Games, Neutrosophic Linear Programming Problem, Mellin's transform.

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1. **Introduction**

Taking the right decision in today's competitive and conflicting world is an arduous affair. Game theory has played a pivotal role in decision making to take right decision and achieve desired goals. In today's real world conflicting scenario, where it is challenging to collect the accurate data for players, game theory provides a strategic mathematical procedure that help players to take the precise and perfect decision even with half-baked, imprecise and vague data. This is the reason why researchers all over the globe are attracted to develop new techniques and horizons in the theory of games.

The notion of theory of games was first introduced by Neumann and Morgenstern [1] by their work that published in 1944. During the classical game theory, the data available used to be accurate and crisp, so the classical set theory served the purpose, where the membership is binary {0, 1}. But classical game theory no longer serves the purpose when the data available is inadequate, imprecise, and vague. To overcome this problem of handling the imprecise, inadequate, and vague data, Prof. Zadeh [2] in 1965, pioneered the trailblazing concept of Fuzzy sets. Since then, numerous extensions and horizons like fuzzy triangular numbers (TFNs), fuzzy trapezoidal numbers (TrFNs), fuzzy pentagonal numbers (PFNs) and many more has been added to the fuzzy set theory by various researchers from all over the world. Li [26–28], Seikh et al. [29] have studied TFNs for their work in developing the theory of games. Jana et al. [30] , Chandra et al. [31], Kumar et al. [32], Bandhopadhyay et al. [33], and Dutta et al. [34] have explored TrFNs for their study of matrix games. Chakraborty et al. [35], Nasir et al. [36], Gajalakshmi et al. [42], and Umamageshwari et al. [43] investigated various properties of PFNs and applied it to various competitive game scenarios and achieved wonderful results of economic and social use.

Pawlak Z. [45, 46] in 1982 presented a novel mathematical instrument called 'Rough set theory' to deal with vague and uncertain information. He, in rough set theory, made use of two sets –lower and upper approximation intervals denoted as LAI and UAI respectively to handle vague and uncertain data. Later various researchers like Jangid et al. [47], Brikaa et al. [48], and Seikh et al. [49] dig deep to combine fuzziness and roughness to get fuzzy rough sets and used it in many types of MGs.

Atanasov [3,4] introduced Intuitionistic fuzzy sets (IFS) by adding non-membership function to the already existing fuzzy sets, to handle the uncertainty present in the available data, in a better way. The concept of the IFS has been used by various researchers [5–14] to investigate uncertainty in game theory using LPP approach.

Fuzzy sets and its generalisations have served well to handle imprecise and incomplete information in game theory, however, they are no longer suitable to handle inconsistent and indeterminate information that exists quiet often in real life situations. To get over this issue, Smarandache [15] invented a very prudent comprehensive framework of neutral logics called 'Neutrosophy', now recognised as a new arm of mathematics. The core theme of Neutrosophy states that beside some degree of truth, each concept possesses some degree of indeterminacy and falsity. Smarandache [16] defined neutrosophic set as a generalisation of IFSs. Smarandache explained indeterminacy in logic of Neutrosophy and clarified that truth, indeterminacy and falsity membership functions are independent of each other. Fuzzy sets and neutrosophic sets are clearly different in their application domain. Fuzzy sets deal with uncertain information i.e., incomplete and imprecise, whereas neutrosophic sets deal with inconsistent and indeterminate information. Neutrosophic sets are viewed from the vision of philosophy and one may find it difficult to apply it in mathematical and scientific problems. To get over this problem Wang et al. [17] defined single-valued neutrosophic sets and gave its various mathematical properties. At the moment, applying neutrosophic sets in game theory is a new thing and is in its initial stage. Now a days it is a very attractive research area for researchers all over the globe. Not much work is available at the moment in this field. However, some researchers like Das et al. [18], Hussain et al. [19], Tamilarasi et al. [20], Das S. K. [21], Seikh et al. [50], Das et al. [51], Bhaumik et al. [52], and Chakraborty et al. [44], have investigated neutrosophic sets in LPP models, integer programming models and MGs.

De-neutrosophication and ranking technique is utmost important while investigating and solving NLPP models. Jangid et al. [22] used a ranking technique by evaluating ambiguity and value of truth, indeterminacy, and falsity membership degree functions using (α, β, γ)-cut of SVNTNs involved in pay-off matrix of a NMG. Mahapatra et al. [23] used de-neutrosophication technique to convert NLPP to crisp LPP using the centroid method. Abdel et al. [24] suggested a novel ranking map to solve fully NLPP with trapezoidal neutrosophic numbers. A new ranking methodology was introduced by Das & Dash [18] to solve NLPP model with mixed parametric constraints. Darehmiraki [25] introduced a parametric de-neutrosophication function for ranking and then solving NLPPs. We in this paper have used a novel ranking technique for SVNTNs using Mellin's transform. [37] introduced a graphical method for solving Neutrosophical nonlinear programming with linear constraints, applicable to various model complexities. [38] reformulated the general model for the optimal distribution of agricultural lands using the concepts of neutrosophic science. In the book [39] discussed industrial engineering and computational intelligence foster intelligent machines for multi-criteria decision-making in smart environments. [40] evaluated the sustainable flue gas treatments in Egypt's steel sector using a new hybrid spherical fuzzy multicriteria decision-making approach. [41] developed a multi-criteria tool to evaluate sustainable battery recycling plant locations, prioritizing environmental factors in Egypt.

This research bridges the gap in neutrosophic game theory by converting neutrosophic matrix games to crisp linear programming, enhancing solution accuracy and reliability. The proposed methodology not only streamlines the process but also enhances the accuracy and reliability of the game's outcomes for both players. We demonstrate the impact of our approach through a real-world case study in the telecommunications sector, showing its potential to yield practical strategies in industry-specific scenarios. However, we must also note that the transition from neutrosophic to crisp values may involve certain trade-offs in terms of capturing the full spectrum of uncertainty inherent in real-world situations. Despite this, the practical benefits of our approach in terms of actionable insights and decision support in complex scenarios hold significant promise.

The structure of the paper has been developed as shown by the following figure -1:

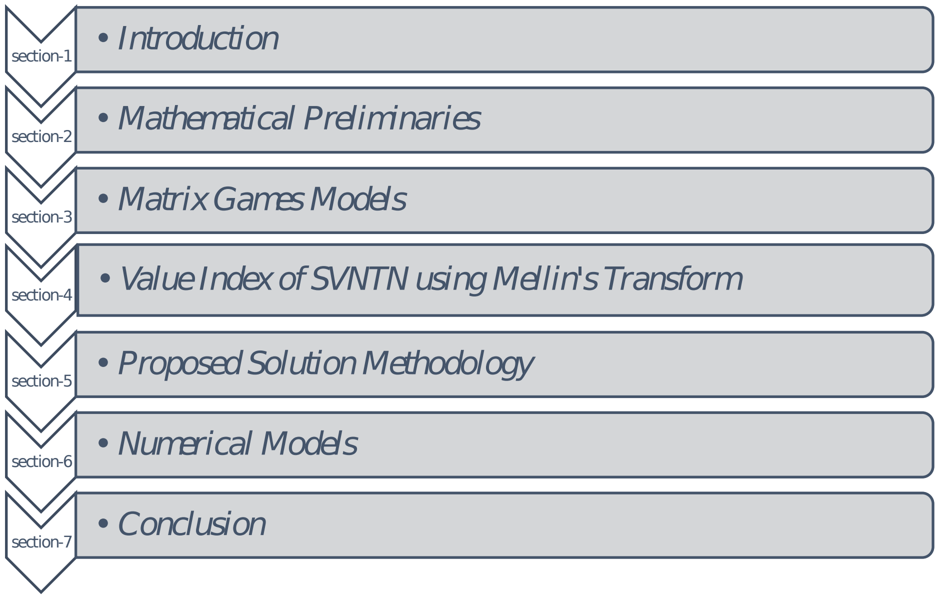


FIGURE 1. Structure of the Paper

2. Mathematical Preliminaries

In the present section, we give some fundamental definitions and symbols that are requisite and will be used throughout this article.

Definition 2.1. (Hussain et al. [19]) Let $X = \{\omega_1, \omega_2, ...\omega_n\}$ be a universe of discourse. A **Neutrosophic Set** ã in X is defined as ã$=\{\langle \omega_i, t_{\tilde{a}}(\omega_i), i_{\tilde{a}}(\omega_i), f_{\tilde{a}}(\omega_i)\rangle : \omega_i \in X\}$ where $t_{\tilde{a}}(\omega_i), i_{\tilde{a}}(\omega_i), f_{\tilde{a}}(\omega_i)$ are truth membership, indeterminacy membership and falsify membership degree mappings respectively with domain X and co-domain $[0, 1]$.

Definition 2.2. (Tamilarasi et al. [20]) A **single valued neutrosophic triangular number** (SVNTN) on $\Re$ (set of reals) is a neutrosophic set, denoted by $\tilde{a}_{SVNTN} = \{(\zeta, \eta, \theta); \rho, \sigma, \tau\}$, whose truth, indeterminacy and falsify functions are respectively written as follows:

$$t_{\tilde{a}_{SVNTN}}(\omega) = \begin{cases} \left(\frac{\omega-\zeta}{\eta-\zeta}\right)\rho & \text{if } \zeta \leq \omega \leq \eta \\ \rho & \text{if } \omega = \eta \\ \left(\frac{\theta-\omega}{\theta-\eta}\right)\rho & \text{if } \eta \leq \omega \leq \theta \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

$$i_{\tilde{a}_{SVNTN}}(\omega) = \begin{cases} \frac{(\eta-\omega)+\sigma(\omega-\zeta)}{\eta-\zeta} & \text{if } \zeta \leq \omega \leq \eta \\ \rho & \text{if } \omega = \eta \\ \frac{(\omega-\eta)+\sigma(\theta-\omega)}{\theta-\eta} & \text{if } \eta \leq \omega \leq \theta \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

$$f_{\tilde{a}_{SVNTN}}(\omega) = \begin{cases} \frac{(\eta-\omega)+\tau(\omega-\zeta)}{\eta-\zeta} & \text{if } \zeta \leq \omega \leq \eta \\ \tau & \text{if } \omega = \eta \\ \frac{(\omega-\eta)+\tau(\theta-\omega)}{\theta-\eta} & \text{if } \eta \leq \omega \leq \theta \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

Where $0 \leq \rho \leq 1$, $0 \leq \sigma \leq 1$, $0 \leq \tau \leq 1$ such that $0 \leq \rho + \sigma + \tau \leq 3$. Here σ, ρ, τ respectively represents maximum truth membership degree, minimum indeterminacy membership degree, minimum falsify membership degree.

Definition 2.3. (Hussain et al. [19]): Let $\tilde{a}^1_{SVNTN} = \{(\zeta^1, \eta^1, \theta^1); \rho^1, \sigma^1, \tau^1\}$ and $\tilde{a}^2_{SVNTN} = \{(\zeta^2, \eta^2, \theta^2); \rho^2, \sigma^2, \tau^2\}$ be two single valued neutrosophic triangular numbers and $\lambda \in \Re$ then some algebraic operations are as follows:

(1) Addition:

$$\tilde{a}^1_{SVNTN} \oplus \tilde{a}^2_{SVNTN} = \{(\zeta^1 + \zeta^2, \eta^1 + \eta^2, \theta^1 + \theta^2); min(\rho^1, \rho^2), max(\sigma^1, \sigma^2), max(\tau^1, \tau^2)\}$$

(2) Negative Image:
$-\tilde{a}^1_{SVNTN} = \{(-\theta^1, -\eta^1, -\zeta^1); \rho^1, \sigma^1, \tau^1\}$

(3) Subtraction:

$$\tilde{a}^1_{SVNTN} \ominus \tilde{a}^2_{SVNTN} = \{(\zeta^1 - \theta^2, \eta^1 - \eta^2, \theta^1 - \zeta^2); min(\rho^1, \rho^2), max(\sigma^1, \sigma^2), max(\tau^1, \tau^2)\}$$

(4) Scalar Product:

$$\lambda\ \tilde{a}^1_{SVNTN} = \begin{cases} \{(\lambda\zeta^1, \lambda\eta^1, \lambda\theta^1); \rho^1, \sigma^1, \tau^1\} & \text{for } \lambda > 0 \\ \{(\lambda\theta^1, \lambda\eta^1, \lambda\zeta^1); \rho^1, \sigma^1, \tau^1\} & \text{for } \lambda < 0 \end{cases}$$

3. **Matrix Games Models**

3.1. *Crisp Matrix Game (CMG)*

A crisp zero-sum two person matrix game denoted by triplet (A, S_1, S_2), where $A = \{a^{jk}\}_{mxn}$ is a real payoff matrix and $S_1 = \{1, 2, ..., m\}$, $S_2 = \{1, 2, ..., n\}$ are pure strategies of player-1 and player-2 respectively. Player-1 is called the maximising player as he plays his pure strategy to maximise his minimum gain and player-2 is called the minimising player as he plays his pure strategy to minimise his maximum loss. This is known as maxmin and minmax principle of matrix game. If the saddle point of the game exist at $(rs)^{th}$ position in the payoff matrix, then a^{rs}, $1 \leq r \leq m$; $1 \leq s \leq n$, is the payoff value for player-1 and its negative is the payoff value for player-2 if they choose to play r^{th} and s^{th} pure strategy respectively. If matrix game (A, S_1, S_2) has no saddle point i.e. $\max_{j \in S_1} \left\{ \min_{k \in S_2} \{a^{jk}\} \right\} \neq \min_{k \in S_2} \left\{ \max_{j \in S_1} \{a^{jk}\} \right\}$, then mixed strategy sets

$$S_1=\left\{P=(p_1,p_2,...,p_m)\in R^m, p_j\geq 0\ \forall\ j=1,2,...,m, and\ \sum_{j=1}^{m}p_j=1\right\}\ and$$

$$S_2=\left\{Q=(q_1,q_2,...,q_n)\in R^n, q_j\geq 0\ \forall\ k=1,2,...,n, and\ \sum_{k=1}^{n}q_k=1\right\}$$

are adopted for player-1 and player-2 respectively.

Here if

$$\max_{P\in S_1}\left\{\min_{Q\in S_2}\left\{\sum_{j=1}^{m}\left(\sum_{k=1}^{n}p_j a^{jk}p_k\right)\right\}\right\}=\min_{Q\in S_2}\left\{\max_{P\in S_1}\left\{\sum_{k=1}^{n}\left(\sum_{j=1}^{m}p_j a^{jk}p_k\right)\right\}\right\}=v^*(say)$$

then v^* is called the value of the game, and $P=(p_1,p_2,...,p_m)\in S_1, Q=(q_1,q_2,...,q_n)\in S_2$ are optimal mixed strategies for player-1 and player-2 respectively.

3.2. *Neutrosophic Matrix Games (NMG)*

If the payoff matrix $\tilde{A}=\{\tilde{a}^{jk}_{SVNTN}\}_{mxn}$ is equiped with single valued neutrosophic triangular numbers $\tilde{a}^{jk}_{SVNTN},\ j=1,2,...,m;\ k=1,2,...,n$, then the game $(\tilde{A},S_1,S_2)$ is called Neutrosophic Triangular matrix game (NTMG). Thus employing the maxmin and minmax principle for NTMG, we get the following mathematical modals for two players respectively

For player-1:

$$\begin{cases} \max_{p_j\in S_1}\left\{\min\left\{\sum_{j=1}^{m}\tilde{a}^{j1}_{SVNTN}\ p_j,\sum_{j=1}^{m}\tilde{a}^{j2}_{SVNTN}\ p_j,...,\sum_{j=1}^{m}\tilde{a}^{jn}_{SVNTN}\ p_j\right\}\right\} \\ s.t., \quad \sum_{j=1}^{m}p_j=1 \\ and \quad p_j\geq 0,\forall\ j=1,2,...,m. \end{cases} \tag{4}$$

For player-2:

$$\begin{cases} \min_{q_k\in S_2}\left\{\max\left\{\sum_{k=1}^{n}\tilde{a}^{1k}_{SVNTN}\ q_k,\sum_{k=1}^{n}\tilde{a}^{2k}_{SVNTN}\ q_k,...,\sum_{k=1}^{n}\tilde{a}^{mk}_{SVNTN}\ q_k\right\}\right\} \\ s.t., \quad \sum_{k=1}^{n}q_k=1 \\ and \quad q_k\geq 0,\forall\ k=1,2,...,n. \end{cases} \tag{5}$$

Now, let $\min\left\{\sum_{j=1}^{m}\tilde{a}^{j1}_{SVNTN}\ p_j,\sum_{j=1}^{m}\tilde{a}^{j2}_{SVNTN}\ p_j,...,\sum_{j=1}^{m}\tilde{a}^{jn}_{SVNTN}\ p_j\right\}=\tilde{u}_{SVNTN}$ (say)
is the minimum expected gain for player-1 and
$\max\left\{\sum_{k=1}^{n}\tilde{a}^{1k}_{SVNTN}\ q_k,\sum_{k=1}^{n}\tilde{a}^{2k}_{SVNTN}\ q_k,...,\sum_{k=1}^{n}\tilde{a}^{mk}_{SVNTN}\ q_k\right\}=\tilde{v}_{SVNTN}$ (say)
is the maximum expected loss for player-2, we get the following two neutrosophic linear programming problem (NLPP) models for the two players:

For player-1: $(\mathrm{NLPP})^{\mathrm{I}}$

$$\begin{cases} Maximise & \tilde{u}_{SVNTN} \\ subject\ to & \sum_{j=1}^{m} \tilde{a}_{SVNTN}^{j1}\ p_j \succeq \tilde{u}_{SVNTN} \\ & \sum_{j=1}^{m} \tilde{a}_{SVNTN}^{j2}\ p_j \succeq \tilde{u}_{SVNTN} \\ & \quad .... \\ & \quad .... \\ & \sum_{j=1}^{m} \tilde{a}_{SVNTN}^{jn}\ p_j \succeq \tilde{u}_{SVNTN} \\ & \sum_{j=1}^{m} p_j = 1, \\ and & p_j \geq 0,\ \forall\ j = 1, 2, ..., m. \end{cases} \tag{6}$$

For player-2: $(\mathrm{NLPP})^{\mathrm{II}}$

$$\begin{cases} Minimise & \tilde{v}_{SVNTN} \\ subject\ to & \sum_{k=1}^{n} \tilde{a}_{SVNTN}^{1k}\ q_k \preceq \tilde{v}_{SVNTN} \\ & \sum_{k=1}^{n} \tilde{a}_{SVNTN}^{2k}\ q_k \preceq \tilde{v}_{SVNTN} \\ & \quad .... \\ & \quad .... \\ & \sum_{k=1}^{n} \tilde{a}_{SVNTN}^{mk}\ q_k \preceq \tilde{v}_{SVNTN} \\ & \sum_{k=1}^{n} q_k = 1, \\ and & q_k \geq 0,\ \forall\ k = 1, 2, ..., n. \end{cases} \tag{7}$$

Here $\tilde{u}_{SVNTN}$ and $\tilde{v}_{SVNTN}$ are SVNTNs representing expected minimum gain and expected maximum loss for player-1 and palyer-2 respectively. The symbols $\succeq$ and $\preceq$ represents the neutrosophic adaptations of order relation $\geq$ and $\leq$ respectively. The above NLPP models for the two players can be restructured as follows-
For player-1: $(\mathrm{NLPP})^{\mathrm{I}}$

$$\begin{cases} Maximise & \tilde{u}_{SVNTN} \\ s.t., & \sum_{j=1}^{m} \tilde{a}_{SVNTN}^{jk}\ p_j \succeq \tilde{u}_{SVNTN}\ \forall\ k = 1, 2, ..., n. \\ & \sum_{j=1}^{m} p_j = 1, \\ and & p_j \geq 0,\ \forall\ j = 1, 2, ..., m. \end{cases} \tag{8}$$

For player-2: $(\text{NLPP})^{\text{II}}$

$$\begin{cases} Minimise & \tilde{v}_{SVNTN} \\ s.t., & \sum_{k=1}^{n} \tilde{a}^{jk}_{SVNTN}\ q_k \preceq \tilde{v}_{SVNTN}\ \forall\ j = 1, 2, ..., m. \\ & \sum_{k=1}^{n} q_k = 1, \\ and & q_k \geq 0,\ \forall\ k = 1, 2, ..., n. \end{cases} \tag{9}$$

4. **Value Index of SVNTN using Mellin's Transform**

Let $\tilde{a}_{SVNTN} = \{(\zeta, \eta, \theta); \rho, \sigma, \tau\}$ be any SVNTN and $t_{\tilde{a}_{SVNTN}}(\omega)$, $i_{\tilde{a}_{SVNTN}}(\omega)$, $f_{\tilde{a}_{SVNTN}}(\omega)$ are associated truth, indeterminacy and falsify membership function respectively. Now we first define probability density function (p.d.f) from truth, indeterminacy and falsify membership function respectively as follows

$$\phi_1(\omega) = k_1 t_{\tilde{a}_{SVNTN}}(\omega),\ \phi_2(\omega) = k_2 i_{\tilde{a}_{SVNTN}}(\omega)\ and\ \phi_3(\omega) = k_3 f_{\tilde{a}_{SVNTN}}(\omega)$$

where $k_1,\ k_2$ and k_3 are constants to be obtained using the property of probability density function i.e.,
$\int_{-\infty}^{\infty} \phi_1(\omega) d\omega = 1, \int_{-\infty}^{\infty} \phi_2(\omega) d\omega = 1, \int_{-\infty}^{\infty} \phi_3(\omega) d\omega = 1$ respectively.

We get

$$k_1 = \frac{2}{(\theta - \zeta)\rho},\ k_2 = \frac{2}{(\theta - \zeta)(1 + \sigma)},\ k_3 = \frac{2}{(\theta - \zeta)(1 + \tau)}. \tag{10}$$

Using $k_1,\ k_2,\ k_3$ in $\phi_1(\omega),\ \phi_2(\omega),\ \phi_3(\omega)$ respectively, we now define $\phi(\omega)$ as the p.d.f corresponding to SVNTN $\tilde{a}_{SVNTN}$ as follows

$$\phi(\omega) = \lambda\phi_1(\omega) + (1 - \lambda)\phi_2(\omega) + (1 - \lambda)\phi_3(\omega),\ (0 \leq \lambda \leq 1) \tag{11}$$

where $\lambda \in [0, 1]$ represents the player's preference information. If $\lambda \in \left[0, \frac{1}{2}\right]$, it means that the player is pessimist i.e. he incurs negative feeling and prefer uncertainity. If $\lambda \in \left]\frac{1}{2}, 1\right[$ it means that the player is optimist i.e. he incurs positive feeling and prefer certainity. If $\lambda = \frac{1}{2}$ the player is indifferent of positive or negative feeling, he is moderate.

Now, the Mellin's Transform $M[\phi(\omega), s]$ of a p.d.f $\phi(\omega)$ is defined as $M[\phi(\omega), s] = \int_0^{\infty} \omega^{s-1}\phi(\omega) d\omega$, provided the integral exists. Using the function $\phi(\omega)$ (equation 11) we get

$$M[\phi(\omega), s] = \int_0^{\infty} \omega^{s-1}[\lambda\phi_1(\omega) + (1 - \lambda)\phi_2(\omega) + (1 - \lambda)\phi_3(\omega)] d\omega \tag{12}$$

Now by taking $s = 2$, Mellin's transform is converted into expected value of associated random variable. Hence we get the de-neutrosophicated value or the expected value of SVNTN $\tilde{a}_{SVNTN} = \{(\zeta, \eta, \theta); \rho, \sigma, \tau\}$.

We obtain for $s = 2$,

$$M[\phi(\omega), 2] = \lambda\left\{\frac{\zeta+\eta+\theta}{3}\right\} + (1-\lambda)\left\{\frac{\sigma(\zeta+\eta+\theta)+2(\zeta+\theta)-\eta}{3(1+\sigma)}\right\} + (1-\lambda)\left\{\frac{\tau(\zeta+\eta+\theta)+2(\zeta+\theta)-\eta}{3(1+\tau)}\right\} \quad (13)$$

Where, $\lambda \in [0,1]$ express the degree of optimism of the player. $\lambda \in \left[0, \frac{1}{2}\right[$ express the pessimist behaviour, $\lambda \in \left]\frac{1}{2}, 1\right[$ express the optimist behaviour and $\lambda = \frac{1}{2}$ express that the player is moderate.

Since $M[\phi(\omega), 2]$ depends on λ, let us denote it by $V(\tilde{a}_{SVNTN}, \lambda)$ and is called the 'Value Index' of single valued neutrosophic triangular number $\tilde{a}_{SVNTN} = \{(\zeta, \eta, \theta); \rho, \sigma, \tau\}$. We write

$$V(\tilde{a}_{SVNTN}, \lambda) = M[\phi(\omega), 2] = \lambda\left\{\frac{\zeta+\eta+\theta}{3}\right\} + (1-\lambda)\left\{\frac{\sigma(\zeta+\eta+\theta)+2(\zeta+\theta)-\eta}{3(1+\sigma)}\right\} + (1-\lambda)\left\{\frac{\tau(\zeta+\eta+\theta)+2(\zeta+\theta)-\eta}{3(1+\tau)}\right\} \quad (14)$$

Proposition-1: For a given $\lambda \in [0,1]$ and a given $\tilde{a}_{SVNTN} = \{(\zeta, \eta, \theta); \rho, \sigma, \tau\}$, $V(\tilde{a}_{SVNTN}, \lambda)$ is a unique real number.

4.1. *De-neutrosophication and Ranking of SVNTN:*

Let $\widetilde{neu}(\Re)$ be the set of all SVNTNs, and $\lambda \in [0,1]$ be a given number. We define a mapping $h_\lambda : \widetilde{neu}(\Re) \to \Re$ such that $h_\lambda(\tilde{a}_{SVNTN}) = V(\tilde{a}_{SVNTN}, \lambda) \quad \forall \quad \tilde{a}_{SVNTN} \in \widetilde{neu}(\Re)$; $\Re$ being the set of real numbers. The mapping h_λ is well-defined and associates each $\tilde{a}_{SVNTN} \in \widetilde{neu}(\Re)$ to a unique real number (proposition-1) where the order relations exist naturally. h_λ is called a de-neutrosophication function and used to rank SVNTNs as detailed out in proposition-2 below.

Proposition-2: Let $\tilde{a}_{SVNTN} = \{(\zeta^1, \eta^1, \theta^1); \rho^1, \sigma^1, \tau^1\}$ and $\tilde{b}_{SVNTN} = \{(\zeta^2, \eta^2, \theta^2); \rho^2, \sigma^2, \tau^2\}$ be SVNTNs and $\lambda \in [0,1]$, then the ranking order relation between the two SVNTNs are defined as follows

(1) $\tilde{a}_{SVNTN} \preceq \tilde{b}_{SVNTN} \Leftrightarrow h_\lambda(\tilde{a}_{SVNTN}, \lambda) \leq h_\lambda(\tilde{b}_{SVNTN}, \lambda)$
(2) $\tilde{a}_{SVNTN} \succeq \tilde{b}_{SVNTN} \Leftrightarrow h_\lambda(\tilde{a}_{SVNTN}, \lambda) \geq h_\lambda(\tilde{b}_{SVNTN}, \lambda)$
(3) $\tilde{a}_{SVNTN} \approx \tilde{b}_{SVNTN} \Leftrightarrow h_\lambda(\tilde{a}_{SVNTN}, \lambda) = h_\lambda(\tilde{b}_{SVNTN}, \lambda)$

Where the symbols $\preceq$, $\succeq$ and $\approx$ represents the neutrosophic adaptations of order relation $\leq$, $\geq$ and $=$ respectively.

5. Proposed Solution Methodology

In this section, we detail out a step-wise solution methodology we propose to solve any NTMG. The steps of our proposed solution methodology are as follows:

Step-1: Write the respective neutrosophic linear programming problem (NLPP), i.e. equations (8) and (9), for the two players respectively.

Step-2: Write the de-neutrosophic version of NLPPs obtained in step-1 by using the value index of all SVNTNs involved. We get the following respective crisp linear programming problems (CLPPs) for the two players.

For player-1:$(\text{CLPP})^{\text{I}}$

$$\begin{cases} Maximise & V(\tilde{u}_{SVNTN},\lambda) \\ s.t., & \sum_{j=1}^{m} V(\tilde{a}^{jk}_{SVNTN},\lambda)p_j \geq V(\tilde{u}_{SVNTN},\lambda)\ \forall\ k=1,2,...,n. \\ & \sum_{j=1}^{m} p_j = 1, \\ and & p_j \geq 0,\ \forall\ j=1,2,...,m. \end{cases} \tag{15}$$

For player-2: $(\text{CLPP})^{\text{II}}$

$$\begin{cases} Minimise & V(\tilde{v}_{SVNTN},\lambda) \\ s.t., & \sum_{k=1}^{n} V(\tilde{a}^{jk}_{SVNTN},\lambda)q_k \leq V(\tilde{v}_{SVNTN},\lambda)\ \forall\ j=1,2,...,m. \\ & \sum_{k=1}^{n} q_k = 1, \\ and & q_k \geq 0,\ \forall\ k=1,2,...,n. \end{cases} \tag{16}$$

Step-3: Use the formula for value index, i.e. equation-(14), and write the CLPPs for various values of $\lambda \in [0,1]$ for both players.

Step-4: Solve these CLPPs by simplex method to get optimal mixed strategies and the optimal value of the game for both players.

5.1. *Flowchart*

For an easy understanding, a visual representation of the proposed solution methodology has been depicted by the flow-chart in Figure-2 below.

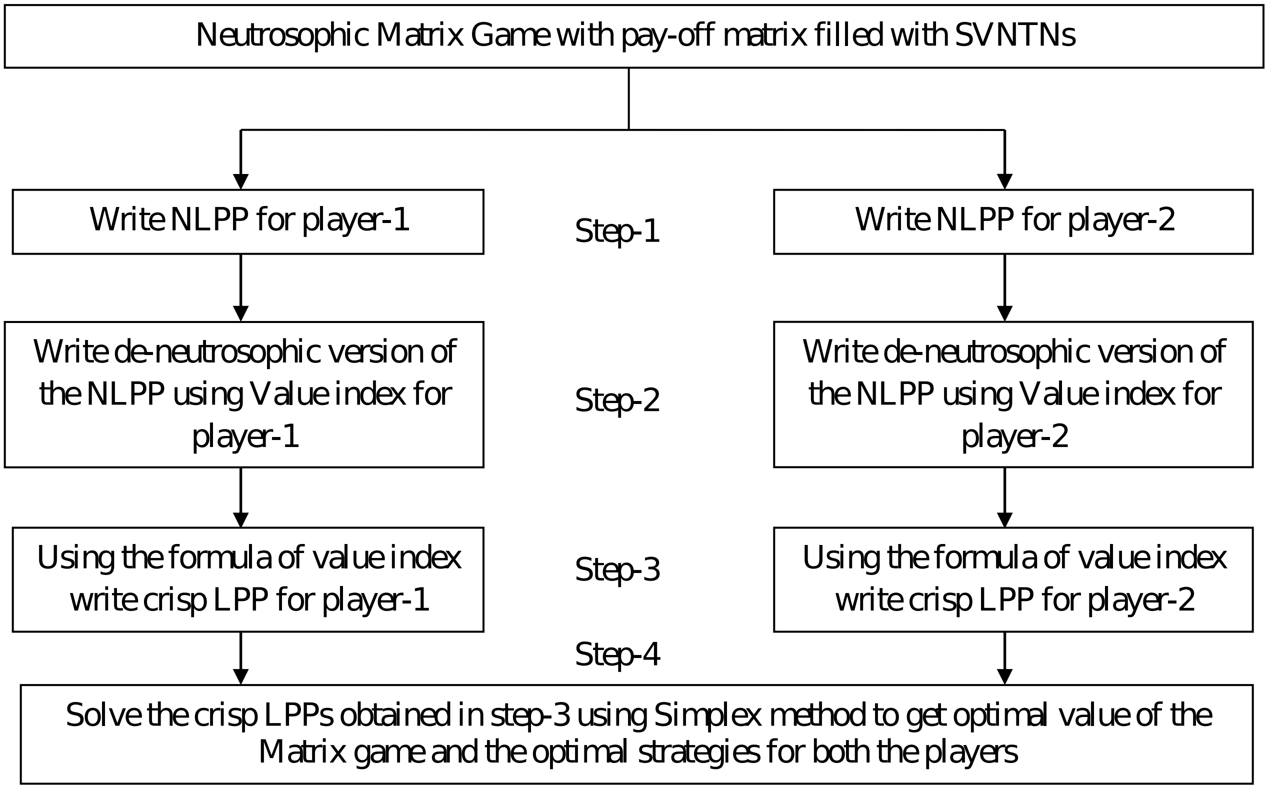


FIGURE 2. Flow Chart of the Proposed Solution Methodology

6. Numerical Models

In this section of our work, we show the validity and applicability of our solution methodology by giving the solution procedure of two NTMG examples. In example-1 we took a simple case of 2x2 pay-off matrix of a NTMG from the work of Jangid et al. [22]. We solve it by our method and then discuss, analyse, and compare their results with the our results using Tables-(1, 2) and a histogram (Figure-3).

In example-2 we consider a real-world case study from telecom sector by taking a 3x3 pay-off matrix of strategies adopted by the companies to capture the market share in the target area. Strategies floated by the companies are represented by SVNTNs. Results obtained are discussed and analysed by means of a graph in Figure-4 and Tables-(3, 4, 5).

6.1. *Example-1:(Jangid et al. [22])*

Let NTMG $(\tilde{A}, S_1, S_2) = \begin{bmatrix} \tilde{a}^{11}_{SVNTN} & \tilde{a}^{12}_{SVNTN} \\ \tilde{a}^{21}_{SVNTN} & \tilde{a}^{22}_{SVNTN} \end{bmatrix}$, where SVNTNs $\tilde{a}^{jk}_{SVNTN}$ are as follows-

$\tilde{a}^{11}_{SVNTN} = \widehat{180} = \{(175, 180, 190); 0.6, 0.4, 0.2\}$,
$\tilde{a}^{12}_{SVNTN} = \widehat{156} = \{(150, 156, 158); 0.6, 0.35, 0.1\}$,

$\tilde{a}^{21}_{SVNTN} = \widehat{90} = \{(80, 90, 100); 0.9, 0.5, 0.1\}$,
$\tilde{a}^{22}_{SVNTN} = \widehat{180} = \{(175, 180, 190); 0.6, 0.4, 0.2\}$

Solution Procedure: Let (p_1, p_2) and (q_1, q_2) are the optimal strategies and $\tilde{u}_{SVNTN}$, $\tilde{v}_{SVNTN}$ are optimal SVNTN values of the game for player-1 and player-2 respectively, then NLPPs for the two players are written as follows-

$$\text{For player-1 } (\text{NLPP})^{\text{I}} : \begin{cases} Max & \tilde{u}_{SVNTN} \\ s.t., & \widehat{180}\ p_1 + \widehat{90}\ p_2 \succeq \tilde{u}_{SVNTN} \\ & \widehat{156}\ p_1 + \widehat{180}\ p_2 \succeq \tilde{u}_{SVNTN} \\ & p_1 + p_2 = 1, \\ and & p_1,\ p_2 \geq 0. \end{cases} \tag{17}$$

$$\text{For player-2 } (\text{NLPP})^{\text{II}} : \begin{cases} Min & \tilde{v}_{SVNTN} \\ s.t., & \widehat{180}\ q_1 + \widehat{156}\ q_2 \preceq \tilde{v}_{SVNTN} \\ & \widehat{90}\ q_1 + \widehat{180}\ q_2 \preceq \tilde{u}_{SVNTN} \\ & q_1 + q_2 = 1, \\ and & q_1,\ q_2 \geq 0. \end{cases} \tag{18}$$

For de-neutrosophication of above NLPP models, we apply the value index of all the SVNTNs, we get the following CLPPs for both the players

For player-1 $(\text{CLPP})^{\text{I}}$:

$$\begin{cases} Max & V(\tilde{u}_{SVNTN}, \lambda) \\ s.t., & V(\widehat{180}, \lambda)p_1 + V(\widehat{90}, \lambda)p_2 \geq V(\tilde{u}_{SVNTN}, \lambda) \\ & V(\widehat{156}, \lambda)p_1 + V(\widehat{180}, \lambda)p_2 \geq V(\tilde{u}_{SVNTN}, \lambda) \\ & p_1 + p_2 = 1, \\ and & p_1,\ p_2 \geq 0. \end{cases} \tag{19}$$

For player-2 $(\text{CLPP})^{\text{II}}$:

$$\begin{cases} Min & V(\tilde{v}_{SVNTN}, \lambda) \\ s.t., & V(\widehat{180}, \lambda)q_1 + V(\widehat{156}, \lambda)q_2 \leq V(\tilde{v}_{SVNTN}, \lambda) \\ & V(\widehat{90}, \lambda)q_1 + V(\widehat{180}, \lambda)q_2 \leq V(\tilde{v}_{SVNTN}, \lambda) \\ & q_1 + q_2 = 1, \\ and & q_1,\ q_2 \geq 0. \end{cases} \tag{20}$$

The value index of all the different SVNTNs involved are calculated using the formulla (equation-14) explained in section-4. They are given as follows:

$V(\widehat{180}, \lambda) = (365.9126 - 184.246\lambda)$,

$V(\widehat{156}, \lambda) = (307.1335 - 152.4669\lambda)$, and

$V(\widehat{90}, \lambda) = (180 - 90\lambda)$

Using these value indexes we get the following crisp LPPs for the two players

For player-1 $(\text{CLPP})^{\text{I}}$:

$$\begin{cases} Max & V(\tilde{u}_{SVNTN}, \lambda) = u(say) \\ s.t., & (365.9126 - 184.246\lambda)p_1 + (180 - 90\lambda)p_2 \geq u \\ & (307.1335 - 152.4669\lambda)p_1 + (365.9126 - 184.246\lambda)p_2 \geq u \\ & p_1 + p_2 = 1, \\ and & p_1,\ p_2 \geq 0. \end{cases} \tag{21}$$

For player-2 $(\text{CLPP})^{\text{II}}$:

$$\begin{cases} Min & V(\tilde{v}_{SVNTN}, \lambda) = v(say) \\ s.t., & (365.9126 - 184.246\lambda)q_1 + (307.1335 - 152.4669\lambda)q_2 \leq v \\ & (180 - 90\lambda)q_1 + (365.9126 - 184.246\lambda)q_2 \leq v \\ & q_1 + q_2 = 1, \\ and & q_1,\ q_2 \geq 0. \end{cases} \tag{22}$$

For various values of optimism degree λ, the value index of SVNTNs are calculated using the formula explained in section-4 (equation 14), and are given in Table-1 below.

TABLE 1. Value index of SVNTNs for different values of optimism degree λ

λ	$V(\widehat{180}, \lambda)$	$V(\widehat{156}, \lambda)$	$V(\widehat{90}, \lambda)$
0.0	365.9126	307.1335	180
0.1	347.4880	291.8868	171
0.2	329.0634	276.6401	162
0.3	310.6388	261.3934	153
0.4	292.2142	246.1467	144
0.5	273.7896	230.9000	135
0.6	255.3650	215.6533	126
0.7	236.9404	200.4066	117
0.8	218.5158	185.1599	108
0.9	200.0912	169.9132	99
1.0	181.6666	154.6666	90

Using the values given in Table 1, optimal solutions for various degree of optimism λ are obtained by solving CLPPs for player-1 and player-2 equations 21 and 22 and are given in Table-2 below.

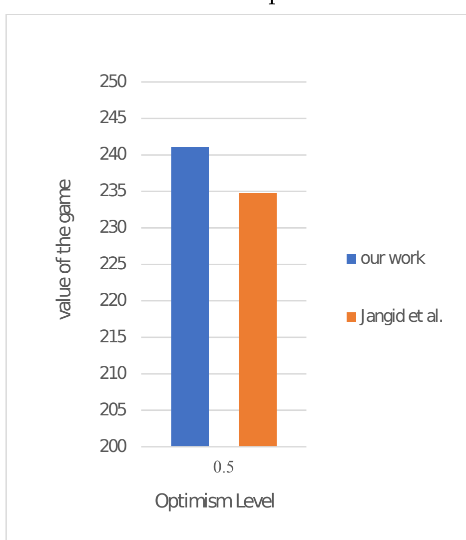
Table 2. Optimal Solution for Player-1 at different values of optimism degree λ

λ	p_1	p_2	q_1	q_1	$Max(u)$
0.0	0.7598	0.2402	0.2402	0.7598	305.2071
0.2	0.7612	0.2388	0.2388	0.7612	289.1611
0.3	0.7620	0.2380	0.2380	0.7620	273.1155
0.4	0.7629	0.2371	0.2371	0.7629	257.0701
0.5	0.7639	0.2361	0.2361	0.7639	241.0251
0.6	0.7651	0.2349	0.2349	0.7651	224.9805
0.7	0.7665	0.2335	0.2335	0.7665	208.9366
0.8	0.7682	0.2318	0.2318	0.7682	192.8933
0.9	0.7701	0.2299	0.2299	0.7701	176.8509
1.0	0.7725	0.2296	0.2296	0.7725	160.8099

6.1.1. *Discussion and Comparison of Results of Example-1*

Our solution results for various values of degree of optimism are given in Table-2 above. Results show that value of the game decreases from 321.2532 to 160.8099 as the degree of optimism increases from 0.0 to 1.0, it means the value of the game is inversely proportional to the degree of optimism of the incumbent player. It righty suggests that it is not wise to take decisions with high level of optimism. It is better to be more realistic than too much optimistic. For a better understanding and analysis, the graphical representation of obtained optimal values 'u' against different values of the degree of optimism 'λ' for player-1 is given in Figure-4.

Figure 3. Comparison of Works



Jangid et al. [22] with their solution methodology have solved only for $\lambda = \frac{1}{2}$ in their work. They have got SVNTN $\langle(152.22, 158.1312, 160.8416); 0.6, 0.4, 0.2\rangle$ as the value of the game for player-1, its value index can be calculated as 234.7706 using formula explained in section-4. Whereas our approach gives 241.0251 as the optimal value of the game for $\lambda = \frac{1}{2}$ (refer Table-2). So, our methodology yields better results for given optimism level, comparison of our result with Jangid et al. [22] can be seen in the histogram (Figure-3). Also, their method is difficult on calculations so they have calculated it only for $\lambda = \frac{1}{2}$, whereas our is an easy procedure, we have done it for various values of λ varying from 0 to 1.

6.2. *Example-2:(A telecom sector case-study)*

Nowadays it is impossible to think a life without a high-speed internet connection in your mobile handset or a high speed wi-fi internet connection at your home and at your workplace. In this regard the launch of fifth generation (5G) network recently has brought a revolution in India. Presently the number of mobile network subscribers in India is around 449 million and this number is surely going to increase with launch of 5G network. The two major telecom operators in India, Airtel (C_1) and Vodafone (C_2) (say), want to take advantage of this situation and each of them aim to increase the number of subscribers than the other. The two companies have fixed number of costumers and each of them want to add new costumers by porting to one from the other or by adding new subscribers. They make the following strategies to lure more costumers –

Strategy-I: 'Reducing the tariff of their data plans per GB'

Strategy-II: 'Giving free soaps like hotstar, amazon prime etc with their data plan'

Strategy-III: 'Advertising through print, electronic and social media'

The market research wing (MRW) experts of the two companies cannot precisely predict the increase in the number of costumers because of the uncertainty and indeterminacy that is always present in the large telecom market. They can only provide some estimated data with some amount of uncertainty and indeterminacy involved in it. This competitive situation between the two companies can be presented by means of a matrix game (MG) with payoff matrix equipped with SVNTNs. Supposing that the MRW experts of the two companies after analysing the collected data through some survey and their expertise, presented the following pay-off matrix on the number of costumers. (All numbers are supposed to be multiplied by 1000).

$$\tilde{E} = \begin{array}{c} \\ I \\ II \\ III \end{array} \begin{array}{c} \begin{array}{ccc} \quad I \quad & \quad II \quad & \quad III \quad \end{array} \\ \begin{bmatrix} \langle(176,180,183);.6,.5,.2\rangle & \langle(83,90,96);.8,.4,.2\rangle & \langle(110,120,133);.9,.5,.1\rangle \\ \langle(87,89,92);.6,.4,.2\rangle & \langle(176,180,183);.6,.5,.2\rangle & \langle(118,125,130);.7,.5,.3\rangle \\ \langle(118,125,130);.7,.5,.3\rangle & \langle(145,150,153);.8,.4,.1\rangle & \langle(83,90,96);.8,.4,.2\rangle \end{bmatrix} \end{array}$$

$$= \begin{array}{c} \\ I \\ II \\ III \end{array} \begin{array}{c} \begin{array}{ccc} I & II & III \end{array} \\ \begin{bmatrix} \langle\widehat{180}\rangle & \langle\widehat{90}\rangle & \langle\widehat{120}\rangle \\ \langle\widehat{89}\rangle & \langle\widehat{180}\rangle & \langle\widehat{125}\rangle \\ \langle\widehat{125}\rangle & \langle\widehat{150}\rangle & \langle\widehat{90}\rangle \end{bmatrix} \end{array} = \begin{bmatrix} \tilde{a}^{jk}_{SVNTN} \end{bmatrix} \ (say) \ j = 1,2,3; \ k = 1,2,3.$$

Where, $\tilde{a}^{12}_{SVNTN} = \langle\widehat{90}\rangle = \langle(83,90,96);.8,.4,.2\rangle$ means that the company C_1 (Player-1) will get an increase of 90 units in its customer base if C_1 sticks to apply strategy-I (i.e., 'Reducing the tariff of their data plans per GB') and if company C_2 sticks to apply strategy-II (i.e., 'Giving free soaps like hotstar, amazon prime etc with their data'). MRW experts are 80% positive about it, 20% they are not positive and they remain indeterminate by 40% about the increase.

All other SVNTNs can be explained similarly.

Solution Procedure: let (p_1, p_2, p_3) and (q_1, q_2, q_3) are the optimal strategies and $\tilde{u}_{SVNTN}$, $\tilde{v}_{SVNTN}$ are optimal SVNTN values of the game for company C_1 and company C_2 respectively, then NLPPs for the two players (C_1 & C_2) are written as follows-

For player-1 $(\text{NLPP})^{\text{I}}$:

$$\begin{cases} Max & \tilde{u}_{SVNTN} \\ s.t., & \tilde{a}^{11}_{SVNTN}\ p_1 + \tilde{a}^{21}_{SVNTN}\ p_2 + \tilde{a}^{31}_{SVNTN}\ p_3 \succeq \tilde{u}_{SVNTN} \\ & \tilde{a}^{12}_{SVNTN}\ p_1 + \tilde{a}^{22}_{SVNTN}\ p_2 + \tilde{a}^{32}_{SVNTN}\ p_3 \succeq \tilde{u}_{SVNTN} \\ & \tilde{a}^{13}_{SVNTN}\ p_1 + \tilde{a}^{23}_{SVNTN}\ p_2 + \tilde{a}^{33}_{SVNTN}\ p_3 \succeq \tilde{u}_{SVNTN} \\ & p_1 + p_2 + p_3 = 1, \\ and & p_1,\ p_2,\ p_3 \geq 0. \end{cases} \tag{23}$$

For player-2 $(\text{NLPP})^{\text{II}}$:

$$\begin{cases} Min & \tilde{v}_{SVNTN} \\ s.t., & \tilde{a}^{11}_{SVNTN}\ q_1 + \tilde{a}^{12}_{SVNTN}\ q_2 + \tilde{a}^{13}_{SVNTN}\ q_3 \preceq \tilde{v}_{SVNTN} \\ & \tilde{a}^{21}_{SVNTN}\ q_1 + \tilde{a}^{22}_{SVNTN}\ q_2 + \tilde{a}^{23}_{SVNTN}\ q_3 \preceq \tilde{v}_{SVNTN} \\ & \tilde{a}^{31}_{SVNTN}\ q_1 + \tilde{a}^{32}_{SVNTN}\ q_2 + \tilde{a}^{33}_{SVNTN}\ q_3 \preceq \tilde{v}_{SVNTN} \\ & q_1 + q_2 + q_3 = 1, \\ and & q_1,\ q_2,\ q_3 \geq 0. \end{cases} \tag{24}$$

For de-neutrosophication of above NLPP models, we apply the value index of all the SVNTNs. we get the following CLPPs for the two players

For player-1 $(\text{CLPP})^{\text{I}}$:

$$\begin{cases} Max & V(\tilde{u}_{SVNTN}, \lambda) = u(say) \\ s.t., & V(\widehat{180}, \lambda)\ p_1 + V(\widehat{89}, \lambda)\ p_2 + V(\widehat{125}, \lambda)\ p_3 \geq u \\ & V(\widehat{90}, \lambda)\ p_1 + V(\widehat{180}, \lambda)\ p_2 + V(\widehat{150}, \lambda)\ p_3 \geq u \\ & V(\widehat{120}, \lambda)\ p_1 + V(\widehat{125}, \lambda)\ p_2 + V(\widehat{90}, \lambda)\ p_3 \geq u \\ & p_1 + p_2 + p_3 = 1, \\ and & p_1,\ p_2,\ p_3 \geq 0. \end{cases} \tag{25}$$

For player-2 $(\text{CLPP})^{\text{II}}$:

$$\begin{cases} Min & V(\tilde{v}_{SVNTN},\lambda)=v(say) \\ s.t., & V(\widehat{180},\lambda)\ q_1+V(\widehat{90},\lambda)\ q_2+V(\widehat{120},\lambda)\ q_3\le v \\ & V(\widehat{89},\lambda)\ q_1+V(\widehat{180},\lambda)\ q_2+V(\widehat{125},\lambda)\ q_3\le v \\ & V(\widehat{125},\lambda)\ q_1+V(\widehat{150},\lambda)\ q_2+V(\widehat{90},\lambda)\ q_3\le v \\ & q_1+q_2+q_3=1, \\ and & q_1,\ q_2,\ q_3\ge 0. \end{cases} \tag{26}$$

The value index of all the distinct SVNTNs involved are calculated as
$V(\tilde{a}^{11}_{SVNTN},\lambda)=V(\widehat{180},\lambda)=(378.8332–199.1666\lambda)=V(\tilde{a}^{22}_{SVNTN},\lambda)$;
$V(\tilde{a}^{12}_{SVNTN},\lambda)=V(\widehat{90},\lambda)=(178.8173–89.1507\lambda)=V(\tilde{a}^{33}_{SVNTN},\lambda)$;
$V(\tilde{a}^{13}_{SVNTN},\lambda)=V(\widehat{120},\lambda)=(243.5756–122.5756\lambda)$;
$V(\tilde{a}^{21}_{SVNTN},\lambda)=V(\widehat{89},\lambda)=(179.1825–89.8492\lambda)$;
$V(\tilde{a}^{23}_{SVNTN},\lambda)=V(\widehat{125},\lambda)=(247.7093–123.3760\lambda)=V(\tilde{a}^{31}_{SVNTN},\lambda)$;
$V(\tilde{a}^{32}_{SVNTN},\lambda)=V(\widehat{150},\lambda)=(297.5843–148.2510\lambda)$;
Using these value indexes, we get the following CLPPs for the two players
For player-1 $(\text{CLPP})^{\text{I}}$:

$$\begin{cases} Max & V(\tilde{u}_{SVNTN},\lambda)=u(say) \\ s.t., & (378.8332–199.1666\lambda)\ p_1+(179.1825–89.8492\lambda)\ p_2+(247.7093–123.3760\lambda)\ p_3\ge u \\ & (178.8173–89.1507\lambda)\ p_1+(378.8332–199.1666\lambda)\ p_2+(297.5843–148.2510\lambda)\ p_3\ge u \\ & (243.5756–122.5756\lambda)\ p_1+(247.7093–123.3760\lambda)\ p_2+(178.8173–89.1507\lambda)\ p_3\ge u \\ & p_1+p_2+p_3=1, \\ and & p_1,\ p_2,\ p_3\ge 0. \end{cases} \tag{27}$$

For player-2 $(\text{CLPP})^{\text{II}}$:

$$\begin{cases} Min & V(\tilde{v}_{SVNTN},\lambda)=v(say) \\ s.t., & (378.8332–199.1666\lambda)\ q_1+(178.8173–89.1507\lambda)\ q_2+(243.5756–122.5756\lambda)\ q_3\le v \\ & (179.1825–89.8492\lambda)\ q_1+(378.8332–199.1666\lambda)\ q_2+(247.7093–123.3760\lambda)\ q_3\le v \\ & (247.7093–123.3760\lambda)\ q_1+(297.5843–148.2510\lambda)\ q_2+(178.8173–89.1507\lambda)\ q_3\le v \\ & q_1+q_2+q_3=1, \\ and & q_1,\ q_2,\ q_3\ge 0. \end{cases} \tag{28}$$

The value index of distinct SVNTNs are calculated using the proposed method for various degree of optimism λ, and are given in table-3 below.

Table 3. Value index of SVNTNs for different values of optimism degree λ

λ	$V(\widehat{180},\lambda)$	$V(\widehat{90},\lambda)$	$V(\widehat{120},\lambda)$	$V(\widehat{89},\lambda)$	$V(\widehat{125},\lambda)$	$V(\widehat{150},\lambda)$
0.0	378.8332	178.8173	243.5756	179.1825	247.7093	297.5843
0.1	358.9165	169.9022	231.3180	170.1975	235.3717	282.7592
0.2	338.9998	160.9871	219.0604	161.2126	223.0341	267.9341
0.3	319.0832	152.0720	206.8029	152.2277	210.6965	253.1090
0.4	299.1665	143.1570	194.5453	143.2428	198.3589	238.2839
0.5	279.2499	134.2419	182.2878	134.2579	186.0213	223.4588
0.6	259.3332	125.3268	170.0302	125.2729	173.6837	208.6337
0.7	239.4165	116.4118	157.7726	116.2880	161.3461	193.8086
0.8	219.4999	107.4967	145.5151	107.3031	149.0085	178.9835
0.9	199.5832	98.5816	133.2575	98.3182	136.6709	164.1584
1.0	179.6666	89.6666	121.0000	89.3333	124.3333	149.3333

Using the values given in Table-3, optimal solutions for various degree of optimism λ are obtained by solving CLPP for Player-1 (equation-27) and are given in table-4 below.

Table 4. Optimal Solution For Player-1 for Different Values of Optimism Degree λ

λ	p_1	p_2	p_3	$Max(u)$
0.0	0.3363	0.6637	0.0000	246.3193
0.1	0.3381	0.6619	0.0000	234.0012
0.2	0.3401	0.6599	0.0000	221.6825
0.3	0.3424	0.6576	0.0000	209.3632
0.4	0.3450	0.6550	0.0000	197.0430
0.5	0.3480	0.6520	0.0000	184.7219
0.6	0.3515	0.6485	0.0000	172.3994
0.7	0.3556	0.6444	0.0000	160.0753
0.8	0.3605	0.6395	0.0000	147.7492
0.9	0.3664	0.6336	0.0000	135.4203
1.0	0.3737	0.6263	0.0000	123.0878

Using the values given in Table-3, optimal solutions for various degree of optimism λ are obtained by solving CLPP for Player-2 (equation-28) and are given in table-5 below.

Table 5. Optimal Solution For Player-2 for Different Values of Optimism Degree λ

λ	q_1	q_2	q_3	$Max(v)$
0.0	0.0203	0.0000	0.9797	246.3199
0.1	0.0210	0.0000	0.9790	234.0012
0.2	0.0219	0.0000	0.9781	221.6825
0.3	0.0228	0.0000	0.9772	209.3632
0.4	0.0239	0.0000	0.9761	197.0430
0.5	0.0251	0.0000	0.9749	184.7219
0.6	0.0265	0.0000	0.9735	172.3994
0.7	0.0282	0.0000	0.9718	160.0753
0.8	0.0302	0.0000	0.9698	147.7492
0.9	0.0326	0.0000	0.9674	135.4203
1.0	0.0356	0.0000	0.9644	123.0878

6.2.1. *Conclusive Words on the Results of Example-2:*

The solution results for various values of degree of optimism for the incumbent player i.e., company C_1 (Airtel), are given in Table-4 above. Results show that value of the game decreases from 246.3193 to 123.0878 as the degree of optimism increases from 0.0 to 1.0, it means the value of the game is inversely proportional to the degree of optimism of the incumbent player. The results of example-2 almost follow the same pattern as of example-1. This can be observed from the graphical representation in Figure-4 of obtained optimal values 'u' for incumbent player-1 against different values of the degree of optimism 'λ' for both example-1 & 2.

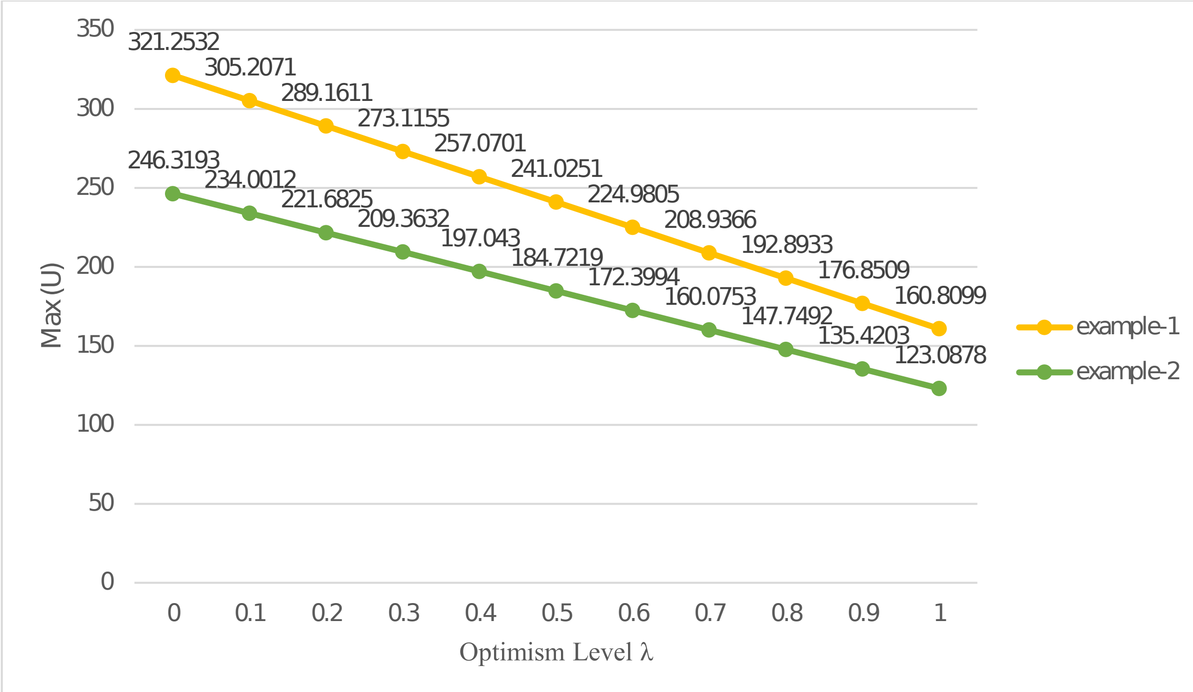


FIGURE 4. Value of the Game Against Degree of Optimism λ

7. Discussion:

In the zero sum NMG, the optimal expected loss of player-2 is equal to the optimal expected gain of player-1, it can be observed from Table-2 in Example-1 and Table-4 & 5 in Example-2. A graphical representation of optimal values against different values of degree of optimism λ is given in Figure-4 for both Example-1 and Example-2. As all the results obtained by our solution methodology are crisp, they are more reliable and trustworthy. Analysing the results of our work it can be summarised that the optimum value of game for player-1 decreases as degree of optimism 'λ' increases in interval [0, 1]. So we can say that optimal value obtained for player-1 is inversely proportional to his degree of optimism, i.e., the more optimistic you are, and the more you may lose. So, we can conclude that it good to be moderate rather than over optimistic.

8. Conclusion:

This study introduces an efficient de-neutrosophication method that leverages Mellin's transform to derive precise values from Single-Valued Neutrosophic Triangular Numbers (SVNTNs), significantly enhancing decision-making processes in Neutrosophic Matrix-Game strategies (NTMGs). We have demonstrated the efficacy of this technique through detailed numerical examples. By transforming Neutrosophic Linear Programming Problems (NLPPs) into Crisp Linear Programming Problems (CLPPs), and adjusting for varying degrees of optimism, we have utilized the TORA-2.0 software to achieve optimal solutions that promise to aid competitive players in the industrial sector in making more informed and economically beneficial decisions. We aim to broaden the scope of our research to encompass a wider array of neutrosophic numbers, including but not limited to trapezoidal, pentagonal, interval-valued, bi-polar, and spherical neutrosophic numbers. This expansion is anticipated to address more complex and diverse decision-making scenarios, offering a comprehensive toolkit for both theoretical exploration and practical application in the field of neutrosophic decision-making and optimization.

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An Enhanced Generalized Neutrosophic Number and its role in Multi-Criteria Decision-Making Challenges

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Abstract. In this article, we have proposed an ordering technique for Neutrosophic numbers with non-linear functions. Consequently, the non-linear functions overcome the limitations of linear function approaches by giving an enhanced framework for handling and modeling uncertainty. Hence, this study presents the Generalized Parabolic Single Valued Neutrosophic Number (GPSVNN) to address the uncertainties in Multi-Criteria Decision-Making (MCDM) circumstances. GPSVNN can handle uncertainty and perform arithmetic operations to deal with MCDM through the (α, β, γ)-cut technique. The computation of the (α, β, γ)-cut of the neutrosophic number is reduced by defining the "value" and "ambiguity." As a result, it becomes more systematic when the complicated computations using the (α, β, γ)-cut approach are carried out. A novel ordering approach has been developed in this study by incorporating the "value" and "ambiguity" of GPSVNN. Finally, we have given an example using GPSVNN in a life satisfaction survey to show its applicability.

Keywords: Generalized Parabolic Single Valued Neutrosophic Number(GPSVNN); Arithmetic Operators of GPSVNN; Values and Ambiguities of GPSVNN & Mutli-criteria Decision making problem.

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1. Introduction

Handling data that contain uncertainty and dealing with nonlinearity has become vital in numerous applications such as facial pattern recognition, transmission systems, knowledge-based models for risk assessment, stock trading, etc. Information derived from computational perception and cognition, which is unclear, imprecise, ambiguous, partially true, or lacking specific limits, can be dealt with the help of fuzzy logic. Lotfi A. Zadeh [1] initially proposed the concept of fuzzy sets in 1965. In [2, 3] fuzzy logic in multi-criteria decision-making using the concept of fuzzy numbers have been applied. The arithmetic operations for generalized parabolic fuzzy numbers and its applications were explored in [4]. In [5], the authors address the mF Dombi weighted averaging and geometric operators to solve multi-criteria decision-making problems that utilize mF information under M-polar fuzzy sets.

Chakraborty et al. [6] demonstrated the hexagonal fuzzy number and its characteristic representation, ranking, defuzzification method, and application in the manufacturing inventory management problem. By expanding the concept of evidence theory, Krishankumar [7] has suggested a unique ranking mechanism under the probabilistic hesitant fuzzy set.The fuzzy set gives one index to represent both membership and non-membership degrees.

The fuzzy set cannot express its independence. Atanassov [8] proposed the idea of intuitionistic fuzzy sets to solve this problem. Employing intuitionistic fuzzy logic enables the resolution of challenging decision-making problems. Many researchers [9–13] have applied various intuitionistic fuzzy numbers to multi-criteria decision-making problems. The study of modelling uncertainty is evolving rapidly. Researchers have previously conducted different significant and progressive investigations, and there are numerous approaches, including fuzzy and intuitionistic fuzzy sets, to handle these uncertainties in modelling. These problems, which apply to real-world problems, cannot deal with all forms of uncertainty, such as ambiguous and inconsistent information. Smarandache [14, 15] initiated neutrosophic theory, which further generalizes fuzzy and intuitionistic fuzzy sets. In [16], the authors defined a particular case of a neutrosophic set called a single-valued neutrosophic set and set-theoretic operators. Chen and Jiqian [17] introduced the Dombi operations of t-norm and t-conorm, and they benefit from being very flexible concerning the operational parameters. and to solve multi-criteria decision-making problems, in [18] a new tool have been developed, that considers the bipolar trapezoidal neutrosophic and the Dombi operators. In [19–21], investigated trapezoidal neutrosophic numbers and its applicability. Chakraborty [22–24] developed the de-neutrosophication approach using the elimination area method as a manifestation of the linear pentagonal neutrosophic number. In [25], a decision-making strategy is described by applying similarity measures based on distance measures. Paulraj S. [26] presented an expansion of single-valued trapezoidal neutrosophic ordered weighted harmonic averaging. Researchers widely use a proactive green supply chain management strategy in [27]. Janani [28] and Ramya [29] presented a perceptive investigation that expands on Bipolar Pythagorean refined set and Pythagorean Neutrosophic Hypersoft Sets, emphasizing the essential features. Many researchers [30–40] have used different neutrosophic numbers to deal with various multi-criteria decision-making problems.

The increasing complexity and unpredictability of decision-making situations in several fields need innovative mathematical frameworks which could effectively handle these challenges. However, there may be some restrictions due to insufficient or lacking quality of the currently available data. Sometimes, using linear functions is inadequate for the consideration of uncertainty. Therefore, the non-linear functions provide an improved framework for managing and modeling uncertainty. Hence, the non-linearity in the Neutrosophic numbers enhances its applicability range. In this study, we explore the Generalized Parabolic Single Valued Neutrosophic Number (GPSVNN), a novel form of non-linear neutrosophic number.

Contributions:

- We have introduced a new type of non-linear neutrosophic number called Generalized Parabolic Single Valued Neutrosophic Number.
- This study develops a novel ordering method by incorporating the ”value” and ”ambiguity” of these Neutrosophic numbers.
- By defining the ”value” and ”ambiguity” of these Neutrosophic numbers, significantly reduces the requirement to compute the (α, β, γ) -cut of the neutrosophic number. Consequently, it becomes more systematic when the tedious calculations employing the (α, β, γ)-cut approach are performed.
- Instead of computing over the complete integration range, the value and ambiguity are computed at (α, β, γ)- levels. These levels are referred to as flexibility parameters because they enable decision-makers to act at different stages of the decision-making process.

The paper is structured as follows. A detailed literature study and introduction are discussed in the first section, and essential preliminary remarks are presented in the second section. The definition of Generalized Parabolic Single-Valued Neutrosophic Numbers (GPSVNN), along with their arithmetic operators, values, and ambiguities, are provided in the following part. In [41], the study addressed the ranking of inhabitants' satisfaction levels with municipal services. Twenty municipal services from the Life Satisfaction Survey (LSS), conducted annually by the Turkish Statistical Institution, are considered possibilities for this purpose. Additionally, the 2014–2019 period was used as a set of criteria when evaluating the inhabitants' contentment, in addition to the previous year. The researchers transformed the participant responses in the dataset into Picture Fuzzy Numbers (PFNs) with four parameters to analyze the impact of all opinions on the decision-making process. (positive, neutral, negative, and refusal). Finally, they used PFNs arithmetic operators and evaluated the results using the VIKOR (VIseKriterijumska Optimizacija I Kompromisno Resenje) technique. In this scenario, we offered the choice within the neutrosophic environment for the same problem, indicating that the opinion type is expressed in GPSVNN. Using its values and ambiguity, we have ranked the alternative from 2014 to 2017.

2. Preliminaries

Definition 2.1. [14] A neutrosophic set A on a universal set X is defined as $A = \{\langle T_A(x), I_A(x), F_A(x)\rangle : x \in X\}$, where T_A, I_A ,F_A : $X \rightarrow]0^-, 1[^+$, represents the degree of membership, degree of indeterministic, and degree of non-membership respectively of the element $x \in X$, such that $0^- \leq supT_A(x) + supI_A(x) + supF_A(x) \leq 3^+$.

Definition 2.2. [16] A single valued neutrosophic set A on a universal set X is defined as $A = \{\langle T_A(x), I_A(x), F_A(x)\rangle : x \in X\}$, where T_A, I_A ,$F_A : X \to [0,1]$, represents the degree of membership, degree of indeterministic, and degree of non-membership respectively of the element $x \in X$, such that $0 \leq supT_A(x) + supI_A(x) + supF_A(x) \leq 3$.

Definition 2.3. [24] A Single Valued Neutrosophic Number (SVNN) $\tilde{a} = \langle T_{\tilde{a}}, I_{\tilde{a}}, F_{\tilde{a}}\rangle$, in the set of real numbers R with truth-membership function $T_{\tilde{a}}$, indeterminacy-membership function $I_{\tilde{a}}$ and falsity-membership function $F_{\tilde{a}}$, is defined as

$$T_{\tilde{a}}(x) = \begin{cases} f_{\tilde{a}}(x) & , \text{ if } a_1 \leq x < b_1 \\ 1 & , \text{ if } b_1 \leq x < c_1 \\ g_{\tilde{a}}(x) & , \text{ if } c_1 \leq x < d_1 \\ 0 & , \text{ otherwise} \end{cases}, I_{\tilde{a}}(x) = \begin{cases} l_{\tilde{a}}(x) & , \text{ if } a_2 \leq x < b_2 \\ 0 & , \text{ if } b_2 \leq x < c_2 \\ m_{\tilde{x}}(x) & , \text{ if } c_2 \leq x < d_2 \\ 1 & , \text{ otherwise} \end{cases} \text{ and}$$

$$F_{\tilde{a}}(x) = \begin{cases} h_{\tilde{a}}(x) & , \text{ if } a_3 \leq x < b_3 \\ 0 & , \text{ if } b_3 \leq x < c_3 \\ k_{\tilde{x}}(x) & , \text{ if } c_3 \leq x < d_3 \\ 1 & , \text{ otherwise} \end{cases}$$ respectively, where $0 \leq T_{\tilde{a}} + I_{\tilde{a}} + F_{\tilde{a}} \leq 3$ and $a_i, b_i, c_i, di \in \mathbb{R}$,

$a_i \leq b_i \leq c_i \leq d_i$ where $i = 1, 2, 3$ and the functions $f_{\tilde{a}}, g_{\tilde{a}}, l_{\tilde{a}}, m_{\tilde{a}}, h_{\tilde{a}}, k_{\tilde{a}}$:$\mathbb{R} \to [0, 1]$.

The functions, $f_{\tilde{a}}, m_{\tilde{a}}, k_{\tilde{a}}$ are non-decreasing continuous function and $g_{\tilde{a}}, l_{\tilde{a}}, h_{\tilde{a}}$ are non-increasing continuous function. SVNN is also denoted by $\tilde{a} = \langle (a_1, b_1, c_1, d_1), (a_2, b_2, c_2, d_2), (a_3, b_3, c_3, d_3)\rangle$

Definition 2.4. A Single Valued Neutrosophic Number defined on the set of real numbers $\mathbb{R}$ is said to be Generalized Single Valued Neutrosophic Number (GSVNN) $G_{\tilde{a}} = \langle T_{G\tilde{a}}, I_{G\tilde{a}}, F_{G\tilde{a}}; \omega, \rho, \delta\rangle$, with truth-membership function $T_{G\tilde{a}}(x)$, indeterminacy-membership function $I_{G\tilde{a}}(x)$ and falsity-membership function $F_{G\tilde{a}}(x)$ has the following characteristics.

(1) $T_{G\tilde{a}}, I_{G\tilde{a}}, F_{G\tilde{a}}\ \mathbb{R} \to [0, 1]$.
(2) $T_{G\tilde{a}} = 0, I_{G\tilde{a}} = 1, F_{G\tilde{a}} = 1 for\ \ all$ x $\in (-\infty, a_i] \cup [d_i, \infty)$.
(3) $T_{G\tilde{a}}(x)$ is strictly increasing on $[a_1, b_1]$ and $T_{G\tilde{a}}(x)$ is strictly decreasing on $[c_1, d_1]$.
$I_{G\tilde{a}}(x)$ is strictly decreasing on $[a_2, b_2]$ and $I_{G\tilde{a}}(x)$ is strictly increasing on $[c_2, d_2]$.
$F_{G\tilde{a}}(x)$ is strictly decreasing on $[a_3, b_3]$ and $F_{G\tilde{a}}(x)$ is strictly increasing on $[c_3, d_3]$.
(4) $T_{G\tilde{a}}(x) = \omega$ for all $x \in [b_1, c_1]$ where $0 < \omega \leq 1$. $I_{G\tilde{a}}(x) = \rho$ for all $x \in [b_2, c_2]$ where $0 \leq \rho < 1$. $F_{G\tilde{a}}(x) = \delta$ for all $x \in [b_2, c_2]$ where $0 \leq \delta < 1$.

3. **A Generalized Parabolic Single Valued Neutrosophic Number(GPSVNN)**

Definition 3.1. A Generalized Parabolic Single Valued Neutrosophic Number (GPSVNN) ,
$\tilde{A} = \langle (T_{\tilde{A}}; \omega), (I_{\tilde{A}}; \rho), (F_{\tilde{A}}; \delta)\rangle$, is a Neutrosophic set on real number $\mathbb{R}$ with truth-membership function $T_{\tilde{A}}$, indeterminacy-membership function $I_{\tilde{A}}$ and falsity-membership function $F_{\tilde{A}}$, is defined as

$$T_{\tilde{A}}(x) = \begin{cases} \omega(\frac{x-a_1}{b_1-a_1})^2 & ; x \in [a_1, b_1) \\ \omega & ; x \in [b_1, c_1) \\ \omega(\frac{d_1-x}{d_1-c_1})^2 & ; x \in [c_1, d_1) \\ 0 & ; otherwise \end{cases}, I_{\tilde{A}}(x) = \begin{cases} 1-(\frac{x-a_2}{b_2-a_2})^2(1-\rho) & ; x \in [a_2, b_2) \\ \rho & ; x \in [b_2, c_2) \\ 1-(\frac{d_2-x}{d_2-c_2})^2(1-\rho) & ; x \in [c_2, d_2) \\ 1 & ; otherwise \end{cases} \text{ and}$$

$$F_{\tilde{A}}(x) = \begin{cases} 1-(\frac{x-a_3}{b_3-a_3}^2)(1-\delta) & ; x \in [a_3, b_3) \\ \delta & ; x \in [b_3, c_3) \\ 1-(\frac{d_3-x}{d_3-c_3})^2(1-\delta) & ; x \in [c_3, d_3) \\ 1 & ; otherwise \end{cases}$$

where $0 \leq T_{\tilde{A}} + I_{\tilde{A}} + F_{\tilde{A}} \leq 3$, $0 < \omega \leq 1$, $0 \leq \rho < 1$, $0 \leq \delta < 1$ and $a_i, b_i, c_i, di \in \mathbb{R}$, $a_i \leq b_i \leq c_i \leq d_i$ where $i = 1, 2, 3$.

Note:1 GPSVNN is also denoted by

(1) $\tilde{A} = \langle (a_1, b_1, c_1, d_1; \omega), (a_2, b_2, c_2, d_2; \rho), (a_3, b_3, c_3, d_3; \delta) \rangle$

(2) $\tilde{A} = \langle (a, b, c, d)\ ; \omega, \rho, \delta \rangle$(If we consider the same values for the truth,falsity and indeterminacy membership).

Definition 3.2. The (α, β, γ) - cut of GPSVNN defined as $\tilde{A}^{(\alpha,\beta,\gamma)} = \{x | T_{\tilde{A}}(x) \geq \alpha, I_{\tilde{A}}(x) \leq \beta, F_{\tilde{A}}(x) \leq \gamma\}$, where $\alpha \in [0, \omega], \beta \in [\rho, 1], \gamma \in [\delta, 1]$ such that $\alpha + \beta + \gamma \leq 3$, ie., $\tilde{A}^{\alpha,\beta,\gamma} = \langle \tilde{A}^{\alpha}, \tilde{A}^{\beta}, \tilde{A}^{\gamma} \rangle$, where $\tilde{A}^{\alpha} = [\, a_1 + (b_1 - a_1)\sqrt{\alpha/\omega}, d_1 - (d_1 - c_1)\sqrt{\alpha/\omega}\,] = [L^{\alpha}, U^{\alpha}]$
$\tilde{A}^{\beta} = [\, a_2 + (b_2 - a_2)\sqrt{(1-\beta)/(1-\rho)}, d_2 - (d_2 - c_2)\sqrt{(1-\beta)/(1-\rho)}\,] = [L'^{\alpha}, U'^{\alpha}]$
$\tilde{A}^{\gamma} = [\, a_3 + (b_3 - a_3)\sqrt{(1-\gamma)/(1-\delta)}, d_3 - (d_3 - c_3)\sqrt{(1-\beta)/(1-\delta)}\,] = [L''^{\alpha}, U''^{\alpha}]$

Definition 3.3. A Parabolic Single Valued Neutrosophic Number (PSVNN) , $\tilde{A} = \langle T_{\tilde{A}}), I_{\tilde{A}}, F_{\tilde{A}} \rangle$, is a Neutrosophic set on real number $\mathbb{R}$ with truth-membership function $T_{\tilde{A}}$, indeterminacy-membership function $I_{\tilde{A}}$ and falsity-membership function $F_{\tilde{A}}$, is defined as,

$$T_{\tilde{A}}(x) = \begin{cases} (\frac{x-a_1}{b_1-a_1})^2 & ; x \in [a_1, b_1) \\ 1 & ; x \in [b_1, c_1) \\ (\frac{d_1-x}{d_1-c_1})^2 & ; x \in [c_1, d_1) \\ 0 & ; otherwise \end{cases}, I_{\tilde{A}}(x) = \begin{cases} 1-(\frac{x-a_2}{b_2-a_2})^2 & ; x \in [a_2, b_2) \\ 0 & ; x \in [b_2, c_2) \\ 1-(\frac{d_2-x}{d_2-c_2})^2 & ; x \in [c_2, d_2) \\ 1 & ; otherwise \end{cases} \text{ and}$$

$$F_{\tilde{A}}(x) = \begin{cases} 1-(\frac{x-a_3}{b_3-a_3}^2) & ; x \in [a_3, b_3) \\ 0 & ; x \in [b_3, c_3) \\ 1-(\frac{d_3-x}{d_3-c_3})^2 & ; x \in [c_3, d_3) \\ 1 & ; otherwise \end{cases}$$

where $0 \leq T_{\tilde{A}} + I_{\tilde{A}} + F_{\tilde{A}} \leq 3$, $a_i, b_i, c_i, di \in \mathbb{R}$, $a_i \leq b_i \leq c_i \leq d_i$ where $i = 1, 2, 3$.

Arithmetic Operators of GPSVNN

Definition 3.4. Let $\tilde{A}$ and $\tilde{B}$ are the two GPSVNN , then we define the arithmetic operators for
$\tilde{A} = \langle(a_1, b_1, c_1, d_1; \omega_1), (a_2, b_2, c_2, d_2; \rho_1), (a_3, b_3, c_3, d_3; \delta_1)\rangle$ and
$\tilde{B} = \langle(a'_1, b'_1, c'_1, d'_1; \omega_2), (a'_2, b'_2, c'_2, d'_2; \rho_2), (a'_3, b'_3, c'_3, d'_3; \delta_2)\rangle$ as follows.
where, $\omega = min\{\omega_1, \omega_2\}$, $\rho = max\{\rho_1, \rho_2\}$ and $\delta = max\{\delta_1, \delta_2\}$
(We denote $\wedge$ for min and $\vee$ for max.)

1. The addition of GPSVNN's $\tilde{A} + \tilde{B} = \tilde{C}$ is

$$T_{\tilde{C}}(x) = \begin{cases} \omega\left(\frac{x-(a_1+a'_1)}{(b_1+b'_1)-(a_1+a'_1)}\right)^2 & ; x \in [a_1 + a'_1, b_1 + b'_1) \\ \omega & ; x \in [b_1 + b'_1, c_1 + c'_1) \\ \omega\left(\frac{(d_1+d'_1)-x}{(d_1+d'_1)-(c_1+c'_1)}\right)^2 & ; x \in [c_1 + c'_1, d_1 + d'_1) \\ 0 & ; otherwise \end{cases},$$

$$I_{\tilde{C}}(x) = \begin{cases} 1 - \left(\frac{x-(a_2+a'_2)}{(b_2+b'_2)-(a_2+a'_2)}\right)^2(1-\rho) & ; x \in [a_2 + a'_2, b_2 + b'_2) \\ \rho & ; x \in [b_2 + b'_2, c_2 + c'_2) \\ 1 - \left(\frac{(d_2+d'_2)-x}{(d_2+d'_2)-(c_2+c'_2)}\right)^2(1-\rho) & ; x \in [c_2 + c'_2, d_2 + d'_2) \\ 1 & ; otherwise \end{cases}$$

and $$F_{\tilde{C}}(x) = \begin{cases} 1 - \left(\frac{x-(a_3+a'_2)}{(b_3+b'_3)-(a_3+a'_3)}\right)^2(1-\delta) & ; x \in [a_3 + a'_3, b_3 + b'_3) \\ \delta & ; x \in [b_3 + b'_3, c_3 + c'_3) \\ 1 - \left(\frac{(d_3+d'_3)-x}{(d_3+d'_3)-(c_3+c'_3)}\right)^2(1-\delta) & ; x \in [c_3 + c'_3, d_3 + d'_3) \\ 1 & ; otherwise \end{cases}$$

2.The subtraction of GPSVNN's $\tilde{A} - \tilde{B} = \tilde{C}$ is

$$T_{\tilde{C}}(x) = \begin{cases} \omega\left(\frac{x-(a_1-d'_1)}{(b_1-c'_1)-(a_1-d'_1}\right)^2 & ; x \in [a_1 - d'_1, b_1 - c'_1) \\ \omega & ; x \in [b_1 - c'_1, c_1 - b'_1) \\ \omega\left(\frac{(d_1-a'_1)-x}{(d_1-a'_1)-(c_1-b'_1)}\right)^2 & ; x \in [c_1 - b'_1, d_1 - a'_1) \\ 0 & ; otherwise \end{cases},$$

$$I_{\tilde{C}}(x) = \begin{cases} 1 - \left(\frac{x-(a_2-d'_2)}{(b_2-c'_2)-(a_2-d'_2)}\right)^2(1-\rho) & ; x \in [a_2 - d'_2, b_2 - c'_2) \\ \rho & ; x \in [b_2 - c'_2, c_2 - b'_2) \\ 1 - \left(\frac{(d_2-a'_2)-x}{(d_2-a'_2)-(c_2-b'_2)}\right)^2(1-\rho) & ; x \in [c_2 - b'_2, d_2 - a'_2) \\ 1 & ; otherwise \end{cases}$$

$$F_{\tilde{C}}(x) = \begin{cases} 1 - \left(\frac{x-(a_3-d_3')}{(b_3-c_3')-(a_3-d_3')}\right)^2(1-\delta) & ; x \in [a_3-d_3', b_3-c_3') \\ \delta & ; x \in [b_3-c_3', c_3-b_3') \\ 1 - \left(\frac{(d_3-a_3')-x}{(d_3-a_3')-(c_3-b_3')}\right)^2(1-\delta) & ; x \in [c_3-b_3', d_3-a_3') \\ 1 & ; otherwise \end{cases}$$

3. The multiplication of GPSVNN's $\tilde{A} * \tilde{B} = \tilde{C}$ is

$$T_{\tilde{C}}(x) = \begin{cases} \omega(\frac{x-p_1}{p_2-p_1})^2 & ; x \in [p_1, p_2) \\ \omega & ; x \in [p_2, p_3) \\ \omega(\frac{p_4-x}{p_4-p_3})^2 & ; x \in [p_3, p_4) \\ 0 & ; otherwise \end{cases}, I_{\tilde{C}}(x) = \begin{cases} 1-(\frac{x-q_1}{q_2-q_1})^2(1-\rho) & ; x \in [q_1, q_2) \\ \rho & ; x \in [q_2, q_3) \\ 1-(\frac{q_4-x}{q_4-q_3})^2(1-\rho) & ; x \in [q_3, q_4) \\ 1 & ; otherwise \end{cases}$$

and $F_{\tilde{C}}(x) = \begin{cases} 1-(\frac{x-r_1}{r_2-r_1})^2(1-\delta) & ; x \in [r_1, r_2) \\ \delta & ; x \in [r_2, r_3) \\ 1-(\frac{r_4-x}{r_4-r_3})^2(1-\delta) & ; x \in [r_3, r_4) \\ 1 & ; otherwise \end{cases}$

where $p_1 = min\{a_1 * a_1', a_1 * d_1', d_1 * a_1', d_1 * d_1'\}$, $p_2 = min\{b_1 * b_1', b_1 * c_1', c_1 * b_1', c_1 * c_1'\}$
$p_3 = max\{b_1 * b_1', b_1 * c_1', c_1 * b_1', c_1 * c_1'\}$, $p_4 = max\{a_1 * a_1', a_1 * d_1', d_1 * a_1', d_1 * d_1'\}$
$q_1 = min\{a_2 * a_2', a_2 * d_2', d_2 * a_2', d_2 * d_2'\}$, $q_2 = min\{b_2 * b_2', b_2 * c_2', c_2 * b_2', c_2 * c_2'\}$
$q_3 = max\{b_2 * b_2', b_2 * c_2', c_2 * b_2', c_2 * c_2'\}$, $q_4 = max\{a_2 * a_2', a_2 * d_2', d_2 * a_2', d_2 * d_2'\}$
$r_1 = min\{a_3 * a_3', a_3 * d_3', d_3 * a_3', d_3 * d_3'\}$, $r_2 = min\{b_3 * b_3', b_3 * c_3', c_3 * b_3', c_3 * c_3'\}$
$r_3 = max\{b_3 * b_3', b_3 * c_3', c_3 * b_3', c_3 * c_3'\}$, $r_4 = max\{a_3 * a_3', a_3 * d_3', d_3 * a_3', d_3 * d_3'\}$.

4. Inverse of GPSVNN

Consider the GPSVN-number, $\tilde{B} = \langle(a_1', b_1', c_1', d_1'; \omega), (a_2', b_2', c_2', d_2'; \rho), (a_3', b_3', c_3', d_3'; \delta)\rangle$.

The inverse of this GPSVN-number is,

$\frac{1}{\tilde{B}} = \langle(\frac{1}{d_1'}, \frac{1}{c_1'}, \frac{1}{b_1'}, \frac{1}{a_1'}; \omega), (\frac{1}{d_2'}, \frac{1}{c_2'}, \frac{1}{b_2'}, \frac{1}{a_2'}; \rho), (\frac{1}{d_3'}, \frac{1}{c_3'}, \frac{1}{b_3'}, \frac{1}{a_3'}; \delta)\rangle$, $0 \notin [a_i', d_i']$, where i = 1,2,3.

5. Division of GPSVNN The division of $\tilde{A}/\tilde{B}$ can be defined as the multiplication of two GPSVNN $\tilde{A} * \frac{1}{\tilde{B}} = \tilde{C}$,

$$T_{\tilde{C}}(x) = \begin{cases} \omega(\frac{x-p_1}{p_2-p_1})^2 & ; x \in [p_1, p_2) \\ \omega & ; x \in [p_2, p_3) \\ \omega(\frac{p_4-x}{p_4-p_3})^2 & ; x \in [p_3, p_4) \\ 0 & ; otherwise \end{cases},$$

$$I_{\tilde{C}}(x) = \begin{cases} 1-(\frac{x-q_1}{q_2-q_1})^2(1-\rho) & ; x \in [q_1, q_2) \\ \rho & ; x \in [q_2, q_3) \\ 1-(\frac{q_4-x}{q_4-q_3})^2(1-\rho) & ; x \in [q_3, q_4) \\ 1 & ; otherwise \end{cases}$$

and $F_{\tilde{C}}(x) = \begin{cases} 1-(\frac{x-r_1}{r_2-r_1})^2(1-\delta) & ; x \in [r_1, r_2) \\ \delta & ; x \in [r_2, r_3) \\ 1-(\frac{r_4-x}{r_4-r_3})^2(1-\delta) & ; x \in [r_3, r_4) \\ 1 & ; otherwise \end{cases}$

where $p_1 = min\{\frac{a_1}{d_1'}, \frac{a_1}{a_1'}, \frac{d_1}{d_1'}, \frac{d_1}{a_1'}\}$, $p_2 = min\{\frac{b_1}{c_1'}, \frac{b_1}{b_1'}, \frac{c_1}{c_1'}, \frac{c_1}{b_1'}\}$, $p_3 = max\{\frac{b_1}{c_1'}, \frac{b_1}{b_1'}, \frac{c_1}{c_1'}, \frac{c_1}{b_1'}\}$, $p_4 = max\{\frac{a_1}{d_1'}, \frac{a_1}{a_1'}, \frac{d_1}{d_1'}, \frac{d_1}{a_1'}\}$, $q_1 = min\{\frac{a_2}{d_2'}, \frac{a_2}{a_2'}, \frac{d_2}{d_2'}, \frac{d_2}{a_2'}\}$, $q_2 = min\{\frac{b_2}{c_2'}, \frac{b_2}{b_2'}, \frac{c_2}{c_2'}, \frac{c_2}{b_2'}\}$, $q_3 = max\{\frac{b_2}{c_2'}, \frac{b_2}{b_2'}, \frac{c_2}{c_2'}, \frac{c_2}{b_2'}\}$, $q_4 = max\{\frac{a_2}{d_2'}, \frac{a_2}{a_2'}, \frac{d_2}{d_2'}, \frac{d_2}{a_2'}\}$, $r_1 = min\{\frac{a_3}{d_3'}, \frac{a_3}{a_3'}, \frac{d_3}{d_3'}, \frac{d_3}{a_3'}\}$, $r_2 = min\{\frac{b_3}{c_3'}, \frac{b_3}{b_3'}, \frac{c_3}{c_3'}, \frac{c_3}{b_3'}\}$, $r_3 = max\{\frac{b_3}{c_3'}, \frac{b_3}{b_3'}, \frac{c_3}{c_3'}, \frac{c_3}{b_3'}\}$, $r_4 = max\{\frac{a_3}{d_3'}, \frac{a_3}{a_3'}, \frac{d_3}{d_3'}, \frac{d_3}{a_3'}\}$.

Example 3.5. $\tilde{A} = \langle(3, 5, 8, 12); 0.2, 0.3, 0.5\rangle$ and $\tilde{B} = \langle(-7, -5, 6, 7); 0.2, 0.3, 0.5\rangle$ then

(1) $\tilde{A} + \tilde{B}$ **is**

$$T_{\tilde{A}}(x) = \begin{cases} 0.2(\frac{x+4}{4})^2 & ; x \in [-4, 0) \\ 0.2 & ; x \in [0, 14) \\ 0.2(\frac{19-x}{5})^2 & ; x \in [14, 19) \\ 0 & ; otherwise \end{cases}, I_{\tilde{A}}(x) = \begin{cases} 1-(\frac{x+4}{4})^2(0.7) & ; x \in [-4, 0) \\ 0.3 & ; x \in [0, 14) \\ 1-(\frac{19-x}{5})^2(0.7) & ; x \in [14, 19) \\ 1 & ; otherwise \end{cases}$$

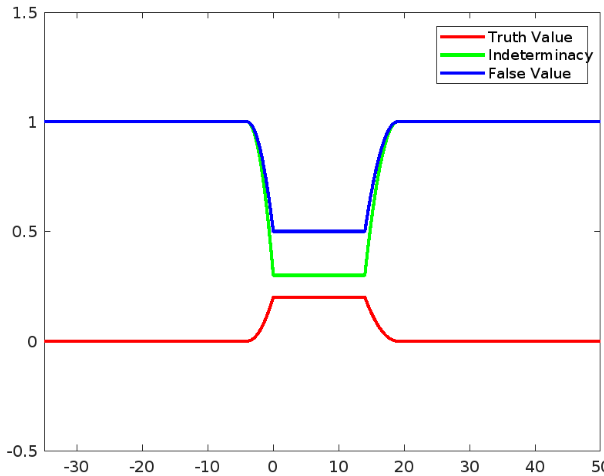
and $F_{\tilde{A}}(x) = \begin{cases} 1-(\frac{x+4}{4})^2(0.5) & ; x \in [-4, 0) \\ 0.5 & ; x \in [0, 14) \\ 1-(\frac{19-x}{5})^2(0.5) & ; x \in [14, 19) \\ 1 & ; otherwise \end{cases}$

Example 3.6. $\tilde{A} = \langle(3, 5, 8, 12); 0.2, 0.3, 0.5\rangle$ and $\tilde{B} = \langle(1, 2, 3, 4); 0.2, 0.3, 0.5\rangle$ then $\tilde{A}/\tilde{B}$ is

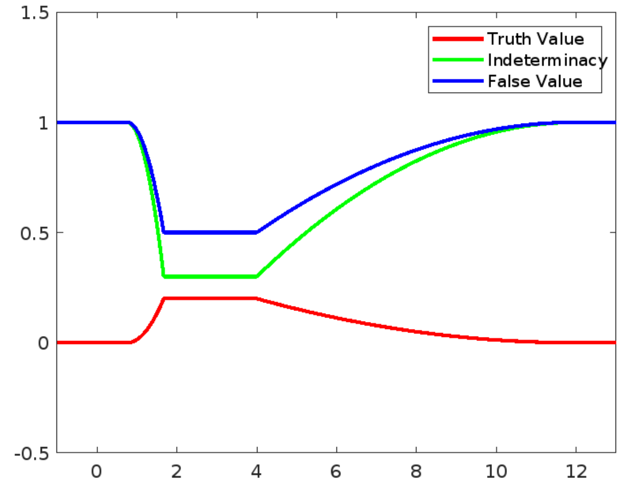
$$T_{\tilde{A}}(x) = \begin{cases} 0.2(\frac{x-0.75}{0.92})^2 & ; x \in [0.75, 1.67) \\ 0.2 & ; x \in [1.67, 4) \\ 0.2(\frac{12-x}{8})^2 & ; x \in [4, 12) \\ 0 & ; otherwise \end{cases}, I_{\tilde{A}}(x) = \begin{cases} 1-(\frac{x-0.75}{0.92})^2(0.7) & ; x \in [0.75, 1.67) \\ 0.3 & ; x \in [1.67, 4) \\ 1-(\frac{12-x}{8})^2(0.7) & ; x \in [4, 12) \\ 1 & ; otherwise \end{cases}$$

and $F_{\tilde{A}}(x) = \begin{cases} 1-(\frac{x-0.75}{0.92})^2(0.5) & ; x \in [0.75, 1.67) \\ 0.5 & ; x \in [1.67, 4) \\ 1-(\frac{12-x}{8})^2(0.5) & ; x \in [4, 12) \\ 1 & ; otherwise \end{cases}$

The graphical interpretation of Example 3.5 and 3.6 are given below.



(A) Addition



(B) Division

3.1. *Value and Ambiguity*

Definition 3.7. Let $\tilde{A} = \langle (a_1, b_1, c_1, d_1; \omega), (a_2, b_2, c_2, d_2; \rho), (a_3, b_3, c_3, d_3; \delta) \rangle$ be a GPSVNN . Then (α, β, γ)-cut set of the GPSVNN are $\tilde{A}^\alpha = [L^\alpha, U^\alpha]$, $\tilde{A}^\beta = [L'^\alpha, U'^\alpha]$ and $\tilde{A}^\gamma = [L''^\alpha, U''^\alpha]$ respectively. Then the **Values** of GPSVNN are defined as,
$\mathcal{V}(\tilde{A}^\alpha) = \int_0^\omega (L^\alpha + U^\alpha) f(\alpha)\, d\alpha$ where, $f(\alpha) \in [0,1] (\alpha \in [0, \omega])$, $f(0) = 0$ and $f(\alpha)$ is monotonic and non-decreasing of $\alpha \in [0, \omega]$.
$\mathcal{V}(\tilde{A}^\beta) = \int_\rho^1 (L'^\alpha + U'^\alpha) g(\beta)\, d\beta$ where , $g(\beta) \in [0,1] (\beta \in [\rho, 1])$, $g(1) = 0$ and $g(\beta)$ is monotonic and non-increasing of $\beta \in [\rho, 1]$.
$\mathcal{V}(\tilde{A}^\gamma) = \int_\delta^1 (L''^\alpha + U''^\alpha) h(\gamma)\, d\gamma$ where , $h(\gamma) \in [0,1] (\gamma \in [\delta, 1])$, $h(1) = 0$ and $h(\gamma)$ is monotonic and non-increasing of $\gamma \in [\delta, 1]$.

Definition 3.8. Let $\tilde{A} = \langle (a_1, b_1, c_1, d_1; \omega), (a_2, b_2, c_2, d_2; \rho), (a_3, b_3, c_3, d_3; \delta) \rangle$ be a GPSVNN . Then (α, β, γ)-cut set of the GPSVNN $\tilde{A}^\alpha = [L^\alpha, U^\alpha]$, $\tilde{A}^\beta = [L'^\alpha, U'^\alpha]$ and $\tilde{A}^\gamma = [L''^\alpha, U''^\alpha]$ are respectively. Then the **Ambiguties** of GPSVNN are defined as,
$\mathcal{A}(\tilde{A}^\alpha) = \int_0^\omega (U^\alpha - L^\alpha) f(\alpha)\, d\alpha$ where , $f(\alpha) \in [0,1] (\alpha \in [0, \omega])$, $f(0) = 0$ and $f(\alpha)$ is monotonic and non-decreasing of $\alpha \in [0, \omega]$
$\mathcal{A}(\tilde{A}^\beta) = \int_\rho^1 (U'^\alpha - L'^\alpha) g(\beta)\, d\beta$ where , $g(\beta) \in [0,1] (\beta \in [\rho, 1])$, $g(1) = 0$ and $g(\beta)$ is monotonic and non-increasing of $\beta \in [\rho, 1]$
$\mathcal{A}(\tilde{a}^\gamma) = \int_\delta^1 (U''^\alpha - L''^\alpha) h(\gamma)\, d\gamma$ where , $h(\gamma) \in [0,1] (\gamma \in [\delta, 1])$, $h(1) = 0$ and $h(\gamma)$ is monotonic and non-increasing of $\gamma \in [\delta, 1]$

Result 3.9. *Let $\tilde{A} = \langle (a_1, b_1, c_1, d_1; \omega), (a_2, b_2, c_2, d_2; \rho), (a_3, b_3, c_3, d_3; \delta) \rangle$ be a GPSVNN . Then (α, β, γ)-cut set of the GPSVNN $\tilde{A}^\alpha = [L^\alpha, U^\alpha]$, $\tilde{A}^\beta = [L'^\alpha, U'^\alpha]$ and $\tilde{A}^\gamma = [L''^\alpha, U''^\alpha]$ are respectively. Then, for the truth membership,*
$\tilde{A}^\alpha = [L^\alpha, U^\alpha] = [a_1 + (b_1 - a_1)\sqrt{\alpha/\omega}, d_1 - (d_1 - c_1)\sqrt{\alpha/\omega}]$ *where* $\alpha \in [0, \omega]$.

If $f(\alpha) = \alpha$,we obtain value and ambiguity as,

$$\begin{aligned}\mathcal{V}(\tilde{A}^{\alpha}) &= \int_0^{\omega}\Big[a_1 + (b_1 - a_1)\sqrt{\alpha/\omega} + d_1 - (d_1 - c_1)\sqrt{\alpha/\omega}\Big]\alpha d\alpha \\ &= \Big[\frac{\omega^2}{2}(a_1 + d_1) + \frac{2\omega^2}{5}(b_1 - a_1 - d_1 + c_1)\Big] \\ &= \frac{\omega^2}{10}(a_1 + d_1 + 4b_1 + 4c_1).\end{aligned}$$

$$\begin{aligned}\mathcal{A}(\tilde{A}^{\alpha}) &= \int_0^{\omega}\Big[d_1 - (d_1 - c_1)\sqrt{\alpha/\omega} - a_1 - (b_1 - a_1)\sqrt{\alpha/\omega}\Big]\alpha d\alpha \\ &= \Big[\frac{\omega^2}{2}(d_1 - a_1) - \frac{2\omega^2}{5}(d_1 - c_1 + b_1 - a_1)\Big] \\ &= \frac{\omega^2}{10}(d_1 - a_1 - 4b_1 + 4c_1).\end{aligned}$$

For the indeterminancy membership ,
$\tilde{A}^{\beta} = [L'^{\alpha}, U'^{\alpha}] = [a_2 + (b_2 - a_2)\sqrt{(1-\beta)/(1-\rho)}, d_2 - (d_2 - c)_2\sqrt{(1-\beta)/(1-\rho)}]$ *where* $\beta \in [\rho, 1]$. *If $g(\rho) = (1 - \rho)$, we obtain value and ambiguity as,*

$$\begin{aligned}\mathcal{V}(\tilde{A}^{\beta}) &= [\int_{\rho}^{1}\Big[a_2 + (b_2 - a_2)\sqrt{(1-\beta)/(1-\rho)} + d_2 - (d_2 - c_2)\sqrt{(1-\beta)/(1-\rho)}\Big](1-\beta)d\beta \\ &= [\Big[\frac{(1-\rho)^2}{2}(a_2 + d_2) + \frac{2(1-\rho)^2}{5}(b_2 - a_2 - d_2 + c_2)\Big] = \frac{(1-\rho)^2}{10}(a_2 + d_2 + 4b_2 + 4c_2).\end{aligned}$$

$$\begin{aligned}\mathcal{A}(\tilde{A}^{\beta}) &= [\int_{\rho}^{1}\Big[d_2 - (d_2 - c_2)\sqrt{(1-\beta)/(1-\rho)} - a_2 - (b_2 - a_2)\sqrt{(1-\beta)/(1-\rho)}\Big](1-\beta)d\beta \\ &= [\Big[\frac{(1-\rho)^2}{2}(d_2 - a_2) - \frac{2\omega^2}{5}(d_2 - c_2 + b_2 - a_2)\Big] = \frac{(1-\rho)^2}{10}(d_2 - a_2 - 4b_2 + 4c_2).\end{aligned}$$

For the falsity membership ,
$\tilde{A}^{\gamma} = [L''^{\alpha}, U''^{\alpha}] = [a_3 + (b_3 - a_3)\sqrt{(1-\gamma)/(1-\delta)}, d_3 - (d_3 - c_3)\sqrt{(1-\gamma)/(1-\delta)}\Big]$ *where* $\gamma \in [\delta, 1]$. *If $h(\delta) = (1 - \delta)$,we obtain value and ambiguity as,*

$$\begin{aligned}\mathcal{V}(\tilde{A}^{\gamma}) &= [\int_{\delta}^{1}\Big[a_3 + (b_3 - a_3)\sqrt{(1-\gamma)/(1-\delta)} + d_3 - (d_3 - c_3)\sqrt{(1-\gamma)/(1-\delta)}\Big](1-\gamma)d\gamma \\ &= [\Big[\frac{(1-\delta)^2}{2}(a_3 + d_3) + \frac{2(1-\delta)^2}{5}(b_3 - a_3 - d_3 + c_3)\Big] = \frac{(1-\delta)^2}{10}(a_3 + d_3 + 4b_3 + 4c_3).\end{aligned}$$

$$\begin{aligned}\mathcal{A}(\tilde{A}^{\gamma}) &= [\int_{\delta}^{1}\Big[d_3 - (d_3 - c_3)\sqrt{(1-\gamma)/(1-\delta)} - a_3 - (b_3 - a_3)\sqrt{(1-\gamma)/(1-\delta)}\Big](1-\gamma)d\gamma \\ &= [\Big[\frac{(1-\delta)^2}{2}(d_3 - a_3) - \frac{2\omega^2}{5}(d_3 - c_3 + b_3 - a_3)\Big] = \frac{(1-\delta)^2}{10}(d_3 - a_3 - 4b_3 + 4c_3).\end{aligned}$$

Definition 3.10. Let $\tilde{A} = \langle(a_1, b_1, c_1, d_1; \omega), (a_2, b_2, c_2, d_2; \rho), (a_3, b_3, c_3, d_3; \delta)\rangle$ be a GPSVNN. The weighted value and ambiguity for $\lambda \in [0, 1]$ are,

$\mathcal{V}_\lambda(\tilde{A}) = \lambda\mathcal{V}(\tilde{A}^\alpha) + (1-\lambda)\mathcal{V}(\tilde{A}^\beta) + (1-\lambda)\mathcal{V}(\tilde{A}^\gamma)$.
$\mathcal{A}_\lambda(\tilde{A}) = \lambda\mathcal{A}(\tilde{A}^\alpha) + (1-\lambda)\mathcal{A}(\tilde{A}^\beta) + (1-\lambda)\mathcal{A}(\tilde{A}^\gamma)$.

Note: When $\lambda = 0$, it marks the preference for uncertainty, On the other hand,when $\lambda = 1$ it is considered with strongly preferred certainty.

Definition 3.11. (Ranking Order) Let $\tilde{A}$ and $\tilde{B}$ be two GPSVNN and $\lambda \in [0,1]$. For weighted values and ambiguities of the GPSVNN $\tilde{A}$ and $\tilde{B}$. The ranking order of $\tilde{A}$ and $\tilde{B}$ is defined as,

(1) If $\mathcal{V}_\lambda(\tilde{A}) > \mathcal{V}_\lambda(\tilde{B})$, then $\tilde{A} > \tilde{B}$.
(2) If $\mathcal{V}_\lambda(\tilde{A}) < \mathcal{V}_\lambda(\tilde{B})$, then $\tilde{A} < \tilde{B}$
(3) If $\mathcal{V}_\lambda(\tilde{A}) = \mathcal{V}_\lambda(\tilde{B})$, then

- If $\mathcal{A}_\lambda(\tilde{A}) = \mathcal{A}_\lambda(\tilde{B})$, then $\tilde{A} = \tilde{B}$.
- If $\mathcal{A}_\lambda(\tilde{A}) > \mathcal{A}_\lambda(\tilde{B})$, then $\tilde{A} > \tilde{B}$.
- If $\mathcal{A}_\lambda(\tilde{A}) < \mathcal{A}_\lambda(\tilde{B})$, then $\tilde{A} < \tilde{B}$.

Example : $\tilde{A} = \langle(3,5,8,12); 0.2, 0.3, 0.4\rangle$ and $\tilde{B} = \langle(-7,-5,6,7); 0.5, 0.4, 0.3\rangle$. Then the ranking for between these two numbers are .

$$\begin{aligned}\mathcal{V}_\lambda(\tilde{A}) &= \frac{3+12+20+32}{10}[\lambda(0.2^2) + (1-\lambda)(1-0.3)^2 + (1-\lambda)(1-0.4)^2]\\ &= 6.7[0.85 - 0.81\lambda] = 5.70 - 5.43\lambda.\end{aligned}$$

$$\begin{aligned}\mathcal{V}_\lambda(\tilde{B}) &= \frac{-7+7-20+24}{10}[\lambda(0.5^2) + (1-\lambda)(1-0.4)^2 + (1-\lambda)(1-0.3)^2]\\ &= 0.4[0.85 - 0.60\lambda] = 0.34 - 0.24\lambda.\end{aligned}$$

When $\lambda = 0, \mathcal{V}_\lambda(\tilde{A}) = 5.70$ and $\mathcal{V}_\lambda(\tilde{B}) = 0.34$
When $\lambda = 1, \mathcal{V}_\lambda(\tilde{A}) = 0.27$ and $\mathcal{V}_\lambda(\tilde{B}) = 0.10$.
Also for all values of λ between 0 and 1 $\mathcal{V}_\lambda(\tilde{A}) > \mathcal{V}_\lambda(\tilde{B})$. then the ranking order of the numbers $\tilde{A}$ and $\tilde{B}$ is $\tilde{A} > \tilde{B}$.

Theorem 3.12. *Let $\tilde{A} = \langle(a_1, b_1, c_1, d_1; \omega), (a_2, b_2, c_2, d_2; \rho), (a_3, b_3, c_3, d_3; \delta)\rangle$ and $\tilde{B} = \langle(a'_1, b'_1, c'_1, d'_1; \omega_2), (a'_2, b'_2, c'_2, d'_2; \rho), (a'_3, b'_3, c'_3, d'_3; \delta_2)\rangle$ be the two GPSVNN , $\lambda \in [0,1]$ and $k \in \mathbb{R}$ then (i) $\mathcal{V}_\lambda(\tilde{A}+\tilde{B}) = \mathcal{V}_\lambda(\tilde{A}) + \mathcal{V}_\lambda(\tilde{B})$ (ii)$V_\lambda(k\tilde{A}) = kV_\lambda(\tilde{A})$.*

Proof:

$$\begin{aligned}(i)\mathcal{V}_\lambda(\tilde{A}+\tilde{B}) &= \lambda\mathcal{V}(\tilde{A}^\alpha + \tilde{B}^\alpha) + (1-\lambda)\mathcal{V}(\tilde{A}^\beta + \tilde{B}^\beta) + (1-\lambda)\mathcal{V}(\tilde{A}^\gamma + \tilde{B}^\gamma)\\ &= \lambda\mathcal{V}(\tilde{A}^\alpha) + \lambda\mathcal{V}(\tilde{B}^\alpha) + (1-\lambda)\mathcal{V}(\tilde{A}^\beta) + (1-\lambda)\mathcal{V}(\tilde{B}^\beta) + (1-\lambda)\mathcal{V}(\tilde{A}^\gamma) + (1-\lambda)\mathcal{V}(\tilde{B}^\gamma)\\ &= \mathcal{V}_\lambda(\tilde{A}) + \mathcal{V}_\lambda(\tilde{B}).\end{aligned}$$

$$(ii)\mathcal{V}_\lambda(k\tilde{A}) = \lambda\mathcal{V}(k\tilde{A}^\alpha) + (1-\lambda)\mathcal{V}(k\tilde{A}^\beta) + (1-\lambda)\mathcal{V}(k\tilde{A}^\gamma) = k[\lambda\mathcal{V}(\tilde{A}^\alpha) + (1-\lambda)\mathcal{V}(\tilde{A}^\beta) + (1-\lambda)\mathcal{V}(\tilde{A}^\gamma)]$$
$$= k\mathcal{V}_\lambda(\tilde{A}).$$

Theorem 3.13. *Let* $\tilde{A} = \langle(a_1, b_1, c_1, d_1; \omega), (a_2, b_2, c_2, d_2; \rho), (a_3, b_3, c_3, d_3; \delta)\rangle$ *and* $\tilde{B} = \langle(a_1', b_1', c_1', d_1'; \omega_2), (a_2', b_2', c_2', d_2'; \rho), (a_3', b_3', c_3', d_3'; \delta_2)\rangle$ *be the two GPSVNN ,* $\lambda \in [0,1]$ *and* $k \in \mathbb{R}$ *then* *(i)*$\mathcal{A}_\lambda(\tilde{A}+\tilde{B}) = \mathcal{A}_\lambda(\tilde{B}) + \mathcal{A}_\lambda(\tilde{B})$ *(ii)* $\mathcal{A}_\lambda(k\tilde{A}) = k\mathcal{A}_\lambda(\tilde{A})$.

Proof:

$$\begin{aligned}(i)\mathcal{A}_\lambda(\tilde{A}+\tilde{B}) &= \lambda\mathcal{A}(\tilde{A}^\alpha + \tilde{B}^\alpha) + (1-\lambda)\mathcal{A}(\tilde{A}^\beta + \tilde{B}^\beta) + (1-\lambda)\mathcal{A}(\tilde{A}^\gamma + \tilde{B}^\gamma)\\ &= \lambda\mathcal{A}(\tilde{A}^\alpha) + \lambda\mathcal{A}(\tilde{B}^\alpha) + (1-\lambda)\mathcal{A}(\tilde{A}^\beta) + (1-\lambda)\mathcal{A}(\tilde{B}^\beta) + (1-\lambda)\mathcal{A}(\tilde{A}^\gamma) + (1-\lambda)\mathcal{A}(\tilde{B}^\gamma)\\ &= \mathcal{A}_\lambda(\tilde{A}) + \mathcal{A}_\lambda(\tilde{B}).\end{aligned}$$

$$(ii)\mathcal{A}_\lambda(k\tilde{A}) = \lambda\mathcal{A}(k\tilde{A}^\alpha) + (1-\lambda)\mathcal{A}(k\tilde{A}^\beta) + (1-\lambda)\mathcal{A}(k\tilde{A}^\gamma) = k[\lambda\mathcal{A}(\tilde{A}^\alpha) + (1-\lambda)\mathcal{A}(\tilde{A}^\beta) + (1-\lambda)\mathcal{A}(\tilde{A}^\gamma)]$$
$$= kA_\lambda(\tilde{A}).$$

Theorem 3.14. *Suppose* $\tilde{A}, \tilde{B}$ *and* $\tilde{C}$ *are any GPSVNN , where* $\omega_1 = \omega_2$ *,* $\rho_1 = \rho_2$ *and* $\delta_1 = \delta_2$ *. If* $\tilde{A} > \tilde{B}$*, then* $(\tilde{A}+\tilde{C}) > (\tilde{B}+\tilde{C})$.

Proof:

$$\begin{aligned}\mathcal{V}(\tilde{A}^\alpha + \tilde{C}^\alpha) &= \int_0^{\omega_1\wedge\omega_3} [L^\alpha_{\tilde{A}} + U^\alpha_{\tilde{A}} + L^\alpha_{\tilde{C}} + U^\alpha_{\tilde{C}}] f(\alpha)d\alpha\\ &= \int_0^{\omega_1\wedge\omega_3} [L^\alpha_{\tilde{A}} + U^\alpha_{\tilde{A}}] f(\alpha)d\alpha + \int_0^{\omega_1\wedge\omega_3} [L^\alpha_{\tilde{C}} + U^\alpha_{\tilde{C}}] f(\alpha)d\alpha\\ \mathcal{V}(\tilde{B}^\alpha + \tilde{C}^\alpha) &= \int_0^{\omega_2\wedge\omega_3} [L^\alpha_{\tilde{B}} + U^\alpha_{\tilde{B}} + L^\alpha_{\tilde{C}} + U^\alpha_{\tilde{C}}] f(\alpha)d\alpha\\ &= \int_0^{\omega_2\wedge\omega_3} [L^\alpha_{\tilde{B}} + U^\alpha_{\tilde{B}}] f(\alpha)d\alpha + \int_0^{\omega_2\wedge\omega_3} [L^\alpha_{\tilde{C}} + U^\alpha_{\tilde{C}}] f(\alpha)d\alpha\end{aligned}$$

From the conditions , $\tilde{A} > \tilde{B}$ *and* $\omega_1 = \omega_2$*, we have,*
$\int_0^{\omega_1\wedge\omega_3} [L^\alpha_{\tilde{A}} + U^\alpha_{\tilde{A}}] f(\alpha)d\alpha > \int_0^{\omega_2\wedge\omega_3} [L^\alpha_{\tilde{B}} + U^\alpha_{\tilde{B}}] f(\alpha)d\alpha$

$$\implies \mathcal{V}(\tilde{A}^\alpha + \tilde{C}^\alpha) > \mathcal{V}(\tilde{B}^\alpha + \tilde{C}^\alpha) \tag{1}$$

$$\begin{aligned}\mathcal{V}(\tilde{A}^\beta + \tilde{C}^\beta) &= \int_{\rho_1\vee\rho_3}^1 [L'^\beta_{\tilde{A}} + U'^\beta_{\tilde{A}} + L'^\beta_{\tilde{C}} + U'^\beta_{\tilde{C}}] g'^\beta d\beta\\ &= \int_{\rho_1\vee\rho_3}^1 [L'^\beta_{\tilde{A}} + U'^\beta_{\tilde{A}}] g'^\beta d\beta + \int_{\rho_1\vee\rho_3}^1 [L'^\beta_{\tilde{C}} + U'^\beta_{\tilde{C}}] g'^\beta d\beta\\ \mathcal{V}(\tilde{B}^\beta + \tilde{C}^\beta) &= \int_{\rho_2\vee\rho_3}^1 [L'^\beta_{\tilde{B}} + U'^\beta_{\tilde{B}} + L'^\beta_{\tilde{C}} + U'^\beta_{\tilde{C}}] g'^\beta d\beta\\ &= \int_{\rho_2\vee\rho_3}^1 [L'^\beta_{\tilde{B}} + U'^\beta_{\tilde{B}}] g'^\beta d\beta + \int_{\rho_2\vee\rho_3}^1 [L'^\beta_{\tilde{C}} + U'^\beta_{\tilde{C}}] g'^\beta d\beta\end{aligned}$$

From the conditions , $\tilde{A} > \tilde{B}$ *and* $\rho_1 = \rho_2$*, we have,*
$\int_{\rho_1 \vee \rho_3}^{1} [L'^{\beta}_{\tilde{A}} + U'^{\beta}_{\tilde{A}}] g'^{\beta} d\beta > \int_{\rho_2 \vee \rho_3}^{1} [L'^{\beta}_{\tilde{B}} + U'^{\beta}_{\tilde{B}}] g'^{\beta} d\beta$

$$\implies \mathcal{V}(\tilde{A}^{\beta} + \tilde{C}^{\beta}) > \mathcal{V}(\tilde{B}^{\beta} + \tilde{C}^{\beta}) \tag{2}$$

$$\begin{aligned}
\mathcal{V}(\tilde{A}^{\gamma} + \tilde{C}^{\gamma}) &= \int_{\delta_1 \vee \delta_3}^{1} [L''^{\gamma}_{\tilde{A}} + U''^{\gamma}_{\tilde{A}} + L''^{\gamma}_{\tilde{C}} + U''^{\gamma}_{\tilde{C}}] h(\gamma) d\gamma \\
&= \int_{\delta_1 \vee \delta_3}^{1} [L''^{\gamma}_{\tilde{A}} + U''^{\gamma}_{\tilde{A}}] h(\gamma) d\gamma + \int_{\delta_1 \vee \delta_3}^{1} [L''^{\gamma}_{\tilde{C}} + U''^{\gamma}_{\tilde{C}}] h(\gamma) d\gamma \\
\mathcal{V}(\tilde{B}^{\gamma} + \tilde{C}^{\gamma}) &= \int_{\delta_2 \vee \delta_3}^{1} [L''^{\gamma}_{\tilde{B}} + U''^{\gamma}_{\tilde{B}} + L''^{\gamma}_{\tilde{C}} + U''^{\gamma}_{\tilde{C}}] h(\gamma) d\gamma \\
&= \int_{\delta_2 \vee \delta_3}^{1} [L''^{\gamma}_{\tilde{B}} + U''^{\gamma}_{\tilde{B}}] h(\gamma) d\gamma + \int_{\delta_2 \vee \delta_3}^{1} [L''^{\gamma}_{\tilde{C}} + U''^{\gamma}_{\tilde{C}}] h(\gamma) d\gamma
\end{aligned}$$

From the conditions , $\tilde{A} > \tilde{B}$ *and* $\gamma_1 = \gamma_2$*, we have,*
$\int_{\gamma_1 \vee \gamma_3}^{1} [L''^{\gamma}_{\tilde{A}} + U''^{\gamma}_{\tilde{A}}] h(\gamma) d\gamma > \int_{\gamma_2 \vee \gamma_3}^{1} [L''^{\gamma}_{\tilde{B}} + U''^{\gamma}_{\tilde{B}}] h(\gamma) d\gamma$

$$\implies \mathcal{V}(\tilde{A}^{\gamma} + \tilde{C}^{\gamma}) > \mathcal{V}(\tilde{B}^{\gamma} + \tilde{C}^{\gamma}) \tag{3}$$

by the combining equation (1), (2) and (3) the following inequality is always valid for any $\lambda \in [0, 1]$,

$$\begin{gathered}
\lambda \mathcal{V}(\tilde{A}^{\alpha} + \tilde{C}^{\alpha}) + (1-\lambda)\mathcal{V}(\tilde{A}^{\beta} + \tilde{C}^{\beta}) + (1-\lambda)\mathcal{V}(\tilde{A}^{\gamma} + \tilde{C}^{\gamma}) > \\
\lambda \mathcal{V}(\tilde{B}^{\alpha} + \tilde{C}^{\alpha}) + (1-\lambda)\mathcal{V}(\tilde{B}^{\beta} + \tilde{C}^{\beta}) + (1-\lambda)\mathcal{V}(\tilde{B}^{\gamma} + \tilde{C}^{\gamma}).
\end{gathered}$$

Therefore $\mathcal{V}_{\lambda}(\tilde{A} + \tilde{C}) > \mathcal{V}_{\lambda}(\tilde{B} + \tilde{C})$*, and from the definition,* $(\tilde{A} + \tilde{C}) > (\tilde{B} + \tilde{C})$

Theorem 3.15. *Suppose that* $\tilde{A} = \langle (a_1, b_1, c_1, d_1; \omega), (a_2, b_2, c_2, d_2; \rho), (a_3, b_3, c_3, d_3; \delta) \rangle$ *and* $\tilde{B} = \langle (a'_1, b'_1, c'_1, d'_1; \omega_2), (a'_2, b'_2, c'_2, d'_2; \rho), (a'_3, b'_3, c'_3, d'_3; \delta_2) \rangle$ *be the two GPSVNN with* $\omega_1 = \omega_2$, $\rho_1 = \rho_2$ *and* $\delta_1 = \delta_2$*. If* $a_i > d'_i$*, where i=1,2,3, then* $\tilde{A} > \tilde{B}$*.*

Proof: *As we have ,* $\omega_1 = \omega_2$ *and* $a_1 > d'_1$ *,*

$$\mathcal{V}(\tilde{A}^{\alpha}) = \int_{0}^{\omega_1} [L^{\alpha} + U^{\alpha}] f(\alpha) d\alpha \geq 2a_1 \int_{0}^{\omega_1} f(\alpha) d\alpha$$

$$\mathcal{V}(\tilde{B}^{\alpha}) = \int_{0}^{\omega_2} [L^{\alpha} + U^{\alpha}] f(\alpha) d\alpha \leq 2d_1 \int_{0}^{\omega_2} f(\alpha) d\alpha$$

$\int_0^{\omega_1} f(\alpha) d\alpha = \int_0^{\omega_2} f(\alpha) d\alpha$*, we have,* $a_1 > d'_1$

$$\mathcal{V}(\tilde{A}^{\alpha}) \geq 2a_1 \geq 2d'_1 \geq \mathcal{V}(\tilde{B}^{\alpha}). \tag{4}$$

As we have , $\rho_1 = \rho_2$ *and* $a_2 > d'_2$ *,*

$$\mathcal{V}(\tilde{A}^{\beta}) = \int_{\rho_1}^{1} [L'^{\alpha} + U'^{\alpha}] g d\beta \geq 2a_2 \int_{\rho_2}^{1} g(\beta) d\beta$$

$$\mathcal{V}(\tilde{B}^{\beta}) = \int_{\rho_2}^{1} [L'^{\alpha} + U'^{\alpha}] g(\beta) d\beta \leq 2d_2' \int_{\rho_2}^{1} g(\beta) d\beta$$

$\int_{\rho_1}^{1} g(\beta)d\beta = \int_{\rho_2}^{1} g(\beta)d\beta$, *we have,* $a_2 > d_2'$.

$$\mathcal{V}(\tilde{A}^{\beta}) \geq 2a_2 \geq 2d_2' \geq \mathcal{V}(\tilde{B}^{\beta}). \tag{5}$$

As we have , $\delta_1 = \delta_2$ *and* $a_3 > d_3'$ *,*

$$\mathcal{V}(\tilde{A}^{\gamma}) = [\int_{\delta_1}^{1} [L''^{\alpha} + U''^{\alpha}] h(\gamma) d\gamma \geq 2a_3 [\int_{\delta_2}^{1} h(\gamma) d\gamma$$

$$\mathcal{V}(\tilde{B}^{\gamma}) = \int_{\delta_2}^{1} [L''^{\alpha} + U''^{\alpha}] h(\gamma) d\gamma \leq 2d_3' \int_{\delta_2}^{1} h(\gamma) d\gamma$$

$\int_{\delta_1}^{1} h(\gamma)d\gamma = \int_{\delta_2}^{1} h(\gamma)d\gamma$ *we have ,* $a_3 > d_3'$

$$\mathcal{V}(\tilde{A}^{\gamma}) \geq 2a_3 \geq 2d_3' \geq \mathcal{V}(\tilde{B}^{\gamma}) \tag{6}$$

According to the definition, from the equations (4), (5) and (6), we have ,
$\lambda\mathcal{V}(\tilde{A}^{\alpha}) + (1-\lambda)\mathcal{V}(\tilde{A}^{\beta}) + (1-\lambda)\mathcal{V}(\tilde{A}^{\gamma}) > \lambda\mathcal{V}(\tilde{B}^{\alpha}) + (1-\lambda)\mathcal{V}(\tilde{B}^{\beta}) + (1-\lambda)\mathcal{V}(\tilde{B}^{\gamma})$.
Therefore, from the definition, $\tilde{A} > \tilde{B}$.

4. **Application of GPSVNN**

An algorithm for the GPSVN-numbers multi-criteria decision-making method as follows; Let $S_i = \{S_1, S_2, ...S_m\}$ be the set of alternatives , $T_j = \{T_1, T_2, ...T_n\}$ be the set of criteria and $\{[\tilde{A_{ij}}] = \langle (a_{1ij}, b_{1ij}, c_{1ij}, d_{1ij}; \omega_{ij}), (a_{2ij}, b_{2ij}, c_{2ij}, d_{2ij}; \rho_{ij}), (a_{3ij}, b_{3ij}, c_{3ij}, d_{3ij}; \delta_{ij}) \rangle$ be the GPSVN-numbers.

Step 1: Construct the decision-making matrix, G=$[\tilde{A_{ij}}]_{m*n}$ using GPSVNN.

Step 2: Compute the normalised decision-making matrix, N = $[\tilde{n_{ij}}]_{m*n}$ of G, for $[\tilde{n_{ij}}]_{m*n} = \langle \big(\frac{a_{1ij}}{d_1{}^+}, \frac{b_{1ij}}{d_1{}^+}, \frac{c_{1ij}}{d_1{}^+}, \frac{d_{1ij}}{d_1{}^+}; \omega_{ij}\big), \big(\frac{a_{2ij}}{d_2{}^+}, \frac{b_{2ij}}{d_2{}^+}, \frac{c_{2ij}}{d_2{}^+}, \frac{d_{2ij}}{d_2{}^+}; \rho_{ij}\big), \big(\frac{a_{3ij}}{d_3{}^+}, \frac{b_{3ij}}{d_3{}^+}, \frac{c_{3ij}}{d_3{}^+}, \frac{d_{3ij}}{d_3{}^+}; \delta_{ij}\big) \rangle$, where d^+ = $max\{d_{ij}\}$.

Step 3: Compute the T = $[t_{ij}]_{m*n}$ of N,where $[t_{ij}]_{m*n}$ = $w_i * \tilde{r_{ij}}$,(should satisfy the normalized condition , $w_i = [0, 1], \sum_{i=1}^{\infty} w_i = 1$).

Step 4: Compute the comprehensive values $\tilde{C_i}$ as, $\tilde{C_i} = \sum_{j=1}^{\infty}[t_{ij}]$.

Step 5: Determine the increasing order of $\tilde{C_i}$.

Step 6: Rank the alternatives s_i according to the C_i and select the best and worst alternatives.

Application of GPSVNN in Multi-Criteria Decision Making:

Since 2003, the Life Satisfaction Survey (LSS) has been performed by the Turkish Statistics Institute (TUIK). LSS is essential to gauge how happy people feel in general, how they regard other people, how satisfied they are with their primary living conditions, and how comfortable they are with public services. Evaluation of the surveys using statistical techniques has been the primary focus of studies to ascertain the quality of municipal service in Turkey. They used the image fuzzy vikor approach to evaluate it in that regard. In this case, GPSVNN was employed. Additionally, the twenty choices' weight vector may be expressed as follows:

$w = (0.03, 0.08, 0.04, 0.02, 0.06, 0.05, 0.01, 0.07, 0.09, 0.04, 0.06, 0.07, 0.05, 0.04, 0.01, 0.02, 0.03, 0.1, 0.08, 0.05)^T$

The below numbers are choosed randomly in between the interval [0,10], which are GPSVN-numbers for the twenty municipal service alternatives $(S_1, S_2, ..., S_{20})$ with the four criteria as (T_1, T_2, T_3, T_4) . To find the best and worst alternatives in those municipal services, to improve the society and to award which have given it's best service.

TABLE 1. Alternative Sets.

S_i	**Service Alternative**	S_i	**Service Alternative**
S_1	Garbage and environmental cleanliness	S_2	Drainage
S_3	Drinking water	S_4	Public transport
S_5	Municipal police	S_6	Road and pavement construction
S_7	Parks and gardens	S_8	Minimization of noise and air pollution
S_9	Health, fitness center facilities	S_{10}	Zoning and city planning
S_{11}	Arrangements for the disabled	S_{12}	Social aids
S_{13}	Cultural activities	S_{14}	Public education centers
S_{15}	Street and road lighting	S_{16}	Cleanliness
S_{17}	Fire-fighting	S_{18}	Graveyard
S_{19}	Address information systems	S_{20}	Control of food producing facilities

TABLE 2. Decision-Matrix using GPSVNN

G	2014	2015	2016	2017
S_1	<(1.8,3.0, 4.2,7.1;0.4), (2.6,2.9,5.2,6.7;0.7), (4.6,5.5,6.9,7.2;0.2)>	<(1.5,3.6,4.2,7.3;0.5), (1.4,3.1,4.4,5.6;0.8), (0.7,3.2,4.7,8.9;0.3)>	<(3.5,4.3,6.0,7.4;0.8), (0.1,0.9,3.3,5.1;0.6), (1.8,3.0,4.2,5.7;0.3)>	<(0.1,0.9,1.7,7.3;0.5), (0.6,2.5,3.9,4.3;0.2), (5.4,6.9,8.5,9.7;0.7)>
S_2	<(1.8,2.5,2.9,6.6;0.6), (1.0,1.9,2.6,3.1;0.2), (1.9,3.3,5.0,8.1;0.6)>	<(4.5,6.7,8.3,9.1;0.7), (0.9,1,1.5,2.7;0.5), (0.6,1.2,1.8,3.1;0.4)>	<(1.5,3.6,5.8,8.1;0.5), (6.2,7.7, 9.1,10;0.2), (0.2,0.4,1.1,1.8;0.7)>	<(5.1,6.6,8.4,9.0;0.5), (1.7,2.9,4.7,6.6;0.8), (1.1,1.2,2.3,5.5;0.3)>
S_3	<(2.0,3.9,7.0,8.8;0.3), (0.7,3.6,4.3,9.0;0.5), (5.5,6.6,7.7,8.8;0.2)>	<(2.3,2.7,3.4,5.1;0.8), (1.0,3.6,6.2,7.2;0.6), (2.0,3.9,4.4,5.7;0.3)>	<(3.7, 5.0,6.2,7.5;0.4), (0.2,1.7,3.0,3.1;0.7), (0.9,1.5,3.4,4.7;0.5)>	<(3.0,4.5,6.9,7.5;0.2), (3.1,6.3,7.3,9.5;0.7), (0.1,0.3,1.1,7.5;0.6)>
S_4	<(0.9,1.2,2.1,5.9;0.7), (1.8,3.0,4.2,7.1;0.2), (0.6,2.5,2.9,5.9;0.3)>	<(0.1,0.7,1.1,1.4;0.3), (5.3,7.3,8.7,10;0.1), (4.4,4.5,4.7,4.9;0.3)>	<(6.3,7.5,8.0,9.9;0.3), (0.2,0.3,0.4,0.5;0.7), (0.1,0.2,0.3,0.4;0.5)>	<(6.2,6.9,7.5,9.9;0.7), (1.8,3.0,4.2,5.1;0.4), (0.2,0.7,8.1 10;0.3)>
S_5	<(1.1,1.9,2.6,5.4;0.6), (5.4,5.9,6.6,7.1;0.2), (3.1,6.7,7.1,7.9;0.2)>	<(0.5,1.8,3.9,5.5;0.7), (1.1,2.9,5.2,7.7;0.2), (5.1,6.7,7.1,7.9;0.3)>	<(0.8,1.1,2.2,2.6;0.6), (5.4,6.2,7.9,8.3;0.9), (4.6,5.5,6.9,7.2;0.8)>	<(2.9,3.7,5.9,8.1;0.3), (0.3,1.1,3.4,6.9;0.4), (1.1,2.0,2.8,3.0;0.2)>
S_6	<(0.5,1.0,2.9,5.6;0.4), (0.2,0.5,0.7,2.3;0.1), (1.2,4.3,5.0,6.7;0.3)>	<(0.7,1.2,2.7,5.6;0.4), (2.3,2.7,3.4,5.1;0.3), (3.4,5.2,6.2,8.7;0.5)>	<(0.3,1.5,4.3,7.3;0.4), (4.7,6.9,7.3,8.9;0.1), (3.4,5.2,6.6,7.7;0.3)>	<(0.4,1.2,3.0,5.4;0.6), (0.4,1.8,4.7,5.7;0.4), (2.3,5.6,8.5,9.8;0.3)>
S_7	<(0.9,1.0,2.7,5.4;0.5), (2.0,3.9,7.0,8.8;0.2), (1.8,3.0,4.2,7.1;0.4)>	<(4.7,6.9,7.3,8.5;0.7), (0.6,2.5,2.9,3.3;0.8), (0.2,,0.4,1.8,2.6;0.9)>	<(2.4,3.5,5.8,6.3;0.3), (4.4,5.2,6.7,7.8;0.6), (1.5,3.6,4.8,6.1;0.5)>	<(0.9,1.2,2.1,3.9;0.5), (2.7,5.7,6.2,6.9;0.2), (0.6,1.2,2.8,5.4;0.7)>
S_8	<(0.6,2.2,2.6,4.2;0.6), (0.2,1.2,2.0,5.4;0.3), (0.4,1.7,3.3,9.0;0.5)>	<(1.2,3.0,4.6,5.9;0.5), (2.7,5.2,6.7,7.9;0.3), (0.7,3.9,4.3,6.2;0.2)>	<(5.7,6.3,7.1,9.5;0.2), (0.6,1.4,1.8,2.3;0.4), (2.6,3.9,4.5,5.6;0.5)>	<(4.1,7.3,8.8,9.5;0.3), (0.5,1.1,2.6,3.5;0.5), (5.5,7.7,8.3,9.9;0.2)>
S_9	<(1.7,2.8,4.5,8.5;0.5), (0.4,1.2,3.0,5.4;0.2), (1.4,3.1,4.4,7.6;0.7)>	<(1.1,1.6,2.7,4.6;0.6), (3.4,6.9,7.3,9.3;0.9), (4.1,6.1,7.3,8.1;0.8)>	<(1.0,1.5,2.4,3.1;0.2), (4.5,6.7,8.3,9.4;0.6), (1.4,3.1,4.4,5.9;0.2)>	<(7.4,8.5,9.6,9.9;0.6), (0.9,1.0,1.5,2.7;0.2), (0.1,0.5,0.7,2.3;0.3)>
S_{10}	<(1.1,3.6,3.9,8.0;0.7), (1.8,3.9,5.7,9.0;0.4), (4.4,6.9,8.5,9.7;0.6)>	<(7.6,8.1,9.0,9.7;0.2), (0.9,1.5,3.4,4.3;0.4), (0.2,0.5,0.7,0.8;0.4)>	<(2.4,3.5,4.3,6.0;0.5), (3.4,5.5,7.8,9.0;0.2), (1.2,1.8,2.2,2.3;0.7)>	<(0.2,3.1,7.1,9.2;0.8), (1.5,3.7,3.8,4.2;0.6), (1.0,1.9,2.6,3.1;0.3)>
S_{11}	<(2.0,2.7,5.4,9.4;0.8), (1.1,1.9,2.6,5.4;0.5), (0.7,3.9,4.3,9.0;0.4)>	<(6.2,6.9,7.8,9.1;0.4), (2.8,3.0,4.2,5.3;0.7), (0.4,0.9,1.7,4.4;0.3)>	<(5.1,6.6,8.3,9.3;0.1), (1.5,1.5,3.5,6.3;0.3), (2.0,3.9,4.4,5.7;0.6)>	<(1.2,3.0,4.6,5.9;0.4), (0.8,4.4,6.2,8.1;0.5), (1.3,3.9,7.4,8.9;0.4)>
S_{12}	<(0.5,1.6,2.6,8.5;0.8), (1.8,2.5,2.9,6.6;0.1), (0.9,5.5,7.7,8.1;0.2)>	<(2.3,4.4,5.6,6.7;0.5), (2.0,3.4,5.7,8.4;0.2), (0.2,0.4,1.1,1.8;0.7)>	<(0.9,1.3,2.2,5.6;0.4), (3.1,6.3,7.3,9.5;0.1), (1.4,5.2,6.2,6.9;0.3)>	<(1.1,1.3,2.2,5.4;0.2), (2.6,3.9,4.5,5.6;0.4), (5.9,6.7,7.7,8.8;0.5)>
S_{13}	<(1.0,4.2,5.7,10;0.7), (5.3,7.3,8.7,9.1;0.2), (1.1,3.6,3.9,8.0;0.6)>	<(3.0,4.1,5.5,8.3;0.3), (1.8,3.0,4.2,5.1;0.5), (2.3,2.7,3.4,5.1;0.2)>	<(0.1,0.2,2.2,5.3;0.8), (5.9,6.7,7.9,8.8;0.5), (0.6,1.1,2.3,2.7;0.4)>	<(0.8,1.1,2.2,2.6;0.9), (2.8,3.1,5.3,5.6;0.1), (3.4,5.5,7.8,9.0;0.2)>
S_{14}	<(2.7,2.9,3.3,5.1;0.6), (1.2,4.3,5.0,7.1;0.2), (4.4,6.9,8.5,9.7;0.6)>	<(0.8,1.8,3.2,4.5;0.8), (5.4,5.9,6.6,7.1;0.5), (5.3,7.3,8.7,9.1;0.4)>	<(0.8,1.8,2.7,3.5;0.1), (2.3,7.8,8.3,8.9;0.2), (3.1,3.6,5.0,6.2;0.2)>	<(0.9,1.8,2.8,5.5;0.5), (0.8,1.0,2.7,5.4;0.2), (0.9,1.0,2.8,5.3;0.2)>
S_{15}	<(3.6,5.0,6.7,7.1;0.7), (0.5,1.6,2.6,3.5;0.2), (0.9,1.2,2.1,3.9;0.8)>	<(1.4,3.7,5.6,7.3;0.7), (0.9,1.0,2.7,5.4;0.3), (1.8,2.5,4.3,5.1;0.6)>	<(3.0,4.3,4.9,5.7;0.7), (0.3,1.1,3.4,6.9;0.2), (2.8,3.0,4.2,5.3;0.6)>	<(1.4,3.3,4.7,8.2;0.3), (5.3,6.2,7.7,9.9;0.3), (3.1,3.6,5.0,6.2;0.4)>
S_{16}	<(2.8,4.8,5.9,9.4;0.3), (1.9,2.7,3.9,5.6;0.4), (0.4,1.2,3.0,5.4;0.5)>	<(0.3,1.2,1.5,2.7;0.5), (4.4,6.9,8.5,9.0;0.3), (3.6,3.9,5.5,6.9;0.2)>	<(1.4,2.9,4.3,4.8;0.6), (0.3,1 1.4,7.4;0.2), (5.3,7.3,8.5,9.9;0.5)>	<(4.6,5.7,9.5,9.7;0.8), (1.2,1.8,2.2,2.3;0.5), (0.3,2.0,3.1,3.3;0.5)>
S_{17}	<(3.7,4.6,7.3,9.4;0.2), (0.5,1.0,2.9,3.6;0.7), (0.7,2.2,3.6,4.3;0.8)>	<(0.4,1.2,2.9,3.3;0.8), (2.0,3.4,5.5,7.1;0.5), (7.6,8.1,9.0,9.7;0.4)>	<(0.5,0.6,1.8,7.1;0.7), (1.3,1.6,2.3,4.8;0.4), (5.6,5.9,6.1,7.7;0.5)>	<(0.5,0.6,1.7,7.0;0.5), (0.4,0.7,2.0,6.9;0.1), (0.5,0.7,1.8,7.0;0.4)>
S_{18}	<(0.6,1.4,1.8,3.7;0.5), (5.4,5.9,6.6,7.1;0.3), (1.0,1.9,2.6,3.1;0.3)>	<(0.6,2.7,4.5,7.4;0.1), (1.2,4.3,5.0,6.7;0.2), (1.7,2.8,4.5,7.3;0.2)>	<(5.6,5.9,6.0,6.1;0.5), (0.3,1.4,5.0,7.8;0.2), (3.1,4.9,5.7,7.3;0.7)>	<(2.4,3.5,4.8,5.0;0.4), (0.6,1.4,1.8,2.3;0.3), (3.1,3.6,5.0,6.2;0.1)>
S_{19}	<(1.2,3.7,7.3,8.0;0.4), (0.2,0.4,1.1,1.8;0.2), (1.2,3.1,5.5,5.9;0.7)>	<(1.0,3.9,4.2,5.3;0.7), (2.7,4.6,7.8,9.0;0.4), (3.4,4.2,5.7,7.3;0.8)>	<(0.2,1.3,1.7,3.6;0.6), (0.2,0.5,0.7,2.3;0.5), (1.2,3.6,5.5,6.9;0.7)>	<(2.3,7.8,8.3,8.9;0.5), (0.8,1.8,7.3,8.5;0.1), (0.6,1.1,2.3,2.7;0.9)>
S_{20}	<(2.9,3.5,3.9,4.7;0.8), (1.9,3.0,5.4,9.2;0.6), (3.5,4.3,6.1,7.7;0.7)>	<(4.2,5.3,6.7,7.1;0.8), (1.8,3.9,4.7,5.0;0.1), (0.2,0.7,1.8,2.0;0.4)>	<(1.5,3.5,4.2,8.0;0.6), (1.5,2.3,3.1,5.9;0.8), (6.3,9.0,9.4,9.5;0.3)>	<(1.0,1.5,3.6,3.9;0.1), (1.1,1.8,3.5,3.6;0.3), (1.0,1.7,3.5,3.8;0.8)>

TABLE 3. Normalized-Matrix

N	2014	2015	2016	2017
S_1	<(0.18,0.30,0.42,0.71;0.4), (0.26,0.29,0.52,0.67;0.7), (0.46,0.55,0.69,0.72;0.2)>	<(0.15,0.36,0.42,0.73;0.5), (0.14,0.31,0.44,0.56;0.8), (0.07,0.32,0.47,0.89;0.3)>	<(0.35,0.43,0.60,0.74;0.8), (0.01,0.09,0.33,0.51;0.6), (0.18,0.30,0.42,0.57;0.3)>	<(0.01,0.09,0.17,0.73;0.5), (0.06,0.25,0.39,0.43;0.2), (0.54,0.69,0.85,0.97;0.7)>
S_2	<(0.18,0.25,0.29,0.66;0.6), (0.10,0.19,0.26,0.31;0.2), (0.19,0.33,0.50,0.81;0.6)>	<(0.45,0.67,0.83,0.91;0.7), (0.09,0.10,0.15,0.27;0.5), (0.06,0.12,0.18,0.31;0.4)>	<(0.15,0.36,0.58,0.81;0.5), (0.62,0.77,0.91,1.00;0.2), (0.02,0.04,0.11,0.18;0.7)>	<(0.51,0.66,0.84,0.90;0.5), (0.17,0.29,0.47,0.66;0.8), (0.11,0.12,0.23,0.55;0.3)>
S_3	<(0.20,,0.39,0.70,0.88;0.3), (0.07,0.36,0.43,0.90;0.5), (0.55,,0.66,0.77,0.88;0.2)>	<(0.23,0.27,0.34,0.51;0.8), (0.10,0.36,0.62,0.72;0.6), (0.20,0.39,0.44,0.57;0.3)>	<(0.37,0.50,0.62,0.75;0.4), (0.02,0.17,0.30,0.31;0.7), (0.09,0.15,0.34,0.47;0.5)>	<(0.30,0.45,0.69,0.75;0.2), (0.31,0.63,0.73,0.95;0.7), (0.01,0.03,0.11,0.75;0.6)>
S_4	<(0.09,0.12,0.21,0.59;0.7), (0.18,0.30,0.42,0.71;0.2), (0.06,0.25,0.29,0.59;0.3)>	<(0.01,0.07,0.11,0.14;0.3), (0.53,0.73,0.87,1.00;0.1), (0.44,0.45,0.47,0.49;0.3)>	<(0.63,0.75,0.80,0.99;0.3), (0.02,0.03,0.04,0.05;0.7), (0.01,0.02,0.03,0.04;0.5)>	<(0.62,0.69,0.75,0.99;0.7), (0.18,0.30,0.42,0.51;0.4), (0.02,0.07,0.81,1.00;0.3)>
S_5	<(0.11,0.19,0.26,0.54;0.6), (0.54,0.59,0.66,0.71;0.2), (0.31,0.67,0.71,0.79;0.2)>	<(0.05,0.18,0.39,0.55;0.7), (0.11,0.29,0.52,0.77;0.2), (0.51,0.67,0.71,0.79;0.3)>	<(0.08,0.11,0.22,0.26;0.6), (0.54,0.62,0.79,0.83;0.9), (0.46,0.55,0.69,0.72;0.8)>	<(0.29,0.37,0.59,0.81;0.3), (0.03,0.11,0.34,0.69;0.4), (0.11,0.20,0.28,0.30;0.2)>
S_6	<(0.05,0.10,0.29,0.56;0.4), (0.02,0.05,0.07,0.23;0.1), (0.12,0.43,0.50,0.67;0.3)>	<(0.07,0.12,0.27,0.56;0.4), (0.23,0.27,0.34,0.51;0.3), (0.34,0.52,0.62,0.87;0.5)>	<(0.03,0.15,0.43,0.73;0.4), (0.47,0.69,0.73,0.89;0.1), (0.34,0.52,0.66,0.77;0.3)>	<(0.04,0.12,0.30,0.54;0.6), (0.04,0.18,0.47,0.57;0.4), (0.23,0.56,0.85,0.98;0.3)>
S_7	<(0.09,0.10,0.27,0.54;0.5), (0.20,0.39,0.70,0.88;0.2), (0.18,0.30,0.42,0.71;0.4)>	<(0.47,0.69,0.73,0.85;0.7), (0.06,0.25,0.29,0.33;0.8), (0.02,0.04,0.18,0.26;0.9)>	<(0.24,0.35,0.58,0.63;0.3), (0.44,0.52,0.67,0.78;0.6), (0.15,0.36,0.48,0.61;0.5)>	<(0.09,0.12,0.21,0.39;0.5), (0.27,0.57,0.62,0.69;0.2), (0.06,0.12,0.28,0.54;0.7)>
S_8	<(0.06,0.22,0.26,0.42;0.6), (0.02,0.12,0.20,0.54;0.3), (0.04,0.17,0.33,0.90;0.5)>	<(0.12,0.30,0.46,0.59;0.5), (0.27,0.52,0.67,0.79;0.3), (0.07,0.39,0.43,0.62;0.2)>	<(0.57,0.63,0.71,0.95;0.2), (0.06,0.14,0.18,0.23;0.4), (0.26,0.39,0.45,0.56;0.5)>	<(0.41,0.73,0.88,0.95;0.3), (0.05,0.11,0.26,0.35;0.5), (0.55,0.77,0.83,0.99;0.2)>
S_9	<(0.17,0.28,0.45,0.85;0.5), (0.04,0.12,0.30,0.54;0.2), (0.14,0.31,0.44,0.76;0.7)>	<(0.11,0.16,0.27,0.46;0.6), (0.34,0.69,0.73,0.93;0.9), (0.41,0.61,0.73,0.81;0.8)>	<(0.10,0.15,0.24,0.31;0.2), (0.45,0.67,0.83,0.94;0.6), (0.14,0.31,0.44,0.59;0.2)>	<(0.74,0.85,0.96,0.99;0.6), (0.09,0.10,0.15,0.27;0.2), (0.01,0.05,0.07,0.23;0.3)>
S_{10}	<(0.11,0.36,0.39,0.80;0.7), (0.18,0.39,0.57,0.90;0.4), (0.44,0.69,0.85,0.97;0.6)>	<(0.76,0.81,0.90,0.97;0.2), (0.09,0.15,0.34,0.43;0.4), (0.02,0.05,0.07,0.08;0.4)>	<(0.24,0.35,0.43,0.60;0.5), (0.34,0.55,0.78,0.90;0.2), (0.12,0.18,0.22,0.23;0.7)>	<(0.02,0.31,0.71,0.92;0.8), (0.15,0.37,0.38,0.42;0.6), (0.10,0.19,0.26,0.31;0.3)>
S_{11}	<(0.20,0.27,0.54,0.94;0.8), (0.11,0.19,0.26,0.54;0.5), (0.07,0.39,0.43,0.90;0.4)>	<(0.62,0.69,0.78,0.91;0.4), (0.28,0.30,0.42,0.53;0.7), (0.04,0.09,0.17,0.44;0.3)>	<(0.51,0.66,0.83,0.93;0.1), (0.15,0.15,0.35,0.63;0.3), (0.20,0.39,0.44,0.57;0.6)>	<(0.12,0.30,0.46,0.59;0.4), (0.08,0.44,0.62,0.81;0.5), (0.13,0.39,0.74,0.89;0.4)>
S_{12}	<(0.05,0.16,0.26,0.85;0.8), (0.18,0.25,0.29,0.66;0.1), (0.09,0.55,0.77,0.81;0.2)>	<(0.23,0.44,0.56,0.67;0.5), (0.20,0.34,0.57,0.84;0.2), (0.02,0.04,0.11,0.18;0.7)>	<(0.09,0.13,0.22,0.56;0.4), (0.31,0.63,0.73,0.95;0.1), (0.14,0.52,0.62,0.69;0.3)>	<(0.11,0.13,0.22,0.54;0.2), (0.26,0.39,0.45,0.56;0.4), (0.59,0.67,0.77,0.88;0.5)>
S_{13}	<(0.10,0.42,0.57,1.00;0.7), (0.53,0.73,0.87,0.91;0.2), (0.11,0.36,0.39,0.80;0.6)>	<(0.30,0.41,0.55,0.83;0.3), (0.18,0.30,0.42,0.51;0.5), (0.23,0.27,0.34,0.51;0.2)>	<(0.01,0.02,0.22,0.53;0.8), (0.59,0.67,0.79,0.88;0.5), (0.06,0.11,0.23,0.27;0.4)>	<(0.08,0.11,0.22,0.26;0.9), (0.28,0.31,0.53,0.56;0.1), (0.34,0.55,0.78,0.90;0.2)>
S_{14}	<(0.27,0.29,0.33,0.51;0.6), (0.12,0.43,0.50,0.71;0.2), (0.44,0.69,0.85,0.97;0.6)>	<(0.08,0.18,0.32,0.45;0.8), (0.54,0.59,0.66,0.71;0.5), (0.53,0.73,0.87,0.91;0.4)>	<(0.08,0.18,0.27,0.35;0.1), (0.23,0.78,0.83,0.89;0.2), (0.31,0.36,0.50 0.62;0.2)>	<(0.09,0.18,0.28,0.55;0.5), (0.08,0.10,0.27,0.54;0.2), (0.09,0.10,0.28,0.53;0.2)>
S_{15}	<(0.36,0.50,0.67,0.71;0.7), (0.05,0.16,0.26,0.35;0.2), (0.09,0.12,0.21,0.39;0.8)>	<(0.14,0.37,0.56,0.73;0.7), (0.09,0.10,0.27,0.54;0.3), (0.18,0.25,0.43,0.51;0.6)>	<(0.30,0.43,0.49,0.57;0.7), (0.03,0.11,0.34,0.69;0.2), (0.28,0.30,0.42,0.53;0.6)>	<(0.14,0.33,0.47,0.82;0.3), (0.53,0.62,0.77,0.99;0.3), (0.31,0.36,0.50,0.62;0.4)>
S_{16}	<(0.28,0.48,0.59,0.94;0.3), (0.19,0.27,0.39,0.56;0.4), (0.04,0.12,0.30,0.54;0.5)>	<(0.03,0.12,0.15,0.27;0.5), (0.44,0.69,0.85,0.90;0.3), (0.36,0.39,0.55,0.69;0.2)>	<(0.14,0.29,0.43,0.48;0.6), (0.03,0.10,0.14,0.74;0.2), (0.53,0.73,0.85,0.99;0.5)>	<(0.46,0.57,0.95,0.97;0.8), (0.12,0.18,0.22,0.23;0.5), (0.03,0.20,0.31,0.33;0.5)>
S_{17}	<(0.37,0.46,0.73,0.94;0.2), (0.05,0.10,0.29,0.36;0.7), (0.07,0.22,0.36,0.43;0.8)>	<(0.04,0.12,0.29,0.33;0.8), (0.20,0.34,0.55,0.71;0.5), (0.76,0.81,0.90,0.97;0.4)>	<(0.05,0.06,0.18,0.71;0.7), (0.13,0.16,0.23,0.48;0.4), (0.56,0.59,0.61,0.77;0.5)>	<(0.05,0.06,0.17,0.70;0.5), (0.04,0.07,0.20,0.69;0.1), (0.05,0.07,0.18,0.70;0.4)>
S_{18}	<(0.06,0.14,0.18,0.37;0.5), (0.54,0.59,0.66,0.71;0.3), (0.10,0.19,0.26,0.31;0.3)>	<(0.06,0.27,0.45,0.74;0.1), (0.12,0.43,0.50,0.67;0.2), (0.17,0.28,0.45,0.73;0.2)>	<(0.56,0.59,0.60,0.61;0.5), (0.03,0.14,0.50,0.78;0.2), (0.31,0.49,0.57,0.73;0.7)>	<(0.24,0.35,0.48,0.50;0.4), (0.06,0.14,0.18,0.23;0.3), (0.31,0.36,0.50,0.62;0.1)>
S_{19}	<(0.12,0.37,0.73,0.80;0.4), (0.02,0.04,0.11,0.18;0.2), (0.12,0.31,0.55,0.59;0.7)>	<(0.10,0.39,0.42,0.53;0.7), (0.27,0.46,0.78,0.90;0.4), (0.34,0.42,0.57,0.73;0.8)>	<(0.02,0.13,0.17,0.36;0.6), (0.02,0.05,0.07,0.23;0.5), (0.12,0.36,0.55,0.69;0.7)>	<(0.23,0.78,0.83,0.89;0.5), (0.08,0.18,0.73,0.85;0.1), (0.06,0.11,0.23,0.27;0.9)>
S_{20}	<(0.29,0.35,0.39,0.47;0.8), (0.19,0.30,0.54,0.92;0.6), (0.35,0.43,0.61,0.77;0.7)>	<(0.42,0.53,0.67,0.71;0.8), (0.18,0.39,0.47,0.50;0.1), (0.02,0.07,0.18,0.20;0.4)>	<(0.15,0.35,0.42,0.80;0.6), (0.15,0.23,0.31,0.59;0.8), (0.63,0.90,0.94,0.95;0.3)>	<(0.10,0.15,0.36,0.39;0.1), (0.11,0.18,0.35,0.36;0.3), (0.10,0.17,0.35,0.38;0.8)>

TABLE 4. $T = w_i * r_{ij}$

T	2014	2015	2016	2017
S_1	<(0.0054,0.0090,0.0126,0.0213;0.4), (0.0078,0.0087,0.0156,0.0201;0.7), (0.0138,0.0165,0.0207,0.0216;0.2)>	<(0.0045,0.0108,0.0126,0.0219;0.5), (0.0042,0.0093,0.0132,0.0168;0.8), (0.0021,0.0096,0.0141,0.0267;0.3)>	<(0.0105,0.0129,0.0180,0.0222;0.8), (0.0003,0.0027,0.0099,0.0153;0.6), (0.0054,0.0090,0.0126,0.0171;0.3)>	<(0.0003,0.0027,0.0051,0.0219;0.5), (0.0018,0.0075,0.0117,0.0129;0.2), (0.0162,0.0207,0.0255,0.0291;0.7)>
S_2	<(0.0144,0.0200,0.0232, 0.0528;0.6), (0.0080,0.0152,0.0208,0.0248;0.2), (0.0152,0.0264,0.0400,0.0648;0.6)>	<(0.0360,0.0536,0.0664,0.0728;0.7), (0.0072,0.0080,0.0120,0.0216;0.5), (0.0048,0.0096,0.0144,0.0248;0.4)>	<(0.0120,0.0288,0.0464,0.0648;0.5), (0.0496,0.0616,0.0728,0.0800;0.2), (0.0016,0.0032,0.0088,0.0144;0.7)>	<(0.0408,0.0528,0.0672,0.0720;0.5), (0.0136,0.0232,0.0376,0.0528;0.8), (0.0088,0.0096,0.0184,0.0440;0.3)>
S_3	<(0.0080,0.0156,0.0280,0.0352;0.3), (0.0028,0.0144,0.0172,0.0360;0.5), (0.0220,0.0264,0.0308,0.0352;0.2)>	<(0.0092,0.0108,0.0136,0.0204;0.8), (0.0040,0.0144,0.0248,0.0288;0.6), (0.0080,0.0156,0.0176,0.0228;0.3)>	<(0.0148,0.0200,0.0248,0.0300;0.4), (0.0008,0.0068,0.0120,0.0124;0.7), (0.0036,0.0060,0.0136,0.0188;0.5)>	<(0.0120,0.0180,0.0276,0.0300;0.2), (0.0124,0.0252,0.0292,0.0380;0.7), (0.0004,0.0012,0.0044,0.0300;0.6)>
S_4	<(0.0018,0.0024,0.0042,0.0118;0.7), (0.0036,0.0060,0.0084,0.0142;0.2), (0.0012,0.0050,0.0058,0.0118;0.3)>	<(0.0002,0.0014,0.0022,0.0028;0.3), (0.0106,0.0146,0.0174,0.0200;0.1), (0.0088,0.0090,0.0094,0.0098;0.3)>	<(0.0126,0.0150,0.0160,0.0198;0.3), (0.0004,0.0006,0.0008,0.001;0.7), (0.0002, 0.0004, 0.0006, 0.0008;0.5)>	<(0.0124,0.0138,0.0150,0.0198;0.7), (0.0036,0.0060,0.0084,0.0102;0.4), (0.0004,0.0014,0.0162,0.0200;0.3)>
S_5	<(0.0066,0.0114,0.0156,0.0324;0.6), (0.0324,0.0354,0.0396,0.0426;0.2), (0.0186,0.0402,0.0426,0.0474;0.2)>	<(0.0030,0.0108,0.0234,0.0330;0.7), (0.0066,0.0174,0.0312,0.0462;0.2), (0.0306,0.0402,0.0426,0.0474;0.3)>	<(0.0048,0.0066,0.0132,0.0156;0.6), (0.0324,0.0372,0.0474,0.0498;0.9), (0.0276,0.0330,0.0414,0.0432;0.8)>	<(0.0174,0.0222,0.0354,0.0486;0.3), (0.0018,0.0066,0.0204,0.0414;0.4), (0.0066,0.0120,0.0168,0.0180;0.2)>
S_6	<(0.0025,0.0050,0.0145,0.0280;0.4), (0.0010,0.0025,0.0035,0.0115;0.1), (0.0060,0.0215,0.0250,0.0335;0.3)>	<(0.0035,0.0060,0.0135,0.0280;0.4), (0.0115,0.0135,0.0170,0.0255;0.3), (0.0170,0.0260,0.0310,0.0435;0.5)>	<(0.0015,0.0075,0.0215,0.0365;0.4), (0.0235,0.0345,0.0365,0.0445;0.1), (0.0170,0.0260,0.0330,0.0385;0.3)>	<(0.0020,0.0060,0.0150,0.0270;0.6), (0.0020,0.0090,0.0235,0.0285;0.4), (0.0115,0.0280,0.0425,0.0490;0.3)>
S_7	<(0.0009,0.0010,0.0027,0.0054;0.5), (0.0020,0.0039,0.0070,0.0088;0.2), (0.0018,0.0030,0.0042,0.0071;0.4)>	<(0.0047,0.0069,0.0073,0.0085;0.7), (0.0006,0.0025,0.0029,0.0033;0.8), (0.0002,0.0004,0.0018,0.0026;0.9)>	<(0.0024,0.0035,0.0058,0.0063;0.3), (0.0044,0.0052,0.0067,0.0078;0.6), (0.0015,0.0036,0.0048,0.0061;0.5)>	<(0.0009,0.0012,0.0021,0.0039;0.5), (0.0027,0.0057,0.0062,0.0069;0.2), (0.0006,0.0012,0.0028,0.0054;0.7)>
S_8	<(0.0042,0.0154,0.0182,0.0294;0.6), (0.0014,0.0084,0.0140,0.0378;0.3), (0.0028,0.0119,0.0231,0.0630;0.5)>	<(0.0084,0.0210,0.0322,0.0413;0.5), (0.0189,0.0364,0.0469,0.0553;0.3), (0.0049,0.0273,0.0301,0.0434;0.2)>	<(0.0399,0.0441,0.0497,0.0665;0.2), (0.0042,0.0098,0.0126,0.0161;0.4), (0.0182,0.0273,0.0315,0.0392;0.5)>	<(0.0287,0.0511,0.0616,0.0665;0.3), (0.0035,0.0077,0.0182,0.0245;0.5), (0.0385,0.0539,0.0581,0.0693;0.2)>
S_9	<(0.0153,0.0252,0.0405,0.0765;0.5), (0.0036,0.0108,0.0270,0.0486;0.2), (0.0126,0.0279,0.0396,0.0684;0.7)>	<(0.0099,0.0144,0.0243,0.0414;0.6), (0.0306,0.0621,0.0657,0.0837;0.9), (0.0369,0.0549,0.0657,0.0729;0.8)>	<(0.0090,0.0135,0.0216,0.0279;0.2), (0.0405,0.0603,0.0747,0.0846;0.6), (0.0126,0.0279,0.0396,0.0531;0.2)>	<(0.0666,0.0765,0.0864,0.0891;0.6), (0.0081,0.0090,0.0135,0.0243;0.2), (0.0009,0.0045,0.0063,0.0207;0.3)>
S_{10}	<(0.0044,0.0144,0.0156,0.0320;0.7), (0.0072,0.0156,0.0228,0.0360;0.4), (0.0176,0.0276,0.0340,0.0388;0.6)>	<(0.0304,0.0324,0.0360,0.0388;0.2), (0.0036,0.0060,0.0136,0.0172;0.4), (0.0008,0.0020,0.0028,0.0032;0.4)>	<(0.0096,0.0140,0.0172,0.0240;0.5), (0.0136,0.0220,0.0312,0.0360;0.2), (0.0048,0.0072,0.0088,0.0092;0.7)>	<(0.0008,0.0124,0.0284,0.0368;0.8), (0.0060,0.0148,0.0152,0.0168;0.6), (0.0040,0.0076,0.0104,0.0124;0.3)>
S_{11}	<(0.0120,0.0162,0.0324,0.0564;0.8), (0.0066,0.0114,0.0156,0.0324;0.5), (0.0042,0.0234,0.0258,0.0540;0.4)>	<(0.0372,0.0414,0.0468,0.0546;0.4), (0.0168,0.0180,0.0252,0.0318;0.7), (0.0024,0.0054,0.0102,0.0264;0.3)>	<(0.0306,0.0396,0.0498,0.0558;0.1), (0.0090,0.0090,0.0210,0.0378;0.3), (0.0120,0.0234,0.0264,0.0342;0.6)>	<(0.0072,0.0180,0.0276,0.0354;0.4), (0.0048,0.0264,0.0372,0.0486;0.5), (0.0078,0.0234,0.0444,0.0534;0.4)>
S_{12}	<(0.0035,0.0112,0.0182,0.0595;0.8), (0.0126,0.0175,0.0203,0.0462;0.1), (0.0063,0.0385,0.0539,0.0567;0.2)>	<(0.0161,0.0308,0.0392,0.0469;0.5), (0.0140,0.0238,0.0399,0.0588;0.2), (0.0014,0.0028,0.0077,0.0126;0.7)>	<(0.0063,0.0091,0.0154,0.0392;0.4), (0.0217,0.0441,0.0511,0.0665;0.1), (0.0098,0.0364,0.0434,0.0483;0.3)>	<(0.0077,0.0091,0.0154,0.0378;0.2), (0.0182,0.0273,0.0315,0.0392;0.4), (0.0413,0.0469,0.0539,0.0616;0.5)>
S_{13}	<(0.0050,0.0210,0.0285,0.0500;0.7), (0.0265,0.0365,0.0435,0.0455;0.2), (0.0055,0.0180,0.0195,0.0400;0.6)>	<(0.0150,0.0205,0.0275,0.0415;0.3), (0.0090,0.0150,0.0210,0.0255;0.5), (0.0115,0.0135,0.0170,0.0255;0.2)>	<(0.0005,0.0010,0.0110,0.0265;0.8), (0.0295,0.0335,0.0395,0.0440;0.5), (0.0030,0.0055,0.0115,0.0135;0.4)>	<(0.0040,0.0055,0.0110,0.0130;0.9), (0.0140,0.0155,0.0265,0.0280;0.1), (0.0170,0.0275,0.0390,0.0450;0.2)>
S_{14}	<(0.0108,0.0116,0.0132,0.0204;0.6), (0.0048,0.0172,0.0200,0.0284;0.2), (0.0176,0.0276,0.0340,0.0388;0.6)>	<(0.0032,0.0072,0.0128,0.0180;0.8), (0.0216,0.0236,0.0264,0.0284;0.5), (0.0212,0.0292,0.0348,0.0364;0.4)>	<(0.0032,0.0072,0.0108,0.0140;0.1), (0.0092,0.0312,0.0332,0.0356;0.2), (0.0124,0.0144,0.0200,0.0248;0.2)>	<(0.0036,0.0072,0.0112,0.0220;0.5), (0.0032,0.0040,0.0108,0.0216;0.2), (0.0036,0.0040,0.0112,0.0212;0.2)>
S_{15}	<(0.0036,0.0050,0.0067,0.0071;0.7), (0.0005,0.0016,0.0026,0.0035;0.2), (0.0009,0.0012,0.0021,0.0039;0.8)>	<(0.0014,0.0037,0.0056,0.0073;0.7), (0.0009,0.0010,0.0027,0.0054;0.3), (0.0018,0.0025,0.0043,0.0051;0.6)>	<(0.0030,0.0043,0.0049,0.0057;0.7), (0.0003,0.0011,0.0034,0.0069;0.2), (0.0028,0.0030,0.0042,0.0053;0.6)>	<(0.0014,0.0033,0.0047,0.0082;0.3), (0.0053,0.0062,0.0077,0.0099;0.3), (0.0031,0.0036,0.0050,0.0062;0.4)>
S_{16}	<(0.0056,0.0096,0.0118,0.0188;0.3), (0.0038,0.0054,0.0078,0.0112;0.4), (0.0008,0.0024,0.0060,0.0108;0.5)>	<(0.0006,0.0024,0.0030,0.0054;0.5), (0.0088,0.0138,0.0170,0.0180;0.3), (0.0072,0.0078,0.0110,0.0138;0.2)>	<(0.0028,0.0058,0.0086,0.0096;0.6), (0.0006,0.0020,0.0028,0.0148;0.2), (0.0106,0.0146,0.0170,0.0198;0.5)>	<(0.0092,0.0114,0.0190,0.0194;0.8), (0.0024,0.0036,0.0044,0.0046;0.5), (0.0006,0.0040,0.0062,0.0066;0.5)>
S_{17}	<(0.0111,0.0138,0.0219,0.0282;0.2), (0.0015,0.0030,0.0087,0.0108;0.7), (0.0021,0.0066,0.0108,0.0129;0.8)>	<(0.0012,0.0036,0.0087,0.0099;0.8), (0.0060,0.0102,0.0165,0.0213;0.5), (0.0228,0.0243,0.0270,0.0291;0.4)>	<(0.0015,0.0018,0.0054,0.0213;0.7), (0.0039,0.0048,0.0069,0.0144;0.4), (0.0168,0.0177,0.0183,0.0231;0.5)>	<(0.0015,0.0018,0.0051,0.0210;0.5), (0.0012,0.0021,0.0060,0.0207;0.1), (0.0015,0.0021,0.0054,0.0210;0.4)>
S_{18}	<(0.0060,0.0140,0.0180,0.0370;0.5), (0.0540,0.0590,0.0660,0.0710;0.3), (0.0100,0.0190,0.0260,0.0310;0.3)>	<(0.0060,0.0270,0.0450,0.0740;0.1), (0.0120,0.0430,0.0500,0.0670;0.2), (0.0170,0.0280,0.0450,0.07300.2)>	<(0.0560,0.0590,0.0600,0.0610;0.5), (0.0030,0.0140,0.0500,0.0780;0.2), (0.0310,0.0490,0.0570,0.0730;0.7)>	<(0.0240,0.0350,0.0480,0.0500;0.4), (0.0060,0.0140,0.0180,0.0230;0.3), (0.0310,0.0360,0.0500,0.0620;0.1)>
S_{19}	<(0.0096,0.0296,0.0584,0.0640;0.4), (0.0016,0.0032,0.0088,0.0144;0.2), (0.0096,0.0248,0.0440,0.0472;0.7)>	<(0.0080,0.0312,0.0336,0.0424;0.7), (0.0216,0.0368,0.0624,0.0720;0.4), (0.0272,0.0336,0.0456,0.0584;0.8)>	<(0.0016,0.0104,0.0136,0.0288;0.6), (0.0016,0.0040,0.0056,0.0184;0.5), (0.0096,0.0288,0.0440,0.0552;0.7)>	<(0.0184,0.0624,0.0664,0.0712;0.5), (0.0064,0.0144,0.0584,0.0680;0.1), (0.0048,0.0088,0.0184,0.0216;0.9)>
S_{20}	<(0.0145,0.0175,0.0195,0.0235;0.8), (0.0095,0.0150,0.0270,0.0460;0.6), (0.0175,0.0215,0.0305,0.0385;0.7)>	<(0.0210,0.0265,0.0335,0.0355;0.8), (0.0090,0.0195,0.0235,0.0250;0.1), (0.0010,0.0035,0.0090,0.0100;0.4)>	<(0.0075,0.0175,0.0210,0.0400;0.6), (0.0075,0.0115,0.0155,0.0295;0.8), (0.0315,0.0450,0.0470,0.0475;0.3)>	<(0.0050,0.0075,0.0180,0.0195;0.1), (0.0055,0.0090,0.0175,0.0180;0.3), (0.0050,0.0085,0.0175,0.0190;0.8)>

TABLE 5. Comprehensive Values

C	Comprehensive Values
C_1	<(0.0207,0.0354,0.0483,0.0873;0.4),(0.0141,0.0282,0.0504,0.0651;0.8),(0.0375,0.0558,0.0729,0.0945;0.7)>
C_2	<(0.1032,0.1552,0.2032,0.2624;0.5),(0.0784,0.1080,0.1432,0.1792;0.8),(0.0304,0.0488,0.0816,0.1480;0.7)>
C_3	<(0.0440,0.0644,0.0940,0.1156;0.2),(0.0200,0.0608,0.0832,0.1152;0.7),(0.0340,0.0492,0.0664,0.1068;0.6)>
C_4	<(0.0270,0.0326,0.0374,0.0542;0.3),(0.0182,0.0272,0.0350,0.0454;0.7),(0.0106,0.0158,0.0320,0.0424;0.5)>
C_5	<(0.0318,0.0510,0.0876,0.1296;0.3),(0.0732,0.0966,0.1386,0.1800;0.9),(0.0834,0.1254,0.1434,0.1560;0.8)>
C_6	<(0.0095,0.0245,0.0645,0.1195;0.4),(0.0380,0.0595,0.0805,0.1100;0.4),(0.0515,0.1015,0.1315,0.1645;0.5)>
C_7	<(0.0089,0.0126,0.0179,0.0241;0.3),(0.0097,0.0173,0.0228,0.0268;0.8),(0.0041,0.0082,0.0136,0.0212;0.9)>
C_8	<(0.0812, 0.1316,0.1617,0.2037;0.2),(0.0280,0.0623,0.0917,0.1337;0.5),(0.0644,0.1204,0.1428,0.2149;0.4)>
C_9	<(0.1008,0.1296,0.1728,0.2349;0.2),(0.0828,0.1422,0.1809,0.2412;0.9),(0.0630,0.1152,0.1512,0.2151;0.8)>
C_{10}	<(0.0452,0.0732,0.0972,0.1316;0.2),(0.0304,0.0584,0.0828,0.1060;0.6),(0.0272,0.0444,0.0560,0.0636;0.7)>
C_{11}	<(0.0870,0.1152,0.1566,0.2022;0.1),(0.0372,0.0648,0.0990,0.1506;0.7),(0.0264,0.0756,0.1068,0.1680;0.6)>
C_{12}	<(0.0336,0.0602,0.0882,0.1834;0.2),(0.0665,0.1127,0.1428,0.2107;0.4),(0.0588,0.1246,0.1589,0.1792;0.7)>
C_{13}	<(0.0245,0.0480,0.0780,0.1310;0.3),(0.0790,0.1005,0.1305,0.1430;0.5),(0.0370,0.0645,0.0870,0.1240;0.6)>
C_{14}	<(0.0208,0.0332,0.0480,0.0744;0.1),(0.0388,0.0760,0.0904,0.1140;0.5),(0.0548,0.0752,0.1000,0.1212;0.6)>
C_{15}	<(0.0094,0.0163,0.0219,0.0283;0.3),(0.0070,0.0099,0.0164,0.0257;0.3),(0.0086,0.0103,0.0156,0.0205;0.8)>
C_{16}	<(0.0182,0.0292,0.0424,0.0532;0.3),(0.0156,0.0248,0.0320,0.0486;0.5),(0.0192,0.0288,0.0402,0.0510;0.5)>
C_{17}	<(0.0153,0.0210,0.0411,0.0804;0.2),(0.0126,0.0201,0.0381,0.0672;0.7),(0.0432,0.0507,0.0615,0.0861;0.8)>
C_{19}	<(0.0376,0.1336,0.1720,0.2064;0.4),(0.0312,0.0584,0.1352,0.1728;0.5),(0.0512,0.0960,0.1520,0.1824;0.9)>
C_{20}	<(0.0480,0.0690,0.0920,0.1185;0.1),(0.0315,0.0550,0.0835,0.1185;0.8),(0.0550,0.0785,0.1040,0.1150;0.8)>

TABLE 6. Values and Ambiguities of the alternatives

Values	Ambiguities
$V_1 = 0.0074 - 0.0003\lambda$	$A_1 = 0.0017 + .0002\lambda$
$V_2 = 0.0113 + 0.0337\lambda$	$A_2 = 0.0032 + 0.0056\lambda$
$V_3 = 0.0161 - 0.0129\lambda$	$A_3 = 0.004 - 0.0032\lambda$
$V_4 = 0.0089 - 0.0056\lambda$	$A_4 = 0.0029 - 0.0025\lambda$
$V_5 = 0.0065 - 0.0001\lambda$	$A_5 = 0.0009 + 0.0013\lambda$
$V_6 = 0.0542 - 0.0464\lambda$	$A_6 = 0.0114 - 0.0071\lambda$
$V_7 = 0.0009 + 0.0005\lambda$	$A_7 = 0.0002 + 0.0001\lambda$
$V_8 = 0.0674 - 0.0616\lambda$	$A_8 = 0.0142 - 0.0132\lambda$
$V_9 = 0.007 - 0.0008\lambda$	$A_9 = 0.0015 - 0.0003\lambda$
$V_{10} = 0.0156 - 0.0122\lambda$	$A_{10} = 0.0035 - 0.0028\lambda$
$V_{11} = 0.0224 - 0.0210\lambda$	$A_{11} = 0.0066 - 0.0063\lambda$
$V_{12} = 0.0591 - 0.0559\lambda$	$A_{12} = 0.0118 - 0.0108\lambda$
$V_{13} = 0.0410 - 0.0351\lambda$	$A_{13} = 0.0074 - 0.0054\lambda$
$V_{14} = 0.0345 - 0.0341\lambda$	$A_{14} = 0.0059 - 0.0058\lambda$
$V_{15} = 0.0073 - 0.0056\lambda$	$A_{15} = 0.0023 - 0.00196\lambda$
$V_{16} = 0.0160 - 0.0128\lambda$	$A_{16} = 0.0034 - 0.0026\lambda$
$V_{17} = 0.0051 - 0.0037\lambda$	$A_{17} = 0.0014 - 0.00086\lambda$
$V_{18} = 0.0910 - 0.0895\lambda$	$A_{18} = 0.0216 - 0.0213\lambda$
$V_{19} = 0.0257 - 0.0022\lambda$	$A_{19} = 0.0116 - 0.0064\lambda$
$V_{20} = 0.0064 - 0.0056\lambda$	$A_{20} = 0.0015 - 0.0013\lambda$

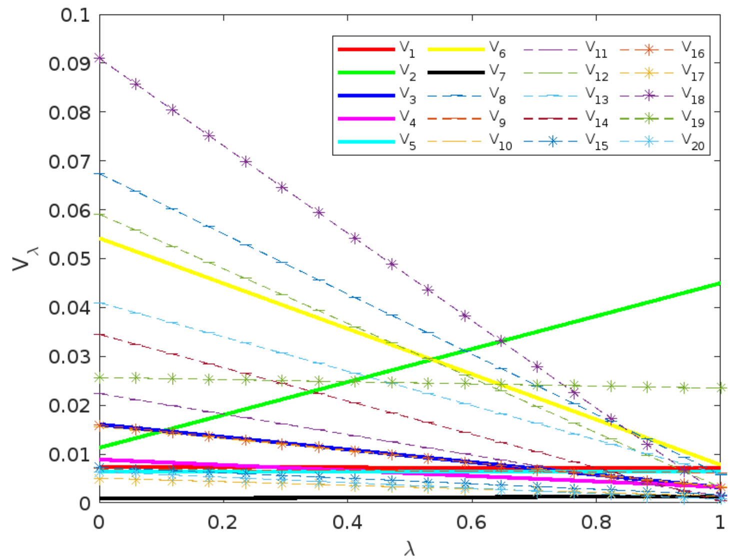


FIGURE 2. Value

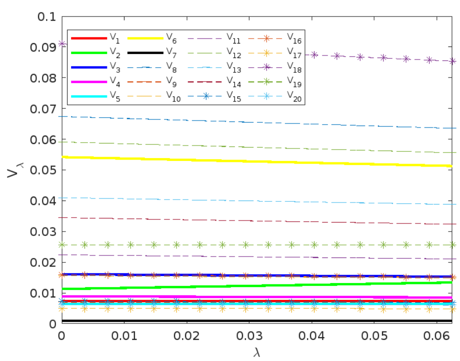


FIGURE 3. Value-$\lambda \in (0, 0.0625)$

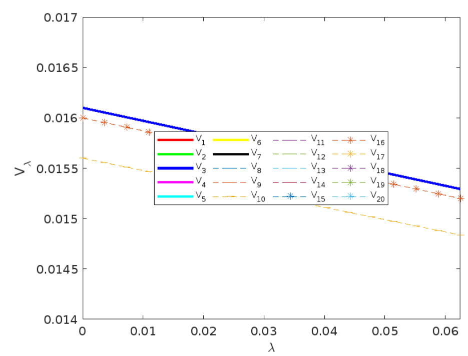


FIGURE 4. Value-$\lambda \in (0, 0.0625)$

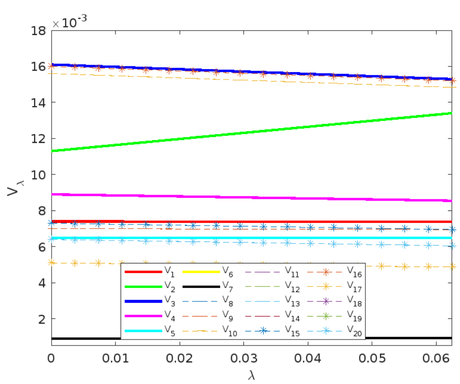


FIGURE 5. Value-$\lambda \in (0, 0.0625)$

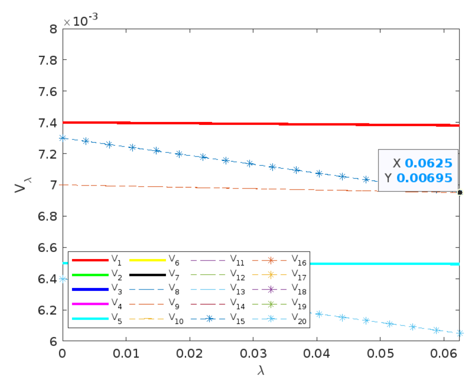


FIGURE 6. Value-$\lambda \in (0, 0.0625)$

The graphical representation of the values is shown in the figure 2 . The intersection of lines denotes that the values of GPSVNN are same at the value λ. First we compare the values for $\lambda \in [0, 0.0625]$, we split the graph by comparing values of the alternatives. Finally, we calculate the ambuiguity at the point of intersection for $\lambda = 0.0625$. The ranking of values are given in the graphs 3 ,4, 5 and 6 . At $\lambda = 0.0625$ we calculate the ambiguity $A_9 = 0.001494$ and $A_{15} = 0.002210$ in which V_9 and V_{15} are intersecting. Hence the ranking of the alternatives for $\lambda \in [0, 0.0625]$ are $S_{18} > S_8 > S_{12} > S_6 > S_{13} > S_{14} > S_{19} > S_{11} > S_3 > S_{16} > S_{10} > S_2 > S_4 > S_1 > S_{15} > S_9 > S_5 > S_{20} > S_{17} > S_7$. If $\lambda \in [0.6220, 0.6300]$, at $\lambda = 0.6220$ V_{12} and V_{19} are intersecting. Hence we calculate the ambiguity at 0.6220 for the alternatives. ie $A_{19} = 0.007602$ and $A_{12} = 0.0050824$. Then the order of the alternatives are $S_{18} > S_2 > S_8 > S_6 > S_{19} > S_{12} > S_{13} > S_{14} > S_{11} > S_3 > S_{16} > S_{10} > S_1 > S_9 > S_5 > S_4 > S_{15} > S_{20} > S_{17} > S_7$. If $\lambda \in [0.9976, 0.9980]$ at $\lambda = 0.9976$ V_{15} and V_{18} and at $\lambda = 0.9980$ V_4 and V_{12} are intersecting. Hence we calculate the ambiguity at λ=0.9976 for the alternatives, ie $A_{15} = 0.00037640568$ and $A_{18} = 0.0003251824$ and at λ=0.9980 for the alternatives, ie $A_4 = 0.000422616$ and $A_{12} = 0.0010216$. Then the ranking is $S_2 > S_{19} > S_6 > S_1 > S_5 > S_9 > S_{13} > S_8 > S_{10} > S_{12} > S_4 > S_3 > S_{16} > S_{15} > S_{18} > S_{11} > S_{17} > S_7 > S_{20} > S_{14}$. At $\lambda \in (0.9980, 1)$ the ranking is $S_2 > S_{19} > S_6 > S_1 > S_5 > S_9 > S_{13} > S_8 > S_{10} > S_4 > S_{12} > S_3 > S_{16} > S_{15} > S_{18} > S_{11} > S_{17} > S_7 > S_{20} > S_{14}$. At $\lambda = 1$ we calculate the ambiguties for the intersecting values and the ranking order is $S_2 > S_{19} > S_6 > S_1 > S_5 > S_9 > S_{13} > S_8 > S_{10} > S_4 > S_{12} > S_{16} > S_3 > S_{15} > S_{18} > S_{11} > S_7 > S_{17} > S_{20} > S_{14}$. The ranking order is related to the weight $\lambda \in [0, 1]$.

5. Conclusion

In this research article, the concept of Generalized Parabolic Single-Valued Neutroposophic Number (GPSVNN) has been developed. We have defined the (α, β, γ)-cut of GPSVNN. Also, the arithmetic operators of these numbers are discussed and illustrated using graphical representation. A demonstration of the De-Neutrosophication method utilising values and ambiguities has been introduced here for the conversion of a GPSVNN into a real number. Further, this result is applied in the ranking of the satisfaction levels of citizens in municipal services. For this purpose, 20 municipal services included in the Life Satisfaction Survey (LSS) that the Turkish Statistical Institution regularly applies every year are considered as alternatives. In addition, the satisfaction of citizens was evaluated for the period of 2014–2017. To analyse the effect of all opinion types on the decision process, the participant responses constituting the dataset of GPSVNN and these years were considered as a set of criteria. We have utilised the values and ambiguities to evaluate the citizens' satisfaction levels with the municipality's services. Finally, the best and worst alternatives were chosen by ranking the alternatives.

In the future, researchers can develop algorithms using GPSVNN in various fields like image processing problems, pattern recognition problems, cloud computing problems, and other mathematical modelling problems involving uncertainty and nonlinearity.

Appendix A

In this section, we have given the MATLAB code for calculating the Normalized values, Comprehensive values, Values and Ambiguities for the alternatives. The matrix A1 to A4 denotes the Truth-membership, A5-A8 represents the Indeterminacy membership and A9-A12 for Falsity membership for the four alternatives, respectively. W1 represents the weight of each criterion of the alternatives. $OMEGA = min(\omega_{ij})$, $RHO = max(\rho_{ij})$ $DELTA = max(\delta_{ij})$.

```
A1=input('Matrix A1');
A2=input('Matrix A2');
A3=input('Matrix A3');
A4=input('Matrix A4');
A5=input('Matrix A5');
A6=input('Matrix A6');
A7=input('Matrix A7');
A8=input('Matrix A8');
A9=input('Matrix A9');
A10=input('Matrix A10');
A11=input('Matrix A11');
A12=input('Matrix A12');
W1=input('Enter W1');
OMEGA=input('Enter OMEGA');
RHO=input('Enter RHO');
DELTA=input('Enter DELTA');
N1=A1/10
N2=A2/10
N3=A3/10
N4=A4/10
N5=A5/10
N6=A6/10
N7=A7/10
N8=A8/10
N9=A9/10
N10=A10/10
N11=A11/10
N12=A12/10
C1=N1*W1
C2=N2*W1
```

C3=N3∗W1

C4=N4∗W1

C5=N5∗W1

C6=N6∗W1

C7=N7∗W1

C8=N8∗W1

C9=N9∗W1

C10=N10∗W1

C11=N11∗W1

C12=N12∗W1

D1=C1+C2+C3+C4

D2=C5+C6+C7+C8

D3=C9+C10+C11+C12

VALUES1=OMEGA^2/10∗(D1(1)+4∗D1(2)+4∗D1(3)+D1(4))

VALUES2=(1−RHO)^2/10∗(D2(1)+4∗D2(2)+4∗D2(3)+D2(4))

VALUES3=(1−DELTA)^2/10∗(D3(1)+4∗D3(2)+4∗D3(3)+D3(4))

AM1=OMEGA^2/10∗(−D1(1)−4∗D1(2)+4∗D1(3)+D1(4))

AM2=(1−RHO)^2/10∗(−D2(1)−4∗D2(2)+4∗D2(3)+D2(4))

AM3=(1−DELTA)^2/10∗(−D3(1)−4∗D3(2)+4∗D3(3)+D3(4))

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Text Analysis Using Morphological Operations on a Neutrosophic Text Hypergraph

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Abstract. Due to the rise in the operation of platforms on social media, there is more opportunity for users to post content online, out of which some tend to be hate speech. Hate speech is found in almost all domains like sports, politics, religion, government affairs, and personal matters. Its detection and removal from platforms like Twitter, Facebook, etc. are tedious. Over the years, a lot of methods have evolved in this area most of which are more time-consuming machine learning methods. Our objective is to find a better method that considers indeterminacy at the word level and sentence level for the detection and removal of hate speech using fuzzy logic applied to Neutrosophic hypergraphs. A neutrosophic hypergraph is a kind of hypergraph where each node and hyperedge has three associated membership functions namely Indeterminacy, Truth and Falsity. Our work has successfully modeled Text documents into neutrosophic hypergraphs and morphological operators like dilation, erosion etc. are applied to it. Using these operations further operators like thinning, thickening, hit-or-miss, and skeletoning are applied. Finally hate speech is identified and removed. This a novel method in this area. The system is tested with Twitter tweets and the results are promising with an accuracy of 88%.

Keywords: neutrosophic; hypergraph; morphology ; hate speech

1. Introduction

Since lakhs of contents are posted every day on social media platforms, there is more chance for it to be against the rules of a government, religion, and mostly the society. Filtering the contents and making it suitable for everyone to read it is a tremendous job. In most cases, contents are manually detected after mass protest and are removed or deactivated. Since the readability and reachability of the social media content are higher when compared to the printing media or visual media, there should be good and efficient methods for identifying and removing hate speech. Our system has made efforts in this area by applying the concepts of neutrosophic hypergraphs and their morphological operators.

Many real-life problems were solved by modeling hypergraphs with neutrosophic sets and logic. Neutrosophic sets are used to deal with uncertainties in such problems. Neutrosophic sets are to deal with this indeterminacy. Morphological operators like dilation, erosion, thickening, thinning, and skeletoning are useful for various text analysis operations which are discussed in this paper with the main focus on hate speech detection and removal.

1.1. *Related works*

The proposed work in this paper has applied neutrosophic hypergraph operations for hate speech detection. There has been a lot of research work on detecting hate speech that is mostly seen in social media. In order to classify hate speech, a novel method namely H-CovBi-Caps [1] was implemented that is a deep learning model based on coventional, BiGRU and Capsule model. Evaluation of this model was done using balanced and unbalanced Twitter data sets. This method gave a recall of 0.80 and f-score of 0.84. Another method used natural language processing strategies and data analysis to make providers of social media responsive to hate speech content [2]. The authors claim that they can surpass the state-of-the-art approach in terms of precision, recall, and F1 scores by approximately 10%. There have been works that focus on the lack of transparency and bias experienced by various hate speech detection and mitigation systems [3]. Using SAS Enterprise Miner's Text Analytics [4], the authors demonstrated how to consider the information in the tweets to classify them as hostile. The tweets were subjected to preprocessing and models were applied and analyzed. The authors claim adequate accuracy. Different approaches to hate speech detection are discussed and compared in a survey [5], where the authors have considered various data sets, features, and machine learning models for comparison.

In all these methods even though the authors claim good accuracy, the works lag a proper mathematical modeling and representation. They have the disadvantage of being costly in terms of time and resources. Since our work is concentrating on finding a solution for this with the help of neutrosophic hypergraphs, let us see some works already done on hypergraphs.

The perspective of a single-valued neutrosophic set, its complement, union, difference, properties of set-theoretic operators etc. was introduced in [6]. The structure of a system can be studied by using hypergraph [7] which is the generalization of a graph. A detailed study of fuzzy graphs and fuzzy hypergraphs [8] and related extensions [9], discusses mathematical models of hypergraphs like intuitionistic, complex, m-polar fuzzy, bi-polar, Pythagorean and q-rung ortho-pair hypergraphs, and also neutrosophic hypergraphs like single-valued, bi-polar and complex. Graph morphology [10] extracted the structural information from graphs using structuring graphs. Lattice structure on hypergraphs are developed and morphological operators [11], [12] are defined by using vertex-hyperedge correspondence. Also, the classical notion of a dilation/erosion of a subset of vertices is extended to sub-hypergraphs. Several opening, closing and alternate sequential filters are also proposed. Morphology applied on Intuitionistic fuzzy hypergraphs are discussed in [13], [14], [15]. Text summarization using morphological filter [16] is done on intuitionistic fuzzy hypergraphs. Crime Analysis [17] done with the application of graph morphology has successfully tracked the crime rate in various areas. More than 200 neutrosophic graphs [18] are discussed, particularly the bipartite neutrosophic graphs, neutrosophic tree and directed neutrosophic graphs applied in cognitive maps, relational maps and relational equations. The perception of neutrosophic incidence graphs that are single-valued, their cut vertex, blocks and bridges are discussed in [19]. The paper has discussed the neutrosophic incidence graphs and their vertex, edge and pair connectivity. Neutrosophic logic and connectors [20] based on set operations are also defined. The idea of constant single-valued neutrosophic graph (CSVNG) [21], which is the modified form of a single-valued neutrosophic graph has also evolved. The authors applied it to Wi-Fi systems and also discussed the consequences. A methodology of decision-making with multiple criteria [22] applied with a neutrosophic set was developed to handle uncertain data, and the authors have used it in the Logistics Service Sector. A novel adaptable method [23] was used with eleven criteria and ten solar panels in PV which used a neutrosophic set to deal with vague data. Another work in IoT [24] intended to introduce a weight product method based on the neutrosophic framework for the assessment of IoT-based cities that are sustainable and smart. The notion of Fermatean neutrosophic dombi fuzzy graph [25] was initiated which constructed the cartesian, direct, composition of such graphs. A neutrosophic method using type-2 neutrosophic numbers [26] was used in the field of study of risks in power plants. A hybrid approach to decision-making using many criteria under a spherical fuzzy environment was introduced in [27].

Most of the hate speech detection methods developed failed to address the ambiguity aspect of it, our method has included an indeterminacy parameter with every word and every sentence. Even though there are many applications with hypergraphs in the area of image processing, networks, text data etc., the proposed method is the first work that has done hate speech detection and removal of it using a neutrosophic text hypergraph. The preliminaries of the neutrosophic hypergraph and the morphological operations are given in sections 1.2 to 1.4. The section 2 focuses on how a document is converted to a neutrosophic hypergraph and how

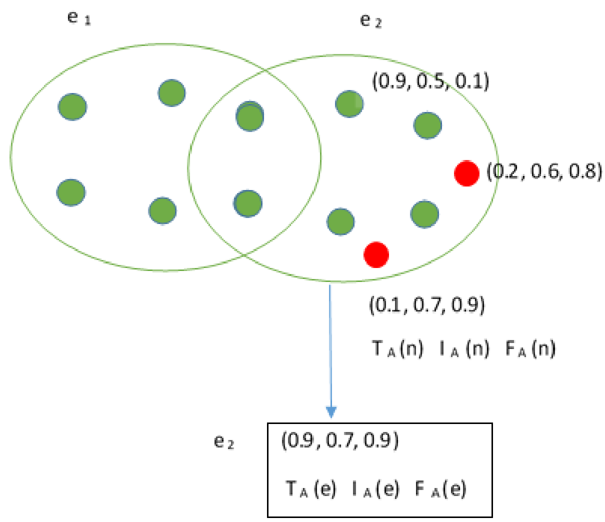

FIGURE 1. Neutrosophic graph with membership degree

operations like hit-or-miss, skeletoning etc. are applied to it. Section 3 deals with operations like thinning and thickening. Section 4 shows how hate speech detection is done using the operations discussed in sections 3 and 4. Finally, section 5 gives a detailed result analysis.

1.2. *Preliminaries*

Let a neutrosophic hypergraph be defined as $H = (H^n, H^e)$ and is shown in Figure 1, where H^n is the collection of nodes and H^e is the collection of hyperedges. For every n in H^n; $F(A) \in [0,1]$, $I_A(n) \in [0,1]$, $T_A(n) \in [0,1]$, and $I_A(n) + T_A(n) + F_A(n) <= 3$, where $I_A(n)$, $T_A(n)$ and $F_A(n)$ are the indeterminacy, truth and falsity value respectively. Set A which is a neutrosophic set in $H^n = \{(n, I_A(n), T_A(n), F_A(n)); n \in H^n\}$. Likewise for every e in H^e, $T_A(e) \in [0,1]$, $I_A(e) \in [0,1]$,$F_A(e) \in [0,1]$ and $I_A(e) + T_A(e) + F_A(e) <= 3$, where $I_A(e)$ is the indeterminacy value, $T_A(e)$ is the truth value, and $F_A(e)$ is the falsity value. A neutrosophic set B in $H^e = \{(e, I_A(e), T_A(e), F_A(e)); e \in H^e\}$. The edge membership degree, $(I_A(e), T_A(e), F_A(e))$ is defined as the maximum of respective membership degrees of the nodes and is given by

$$T_A(e) = \vee T_A(n); \forall_n \in e \tag{1}$$

$$I_A(e) = \vee I_A(n); \forall_n \in e \tag{2}$$

$$F_A(e) = \vee F_A(n); \forall_n \in e \tag{3}$$

1.3. *Special cases of membership values*

- case 1: $[1, 1, 0]$ In weather prediction during the rainy season, the truth value of rain and the indeterminacy is 1. Non-occurrence of rain tends to 0.

- case 2: $[0, 1, 1]$ In the case of the Nipah virus attack, based on previous experiences in past years, the possibility of a patient being alive is 0. But there is an indeterminacy due to the nature of the virus and the falsity of death is 1. Since indeterminacy is 1, the converse may also happen violating the history and we may get [1,1,0].
- case 3: $[1, 0, 1]$ At a particular point of time of hartal or strike, there is a chance of crime or not. Hence at instance t, the truth value is 1 also falsity can be 1.
- case 4: $[0, 0, 0]$ In the case of cancer patients, the region not affected by cancer need not be considered for treatment. For this region the indeterminacy is 0, the Truth value is 0 and there is no doubt in the falsity of the disease.
- case 5: $[1, 1, 1]$ This is a chaotic situation where all the values are 1. In the case of a Tornado, since the system is chaotic, the occurrence of Tornado, the indeterminacy and the falsity is 1.

1.4. *Applying morphological operators*

Let $\{H_{NF}, H^n, H^e, (\mu_n, \gamma_n, \kappa_n), (\mu_e, \gamma_e, \kappa_e)\}$ be a neutrosophic hypergraph, where γ_n is the non-membership degree, μ_n is the membership degree, and κ_n is the indeterminacy degree defined on a collection of nodes. Let membership degree μ_e, non-membership degree γ_e and indeterminacy degree κ_e be defined on a collection of hyperedges of the neutrosophic hypergraph. Here the sum of $(\mu_n, \gamma_n, \kappa_n) <= 3$. Also μ_e is the supremum of μ_n, γ_e is the supremum of γ_n and κ_e is the supremum of κ_n.

1.4.1. $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\omega})$ *cut of a neutrosophic fuzzy hypergraph*

The (α, β, ω) cut of a neutrosophic hypergraph H_{NF} is the crisp set of nodes given by $X_{NF} = H_{\alpha,\beta,\omega}/\alpha >= m, \beta >= n, \omega >= k$ which retrieves a sub hypergraph of H_{NF}. Once we have H_{NF}, the parent graph and X_{NF} as its sub-graph, we can define many morphological operators adjunction, erosion, dilation, closing, and opening filters on it. Figure 2(a) shows a parent neutrosophic hypergraph and Figure 2(b) shows a sub-graph obtained by (α, β, ω) cut. All the following morphological operations are defined for this parent and sub-hypergraph.

1.4.2. *Dilation of* X_{NF}

The dilation operation can be done to concerning nodes or concerning edges. Dilation concerning nodes can be written as follows:-

$$\delta^n(X_{NF}) = \{n/n \in X_{NF}\} \tag{4}$$

As per e.q(4), it is the collection of nodes present in the sub-hypergraph X_{NF}. The dilation concerning edges can be written as follows:-

$$\delta^e(X^n_{NF}) = \{e/e \in H_{NF}/n \in X^e\} \tag{5}$$

As per e.q(5), it includes all edges in H_{NF} such that it contains at least one node in X^e. Both the dilations are shown in Figure 2(c) and Figure 2(d).

1.4.3. ***Erosion of*** X_{NF}

The erosion operator can be applied in two ways. It can be either concerning nodes or concerning hyperedges. Erosion concerning nodes is written as the following:-

$$\varepsilon^n(X^e_{NF}) = \{n \in X_{NF}/n \notin X^{e'}_{NF}; X^{e'}_{NF} = H^e_{NF} - X^e_{NF}\} \tag{6}$$

According to e.q(6), erosion concerning nodes is defined as the collection of nodes in X_{NF} which are not present in its complement graph. This is shown in Figure. 2(e). Erosion concerning hyperedges is the collection of edges consisting of nodes of X_{NF} only. It can be written as the following:-

$$\varepsilon^e(X^n_{NF}) = \{e \in X_{NF}/\forall_{n\in e} n \notin X^{e'}_{NF}\} \tag{7}$$

This is shown in Figure 2(f).

1.4.4. ***Adjunction of*** X_{NF}

We can say that $(\varepsilon^e, \delta^n)$ are adjunctions iff

$$X^e_{NF} \subseteq \varepsilon^e(Y^n_{NF}) \tag{8}$$

$$\delta^n(X^e_{NF}) \subseteq Y^n_{NF}; X_{NF} \subseteq Y_{NF} \tag{9}$$

1.4.5. ***Morphological Opening and Closing***

The morphological opening is of two types:-

- Opening w.r.to edge(γ_e)

  This Morphological opening

  $$\gamma_e = \delta^e(e^n(X^e_{NF})) \tag{10}$$

  is a composition of the form $\delta \circ \varepsilon$ which gives edges in X_{NF} by applying e.q(6) followed by e.q(5).

- Opening w.r.to node(γ_n)
  This Morphological opening of X_{NF} is

$$\gamma_n = \delta^n(\varepsilon^e(X^n_{NF})) \tag{11}$$

  which is a composition of $\delta \circ \varepsilon$ obtained by applying e.q(7) followed by e.q(4).

- Closing w.r.to edge
  This Morphological closing is the set of edges in X_{NF}

$$\phi_e = \varepsilon^e(\delta^n(X^e_{NF})) \tag{12}$$

  which is a composition of $\varepsilon \circ \delta$ obtained by applying e.q(4) followed by e.q(7).

- Closing w.r.to node
  This Morphological closing

$$\phi_n = \varepsilon^n(\delta^e(X^n_{NF})) \tag{13}$$

  is the set of nodes in X_{NF} which is a composition of $\varepsilon \circ \delta$ obtained by applying e.q(5) followed by e.q(6)

Repeated application of opening as well as closing operations as mentioned in e.q(10) to e.q(13) results in the same hypergraph. Such operators are called filters. They are shown in Figures 2(g) to 2(j).

2. **Materials and Methods**

2.1. *Skeleton operation with dilation w.r.to edge*

Dilation related to edge is defined as the collection of all edges retrieved from the parent graph H, which contains all nodes in sub-hypergraph X. It can be written as $\delta^e(X^n)$. The skeleton operation on a graph H, can be defined as

$$S(H) = H - (\delta^e(X^n))^k \tag{14}$$

Let H be the hypergraph related to text pertaining to the *sports* domain. Some of the words related to specific sports domains are given in Table 1. Let X_1 be the sub-hypergraph of H, which is obtained by taking the words related to *cricket*. By applying dilation w.r.to edge, $\delta^e(X^n_1)$, we get all the text related to *cricket*. On applying $S(H_1) = H - (\delta^e(X^n_1))$, we get a minimal skeleton of *sports* devoid of *cricket*. Now take X_2 = set of words related to $football$. On applying $\delta^e(X^n_2)$, we get all the text related to $football$. Thus $S(H_2) = S(H_1) - (\delta^e(X^n_1))$ will give us the text devoid of $football$. On repeating this K times we get the skeleton of sports

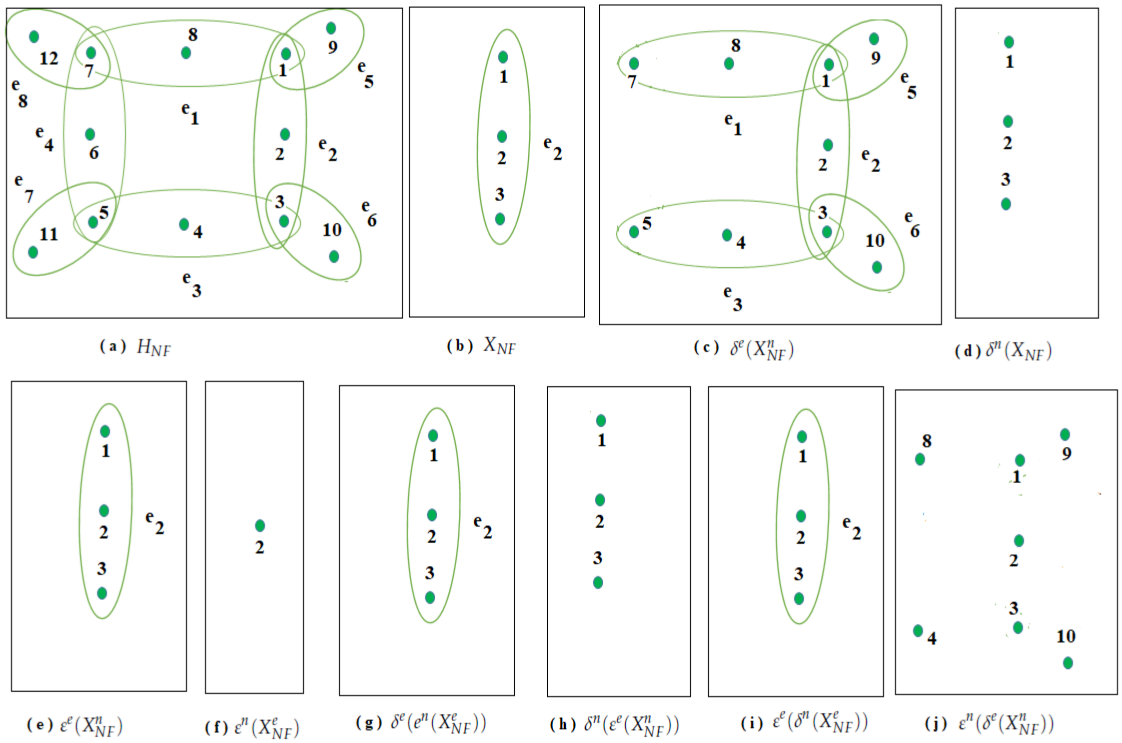

FIGURE 2. Result of morphological operations on a neutrosophic hypergraph

which is devoid of specific sports areas. As a byproduct of this, we get many sub-hypergraphs of H.

2.1.1. *Illustration*

Consider the text given in Figure 3 with words numbered. A hypergraph can be drawn by considering unique words as nodes and sentences as hyperedges. It can be made neutrosophic by giving three degrees to each word based on a criteria. Some of the words in the sports field and the criteria are shown in Table 2. If there are common words across sentences, then edges will overlap as shown in Figure 4. We consider words related to *cricket* first and then apply dilation $\delta^e(X_1^n)$. Let X_1 be a sub-hypergraph that consists of words in the *cricket* domain. This dilation is a conditional dilation, which selects the statements in the original text which consists of words in the *cricket* domain. It is subtracted from the hypergraph to get the skeleton $S(H_1)$. The first skeleton obtained is shown in Figure 5. Now select sub-hypergraph X_2 which is the set of words in the *football* domain. Apply dilation and select the sentences in the original text related to the *football* domain. On subtracting this we get the next level skeleton which is shown in Figure 6 and Algorithm 1.

TABLE 1. Words related to specific sports

cricket	**football**
cricket	striker
ICC	Manchester

TABLE 2. Criteria for giving degrees $T_A(n), I_A(n), F_A(n)$ to the words

Words	$\boldsymbol{T_A(n)}$	$\boldsymbol{I_A(n)}$	$\boldsymbol{F_A(n)}$	***Criteria***
Cricket	0.9	0.1	0.3	Cricket is a sports game only in a few countries in the world. Even though it is a sports game, it is not seen in the Olympics. So $F_A(n) = 0.3$ and $I_A(n) = 0.1$. Indeterminacy is less since it is related to sports
ICC				-do-
Tournament				-do-
Football	1.0	0	0	Indeterminacy is 0, falsity is 0. since it is a sports event and seen in Olympics
Manchester				-do-
Olympics				-do-
Badminton				-do-
Game				-do-
Sachin	0.8	0.3	0.3	Depends on person to person and also value varies from person to person. When compared to very popular persons in Football, there is a bit more level of indeterminacy for *Sachin* for being identified as a sports person.
Stages	0.5	0.5	0.5	This is a word have medium value for all the degrees
Season				-do-
Performance				-do-

FIGURE 3. Sample text for skeletoning

2.1.2. *Algorithm: Skeletoning*

Algorithm 1: Skeleton creation of a text hypergraph

Data: Hypergraph

Result: Skeleton

Create a text hypergraph H_τ;

$i = 1$;

repeat

  Create sub-hypergraph X_i of H_τ;

  Apply the dilation $\delta^e(X_i^n)$;

  Find the skeleton $S(H_i) = H_\tau - \delta^e(X_i^n)$;

  $H_\tau = S(H_i)$;

  $i = i + 1$;

until $X_i = \phi$ *or* $i = k$;

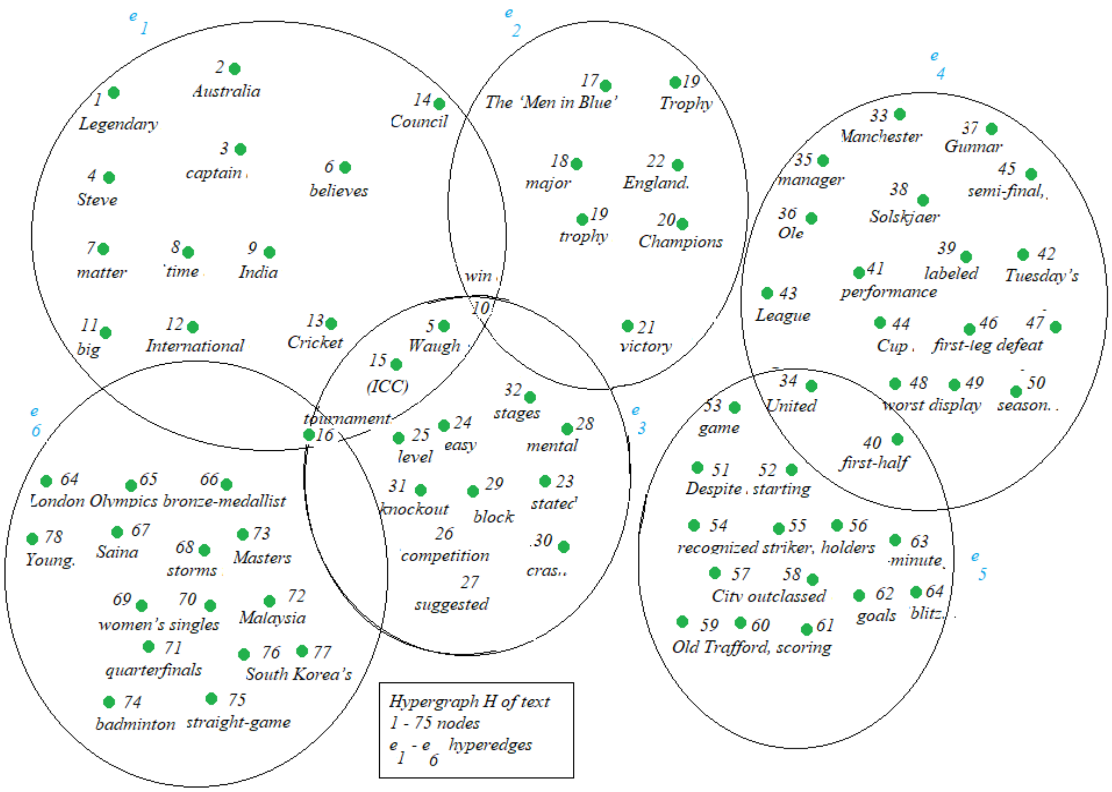

FIGURE 4. Hypergraph formed from text in Figure 3

2.2. *Skeleton operation with dilation related to node*

Dilation related to node which is written as $\delta^n(X^e)$ is defined as the set of nodes in X^e of H. On applying $H - (\delta^n(X^e))$ we get the skeleton of H w.r.to nodes. We can further apply skeleton operation by varying X.

2.2.1. *Illustration*

Let us take the same example given in Figure 4. Let $X_1 = e_1, e_3$ as shown in Figure 4. Now when $k = 1$, the skeleton operation $S(H_1) = H - (\delta^n(X_1)^e))$ results in Figure 5. Now let X_2= set of sentences related to $football$. On applying skeleton operation $S(H_2) = S(H_1) - \delta^n(X_2)^e)$, we get the graph shown in Figure 6. When $k = 1$, we get the maximal skeleton. When k increases the thinning nature of the skeleton increases and we get the minimal skeleton as shown in Figure 7 and Figure 8.

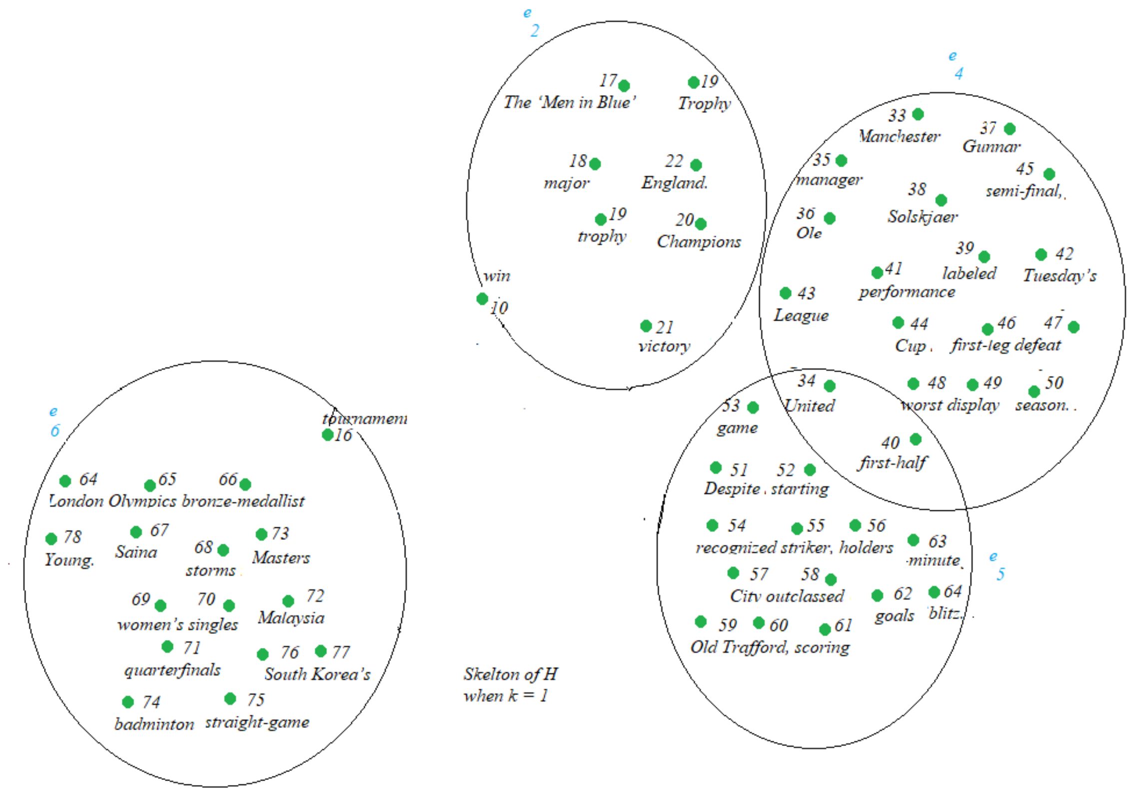

Figure 5. Skeleton of H w.r.to edge when k=1

Similarly, skeleton operation can be done with erosion w.r.to edge which can be defined as

$$S(H) = H - (\varepsilon^e(X^n)) \tag{15}$$

where $\varepsilon^e(X^n)$ is defined as the collection of hyperedges containing only nodes in X^n. Skeleton operation using erosion related to node is defined as

$$S(H) = H - (\varepsilon^n(X^e)) \tag{16}$$

where $\varepsilon^n(X^e)$ is defined as the collection of nodes in X^n, which are only seen in X and not in the complement of X.

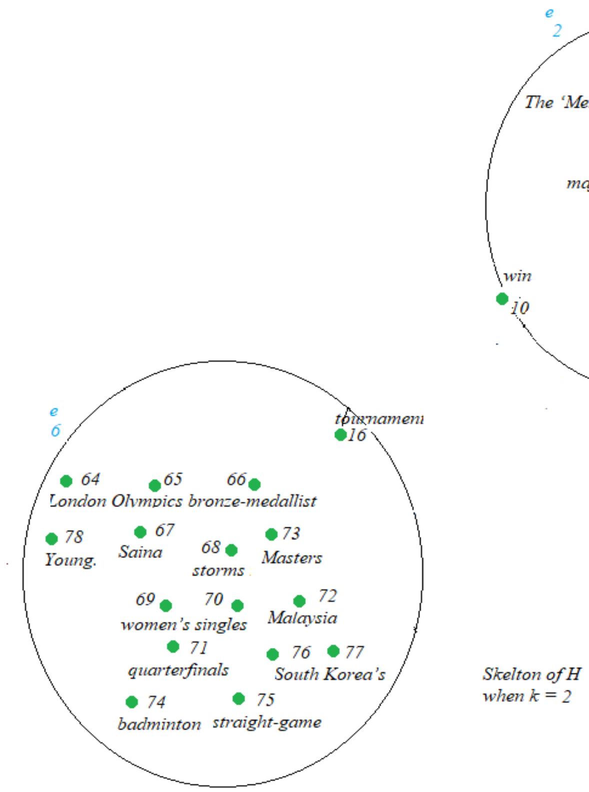

FIGURE 6. Skeleton of H w.r.to edge when k=2

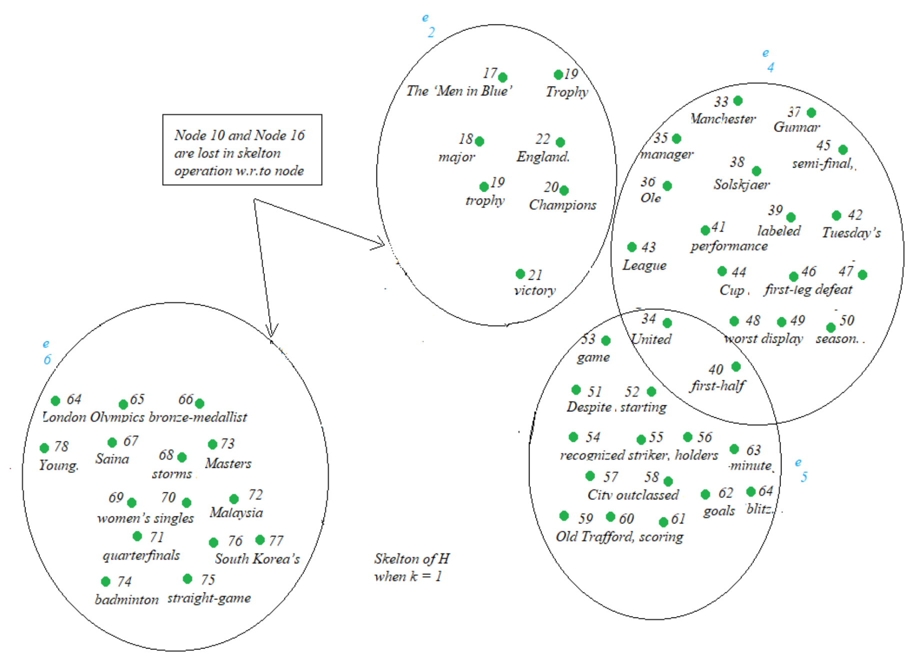

FIGURE 7. Skeleton of H w.r.to node when k=1

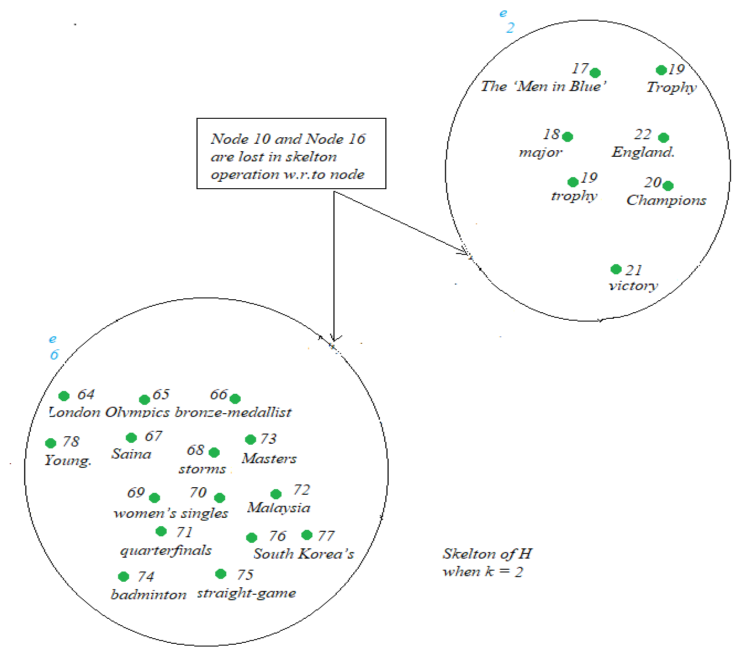

FIGURE 8. Skeleton of H w.r.to node when k=2

2.3. *Hit-or-miss algorithm w.r.to dilation*

Consider the text related to *Sachin Tendulkar*. He is there in the field of *cricket*, *football* and *politics*. Let us take the set of words related to *sachin* and *cricket* as n_{sc}, the set of words related to *sachin* and *football* as n_{sf} and the words related to *sachin* and *politics* as n_{sp}. Here the word with highest priority is MP(Member of Parliament) which comes with in n_{sp}. Now set $A = n_{sc} \cup n_{sf} \cup n_{sp}$. Now let us a take a window W of n_{sp} which is the neighbourhood of n_{sp} obtained as $\delta^e(n_{sp})$. This can be defined as the set of social service and charity activities done by *sachin* while he is an MP. The hypergraph for the above can be shown in the Figure 9.

Here $A = n_{sc} \cup n_{sf} \cup n_{sp}$. which is shown in Figure 9. Let X = Text related to *sachin* while he is an MP. Here MP is the node with the highest priority. Let it be named as p_{high}. Now $X = n_{sp}$. Let W, be the window of X as shown in Figure 9. The hit-or-miss operation of the hypergraph is defined as

$$HM(H) = (A\varepsilon X) \cap (A'\varepsilon(W - X)) \tag{17}$$

and the same is shown in Figure 10 and the method is shown in Algorithm 2.

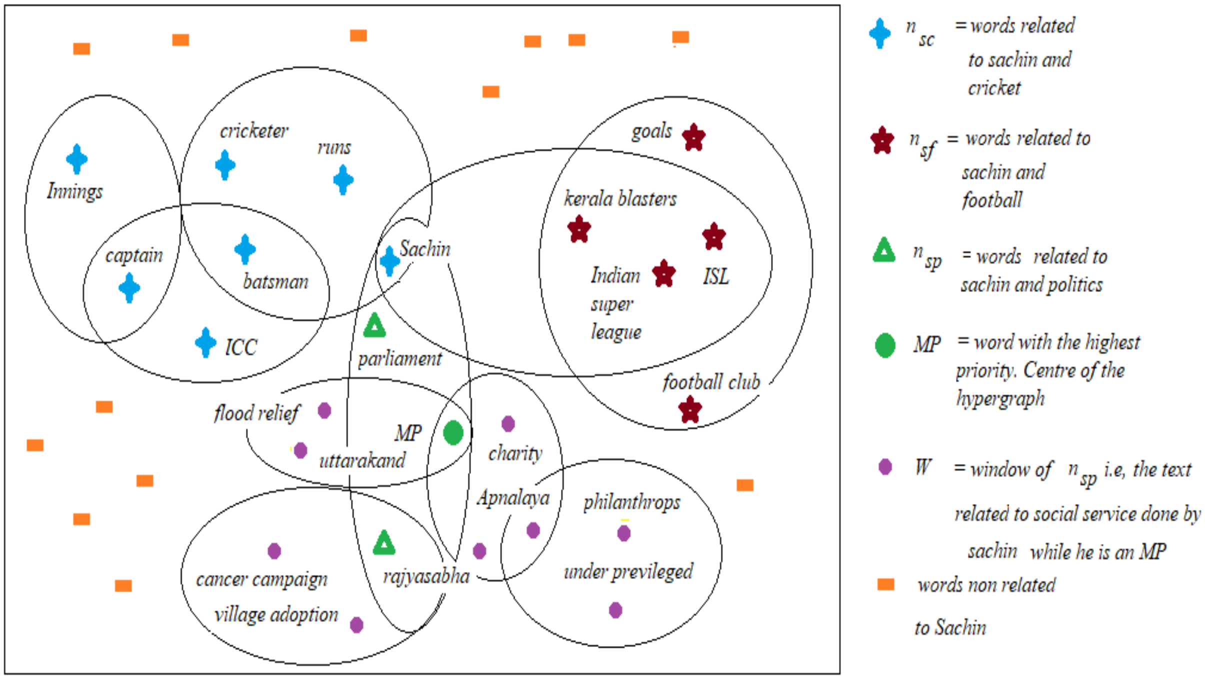

FIGURE 9. Parent Hypergraph H of text, which contains text related to *sachin*

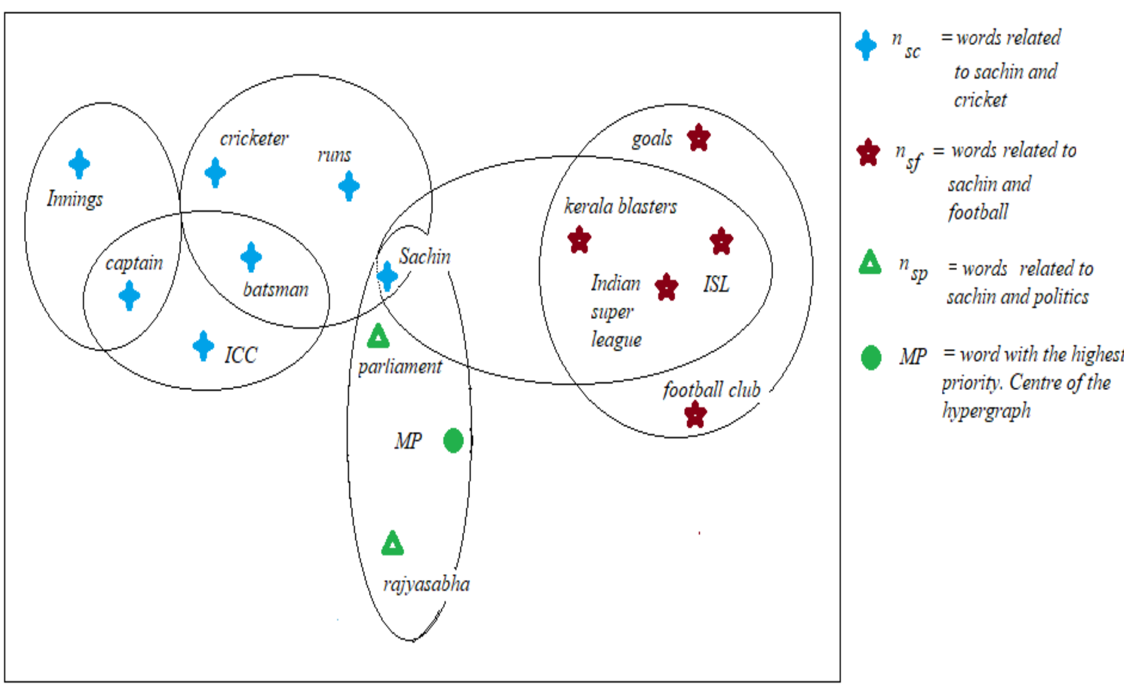

FIGURE 10. Text hypergraph $A = n_{sc} \cup n_{sf} \cup n_{sp}$ related to *sachin*

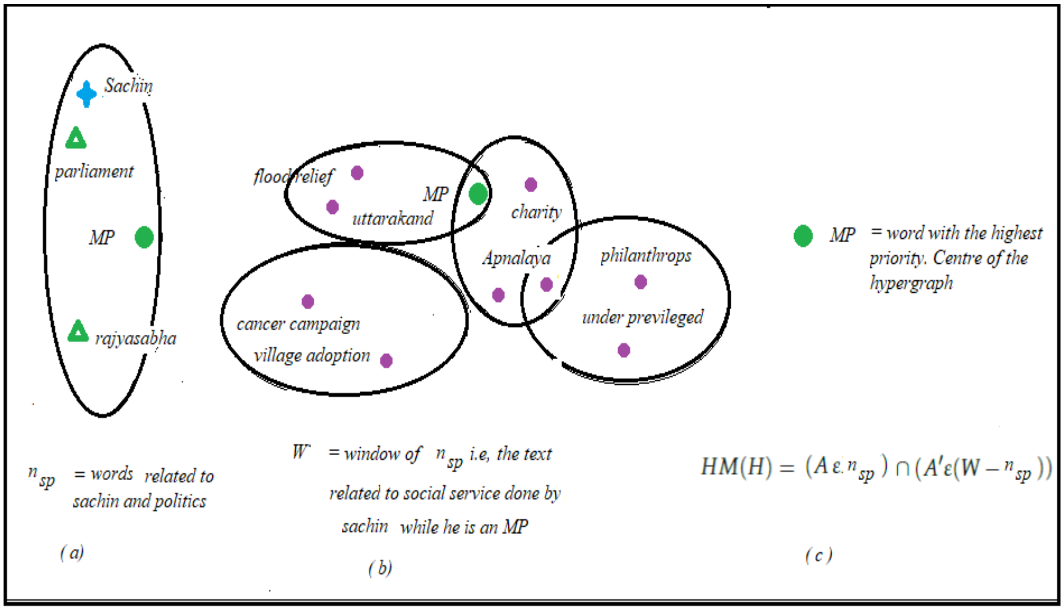

FIGURE 11. Result of hit-or-miss operation on text hypergraph

2.3.1. *Hit-miss-algorithm using dilation and erosion*

Algorithm 2: Hit-or-miss algorithm to find the required information node

Data: Text τ

Result: Hit node $H_\tau t_n$

Create a text hypergraph H_τ as given in Figure 4.;

$i = 1$;

Create sub-hypergraphs n_i, such that node p is common;

Let $A = \cup_{i=1}^{m} n_i$;

Let p_{high} be the node which is the origin of the sub-hypergraph where the node priority > 0.9;

repeat

  Find $A \ \varepsilon \ n_i$;

  Calculate the neighbourhood window $W_i = \delta^e(n_i)$;

  Obtain $W_i - n_i$;

  Compute $A' \ \varepsilon \ (W_i - n_i)$;

  Derive hit node $H_\tau t_n = (\mathrm{A} \ \varepsilon \ \mathrm{n}_i) \cap (A' \varepsilon \ (W_i - n_i))$;

until $i = m$ *or* $H_\tau t_n = p_{high}$;

3. Thinning and Thickening

3.1. *Thinning using hypergraph operations*

Thinning operation can be applied to a hypergraph H_τ by taking sub-hypergraph A and taking a hit node $H_\tau t_n$. As per Figure 9, hypergraph H_τ is the hypergraph related to *sports*, and sub-hypergraph A is the text related to *Sachin*. Dilation with respect to the edge is done for the hit node as $\delta^e(H_\tau t_n)$. All those edges obtained as part of this dilation are removed from the hypergraph A. The algorithm for the same is shown in Algorithm 3. The result of the thinning operation with respect to hit node MP is given in Figure 12. Hit nodes can be varied and thinning can be repeatedly done. Thinning with respect to the hit node *Sachin* is given in Figure 13.

Algorithm 3: Thinning algorithm on a text hypergraph

Data: Text τ and hit nodes $H_\tau t_k$; where $k = 1$ to q

Result: Sub-hypergraph $T^k(H_\tau)$ after thinning

Create a text hypergraph H_τ with the text τ as given in Figure 4.;

$i = 1$;

Create sub-hypergraphs n_i, such that node p is common;

Let $A = \cup_{i=1}^{m} n_i$;

Let p_{high} be the node which is the origin of the sub-hypergraph where the node priority > 0.9;

repeat

  Find $A(n_i) = A \ \varepsilon \ n_i$;

  Calculate the neighbourhood window $W_i = \delta^e(n_i)$;

  Obtain $B_i = W_i - n_i$;

  Compute $A(B_i) = A' \ \varepsilon \ (W_i - n_i)$;

until $i = m$;

$k = 1$;

$T^1(H_\tau) = H_\tau$;

repeat

  $i = 1$;

  repeat

    Derive hit node $H_\tau t_k = A(n_i) \cap A(B_i)$;

    $i = i + 1$;

  until $i = m$ *or* $H_\tau t_k = p_{high}$;

  Derive sub-hypergraph $T^k(H_\tau) = T^{k-1}(H_\tau) - \delta^e(H_\tau t_k)$;

  $k = k + 1$;

until $k = q$ *or* $T^k(H_\tau) = T^{k-1}(H_\tau)$;

;

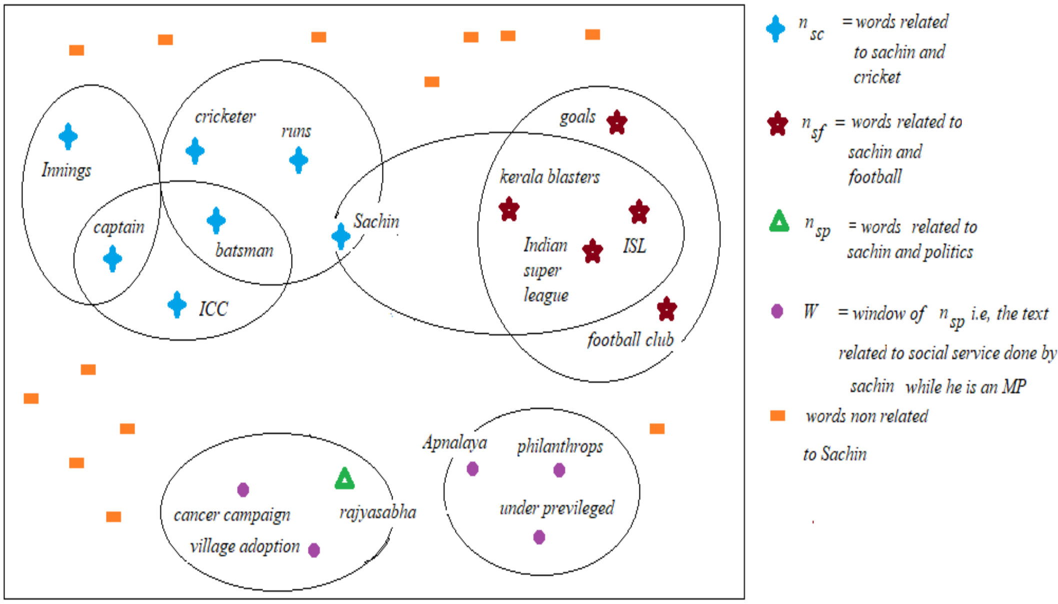

FIGURE 12. Result of Thinning operation on text hypergraph, when hit node $= MP$

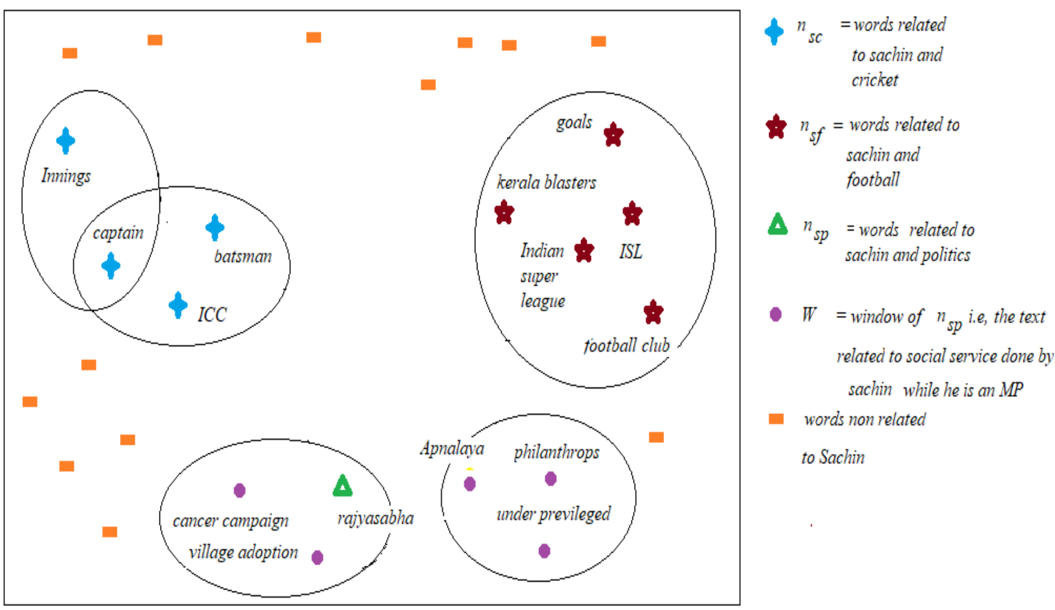

FIGURE 13. Result of Thinning operation on text hypergraph, when hit node $= Sachin$

3.2. *Thickening operation of text hypergraph using dilation*

Given the parent neutrosophic hypergraph H, find A which is the sub-hypergraph of more truth value. Thickening is done by taking the complement of A and applying its thinning. After thinning A', take its complement to get a thickening of A. The result of thinning of A',

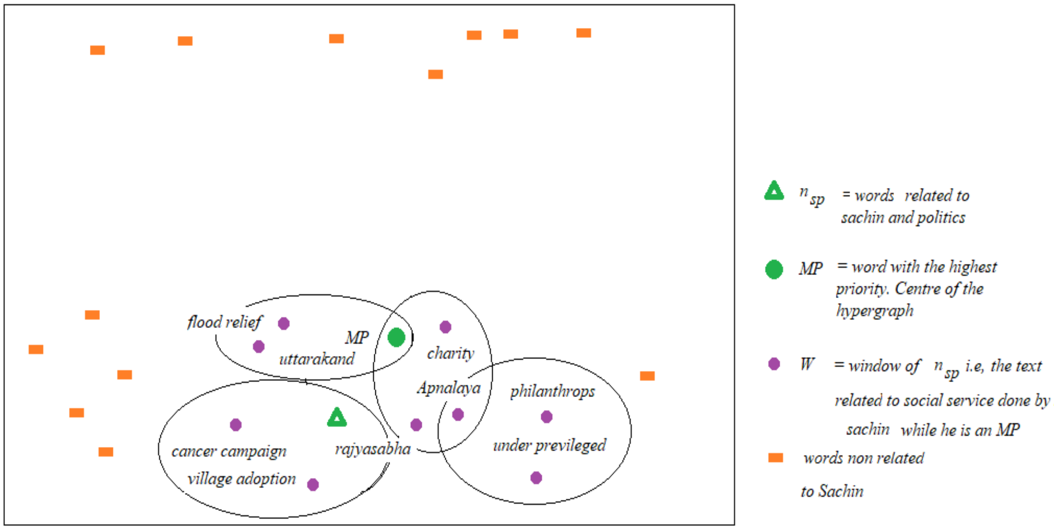

Figure 14. Complement of A with respect to Figure. 9.

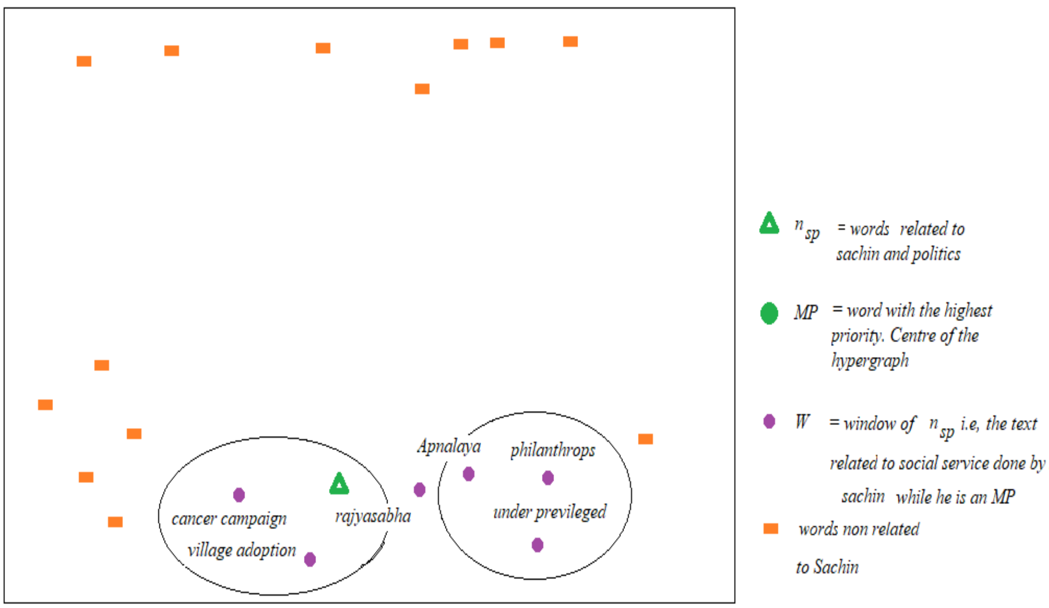

Figure 15. Thinning of A'

when hit node $= MP$ is given in Figure 15. The result of thickening of A, by thinning A' and eliminating disconnected components is given in Figure 16. The algorithm for the same is shown in Algorithm 4.

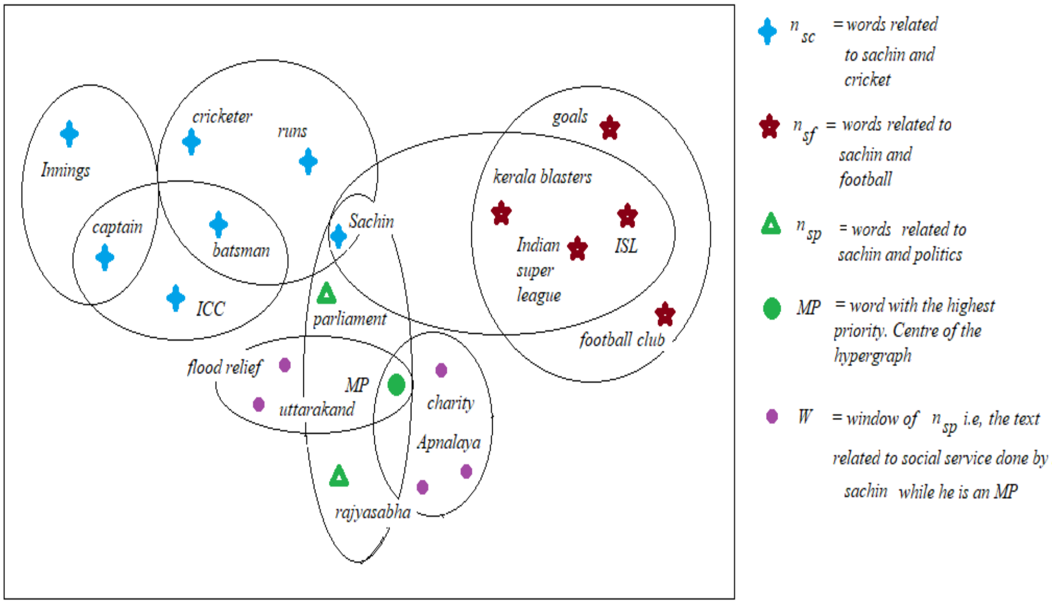

FIGURE 16. Thickening of A

3.3. *Algebra of morphological operators*

Proposition 1: Let H_1 and H_2 be the neutrosophic sub-hypergraphs, then

$$S(H_1 \cup H_2) = S(H_1) \cup S(H_2) \tag{18}$$

Proof: Let $e \in S(H_1 \cup H_2)$, i.e., $e \in (H_1 \cup H_2) - (\delta^e(X^n))^k$ i.e., $e \in (H_1 \cup H_2)$ and $e \notin (\delta^e(X^n))^k$ i.e., $e \in H_1$ and $e \notin (\delta^e(X^n))^k$ or $e \in H_2$ and $e \notin (\delta^e(X^n))^k$ i.e.,$e \in (H_1 - (\delta^e(X^n))^k$ or $e \in (H_2 - (\delta^e(X^n))^k$ i.e., $e \in S(H_1) \cup S(H_2)$. Therefore $S(H_1 \cup H_2) = S(H_1) \cup S(H_2)$.

Proposition 2: Let H_1 and H_2 be the neutrosophic sub hypergraphs, then

$$S(H_1 \cap H_2) = S(H_1) \cap S(H_2) \tag{19}$$

Proof: Let $e \in S(H_1 \cap H_2)$, i.e., $e \in (H_1 \cap H_2) - (\delta^e(X^n))^k$ i.e., $e \in (H_1 \cap H_2)$ and $e \notin (\delta^e(X^n))^k$ i.e., $e \in H_1$ and $e \notin (\delta^e(X^n))^k$ also $e \in H_2$ and $e \notin (\delta^e(X^n))^k$ i.e.,$e \in (H_1 - (\delta^e(X^n))^k$ also $e \in (H_2 - (\delta^e(X^n))^k$ i.e., $e \in S(H_1) \cap S(H_2)$. Therefore $S(H_1 \cap H_2) = S(H_1) \cap S(H_2)$.

Definition: Let neutrosophic hypergraph be H ,sub hypergraph of H be X, $S(H)$ be the skeleton of H obtained as per e.q.(14), then a dilated skeleton $\delta^e(S(H))$ is defined as

$$\delta^e(S(H)) = \{e/e \in N(S(H)); n/n \in e\} \tag{20}$$

where $N(S(H))$ is the neighbourhood of $S(H)$.

Proposition 3: Let $S(H_1)$ be the skeleton of H_1, $S(H_2)$ be the skeleton of H_2, where H_1 and H_2 be the sub hypergraphs of H, then

$$\delta^e(S(H_1) \cup S(H_2)) = \delta^e(S(H_1)) \cup \delta^e(S(H_2)) \tag{21}$$

Algorithm 4: Thickening algorithm on a text hypergraph

Data: Text τ and hit nodes $H_\tau t_k$; where $k = 1$ to q

Result: Sub-hypergraph $Th^k(H_\tau)$ after thickening

Create a text hypergraph H_τ as given in Figure 4.;

$i = 1$;

Create sub-hypergraphs n_i, such that node p is common;

Let $A = \cup_{i=1}^{m} n_i$;

Let p_{low} be the node which is the origin of the A' where the node priority < 0.2;

Create sub-hypergraphs x_i in A' where p_{low} is present.

repeat

  Find $A'(x_i) = A' \; \varepsilon \; x_i$;

  Calculate the neighbourhood window $W_i = \delta^e(x_i)$;

  Obtain $B_i = W_i - x_i$;

  Compute $A'(B_i) = A \; \varepsilon \; (W_i - x_i)$;

until $i = m$;

$k = 1$;

$T^1(H_\tau) = H_\tau$;

repeat

  $i = 1$;

  repeat

    Derive hit node $H_\tau t_k = A'(x_i) \cap A'(B_i)$;

    $i = i + 1$;

  until $i = m$ *or* $H_\tau t_k = p_{low}$;

  Derive sub-hypergraph $T^k(H_\tau) = T^{k-1}(H_\tau) - \delta^e(H_\tau t_k)$;

  $k = k + 1$;

until $k = q$ *or* $T^k(H_\tau) \; = T^{k-1}(H_\tau)$;

;

Find $Th^k(H_\tau) = H_\tau - T^k(H_\tau)$

Proof: According to the definition of dilated skeleton, $\delta^e(S(H_1) \cup S(H_2))$ can be written as $\{e/e \in N(S(H_1)); n/n \in e\}$ or $\{e/e \in N(S(H_2)); n/n \in e\}$
$= \delta^e(S(H_1)) \cup \delta^e(S(H_2))$
similarly we can write $\delta^e(S(H_1) \cap S(H_2)) = \delta^e(S(H_1)) \cap \delta^e(S(H_2))$

Proposition 4: Let $S(H_1)$ be the skeleton of H_1 which is the sub hypergraph H, $S(H_2)$ be the skeleton of H_2 which is sub hypergraph of H, then De morgan's law

$$(S(H_1) \cup S(H_2))^c = (S(H_1))^c \cap (S(H_2))^c \tag{22}$$

holds here

Proof: Let $e \in (S(H_1) \cup S(H_2))^c$. i.e., $e \notin (S(H_1) \cup S(H_2))$. i.e., $e \notin (H_1 - (\delta^e(X^n))^k) \cup (H_2 - (\delta^e(X^n))^k)$. i.e., $e \notin (H_1 - (\delta^e(X^n))^k)$ or $e \notin (H_2 - (\delta^e(X^n))^k)$ i.e., $e \in (H_1 - (\delta^e(X^n))^k)^c$ or $e \in (H_2 - (\delta^e(X^n))^k)^c$. i.e., $e \in (S(H_1))^c$ and $e \in (S(H_2))^c$

Theorem 5: Let a neutrosophic hypergraph be represented using H, $S(H)$ be the skeleton of H, H_1,H_2 be two sub hypergraphs of the neutrosophic hypergraph H then

$$S(S(H)) = S(H) \tag{23}$$

$$S(H_1) \cup S(H_2) = S(H_2) \cup S(H_1) \tag{24}$$

$$S(H_1) \cup (S(H_2) \cap S(H_3)) = (S(H_1) \cup S(H_2)) \cap (S(H_1) \cup S(H_2)) \tag{25}$$

E.q(18) to E.q(25) give a clear picture of the algebra of skeleton operation.

4. Applications

There are many applications in the field of text analysis using the various operations discussed so far namely thinning, thickening, skeltoning, hit-or-miss operation etc. In this paper, we have applied it in identifying the hate speech in a text and removing it. The system architecture is shown in Figure 17, where the input text is subjected to preprocessing like splitting into sentences and sentences further into words. Stop words are removed from the set of words as they do not contribute to the meaning of the sentence. A neutrosophic hypergraph is constructed out of this by modeling sentences as edges and words as nodes. Lukasiewicz's fuzzy implication is applied as given in Figure 18 and Algorithm 5.

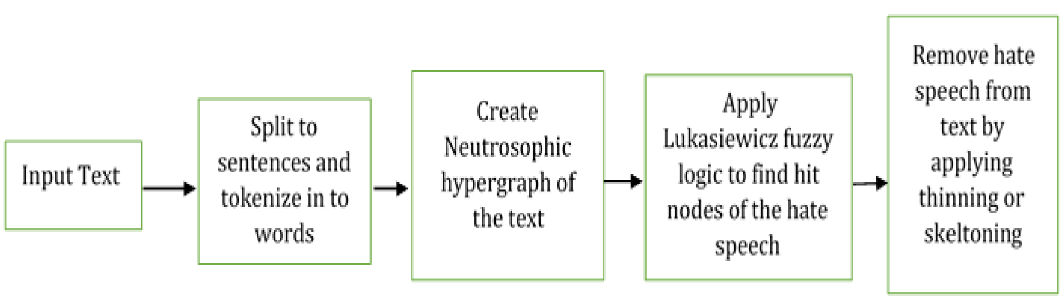

FIGURE 17. System architecture

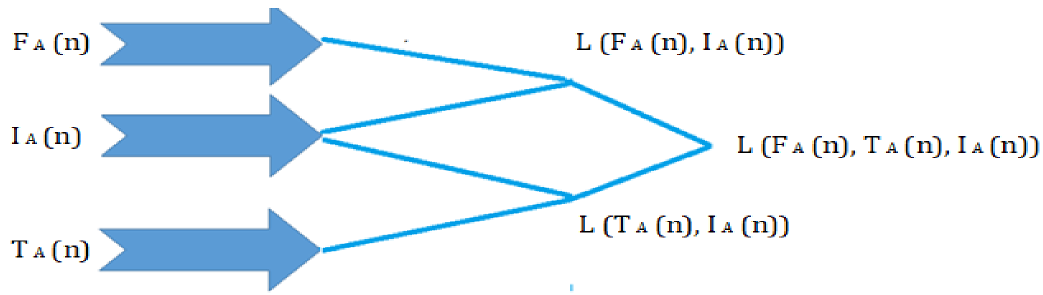

FIGURE 18. Lukasiewicz implication for hatred

Algorithm 5: Algorithm 5: Hate speech detection

Data: Hate Speech Detection Method using fuzzy neutrosophic hypergraph

Result: Tweets devoid of hate speech

1. Nodes and edges of the hypergraph represent the words and sentences of the document. Weights are assigned as given in section 1.2. $T_A(n)$ denotes hatred measure of a word, $I_A(n)$ denotes the uncertainty and $F_A(n)$ shows the perfectness measure;
2. Now Lukasiewicz's implication is applied to these measures as shown in Figure 17.
   - Case 1: [0,0,1] – Not a hate word. The Lukasiewicz implication would be
     $f \implies (F_A(n), I_A(n)) = min\{1, 1 - F_A(n), I_A(n)\} = min\{1, 1 - 1 + 0\} = 0$
     $f \implies (T_A(n), I_A(n)) = min\{1, 1 - T_A(n), I_A(n)\} = min\{1, 1 - 0 + 0\} = 1$
     $f \implies (F_A(n), T_A(n), I_A(n)) =$
     $min\{1, 1 - f \implies (T_A(n), I_A(n)) + f \implies (F_A(n), I_A(n))\} = min\{1, 1 - 1 + 0\} = 0$
   - Case 2: [1,0,0] – Definitely, it is a hate word. The Lukasiewicz implication for this case would be $f \implies (F_A(n), I_A(n)) = min\{1, 1 - F_A(n), I_A(n)\} =$
     $min\{1, 1 - 0 + 0\} = 1$
     $f \implies (T_A(n), I_A(n)) = min\{1, 1 - T_A(n), I_A(n)\} = min\{1, 1 - 1 + 0\} = 0$
     $f \implies (F_A(n), T_A(n), I_A(n)) = min\{1, 1 - 0 + 1\} = 1$
   - Case 3: [1, 0.5,0] – Depends on circumstances even though a Hate word.
     $f \implies (F_A(n), I_A(n)) = min\{1, 1 - F_A(n), I_A(n)\} = min\{1, 1 - 0 + 1\} = 1$
     $f \implies (T_A(n), I_A(n)) = min\{1, 1 - T_A(n), I_A(n)\} = min\{1, 1 - 0.5 + 1\} = 1$
     $f \implies (F_A(n), T_A(n), I_A(n)) = min\{1, 1 - 1 + 1\} = 1$
   - Case 4: [0.5, 1, 0] – High indeterminacy, can be a hate word.
     $f \implies (F_A(n), I_A(n)) = min\{1, 1 - F_A(n), I_A(n)\} = min\{1, 1 - 0 + 0.5\} = 1$
     $f \implies (T_A(n), I_A(n)) = min\{1, 1 - T_A(n), I_A(n)\} = min\{1, 1 - 1 + 0.5\} = 0.5$
     $f \implies (F_A(n), T_A(n), I_A(n)) = min\{1, 1 - 0.5 + 1\} = 1$
   - Case 5: [0, 0.5, 1] – Depends on circumstances even though a non-hate word.
     $f \implies (F_A(n), I_A(n)) = min\{1, 1 - F_A(n), I_A(n)\} = min\{1, 1 - 1 + 0.5\} = 0.5$
     $f \implies (T_A(n), I_A(n)) = min\{1, 1 - T_A(n), I_A(n)\} = min\{1, 1 - 0 + 0.5\} = 1$
     $f \implies (F_A(n), T_A(n), I_A(n)) = min\{1, 1 - 1 + 0.5\} = 0.5$

Algorithm 6: Hate speech detection .....continuation of Algorithm 5

3. Assign for each edge e in H^e, $T_A(e)[0,1]$, $I_A(e)[0,1]$, $F_A(e)[0,1]$ and $T_A(e) + I_A(e) + F_A(e) <= 3.$;
4. T$_A(e)$ is as per e.q(1) and $I_A(e), F_A(e)$ is given by

$$I_A(e) = avg(I_A(n)); \forall n \in e \tag{26}$$

$$F_A(e) = avg(F_A(n)); \forall n \in e \tag{27}$$

5. Create sub-hypergraph X by applying (α, β, γ) cut such that $T_A(e) >= 0.5$, $I_A(e) >= 0.3$ and $F_A(e) >= 0$;
6. Create a sub-hypergraph A by applying higher level (α, β, γ) cut such that $T_A(e) >= 0.8$, $I_A(e) >= 0.3$, $F_A(e) >= 0$;
7. Apply the morphological operations on H with X
   - $\delta^e(X^n)$ – dilation of X, pertaining edges.This takes all words in X and fetches all sentences that contain minimum of one such word.
   - $\delta^n(X^e)$ – dilation of X pertaining to nodes. This operation takes all sentences in X and retrieves all words in those sentences.
   - $\varepsilon^e(X^n)$ – is an erosion of X pertaining to edges. This operation takes all words in X and retrieves all sentences that contain X^n only.
   - $\varepsilon^n(X^e)$ – is erosion of X pertaining to nodes. This retrieves all words in X and not in X'.
8. Hate speech can be removed in two ways as follows:-
   - Apply skeleton operation $S(H) = H{-}(\delta^e(X^n))^k$. Here hate speech is eliminated from the tweets.
   - Implement Thinning
     - Obtain $Hit - or - Miss(H, A) = A\varepsilon X \cap A'\varepsilon(W{-}X)$ where X is dilated to get W. This operation generates intense hate words.
     - Obtain $H - \delta^e(Hit - or - Miss(H, A))$
9. The sentences obtained after step 8 give tweets without hate speech.

A variation of this method without Lukasiewicz implication is seen in [28]. As per the system architecture shown in Figure 17, Twitter tweets are collected using Twitter APIs, text cleaning is done to remove irrelevant information such as URLs, emojis, hashtags, and punctuation marks. After preprocessing, tokenization is applied to split into words and stop word removal is done. Once the words are separated and stop words are removed, as mentioned in the above algorithm, words are given three membership values namely indeterminacy, truth and falsity. Sentences are also assigned with these three membership values. The truth value of a sentence will be the maximum truth value of the words in it. The indeterminacy value of the sentence will be the average of the indeterminacy values of all the words in it. The falsity value of the sentence will be the average of the falsity values of all the words in it. Once a neutrosophic hypergraph(H) is created with these three values for the edges and nodes an alpha, beta, and gamma cut is applied to it to create a sub-hypergraph(X) which retrieves the sentences which are more likely to have hate speech. Morphological operations namely erosion and dilation are applied with this X on H which gives various query results as mentioned in the algorithm. Applying dilation k times with X and subtracting it from H will result in a skeleton of tweets devoid of hate speech. Hit-or-miss operation is also applied which results in retrieval of most hate words. Applying dilation of these words and subtracting it from H gives thinning.

5. Result Analysis

The system is implemented using Python. The data set used in this system is Twitter data(tweets) from which the hate tweets are identified and removed. Results are analyzed using various measures namely:-

- t_p = true positives = Number of tweets which actually consist of hatred words and are classified as hate tweets.
- t_n = true negatives = Number of tweets that do not consist of hatred words and are classified as non-hate tweets.
- f_p = false positives = Number of tweets which are actually non-hate tweets but classified as hate tweets.
- f_n = false negatives = Number of tweets which are actually hate tweets but classified as non-hate tweets.

Further, using the above values we calculate the measures like recall, miss rate, false positive rate, true negative rate, false omission rate, positive predictive value, negative likelihood ratio, negative predictive value, positive likelihood ratio, false discovery rate, accuracy etc. According to our proposed system, recall or sensitivity is the ratio of hate sentences identified by the system to the total hate sentences in the input data set. Our system has shown a better value of 0.87. Precision or specificity is the ratio of non-hate sentences identified by the system to the total number of non-hate sentences in the data set, where our system reported 89% results. The false positive ratio is the ratio between the number of non-hate sentences wrongly identified as hate sentences and the count of non-hate sentences. The system showed a pretty less false positive rate of 0.11. Positive Predictive Value (92%) shows how many are hate out of hate sentences identified by the system. Similarly, other values are also calculated and tabulated in Table 3. Data set 1 is of size 500, Data set 2 is of size 1000 and Data set 3 is of size 5000.

Table 3. Result Analysis of the proposed system

Parameter	Data set 1	Data set 2	Data set 3
t_p	0.83	0.95	0.87
t_n	0.88	0.75	0.92
f_p	0.13	0.09	0.03
f_n	0.15	0.11	0.42
Recall	0.85	0.89	0.87
Precision	0.87	0.89	0.92
Miss rate(FNR)	0.153	0.103	0.132
False Positive Rate	0.129	0.107	0.083
True Negative Rate	0.871	0.893	0.92
Positive Predictive Value	0.865	0.913	0.97
False Omission Rate	0.145	0.128	0.313
LR+	6.59	8.09	10.8
LR-	0.024	0.124	0.054
Accuracy	0.86	0.898	0.88
False Discovery Rate	0.135	0.098	0.029
Negative Predictive Value	0.854	0.87	0.69

6. Conclusions

In this work, we have done a detailed study of various neutrosophic morphological operators like hit-or-miss, thickening, thinning, skeleton etc. This a novel method of representing text as a neutrosophic hypergraph and Illustration of these operators on it. Also, their algorithms are implemented with text as input. As an application of the proposed work, we have applied it to hate speech detection in Twitter tweets and got an accuracy of 88%. It is observed that various compositions of neutrosophic morphological operators may give various results of text analysis. Such a study is very useful for categorizing text with respect to key information provided to the system. This is a novel method for extracting relevant information from text or a document. It is possible to extend the work by analyzing various properties of neutrosophic hypergraphs. Neutrosophic logic has a very important part in the construction of inference systems where connectors like Sheffors and Pierce's connectors may be useful. Since optimality is a major concern in every problem, constructing operators that satisfy various optimality conditions is a future work. Such new operators can be used for the comparison of various data sets and multi-classification of extracted information. This work can be extended to the area of proper fertilizer applications in the area of agriculture, team selection in sports, educational admission systems, and pandemic spread detection and isolation of people.

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Pentapartitioned Neutrosophic Fuzzy Optimization Method for Multi-objective Reliability Optimization Problem

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Abstract: Fuzzy logic is an important mathematical tool that deals with uncertainty and imprecision in decision-making processes. The prevalent frameworks, known as neutrosophic sets, study the connection of neutralities with various ideational spectra in addition to generalizing concept of fuzzy sets. Using a penta-partitioned neutrosophic fuzzy environment, a novel optimization technique is proposed in this study. Proposed optimization technique is an expansion of fuzzy optimization, intuitionistic fuzzy optimization (IFO), single-valued neutrosophic optimization (NSO) and four valued neutrosophic optimization (FVNO). Here, the neutrosophic set's indeterminacy term is broken down into three components: contradiction (*C*), unknown (*U*), ignorance (*I*). To demonstrate the applicability and effectiveness of the suggested approach a numerical example is solved and the outcomes are contrasted with those of other methods already in use by cumulative percentage gap and sum of optimal values. Finally, a multi objective reliability optimization model of LCD display unit is solved by this method.

Keywords: Reliability; Neutrosophic set; Neutrosophic optimization; Pentapartitioned neutrosophic optimization.

1.Introduction

Recent years have seen a rise in interest in the topic of reliability optimization, which aims to enhance the performance and dependability of complex systems. Reliability optimization involves making decisions regarding system design, maintenance, and resource allocation to improve the system's ability to function effectively and consistently in various operating conditions. However, traditional reliability optimization approaches often encounter challenges when dealing with multiple conflicting objectives, such as maximizing system performance while minimizing costs or minimizing failure rates while maximizing system availability. In literature reliability optimization models are solved using various exact, heuristic and metaheuristic methods. For example, Misra [1] described usage of the integer programming technique before introducing [2] the use of the maximum principle and lagrange multipliers to solve reliability optimization problems. Sakawa [3] has presented multi-objective reliability allocation problem utilizing surrogate worth trading strategies to minimize system cost while maximize system reliability. A method using parametric programming was presented by Chern and Jan [4]. W.kuo, V.R.Prsad [5] solved system reliability optimization problem using some heuristic and metaheuristic algorithms Kuo et. al [6] presented some fundamental method and its application by solving reliability optimization model.

Uncertainty and ambiguity are very practical issue in real life mathematical problems. To address these challenges, researchers have explored the application of advanced mathematical tools and techniques, including fuzzy sets and optimization methods, to multi-objective reliability optimization problems. In 1965, Zadeh [7] invented the fuzzy set (FS). Fuzzy sets deals with one membership value in [0,1] ,but sometimes uncertainty is not properly expressed by single membership value. So, considering the membership as an interval in [0,1] the interval-valued fuzzy set (IVFS) [8] was invented. In some situations, only membership is insufficient to fully convey the uncertainty, so non-membership value also required to clarify the vagueness. That is outside the purview of FS and IVFS. Atanassov [9] first suggested intuitionistic fuzzy sets (IFS) in 1986, expands beyond the scope of IVFS and FS. IFS introduce the notion of considering the total of membership and non-membership values, ensuring that the sum $(\leq)1$.

In 1998, Smarandache [10] introduced neutrosophic sets (NSs) extending from FS, IFS, hesitant fuzzy sets, and IVFS, to handle uncertain information encountered in real-world situations. Neutrosophic sets serve as a valuable mathematical tool for addressing ambiguous and conflicting information. They consist of three independent components: truth, falsity, and indeterminacy membership. However, applying neutrosophic fuzzy sets in practical scenarios presents challenges due to the presence of both standard and non-standard intervals of membership values. To address these challenges, Wang et al. [11, 12] invented single-valued NSs and interval-valued NSs, enabling the application of NSs to real-world problems. In an effort to generalize neutrosophic sets further, F. Smarandache [13] introduced n-Valued neutrosophic logic through categorizing truth, indeterminacy, and falsity into n types. Subsequently, Freen et al. [14] defined four-valued neutrosophic set (FVNS) by refining the indeterminacy term into unknown and contradiction. Expanding on this concept, Mallik and Pramanik [15] introduced the penta-partitioned neutrosophic set, which splits the indeterminacy term into contradiction, unknown, and ignorance.

In a wide range of areas, optimization methods are crucial for addressing a variety of practical problems and decision-making problems. In last few decades fuzzy optimization [16] is very efficient tool as it deals with the ambiguity and uncertainty of real-life problems. Bellman and Zadeh [17] first introduced decisions, goals and constraints in fuzzy. As an extension of this work Zimmerman [18] introduced fuzzy programming method. To deal with the non-membership of an information Angelov [19] invented intuitionistic fuzzy optimization method. It's interesting to observe that there are several optimization problems that require a collection of membership grades rather than a single grade of membership since experts' estimates of the optimization's parameters vary significantly. Considering this a multi objective optimization problem (MOOP) is solved by Bharati [20] in hesitant fuzzy environment. To solve a MOOP, Sarkar et al. [21] applied the multi-objective neutrosophic optimization algorithm. Abdel-Basset et al. [22] introduced neutrosophic goal programming approach. The integer programming problem was proposed by Mohamed, Mai et al. [23] in triangular neutrosophic environment. Group decision-making problem was solved by Abdel-Basset et al. [24] utilizing triangular neutrosophic weighted aggregation operator. In an IVNSs framework, Garg [25] has presented a nonlinear programming method to solve MCDM problems.

This method provides a systematic approach for tackling decision-making challenges in uncertain and ambiguous environments. Recently, Freen et al. [14] proposed FVRNO method to solve MOOP and applied it to car-side impact and riser design problems. Recently, in various field the concept of Penta partitioned neutrosophic graph (PNG) is used to find the optimal path using Penta partitioned neutrosophic set. Quek, Shio Gai, et al. [26] used the concept of PNG to find the safest path of travel and stay to reduce the spread of COVID-19. Broumi, Said, et al [27] used PNG to solve MCDM problem. Das, Suman, et al [28,29] introduced single valued bipolar PNS and its application by solving MADM problem, and author also presented single valued PNG and solution strategy to MCDM problem.

In formation of a system design, reliability optimization is one of the crucial jobs. Finding the most effective way to raise system reliability in limited resource has always been the reliability engineer's main objective. There are several parameters in the MOOP that are constantly vague and ambiguous in nature for ambiguity in decision makers judgments. Fuzzy technique is used to analyze this in MOOP to manage such kind of nature. Fuzzy non-linear programming was utilized by Park [30] for the reliability apportionment problem of series system. Fuzzy global optimization reliability model was utilized by Ravi et al. [31]. To address the reliability optimization problem, Huang [32] suggested a multi-objective fuzzy optimization approach. Later, the intuitionistic fuzzy optimization approach [19] is used in a variety of study areas in reliability optimization problem. Mahapatra et al. [33] used IFO methods to solve reliability optimization model. To address the problem of multi-objective reliability optimization IFO method was applied in interval environments by Garg et al [34]. Islam and Kundu [35] applied NSO technique to solve the reliability optimization of LCD display unit. As far as known to us, there isn't a research paper in the literature that addresses how to solve a MOOP in a pentapartitioned neutrosophic environment.

In this article, a penta-partitioned neutrosophic fuzzy environment is used to suggest a multi-objective optimization technique. To show that the suggested strategy is effective a nonlinear MOOP is solved and the outcomes are compared against those of other techniques already in use. Also, this method is applied to solve the reliability optimization model of LCD display unit and the result is compared with four valued refined optimization method. Remaining part is arranged as: the definition of fuzzy set, its extension and properties are discussed in section 2. The Proposed penta-partitioned neutrosophic fuzzy optimization technique and computational algorithm is explained in section 3. In section 4 a numerical example is solved by developed method. Reliability model of LCD display unit is shown in section 5. In Section 6, results and discussion are presented. Finally, in section 7. conclusion and future works are discussed.

2. **Preliminaries**

Definition 1. Fuzzy set (FS) [7]

E be the universal set, then the FS $\tilde{F}$ on the set E is defined as $\tilde{F} = \{(e, \mu_{\tilde{F}}(e)) \mid e \in E\}$, where $\mu_{\tilde{F}}: E \to [0,1]$ is membership function on E.

Example: Consider set of number $E = \{1,2,3,4,5,6\}$, fuzzy set $\tilde{F}$ is number closed to 4. Then we can define $\tilde{F} = \{(1,0), (2,.2), (3,.6), (4,1), (5,.5), (6,.3)\}$.

Definition 2. Intuitionistic Fuzzy Set (IFS) [9]

E is the universal set, the IFS $\tilde{I}$ on E is the collection of order triplets $\tilde{I} = \{(e, \mu_{\tilde{I}}(e), \vartheta_{\tilde{I}}(e))| \ e \in E\}$, where $\mu_{\tilde{I}}, \vartheta_{\tilde{I}}: E \to [0,1]$ represent membership, non-membership function on E, $0 \leq \mu_{\tilde{I}}(e) + \vartheta_{\tilde{I}}(e) \leq 1$ for all $e \in E$.

Here the function $\pi_{\tilde{I}}(e) = (1 - \mu_{\tilde{I}}(e) - \vartheta_{\tilde{I}}(e))$ is the hesitancy degree for each $e \in E$.

Example:

If a company produce three products $E = \{p_1, p_2, p_3\}$, there be three opinions on these products, "good (membership)", "bad(non-membership)", "no idea (hesitancy)". Then the intuitionistic fuzzy set $\tilde{I} = \{(p_1, .6, .3), (p_2, .7, .25), (p_3, .8, .16)\}$. Here $\pi_{\tilde{I}}(p_1) = .1, \pi_{\tilde{I}}(p_2) = .05, \pi_{\tilde{I}}(p_1) = .04$.

Definition 3. Neutrosophic fuzzy set (NSs) [10]

E is the universal set. NSs on E is $\widetilde{N} = \{(e, T_{\widetilde{N}}(e), I_{\widetilde{N}}(e), F_{\widetilde{N}}(e))| \ e \in E\}$, here $T_{\widetilde{N}}(e), I_{\widetilde{N}}(e), F_{\widetilde{N}}(e)$ are subsets of $]0^-, 1^+[$ which represent truth, indeterminacy and falsity membership on E and $0^- \leq SupT_{\widetilde{N}}(e) + SupI_{\widetilde{N}}(e) + SupF_{\widetilde{N}}(e) \leq 3^+$ for all $e \in E$. In real life, the application of NS is difficult because the membership values are subsets of $]0^-, 1^+[$.

Definition 4. Single valued neutrosophic set (SVNSs) [11]

In SVNSs, for all $e \in E$ (universal set) the set $\widetilde{SN}$ is characterized by $T_{\widetilde{SN}}(e), I_{\widetilde{SN}}(e), F_{\widetilde{SN}}(e)$, each takes single value in $[0,1]$, $0 \leq T_{\widetilde{SN}}(e) + I_{\widetilde{SN}}(e) + F_{\widetilde{SN}}(e) \leq 3$.Where,

$$\widetilde{SN} = \{(e, T_{\widetilde{SN}}(e), I_{\widetilde{SN}}(e), F_{\widetilde{SN}}(e)): e \in E\}.$$

Example: Suppose a phone company launch a phone, customers may review the phone on the basis of $E = \{e_1 = price, e_2 = ram \ \ and \ \ storage, e_3 = battery \ \ service, e_4 = camera\}$. The customers opinion on each criterion be positive (truth degree), Indeterminate, negative (falsity degree). Then the set $\widetilde{SN}$ on E as : $\widetilde{SN} = \{(e_1, .7, .5, .4), (e_2, .5, .6, .3), (e_3, .3, .4, .8), (e_4, .8, .3, .4)\}$.

Definition 5. Four-valued neutrosophic set (FVNS) [14]

By splitting indeterminacy in two ways, there are two types of FVNS. For one of such FVNS, indeterminacy is split into unknown (U) and contradiction (C), where $C = T \wedge F$. The values of T, U, C and F are the function from E to $[0,1]$ and $0 \leq T_{\widetilde{FN}}(e) + U_{\widetilde{FN}}(e) + C_{\widetilde{FN}}(e) + F_{\widetilde{FN}}(e) \leq 4$. Thus, this type of FVNS is

$$\widetilde{FN} = \{(e, T_{\widetilde{FN}}(e), I_{\widetilde{FN}}(e), C_{\widetilde{FN}}(e) F_{\widetilde{FN}}(e)): e \in E\}.$$

For another type of FVNS, here the indeterminacy split into two parts, Ignorance (G) and contradiction (C), where $C = T \wedge F$ and $G = T \vee F$. The values of T, G, C and F are the function from E to $[0,1]$ and $0 \leq T_{\widetilde{FN}}(e) + C_{\widetilde{FN}}(e) + G_{\widetilde{FN}}(e) + F_{\widetilde{FN}}(e) \leq 4$. Thus, this type of FVNS is

$$\widetilde{FN} = \{(e, T_{\widetilde{FN}}(e), C_{\widetilde{FN}}(e), G_{\widetilde{FN}}(e) F_{\widetilde{FN}}(e)): e \in E\}.$$

Example of FVNS:

Consider a criterion set $E = \{e_1, e_2, e_3, e_4\}$. There be four types of opinion for each criterion, such as "truth, contradiction, unknown, falsity" or "truth, ignorance, contradiction, falsity", where each degree in $[0,1]$. Then we can construct FVNS as

$$X = \frac{\langle 0.6, 0.3, 0.5, 0.4 \rangle}{e_1} + \frac{\langle 0.5, 0.3, 0.7, 0.4 \rangle}{e_2} + \frac{\langle 0.7, 0.3, 0.4, 0.2 \rangle}{e_3} + \frac{\langle 0.8, 0.3, 0.2, 0.1 \rangle}{e_4}$$

Definition 6: Penta-partitioned neutrosophic set (PNS) [15]

PNS was defined by Rama Mallick and Surapati Pramanik using the concepts of n-valued neutrosophic set. Here indeterminacy divided into ignorance, contradiction, and unknown (U, G, C). This is how PNS is defined:

E be a universal set. PNS, $\widetilde{PN}$ over E is the combination of Truth($T_{\widetilde{PN}}$), unknown($U_{\widetilde{PN}}$),ignorance ($G_{\widetilde{PN}}$), contradiction($C_{\widetilde{PN}}$), falsity($F_{\widetilde{PN}}$) memberships which are in [0,1]for all $e \in E$ and

$0 \leq T_{\widetilde{PN}}(e) + C_{\widetilde{PN}}(e) + G_{\widetilde{PN}}(e) + U_{\widetilde{PN}}(e) + F_{\widetilde{PN}}(e) \leq 5$.

2.1 ***Basic properties***

Definition 7. [15] P_1 and P_2 be two PNSs over E then $P_1 \subseteq P_2$ iff $T_{P_1}(e) \leq T_{P_2}(e), C_{P_1}(e) \leq C_{P_2}(e), G_{P_1}(e) \geq G_{P_2}(e), U_{P_1}(e) \geq U_{P_2}(e)$ and $F_{P_1}(e) \geq F_{P_2}(e)$ for all $e \in E$.

Definition 8. [15] The complement of PNS P is denoted by P^c and is defined by:

$P = \{(T_P(e), C_P(e), G_P(e), U_P(e), F_P(e)) | e \in E\}$, then

$$P^c = \{(F_P(e), U_P(e), 1 - G_P(e), C_P(e), T_P(e)) \mid e \in E\}$$

i.e, $T_{P^c}(e) = F_P(e), C_{P^c}(e) = U_P(e), G_{P^c}(e) = 1 - G_P(e), U_{P^c}(e) = C_P(e), F_{P^c}(e) = T_P(e)$ for all $e \in E$

Definition 9. [15] P_1 and P_2 be two PNSs. Then $P_1 \cup P_2$ and $P_1 \cap P_2$ is defined by:

$$P_1 \cup P_2 = \left\{ \left(\begin{matrix} \max\left(T_{P_1}(e), T_{P_2}(e)\right), \max\left(C_{P_1}(e), C_{P_2}(e)\right), \min\left(G_{P_1}(e), G_{P_2}(e)\right), \\ \min\left(U_{P_1}(e), U_{P_2}(e)\right), \min\left(F_{P_1}(e), F_{P_2}(e)\right) \end{matrix} \right) \middle| e \in E \right\}$$

$$P_1 \cap P_2 = \left\{ \left(\begin{matrix} \min\left(T_{P_1}(e), T_{P_2}(e)\right), \min\left(C_{P_1}(e), C_{P_2}(e)\right), \max\left(G_{P_1}(e), G_{P_2}(e)\right), \\ \max\left(U_{P_1}(e), U_{P_2}(e)\right), \max\left(F_{P_1}(e), F_{P_2}(e)\right) \end{matrix} \right) \middle| e \in E \right\}$$

2.2. ***Example of PNS:***

Suppose a company have manufactured a car. The quality of the car is determined by some domain experts over the set of criterions $E = \{e_1, e_2, e_3\}$, where e_1=reliability, e_2= fuel consumption, e_3= cost. The question to the domain experts is "is the car is good?". There may be the five types of degrees of opinions in [0,1] under each category, which are "good", "contradictory", "Ignorance", "Unknown", "Bad". P and Q two PNSs, which are opinion of two experts on W, are defined by:

$$P = \frac{\langle 0.6,0.3,0.3,0.5,0.4 \rangle}{e_1} + \frac{\langle 0.5,0.3,0.7,0.4,0.2 \rangle}{e_2} + \frac{\langle 0.7,0.3,0.4,0.2,0.2 \rangle}{e_3}$$
$$Q = \frac{\langle 0.4,0.7,0.3,0.5,0.6 \rangle}{e_1} + \frac{\langle 0.2,0.8,0.3,0.5,0.7 \rangle}{e_2} + \frac{\langle 0.3,0.6,0.8,0.5,0.6 \rangle}{e_3}.$$

Then we have,

$$P^C = \langle 0.4,0.5,0.7,0.3,0.6 \rangle / e_1 + \langle 0.2,0.4,0.3,0.3,0.5 \rangle / e_2 + \langle 0.2,0.2,0.6,0.3,0.7 \rangle / e_3$$
$$P \cup Q = \langle 0.6,0.7,0.3,0.5,0.4 \rangle / e_1 + \langle 0.5,0.8,0.3,0.4,0.2 \rangle / e_2 + \langle 0.7,0.6,0.4,0.2,0.2 \rangle / e_3$$
$$P \cap Q = \langle 0.4,0.3,0.3,0.5,0.6 \rangle / e_1 + \langle 0.2,0.3,0.7,0.5,0.7 \rangle / e_2 + \langle 0.3,0.3,0.8,0.5,0.6 \rangle / e_3$$

3. Proposed penta-partitioned neutrosophic fuzzy optimization technique

If we take a look at a multi-objective optimization problem (MOOP),

$$\text{Minimize } \{Z_i(w)\} \qquad i = 1, \dots, m.$$

Subject to

$$f_j(w) \le b_j \qquad j = 1, \dots, n. \tag{1}$$

$$w \ge 0$$

where $Z_i(w)$ are m objectives, $f_j(w)$ are the n constraints, w are decision variables, and m and n presents number of objectives and constraints respectively. $\tilde{D}$ is the decision set, which combines penta-partitioned neurotrophic goals $(\tilde{O}_i)$ and constraints $(\tilde{L}_j)$, is defined by:

$$\tilde{D} = \left(\cap_{i=1}^{m} \tilde{O}_i\right) \cap \left(\cap_{j=1}^{n} \tilde{L}_j\right) = \{w, T_{\tilde{D}}(w), C_{\tilde{D}}(w), G_{\tilde{D}}(w), U_{\tilde{D}}(w), F_{\tilde{D}}(w))\}$$

Where $w \in W$.

$$T_{\tilde{D}}(w) = \min\{T_{\tilde{O}_1}(w), T_{\tilde{O}_2}(w), \dots, T_{\tilde{O}_m}(w); T_{\tilde{L}_1}(w), T_{\tilde{L}_2}(w), \dots, T_{\tilde{L}_n}(w)\} = A$$
$$C_{\tilde{D}}(w) = \min\{C_{\tilde{O}_1}(w), C_{\tilde{O}_2}(w), \dots, C_{\tilde{O}_m}(w); C_{\tilde{L}_1}(w), C_{\tilde{L}_2}(w), \dots, C_{\tilde{L}_n}(w)\} = B$$
$$G_{\tilde{D}}(w) = \max\{G_{\tilde{O}_1}(w), G_{\tilde{O}_2}(w), \dots, G_{\tilde{O}_m}(w); G_{\tilde{L}_1}(w), G_{\tilde{L}_2}(w), \dots, G_{\tilde{L}_n}(w)\} = C$$
$$U_{\tilde{D}}(w) = \max\{U_{\tilde{O}_1}(w), U_{\tilde{O}_2}(w), \dots, U_{\tilde{O}_m}(w); U_{\tilde{L}_1}(w), U_{\tilde{L}_2}(w), \dots, U_{\tilde{L}_n}(w)\} = D$$
$$F_{\tilde{D}}(w) = \max\{F_{\tilde{O}_1}(w), F_{\tilde{O}_2}(w), \dots, F_{\tilde{O}_m}(w); F_{\tilde{L}_1}(w), F_{\tilde{L}_2}(w), \dots, F_{\tilde{L}_n}(w)\} = E$$

Where $T_{\tilde{D}}, C_{\tilde{D}}, G_{\tilde{D}}$, $U_{\tilde{D}}$ and $F_{\tilde{D}}$ presents the truth, contradiction, ignorance, unknown and falsity degree of membership of penta-partitioned neutrosophic decision set, respectively. Now using PNO, the above problem (1) is reformulated into a MOOP as:

$Max\ A, \quad Max\ B, \quad Min\ C, \quad Min\ D, \quad Min\ E.$

Subject to,

$$T_{\tilde{O}_i}(w) \ge A, \quad T_{\tilde{L}_j}(w) \ge A$$
$$C_{\tilde{O}_i}(w) \ge B, \quad C_{\tilde{L}_j}(w) \ge B$$
$$G_{\tilde{O}_i}(w) \le C, \quad G_{\tilde{L}_j}(w) \le C$$
$$U_{\tilde{O}_i}(w) \le D, \quad U_{\tilde{L}_j}(w) \le D$$
$$F_{\tilde{O}_i}(w) \le E, \quad F_{\tilde{L}_j}(w) \le E$$
$$A \ge B, A \ge C, A \ge D, A \ge E$$
$$0 \le A + B + C + D + E \le 5$$
$$A, B, C, D, E \in [0,1], \ i = 1, \dots, m$$
$$f_j(w) \le b_j, \quad w \ge 0, \quad j = 1, \dots, n. \tag{2}$$

Computational method:

Step 1: Each objective function is solved individually ignoring the others subject to the constraints.

Step 2: Determine the value of other objective functions at the point where the best value of the individual objective function occurs.

Step 3: Using above two steps, construct pay-off matrix:

$$\begin{bmatrix} Z_1^*(w_1) & Z_2(w_1) & \cdots & Z_m(w_1) \\ Z_1(w_2) & Z_2^*(w_2) & \cdots & Z_m(w_2) \\ \vdots & \vdots & \ddots & \vdots \\ Z_1(w_m) & Z_2(w_m) & \cdots & Z_m^*(w_m) \end{bmatrix}.$$

Step 4: Find lower bound L_m^T, upper bound U_m^T of truth membership of each $Z_m(w)$ by,

$U_m^T =$

$max\{Z_m(w_i)\}$ and $L_m^T = min\{Z_m(w_i)\}$, $i = 1,2,\dots,m$.

Lower bound L_m^C and upper bound U_m^C for contradiction membership of objective functions $Z_m(w)$ are,

$$L_m^C = L_m^T \quad \text{and} \quad U_m^C = L_m^T + q_m(U_m^T - L_m^T),$$

lower bound L_m^G , Upper bound U_m^G for Ignorance membership of objectives $Z_m(w)$ are

$$U_m^G = U_m^T \quad \text{and} \quad L_m^G = L_m^T + r_m(U_m^T - L_m^T),$$

The upper bounds U_m^U and lower bounds L_m^U for unknown membership function of objectives are,

$$U_m^U = U_m^T \quad \text{and} \quad L_m^U = L_m^T + s_m(U_m^T - L_m^T),$$

The upper bounds U_m^F and lower bounds L_m^F of falsity membership function of objectives are,

$$U_m^F = U_m^T \quad \text{and} \quad L_m^F = L_m^T + t_m(U_m^T - L_m^T),$$

where $q_m, r_m, s_m, t_m \in (0,1)$.

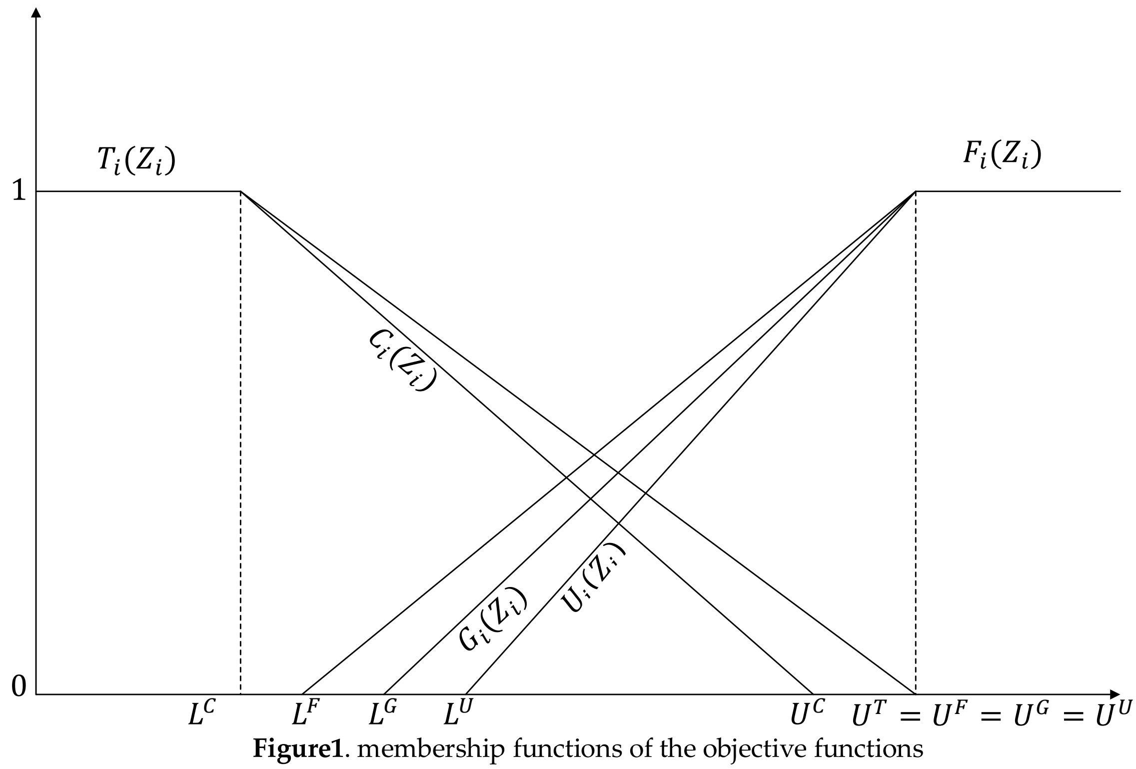

Figure1. membership functions of the objective functions

Step 5: In this step, truth, contradiction, ignorance, unknown, falsity membership functions are:

$$T_m(Z_m(w)) = \begin{cases} 1 & Z_m(w) \le \mathrm{L}_m^T \\ \dfrac{\mathrm{U}_m^T - Z_m(w)}{\mathrm{U}_m^T - \mathrm{L}_m^T} & \mathrm{L}_m^T \le Z_m(w) \le \mathrm{U}_m^T \\ 0 & Z_m(w) \ge \mathrm{U}_m^T \end{cases}$$

$$C_m(Z_m(w)) = \begin{cases} 1 & Z_m(w) \le \mathrm{L}_m^C \\ \dfrac{\mathrm{U}_m^C - Z_m(w)}{\mathrm{U}_m^U - \mathrm{L}_m^C} & \mathrm{L}_m^C \le Z_m(w) \le \mathrm{U}_m^C \\ 0 & Z_m(w) \ge \mathrm{U}_m^U \end{cases}$$

$$G_m(Z_m(w)) = \begin{cases} 0 & Z_m(w) \le \mathrm{L}_m^G \\ \dfrac{Z_m(w) - \mathrm{L}_m^G}{\mathrm{U}_m^G - \mathrm{L}_m^G} & \mathrm{L}_m^G \le Z_m(w) \le \mathrm{U}_m^G \\ 1 & Z_m(w) \ge \mathrm{U}_m^G \end{cases}$$

$$U_m\big(Z_m(w)\big) = \begin{cases} 0 & Z_m(w) \le \mathrm{L}_m^U \\ \dfrac{Z_m(w) - \mathrm{L}_m^U}{\mathrm{U}_m^U - \mathrm{L}_m^U} & \mathrm{L}_m^U \le Z_m(w) \le \mathrm{U}_m^U \\ 1 & Z_m(w) \ge \mathrm{U}_m^U \end{cases}$$

$$F_m(Z_m(w)) = \begin{cases} 0 & Z_m(w) \le \mathrm{L}_m^F \\ \dfrac{Z_m(w) - \mathrm{L}_m^F}{\mathrm{U}_m^F - \mathrm{L}_m^F} & \mathrm{L}_m^F \le Z_m(w) \le \mathrm{U}_m^F \\ 1 & Z_m(w) \ge \mathrm{U}_m^F \end{cases}$$

Step 6: Now PNO method for MOOP is presented by max-min method as:

Max $(A + B - C - D - E)$,

Subject to

$$T_m\big(Z_m(w)\big) \ge A$$
$$C_m\big(Z_m(w)\big) \ge B$$
$$G_m\big(Z_m(w)\big) \le C$$
$$U_m\big(Z_m(w)\big) \le D$$
$$F_m\big(Z_m(w)\big) \le E$$

$f_j(w) \le b_j, \quad w \ge$

$0, \quad j = 1, \dots, n$

with, $\quad 0 \le A + B + C + D + E \le 5,$

$A \ge B, A \ge C, A \ge D, A \ge E \quad , \qquad A, B, C, D, E \in [0,1]$

(3)

This equivalent to:

Max $(A + B - C - D - E)$

Subject to

$$Z_m(w) + (\mathrm{U}_m^T - \mathrm{L}_m^T) \cdot A \leq \mathrm{U}_m^T$$
$$Z_m(w) + (\mathrm{U}_m^C - \mathrm{L}_m^C) \cdot B \leq \mathrm{U}_m^C$$
$$Z_m(w) - (\mathrm{U}_m^G - \mathrm{L}_m^G) \cdot C \leq \mathrm{L}_m^G$$
$$Z_m(w) - (\mathrm{U}_m^U - \mathrm{L}_m^U) \cdot D \leq \mathrm{L}_m^U$$
$$Z_m(w) - (\mathrm{U}_m^F - \mathrm{L}_m^F) \cdot E \leq \mathrm{L}_m^F$$
$$f_j(w) \leq b_j, \quad w \geq 0, \quad j = 1, \dots, n$$

For all m objectives

$0 \leq A + B + C + D + E \leq 5,$ $A \geq B, A \geq C, A \geq D, A \geq E$ $A, B, C, D, E \in [0,1]$

(4)

4. Numerical example [14]

Consider the following MOOP:

$$\text{Min}\, Z_1(x_1, x_2) = x_1^{-1} x_2^{-2},$$
$$\text{Min}\, Z_2(x_1, x_2) = 2x_1^{-2} x_2^{-3},$$

Subject to $x_1 + x_2 \leq 1.$

$x_1, x_2 \geq 0$ (5)

	Z_1	Z_2
X^1	6.75	60.78
X^2	6.94	57.87

Step 1: Solving the above objective functions individually ignoring other objective subject to the constraint, we get the optimal values $Z_1^*(X^1) = 6.75$ at the point $X^1 = (.33, .67)$ and $Z_2^*(X^2) = 57.87$ at the point $X^2 = (.4, .6)$.

Step 2: At the point of optimal the values of other objectives have calculated. Here $Z_1(X^2) = 6.94$ and $Z_2(X^1) = 60.78$.

Step 3: The pay-off matrix is:

Step 4: Calculate upper and lower bound of membership functions corresponding to each objective function: $\mathrm{L}_1^T = 6.75,\ \mathrm{U}_1^T = 6.94$

$$\mathrm{L}_1^C = 6.75, \quad \mathrm{U}_1^C = 6.75 + 0.19 \times q_1 = 6.902$$
$$\mathrm{L}_1^G = 6.75 + 0.19 \times r_1 = 6.7975, \quad \mathrm{U}_1^G = 6.94$$
$$\mathrm{L}_1^U = 6.75 + 0.19 \times s_1 = 6.807, \quad \mathrm{U}_1^U = 6.94$$
$$\mathrm{L}_1^F = 6.75 + 0.19 \times t_1 = 6.788, \quad \mathrm{U}_1^F = 6.94$$

$$\mathrm{L}_2^T = 57.87, \ \mathrm{U}_2^T = 60.78$$
$$\mathrm{L}_2^C = 57.87, \quad \mathrm{U}_2^C = 57.87 + 2.91 \times q_2 = 60.489$$
$$\mathrm{L}_2^G = 57.87 + 2.91 \times r_2 = 58.3065, \quad \mathrm{U}_2^G = 60.78$$
$$\mathrm{L}_2^U = 57.87 + 2.91 \times s_2 = 58.452, \quad \mathrm{U}_2^U = 60.78$$
$$\mathrm{L}_2^F = 57.87 + 2.91 \times t_2 = 58.161, \quad \mathrm{U}_2^F = 60.78$$

where $q_1 = 0.800,\ r_1 = 0.250,\ s_1 = 0.300,\ t_1 = 0.200,\ q_2 = 0.900,\ r_2 = 0.150,\ s_2 = 0.200,\ t_2 = 0.100$.

Step 5: Now, membership functions of $T, C, U, G, and\ F$ can be defined as:

$$T_1(x_1{}^{-1}x_2^{-2}) = \begin{cases} 1 & x_1^{-1}x_2^{-2} \le 6.75 \\ \dfrac{6.94 - x_1^{-1}x_2^{-2}}{6.94 - 6.75} & 6.75 \le x_1^{-1}x_2^{-2} \le 6.94 \\ 0 & x_1^{-1}x_2^{-2} \ge 6.94 \end{cases}$$

$$T_2(2x_1^{-2}x_2^{-3}) = \begin{cases} 1 & 2x_1^{-2}x_2^{-3} \le 57.87 \\ \dfrac{60.78 - 2x_1^{-2}x_2^{-3}}{60.78 - 57.87} & 57.87 \le 2x_1^{-2}x_2^{-3} \le 60.78 \\ 0 & x_1^{-1}x_2{}^{-2} \ge 60.78 \end{cases}$$

$$C_1(x_1{}^{-1}x_2^{-2}) = \begin{cases} 1 & x_1^{-1}x_2^{-2} \le 6.75 \\ \dfrac{6.902 - x_1^{-1}x_2^{-2}}{6.94 - 6.75} & 6.75 \le x_1^{-1}x_2^{-2} \le 6.902 \\ 0 & x_1^{-1}x_2^{-2} \ge 6.902 \end{cases}$$

$$C_2(2x_1^{-2}x_2^{-3}) = \begin{cases} 1 & 2x_1^{-2}x_2^{-3} \le 57.87 \\ \dfrac{60.489 - 2x_1^{-2}x_2^{-3}}{60.489 - 57.87} & 57.87 \le 2x_1^{-2}x_2^{-3} \le 60.489 \\ 0 & x_1^{-1}x_2{}^{-2} \ge 60.78 \end{cases}$$

$$G_1(x_1{}^{-1}x_2^{-2}) = \begin{cases} 0 & x_1{}^{-1}x_2^{-2} \le 6.7975 \\ \dfrac{x_1{}^{-1}x_2^{-2} - 6.7975}{6.94 - 6.7975} & 6.7975 \le x_1{}^{-1}x_2^{-2} \le 6.94 \\ 1 & x_1{}^{-1}x_2^{-2} \ge 6.94 \end{cases}$$

$$G_2\left(2x_1{}^{-2}x_2^{-3}\right) = \begin{cases} 0 & 2x_1^{-2}x_2^{-3} \le 58.3065 \\ \dfrac{2x_1^{-2}x_2^{-3} - 58.3065}{60.78 - 58.3065} & 58.3065 \le 2x_1^{-2}x_2^{-3} \le 60.78 \\ 1 & 2_1^{-2}x_2^{-3} \ge 60.78 \end{cases}$$

$$U_1(x_1{}^{-1}x_2^{-2}) = \begin{cases} 0 & x_1{}^{-1}x_2^{-2} \leq 6.807 \\ \dfrac{x_1{}^{-1}x_2^{-2} - 6.807}{6.94 - 6.807} & 6.807 \leq x_1{}^{-1}x_2^{-2} \leq 6.94 \\ 1 & x_1{}^{-1}x_2^{-2} \geq 6.94 \end{cases}$$

$$U_2\left(2x_1{}^{-2}x_2^{-3}\right) = \begin{cases} 0 & 2x_1^{-2}x_2^{-3} \leq 58.452 \\ \dfrac{2x_1^{-2}x_2^{-3} - 58.452}{60.78 - 58.452} & 58.452 \leq 2x_1^{-2}x_2^{-3} \leq 60.78 \\ 1 & 2_1^{-2}x_2^{-3} \geq 60.78 \end{cases}$$

$$F_1(x_1{}^{-1}x_2^{-2}) = \begin{cases} 0 & x_1{}^{-1}x_2^{-2} \leq 6.788 \\ \dfrac{x_1{}^{-1}x_2^{-2} - 6.788}{6.94 - 6.788} & 6.788 \leq x_1{}^{-1}x_2^{-2} \leq 6.94 \\ 1 & x_1{}^{-1}x_2^{-2} \geq 6.94 \end{cases}$$

$$F_2\left(2x_1{}^{-2}x_2^{-3}\right) = \begin{cases} 0 & 2x_1^{-2}x_2^{-3} \leq 58.161 \\ \dfrac{2x_1^{-2}x_2^{-3} - 58.161}{60.78 - 58.161} & 58.161 \leq 2x_1^{-2}x_2^{-3} \leq 60.78 \\ 1 & 2_1^{-2}x_2^{-3} \geq 60.78 \end{cases}$$

Step 6: The above problem in PNS is now

$$\text{Max} \quad (A + B - C - D - E)$$

Subject to

$$x_1^{-1}x_2^{-2} + (0.19)A \leq 6.94$$
$$2x_1^{-2}x_2^{-3} + (2.19)A \leq 60.78$$
$$x_1^{-1}x_2^{-2} + (0.152)B \leq 6.902$$
$$2x_1^{-2}x_2^{-3} + (2.619)B \leq 60.489$$
$$x_1^{-1}x_2^{-2} - (0.1425)C \leq 6.7975$$
$$2x_1^{-2}x_2^{-3} - (2.4735)C \leq 58.3065$$
$$x_1^{-1}x_2^{-2} - (0.133)D \leq 6.807$$
$$2x_1^{-2}x_2^{-3} - (2.328)D \leq 58.452$$
$$x_1^{-1}x_2^{-2} - (0.152)E \leq 6.788$$
$$2x_1^{-2}x_2^{-3} - (2.619)E \leq 58.161$$
$$x_1 + x_2 \leq 1$$
$$0 \leq A, B, C, D, E \leq 1 \quad and \quad A \geq B, C, D, E.$$
$$x_1, x_2 \geq 0$$

(6)

The outcomes of the suggested approach and comparison with alternative approaches, IFO, NSO and FVRNO using LINGO software are shown by table 1 and table 2 in results and discussion section.

5. Application of proposed method on multi-objective reliability optimization model [35]

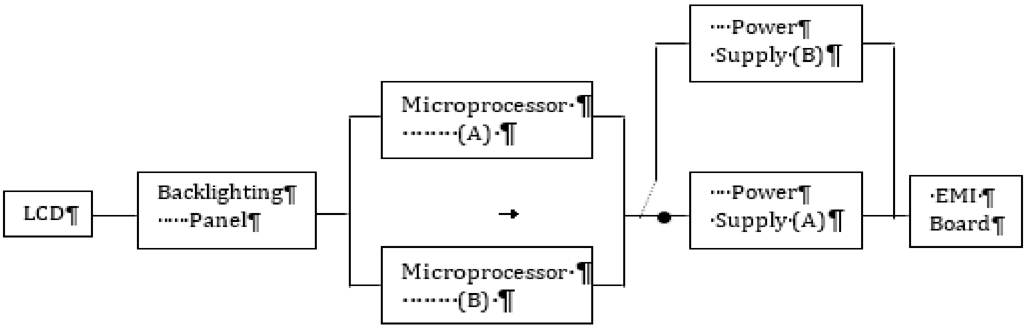

Figure2. LCD display unit

The multi-objective reliability optimization model of the LCD display unit is as follows:

$$\text{Max}\, R(\mathrm{r}) = \mathrm{r}_1\left(r_2^{10} + 10r_2^9(1-r_2)\right)(1-(1-r_3)^2)\left(r_4 + r_4 \ln\left(\frac{1}{r_4}\right)\right)\mathrm{r}_5$$

$$\text{Min}\, C(\mathrm{r}) = \sum_{j=1}^{5} c_j \left[\tan\left(\frac{\pi}{2} r_j\right)\right]^{a_j}$$

$$\text{Subject to, } V(r) = \sum_{j=1}^{5} v_j r_j^{b_j} \le V_{max}$$

$$0.5 \le r_j \le 1 \qquad j = 1,2,\ldots,5 \qquad (7)$$

This problem (7) is equivalent to:

$$\text{Min } R'(r) = 1 - R(r) \text{ and Min } C(r), \text{ subject to same constraints as above.} \qquad (8)$$

6. Results and discussion

Table1. results for problem (1) by sum of optimal objective values.

Optimization Methods	Optimal decision variables (x_1^*, x_2^*)	Optimal value of objectives (Z_1^*, Z_2^*)	Sum of the optimal objective values $Z = (Z_1^* + Z_2^*)$
IFO	$x_1^* = .3659009$, $x_2^* = .6356811$	$Z_1^* = 6.797078$ $Z_2^* = 58.79110$	$Z = 65.588178$
NSO	$x_1^* = .3635224$, $x_2^* = .6364776$	$Z_1^* = 6.790513$ $Z_2^* = 58.68732$	$Z = 65.477833$
FVRNO	$x_1^* = .365902$, $x_2^* = .634098$	$Z_1^* = 6.797081071$ $Z_2^* = 58.59104971$	$Z = 65.3881308$
PNO (proposed method)	$x_1^* = .3688571$, $x_2^* = .6311429$	$Z_1^* = 6.8059139$ $Z_2^* = 58.4696639$	$Z = 65.2755778$

In Table (1), we have shown that sum of the optimal objective values by IFO is 65.588178, by NSO is 65.477833, by FVRNO is 65.3881308. Here by the proposed method, the same is 65.2755778. since the both objectives of problem (5) are minimization type, we can conclude the proposed method is better.

Percentage gap= $\left|\frac{Achieved\ \ value-Best\ \ Value}{Achieved\ \ value}\right| \times 100\%$

Table 2. results for problem (1) by percentage gap.

Optimization Methods	Percentage gap of Z_1^*	Percentage gap of Z_2^*	Total percentage gap
IFO	0.0965856	0.546742789	0.643328389
NSO	0	0.370874151	0.370874151
FVRNO	0.0966308	0.207174663	0.303805463
PNO	0.2262870	0	0.2262870

From Table 2, we have shown that the total percentage gap by IFO, NSO, FVRNO and PNO are 0.643328389, 0.370874151, 0.303805463, 0.2262870 respectively. So, the developed method is better in the view of percentage gap. Graphical presentation of the results of problem (5) by total optimal values of objectives and total percentage gap are presented in Figure 3 and Figure 4.

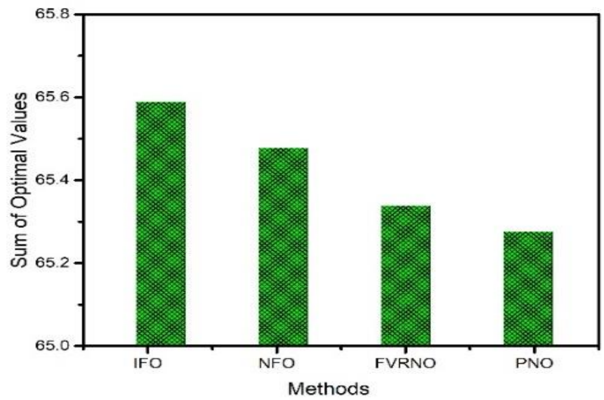

Figure 3. Comparison of the developed method with other by total optimal values

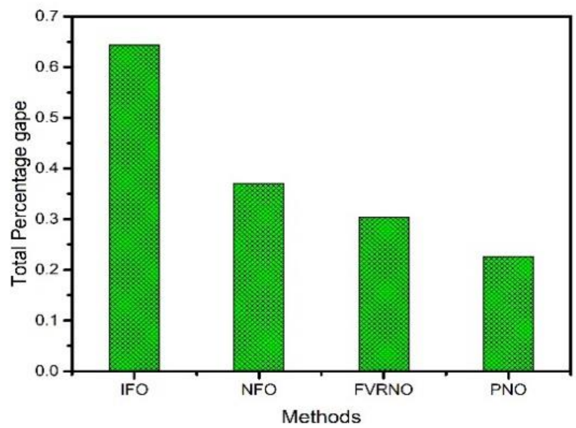

Figure 4. Comparison of the developed method with other by total percentage gap

Table 3. Data used for the reliability optimization model.

c_1	c_2	c_3	c_4	c_5	v_1	v_2	v_3	v_4	v_5	$a_j(\forall j)$	$b_j(\forall j)$	$V_{\max}$
28	30	32	35	29	6	4.5	3.75	3.5	7	0.4	1	24

Pay-off matrix is:

	R'	C
R^1	0.01478132	5196.368
R^2	0.9982949	154

Table 4. Optimal solutions by FVRNO and PNO methods

Methods	r_1	r_2	r_3	r_4	r_5	R^*	C^*
FVRNO	0.9762522	0.9809396	0.9096640	0.8776727	0.9756593	0.923499	470.4295
PNO	0.9725617	0.9791286	0.9015574	0.8669358	0.9718783	0.91108	449.5225

Table 5. Efficiency of the proposed method by total percentage gap

Methods	percentage gap of R^*	percentage gap of C^*	Total percentage gap
FVRNO	0	4.4442366	4.442366
PNO	1.36310752	0	1.36310752

In Table 4the outcomes of the suggested approach (PNO) and FVRNO solving the problem (7) are shown. The comparison of the result is presented by percentage gap in table 5. From table 5, we can conclude that the result obtained by PNO method is better than FVRNO. The graphical presentation of the result of percentage gap is presented by Figure 5.

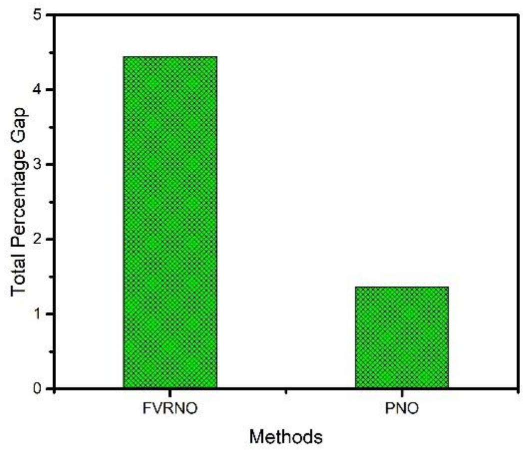

Figure 5. Comparison of the proposed method with other by total percentage gap

7. Conclusion and future directions

Using a penta-partitioned neutrosophic fuzzy environment, we have suggested a new computational approach in this article. A well-known example is solved to show the efficiency of developed method and the results are compared with other existing methods such as IFO, NFO, FVRNO by sum of optimal values and total percentage gap in table 1 and table 2. We have also applied this method to solve multi objective reliability optimization model (LCD display unit) by maximizing the system reliability and minimizing system cost and the results are compared with FVRNO by total percentage gap in table 5. We could deduce from the results that the suggested approach is effective and more flexible than those already in use.

In future, we can apply this method to inventory model, transportation problem, portfolio selection model etc. considering various fuzzy parameters.

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Neutrosophic Structure of the Geometric Model with Applications

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Abstract: In practical scenarios, it is common to encounter fuzzy data that contains numerous imprecise observations. The uncertainty associated with this type of data often leads to the use of interval statistical measures and the proposal of neutrosophic versions of probability distributions to better handle such data. We present a unique methodology that is based on the maximum likelihood approach and neutrosophic approach for estimating parameter of the proposed neutrosophic geometric distribution (NGD). The proposed methodology is supported by key likelihood inference results. The proposed distribution is specifically designed to handle variables with imprecise observation, hence effectively addressing a wide range of situations often encountered in the analysis of uncertain data. To evaluate the efficacy of the proposed neutrosophic model, we have carried out a comprehensive simulation experiment that rigorously examined the performance of the proposed model. The practical utility of NGD in the analysis of incomplete data is further exemplified through real-world applications.

Keywords: Neutrosophic logic, uncertain analysis, probability model, estimation, simulation

1. Introduction

Statistical distributions are a powerful tool for describing and predicting real-world events. The geometric distribution is possibly the most common distribution in statistical applications [1] . The geometric distribution is widely employed in various domains such as finance, investment, scientific research, and engineering, making it the most frequently utilized distribution [2]. The geometric distribution is a discrete probability distribution that is commonly employed to model the probability of attaining success in a sequence of independent trials with two possible outcomes [3]. Through the use of geometric distribution, it becomes possible to ascertain the likelihood of attaining success subsequent to a designated quantity of attempts [4]. The geometric distribution exhibits a multitude of uses in practical, real-world situations. As an illustration, it can be employed to simulate the quantity of endeavors required to achieve win in a game of probability or the quantity of unsuccessful tries prior to attaining success in a manufacturing procedure [5].

The geometric distribution is also used in banking to figure out how likely it is that a loan will not be paid back or how many trades are needed to make a profit [6], [7]. In the field of epidemiology, geometric distribution can also be used to model how many contacts a person with a disease has before they spread it to other people [8]. Additionally, it can be used in telecommunications to determine how many tries are needed to make a call in a busy network [9].

The geometric distribution is an important part of probability theory and has been studied a lot for its uses in many different areas [10]. Figuring out the chance of getting the first victory after a certain number of tries is what the geometric probability mass function is based on [11]. Well-known scientists like Feller [2] and Ross [3] have spent a great deal of time studying and exploring this idea. They have come up with detailed explanations and studies of its properties. In queuing theory, the geometric distribution is a key tool for finding out how long people will have to wait. Kleinrock's efforts [4] have shown that this can be used.

Barlow and Proschan [2] employ this probability distribution within the domain of reliability engineering to examine the duration required for the initial failure occurrence in systems. Furthermore, researchers in the field of epidemiology, such as Thelwell et al. [12], employ this tool as a means to get valuable understanding regarding the intricacies of disease transmission. The research conducted by Mandelbrot emphasises the importance of the Geometric distribution in the assessment of financial risk [13]. Furthermore, Preston's research delves into the use of this concept in the field of environmental science, namely in the modelling of species abundance [14]. The geometric distribution is widely employed in many disciplines, including information theory[15] , machine learning for pattern identification [10], game theory for strategic interactions [16], and educational research for comprehending learning patterns [17]–[20].

Fuzzy sets serve as the fundamental construct underlying the notion of fuzzy set theory. The notion of fuzzy sets is a crucial aspect within the framework of fuzzy set theory [21]. Fuzzy sets are mathematical constructs that enable the incorporation of partial membership or degrees of truth inside their representations [22]. The aforementioned frameworks offer a versatile structure for addressing ambiguity and imprecision across many domains, including but not limited to artificial intelligence, decision-making, and pattern recognition [23]. The integration of fuzzy sets within the framework of fuzzy set theory enables a more sophisticated and authentic methodology for modelling intricate systems and representing imprecise data [23]–[26]. The use of fuzzy set theory enables a more detailed modelling of complex systems, allowing for effective capture of imprecise information. Fuzzy control has been effectively employed in the automobile sector to regulate diverse systems, including automatic gearbox, suspension, engine, temperature control, and antilock brakes [27]. Furthermore, washing machines employ fuzzy control algorithms to adapt their washing approach according on several criteria, including the detected degree of filth, kind of cloth, size of the load, and water level [28]. The neutrosophy idea, initially proposed by Smarandache, is increasingly being recognised and used due to its capacity to offer a more adaptable and all-encompassing approach in addressing uncertainty and imprecision within the context of data analysis [29]. Neutrosophic statistics provide an expanded range of options for the representation and analysis of data, hence enabling to achievement of enhanced precision and dependability in the obtained outcomes [30], [31]. This strategy demonstrates significant use in scenarios when conventional statistical methods prove inadequate, consequently gaining greater popularity within the discipline of uncertain data analysis [32]–[35]. The proposal of NGD in this work is driven by the recognition of the significant role geometric distribution plays in statistical applications. Its wide applicability and the prevalence of uncertainty in real data make NGD an important consideration.

The proposed distribution and its key characteristics are described in Section 2. The estimation procedure for unknown parameters under the neutrosophic logic is presented in Section 3. In Section 4, the quantile function of the proposed model is formulated and the procedure for simulating data is explained. The significance of theoretical findings is concisely explained by analyzing a real-world examples in Section 5. Finally, Section 6 provides the final remarks of the study.

2. Proposed Model

This section presents a summary statistic of the proposed model and describes some of its important functions. The summary statistics of the proposed model provide a concise overview of its key characteristics. Additionally, the description of important functions commonly used in applied probability distribution theory helps to understand how the model can be utilized in practical applications. The geometric distribution holds significant importance in the field of statistics, being one of the fundamental distributions.

The formula provided below represents the neutrosophic probability density function (DF_n).

$$g_n(\mathcal{X}) = \mathcal{P}_n(1-\mathcal{P}_n)^{\mathcal{X}} \; ; \; \mathcal{X} \geq 0 \quad (1)$$

where $0 < \mathcal{P}_n = [\mathcal{P}_l, \mathcal{P}_u] < 1$ is the neutrosophic parameter of the NGD. To calculate the probability of waiting exactly r trials before the first successful event, we need to know the probability of success in a single trial ($\mathcal{P}_n$). The probability of failure (q_n) can be calculated as 1 minus $\mathcal{P}_n$. This scenario is known as a special case of the negative binomial distribution. It should be noted that the suggested model differs from the existing framework of the geometric model, where the parameter is precisely determined. The suggested model becomes equal to the classical model, when the indeterminate portion of the suggested model is zero, i.e., $\mathcal{P}_l = \mathcal{P}_u = \mathcal{P}$. The neutrosophic probability density function, often denoted as DF_n, is a mathematical function that describes the likelihood of a neutrosophic random variable taking on a particular interval value due to imprecision in $\mathcal{P}_n$. It provides valuable information about the distribution of the neutrosophic variable and can be used to calculate probabilities of different outcomes. Based on (1), the NGD is depicted in Figure1.

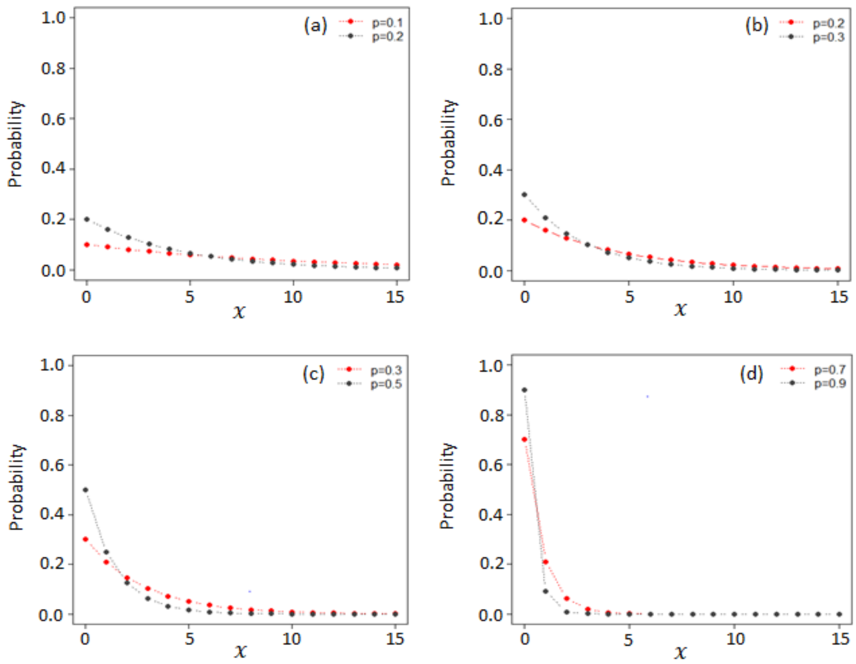

Figure 1: Density plots of the proposed NGD with different vague values of parameter

Figure 1 illustrates that there is a distinct interval probability for every value of the random variable $\mathcal{X}$.As illustrated in Figure 1(a), for instance, $\mathcal{P}_n = [0.1, 0.2]$ approximation for $\mathcal{X} = 1$, and the same is true for other values. The graph of DF_n shows that the likelihood of different outcomes occurring within a given range. It provides a visual representation of the probability of each possible outcome. By examining the shape and characteristics of the DF_n, one can gain insights into the likelihood and spread of values within the distribution. The neutrosophic probability mass function (PMF_n) of any density is another fascinating feature of probability theory applications. To describe the distribution of a discrete random variable, we can use the PMF_n. This function assigns probabilities to each possible value that the random variable can take. The $PMF_n$is a cooperatively linked variant of the DF_n and may be calculated as:

$$\mathcal{G}_n(\mathcal{X}) = 1 - (1 - \mathcal{P}_n)^{\mathcal{X}} \quad (2)$$

It should be noted that the PMF_n can be applied to any real number in the set R. However, if an argument does not belong to the possible values that the variable can take (i.e., the support of the sample space), then the PMF_n will have a value of zero. Conversely, if an argument does belong to the support of the sample space, then the PMF_n will have a positive value. This means that the PMF_n assigns probabilities to specific values within the sample space. It is important to note that the sum of all the probabilities assigned by the PMF_n must equal 1. The graph of PMF_n with imprecise values of NGD with different interval values of $\mathcal{P}_n$ is shown in Figure 2.

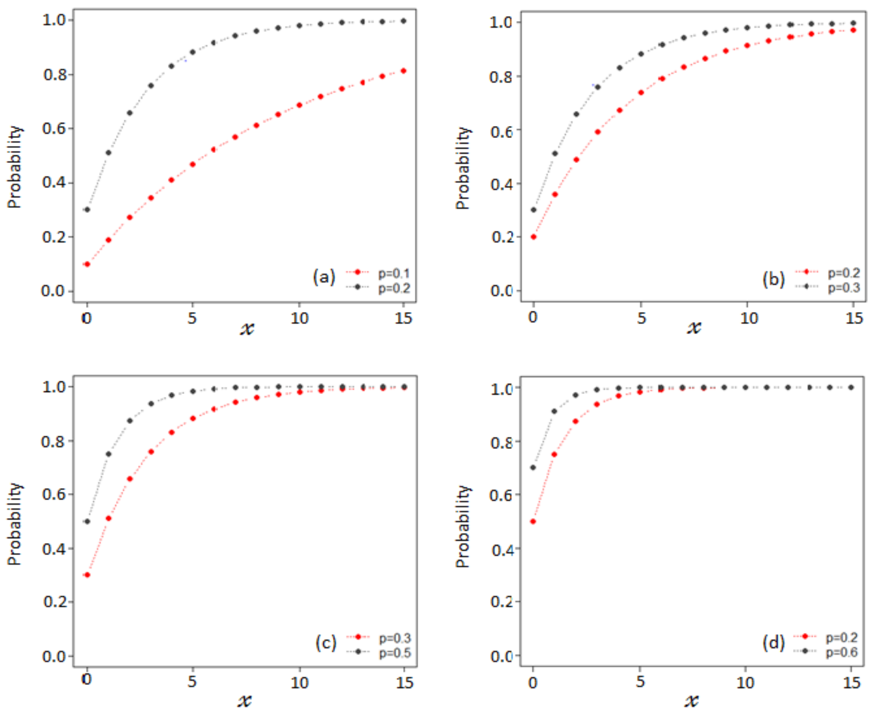

Figure 2: The graph of PMF_n of the proposed model

The PMF_n graph provides a visual representation of the probabilities linked to neutrosophic random variable. This graph illustrates discrete outcomes on the horizontal axis and their corresponding neutrosophic probability on the vertical axis. Each data point on the graph represents the probability of a certain result, with taller hight indicating more likely events. Importantly, the total of all probabilities shown on the graph equals one. The peaks spots on the graph depict the most probable occurrences, providing a distinct comparative examination of the likelihood of various events. The graph's discrete form, characterized by distinct double points, sets it apart from the classical plot of the geometric distribution. The PMF_n graph is a useful tool for comprehending and forecasting the unpredictability linked to discrete events in statistical research.

The suggested model's survival function can be described as follows in the neutrosophic framework: In the given statistical approach, the survival function plays a significant role in determining the probability of an individual's life surviving for a specific duration. Referred to as the survival rate, this function can be defined within the neutrosophic framework according to suggested model as:

$$\mathcal{S}_n(\mathcal{X}) = (1-\mathcal{P}_n)^{\mathcal{X}} \tag{3}$$

The graph of the survival function which is also known as reliability function is depicted in Figure 3.

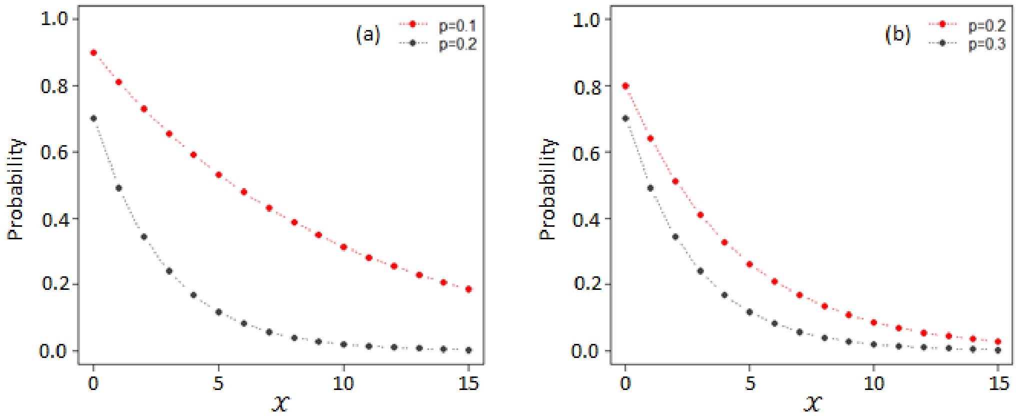

Figure 3: The survival function of the suggested NGD

The neutrosophic hazard function (HF_n), often known as the impending failure rate, is another important function in reliability analysis. For the given model, it is the ratio of the survival and density functions, which may be computed as follows:

$$\hbar_n(\mathcal{X}) = \frac{g_n(x)}{\mathcal{S}_n(x)} = \mathcal{P}_n \tag{4}$$

The function $\hbar_n(x)$ calculates an individual or item failure probability over a short period of time. The HF_n may increase, decrease, stay constant, or reflect a more complex process. In this way the suggested model is memoryless in the family of discrete probability distribution like the exponential distribution in the class of continuous distributions.

Several theorems can be used to establish statistical properties of the proposed distribution. Some of these theorems include the derivations of important statistical measures in neutrosophic framework that can help to understand the behavior of the distribution for analyzing the vague dataset. These theorems provide a solid foundation for making reliable inferences and drawing meaningful conclusions.

Theorem 1 If x follows the NGD then $E(\mathcal{X}) = \frac{1-\mathcal{P}_n}{\mathcal{P}_n}$

Proof: By definition, the mean of the NGD is given by:

$$E(\mathcal{X}) = \sum_{\mathcal{X}=0}^{\infty}(1-\mathcal{P}_n)^{\mathcal{X}}\mathcal{P}_n\mathcal{X}$$
$$= (1-\mathcal{P}_n)\mathcal{P}_n\sum_{\mathcal{X}=0}^{\infty}(1-\mathcal{P}_n)^{\mathcal{X}-1}\mathcal{X}$$
$$= [(1-\mathcal{P}_l)\mathcal{P}_l\sum_{\mathcal{X}=0}^{\infty}(1-\mathcal{P}_l)^{\mathcal{X}-1}\mathcal{X},\ (1-\mathcal{P}_{u.})\mathcal{P}_u\sum_{\mathcal{X}=0}^{\infty}(1-\mathcal{P}_u)^{\mathcal{X}-1}\mathcal{X}] \tag{5}$$

Equation (5) further yielded:

$$(1-\mathcal{P}_l)\mathcal{P}_l\sum_{\mathcal{X}=0}^{\infty}(1-\mathcal{P}_l)^{\mathcal{X}-1}\mathcal{X} = \frac{1-\mathcal{P}_l}{\mathcal{P}_l}$$

and

$$(1-\mathcal{P}_{u.})\mathcal{P}_u\sum_{\mathcal{X}=0}^{\infty}(1-\mathcal{P}_u)^{\mathcal{X}-1}\mathcal{X} = \frac{1-\mathcal{P}_u}{\mathcal{P}_u}$$

So,

$\left[\frac{1-\mathcal{P}_l}{\mathcal{P}_l}, \frac{1-\mathcal{P}_u}{\mathcal{P}_u}\right] = \frac{1-\mathcal{P}_n}{\mathcal{P}_n}$, hence proved.

Theorem 2 If x follows the NGD, then $\tilde{V}_n(\mathcal{X}) = \frac{1-\mathcal{P}_n}{\mathcal{P}_n^2}$ is the variance of the proposed model.

Proof: The variance of the NGD is given by:

$$\tilde{V}_n(x) = E(\mathcal{X}^2) - [E(\mathcal{X})]^2 \quad (6)$$

Now

$$E(\mathcal{X}^2) = \sum_{\mathcal{X}=0}^{\infty}(1-\mathcal{P}_n)^{\mathcal{X}}\mathcal{P}_n\mathcal{X}^2$$
$$= [(1-\mathcal{P}_{u.})^2\mathcal{P}_l\sum_{\mathcal{X}=0}^{\infty}(1-\mathcal{P}_l)^{\mathcal{X}-1}\mathcal{X}^2,\ (1-\mathcal{P}_{u.})^2\mathcal{P}_u\sum_{\mathcal{X}=0}^{\infty}(1-\mathcal{P}_u)^{\mathcal{X}-1}\mathcal{X}^2] \quad (7)$$

Simplification of (7) provided:

$$\left[\frac{2-3\mathcal{P}_l+\mathcal{P}_l^2}{\mathcal{P}_l^2},\ \frac{2-3\mathcal{P}_u+\mathcal{P}_u^2}{\mathcal{P}_u^2}\right] = \frac{2-3\mathcal{P}_n+\mathcal{P}_n^2}{\mathcal{P}_n^2}$$

Thus (6) becomes:

$$\tilde{V}_n(x) = \left[\frac{1-\mathcal{P}_l}{\mathcal{P}_l^2},\ \frac{1-\mathcal{P}_u}{\mathcal{P}_u^2}\right] = \frac{1-\mathcal{P}_n}{\mathcal{P}_n^2}$$

Theorem 3 Show that k^{th} moment of the NGD is $\frac{\mathcal{P}_n}{1-(1-\mathcal{P}_n)e^k}$

Proof: By definition the k^{th} moment of the NGD is given by:

$$\mu_{kn} = \sum_{\mathcal{X}=0}^{\infty} e^{k\mathcal{X}}(1-\mathcal{P}_n)^{\mathcal{X}}\mathcal{P}_n$$
$$= \mathcal{P}_n\sum_{\mathcal{X}=0}^{\infty}[e^k(1-\mathcal{P}_n)]^{\mathcal{X}}$$
$$= [\mathcal{P}_l\sum_{\mathcal{X}=0}^{\infty}[e^k(1-\mathcal{P}_l)]^{\mathcal{X}},\ \mathcal{P}_u\sum_{\mathcal{X}=0}^{\infty}[e^k(1-\mathcal{P}_u)]^{\mathcal{X}}] \quad (8)$$

From (8), we can write;

$$\mathcal{P}_l\sum_{\mathcal{X}=0}^{\infty}[e^k(1-\mathcal{P}_l)]^{\mathcal{X}} = \frac{\mathcal{P}_l}{1-(1-\mathcal{P}_l)e^k}$$

and

$$\mathcal{P}_u\sum_{\mathcal{X}=0}^{\infty}[e^k(1-\mathcal{P}_u)]^{\mathcal{X}} = \frac{\mathcal{P}_u}{1-(1-\mathcal{P}_u)e^k}$$

Hence

$$\mu_{kn} = \left[\frac{\mathcal{P}_l}{1-(1-\mathcal{P}_l)e^k}, \frac{\mathcal{P}_u}{1-(1-\mathcal{P}_u)e^k}\right] = \frac{\mathcal{P}_n}{1-(1-\mathcal{P}_n)e^k}$$ is required result.

$where\ k = 1,2,3,\ldots$ is a general expression for the kth row moment about the origin of the NGD. By using the following relations, moments about the mean for NGD can be derived as:

$$\mu'_{1n} = \mu_{1n} = \frac{1-\mathcal{P}_n}{\mathcal{P}_n}$$
$$\mu'_{2n} = \mu_{2n} - (\mu_{1n})^2 = \frac{1-\mathcal{P}_n}{\mathcal{P}_n^2}$$
$$\mu'_{3n} = \mu_{3n} - 3\mu_{2n}\mu_{1n} + 2(\mu_{1n})^3 = (1-\mathcal{P}_n)(1+(1-\mathcal{P}_n))\mathcal{P}_n$$
$$\mu'_{4n} = \mu_{4N} - 4\mu_{3n}\mu_{1n} + 6\mu_{2n}{\mu_{1n}}^2 - 3{\mu_{1n}}^4 = \left(\frac{9(1-\mathcal{P}_n^2)}{\mathcal{P}_n^4}\right) + \left(\frac{1-\mathcal{P}_n}{\mathcal{P}_n^2}\right)$$

Theorem 4 The coefficient of skewness of the NGD is $\frac{(1+(1-\mathcal{P}_n))}{(1-\mathcal{P}_n)^{1/2}}$

Proof: By definition, the coefficient of skewness for NGD is given by:

$$\alpha_3 = \frac{\mu'_{3n}}{(\mu'_{2n})^{3/2}} \quad (9)$$

Where $\mu'_{3n} = (1-\mathcal{P}_n)(1+(1-\mathcal{P}_n))\mathcal{P}_n$ and $\mu'_{2n} = \frac{1-\mathcal{P}_n}{\mathcal{P}_n^2}$

Substituting in (9) yielded;

$$\alpha_3 = \frac{(1+(1-\mathcal{P}_n))}{(1-\mathcal{P}_n)^{1/2}}$$

where $\alpha_3 \in [\alpha_l, \alpha_u]$.

Theorem 5 Show that the coefficient of kurtosis for NGD is $\left(9 + \mathcal{P}_n^2 / 1-\mathcal{P}_n\right)$

Proof: By definition, the coefficient of kurtosis is given by:

$$\alpha_4 = \frac{\mu'_{4n}}{{\mu'_{2n}}^2} \quad (10)$$

Where $\mu'_{4n} = \left(\frac{9(1-\mathcal{P}_n^2)}{\mathcal{P}_n^4}\right) + \left(\frac{1-\mathcal{P}_n}{\mathcal{P}_n^2}\right)$ and $\mu'_{2n} = \frac{1-\mathcal{P}_n}{\mathcal{P}_n^2}$

Substituting in (10) yielded:
$\alpha_4 = \left(9 + \mathcal{P}_n^2 \big/ 1 - \mathcal{P}_n\right)$
where $\alpha_4 = [\alpha_l, \alpha_u]$.
In the same way, other important distributional properties can also be explored through the neutrosophic framework. These properties offer a comprehensive approach to analyzing uncertainties and vagueness.

3. Estimation Procedure

The maximum likelihood estimate (MLE) is a widely used method in many real-world applications. It aims to determine the parameter value(s) that provide the highest probability of the observed data occurring. In uncertain environments, MLE differs from the classical approach as it provides interval estimates of neutrosophic parameters instead of a single point estimate. This allows for a more comprehensive representation of uncertainty and variability in the data. By providing interval estimates, MLE under the neutrosophic structure accounts for the inherent ambiguity and imprecision present in uncertain environments, making it a valuable tool in decision-making processes. In this part, a well-known MLE technique is used to determine the neutrosophic parameter of the proposed NGD. The ML technique is defined by considering the parameters unknown and calculating the joint density of all observations is a dataset that are assumed to be identical and dispersed independently. Once the likelihood of the NGD is established, maxima of the function are determined. These ML estimators are essential in the statistical viewpoint because of minimal variance and asymptotic unbiasedness properties. Let $y_1, y_2, \ldots, y_k$ are identical and independently observations from the k subjects which follow the parametric model given in (1) then the joint density is given by:

$$\begin{aligned}\mathcal{L}(\mathcal{P}_n|\ \mathcal{X}) &= \textstyle\prod_{i=1}^{k} g_n(\mathcal{X}|\mathcal{P}_n)\\ &= \textstyle\prod_{i=1}^{k} \mathcal{P}_n(1-\mathcal{P}_n)^{\mathcal{X}_i}\\ &= \mathcal{P}_n \textstyle\prod_{i=1}^{k}(1-\mathcal{P}_n)^{\mathcal{X}_i} \qquad (11)\end{aligned}$$

Taking the logarithm of (11) and symbolizing it by $\omega_n(\mathcal{T}_i|\mathcal{P}_n)$,

$$\omega_n(\mathcal{T}_i|\mathcal{P}_n) = log\left[\mathcal{P}_n \textstyle\prod_{i=1}^{k}(1-\mathcal{P}_n)^{\mathcal{X}_i}\right] \qquad (12)$$

Simplification of (12) yielded;

$$\omega_n(\mathcal{T}_i|\mathcal{P}_n) = klog(\mathcal{P}_n) + \left(\textstyle\sum_1^k \mathcal{X}_i - k\right)\log(1-\mathcal{P}_n) \qquad (13)$$

Partially differentiating (13) by unknown values and equating to zero implies:

$$\left[\frac{\delta\omega_n(\mathcal{X}_i,\mathcal{P}_n)}{\delta\mathcal{P}_n}\right] = 0 \qquad (14)$$

Further solution of (14) provides the following estimates for unknown parameter of the NGD

$$\hat{\mathcal{P}}_n = \frac{k}{(\sum_1^k \mathcal{X}_i)} \qquad (15)$$

Note that $\hat{\mathcal{P}}_n$ will be interval forms because of imprecise sample data.
This aligns with intuition because when observing a geometric random variable across k trials, the total number of successes observed is represented by the sum of individual trial outcomes $\sum_1^k \mathcal{X}_i$. By calculating the ratio of the number of successes to the total number of trials, we can estimate the probability $\mathcal{P}_n$. It is crucial to note that the maximum likelihood estimator (MLE) can be considered as a random variable since it is based on random data. Consequently, the MLE inherits the randomness of the underlying dataset from which it is derived. Let us take an example where we see that how the MLE estimation can be performed. We consider a situation where we assume that a manufacturing process that produce some specific items. We want to model the number of attempts needed to produce a defect produced by a manufacturing machine. In a sample of 10 attempts, we can record the number of attempts it took to produce a defective item for each attempt. For example, in this case, the recorded attempts are:

2, 5, [1,2], 3, [4,5] 2, 1, [6,7] 2, [5, 6]

Here some values such as [1,2], [4,5], [6,7] and [5,6] are imprecise. Here the value for instance [4,5] means that position of the defective item is not clearly defined. The same holds for other imprecise items. Now this data the unknown neutrosophic parameter can be estimated as:

The above data can further be written as:

[2, 2], [5, 5], [1, 2], [3, 3], [4, 5], [2, 2], [1, 1], [6, 7], [2, 2], [5, 6]

By using (15) the $\mathcal{P}_n$ can be estimated as:

$$\hat{\mathcal{P}}_n = \frac{10}{[\sum_1^k \mathcal{X}_{il}, \sum_1^k \mathcal{X}_{iu}]}$$

where $\sum_1^k \mathcal{X}_{il}$ and $\sum_1^k \mathcal{X}_{iu}$ are lower and upper values of the neutrosophic data.

Thus,

$$\hat{\mathcal{P}}_n = \frac{10}{[31,35]} \cong [0.28, 0.32]$$

Hence the estimated imprecise value lies between 0.28 and 0.32.

4. Random Data Generation

We may require information on the number of trials needed to achieve a 25%, 50% or 75% probability of success occurrence. For example in a production line where there is a 5% defective rate, we aim to determine the minimum number of inspections, denoted as a, required to ensure that the probability of observing at least one defective item reaches or exceeds 50%.

To find a such that

$$p(\mathcal{X} \leq a) \geq 0.50$$

where a is known as the 50% quantile of geometric distribution.

Generally, k percentile provides the minimum interval value of a such that

$$p(\mathcal{X} \leq a) \geq k/100 \tag{16}$$

Equation (16) can be expressed as:

$$1 - (1 - \mathcal{P}_n)^a \geq \frac{k}{100} \tag{17}$$

Further simplification of (17) yields:

$$a \leq \frac{\ln(1 - \frac{k}{100})}{\ln((1 - \mathcal{P}_n))} \tag{18}$$

Solution of (8) provides the minimum interval value of a. For example the 50% quantile for defective rate $\mathcal{P}_n = [0.1, 0.15]$ can be found utilizing (8) as:

$$a \leq \frac{\ln(1 - 0.5)}{\ln(1 - [0.1, 0.15])} \cong [4, 6]$$

This means that there is at least 50% chance to get the first success in the trial interval [4, 6]. In general the inverse distribution can be used to produce random neutrosophic variable from the model as:

$$\mathcal{G}_n(\mathcal{X})^{-1} = \frac{ln(1-u)}{ln(1-\mathcal{P}_n)}; \quad 0 < u < 1. \tag{19}$$

The (19) based on inverse transformation method and can used to generate random data from the proposed NGD.

By taking the value $\mathcal{P}_n = [0.2, 0.4]$, exact mean and variance from Theorem 1 and Theorem 2 can be calculated as follows:

$$E(\mathcal{X}) = \frac{1 - \mathcal{P}_n}{\mathcal{P}_n}$$

$$= \frac{1 - [0.2, 0.4]}{[0.2, 0.4]}$$

$$E(\mathcal{X}) = [1.5, 4]$$

$$V(\mathcal{X}) = \frac{1 - \mathcal{P}_n}{\mathcal{P}_n{}^2}$$

$$= \frac{1-[0.2,0.4]}{[0.2,0.4]^2}$$

$V(\mathcal{X}) = [3.75, 20]$

Thus exact mean and variance of the proposed distribution by considering $\mathcal{P}_n = [0.2,0.4]$ are $[1.5, 4]$ and $[3.75, 20]$ respectively. Now we will see that our simulation results are also in close approximation to exact values.

The study uses a larger sequence of random numbers generated from 10,000 Monte Carlo simulations to estimate the parameter of a proposed model. The parameter range considered in the study is between 0.2 and 0.4. To obtain simulated results, a program written in R is utilized. Additionally, the program utilizes the "moments" package to analyze moment-based characteristics of the proposed distribution. The larger sequence of random numbers generated from 10,000 Monte Carlo simulations allows for a more accurate estimation of the parameter in the proposed model. By considering a parameter range between 0.2 and 0.4, the study ensures a comprehensive analysis of the distribution's characteristics. Table 1 displays the estimation results of the NGD parameter using the generated simulated data.

Table 1: Summary statistics of the NGD based on simulated data.

Properties	Estimated values
MLE Estimate	[0.20, 0.40]
Mean	[2.49, 4.98]
Variance	[3.73, 19.93]
Skewness	[2.05, 2.00]
Quartile 1	[1, 2]
Quartile 2	[2, 4]
Quartile 3	[3, 7]

The results in Table 1 show that due to uncertainty in the parameter of NGD, the characteristics of the distribution are interval based and imprecise. Furthermore, the simulated results closely approximate the true characteristics of the distribution.

5. Real Data Applications

In this section, some numerical examples have been considered to illustrate the application of the concepts discussed in this work. These examples serve to provide a practical understanding of how the concepts can be applied in real-life scenarios. By showcasing numerical calculations and their corresponding interpretations, readers can better grasp the significance and implications of the discussed concepts.

Example1: Assume that a production machine has a faulty rate ranging from 5% to 8%. Considering the unknown defective rate of the machine's products, which ranges from$\mathcal{P}_n = [0.05, 0.08]$, what is the probability range for the occurrence of the first defective item in the third inspection?

Given the defective rate $\mathcal{P}_n = [0.05, 0.08]$

Let $\mathcal{X}$ be the neutrosophic random variable which denotes the number of defective items produced by the machine.

Neutrosophically, the defective rate is in the range $\mathcal{P}_n = [0.05, 0.08]$, signifying the uncertainty or imprecision in the defective rate of the machine's products.

The probability of the first defective occurring in the third item can be calculated under this interval probability as:

$p(\mathcal{X} = 3) \quad = [0.05, 0.08]\big[1 - [0.05, 0.08]\big]^2$
$= [0.05, 0.08][0.8464, 0.9025]$
$= [0.042, 0.072]$

By evaluating these expressions we found the probability range of [0.042,0.072] for the occurrence of the first defective item in the third position. This range takes into consideration the imprecision or uncertainty in the defective rate, which falls between 5% and 8%.

Example 2: There is an estimated possibility of [0.4, 0.6] in a specific Malaysian city that a randomly selected individual owns a motorcycle. What is the likelihood that the first motorbike owner to be encountered among the first four people interviewed in this city will be the fourth person interviewed?

We must take into account this interval in the neutrosophic context, where the estimated likelihood of owning a motorcycle is between 0.4 and 0.6 (i.e., $\mathcal{P}_n = [0.4, 0.6]$), to determine the range of probabilities for the occurrence.

Let $\mathcal{X}$ be the random variable that denotes the number of people having motorbike.

$p(\mathcal{X} = 4) = [0.4, 0.6]\big[1 - [0.4, 0.6]\big]^3$
$= [0.4, 0.6][0.064, 0.216]$
$= [0.0256, 0.1296]$

Thus, assessing these expressions according to neutrosophic arithmetic rules will yield a range of probability $[0.0256, 0.1296]$, taking into consideration the imprecision or uncertainty in the estimated likelihood of motorcycle ownership between 0.4 and 0.6, in the event that the fourth interviewee is the first to have a motorcycle.

Example 3: Calculate the probability of a student pilot passing the written test for a private pilot's license on their third attempt, assuming that the probability of passing the test is between 0.2 and 0.3. Let $\mathcal{X}$ be the neutrosophic random variable that denotes the number of attempts a student makes to pass this test.

Now the required probability can be obtained as:

$p(\mathcal{X} = 4) = [0.2, 0.3]\big[1 - [0.2, 0.3]\big]^2$
$= [0.2, 0.3][0.49, 0.64]$
$= [0.098, 0.192]$

Example 4: What is the neutrosophic probability of encountering the first defective product within the initial six inspections, given a defective rate ranging from 0.03 to 0.05?

To solve this problem, we need to find involve the neutrosophic distribution function as described in (3).

$p(\mathcal{X} \leq 6) = 1 - [1 - \mathcal{P}_n]^6$

where $\mathcal{P}_n = [0.03, 0.05]$

Now

$p(\mathcal{X} \leq 6) = 1 - \big[1 - [0.03, 0.05]\big]^6$
$= 1 - [0.95, 0.97]^6$
$= [0.167, 0.265]$

Based on the provided imprecise defective rate (0.03 to 0.05), the neutrosophic probability of first defective item out of six inspected items fall between 0.0167 and 0.265. This range indicates that there is a relatively low probability of encountering the first defective item, but it is not entirely unlikely.

6. Concluding Remarks

The neutrosophic geometric distribution (NGD) is a revolutionary framework that has been introduced in this research. It is derived from the classical geometric distribution and aims to handle imprecise data analysis. By doing so, it offers a reliable and generalized method for conducting modern statistical investigations for another class of data. We have extensively examined the basic characteristics of the NGD in a neutrosophic setting and clarified its essential reliability functions. To make it more useful in real-life situations, we have devised most method of the ML estimation. The effectiveness of this technique in determining the NGD parameters has been demonstrated through several numerical instances, proving its applicability in real-world situations. Furthermore, we have focused on developing the NGD's quantile function by the inverse cumulative function method. This function enabled us to generate simulation data, serving as a valuable tool for estimating parameter and providing insightful summary statistics on the behavior of the proposed model. We have considered real-life situations to demonstrate the application of NGD and enhance the comprehension of its theoretical concepts.

Furthermore, our research acts as a connection between classical structures and the innovative neutrosophic framework, enabling future developments in extending geometric distribution to neutrosophic domain and exploring its diverse applications.

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Neutrosophic hybrid structures and neutrosophic hybrid matrices

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Abstract. Motivated by the theory of hybrid structures, our aim in this paper is to introduce the notion of hybrid matrices and bring out an application. Some operations on hybrid matrices are discussed. The notion of hybrid matrices is then generalized by introducing the novel idea of neutrosophic hybrid matrices. Some interesting operations and results on neutrosophic hybrid matrices are presented. As an application a multi-criteria decision making (MCDM) problem is presented together with an algorithm and example. The new method is compared with the existing one to exibit its efficiency.

Keywords: soft set; soft matrix; hybrid structure; hybrid matrix; neutrosophic hybrid structure; neutrosophic hybrid matrix

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1. **Introduction**

An innovative idea of soft set theory has been efficiently developed by Molodtsov [15]. This tool has wide scope of applications in several fields such as engineering, medicine, sciences and mathematical modelling. He identified that the classical and recent theories play vital role in the study of uncertainty. However with the rapidly growing quantity and type of uncertainties, these ideas have their own hurdles and drawbacks as given in Molodtsov [15]. Recent applications of soft sets, introduction to soft matrices and their developments can be viewed in the articles Çağman et al., Maji et al., Mondal et al., Vijayabalaji et al. ([5], [14], [16], [20]).

Neutrosophic set is a modern tool in mathematics extensively used for problems containing imprecise, indeterminant and inconsistent data. This novel idea was initiated by Smarandache [19]. This is a generalized concept of fuzzy set theory by Zadeh [23] and intuitionistic fuzzy set by Atanassov [3]. It is established that neutrosophic sets produce more accurate results than those obtained by using intuitionistic fuzzy sets or fuzzy sets.

Maji [12] has further generalized the new concept of neutrosophic set to neutrosophic soft set. The notion of neutrosophic soft matrix was developed by Deli et a.l [7]. The novelty of neutrosophic set is that it comprises of three various membership functions namely a truth, an indeterminacy and a falsity membership functions. Jun [10] applied soft set theory to BCK/BCI algebra. A remarkable theory of hybrid structure was by introduced Jun et al. [11]. The novelty of this structure is that it combines soft set with its grade. An algorithm to exhibit the application of neutrosophic hybrid matrix is also provided.

So far no systematic development has been made in the theory of hybrid matrix using hybrid structure. Our main motivation is to present the notion of hybrid matrices and study their properties. We then intend to generalize the idea of hybrid matrices to neutrosophic hybrid matrices using neutrosphic structure as a tool.

In this paper, some preliminaries about soft set, soft matrix, hybrid structure and some operations between two hybrid structures are provided in section 2. Also we define complement of a hybrid structure, cartesian product and hybrid relation between two hybrid structures and introduce the concept of the hybrid matrices and various types of hybrid matrices with suitable examples in section 3. We introduce various operations on hybrid matrices and some properties of hybrid matrices are also studied in section 4. In section 5 we define neutrosophic hybrid structure and its operations using inception of neutrosophic concepts like neutrosophic set, neutrosophic soft set and neutrosophic soft matrices. Section 6 defines the notion of neutrosophic hybrid matrices involving several operations and we study their properties with suitable examples. A MCDM problem based on neutrosophic hybrid matrix and a comparative analysis of our work with Maji's [13] work is also carried out in section 7.

2. Preliminaries

The basic ideas are presented below. For convenient let us represent $\mathcal{U}$ to be an universe set, $\mathbb{H}$ being a set of parameters and $\mathcal{P}(\mathcal{U})$ representing power set of $\mathcal{U}$ with $\mathbb{A} \subseteq \mathbb{H}$.

Definition 2.1. [15] A pair $(\mathcal{X}, \mathbb{A})$ is called soft set over $\mathcal{U}, \mathcal{X} : \mathbb{A} \longrightarrow \mathcal{P}(\mathcal{U})$.

Definition 2.2. [16] A representation of soft set in matrix form is called as soft matrix.

Definition 2.3. [11] $\mathcal{X}_\lambda = (\mathcal{X}, \lambda) : \mathbb{H} \longrightarrow \mathcal{P}(\mathcal{U}) \times \mathcal{I}, \varrho \longrightarrow (\mathcal{X}(\varrho), \lambda(\varrho))$ is called as hybrid structure where $\mathcal{X} : \mathbb{H} \longrightarrow \mathcal{P}(\mathcal{U}), \lambda : \mathbb{H} \longrightarrow \mathcal{I}$ are mappings and $\mathcal{I}$ is the unit interval $[0, 1]$.

Example 2.4. [11] Let $\mathcal{U} = \{\upsilon_1, \upsilon_2, \upsilon_3, \ldots, \upsilon_{10}\}$ be the universe set and $\mathbb{H} = \{\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5, \varrho_6\}$ be the set of parameters.

TABLE 1. Representation of the hybrid structure $\mathcal{X}_\lambda$

$\mathbb{H}$	$\mathcal{X}$	λ
ϱ_1	$x_1 = \{\upsilon_1, \upsilon_2\}$	0.2
ϱ_2	$x_2 = \{\upsilon_2, \upsilon_3, \upsilon_4, \upsilon_6\}$	0.4
ϱ_3	$x_3 = \{\upsilon_3, \upsilon_5, \upsilon_7\}$	0.1
ϱ_4	$x_4 = \{\upsilon_1, \upsilon_2, \upsilon_6, \upsilon_9\}$	0.9
ϱ_5	$x_5 = \{\upsilon_6, \upsilon_7\}$	0.6
ϱ_6	$x_6 = \{\upsilon_1, \upsilon_2, \upsilon_4\}$	0.8

Definition 2.5. [11] Let $\mathcal{X}_\lambda$ and $\mathcal{Y}_\gamma$ be hybrid structures in $\mathbb{H}$. Then $\mathcal{X}_\lambda \sqcap \mathcal{Y}_\gamma : \mathbb{H} \longrightarrow \mathcal{P}(\mathcal{U}) \times \mathcal{I}, \varrho \longrightarrow ((\mathcal{X}_\lambda \sqcap \mathcal{Y}_\gamma)(\varrho), (\lambda \vee \gamma)(\varrho))$ for all $\varrho \in \mathbb{H}$, is called as the hybrid intersection where

$\mathcal{X}_\lambda \sqcap \mathcal{Y}_\gamma : \mathbb{H} \longrightarrow \mathcal{P}(\mathcal{U}), \varrho \longrightarrow \mathcal{X}(\varrho) \sqcap \mathcal{Y}(\varrho)$

$\vee\gamma : \mathbb{H} \longrightarrow \mathcal{I}, \varrho \longrightarrow \curlyvee\{\lambda(\varrho), \gamma(\varrho)\}$.

Definition 2.6. [11] Let $\mathcal{X}_\lambda$ and $\mathcal{Y}_\gamma$ be hybrid structures in $\mathbb{H}$. Then $\mathcal{X}_\lambda \sqcup \mathcal{Y}_\gamma : \mathbb{H} \longrightarrow \mathcal{P}(\mathcal{U}) \times \mathcal{I}, \varrho \longrightarrow ((\mathcal{X}_\lambda \sqcup \mathcal{Y}_\gamma)(\varrho), (\lambda \wedge \gamma)(\varrho))$ for all $\varrho \in \mathbb{H}$, is called as the hybrid union where

$\mathcal{X}_\lambda \sqcup \mathcal{Y}_\gamma : \mathbb{H} \longrightarrow \mathcal{P}(\mathcal{U}), \varrho \rightarrow \mathcal{X}(\varrho) \sqcup \mathcal{Y}(\varrho)$

$\wedge\gamma : \mathbb{H} \longrightarrow \mathcal{I}, \varrho \longrightarrow \curlywedge\{\lambda(\varrho), \gamma(\varrho)\}$.

Definition 2.7. [7] A mapping $\mathcal{X}_\mathcal{N} : \mathbb{A} \longrightarrow \mathcal{N}(\mathcal{U})$ ia called as neutrosophic soft set over $\mathcal{U}$, $\mathcal{N}(\mathcal{U})$ being the set of all neutrosophic sets in $\mathcal{U}$.

Definition 2.8. [7] Matrix representation of the neutrosophic soft set is called as the neutrosophic soft matrix.

3. **Hybrid matrix and its types**

Inspired by the theory of soft matrices, we introduce the concept of hybrid matrix and its types. Before entering into the notion hybrid matrix we define the complement, cartesian product and relation on hybrid structure as follows.

Definition 3.1. $\mathcal{X}^c_\lambda = (\mathcal{X}^c, \lambda^c) : \mathbb{H}^c \rightarrow \mathcal{P}(\mathcal{U}) \times \mathcal{I}, \varrho \rightarrow (\mathcal{X}^c(\varrho), \lambda^c(\varrho))$ is called the complement of a hybrid structure where $\mathcal{X}^c(\varrho) = \mathcal{U} - \mathcal{X}(\varrho)$ and $\lambda^c(\varrho) = 1 - \lambda(\varrho)$ for all $\varrho \in \neg\mathbb{H}$.

Definition 3.2. Let $\mathcal{X}_\lambda$ and $\mathcal{Y}_\gamma$ be hybrid structures in $\mathbb{H}$. The cartesian product of $\mathcal{X}_\lambda$ and $\mathcal{Y}_\gamma$is:

$$\mathcal{X}_\lambda \times \mathcal{Y}_\gamma = \{\{(\theta,\eta) : \theta \in \mathcal{X}(\varrho), \eta \in \mathcal{Y}(\varrho)\}, \min\{\lambda(\varrho), \gamma(\varrho)\}\}, \text{ for all } \varrho \in \mathbb{H}.$$

Definition 3.3. Given two hybrid structures $\mathcal{X}_\lambda$ and $\mathcal{Y}_\gamma$ in $\mathbb{H}$, then the hybrid relation between $\mathcal{X}_\lambda$ and $\mathcal{Y}_\gamma$is:

$$\mathcal{R} = \{\{(\theta,\eta) : \theta \in \mathcal{X}(\varrho), \eta \in \mathcal{Y}(\varrho)\}, \min\{\lambda(\varrho), \gamma(\varrho)\}\} \subset \mathcal{X}_\lambda \times \mathcal{Y}_\gamma, \text{ for all } \varrho \in \mathbb{H}.$$

Definition 3.4. The hybrid matrix over $(\mathcal{X}_\lambda, \mathbb{H})$ is defined by $[\mathfrak{M}_{\mathfrak{H}}] = [\mathfrak{M}(\mathcal{X}_\lambda, \mathbb{H})] = \mathfrak{M}\left[(\mathcal{X}(\varrho), \lambda(\varrho))\right] = (\mathfrak{m}_{ij})_{m\times n}$, for some $\varrho \in \mathbb{H}$. That is, a hybrid matrix is a matrix whose elements are the elements of the hybrid structure $(\mathcal{X}_\lambda, \mathbb{H})$.

That is, $[\mathfrak{M}_{\mathfrak{H}}] = [\mathfrak{M}(\mathcal{X}_\lambda, \mathbb{H})] = \begin{bmatrix} (x_1, 0.2) & (x_3, 0.1) & (x_4, 0.9) \\ (x_2, 0.4) & (x_6, 0.8) & (x_1, 0.2) \\ (x_3, 0.1) & (x_1, 0.2) & (x_5, 0.6) \\ (x_6, 0.8) & (x_2, 0.4) & (x_3, 0.1) \end{bmatrix}$.

Definition 3.5. Let $[\mathfrak{M}_{\mathfrak{H}}] = [\mathfrak{M}(\mathcal{X}_\lambda, \mathbb{H})] = \mathfrak{M}\left[(\mathcal{X}(\varrho), \lambda(\varrho))\right]$ be a hybrid matrix over a hybrid structure $(\mathcal{X}_\lambda, \mathbb{H})$. Then the zero hybrid matrix is $[\mathfrak{M}_{\mathfrak{H}}] = [0]$ if $\mathcal{X}(\varrho) = \phi, \lambda(\varrho) = 0$, for all $\varrho \in \mathbb{H}$. That is $[\mathfrak{M}_{\mathfrak{H}}] = (\mathfrak{m}_{ij})_{m\times n} = \mathfrak{M}[(\phi, 0)] \forall\ i$ and j.

Definition 3.6. Let $[\mathfrak{M}_{\mathfrak{H}}] = [\mathfrak{M}(\mathcal{X}_\lambda, \mathbb{H})] = \mathfrak{M}\left[(\mathcal{X}(\varrho), \lambda(\varrho))\right]$ be a hybrid matrix over a hybrid structure $(\mathcal{X}_\lambda, \mathbb{H})$. Then the universe hybrid matrix is $[\mathfrak{M}_{\mathfrak{H}}] = [\mathcal{U}]$ if $\mathcal{X}(\alpha) = \mathcal{U}, \lambda(\varrho) = 1$, for all $\varrho \in \mathbb{H}$. That is $[\mathfrak{M}_{\mathfrak{H}}] = (\mathfrak{m}_{ij})_{m\times n} = \mathfrak{M}[(\mathcal{U}, 1)] \forall\ i$ and j.

Definition 3.7. A hybrid row matrix is a matrix with single row.

Example 3.8. An example of hybrid row matrix is $[\mathfrak{M}_{\mathfrak{H}}] = \begin{bmatrix} (x_1, 0.2) & (x_3, 0.1) & (x_4, 0.9) \end{bmatrix}$.

Definition 3.9. A hybrid column matrix is a matrix with single column.

Example 3.10. An example of hybrid column matrix is $[\mathfrak{M}_{\mathfrak{H}}] = \begin{bmatrix} (x_1, 0.2) \\ (x_2, 0.4) \\ (x_3, 0.1) \\ (x_6, 0.8) \end{bmatrix}$.

Definition 3.11. A hybrid matrix with equal number of rows and columns is called hybrid square matrix.

Example 3.12. An example of hybrid square matrix is $[\mathfrak{M}_{\mathfrak{H}}] = \begin{bmatrix} (x_1, 0.2) & (x_3, 0.1) & (x_4, 0.9) \\ (x_2, 0.4) & (x_6, 0.8) & (x_1, 0.2) \\ (x_3, 0.1) & (x_1, 0.2) & (x_5, 0.6) \end{bmatrix}$.

4. Operations on hybrid matrices

Some interesting operations on hybrid matrices are presented in this section.

Definition 4.1. Let $[\mathfrak{M}_{\mathfrak{H}}] = [\mathfrak{M}(\mathcal{P}_\eta, \mathbb{H})] = (\mathfrak{m}_{ij})_{m\times n}$ and $[\mathfrak{N}_{\mathfrak{H}}] = [\mathfrak{N}(\mathcal{Q}_\gamma, \mathbb{H})] = (\mathfrak{n}_{ij})_{m\times n}$ be two hybrid matrices of same order over the hybrid structure $(\mathcal{X}_\lambda, \mathbb{H})$.
Then the AND operation of two hybrid matrices is given below.

$$[\mathfrak{M}_{\mathfrak{H}}]AND[\mathfrak{N}_{\mathfrak{H}}] = [\mathfrak{M}(\mathcal{P}_\eta, \mathbb{H})] \curlywedge [\mathfrak{N}(\mathcal{Q}_\gamma, \mathbb{H})] = [\mathfrak{L}_{\mathfrak{H}}],$$

where $[\mathfrak{L}_{\mathfrak{H}}] = (\mathfrak{l}_{ij})_{m\times n} = [\mathfrak{L}(\mathcal{R}_\nu, \mathbb{H})] = [\mathfrak{L}\left(\mathcal{R}(\varrho), \nu(\varrho)\right)]$
$= [\mathfrak{L}\left(\mathcal{R}(\varrho) = \mathcal{P}(\varrho) \curlywedge \mathcal{Q}(\varrho), \nu(\varrho) = \max\{\eta(\varrho), \gamma(\varrho)\}\right)]$, for some $\varrho \in \mathbb{H}$.

Example 4.2. Let $\mathcal{U} = \{\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4, \upsilon_5\}$ be the universe set and $\mathbb{H} = \{\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5\}$ be the set of parameters.
Let $[\mathfrak{M}_{\mathfrak{H}}] = \begin{bmatrix} (\{\upsilon_1, \upsilon_2\}, 0.2) & (\{\upsilon_3, \upsilon_4, \upsilon_5\}, 0.1) \\ (\{upsilon_1, \upsilon_4, \upsilon_5\}, 0.4) & (\{\upsilon_2, \upsilon_3\}, 0.8) \end{bmatrix}$
and $[\mathfrak{N}_{\mathfrak{H}}] = \begin{bmatrix} (\{\upsilon_3, \upsilon_5\}, 0.1) & (\{\upsilon_1, \upsilon_3\}, 0.6) \\ (\{\upsilon_1, \upsilon_2, \upsilon_3\}, 0.9) & (\{\upsilon_2, \upsilon_3, \upsilon_4\}, 0.4) \end{bmatrix}$.
Then $[\mathfrak{M}_{\mathfrak{H}}]AND[\mathfrak{N}_{\mathfrak{H}}] = \begin{bmatrix} (\phi, 0.2) & (\{\upsilon_3\}, 0.6) \\ (\{\upsilon_1\}, 0.9) & (\{\upsilon_2, \upsilon_3\}, 0.8) \end{bmatrix}$.

Definition 4.3. Let $[\mathfrak{M}_{\mathfrak{H}}] = [\mathfrak{M}(\mathcal{P}_\eta, \mathbb{H})] = (\mathfrak{m}_{ij})_{m\times n}$ and $[\mathfrak{N}_{\mathfrak{H}}] = [\mathfrak{N}(\mathcal{Q}_\gamma, \mathbb{H})] = (\mathfrak{n}_{ij})_{m\times n}$ be two hybrid matrices of same order over the hybrid structure $(\mathcal{X}_\lambda, \mathbb{H})$.
Then the OR operation of two hybrid matrices is given below.

$$[\mathfrak{M}_{\mathfrak{H}}]OR[\mathfrak{N}_{\mathfrak{H}}] = [\mathfrak{M}(\mathcal{P}_\eta, \mathbb{H})] \curlyvee [\mathfrak{N}(\mathcal{Q}_\gamma, \mathbb{H})] = [\mathfrak{L}_{\mathfrak{H}}],$$

where $[\mathfrak{L}_{\mathfrak{H}}] = (\mathfrak{l}_{ij})_{m\times n} = [\mathfrak{L}(\mathcal{R}_\nu, \mathbb{H})] = [\mathfrak{L}\left((\mathcal{R}(\varrho), \nu(\varrho)\right)]$
$= [\mathfrak{L}\left(\mathcal{R}(\varrho) = \mathcal{P}(\varrho) \curlyvee \mathcal{Q}(\varrho), \nu(\varrho) = \min\{\eta(\varrho), \gamma(\varrho)\}\right)]$, for some $\varrho \in \mathbb{H}$.

Example 4.4. Let $\mathcal{U} = \{\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4, \upsilon_5\}$ be the universe set and $\mathbb{H} = \{\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5\}$ be the set of parameters.
Let $[\mathfrak{M}_{\mathfrak{H}}] = \begin{bmatrix} (\{\upsilon_1, \upsilon_2\}, 0.2) & (\{\upsilon_3, \upsilon_4, \upsilon_5\}, 0.1) \\ (\{\upsilon_1, \upsilon_4, \upsilon_5\}, 0.4) & (\{\upsilon_2, \upsilon_3\}, 0.8) \end{bmatrix}$
and $[\mathfrak{N}_{\mathfrak{H}}] = \begin{bmatrix} (\{\upsilon_3, \upsilon_5\}, 0.1) & (\{\upsilon_1, \upsilon_3\}, 0.6) \\ (\{\upsilon_1, \upsilon_2, \upsilon_3\}, 0.9) & (\{\upsilon_2, \upsilon_3, \upsilon_4\}, 0.4) \end{bmatrix}$.
Then $[\mathfrak{M}_{\mathfrak{H}}]OR[\mathfrak{N}_{\mathfrak{H}}] = \begin{bmatrix} (\{\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_5\}, 0.1) & (\{\upsilon_1, \upsilon_3, \upsilon_4, \upsilon_5\}, 0.1) \\ (\mathcal{U}, 0.4) & (\{\upsilon_2, \upsilon_3, \upsilon_4\}, 0.4) \end{bmatrix}$.

Definition 4.5. Let $[\mathfrak{M}_{\mathfrak{H}}] = [\mathfrak{M}(\mathcal{P}_\eta, \mathbb{H})] = (\mathfrak{m}_{ij})_{m\times n}$ be a hybrid matrix over the hybrid structure $(\mathcal{X}_\lambda, \mathbb{H})$. Then $[\mathfrak{M}_{\mathfrak{H}}]^c = \left(\mathfrak{M}^c_{ij}\right)_{m\times n} = [\mathfrak{M}\left(\mathcal{U} - \mathcal{P}(\varrho), 1 - \eta(\varrho)\right)]$,for some $\varrho \in \mathbb{H}$, is called the complement of a hybrid matrix.

Example 4.6. Let $\mathcal{U} = \{\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4, \upsilon_5\}$ be the universe set and $\mathbb{H} = \{\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5\}$ be the set of parameters.

Let $[\mathfrak{M}_{\mathfrak{H}}] = \begin{bmatrix} (\{\upsilon_1, \upsilon_2\}, 0.2) & (\{\upsilon_3, \upsilon_4, \upsilon_5\}, 0.1) \\ (\{\upsilon_1, \upsilon_4, \upsilon_5\}, 0.4) & (\{\upsilon_2, \upsilon_3\}, 0.8) \end{bmatrix}$

and $[\mathfrak{M}_{\mathfrak{H}}]^c = \begin{bmatrix} (\{\upsilon_3, \upsilon_4, \upsilon_5\}, 0.8) & (\{\upsilon_1, \upsilon_2\}, 0.9) \\ (\{\upsilon_2, \upsilon_3\}, 0.6) & (\{\upsilon_1, \upsilon_4, \upsilon_5\}, 0.2) \end{bmatrix}$.

Definition 4.7. Let $[\mathfrak{M}_{\mathfrak{H}}] = [\mathfrak{M}(\mathcal{P}_\eta, \mathbb{H})] = (\mathfrak{m}_{ij})_{m\times n}$ and $[\mathfrak{N}_{\mathfrak{H}}] = [\mathfrak{N}(\mathcal{Q}_\gamma, \mathbb{H})] = (\mathfrak{n}_{ij})_{m\times n}$ be two hybrid matrices over the hybrid structure $(\mathcal{X}_\lambda, \mathbb{H})$.

Then the union operation of two hybrid matrices is given below.

$$[\mathfrak{M}_{\mathfrak{H}}] \sqcup [\mathfrak{N}_{\mathfrak{H}}] = [\mathfrak{M}(\mathcal{P}_\eta, \mathbb{H})] \sqcup [\mathfrak{N}(\mathcal{Q}_\gamma, \mathbb{H})] = [\mathfrak{L}_{\mathfrak{H}}] = (\mathfrak{l}_{ij}).$$

Remark 4.8. $\mathfrak{l}_{ij} = \sqcup_\varrho \mathcal{R}_\nu(\varrho) = \sqcup_\varrho (\mathcal{R}(\varrho), \nu(\varrho))$, where ϱ being the parameter which is common of the i^{th} row of $[\mathfrak{M}_{\mathfrak{H}}]$ and j^{th} column of $[\mathfrak{N}_{\mathfrak{H}}]$ and $\nu(\varrho) = \min\{\eta(\varrho), \gamma(\varrho)\}$.

Example 4.9. Let $\mathcal{U} = \{\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4, \upsilon_5\}$ be the universe set and $\mathbb{H} = \{\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5\}$ be the set of parameters.

$$\text{Let } [\mathfrak{M}_{\mathfrak{H}}] = \begin{bmatrix} (\{\upsilon_2, \upsilon_3\}, 0.8) & (\{\upsilon_4, \upsilon_5\}, 0.1) & (\{\upsilon_2, \upsilon_3, \upsilon_4\}, 0.6) \\ (\{\upsilon_1, \upsilon_3, \upsilon_4\}, 0.4) & (\{\upsilon_2, \upsilon_3\}, 0.8) & (\{\upsilon_1, \upsilon_2, \upsilon_3\}, 0.1) \\ (\{\upsilon_2, \upsilon_3, \upsilon_4\}, 0.6) & (\{\upsilon_1\}, 0.7) & (\{\upsilon_1, \upsilon_2\}, 0.2) \end{bmatrix}$$

$$\text{and } [\mathfrak{N}_{\mathfrak{H}}] = \begin{bmatrix} (\{\upsilon_3, \upsilon_5\}, 0.1) & (\{\upsilon_1, \upsilon_3\}, 0.6) & (\{\upsilon_2\}, 0.5) \\ (\{\upsilon_2, \upsilon_3, \upsilon_4\}, 0.6) & (\{\upsilon, \upsilon_2, \upsilon_3\}, 0.1) & (\{\upsilon_2, \upsilon_3\}, 0.8) \\ (\{\upsilon_4, \upsilon_5\}, 0.1) & (\{\upsilon_2, \upsilon_3\}, 0.8) & (\{\upsilon_1\}, 0.7) \end{bmatrix}.$$

The union of $[\mathfrak{M}_{\mathfrak{H}}]$ and $[\mathfrak{N}_{\mathfrak{H}}]$ is given by,

$$[\mathfrak{M}_{\mathfrak{H}}] \sqcup [\mathfrak{N}_{\mathfrak{H}}] = \begin{bmatrix} (\{\upsilon_2, \upsilon_3, \upsilon_4, \upsilon_5\}, 0.1) & (\{\upsilon_2, \upsilon_3\}, 0.8) & (\{\upsilon_2, \upsilon_3\}, 0.8) \\ \phi & (\{\upsilon_1, \upsilon_2, \upsilon_3\}, 0.1) & (\{\upsilon_2, \upsilon_3\}, 0.8) \\ (\{\upsilon_2, \upsilon_3, \upsilon_4\}, 0.6) & \phi & (\{\upsilon_1\}, 0.7) \end{bmatrix}.$$

Definition 4.10. Let $[\mathfrak{M}_{\mathfrak{H}}] = [\mathfrak{M}(\mathcal{P}_\eta, \mathbb{H})] = (\mathfrak{m}_{ij})_{m\times n}$ and $[\mathfrak{N}_{\mathfrak{H}}] = [\mathfrak{N}(\mathcal{Q}_\gamma, \mathbb{H})] = (\mathfrak{n}_{ij})_{m\times n}$ be two hybrid matrices over the hybrid structure $(\mathcal{X}_\lambda, \mathbb{H})$.

Then the intersection operation of two hybrid matrices is given below.

$$[\mathfrak{M}_{\mathfrak{H}}] \sqcap [\mathfrak{N}_{\mathfrak{H}}] = [\mathfrak{M}(\mathcal{P}_\eta, \mathbb{H})] \sqcap [\mathfrak{N}(\mathcal{Q}_\gamma, \mathbb{H})] = [\mathfrak{L}_{\mathfrak{H}}] = (\mathfrak{l}_{ij}).$$

Remark 4.11. $\mathfrak{l}_{ij} = \sqcap_\varrho \mathcal{R}_\nu(\varrho) = \sqcap_\varrho (\mathcal{R}(\varrho), \nu(\varrho))$, where ϱ being the parameter which is common to the i^{th} row of $[\mathfrak{M}_{\mathfrak{H}}]$ and j^{th} column of $[\mathfrak{N}_{\mathfrak{H}}]$ and $\nu(\varrho) = \max\{\eta(\varrho), \gamma(\varrho)\}$.

Example 4.12. Let $\mathcal{U} = \{\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4, \upsilon_5\}$ be the universe set and $\mathbb{H} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ be the set of parameters.

Let $[\mathfrak{M}_{\mathfrak{H}}] = \begin{bmatrix} (\{\upsilon_2,\upsilon_3\},0.8) & (\{\upsilon_4,\upsilon_5\},0.1) & (\{\upsilon_2,\upsilon_3,\upsilon_4\},0.6) \\ (\{\upsilon_1,\upsilon_3,\upsilon_4\},0.4) & (\{\upsilon_2,\upsilon_3\},0.8) & (\{\upsilon_1,\upsilon_2,\upsilon_3\},0.1) \\ (\{\upsilon_2,\upsilon_3,\upsilon_4\},0.6) & (\{\upsilon_1\},0.7) & (\{\upsilon_1,\upsilon_2\},0.2) \end{bmatrix}$

and $[\mathfrak{N}_{\mathfrak{H}}] = \begin{bmatrix} (\{\upsilon_3,\upsilon_5\},0.1) & (\{\upsilon_1,\upsilon_3\},0.6) & (\{\upsilon_2\},0.5) \\ (\{\upsilon_2,\upsilon_3,\upsilon_4\},0.6) & (\{\upsilon_1,\upsilon_2,\upsilon_3\},0.1) & (\{\upsilon_2,\upsilon_3\},0.8) \\ (\{\upsilon_4,\upsilon_5\},0.1) & (\{\upsilon_2,\upsilon_3\},0.8) & (\{\upsilon_1\},0.7) \end{bmatrix}$.

The intersection of $[\mathfrak{M}_{\mathfrak{H}}]$ and $[\mathfrak{N}_{\mathfrak{H}}]$ is given by,

$$[\mathfrak{M}_{\mathfrak{H}}] \sqcap [\mathfrak{N}_{\mathfrak{H}}] = \begin{bmatrix} \phi & \phi & \phi \\ \phi & (\{\upsilon_3\},0.8) & \phi \\ \phi & \phi & \phi \end{bmatrix}.$$

Theorem 4.13. *Let $[\mathfrak{M}_{\mathfrak{H}}]$ and $[\mathfrak{N}_{\mathfrak{H}}]$ be two hybrid matrices of same order over the hybrid structure $(\mathcal{X}_\lambda, \mathbb{H})$. Then the following results related to the operations hold.*

(1) $[\mathfrak{M}_{\mathfrak{H}}] \curlyvee [\mathfrak{N}_{\mathfrak{H}}] = [\mathfrak{N}_{\mathfrak{H}}] \curlyvee [\mathfrak{M}_{\mathfrak{H}}]$
(2) $[\mathfrak{M}_{\mathfrak{H}}] \curlywedge [\mathfrak{N}_{\mathfrak{H}}] = [\mathfrak{N}_{\mathfrak{H}}] \curlywedge [\mathfrak{M}_{\mathfrak{H}}]$
(3) $([\mathfrak{M}_{\mathfrak{H}}]^c)^c = [\mathfrak{M}_{\mathfrak{H}}]$
(4) $([\mathfrak{M}_{\mathfrak{H}}] \curlyvee [\mathfrak{N}_{\mathfrak{H}}])^c = [\mathfrak{M}_{\mathfrak{H}}]^c \curlywedge [\mathfrak{N}_{\mathfrak{H}}]^c$
(5) $([\mathfrak{M}_{\mathfrak{H}}] \curlywedge [\mathfrak{N}_{\mathfrak{H}}])^c = [\mathfrak{M}_{\mathfrak{H}}]^c \curlyvee [\mathfrak{N}_{\mathfrak{H}}]^c$.

Proof. Let $[\mathfrak{M}_{\mathfrak{H}}] = [\mathfrak{M}(\mathcal{P}_\eta, \mathbb{H})] = (\mathfrak{m}_{ij})_{m\times n}$ and $[\mathfrak{N}_{\mathfrak{H}}] = [\mathfrak{N}(\mathcal{Q}_\gamma, \mathbb{H})] = (\mathfrak{n}_{ij})_{m\times n}$

(1) $[\mathfrak{M}_{\mathfrak{H}}] \curlyvee [\mathfrak{N}_{\mathfrak{H}}] = [\mathfrak{N}_{\mathfrak{H}}] \curlyvee [\mathfrak{M}_{\mathfrak{H}}]$

$$\begin{aligned} [\mathfrak{M}_{\mathfrak{H}}] \curlyvee [\mathfrak{N}_{\mathfrak{H}}] &= [\mathfrak{M}(\mathcal{P}_\eta, \mathbb{H})] \curlyvee [\mathfrak{N}(\mathcal{Q}_\gamma, \mathbb{H})] \\ &= [\mathfrak{M}\left(\mathcal{P}(\varrho), \eta(\varrho)\right)] \curlyvee [\mathfrak{N}\left(\mathcal{Q}(\varrho), \gamma(\varrho)\right)] \\ &= [\mathfrak{L}\left(\mathcal{P}(\varrho) \curlyvee \mathcal{Q}(\varrho), \min\{\eta(\varrho), \gamma(\varrho)\}\right)] \\ &= [\mathfrak{L}\left(\mathcal{Q}(\varrho) \curlyvee \mathcal{P}(\varrho), \min\{\eta(\varrho), \gamma(\varrho)\}\right)] \\ &= [\mathfrak{N}\left(\mathcal{Q}(\varrho), \gamma(\varrho)\right)] \curlyvee [\mathfrak{M}\left(\mathcal{P}(\varrho), \eta(\varrho)\right)] \\ &= [\mathfrak{N}_{\mathfrak{H}}] \curlyvee [\mathfrak{M}_{\mathfrak{H}}]. \end{aligned}$$

(2) $[\mathfrak{M}_{\mathfrak{H}}] \curlywedge [\mathfrak{N}_{\mathfrak{H}}] = [\mathfrak{N}_{\mathfrak{H}}] \curlywedge [\mathfrak{M}_{\mathfrak{H}}]$
Proof is similar to (1).

(3) $([\mathfrak{M}_{\mathfrak{H}}]^c)^c = [\mathfrak{M}_{\mathfrak{H}}]$
Since, $[\mathfrak{M}_{\mathfrak{H}}]^c = \left(\mathfrak{m}_{ij}^c\right) = [\mathfrak{M}\left(\mathcal{U} - \mathcal{P}(\varrho), 1 - \eta(\varrho)\right)]$

$$\begin{aligned} ([\mathfrak{M}_{\mathfrak{H}}]^c)^c &= [\mathfrak{M}\left(\mathcal{U} - \{\mathcal{U} - \mathcal{P}(\varrho)\}, 1 - \{1 - \eta(\varrho)\}\right)] \\ &= [\mathfrak{M}\left(\mathcal{P}(\varrho), \eta(\varrho)\right)] \\ &= [\mathfrak{M}(\mathcal{P}_\eta, \mathbb{H})] \\ &= [\mathfrak{M}_{\mathfrak{H}}]. \end{aligned}$$

(4) $([\mathfrak{M}_{\mathfrak{H}}] \curlyvee [\mathfrak{N}_{\mathfrak{H}}])^c = [\mathfrak{M}_{\mathfrak{H}}]^c \curlywedge [\mathfrak{N}_{\mathfrak{H}}]^c$

Since, $[\mathfrak{M}_{\mathfrak{H}}]^c = \left(\mathfrak{m}_{ij}^c\right) = [\mathfrak{M}\,(\mathcal{U} - \mathcal{P}(\varrho), 1 - \eta(\varrho))]$

and $[\mathfrak{N}_{\mathfrak{H}}]^c = \left(\mathfrak{n}_{ij}^c\right) = [\mathfrak{N}\,(\mathcal{U} - \mathcal{Q}(\varrho), 1 - \gamma(\varrho))]$

$$\begin{aligned}
([\mathfrak{M}_{\mathfrak{H}}] \curlyvee [\mathfrak{N}_{\mathfrak{H}}])^c &= ([\mathfrak{L}\,(\mathcal{P}(\varrho) \curlyvee \mathcal{Q}(\varrho), \min\{\eta(\varrho), \gamma(\varrho)\})])^c \\
&= [\mathfrak{L}\,(\mathcal{U} - \{\mathcal{P}(\varrho) \curlyvee \mathcal{Q}(\varrho)\}, \max\{1 - \eta(\varrho), 1 - \gamma(\varrho)\})] \\
&= [\mathfrak{L}\,(\{\mathcal{U} - \mathcal{P}(\varrho)\} \curlywedge \{\mathcal{U} - \mathcal{Q}(\varrho)\}, \max\{1 - \eta(\varrho), 1 - \gamma(\varrho)\})] \\
&= [\mathfrak{M}\,(\mathcal{U} - \mathcal{P}(\varrho), 1 - \eta(\varrho))] \curlywedge [\mathfrak{N}\,(\mathcal{U} - \mathcal{Q}(\varrho), 1 - \gamma(\varrho))] \\
&= [\mathfrak{M}_{\mathfrak{H}}]^c \curlywedge [\mathfrak{N}_{\mathfrak{H}}]^c.
\end{aligned}$$

(5) $([\mathfrak{M}_{\mathfrak{H}}] \curlywedge [\mathfrak{N}_{\mathfrak{H}}])^c = [\mathfrak{M}_{\mathfrak{H}}]^c \curlyvee [\mathfrak{N}_{\mathfrak{H}}]^c$

Proof is similar to (4).

Theorem 4.14. *Let $[\mathfrak{L}_{\mathfrak{H}}], [\mathfrak{M}_{\mathfrak{H}}]$ and $[\mathfrak{N}_{\mathfrak{H}}]$ be three hybrid matrices of same order over the hybrid structure $(\mathcal{X}_\lambda, \mathbb{H})$. Then the following results related to the operations hold.*

(1) $([\mathfrak{L}_{\mathfrak{H}}] \curlyvee [\mathfrak{M}_{\mathfrak{H}}]) \curlyvee [\mathfrak{N}_{\mathfrak{H}}] = [\mathfrak{L}_{\mathfrak{H}}] \curlyvee ([\mathfrak{M}_{\mathfrak{H}}] \curlyvee [\mathfrak{N}_{\mathfrak{H}}])$
(2) $([\mathfrak{L}_{\mathfrak{H}}] \curlywedge [\mathfrak{M}_{\mathfrak{H}}]) \curlywedge [\mathfrak{N}_{\mathfrak{H}}] = [\mathfrak{L}_{\mathfrak{H}}] \curlywedge ([\mathfrak{M}_{\mathfrak{H}}] \curlywedge [\mathfrak{N}_{\mathfrak{H}}])$
(3) $[\mathfrak{L}_{\mathfrak{H}}] \curlyvee ([\mathfrak{M}_{\mathfrak{H}}] \curlywedge [\mathfrak{N}_{\mathfrak{H}}]) = ([\mathfrak{L}_{\mathfrak{H}}] \curlyvee [\mathfrak{M}_{\mathfrak{H}}]) \curlywedge ([\mathfrak{L}_{\mathfrak{H}}] \curlyvee [\mathfrak{N}_{\mathfrak{H}}])$
(4) $[\mathfrak{L}_{\mathfrak{H}}] \curlywedge ([\mathfrak{M}_{\mathfrak{H}}] \curlyvee [\mathfrak{N}_{\mathfrak{H}}]) = ([\mathfrak{L}_{\mathfrak{H}}] \curlywedge [\mathfrak{M}_{\mathfrak{H}}]) \curlyvee ([\mathfrak{L}_{\mathfrak{H}}] \curlywedge [\mathfrak{N}_{\mathfrak{H}}])$.

Proof. Let $[\mathfrak{L}_{\mathfrak{H}}] = [\mathfrak{M}(\mathcal{P}_\nu, \mathbb{H})] = (\mathfrak{l}_{ij})_{m \times n}$, $[\mathfrak{M}_{\mathfrak{H}}] = [\mathfrak{M}(\mathcal{Q}_\eta, \mathbb{H})] = (\mathfrak{m}_{ij})_{m \times n}$ and $[\mathfrak{N}_{\mathfrak{H}}] = [\mathfrak{N}(\mathcal{R}_\gamma, \mathbb{H})] = (\mathfrak{n}_{ij})_{m \times n}$

(1) $([\mathfrak{L}_{\mathfrak{H}}] \curlyvee [\mathfrak{M}_{\mathfrak{H}}]) \curlyvee [\mathfrak{N}_{\mathfrak{H}}] = [\mathfrak{L}_{\mathfrak{H}}] \curlyvee ([\mathfrak{M}_{\mathfrak{H}}] \curlyvee [\mathfrak{N}_{\mathfrak{H}}])$

$$\begin{aligned}
([\mathfrak{L}_{\mathfrak{H}}] \curlyvee [\mathfrak{M}_{\mathfrak{H}}]) \curlyvee [\mathfrak{N}_{\mathfrak{H}}] &= ([\mathfrak{L}\,(\mathcal{P}(\varrho), \nu(\varrho))] \curlyvee [\mathfrak{M}\,(\mathcal{Q}(\varrho), \eta(\varrho))]) \curlyvee [\mathfrak{N}\,(\mathcal{R}(\varrho), \gamma(\varrho))] \\
&= ([\mathfrak{S}\,(\mathcal{P}(\varrho) \curlyvee \mathcal{Q}(\varrho), \min\{\nu(\varrho), \eta(\varrho)\})]) \curlyvee [\mathfrak{N}\,(\mathcal{R}(\varrho), \gamma(\varrho))] \\
&= [\mathfrak{T}\,(\{\mathcal{P}(\varrho) \curlyvee \mathcal{Q}(\varrho)\} \curlyvee \mathcal{R}(\varrho), \min\{\nu(\varrho), \eta(\varrho), \gamma(\varrho)\})] \\
&= [\mathfrak{T}\,(\mathcal{P}(\varrho) \curlyvee \{\mathcal{Q}(\varrho) \curlyvee \mathcal{R}\varrho)\}, \min\{\nu(\varrho), \eta(\varrho), \gamma(\varrho)\})] \\
&= [\mathfrak{L}\,(\mathcal{P}(\varrho), \nu(\varrho))] \curlyvee ([\mathfrak{S}\,(\mathcal{Q}(\varrho) \curlyvee \mathcal{R}(\varrho), \min\{\eta(\varrho), \gamma(\varrho)\})]) \\
&= [\mathfrak{L}_{\mathfrak{H}}] \curlyvee ([\mathfrak{M}_{\mathfrak{H}}] \curlyvee [\mathfrak{N}_{\mathfrak{H}}]).
\end{aligned}$$

(2) $([\mathfrak{L}_{\mathfrak{H}}] \curlywedge [\mathfrak{M}_{\mathfrak{H}}]) \curlywedge [\mathfrak{N}_{\mathfrak{H}}] = [\mathfrak{L}_{\mathfrak{H}}] \curlywedge ([\mathfrak{M}_{\mathfrak{H}}] \curlywedge [\mathfrak{N}_{\mathfrak{H}}])$

Proof is similar to (1).

(3) $[\mathfrak{L}_{\mathfrak{H}}] \curlyvee ([\mathfrak{M}_{\mathfrak{H}}] \curlywedge [\mathfrak{N}_{\mathfrak{H}}]) = ([\mathfrak{L}_{\mathfrak{H}}] \curlyvee [\mathfrak{M}_{\mathfrak{H}}]) \curlywedge ([\mathfrak{L}_{\mathfrak{H}}] \curlyvee [\mathfrak{N}_{\mathfrak{H}}])$

$$\begin{aligned} LHS &= [\mathfrak{L}_{\mathfrak{H}}] \curlyvee ([\mathfrak{M}_{\mathfrak{H}}] \curlywedge [\mathfrak{N}_{\mathfrak{H}}]) \\ &= [\mathfrak{L}(\mathcal{P}(\varrho), \nu(\varrho))] \curlyvee ([\mathfrak{M}(\mathcal{Q}(\varrho), \eta(\varrho))] \curlywedge [\mathfrak{N}(\mathcal{R}(\varrho), \gamma(\varrho))]) \\ &= [\mathfrak{L}(\mathcal{P}(\varrho), \nu(\varrho))] \curlyvee [\mathfrak{S}(\mathcal{Q}(\varrho) \curlywedge \mathcal{R}(\varrho), \max\{\eta(\varrho), \gamma(\varrho)\})] \\ &= [\mathfrak{T}(\mathcal{P}(\varrho) \curlyvee \{\mathcal{Q}(\varrho) \curlywedge \mathcal{R}(\varrho)\}, \min\{\nu(\varrho), \max\{\eta(\varrho), \gamma(\varrho)\}\})] \\ &= [\mathfrak{T}((\mathcal{P}(\varrho) \curlyvee \mathcal{Q}(\varrho)) \curlywedge (\mathcal{P}(\varrho) \curlyvee \mathcal{R}(\varrho)), \max\{\min\{\nu(\varrho), \eta(\varrho)\}, \min\{\eta(\varrho), \gamma(\varrho)\}\})] \\ &= [\mathfrak{S}((\mathcal{P}(\varrho) \curlyvee \mathcal{Q}(\varrho)), \min\{\nu(\varrho), \eta(\varrho)\})] \curlywedge [\mathfrak{W}((\mathcal{P}(\varrho) \curlyvee \mathcal{R}(\varrho)), \min\{\eta(\varrho), \gamma(\varrho)\})] \\ &= ([\mathfrak{L}_{\mathfrak{H}}] \curlyvee [\mathfrak{M}_{\mathfrak{H}}]) \curlywedge ([\mathfrak{L}_{\mathfrak{H}}] \curlyvee [\mathfrak{N}_{\mathfrak{H}}]). \end{aligned}$$

(4) $[\mathfrak{L}_{\mathfrak{H}}] \curlywedge ([\mathfrak{M}_{\mathfrak{H}}] \curlyvee [\mathfrak{N}_{\mathfrak{H}}]) = ([\mathfrak{L}_{\mathfrak{H}}] \curlywedge [\mathfrak{M}_{\mathfrak{H}}]) \curlyvee ([\mathfrak{L}_{\mathfrak{H}}] \curlywedge [\mathfrak{N}_{\mathfrak{H}}])$

Proof is similar to (3).

Theorem 4.15. *Let $[\mathfrak{L}_{\mathfrak{H}}], [\mathfrak{M}_{\mathfrak{H}}]$ and $[\mathfrak{N}_{\mathfrak{H}}]$ be three hybrid matrices over the hybrid structure $(\mathcal{X}_\lambda, \mathbb{H})$. Then the following results related to the operations hold.*

(1) $([\mathfrak{L}_{\mathfrak{H}}] \sqcup [\mathfrak{M}_{\mathfrak{H}}]) \sqcup [\mathfrak{N}_{\mathfrak{H}}] = [\mathfrak{L}_{\mathfrak{H}}] \sqcup ([\mathfrak{M}_{\mathfrak{H}}] \sqcup [\mathfrak{N}_{\mathfrak{H}}])$

(2) $([\mathfrak{L}_{\mathfrak{H}}] \sqcap [\mathfrak{M}_{\mathfrak{H}}]) \sqcap [\mathfrak{N}_{\mathfrak{H}}] = [\mathfrak{L}_{\mathfrak{H}}] \sqcap ([\mathfrak{M}_{\mathfrak{H}}] \sqcap [\mathfrak{N}_{\mathfrak{H}}])$.

Proof. Let $[\mathfrak{L}_{\mathfrak{H}}] = [\mathfrak{M}(\mathcal{P}_\nu, \mathbb{H})] = (\mathfrak{l}_{ij})$, $[\mathfrak{M}_{\mathfrak{H}}] = [\mathfrak{M}(\mathcal{Q}_\eta, \mathbb{H})] = (\mathfrak{m}_{ij})$ and $[\mathfrak{N}_{\mathfrak{H}}] = [\mathfrak{N}(\mathcal{R}_\gamma, \mathbb{H})] = (\mathfrak{n}_{ij})$

(1) $([\mathfrak{L}_{\mathfrak{H}}] \sqcup [\mathfrak{M}_{\mathfrak{H}}]) \sqcup [\mathfrak{N}_{\mathfrak{H}}] = [\mathfrak{L}_{\mathfrak{H}}] \sqcup ([\mathfrak{M}_{\mathfrak{H}}] \sqcup [\mathfrak{N}_{\mathfrak{H}}])$

Let $[\mathfrak{L}_{\mathfrak{H}}] \sqcup [\mathfrak{M}_{\mathfrak{H}}] = (\mathfrak{r}_{ij})$, then

$$\mathfrak{r}_{ij} = \sqcup_\varrho \mathcal{S}_\tau(\varrho) = \sqcup_\varrho (\mathcal{S}(\varrho), \tau(\varrho))$$

where ϱ being the parameter which is common of the i^{th} row of $[\mathfrak{L}_{\mathfrak{H}}]$ and j^{th} column of $[\mathfrak{M}_{\mathfrak{H}}]$ and $\tau(\varrho) = \min?\{\nu(\varrho), \eta(\varrho)\}$.

Also, let $([\mathfrak{L}_{\mathfrak{H}}] \sqcup [\mathfrak{M}_{\mathfrak{H}}]) \sqcup [\mathfrak{N}_{\mathfrak{H}}] = (\mathfrak{t}_{ij})$

$$\mathfrak{t}_{ij} = \sqcup_\varrho \mathcal{T}_\theta(\varrho) = \sqcup_\varrho (\mathcal{T}(\varrho), \theta(\varrho))$$

where ϱ being the parameter which is common of the i^{th} row of $[\mathfrak{L}_{\mathfrak{H}}] \sqcup [\mathfrak{M}_{\mathfrak{H}}]$ and j^{th} column of $[\mathfrak{N}_{\mathfrak{H}}]$ and $\theta(\varrho) = \min?\{\tau(\varrho), \gamma(\varrho)\}$.

Clearly, the common parameters of i^{th} row of $[\mathfrak{L}_{\mathfrak{H}}] \sqcup [\mathfrak{M}_{\mathfrak{H}}]$ are the parameters of i^{th} row of $[\mathfrak{L}_{\mathfrak{H}}]$.

$$\mathfrak{t}_{ij} = \sqcup_\varrho \mathcal{T}_\theta(\varrho) = \sqcup_\varrho (\mathcal{T}(\varrho), \theta(\varrho))$$

where ϱ being the parameter which is common of the i^{th} row of $[\mathfrak{L}_{\mathfrak{H}}]$ and j^{th} column of $[\mathfrak{N}_{\mathfrak{H}}]$ and $\theta(\varrho) = \min\{\nu(\varrho), \eta(\varrho), \gamma(\varrho)\}$.

Again let $[\mathfrak{M}_{\mathfrak{H}}] \sqcup [\mathfrak{N}_{\mathfrak{H}}] = (\mathfrak{w}_{ij})$, then

$$\mathfrak{w}_{ij} = \sqcup_{\beta}\mathcal{S}_{\tau}(\beta) = \sqcup_{\beta}\left(\mathcal{S}(\beta), \tau(\beta)\right)$$

where β being the parameter which is common of the i^{th} row of $[\mathfrak{M}_{\mathfrak{H}}]$ and j^{th} column of $[\mathfrak{N}_{\mathfrak{H}}]$ and $\tau(\beta) = \min\{\eta(\beta), \gamma(\beta)\}$.

Also, let $[\mathfrak{L}_{\mathfrak{H}}] \sqcup ([\mathfrak{M}_{\mathfrak{H}}] \sqcup [\mathfrak{N}_{\mathfrak{H}}]) = (\mathfrak{u}_{ij})$, then

$$\mathfrak{u}_{ij} = \sqcup_{\beta}\mathcal{T}_{\theta}(\beta) = \sqcup_{\beta}\left(\mathcal{T}(\beta), \theta(\beta)\right)$$

where β being the parameter which is common of the i^{th} row of $[\mathfrak{L}_{\mathfrak{H}}]$ and j^{th} column of $[\mathfrak{M}_{\mathfrak{H}}] \sqcup [\mathfrak{N}_{\mathfrak{H}}]$ and $\theta(\beta) = \min\{\nu(\beta), \tau(\beta)\}$.

Since the common parameters of $j^t h$ column of $[\mathfrak{M}_{\mathfrak{H}}] \sqcup [\mathfrak{N}_{\mathfrak{H}}]$ are the parameters of j^{th} column of $[\mathfrak{N}_{\mathfrak{H}}]$.

$$\mathfrak{u}_{ij} = \sqcup_{\beta}\mathcal{S}_{\theta}(\beta) = \sqcup_{\beta}\left(\mathcal{S}(\beta), \theta(\beta)\right)$$

where β being the parameter which is common of the i^{th} row of $[\mathfrak{L}_{\mathfrak{H}}]$ and j^{th} column of $[\mathfrak{N}_{\mathfrak{H}}]$ and $\theta(\beta) = \min\{\nu(\beta), \eta(\beta), \gamma(\beta)\}$.

Thus, $\mathfrak{s}_{ij} = \mathfrak{u}_{ij}$.

That is, $([\mathfrak{L}_{\mathfrak{H}}] \sqcup [\mathfrak{M}_{\mathfrak{H}}]) \sqcup [\mathfrak{N}_{\mathfrak{H}}] = [\mathfrak{L}_{\mathfrak{H}}] \sqcup ([\mathfrak{M}_{\mathfrak{H}}] \sqcup [\mathfrak{N}_{\mathfrak{H}}])$.

(2) $([\mathfrak{L}_{\mathfrak{H}}] \sqcap [\mathfrak{M}_{\mathfrak{H}}]) \sqcap [\mathfrak{N}_{\mathfrak{H}}] = [\mathfrak{L}_{\mathfrak{H}}] \sqcap ([\mathfrak{M}_{\mathfrak{H}}] \sqcap [\mathfrak{N}_{\mathfrak{H}}])$

Proof is similar to (1).

5. Neutrosophic hybrid structure and its operations

We define the neutrosophic hybrid structure as a generalization of hybrid structure in this section . We study several operations on neutrosophic hybrid structure with necessary examples.

Definition 5.1. $\mathcal{X}_{\widehat{\mathcal{N}}_{\lambda}} = (\mathcal{X}_{\widehat{\mathcal{N}}}, \lambda) : \mathbb{H} \longrightarrow \mathcal{N}(\mathcal{U}) \times \mathcal{I}$,
$\varrho \longrightarrow (< \varrho, (\mathcal{T}_{\mathbb{H}}(\varrho), \mathcal{I}_{\mathbb{H}}(\varrho), \mathcal{F}_{\mathbb{H}}(\varrho)) >, \lambda(\varrho))$ is called as the neutrosophic hybrid structure where $\mathcal{X}_{\widehat{\mathcal{N}}} : \mathbb{H} \longrightarrow \mathcal{N}(\mathcal{U}), \lambda : \mathbb{H} \longrightarrow \mathcal{I}$ are mappings and $\mathcal{I}$ is the unit interval $[0, 1]$.

Example 5.2. Let $\mathcal{U} = \{\upsilon_1, \upsilon_2, \upsilon_3\}$ be the universe set and $\mathbb{H} = \{\varrho_1, \varrho_2, \varrho_3\}$ be the set of parameters. Then

$$\mathcal{X}_{\widehat{\mathcal{N}}}(\varrho_1) = \{(< \upsilon_1, (0.2, 0.6, 0.5) >, 0.4), (< \upsilon_2, (0.3, 0.5, 0.8) >, 0.2), (< \upsilon_3, (0.8, 0.3, 0.6) >, 0.7)\}$$
$$\mathcal{X}_{\widehat{\mathcal{N}}}(\varrho_2) = \{(< \upsilon_1, (0.2, 0.6, 0.3) >, 0.1), (< \upsilon_2, (0.2, 0.5, 0.1) >, 0.6), (< \upsilon_3, (0.9, 0.8, 0.4) >, 0.3)\}$$
$$\mathcal{X}_{\widehat{\mathcal{N}}}(\varrho_3) = \{(< \upsilon_1, (0.3, 0.4, 0.5) >, 0.9), (< \upsilon_2, (0.3, 0.7, 0.1) >, 0.3), (< \upsilon_3, (0.5, 0.4, 0.2) >, 0.4)\}$$

Definition 5.3. Let $\mathcal{X}_{\widehat{\mathcal{N}}_{\lambda}}$ and $\mathcal{Y}_{\widehat{\mathcal{N}}_{\gamma}}$ be two neutrosophic hybrid structures in $\mathbb{H}$. Then their neutrosophic hybrid intersection is $\mathcal{X}_{\widehat{\mathcal{N}}_{\lambda}} \sqcap \mathcal{Y}_{\widehat{\mathcal{N}}_{\gamma}} = \mathcal{K}_{\widehat{\mathcal{N}}_{\nu}}$ where
$\mathcal{K}_{\widehat{\mathcal{N}}_{\nu}}(\varrho) = \left(< \varrho, \left(\mathcal{T}_{\mathcal{K}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{I}_{\mathcal{K}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{F}_{\mathcal{K}_{\widehat{\mathcal{N}}}}(\varrho)\right) >, \nu(\varrho) = \curlyvee\{\lambda(\varrho), \gamma(\varrho)\}\right),$

$\mathcal{T}_{\mathcal{K}_{\widehat{\mathcal{N}}}}(\varrho) = \curlywedge\{\mathcal{T}_{\mathcal{X}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{T}_{\mathcal{Y}_{\widehat{\mathcal{N}}}}(\varrho)\}, \mathcal{I}_{\mathcal{K}_{\widehat{\mathcal{N}}}}(\varrho) = \curlyvee\{\mathcal{I}_{\mathcal{X}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{I}_{\mathcal{Y}_{\widehat{\mathcal{N}}}}(\varrho)\}$ and
$\mathcal{F}_{\mathcal{K}_{\widehat{\mathcal{N}}}}(\varrho) = \curlyvee\{\mathcal{F}_{\mathcal{X}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{F}_{\mathcal{Y}_{\widehat{\mathcal{N}}}}(\varrho)\}$ for all $\varrho \in \mathbb{H}$.

Definition 5.4. Let $\mathcal{X}_{\widehat{\mathcal{N}}_\lambda}$ and $\mathcal{Y}_{\widehat{\mathcal{N}}_\gamma}$ be two neutrosophic hybrid structures in $\mathbb{H}$. Then their neutrosophic hybrid union is $\mathcal{X}_{\widehat{\mathcal{N}}_\lambda} \sqcup \mathcal{Y}_{\widehat{\mathcal{N}}_\gamma} = \mathcal{K}_{\widehat{\mathcal{N}}_\nu}$ where
$\mathcal{K}_{\widehat{\mathcal{N}}_\nu}(\varrho) = \Big(< \varrho, \Big(\mathcal{T}_{\mathcal{K}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{I}_{\mathcal{K}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{F}_{\mathcal{K}_{\widehat{\mathcal{N}}}}(\varrho)\Big) >, \nu(\varrho) = \rfloor\sqcap\nabla\updownarrow\dagger\sqsupseteq\rceil\lceil\}\rceil\{\lambda(\varrho), \gamma(\varrho)\}\Big)$,
$\mathcal{T}_{\mathcal{K}_{\widehat{\mathcal{N}}}}(\varrho) = \curlyvee\{\mathcal{T}_{\mathcal{X}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{T}_{\mathcal{Y}_{\widehat{\mathcal{N}}}}(\varrho)\}, \mathcal{I}_{\mathcal{K}_{\widehat{\mathcal{N}}}}(\varrho) = \curlywedge\{\mathcal{I}_{\mathcal{X}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{I}_{\mathcal{Y}_{\widehat{\mathcal{N}}}}(\varrho)\}$ and
$\mathcal{F}_{\mathcal{K}_{\widehat{\mathcal{N}}}}(\varrho) = \curlywedge\{\mathcal{F}_{\mathcal{X}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{F}_{\mathcal{Y}_{\widehat{\mathcal{N}}}}(\varrho)\}$ for all $\varrho \in \mathbb{H}$.

Definition 5.5. $\mathcal{X}_{\widehat{\mathcal{N}}_\lambda}{}^c(\varrho) = \Big(< \varrho, \Big(\mathcal{F}_{\mathcal{X}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{I}_{\mathcal{X}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{T}_{\mathcal{X}_{\widehat{\mathcal{N}}}}(\varrho)\Big) >, 1 - \lambda(\varrho)\Big)$, for all $\varrho \in \neg\mathbb{H}$ is called the complement of a neutrosophic hybrid structure.

Definition 5.6. Let $\mathcal{X}_{\widehat{\mathcal{N}}_\lambda}$ and $\mathcal{Y}_{\widehat{\mathcal{N}}_\gamma}$ be two neutrosophic hybrid structures in $\mathbb{H}$. The cartesian product of $\mathcal{X}_{\widehat{\mathcal{N}}_\lambda}$ and $\mathcal{Y}_{\widehat{\mathcal{N}}_\gamma}$ is:
$\mathcal{X}_{\widehat{\mathcal{N}}_\lambda} \times \mathcal{Y}_{\widehat{\mathcal{N}}_\gamma} = \big\{\{(\theta, \eta) : \theta \in \mathcal{X}_{\widehat{\mathcal{N}}}(\varrho), \eta \in \mathcal{Y}_{\widehat{\mathcal{N}}}(\varrho)\}, \min\{\lambda(\varrho), \gamma(\varrho)\}\big\}$, for all $\varrho \in \mathbb{H}$.

Definition 5.7. Let $\mathcal{X}_{\widehat{\mathcal{N}}_\lambda}$ and $\mathcal{Y}_{\widehat{\mathcal{N}}_\gamma}$ be two neutrosophic hybrid structures in $\mathbb{H}$ over $\mathcal{N}(\mathcal{U})$. Then the neutrosophic hybrid relation of $\mathcal{X}_{\widehat{\mathcal{N}}_\lambda}$ and $\mathcal{Y}_{\widehat{\mathcal{N}}_\gamma}$ is:
$\mathcal{R} = \big\{\{(\theta, \eta) : \theta \in \mathcal{X}_{\widehat{\mathcal{N}}}(\varrho), \eta \in \mathcal{Y}_{\widehat{\mathcal{N}}}(\varrho)\}, \min\{\lambda(\varrho), \gamma(\varrho)\}\big\} \subset \mathcal{X}_{\widehat{\mathcal{N}}_\lambda} \times \mathcal{Y}_{\widehat{\mathcal{N}}_\gamma}$, for all $\varrho \in \mathbb{H}$.

6. Neutrosophic hybrid matrix and its properties

In this section we define the neutrosophic hybrid matrix as a generalization of hybrid matrix. We also provide various types of neutrosophic hybrid matrices. Some interesting operations on neutrosophic hybrid matrices are also given. For convenience the following notations are used in this section,
$\max\{\mathcal{T}_{\mathcal{A}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{T}_{\mathcal{B}_{\widehat{\mathcal{N}}}}(\varrho)\} = \mathcal{T}_{\vee(\mathcal{A},\mathcal{B})}(\varrho); \min\{\mathcal{T}_{\mathcal{A}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{T}_{\mathcal{B}_{\widehat{\mathcal{N}}}}(\varrho)\} = \mathcal{T}_{\wedge(\mathcal{A},\mathcal{B})}(\varrho)$;
$\max\{\mathcal{I}_{\mathcal{A}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{I}_{\mathcal{B}_{\widehat{\mathcal{N}}}}(\varrho)\} = \mathcal{I}_{\vee(\mathcal{A},\mathcal{B})}(\varrho); \min\{\mathcal{I}_{\mathcal{A}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{I}_{\mathcal{B}_{\widehat{\mathcal{N}}}}(\varrho)\} = \mathcal{I}_{\wedge(\mathcal{A},\mathcal{B})}(\varrho)$;
$\max\{\mathcal{F}_{\mathcal{A}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{F}_{\mathcal{B}_{\widehat{\mathcal{N}}}}(\varrho)\} = \mathcal{F}_{\vee(\mathcal{A},\mathcal{B})}(\varrho); \min\{\mathcal{F}_{\mathcal{A}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{F}_{\mathcal{B}_{\widehat{\mathcal{N}}}}(\varrho)\} = \mathcal{F}_{\wedge(\mathcal{A},\mathcal{B})}(\varrho)$.

Definition 6.1. Let $\Big(\mathcal{X}_{\widehat{\mathcal{N}}_\lambda}, \mathbb{H}\Big)$ be neutrosophic hybrid structure defined over $\mathcal{N}(\mathcal{U})$. Then the neutrosophic hybrid matrix over $(\mathcal{X}_{\widehat{\mathcal{N}}_\lambda}, \mathbb{H})$ is defined by $[\mathfrak{M}_{\widehat{\mathcal{N}}_\mathfrak{H}}] = [\mathfrak{M}(\mathcal{X}_{\widehat{\mathcal{N}}_\lambda}, \mathbb{H})] = \mathfrak{M}\left[\left(\mathcal{X}_{\widehat{\mathcal{N}}}(\varrho), \lambda(\varrho)\right)\right] = (\mathfrak{m}_{ij})_{m\times n}$, for some $\varrho \in \mathbb{H}$. In other words a neutrosophic hybrid matrix is a matrix whose elements are the elements of the neutrosophic hybrid structure $\Big(\mathcal{X}_{\widehat{\mathcal{N}}_\lambda}, \mathbb{H}\Big)$. That is,

$$[\mathfrak{M}_{\widehat{\mathcal{N}}_\mathfrak{H}}] = [\mathfrak{M}(\mathcal{X}_{\widehat{\mathcal{N}}_\lambda}, \mathbb{H})] = \begin{bmatrix} ((0.2, 0.6, 0.5), 0.4) & ((0.3, 0.5, 0.8), 0.2) & ((0.8, 0.3, 0.6), 0.7) \\ ((0.2, 0.6, 0.3), 0.1) & ((0.2, 0.5, 0.1), 0.6) & ((0.9, 0.8, 0.4), 0.3) \\ ((0.3, 0.4, 0.5), 0.9) & ((0.3, 0.7, 0.1), 0.3) & ((0.5, 0.4, 0.2), 0.4) \end{bmatrix}.$$

Definition 6.2. Let $[\mathfrak{M}_{\widehat{\mathcal{N}}_\mathfrak{H}}] = [\mathfrak{M}(\mathcal{X}_{\widehat{\mathcal{N}}_\lambda}, \mathbb{H})] = \mathfrak{M}\left[\left(\mathcal{X}_{\widehat{\mathcal{N}}}(\varrho), \lambda(\varrho)\right)\right]$ be a neutrosophic hybrid matrix over a neutrosophic hybrid structure $\Big(\mathcal{X}_{\widehat{\mathcal{N}}_\lambda}, \mathbb{H}\Big)$. Then the zero neutrosophic hybrid matrix is $[\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] = (\mathfrak{m}_{ij})_{m\times n} = \mathfrak{M}[((0,1,1),0)]$ for all i and j.

That is, $[\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] = \begin{bmatrix} ((0,1,1),0) & ((0,1,1),0) \\ ((0,1,1),0) & ((0,1,1),0) \end{bmatrix}$.

Definition 6.3. Let $[\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] = [\mathfrak{M}(\mathcal{X}_{\widehat{\mathcal{N}}_\lambda}, \mathbb{H})] = \mathfrak{M}\left[\left(\mathcal{X}_{\widehat{\mathcal{N}}}(\varrho), \lambda(\varrho)\right)\right]$ be a neutrosophic hybrid matrix over a neutrosophic hybrid structure $\left(\mathcal{X}_{\widehat{\mathcal{N}}_\lambda}, \mathbb{H}\right)$. Then the universe neutrosophic hybrid matrix is $[\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] = (\mathfrak{m}_{ij})_{m\times n} = \mathfrak{M}[((1,0,0),1)]$ for all i and j.

That is, $[\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] = \begin{bmatrix} ((1,0,0),1) & ((1,0,0),1) \\ ((1,0,0),1) & ((1,0,0),1) \end{bmatrix}$.

Definition 6.4. Let $[\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] = [\mathfrak{M}(\mathcal{X}_{\widehat{\mathcal{N}}_\lambda}, \mathbb{H})] = (\mathfrak{m}_{ij})_{m\times n}$ be a neutrosophic hybrid matrix over a hybrid neutrosophic structure $\left(\mathcal{X}_{\widehat{\mathcal{N}}_\lambda}, \mathbb{H}\right)$ with respect to a universe $\mathcal{N}(\mathcal{U})$. The complement of a neutrosophic hybrid matrix is

$[\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]^c = (\mathfrak{m}^c_{ij})_{m\times n} = \left[\mathfrak{M}\left(\left(\mathcal{F}_{\mathcal{X}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{I}_{\mathcal{X}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{T}_{\mathcal{X}_{\widehat{\mathcal{N}}}}(\varrho)\right), 1-\lambda(\varrho)\right)\right]$, for all $\varrho \in \neg\mathbb{H}$.

Example 6.5. Let $[\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] = \begin{bmatrix} ((0.3,0.5,0.9),0.2) & ((0.6,0.7,0.2),0.7) \\ ((0.8,0.3,0.1),0.8) & ((0.1,0.5,0.8),0.1) \end{bmatrix}$ be a neutrosophic hybrid matrix. Then the complement of a neutrosophic hybrid matrix is

$[\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]^c = \begin{bmatrix} ((0.9,0.5,0.3),0.8) & ((0.2,0.7,0.6),0.3) \\ ((0.1,0.3,0.8),0.2) & ((0.8,0.5,0.1),0.9) \end{bmatrix}$.

Definition 6.6. Let $[\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] = [\mathfrak{M}(\mathcal{P}_{\widehat{\mathcal{N}}_\eta}, \mathbb{H})] = (\mathfrak{m}_{ij})_{m\times n}$ and $[\mathfrak{N}_{\widehat{\mathcal{N}}_{\mathbb{H}}}] = [\mathfrak{N}(\mathcal{Q}_{\widehat{\mathcal{N}}_\gamma}, \mathbb{H})] = (\mathfrak{n}_{ij})_{m\times n}$ be two neutrosophic hybrid matrices of same order over the neutrosophic hybrid structure $\left(\mathcal{X}_{\widehat{\mathcal{N}}_\lambda}, \mathbb{H}\right)$.

Then the union operation of two neutrosophic hybrid matrices is:

$[\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \sqcup [\mathfrak{N}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] = \left[\mathfrak{L}\left(\left(\mathcal{T}_{\mathcal{L}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{I}_{\mathcal{L}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{F}_{\mathcal{L}_{\widehat{\mathcal{N}}}}(\varrho)\right), \min\{\eta(\varrho), \gamma(\varrho)\}\right)\right]$

where $\mathcal{T}_{\mathcal{L}_{\widehat{\mathcal{N}}}}(\varrho) = \mathcal{T}_{\vee(\mathcal{P},\mathcal{Q})}(\varrho), \mathcal{I}_{\mathcal{L}_{\widehat{\mathcal{N}}}}(\varrho) = \mathcal{I}_{\wedge(\mathcal{P},\mathcal{Q})}(\varrho)$ and $\mathcal{F}_{\mathcal{L}_{\widehat{\mathcal{N}}}}(\varrho) = \mathcal{F}_{\wedge(\mathcal{P},\mathcal{Q})}(\varrho)$.

Example 6.7. Let $[\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] = \begin{bmatrix} ((0.3,0.5,0.9),0.2) & ((0.6,0.7,0.2),0.7) \\ ((0.8,0.3,0.1),0.8) & ((0.1,0.5,0.8),0.1) \end{bmatrix}$ and

$[\mathfrak{N}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] = \begin{bmatrix} ((0.6,0.2,0.5),0.4) & ((0.3,0.2,0.5),0.2) \\ ((0.3,0.7,0.2),0.3) & ((0.5,0.8,0.9),0.8) \end{bmatrix}$ be two neutrosophic hybrid matrices.

Then, $[\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \sqcup [\mathfrak{N}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] = \begin{bmatrix} ((0.6,0.2,0.5),0.2) & ((0.6,0.2,0.2),0.2) \\ ((0.8,0.3,0.1),0.3) & ((0.5,0.5,0.8),0.1) \end{bmatrix}$.

Definition 6.8. Let $[\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] = [\mathfrak{M}(\mathcal{P}_{\widehat{\mathcal{N}}_\eta}, \mathbb{H})] = (\mathfrak{m}_{ij})_{m\times n}$ and $[\mathfrak{N}_{\widehat{\mathcal{N}}_{\mathbb{H}}}] = [\mathfrak{N}(\mathcal{Q}_{\widehat{\mathcal{N}}_\gamma}, \mathbb{H})] = (\mathfrak{n}_{ij})_{m\times n}$ be two neutrosophic hybrid matrices of same order over the neutrosophic hybrid structure $\left(\mathcal{X}_{\widehat{\mathcal{N}}_\lambda}, \mathbb{H}\right)$.

Then the intersection operation of two neutrosophic hybrid matrices is:

$[\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \sqcap [\mathfrak{N}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] = \left[\mathfrak{L}\left(\left(\mathcal{T}_{\mathcal{L}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{I}_{\mathcal{L}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{F}_{\mathcal{L}_{\widehat{\mathcal{N}}}}(\varrho)\right), \max\{\eta(\varrho), \gamma(\varrho)\}\right)\right]$

where $\mathcal{T}_{\mathcal{L}_{\widehat{\mathcal{N}}}}(\varrho) = \mathcal{T}_{\wedge(\mathcal{P},\mathcal{Q})}(\varrho), \mathcal{I}_{\mathcal{L}_{\widehat{\mathcal{N}}}}(\varrho) = \mathcal{I}_{\vee(\mathcal{P},\mathcal{Q})}(\varrho)$ and $\mathcal{F}_{\mathcal{L}_{\widehat{\mathcal{N}}}}(\varrho) = \mathcal{F}_{\vee(\mathcal{P},\mathcal{Q})}(\varrho)$.

Example 6.9. Let $[\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] = \begin{bmatrix} ((0.3, 0.5, 0.9), 0.2) & ((0.6, 0.7, 0.2), 0.7) \\ ((0.8, 0.3, 0.1), 0.8) & ((0.1, 0.5, 0.8), 0.1) \end{bmatrix}$ and $[\mathfrak{N}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] = \begin{bmatrix} ((0.6, 0.2, 0.5), 0.4) & ((0.3, 0.2, 0.5), 0.2) \\ ((0.3, 0.7, 0.2), 0.3) & ((0.5, 0.8, 0.9), 0.8) \end{bmatrix}$ be two neutrosophic hybrid matrices. Then, $[\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \sqcap [\mathfrak{N}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] = \begin{bmatrix} ((0.3, 0.5, 0.9), 0.4) & ((0.3, 0.7, 0.5), 0.7) \\ ((0.3, 0.7, 0.2), 0.8) & ((0.1, 0.8, 0.9), 0.8) \end{bmatrix}$.

Theorem 6.10. *Let $[\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]$ and $[\mathfrak{N}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]$ be two neutrosophic hybrid matrices of same order over the hybrid structure $\left(\mathcal{X}_{\widehat{\mathcal{N}}_{\lambda}}, \mathbb{H}\right)$. Then the following results related to the operations hold.*

(1) $[\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \sqcup [\mathfrak{N}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] = [\mathfrak{N}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \sqcup [\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]$
(2) $[\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \sqcap [\mathfrak{N}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] = [\mathfrak{N}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \sqcap [\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]$
(3) $\left([\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]^c\right)^c = [\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]$
(4) $\left([\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \sqcup [\mathfrak{N}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]\right)^c = [\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]^c \sqcap [\mathfrak{N}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]^c$
(5) $\left([\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \sqcap [\mathfrak{N}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]\right)^c = [\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]^c \sqcup [\mathfrak{N}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]^c$.

Proof. Let $[\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] = [\mathfrak{M}(\mathcal{P}_{\widehat{\mathcal{N}}_{\eta}}, \mathbb{H})] = (\mathfrak{m}_{ij})_{m\times n}$ and $[\mathfrak{N}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] = [\mathfrak{N}(\mathcal{Q}_{\widehat{\mathcal{N}}_{\gamma}}, \mathbb{H})] = (\mathfrak{n}_{ij})_{m\times n}$

(1) $[\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \sqcup [\mathfrak{N}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] = [\mathfrak{N}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \sqcup [\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]$

$$\begin{aligned}
[\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \sqcup [\mathfrak{N}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] &= \left[\mathfrak{M}\left(\left(\mathcal{T}_{\mathcal{P}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{I}_{\mathcal{P}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{F}_{\mathcal{P}_{\widehat{\mathcal{N}}}}(\varrho)\right), \eta(\varrho)\right)\right] \\
&\quad \sqcup \left[\mathfrak{N}\left(\left(\mathcal{T}_{\mathcal{Q}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{I}_{\mathcal{Q}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{F}_{\mathcal{Q}_{\widehat{\mathcal{N}}}}(\varrho)\right), \gamma(\varrho)\right)\right] \\
&= \left[\mathfrak{L}\left(\left(\mathcal{T}_{\vee(\mathcal{P},\mathcal{Q})}(\varrho), \mathcal{I}_{\wedge(\mathcal{P},\mathcal{Q})}(\varrho), \mathcal{F}_{\wedge(\mathcal{P},\mathcal{Q})}(\varrho)\right), \min\{\eta(\varrho), \gamma(\varrho)\}\right)\right] \\
&= \left[\mathfrak{L}\left(\left(\mathcal{T}_{\vee(\mathcal{Q},\mathcal{P})}(\varrho), \mathcal{I}_{\wedge(\mathcal{Q},\mathcal{P})}(\varrho), \mathcal{F}_{\wedge(\mathcal{Q},\mathcal{P})}(\varrho)\right), \min\{\eta(\varrho), \gamma(\varrho)\}\right)\right] \\
&= \left[\mathfrak{N}\left(\left(\mathcal{T}_{\mathcal{Q}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{I}_{\mathcal{Q}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{F}_{\mathcal{Q}_{\widehat{\mathcal{N}}}}(\alpha)\right), \gamma(\varrho)\right)\right] \\
&\quad \sqcup \left[\mathfrak{M}\left(\left(\mathcal{T}_{\mathcal{P}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{I}_{\mathcal{P}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{F}_{\mathcal{P}_{\widehat{\mathcal{N}}}}(\varrho)\right), \eta(\alpha)\right)\right] \\
&= [\mathfrak{N}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \sqcup [\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}].
\end{aligned}$$

(2) $[\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \sqcap [\mathfrak{N}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] = [\mathfrak{N}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \sqcap [\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]$
Proof is similar to (1).

(3) $\left([\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]^c\right)^c = [\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]$
Since $[\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]^c = (\mathfrak{m}^c_{ij})_{m\times n} = \left[\mathfrak{M}\left(\left(\mathcal{F}_{\mathcal{P}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{I}_{\mathcal{P}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{T}_{\mathcal{P}_{\widehat{\mathcal{N}}}}(\varrho)\right), 1-\eta(\varrho)\right)\right]$

$$\begin{aligned}
\left([\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathbb{H}}}]^c\right)^c &= \left[\mathfrak{M}\left(\left(\mathcal{F}_{\mathcal{P}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{I}_{\mathcal{P}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{T}_{\mathcal{P}_{\widehat{\mathcal{N}}}}(\varrho)\right), 1-\eta(\varrho)\right)\right]^c \\
&= \left[\mathfrak{M}\left(\left(\mathcal{T}_{\mathcal{P}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{I}_{\mathcal{P}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{F}_{\mathcal{P}_{\widehat{\mathcal{N}}}}(\varrho)\right), 1-\{1-\eta(\varrho)\}\right)\right] \\
&= \left[\mathfrak{M}\left(\left(\mathcal{T}_{\mathcal{P}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{I}_{\mathcal{P}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{F}_{\mathcal{P}_{\widehat{\mathcal{N}}}}(\varrho)\right), \eta(\varrho)\right)\right] \\
&= [\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}].
\end{aligned}$$

(4) $\left([\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \sqcup \mathfrak{N}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]\right)^c = [\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]^c \sqcap [\mathfrak{N}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]^c$

Since $[\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]^c = (\mathfrak{m}^c_{ij})_{m\times n} = \left[\mathfrak{M}\left(\left(\mathcal{F}_{\mathcal{P}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{I}_{\mathcal{P}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{T}_{\mathcal{P}_{\widehat{\mathcal{N}}}}(\varrho)\right), 1-\eta(\varrho)\right)\right]$

and $[\mathfrak{N}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]^c = (\mathfrak{n}^c_{ij})_{m\times n} = \left[\mathfrak{N}\left(\left(\mathcal{F}_{\mathcal{Q}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{I}_{\mathcal{Q}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{T}_{\mathcal{Q}_{\widehat{\mathcal{N}}}}(\varrho)\right), 1-\gamma(\varrho)\right)\right]$

$$\begin{aligned}
\left([\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \sqcup [\mathfrak{N}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]\right)^c &= \left[\mathfrak{L}\left(\left(\mathcal{T}_{\vee(\mathcal{P},\mathcal{Q})}(\varrho), \mathcal{I}_{\wedge(\mathcal{P},\mathcal{Q})}(\varrho), \mathcal{F}_{\wedge(\mathcal{P},\mathcal{Q})}(\varrho)\right), \min\{\eta(\varrho), \gamma(\varrho)\}\right)\right]^c \\
&= \left[\mathfrak{L}\left(\left(\mathcal{F}_{\wedge(\mathcal{P},\mathcal{Q})}(\varrho), \mathcal{I}_{\vee(\mathcal{P},\mathcal{Q})}(\varrho), \mathcal{T}_{\vee(\mathcal{P},\mathcal{Q})}(\varrho)\right), \max\{1-\eta(\varrho), 1-\gamma(\varrho)\}\right)\right] \\
&= \left[\mathfrak{M}\left(\left(\mathcal{F}_{\mathcal{P}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{I}_{\mathcal{P}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{T}_{\mathcal{P}_{\widehat{\mathcal{N}}}}(\varrho)\right), 1-\eta(\varrho)\right)\right] \\
&\quad \sqcap \left[\mathfrak{N}\left(\left(\mathcal{F}_{\mathcal{Q}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{I}_{\mathcal{Q}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{T}_{\mathcal{Q}_{\widehat{\mathcal{N}}}}(\varrho)\right), 1-\gamma(\varrho)\right)\right] \\
&= [\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathbb{H}}}]^c \sqcap [\mathfrak{N}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]^c.
\end{aligned}$$

(5) $\left([\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \sqcap [\mathfrak{N}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]\right)^c = [\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]^c \sqcup [\mathfrak{N}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]^c$

Proof is similar to (4).

Theorem 6.11. *Let $[\mathfrak{L}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}], [\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]$ and $[\mathfrak{N}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]$ be three hybrid matrices of same order over the hybrid structure $\left(\mathcal{X}_{\widehat{\mathcal{N}}_{\lambda}}, \mathbb{H}\right)$. Then the following results related to the operations hold.*

(1) $\left([\mathfrak{L}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \sqcup [\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]\right) \sqcup [\mathfrak{N}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] = [\mathfrak{L}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \sqcup \left([\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \sqcup [\mathfrak{N}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]\right)$

(2) $\left([\mathfrak{L}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \sqcap [\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]\right) \sqcap [\mathfrak{N}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] = [\mathfrak{L}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \sqcap \left([\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \sqcap [\mathfrak{N}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]\right)$

(3) $[\mathfrak{L}\widehat{\mathcal{N}}_{\mathfrak{H}}] \sqcup \left([\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \sqcap [\mathfrak{N}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]\right) = \left([\mathfrak{L}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \sqcup [\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]\right) \sqcap \left([\mathfrak{L}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \sqcup [\mathfrak{N}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]\right)$

(4) $[\mathfrak{L}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \sqcap \left([\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \sqcup [\mathfrak{N}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]\right) = \left([\mathfrak{L}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \sqcap [\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]\right) \left([\mathfrak{L}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \sqcap [\mathfrak{N}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]\right).$

Proof. Let $[\mathfrak{L}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] = [\mathfrak{L}(\mathcal{P}_{\widehat{\mathcal{N}}_{\nu}}, \mathbb{H})] = (\mathfrak{l}_{ij})_{m\times n}, [\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] = [\mathfrak{M}(\mathcal{Q}_{\widehat{\mathcal{N}}_{\eta}}, \mathbb{H})] = (\mathfrak{m}_{ij})_{m\times n}$ and $[\mathfrak{N}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] = [\mathfrak{N}(\mathcal{R}_{\widehat{\mathcal{N}}_{\gamma}}, \mathbb{H})] = (\mathfrak{n}_{ij})_{m\times n}$

(1) $\left([\mathfrak{L}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \sqcup [\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]\right) \sqcup [\mathfrak{N}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] = [\mathfrak{L}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \sqcup \left([\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \sqcup [\mathfrak{N}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]\right)$

$$\begin{aligned}
LHS &= \left([\mathfrak{L}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \sqcup [\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathbb{H}}}]\right) \sqcup [\mathfrak{N}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \\
&= \left[\mathfrak{S}\left(\left(\mathcal{T}_{\vee(\mathcal{P},\mathcal{Q})}(\varrho), \ddot{\mathcal{I}}_{\wedge(\mathcal{P},\mathcal{Q})}(\varrho), \mathcal{F}_{\wedge(\mathcal{P},\mathcal{Q})}(\varrho)\right), \min\{\nu(\varrho), \eta(\varrho)\}\right)\right] \\
&\quad \sqcup \left[\mathfrak{N}\left(\left(\mathcal{T}_{\mathcal{R}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{I}_{\mathcal{R}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{F}_{\mathcal{R}_{\widehat{\mathcal{N}}}}(\varrho)\right), \gamma(\varrho)\right)\right] \\
&= \left[\mathfrak{V}\left(\left(\mathcal{T}_{\vee(\mathcal{P},\mathcal{Q},\mathcal{R})}(\varrho), \mathcal{I}_{\wedge(\mathcal{P},\mathcal{Q},\mathcal{R})}(\varrho), \mathcal{F}_{\wedge(\mathcal{P},\mathcal{Q},\mathcal{R})}(\varrho)\right), \min\{\nu(\varrho), \eta(\varrho), \gamma(\varrho)\}\right)\right] \\
&= \left[\mathfrak{M}\left(\left(\mathcal{T}_{\mathcal{P}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{I}_{\mathcal{P}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{F}_{\mathcal{P}_{\widehat{\mathcal{N}}}}(\varrho)\right), \nu(\varrho)\right)\right] \\
&\quad \sqcup \left[\mathfrak{S}\left(\left(\mathcal{T}_{\vee(\mathcal{Q},\mathcal{R})}(\varrho), \mathcal{I}_{\wedge(\mathcal{Q},\mathcal{R})}(\varrho), \mathcal{F}_{\wedge(\mathcal{Q},\mathcal{R})}(\varrho)\right), \min\{\eta(\varrho), \gamma(\varrho)\}\right)\right] \\
&= [\mathfrak{L}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \sqcup \left([\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \sqcup [\mathfrak{N}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]\right).
\end{aligned}$$

(2) $\left([\mathfrak{L}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \sqcap [\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]\right) \sqcap [\mathfrak{N}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] = [\mathfrak{L}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \sqcap \left([\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \sqcap [\mathfrak{N}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]\right)$

Proof is similar to (1).

(3) $[\mathfrak{L}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \sqcup \left([\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \sqcap [\mathfrak{N}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]\right) = \left([\mathfrak{L}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \sqcup [\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]\right) \sqcap \left([\mathfrak{L}\_\widehat{\mathcal{N}}_{\mathfrak{H}}] \sqcup [\mathfrak{N}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]\right)$

$$
\begin{aligned}
&[\mathfrak{L}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \sqcup \left([\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \sqcap [\mathfrak{N}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]\right) \\
&= \left[\mathfrak{M}\left(\left(\mathcal{T}_{\mathcal{P}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{I}_{\mathcal{P}_{\widehat{\mathcal{N}}}}(\varrho), \mathcal{F}_{\mathcal{P}_{\widehat{\mathcal{N}}}}(\varrho)\right), \nu(\varrho)\right)\right] \\
&\quad \sqcup \left[\mathfrak{S}\left(\left(\mathcal{T}_{\wedge(\mathcal{Q},\mathcal{R})}(\varrho), \mathcal{I}_{\vee(\mathcal{Q},\mathcal{R})}(\varrho), \mathcal{F}_{\vee(\mathcal{Q},\mathcal{R})}(\varrho)\right), \max\{\eta(\varrho), \gamma(\varrho)\}\right)\right] \\
&= \left[\mathfrak{Z}\left(\left(\mathcal{T}_{\vee\{\mathcal{P},\wedge(\mathcal{Q},\mathcal{R})\}}, \mathcal{I}_{\wedge\{\mathcal{P},\vee(\mathcal{Q},\mathcal{R})\}}, \mathcal{F}_{\wedge\{\mathcal{P},\vee(\mathcal{Q},\mathcal{R})\}}\right), \min\{\nu(\varrho), \max\{\eta(\varrho), \gamma(\varrho)\}\}\right)\right] \\
&= \left[\mathfrak{V}\left(\left(\mathcal{T}_{\vee(\mathcal{P},\mathcal{Q})}(\varrho), \mathcal{I}_{\wedge(\mathcal{P},\mathcal{Q})}(\varrho), \mathcal{F}_{\wedge(\mathcal{P},\mathcal{Q})}(\varrho)\right), \min\{\nu(\varrho), \eta(\varrho)\}\right)\right] \\
&\quad \left[\mathfrak{W}\left(\left(\mathcal{T}_{\vee(\mathcal{P},\mathcal{R})}(\varrho), \mathcal{I}_{\wedge(\mathcal{P},\mathcal{R})}(\varrho), \mathcal{F}_{\wedge(\mathcal{P},\mathcal{R})}(\varrho)\right), \min\{\nu(\varrho), \eta(\varrho)\}\right)\right] \\
&= \left([\mathfrak{L}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \sqcup [\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]\right) \sqcap \left([\mathfrak{L}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \sqcup [\mathfrak{N}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]\right).
\end{aligned}
$$

(4) $[\mathfrak{L}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \sqcap \left([\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \sqcup [\mathfrak{N}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]\right) = \left([\mathfrak{L}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \sqcap [\mathfrak{M}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]\right) \sqcup \left([\mathfrak{L}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}] \sqcap [\mathfrak{N}_{\widehat{\mathcal{N}}_{\mathfrak{H}}}]\right)$

Proof is similar to (3).

7. MCDM based on neutrosophic hybrid matrix

This section starts with an algorithm for solving a multi-criteria decision making problem based on neutrosophic hybrid matrices using the notion of comparison matrices. The algorithm is described by a suitable example.

Definition 7.1. Comparison matrix is a matrix whose rows are the different groups $g_1, g_2, \ldots, g_n$ and the columns are the parameters $\varrho_1, \varrho_2, \ldots, \varrho_n$.
The elements of the matrix are calculated by $c_{ij} = (s_{ij} = a + b - c, w_{ij} = d)$, where a, b, c and d are integers calculated as how many times $T_{h_i}(e_j)$ exceeds or equal to $T_{h_k}(e_j), I_{h_i}(e_j)$ exceeds or equal to $I_{h_k}(e_j), F_{h_i}(e_j)$ exceeds or equal to $F_{h_k}(e_j)$and $w_{h_i}(e_j)$ exceeds or equal to $w_{h_k}(e_j)$ for $h_j \neq h_k, \forall h_k \in \mathcal{U}$, respectively.

Definition 7.2. The score of an object g_i is $S_i = \sum_j s_{ij}$. The weightage of an object g_i is $W_i = \sum_j w_{ij}$.

Development in technology is aimed at betterment of life style of people worldwide. Especially technological developments have mixed effects on the study habits and attitudes of student, both good and adverse, that is support and distraction result as a consequence of technological development. We try to analyze the impact of technology on students life using the following algorithm as an MCDM based on neutrosophic hybrid matrices.

7.1. *Algorithm*

The steps of the algorithm for decision making using the construction of a comparison matrix are given below.

Step 1: Identify the possible subsets of the parameter set and neutrosophic hybrid set.

Step 2: Find the neutrosophic hybrid matrix.

Step 3: Compute the comparison matrix of the neutrosophic hybrid matrix.

Step 4: Compute the score S_i? and weightage W_i of g_i.

Also find $S_k = \max ?S_i$ and $W_k = \max ?W_i$.

Step 5: Determine the result, if the scores are equal we consider the weightage.

Example 7.3. We analyze the study habits and attitudes of the student groups from the particular city using the above algorithm.

Step 1: Let $\mathcal{U} = \{g_1, g_2, g_3, g_4, g_5\}$ be the set of group of students. Consider the parameters as changes in student study habits and attitudes like maximum, average and minimum change. That is the parameter set is given by

$\mathbb{H} = \{\varrho_1 = \text{maximum change}, \varrho_2 = \text{average change}, \varrho_3 = \text{minimum change}\}$.

Step 2: Consider the neutrosophic hybrid matrix whose rows are the different group of students $\{g_1, g_2, g_3, g_4, g_5\}$ and the columns are the parameters $\varrho_1, \varrho_2, \varrho_3$.

$$[\mathfrak{M}_{\widehat{\mathcal{N}_{\mathfrak{H}}}}] = \begin{bmatrix} ((0.2,0.6,0.5),0.4) & ((0.3,0.9,0.2),0.1) & ((0.3,0.4,0.5),0.6) \\ ((0.8,0.3,0.9),0.1) & ((0.9,0.8,0.4),0.3) & ((0.1,0.4,0.2),0.4) \\ ((0.2,0.1,0.5),0.5) & ((0.7,0.6,0.9),0.2) & ((0.3,0.7,0.1),0.3) \\ ((0.7,0.4,0.3),1) & ((0.3,0.5,0.8),0.2) & ((0.3,0.4,0.6),0.5) \\ ((0.5,0.4,0.2),0.4) & ((0.2,0.5,0.5),0) & ((0.4,0.6,0.8),0.7) \end{bmatrix}.$$

Step 3: The comparison matrix of the above neutrosophic hybrid matrix is

$$[c_{ij}] = \begin{bmatrix} (2,3) & (6,1) & (1,3) \\ (1,1) & (6,4) & (1,1) \\ (-2,4) & (1,3) & (5,0) \\ (4,1) & (0,3) & (0,2) \\ (4,3) & (-1,0) & (3,4) \end{bmatrix}.$$

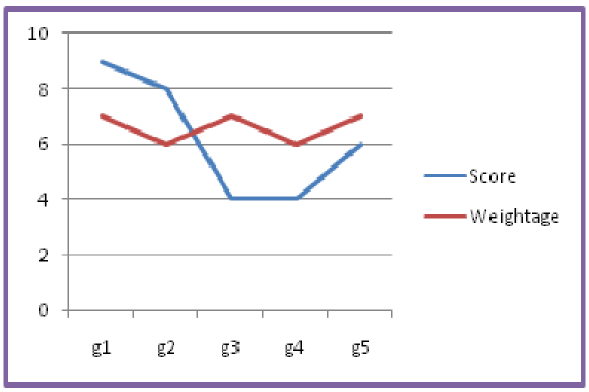
Step 4:

Now we compute the score and weightage for each group g_i,

TABLE 2. Representation of the score and weightage for each group

$\mathbb{H}$	**Score** (S_i)	**Weightage** (W_i)
g_1	9	7
g_2	8	6
g_3	4	7
g_4	4	6
g_5	6	7

The graphical representation of the score and weightage for each group g_i,



Step 5: The maximum score is secured by group 1. That is the group 1 of the students almost adopt the usage of technology. So this group of students have maximum changes in study habit and attitude. Group 3 and group 4 secured minimum score. But weightage of group 4 is less than group 3. So the students of group 4 have minimum changes in study habit and attitude. Rest of the groups are average changes in study habit and attitude.

We compare our result with that of Maji [13]. Both methods give the same scores for each group. In Maji's [13] method the decision becomes random where more than one group have equal scores. This difficulty is overcome in our method using weights in neutrosophic hybrid matrices. This facilitates for choice of better group among the ones with identical score.

8. **Conclusion**

The new notions of hybrid matrices and neutrosophic hybrid matrices are introduced and some of their theoretical properties are studied. We have also developed an algorithm for solving a MCDM problem using neutrosophic hybrid matrices. As future research direction we contemplate to provide more methods for solving multiple criteria decision making (MCDM) problems based on hybrid matrices and neutrosophic hybrid matrices.

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Some Weaker Forms of Bipolar Neutrosophic Nano * Open Sets

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Abstract: In this paper, the weaker forms of open sets in Bipolar Neutrosophic Nano* (BNN*) topology are studied. This topology is defined on a space of bipolar neutrosophic sets with respect to the lower, upper, boundary approximations and the union and intersection of lower and boundary approximations with maximum of 7 elements. The sets $BNN_Q{}^*$ - Preopen, $BNN_Q{}^*$ - Semi open, $BNN_Q{}^*$ - Regular open, $BNN_Q{}^*$ - α - open and $BNN_Q{}^*$ - β - open sets are introduced and their properties are investigated in the corresponding topology in detail and the relationships between them are shown diagrammatically. We proved that, in a BNN^* - topological space $\left(U, \tau_{R_{BNN^*}}(Q)\right)$, the $BNN_Q{}^*$ - open sets of U and for bipolar neutrosophic sets $E \supset \overline{BN}^*(Q)$ with $BNN_Q{}^*cl\left(\overline{BN}^*(Q)\right) = 1_{BNN^*}$ are the only $BNN_Q{}^*$ - α - open sets in U and also the intersection of any two $BNN_Q{}^*$ - α - open sets is $BNN_Q{}^*$ - α - open set in $\left(U, \tau_{R_{BNN^*}}(Q)\right)$. Moreover it is shown that, in U the $BNN_Q{}^*$ - open sets $0_{BNN^*}, 1_{BNN^*}$, $B_{BN}{}^*(Q), BN_1{}^*(Q)$ and $\overline{BN}^*(Q)$ with $BNN_Q{}^*cl\left(\overline{BN}^*(Q)\right) = \left(BN_2{}^*(Q)\right)^C$ are the only $BNN_Q{}^*$ - Regular open sets in U.

Keywords: Nano topology; Neutrosophic set; Bipolar Neutrosophic set; Bipolar Neutrosophic nano topology; Bipolar Neutrosophic nano* topology.

1. Introduction

The concept of fuzzy sets was introduced by Zedeh L. in 1965, which has a single membership grade value attached with each element. Further the generalization of the fuzzy set was made by Atanassov [3] in 1986, known as intuitionistic fuzzy sets. In this set, instead of one membership grade, there is also a non- membership grade attached with each element with a restriction that the sum of these two grades is less than or equal to unity. This concept is useful in the situation of insufficient information. This set is extended to interval valued intuitionistic fuzzy set in 1989 by Atanassov and Gargov [4]. The concept of neutrosophic set is initiated by Smarandache [25] in 1998 which is a generalization of fuzzy sets and intuitionistic fuzzy sets and this set becomes a powerful tool to deal the real life problems with incomplete, indeterminate and inconsistent information. It is characterized by Truth, Indeterminacy and False membership functions and these functions are independent. Salama A.A. and Albowli S.A. [23] introduced Neutrosophic topological spaces. Lee [14] gave an extension of fuzzy sets whose range of membership degree is extended from [0,1] to [-1,1], which is named as bipolar fuzzy set. After that, Deli et. al. [9] defined the concept of bipolar neutrosophic set in 2015.

Many researches have been done in neutrosophic set recently such as in application "Toward Sustainable Emerging Economics based on Industry 5.0: Leveraging Neutrosophic Theory in Appraisal Decision Framework" by Mona Mohamed and Abduallah Gamal, "An Integrated Neutrosophic Regional Management Ranking Method for Agricultural Water Management" by A.Abdel-Monem , A.Nabeeh and M.Abouhawwash, "Towards a Responsive Resilient Supply Chain based on Industry 5.0: A Case Study in Healthcare Systems" by Abduallah Gamal, Amal F.Abd El-Gawad and Mohamed Abouhawwash, "Applications of graph complete degree with bipolar fuzzy information" by soumitra Poulik and Ganesh Ghorai, "Bipolar Neutrosophic Sets and Their Application Based on Multi-Criteria Decision Making Problems" by Irfan Deli, Mumtaz Ali and Florentin Smarandache etc. and in theory "Neutrosophic Pre-open Sets and Pre-closed Sets in Neutrosophic Topology" by Vunnam Venkatewra Rao, "Bipolar neutrosophic soft generalized pre-closed sets and pre-open sets in topological space" by Arulpandy P and Trinita Pricilla M, "Bipolar topological pre-closed neutrosophic sets" by G. Upender Reddy, T. Siva Nageswara Rao, N. Srinivasa Rao and V. Venkateswara Rao. "Bipolar neutrosophic soft generalized pre-continuous mappings" by Arulpandy P and Trinita Pricilla M etc.

Neutrosophic sets were widely used in many topological concepts; in particular, general topology. Most of the general topology concepts were combined with neutrosophic sets and some new topologies were proposed. Lellis Thivagar M. [15] proposed the concept of Nano topology which was defined in terms of approximations and boundary region of a subset of a universe using an equivalence relation on it. In 2022, we defined a topology bipolar neutrosophic nano topology as a combination of nano topology and bipolar neutrosophic set. But in this case, we only get topologies for equivalence relations with independent singleton sets of elements of the universe. We decided to construct a definition to find topologies for each bipolar set irrespective of equivalence relation. Thus, we introduced a topology called Bipolar Neutrosophic nano * topology [10] which consist of maximum 7 elements. In this paper, we introduced and studied some weaker forms of Bipolar neutrosophic nano* open sets $\left(BNN_Q{}^*O\right)$, namely, $BNN_Q{}^*$ -Preopen sets, $BNN_Q{}^*$ -Regular open sets, $BNN_Q{}^*$ -Semi open sets, BNN_Q* - α open sets and BNN_Q* - β open sets. We found the limitations of these open sets with respect to a particular bipolar neutrosophic set and also investigated the properties of them and the relationships between them in detail.

This manuscript is organized as follows: Section 2 contains some basic definitions related to this manuscript. Section 3 consists of weaker forms of bipolar neutrosophic nano* open sets. Sub section 3.1 consists of the properties and results based on bipolar neutrosophic nano* preopen sets. Sub section 3.2 consists of the properties and results based on bipolar neutrosophic nano* semi open sets. Sub section 3.3 consists of the properties and results based on bipolar neutrosophic nano* α open sets. In particular, we proved that, in a BNN^* - topological space $(U, \tau_{R_{BNN^*}}(Q))$, the BNN_Q* open sets of U and for bipolar neutrosophic sets $E \supset \overline{BN}^*(Q)$ with $BNN_Q{}^*cl(\overline{BN}*(Q)) = 1_{BNN^*}$ are the only BNN_Q* - α open sets in U and also the intersection of any two BNN_Q* - α open sets is BNN_Q* - α open set in $(U, \tau_{R_{BNN^*}}(Q))$. Sub section 3.4 consists of the properties and results based on bipolar neutrosophic nano* regular open sets. In this section, it is shown that, in U the BNN_Q* - open sets $0_{BNN^*}, 1_{BNN^*}$, $B_{BN}*(Q), BN_1*(Q)$ and $\overline{BN}*(Q)$ with $BNN_Q{}^*cl(\overline{BN}*(Q)) = (BN_2*(Q))^C$ are the only BNN_Q* - Regular open sets in U. Sub section 3.5 consists of the properties and results based on bipolar neutrosophic nano* β open sets. The properties and relationship between the sets are clearly explained with several examples.

2. Preliminaries

Definition: 2.19 [10] Let U be a nonempty set and R be an equivalence relation on U which is indiscernible. Then U can be divided into disjoint equivalence classes. Let Q be a bipolar neutrosophic set (BNS) in U with the positive degree of true membership η_Q^+, indeterminacy ψ_Q^+ and the false membership function $\xi_Q{}^+$ and the negative degree of true membership η_Q^-, indeterminacy ψ_Q^- and the false membership function $\xi_Q{}^-$, where $\eta_Q^+, \psi_Q^+, \xi_Q^+ : U \to [0,1]$, $\eta_Q^-, \psi_Q^-, \xi_Q^- : U \to [-1,0]$. Then the lower, upper and boundary approximations are respectively given as follows:

(i) $\underline{BN}(Q) = \{\langle q, (\eta^+_{\underline{R}(Q)}(q), \psi^+_{\underline{R}(Q)}(q), \xi^+_{\underline{R}(Q)}(q), \eta^-_{\underline{R}(Q)}(q), \psi^-_{\underline{R}(Q)}(q), \xi^-_{\underline{R}(Q)}(q))\rangle : z \in [q]_R, q \in U\}$.

(ii) $\overline{BN}(Q) = \{\langle q, (\eta^+_{\overline{R}(Q)}(q), \psi^+_{\overline{R}(Q)}(q), \xi^+_{\overline{R}(Q)}(q), \eta^-_{\overline{R}(Q)}(q), \psi^-_{\overline{R}(Q)}(q), \xi^-_{\overline{R}(Q)}(q))\rangle : z \in [q]_R, q \in U\}$.

(iii) $B_{BN}(Q) = \overline{BN}(Q) - \underline{BN}(Q)$. where,

$$\eta_{\underline{R}*(Q)}{}^+(q) = \bigwedge_{z\in[q]_{R^*}} \eta_Q{}^+(z),\ \psi_{\underline{R}*(Q)}{}^+(q) = \bigwedge_{z\in[q]_{R^*}} \psi_Q{}^+(z),\ \xi_{\underline{R}*(Q)}{}^+(q) = \bigvee_{z\in[q]_{R^*}} \xi_Q{}^+(z),$$
$$\eta_{\underline{R}*(Q)}{}^-(q) = \bigvee_{z\in[q]_{R^*}} \eta_Q{}^-(z),\ \psi_{\underline{R}*(Q)}{}^-(q) = \bigvee_{z\in[q]_{R^*}} \psi_Q{}^-(z),\ \xi_{\underline{R}*(Q)}{}^-(q) = \bigwedge_{z\in[q]_{R^*}} \xi_Q{}^-(z),$$
$$\eta_{\overline{R}*(Q)}{}^+(q) = \bigvee_{z\in[q]_{R^*}} \eta_Q{}^+(z),\ \psi_{\overline{R}*(Q)}{}^+(q) = \bigvee_{z\in[q]_{R^*}} \psi_Q{}^+(z),\ \xi_{\overline{R}*(Q)}{}^+(q) = \bigwedge_{z\in[q]_{R^*}} \xi_Q{}^+(z),$$
$$\eta_{\overline{R}*(Q)}{}^-(q) = \bigwedge_{z\in[q]_{R^*}} \eta_Q{}^-(z),\ \psi_{\overline{R}*(Q)}{}^-(q) = \bigwedge_{z\in[q]_{R^*}} \psi_Q{}^-(z),\ \xi_{\overline{R}*(Q)}{}^-(q) = \bigvee_{z\in[q]_{R^*}} \xi_Q{}^-(z)\ .$$

Definition: 2.2 [10] Let U be a nonempty set, R be an equivalence relation on U and let Q be a BNS. The collection $\tau_{R_{BNN}}(Q) = \{0_{BNN}, 1_{BNN}, \underline{BN}(Q), \overline{BN}(Q), B_{BN}(Q)\}$ is called the bipolar neutrosophic nano topology (BNN_Q - topology), if it forms a topology. Then the space $(U, \tau_{R_{BNN}}(Q))$ is called the bipolar neutrosophic nano topological space. The elements of $\tau_{R_{BNN}}(Q)$ are called bipolar neutrosophic nano open sets (BNN_QO).

Remark: 2.3 [10] For every bipolar neutrosophic set, we cannot find a corresponding bipolar neutrosophic nano topology in U. So we defined a topology called Bipolar neutrosophic nano * topology which corresponds to any bipolar neutrosophic set in U with respect to its boundary and approximations.

Definition: 2.4 [10] Let U be a nonempty set and $R*$ be a relation on U, which is indiscernible. Then U can be divided into disjoint equivalence classes. Let Q be a BNS in U with the positive degree of true membership $\eta_Q{}^+$, indeterminacy $\psi_Q{}^+$ and the false membership function $\xi_Q{}^+$ and the negative degree of true membership $\eta_Q{}^-$, indeterminacy $\psi_Q{}^-$ and the false membership function $\xi_Q{}^-$, where, $\eta_Q^+, \psi_Q^+, \xi_Q^+ : U \to [0,1]$, $\eta_Q^-, \psi_Q^-, \xi_Q^- : U \to [-1,0]$. Then

(i) $\underline{BN}*(Q) = \{\langle q, (\eta^+_{\underline{R}*(Q)}(q), \psi^+_{\underline{R}*(Q)}(q), \xi^+_{\underline{R}*(Q)}(q), \eta^-_{\underline{R}*(Q)}(q), \psi^-_{\underline{R}*(Q)}(q), \xi^-_{\underline{R}*(Q)}(q))\rangle : z \in [q]_{R^*}, q \in U\}$ is the lower approximation of Q in respect of $R*$.

(ii) $\overline{BN}*(Q) = \{\langle q, (\eta^+_{\overline{R}*(Q)}(q), \psi^+_{\overline{R}*(Q)}(q), \xi^+_{\overline{R}*(Q)}(q), \eta^-_{\overline{R}*(Q)}(q), \psi^-_{\overline{R}*(Q)}(q), \xi^-_{\overline{R}*(Q)}(q))\rangle : z \in [q]_{R^*}, q \in U\}$ is the upper approximation of Q in respect of $R*$.

(iii) $B_{BN}*(Q) = \overline{BN}*(Q) - \underline{BN}*(Q)$ is the boundary of Q in respect of $R*$.

(iv) $BN_1*(Q) = \underline{BN}*(Q) \cup B_{BN}*(Q)$.

(v) $BN_2*(Q) = \underline{BN}*(Q) \cap B_{BN}*(Q)$. where,

$$\eta_{\underline{R}*(Q)}{}^+(q) = \bigwedge_{z\in[q]_{R^*}} \eta_Q{}^+(z),\ \psi_{\underline{R}*(Q)}{}^+(q) = \bigwedge_{z\in[q]_{R^*}} \psi_Q{}^+(z),\ \xi_{\underline{R}*(Q)}{}^+(q) = \bigvee_{z\in[q]_{R^*}} \xi_Q{}^+(z),$$
$$\eta_{\underline{R}*(Q)}{}^-(q) = \bigvee_{z\in[q]_{R^*}} \eta_Q{}^-(z),\ \psi_{\underline{R}*(Q)}{}^-(q) = \bigvee_{z\in[q]_{R^*}} \psi_Q{}^-(z),\ \xi_{\underline{R}*(Q)}{}^-(q) = \bigwedge_{z\in[q]_{R^*}} \xi_Q{}^-(z),$$
$$\eta_{\overline{R}*(Q)}{}^+(q) = \bigvee_{z\in[q]_{R^*}} \eta_Q{}^+(z),\ \psi_{\overline{R}*(Q)}{}^+(q) = \bigvee_{z\in[q]_{R^*}} \psi_Q{}^+(z),\ \xi_{\overline{R}*(Q)}{}^+(q) = \bigwedge_{z\in[q]_{R^*}} \xi_Q{}^+(z),$$
$$\eta_{\overline{R}*(Q)}{}^-(q) = \bigwedge_{z\in[q]_{R^*}} \eta_Q{}^-(z),\ \psi_{\overline{R}*(Q)}{}^-(q) = \bigwedge_{z\in[q]_{R^*}} \psi_Q{}^-(z),\ \xi_{\overline{R}*(Q)}{}^-(q) = \bigvee_{z\in[q]_{R^*}} \xi_Q{}^-(z)\ .$$

Then the collection $\tau_{R_{BNN}}*(Q)=\{0_{BNN^*},1_{BNN^*},\underline{BN}*(Q),\overline{BN}*(Q),B_{BN}*(Q),BN_1*(Q),BN_2*(Q)\}$ is a topology which is called a bipolar neutrosophic nano$*$ topology (BNN_Q* - topology). The space $(U,\tau_{R_{BNN^*}}(Q))$ is called a bipolar neutrosophic nano$*$ topological space. The elements of $\tau_{\Re_{BNN}}*(Q)$ are bipolar neutrosophic nano $*$ open sets (BNN_Q*O). The complements of these elements are called bipolar neutrosophic nano $*$ closed sets (BNN_Q*C).

Definition: 2.5[10] Let U be a nonempty universe and K and H be the BNS's, where $K=\{\langle q,(\eta_K^+(q),\psi_K^+(q),\xi_K^+(q),\eta_K^-(q),\psi_K^-(q),\xi_K^-(q))\rangle : q\in U\}$ and $H=\{\langle q,(\eta_H^+(q),\psi_H^+(q),\xi_H^+(q),\eta_H^-(q),\psi_H^-(q),\xi_H^-(q))\rangle : q\in U\}$. Then,

(i) the null bipolar neutrosophic nano set is given by $0_{BNN}=\{\langle q,(0,0,1,0,0,-1)\rangle : q\in U\}$.

(ii) the absolute bipolar neutrosophic nano set is given by $1_{BNN}=\{\langle q,(1,1,0,-1,-1,0)\rangle : q\in U\}$.

(iii) $K\subseteq H$ iff $\eta_K^+(q)\le\eta_H^+(q)$, $\psi_K^+(q)\le\psi_H^+(q)$, $\xi_K^+(q)\ge\xi_H^+(q)$,
$\eta_K^-(q)\ge\eta_H^-(q)$, $\psi_K^-(q)\ge\psi_H^-(q)$, $\xi_K^-(q)\le\xi_H^-(q)$.

(iv) $K=H$ iff $K\subseteq H$ and $H\subseteq K$.

(v) $K^C=\{\langle q,(\xi_K^+(q),1-\psi_K^+(q),\eta_K^+(q),\xi_K^-(q),-1-\psi_K^-(q),\eta_K^-(q))\rangle : q\in U\}$.

(vi) $$K\cap H=\left\{\left\langle q,\begin{pmatrix}\eta_K^+(q)\wedge\eta_H^+(q),\psi_K^+(q)\wedge\psi_H^+(q),\xi_K^+(q)\vee\xi_H^+(q),\\ \eta_K^-(q)\vee\eta_H^-(q),\psi_K^-(q)\vee\psi_H^-(q),\xi_K^-(q)\wedge\xi_H^-(q)\end{pmatrix}\right\rangle : q\in U\right\}.$$

(vii) $$K\cup H=\left\{\left\langle q,\begin{pmatrix}\eta_K^+(q)\vee\eta_H^+(q),\psi_K^+(q)\vee\psi_H^+(q),\xi_K^+(q)\wedge\xi_H^+(q),\\ \eta_K^-(q)\wedge\eta_H^-(q),\psi_K^-(q)\wedge\psi_H^-(q),\xi_K^-(q)\vee\xi_H^-(q)\end{pmatrix}\right\rangle : q\in U\right\}.$$

(viii) $$K-H=\left\{\left\langle q,\begin{pmatrix}\min\{\eta_K^+(q),\xi_H^+(q)\},\min\{\psi_K^+(q),1-\psi_H^+(q)\},\max\{\xi_K^+(q),\eta_H^+(q)\},\\ \max\{\eta_K^-(q),\xi_H^-(q),\max\{\psi_K^-(q),-1-\psi_H^-(q),\min\{\xi_K^-(q),\eta_H^-(q)\}\end{pmatrix}\right\rangle : q\in U\right\}.$$

Remark: 2.6 [10] In a BNN^*TS $(U,\tau_{R_{BNN^*}}(Q))$, by definition

$BNN_Q*\text{int}\left((\overline{BN}*(Q))^C\right)=BN_2*(Q)$ or 0_{BNN^*},

$BNN_Q*\text{int}\left((\underline{BN}*(Q))^C\right)=B_{BN}*(Q)$,

$BNN_Q*\text{int}\left((B_{BN}*(Q))^C\right)=BN_1*(Q)$,

$BNN_Q*\text{int}\left((BN_1*(Q))^C\right)=B_{BN}*(Q)$,

$BNN_Q*\text{int}\left((BN_2*(Q))^C\right)=BN_1*(Q)$.

And

$BNN_Q*\text{cl}(\overline{BN}*(Q))=(BN_2*(Q))^C$ or 1_{BNN^*},

$BNN_Q*\text{cl}(\underline{BN}*(Q))=(B_{BN}*(Q))^C$,

$BNN_Q*\text{cl}(B_{BN}*(Q))=(BN_1*(Q))^C$,

$BNN_Q*\text{cl}(BN_1*(Q))=(B_{BN}*(Q))^C$,

$BNN_Q*\text{cl}(BN_2*(Q))=(BN_1*(Q))^C$.

3. Weaker forms of Bipolar Neutrosophic Nano * Topology

In this section, we are going to introduce some of the weaker forms of open sets in Bipolar Neutrosophic Nano* Topology.

*3.1 Bipolar Neutrosophic Nano * Pre-Open Sets*

Definition: 3.1.1 Let E be a bipolar neutrosophic set in a BNN^*-topological space (BNN^*TS) $(U,\tau_{R_{BNN^*}}(Q))$. Then E is said to be BNN_Q^* - pre-open set (BNN_Q^*PO set) of U if $E\subseteq BNN_Q^*\text{ int}(BNN_Q^*\text{cl}(E))$. The complement of BNN_Q^*PO set is called BNN_Q^* - pre-closed set
(BNN_Q^*PC set) of U.

Theorem: 3.1.2 Arbitrary union of BNN_Q^*PO sets in $(U,\tau_{R_{BNN^*}}(Q))$ is BNN_Q^*PO set in U.

Proof. Let $\{E_\alpha\}_{\alpha\in\Omega}$ is a collection of BNN_Q^*PO sets in $(U,\tau_{R_{BNN^*}}(Q))$. For each $\alpha\in\Omega$, $E_\alpha\subseteq BNN_Q^*\text{ int}(BNN_Q^*\text{cl}(E_\alpha))$.

$$\begin{aligned}E_1\cup E_2\cup\ldots\ldots &\subseteq BNN_Q^*\text{ int}(BNN_Q^*\text{cl}(E_1))\cup BNN_Q^*\text{ int}(BNN_Q^*\text{cl}(E_2))\cup\ldots\ldots\ldots\\ &\subseteq BNN_Q^*\text{ int}(BNN_Q^*\text{cl}(E_1)\cup BNN_Q^*\text{cl}(E_2)\cup\ldots\ldots)\\ &\subseteq BNN_Q^*\text{ int}(BNN_Q^*\text{cl}(E_1\cup E_2\cup\ldots.))\end{aligned}$$

Hence $\bigcup_{\alpha\in\Omega}E_\alpha$ is BNN_Q^*PO set in U.

Remark: 3.1.3 The intersection of any two $BNN_Q{}^*PO$ sets need not be a $BNN_Q{}^*PO$ set in U. This is shown in the following example.

Example: 3.1.4 Let $U=\{p_1,p_2\}, U/R=\{\{p_1\},\{p_2\}\}$, $Q=\left\{\begin{matrix}\langle p_1,(0.4,0.8,0.6,-0.5,-0.2,-0.1)\rangle\\ \langle p_2,(0.7,0.5,0.3,-0.2,-0.4,-0.6)\rangle\end{matrix}\right\}$.

$\tau_{R_{BNN^*}}(Q)=\left\{0_{BN^*},1_{BN^*},\left\{\begin{matrix}\langle p_1,(0.4,0.8,0.6,-0.5,-0.2,-0.1)\rangle\\ \langle p_2,(0.7,0.5,0.3,-0.2,-0.4,-0.6)\rangle\end{matrix}\right\},\left\{\begin{matrix}\langle p_1,(0.4,0.2,0.6,-0.1,-0.2,-0.5)\rangle\\ \langle p_2,(0.3,0.5,0.7,-0.2,-0.4,-0.6)\rangle\end{matrix}\right\}\right\}$.

$\left(\tau_{R_{BNN^*}}(Q)\right)^C=\left\{0_{BN^*},1_{BN^*},\left\{\begin{matrix}\langle p_1,(0.6,0.2,0.4,-0.1,-0.8,-0.5)\rangle\\ \langle p_2,(0.3,0.5,0.7,-0.6,-0.6,-0.2)\rangle\end{matrix}\right\},\left\{\begin{matrix}\langle p_1,(0.6,0.8,0.4,-0.5,-0.8,-0.1)\rangle\\ \langle p_2,(0.7,0.5,0.3,-0.6,-0.6,-0.2)\rangle\end{matrix}\right\}\right\}$.

$E_1=\left\{\begin{matrix}\langle p_1,(0.6,0.8,0.3,-0.3,-0.7,-0.4)\rangle\\ \langle p_2,(0.4,0.6,0.6,-0.4,-0.6,-0.6)\rangle\end{matrix}\right\}$, $E_2=\left\{\begin{matrix}\langle p_1,(0.3,0.4,0.6,-0.7,-0.6,-0.3)\rangle\\ \langle p_2,(0.4,0.3,0.5,-0.6,-0.5,-0.3)\rangle\end{matrix}\right\}$ are $BNN_Q{}^*PO$ sets.

$E_1\cap E_2=\left\{\begin{matrix}\langle p_1,(0.3,0.4,0.6,-0.3,-0.6,-0.4)\rangle\\ \langle p_2,(0.4,0.3,0.6,-0.4,-0.5,-0.6)\rangle\end{matrix}\right\}$ and $BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(E_1\cap E_2))=\left\{\begin{matrix}\langle p_1,(0.4,0.8,0.6,-0.5,-0.2,-0.1)\rangle\\ \langle p_2,(0.7,0.5,0.3,-0.2,-0.4,-0.6)\rangle\end{matrix}\right\}$.

Clearly $E_1\cap E_2\not\subset BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(E_1\cap E_2))$. Hence $E_1\cap E_2$ is not $BNN_Q{}^*PO$ set.

Theorem: 3.1.5 Every $BNN_Q{}^*O$ set in $(U,\tau_{R_{BNN^*}}(Q))$ is $BNN_Q{}^*PO$ set in U.

Proof. Let E be $BNN_Q{}^*O$ set in $(U,\tau_{R_{BNN^*}}(Q))$. Then $E=BNN_Q{}^*\operatorname{int}(E)$. Also $E\subseteq BNN_Q{}^*cl(E)$. $BNN_Q{}^*\operatorname{int}(E)\subseteq BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(E))$. $E\subseteq BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(E))$. E is $BNN_Q{}^*PO$ set in U.

Remark: 3.1.6 The following example shows that the converse of the above theorem is not true.

Example: 3.1.7 From example 3.1.4, $E_1=\left\{\begin{matrix}\langle p_1,(0.6,0.8,0.3,-0.3,-0.7,-0.4)\rangle\\ \langle p_2,(0.4,0.6,0.6,-0.4,-0.6,-0.6)\rangle\end{matrix}\right\}$ is $BNN_Q{}^*PO$ set but not $BNN_Q{}^*O$ set.

Theorem: 3.1.8 If $\{E_\alpha\}_{\alpha\in\Omega}$ is a collection of $BNN_Q{}^*PC$ sets in $(U,\tau_{R_{BNN^*}}(Q))$, then $\bigcap_{\alpha\in\Omega}E_\alpha$ is $BNN_Q{}^*PC$ set in U.

Proof. $\{E_\alpha{}^C\}_{\alpha\in\Omega}$ is a collection of $BNN_Q{}^*PO$ sets in $(U,\tau_{R_{BNN^*}}(Q))$. By theorem 3.1.2 and De-Morgan's law $\bigcap_{\alpha\in\Omega}E_\alpha$ is $BNN_Q{}^*PC$ set in U.

Remark: 3.1.9 By remark 3.1.3, the union of any two $BNN_Q{}^*PC$ sets need not be a $BNN_Q{}^*PC$ set in U.

Theorem: 3.1.10 Every $BNN_Q{}^*C$ set in $(U,\tau_{R_{BNN^*}}(Q))$ is $BNN_Q{}^*PC$ set in U.

Proof. Let E be a $BNN_Q{}^*C$ set in $(U,\tau_{R_{BNN^*}}(Q))$. Then $E=BNN_Q{}^*cl(E)$. Also $BNN_Q{}^*\operatorname{int}(E)\subseteq E$. $BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E))\subseteq BNN_Q{}^*cl(E)$. $BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E))\subseteq E$. Hence E is $BNN_Q{}^*PC$ set in U.

*3.2 Bipolar Neutrosophic Nano * Semi Open Sets*

Definition: 3.2.1 Let E be a neutrosophic set in a BNN^*TS $(U,\tau_{R_{BNN^*}}(Q))$. Then E is said to be $BNN_Q{}^*$ - semi-open set ($BNN_Q{}^*SO$ set) of U if $E\subseteq BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E))$. The complement of $BNN_Q{}^*SO$ set is called $BNN_Q{}^*$ - semi-closed set ($BNN_Q{}^*SC$ set) of U.

Theorem: 3.2.2 Arbitrary union of $BNN_Q{}^*SO$ is $BNN_Q{}^*SO$ set in U.

Proof. If $\{E_\alpha\}_{\alpha\in\Omega}$ is a collection of $BNN_Q{}^*SO$ sets in $(U,\tau_{R_{BNN^*}}(Q))$. For each $\alpha\in\Omega$, $E_\alpha\subseteq BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E_\alpha))$.

$$\begin{aligned}E_1\cup E_2\cup......&\subseteq BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E_1))\cup BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E_2))\cup.........\\ &=BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E_1)\cup BNN_Q{}^*\operatorname{int}(E_2)\cup.....)\\ &\subseteq BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E_1\cup E_2\cup....))\end{aligned}$$

Hence $\bigcup_{\alpha\in\Omega}E_\alpha$ is $BNN_Q{}^*SO$ set in U.

Theorem: 3.2.3 Every $BNN_Q{}^*O$ set in $(U,\tau_{R_{BNN^*}}(Q))$ is $BNN_Q{}^*SO$ set in U.

Proof. Let E be $BNN_Q{}^*O$ set in $(U,\tau_{R_{BNN^*}}(Q))$. Then $E=BNN_Q{}^*\operatorname{int}(E)$. $BNN_Q{}^*cl(E)=BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E))$. Also $E\subseteq BNN_Q{}^*cl(E)$. Then $E\subseteq BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E))$. E is $BNN_Q{}^*SO$ set in U.

Remark: 3.2.4 The following example shows that the converse of the above theorem is not true.

Example: 3.2.5 Let $E=\left\{\begin{matrix}\langle p_1,(0.4,0.2,0.6,-0.4,-0.3,-0.5)\rangle\\ \langle p_2,(0.3,0.5,0.7,-0.4,-0.4,-0.5)\rangle\end{matrix}\right\}$.

From example 3.1.4, $BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E))=(BN_1*(Q))^C$. Also $E\subseteq(BN_1*(Q))^C$.

$\therefore$ E is $BNN_Q{}^*SO$ set but not $BNN_Q{}^*O$ set.

Theorem: 3.2.6 Arbitrary intersection of $BNN_Q{}^*SC$ set is $BNN_Q{}^*SC$ set in U.

Proof. Let $\{E_\alpha\}_{\alpha\in\Omega}$ is a collection of $BNN_Q{}^*SC$ sets in $(U,\tau_{R_{BNN^*}}(Q))$. Then $\{E_\alpha{}^C\}_{\alpha\in\Omega}$ is a collection of $BNN_Q{}^*SO$ sets in $(U,\tau_{R_{BNN^*}}(Q))$. By theorem 3.2.2 and by De-Morgan's law, $\bigcap_{\alpha\in\Omega} E_\alpha$ is $BNN_Q{}^*SC$ set in U.

Theorem: 3.2.7 Every $BNN_Q{}^*C$ set in $(U,\tau_{R_{BNN^*}}(Q))$ is $BNN_Q{}^*SC$ set in U.

Proof. Let E be a $BNN_Q{}^*C$ set in $(U,\tau_{R_{BNN^*}}(Q))$. Then $E = BNN_Q{}^*cl(E)$. Also. $BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(E)) = BNN_Q{}^*\operatorname{int}(E)$. $BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(E)) \subseteq E$. Hence E is $BNN_Q{}^*SC$ set in U.

*3.3 Bipolar Neutrosophic Nano * α Open Sets*

Definition: 3.3.1 Let E be a neutrosophic set in a BNN^*TS $(U,\tau_{R_{BNN^*}}(Q))$. Then E is said to be $BNN_Q{}^*$ - α -open set ($BNN_Q{}^*\alpha O$ set) of U if $E \subseteq BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E)))$. The complement of $BNN_Q{}^*\alpha O$ set is called $BNN_Q{}^*$ - α -closed set ($BNN_Q{}^*\alpha C$ set) of U.

Theorem: 3.3.2 Arbitrary union of $BNN_Q{}^*\alpha O$ is $BNN_Q{}^*\alpha O$ set in U.

Proof. Let $\{E_\alpha\}_{\alpha\in\Omega}$ is a collection of $BNN_Q{}^*\alpha O$ sets in $(U,\tau_{R_{BNN^*}}(Q))$. For each $\alpha\in\Omega$, $E_\alpha \subseteq BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E_\alpha)))$.

$$\begin{aligned}
E_1 \cup E_2 \cup \ldots\ldots &\subseteq BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E_1))) \cup BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E_2))) \cup \ldots\ldots\ldots \\
&\subseteq BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E_1) \cup BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E_2)) \cup \ldots\ldots)) \\
&= BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E_1) \cup BNN_Q{}^*\operatorname{int}(E_2) \cup \ldots\ldots)) \\
&\subseteq BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E_1 \cup E_2 \cup \ldots\ldots)))
\end{aligned}$$

Hence $\bigcup_{\alpha\in\Omega} E_\alpha$ is $BNN_Q{}^*\alpha O$ set in U.

Theorem: 3.3.3 In a BNN^*TS $(U,\tau_{R_{BNN^*}}(Q))$, the $BNN_Q{}^*$- open sets of U and for sets $E \supset \overline{BN}^*(Q)$ with $BNN_Q{}^*cl(\overline{BN}^*(Q)) = 1_{BNN^*}$ are the only $BNN_Q{}^*\alpha O$ sets in U.

Proof. Since $BNN_Q{}^*O$ sets are $BNN_Q{}^*\alpha O$, then $0_{BNN^*}, 1_{BNN^*}, \overline{BN}^*(Q), \underline{BN}^*(Q), B_{BN}{}^*(Q), BN_1{}^*(Q), BN_2{}^*(Q)$ are $BNN_Q{}^*\alpha O$ in U. If $E \neq 0_{BNN^*}$ and $E \subset \underline{BN}^*(Q)$, then $BNN_Q{}^*\operatorname{int}(E) = 0_{BNN^*}$, since 0_{BNN^*} is the only $BNN_Q{}^*O$ subset of E. Therefore $BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E))) = 0_{BNN^*}$ and hence E is not $BNN_Q{}^*\alpha O$. If $E \subset B_{BN}{}^*(Q)$, then $BNN^*\operatorname{int}(E) = 0_{BNN^*}$ and hence E is not $BNN_Q{}^*\alpha O$. If $E \subset \overline{BN}^*(Q)$, then $E \subset B_{BN}{}^*(Q)$ and $E \subset \underline{BN}^*(Q)$, hence E is not $BNN_Q{}^*\alpha O$. If $E \supset \overline{BN}^*(Q)$, then $BBN_Q{}^*\operatorname{int}(E) = \overline{BN}^*(Q)$ and hence $BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E))) = BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(\overline{BN}^*(Q))) = BNN_Q{}^*\operatorname{int}(1_{BNN^*}) = 1_{BNN^*}$. Therefore $E \subseteq BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E)))$. E is $BNN_Q{}^*\alpha O$. This will exist only in the case if $BNN_Q{}^*cl(\overline{BN}^*(Q)) = 1_{BNN^*}$. If $E \subset \underline{BN}^*(Q)$ and $E \subset B_{BN}{}^*(Q)$, by definition $E \subset BN_1{}^*(Q)$ and $E \subset BN_2{}^*(Q)$, then in both the cases E is not $BNN_Q{}^*\alpha O$.

Remark: 3.3.4 The following example shows that the case $E \supset \overline{BN}^*(Q)$ in the above theorem in which $BNN_Q{}^*cl(\overline{BN}^*(Q)) \neq 1_{BNN^*}$ is not $BNN_Q{}^*\alpha O$.

Example: 3.3.5 Let $U = \{p_1, p_2, p_3\}, U/R = \{\{p_1, p_3\}, \{p_2\}\}$, $Q = \left\{\begin{matrix}\langle p_1,(0.6,0.5,0.2,-0.3,-0.4,-0.7)\rangle \\ \langle p_2,(0.5,0.6,0.4,-0.4,-0.5,-0.5)\rangle \\ \langle p_3,(0.2,0.3,0.8,-0.5,-0.6,-0.4)\rangle\end{matrix}\right\}$.

$\overline{BN}^*(Q) = \left\{\begin{matrix}\langle p_1,(0.6,0.5,0.2,-0.5,-0.6,-0.4)\rangle \\ \langle p_2,(0.5,0.6,0.4,-0.4,-0.5,-0.5)\rangle \\ \langle p_3,(0.6,0.5,0.2,-0.5,-0.6,-0.4)\rangle\end{matrix}\right\}$, $\underline{BN}^*(Q) = \left\{\begin{matrix}\langle p_1,(0.2,0.3,0.8,-0.3,-0.4,-0.7)\rangle \\ \langle p_2,(0.5,0.6,0.4,-0.4,-0.5,-0.5)\rangle \\ \langle p_3,(0.2,0.3,0.8,-0.3,-0.4,-0.7)\rangle\end{matrix}\right\}$,

$B_{BN}{}^*(Q) = \left\{\begin{matrix}\langle p_1,(0.6,0.5,0.2,-0.5,-0.6,-0.4)\rangle \\ \langle p_2,(0.5,0.6,0.4,-0.4,-0.5,-0.5)\rangle \\ \langle p_3,(0.6,0.5,0.2,-0.5,-0.6,-0.4)\rangle\end{matrix}\right\}$, $BN_1{}^*(Q) = \left\{\begin{matrix}\langle p_1,(0.6,0.5,0.2,-0.5,-0.6,-0.4)\rangle \\ \langle p_2,(0.5,0.6,0.4,-0.4,-0.5,-0.5)\rangle \\ \langle p_3,(0.6,0.5,0.2,-0.5,-0.6,-0.4)\rangle\end{matrix}\right\}$,

$BN_2{}^*(Q) = \left\{\begin{matrix}\langle p_1,(0.2,0.3,0.8,-0.3,-0.4,-0.7)\rangle \\ \langle p_2,(0.4,0.4,0.5,-0.4,-0.5,-0.5)\rangle \\ \langle p_3,(0.2,0.3,0.8,-0.3,-0.4,-0.7)\rangle\end{matrix}\right\}$.

$\tau_{R_{BNN^*(Q)}} = \{0_{BNN^*}, \overline{BN}^*(Q), \underline{BN}^*(Q), B_{BN}{}^*(Q), BN_1{}^*(Q), BN_2{}^*(Q), 1_{BNN^*}\}$.

Let $E=\left\{\begin{matrix}\langle p_1,(0.7,0.6,0.1,-0.6,-0.7,-0.3)\rangle\\ \langle p_2,(0.6,0.7,0.3,-0.5,-0.5,-0.4)\rangle\\ \langle p_3,(0.8,0.6,0.1,-0.7,-0.7,-0.3)\rangle\end{matrix}\right\}\supset\overline{BN}^*(Q)$, then $BNN_Q{}^*int(E)=\overline{BN}^*(Q)$.

$BNN_Q{}^*cl(BNN_Q{}^*int(E))=BNN_Q{}^*cl(\overline{BN}^*(Q))=(BN_2{}^*(Q))^C$.

$BNN_Q{}^*int(BNN_Q{}^*cl(BNN_Q{}^*int(E)))=BNN_Q{}^*int((BN_2{}^*(Q))^C)=\overline{BN}^*(Q)$. $E\not\subset\overline{BN}^*(Q)$, since it contains $\overline{BN}^*(Q)$. Hence E is not $BNN_Q{}^*\alpha O$.

Theorem: 3.3.6 The intersection of any two $BNN_Q{}^*\alpha O$ sets is $BNN_Q{}^*\alpha O$ set in $(U,\tau_{R_{BNN^*}}(Q))$.

Proof. From the above theorem, the $BNN_Q{}^*$ - open sets of U and for sets $E\supset\overline{BN}^*(Q)$ where $BNN_Q{}^*cl(\overline{BN}^*(Q))=1_{BNN^*}$ are the only $BNN_Q{}^*\alpha O$ sets in U. Finite intersection of $BNN_Q{}^*$ - open sets is $BNN_Q{}^*$ - open and hence $BNN_Q{}^*\alpha O$. If $E_1,E_2\supset\overline{BN}^*(Q)$ such that $BNN_Q{}^*int(E_1)=\overline{BN}^*(Q)$, $BNN_Q{}^*int(E_2)=\overline{BN}^*(Q)$ and $BNN_Q{}^*cl(\overline{BN}^*(Q))=1_{BNN^*}$, then

$BNN_Q{}^*int(E_1\cap E_2)=BNN_Q{}^*int(E_1)\cap BNN_Q{}^*int(E_2)=\overline{BN}^*(Q)$.

$BNN_Q{}^*cl(BNN_Q{}^*int(E_1\cap E_2))=BNN_Q{}^*cl(\overline{BN}^*(Q))=1_{BNN^*}$.

$BNN_Q{}^*int(BNN_Q{}^*cl(BNN_Q{}^*int(E_1\cap E_2)))=BNN_Q{}^*int(1_{BNN^*})=1_{BNN^*}$.

Hence the intersection of any two $BNN_Q{}^*\alpha O$ sets is $BNN_Q{}^*\alpha O$ set in U.

Theorem: 3.3.7 Every $BNN_Q{}^*O$ set in $(U,\tau_{R_{BNN^*}}(Q))$ is $BNN_Q{}^*\alpha O$ set in U.

Proof. Let E be $BNN_Q{}^*O$ set in $(U,\tau_{R_{BNN^*}}(Q))$. Then $E=BNN_Q{}^*int(E)$. $BNN_Q{}^*cl(E)=BNN_Q{}^*cl(BNN_Q{}^*int(E))$. Also $E\subseteq BNN_Q{}^*cl(E)$. Then $E\subseteq BNN_Q{}^*cl(BNN_Q{}^*int(E))$.

Now $BNN_Q{}^*int(E)\subseteq BNN_Q{}^*int(BNN_Q{}^*cl(BNN_Q{}^*int(E)))$.

Thus $E\subseteq BNN_Q{}^*int(BNN_Q{}^*cl(BNN_Q{}^*int(E)))$. Hence E is $BNN_Q{}^*\alpha O$ set in U.

Example: 3.3.8 The converse of the above theorem need not be true. For example, let $U=\{p_1,p_2,p_3\}$, $U/R=\{\{p_1,p_2\},\{p_3\}\}$, $Q=\left\{\begin{matrix}\langle p_1,(0.2,0.5,0.6,-0.7,-0.5,-0.2)\rangle\\ \langle p_2,(0.3,0.4,0.6,-0.6,-0.4,-0.3)\rangle\\ \langle p_3,(0.2,0.3,0.8,-0.5,-0.5,-0.4)\rangle\end{matrix}\right\}$.

$\overline{BN}^*(Q)=\left\{\begin{matrix}\langle p_1,(0.3,0.5,0.6,-0.7,-0.5,-0.2)\rangle\\ \langle p_2,(0.3,0.5,0.6,-0.7,-0.5,-0.2)\rangle\\ \langle p_3,(0.4,0.5,0.5,-0.5,-0.5,-0.4)\rangle\end{matrix}\right\}$, $\underline{BN}^*(Q)=\left\{\begin{matrix}\langle p_1,(0.2,0.4,0.6,-0.6,-0.4,-0.3)\rangle\\ \langle p_2,(0.2,0.4,0.6,-0.6,-0.4,-0.3)\rangle\\ \langle p_3,(0.4,0.5,0.5,-0.5,-0.5,-0.4)\rangle\end{matrix}\right\}$,

$B_{BN}{}^*(Q)=\left\{\begin{matrix}\langle p_1,(0.3,0.5,0.6,-0.3,-0.5,-0.6)\rangle\\ \langle p_2,(0.3,0.5,0.6,-0.3,-0.5,-0.6)\rangle\\ \langle p_3,(0.4,0.5,0.5,-0.4,-0.5,-0.5)\rangle\end{matrix}\right\}$, $BN_1{}^*(Q)=\left\{\begin{matrix}\langle p_1,(0.3,0.5,0.6,-0.6,-0.5,-0.3)\rangle\\ \langle p_2,(0.3,0.5,0.6,-0.6,-0.5,-0.3)\rangle\\ \langle p_3,(0.4,0.5,0.5,-0.5,-0.5,-0.4)\rangle\end{matrix}\right\}$,

$BN_2{}^*(Q)=\left\{\begin{matrix}\langle p_1,(0.2,0.4,0.6,-0.3,-0.4,-0.6)\rangle\\ \langle p_2,(0.2,0.4,0.6,-0.3,-0.4,-0.6)\rangle\\ \langle p_3,(0.4,0.5,0.5,-0.4,-0.5,-0.5)\rangle\end{matrix}\right\}$.

$\tau_{R_{BNN}{}^*(Q)}=\{0_{BNN^*},\overline{BN}^*(Q),\underline{BN}^*(Q),B_{BN}{}^*(Q),BN_1{}^*(Q),BN_2{}^*(Q),1_{BNN^*}\}$.

Let $E=\left\{\begin{matrix}\langle p_1,(0.4,0.5,0.5,-0.7,-0.6,-0.2)\rangle\\ \langle p_2,(0.4,0.5,0.5,-0.7,-0.6,-0.2)\rangle\\ \langle p_3,(0.5,0.5,0.3,-0.6,-0.5,-0.2)\rangle\end{matrix}\right\}$, then $BNN_Q{}^*int(BNN_Q{}^*cl(BNN_Q{}^*int(E)))=1_{BNN^*}$.

E is $BNN_Q{}^*\alpha O$ but not $BNN_Q{}^*O$ set.

Theorem: 3.3.9 Arbitrary intersection of $BNN_Q{}^*\alpha C$ sets is $BNN_Q{}^*\alpha C$ set in $(U,\tau_{R_{BNN^*}}(Q))$.

Proof. Let $\{E_\alpha\}_{\alpha\in\Omega}$ is a collection of $BNN_Q{}^*\alpha C$ sets in $(U,\tau_{R_{BNN^*}}(Q))$. Then $\{E_\alpha{}^C\}_{\alpha\in\Omega}$ is a collection of $BNN_Q{}^*\alpha O$ sets in $(U,\tau_{R_{BNN^*}}(Q))$. By theorem 3.3.2 and De-Morgan's law $\bigcap_{\alpha\in\Omega}E_\alpha$ is $BNN_Q{}^*\alpha C$ set in U.

Remark: 3.3.10 By theorem: 3.3.6, union of two $BNN_Q{}^*\alpha C$ sets is a $BNN_Q{}^*\alpha C$ set in U.

Theorem: 3.3.11 Every $BNN_Q{}^*C$ set in $(U,\tau_{R_{BNN^*}}(Q))$ is $BNN_Q{}^*\alpha C$ set in U.

Proof. Let E be a $BNN_Q{}^*C$ set in $(U,\tau_{R_{BNN^*}}(Q))$. Then $BNN_Q{}^*cl(E)=E$. Also $BNN_Q{}^*int(E)\subseteq E$. $BNN_Q{}^*int(BNN_Q{}^*cl(E))=BNN_Q{}^*int(E)\subseteq E$. $BNN_Q{}^*cl(BNN_Q{}^*int(BNN_Q{}^*cl(E)))\subseteq BNN_Q{}^*cl(E)$.

$BNN_Q{}^*cl(BNN_Q{}^*int(BNN_Q{}^*cl(E)))\subseteq E$. Hence E is $BNN_Q{}^*\alpha C$ set in U.

Remark: 3.3.12 The set of all BNN_Q^* - open sets, BNN_Q^* - pre open sets, BNN_Q^* - semi open sets and BNN_Q^* - α open sets of $(U, \tau_{R_{BNN^*}}(Q))$ are denoted by $BNN_Q^*O(U)$, $BNN_Q^*PO(U)$, $BNN_Q^*SO(U)$ and $BNN_Q^*\alpha O(U)$ respectively. The set of all BNN_Q^* - closed sets, BNN_Q^* - pre closed sets, BNN_Q^* - semi closed sets and BNN_Q^* - α closed sets of $(U, \tau_{R_{BNN^*}}(Q))$ are denoted by $BNN_Q^*C(U)$, $BNN_Q^*PC(U)$, $BNN_Q^*SC(U)$ and $BNN_Q^*\alpha C(U)$ respectively.

Theorem: 3.3.13 $BNN_Q^*\alpha O(U) \subseteq BNN_Q^*SO(U)$ in a BNN^*TS $(U, \tau_{R_{BNN^*}}(Q))$.

Proof. If $E \in BNN_Q^*\alpha O(U)$. $E \subseteq BNN_Q^* \operatorname{int}(BNN_Q^* cl(BNN_Q^* \operatorname{int}(E))) \subseteq BNN_Q^* cl(BNN_Q^* \operatorname{int}(E))$.

Then $E \in BNN_Q^*SO(U)$. Hence $BNN_Q^*\alpha O(U) \subseteq BNN_Q^*SO(U)$ in U.

Remark: 3.3.14 The converse of the above theorem need not be true. This is shown in the following example.

Example: 3.3.15 Let $E = \left\{ \begin{array}{l} \langle p_1, (0.4, 0.2, 0.6, -0.1, -0.3, -0.5) \rangle \\ \langle p_2, (0.3, 0.5, 0.7, -0.4, -0.4, -0.5) \rangle \end{array} \right\}$. From example: 3.1.4, E is BNN_Q^*SO but not $BNN_Q^*\alpha O$.

Theorem: 3.3.16 $BNN_Q^*\alpha O(U) \subseteq BNN_Q^*PO(U)$ in a BNN^*TS $(U, \tau_{R_{BNN^*}}(Q))$.

Proof. If $E \in BNN_Q^*\alpha O(U)$. $E \subseteq BNN_Q^* \operatorname{int}(BNN_Q^* cl(BNN_Q^* \operatorname{int}(E)))$.

Since $BNN_Q^* \operatorname{int}(E) \subseteq E$, $E \subseteq BNN_Q^* \operatorname{int}(BNN_Q^* cl(BNN_Q^* \operatorname{int}(E))) \subseteq BNN_Q^* \operatorname{int}(BNN_Q^* cl(E))$.

Then $E \in BNN_Q^*PO(U)$. Hence $BNN_Q^*\alpha O(U) \subseteq BNN_Q^*PO(U)$ in U.

Remark: 3.3.17 The converse of the above theorem need not be true. This is shown in the following example.

Example: 3.3.18 Let $E = \left\{ \begin{array}{l} \langle p_1, (0.6, 0.8, 0.3, -0.3, -0.7, -0.4) \rangle \\ \langle p_2, (0.4, 0.6, 0.6, -0.4, -0.6, -0.6) \rangle \end{array} \right\}$. From example: 3.1.4, E is BNN_Q^*PO but not $BNN_Q^*\alpha O$.

Theorem: 3.3.19 $BNN_Q^*\alpha O(U) = BNN_Q^*PO(U) \cap BNN_Q^*SO(U)$ in a BNN^*TS $(U, \tau_{R_{BNN^*}}(Q))$.

Proof. If $E \in BNN_Q^*\alpha O(U)$, then $E \in BNN_Q^*SO(U)$ and $E \in BNN_Q^*PO(U)$ by theorem 3.3.13 and 3.3.16. This follows that, $E \in BNN_Q^*PO(U) \cap BNN_Q^*SO(U)$. Hence $BNN_Q^*\alpha O(U) \subseteq BNN_Q^*PO(U) \cap BNN_Q^*SO(U)$.

Conversely, if $E \in BNN_Q^*PO(U) \cap BNN_Q^*SO(U)$, then $E \subseteq BNN_Q^* cl(BNN_Q^* \operatorname{int}(U))$ and $E \subseteq BNN_Q^* \operatorname{int}(BNN_Q^* cl(U))$.

Consider $E \subseteq BNN_Q^* cl(BNN_Q^* \operatorname{int}(U))$,

$BNN_Q^* \operatorname{int}(BNN_Q^* cl(E)) \subseteq BNN_Q^* \operatorname{int}\ (BNN_Q^* cl(BNN_Q^* cl(BNN_Q^* \operatorname{int}(U)))) = BNN_Q^* \operatorname{int}(BNN_Q^* cl(BNN_Q^* \operatorname{int}(U)))$.

Then $E \subseteq BNN_Q^* \operatorname{int}(BNN_Q^* cl(BNN_Q^* \operatorname{int}(U))) \Rightarrow E \in BNN_Q^*\alpha O(U)$.

This gives $BNN_Q^*\alpha O(U) \supseteq BNN_Q^*PO(U) \cap BNN_Q^*SO(U)$.

Hence $BNN_Q^*\alpha O(U) = BNN_Q^*PO(U) \cap BNN_Q^*SO(U)$.

Remark: 3.3.20 The following example shows that the BNN_Q^*PO and BNN_Q^*SO sets are independent of each other.

Example: 3.3.21 From example 3.1.4, $E = \left\{ \begin{array}{l} \langle p_1, (0.6, 0.8, 0.3, -0.3, -0.7, -0.4) \rangle \\ \langle p_2, (0.4, 0.6, 0.6, -0.4, -0.6, -0.6) \rangle \end{array} \right\}$ is BNN_Q^*PO but not BNN_Q^*SO. And $E = \left\{ \begin{array}{l} \langle p_1, (0.4, 0.2, 0.6, -0.1, -0.3, -0.5) \rangle \\ \langle p_2, (0.3, 0.5, 0.7, -0.4, -0.4, -0.5) \rangle \end{array} \right\}$ is BNN_Q^*SO but not BNN_Q^*PO.

Theorem: 3.3.22 The union of BNN_Q^*O sets and $BNN_Q^*\alpha O$ sets of $(U, \tau_{R_{BNN^*}}(Q))$ is BNN_Q^*PO.

Proof. Let E be a BNN_Q^*O set and F be a $BNN_Q^*\alpha O$ set in U. Then $BNN_Q^* \operatorname{int}(E) = E$ and $F \subseteq BNN_Q^* \operatorname{int}(BNN_Q^* cl(BNN_Q^* \operatorname{int}(F)))$. Now

$$\begin{aligned} E \cup F &\subseteq BNN_Q^* \operatorname{int}(E) \cup BNN_Q^* \operatorname{int}(BNN_Q^* cl(BNN_Q^* \operatorname{int}(F))) \\ &\subseteq BNN_Q^* \operatorname{int}(E \cup BNN_Q^* cl(BNN_Q^* \operatorname{int}(F))) \\ &\subseteq BNN_Q^* \operatorname{int}(BNN_Q^* cl(E) \cup BNN_Q^* cl(BNN_Q^* \operatorname{int}(F))) \\ &\subseteq BNN_Q^* \operatorname{int}(BNN_Q^* cl(E) \cup BNN_Q^* cl(F)) \\ &\subseteq BNN_Q^* \operatorname{int}(BNN_Q^* cl(E \cup F)). \end{aligned}$$

Hence $E \cup F$ is BNN_Q^*PO.

Theorem: 3.3.23 The union of BNN_Q^*PO sets and $BNN_Q^*\alpha O$ sets of $(U, \tau_{R_{BNN^*}}(Q))$ is BNN_Q^*PO.

Proof. Let E be a BNN_Q^*PO set and F be a $BNN_Q^*\alpha O$ set in U. Then $E \subseteq BNN_Q^* \operatorname{int}(BNN_Q^* cl(E))$ and $F \subseteq BNN_Q^* \operatorname{int}(BNN_Q^* cl(BNN_Q^* \operatorname{int}(F)))$. Now

$$\begin{aligned} E \cup F &\subseteq BNN_Q^* \operatorname{int}(BNN_Q^* cl(E)) \cup BNN_Q^* \operatorname{int}(BNN_Q^* cl(BNN_Q^* \operatorname{int}(F))) \\ &\subseteq BNN_Q^* \operatorname{int}(BNN_Q^* cl(E) \cup BNN_Q^* cl(BNN_Q^* \operatorname{int}(F))) \end{aligned}$$

$\subseteq BNN_Q{}^* \operatorname{int}\left(BNN_Q{}^* cl(E) \cup BNN_Q{}^* cl(F)\right)$

$\subseteq BNN_Q{}^* \operatorname{int}\left(BNN_Q{}^* cl(E \cup F)\right)$. Hence $E \cup F$ is $BNN_Q{}^*PO$.

Theorem: 3.3.24 If E is $BNN_Q{}^*O$ and $BNN_Q{}^*PO$ in $\left(U, \tau_{R_{BNN^*}}(Q)\right)$, then E is $BNN_Q{}^*\alpha O$.

Proof. If E is $BNN_Q{}^*O$ and $BNN_Q{}^*PO$ in U, then $BNN_Q{}^* \operatorname{int}(E) = E$ and $E \subseteq BNN_Q{}^* \operatorname{int}\left(BNN_Q{}^* cl(E)\right)$.

Consider $E = BNN_Q{}^* \operatorname{int}(E) \Rightarrow BNN_Q{}^* cl(E) = BNN_Q{}^* cl\left(BNN_Q{}^* \operatorname{int}(E)\right) \Rightarrow BNN_Q{}^* \operatorname{int}\left(BNN_Q{}^* cl(E)\right) = BNN_Q{}^* \operatorname{int}$ $\left(BNN_Q{}^* cl\left(BNN_Q{}^* \operatorname{int}(E)\right)\right)$. This implies $E \subseteq BNN_Q{}^* \operatorname{int}\left(BNN_Q{}^* cl\left(BNN_Q{}^* \operatorname{int}(E)\right)\right)$. Hence E is $BNN_Q{}^*\alpha O$.

Theorem: 3.3.25 Let E be a BN set in a BNN^*TS $\left(U, \tau_{R_{BNN^*}}(Q)\right)$. If F is a $BNN_Q{}^*SO$ set such that $F \subseteq E \subseteq BNN_Q{}^* \operatorname{int}\left(BNN_Q{}^* cl(F)\right)$, then E is a $BNN_Q{}^*\alpha O$ set.

Proof. Since F is a $BNN_Q{}^*SO$ set, we have $F \subseteq BNN_Q{}^* cl\left(BNN_Q{}^* \operatorname{int}(F)\right)$.

We have $E \subseteq BNN_Q{}^* \operatorname{int}\left(BNN_Q{}^* cl(F)\right) \subseteq BNN_Q{}^* \operatorname{int}\left(BNN_Q{}^* cl\left(BNN_Q{}^* cl\left(BNN_Q{}^* \operatorname{int}(F)\right)\right)\right)$

$= BNN_Q{}^* \operatorname{int}\left(BNN_Q{}^* cl\left(BNN_Q{}^* \operatorname{int}(F)\right)\right) \subseteq BNN_Q{}^* \operatorname{int}\left(BNN_Q{}^* cl\left(BNN_Q{}^* \operatorname{int}(E)\right)\right)$. Hence E is a $BNN_Q{}^*\alpha O$ set.

*3.4 Bipolar Neutrosophic Nano * Regular Open Sets*

Definition: 3.4.1 Let E be a neutrosophic set in BNN^*TS $\left(U, \tau_{R_{BNN^*}}(Q)\right)$. Then E is said to be $BNN_Q{}^*$ - regular-open set ($BNN_Q{}^*RO$ set) of U if $E = BNN_Q{}^* \operatorname{int}\left(BNN_Q{}^* cl(E)\right)$. The complement of $BNN_Q{}^*RO$ set is called $BNN_Q{}^*$ - regular closed set ($BNN_Q{}^*RC$ set) of U.

Theorem: 3.4.2 Every $BNN_Q{}^*RO$ set is $BNN_Q{}^*O$ set in $\left(U, \tau_{R_{BNN^*}}(Q)\right)$.

Proof. If E is $BNN_Q{}^*RO$ in $\left(U, \tau_{R_{BNN^*}}(Q)\right)$, then $E = BNN_Q{}^* \operatorname{int}\left(BNN_Q{}^* cl(E)\right)$. Now $BNN_Q{}^* \operatorname{int}(E) = BNN_Q{}^* \operatorname{int}\left(BNN_Q{}^* \operatorname{int}\left(BNN_Q{}^* cl(E)\right)\right) = BNN_Q{}^* \operatorname{int}\left(BNN_Q{}^* cl(E)\right) = E$. Hence E is $BNN_Q{}^*O$ in U.

Remark: 3.4.3 The converse of the above theorem need not be true. A $BNN_Q{}^*O$ set need not be $BNN_Q{}^*RO$ in $\left(U, \tau_{R_{BNN^*}}(Q)\right)$.

Example: 3.4.4 Let $U = \{p_1, p_2, p_3\}, U/R = \{\{p_1, p_3\}, \{p_2\}\}$, $Q = \left\{\begin{array}{l}\langle p_1, (0.6,0.5,0.2,-0.3,-0.4,-0.7)\rangle \\ \langle p_2, (0.5,0.6,0.4,-0.4,-0.5,-0.5)\rangle \\ \langle p_3, (0.2,0.3,0.8,-0.5,-0.6,-0.4)\rangle\end{array}\right\}$.

$\overline{BN}^*(Q) = \left\{\begin{array}{l}\langle p_1, (0.6,0.5,0.2,-0.5,-0.6,-0.4)\rangle \\ \langle p_2, (0.5,0.6,0.4,-0.4,-0.5,-0.5)\rangle \\ \langle p_3, (0.6,0.5,0.2,-0.5,-0.6,-0.4)\rangle\end{array}\right\}$, $\underline{BN}^*(Q) = \left\{\begin{array}{l}\langle p_1, (0.2,0.3,0.8,-0.3,-0.4,-0.7)\rangle \\ \langle p_2, (0.5,0.6,0.4,-0.4,-0.5,-0.5)\rangle \\ \langle p_3, (0.2,0.3,0.8,-0.3,-0.4,-0.7)\rangle\end{array}\right\}$,

$B_{BN}{}^*(Q) = \left\{\begin{array}{l}\langle p_1, (0.6,0.5,0.2,-0.5,-0.6,-0.4)\rangle \\ \langle p_2, (0.5,0.6,0.4,-0.4,-0.5,-0.5)\rangle \\ \langle p_3, (0.6,0.5,0.2,-0.5,-0.6,-0.4)\rangle\end{array}\right\}$, $BN_1{}^*(Q) = \left\{\begin{array}{l}\langle p_1, (0.6,0.5,0.2,-0.5,-0.6,-0.4)\rangle \\ \langle p_2, (0.5,0.6,0.4,-0.4,-0.5,-0.5)\rangle \\ \langle p_3, (0.6,0.5,0.2,-0.5,-0.6,-0.4)\rangle\end{array}\right\}$,

$BN_2{}^*(Q) = \left\{\begin{array}{l}\langle p_1, (0.2,0.3,0.8,-0.3,-0.4,-0.7)\rangle \\ \langle p_2, (0.4,0.4,0.5,-0.4,-0.5,-0.5)\rangle \\ \langle p_3, (0.2,0.3,0.8,-0.3,-0.4,-0.7)\rangle\end{array}\right\}$. $\tau_{R_{BNN^*(Q)}} = \left\{0_{BNN^*}, \overline{BN}^*(Q), \underline{BN}^*(Q), B_{BN}{}^*(Q), BN_1{}^*(Q), BN_2{}^*(Q), 1_{BNN^*}\right\}$.

$0_{BNN^*}, \overline{BN}^*(Q), B_{BN}{}^*(Q), BN_1{}^*(Q), 1_{BNN}{}^*$ are $BNN_Q{}^*RO$ sets in U.

Theorem: 3.4.5 $BNN_Q{}^*RO$ sets are $BNN_Q{}^*PO$ sets.

Proof. The proof follows from the definitions of $BNN_Q{}^*RO$ and $BNN_Q{}^*PO$ sets.

Remark: 3.4.6 The converse of the above theorem is not true. This is shown in the following example.

Example: 3.4.7 Let $E = \left\{\begin{array}{l}\langle p_1, (0.4,0.4,0.5,-0.2,-0.3,-0.8)\rangle \\ \langle p_2, (0.4,0.5,0.6,-0.3,-0.3,-0.6)\rangle \\ \langle p_3, (0.1,0.2,0.9,-0.4,-0.4,-0.6)\rangle\end{array}\right\}$.

From example: 3.4.4, $BNN_Q{}^* \operatorname{int}\left(BNN_Q{}^* cl(E)\right) = \overline{BN}^*(Q)$. Also $E \subseteq \overline{BN}^*(Q)$.

E is $BNN_Q{}^*PO$, but not $BNN_Q{}^*RO$ set.

Theorem: 3.4.8 $BNN_Q{}^*RO$ sets are $BNN_Q{}^*\alpha O$ sets.

Proof. Since $BNN_Q{}^*RO$ sets are $BNN_Q{}^*O$ and $BNN_Q{}^*O$ sets are $BNN_Q{}^*\alpha O$ sets, the result follows.

Example: 3.4.9 This example shows that the converse of the above theorem is not true.

From example: 3.3.6, E is $BNN_Q{}^*\alpha O$ but not $BNN_Q{}^*RO$.

Theorem: 3.4.10 $BNN_Q{}^*RO$ sets are $BNN_Q{}^*SO$ sets.

Proof. Since $BNN_Q{}^*RO$ sets are $BNN_Q{}^*O$ and $BNN_Q{}^*O$ sets are $BNN_Q{}^*SO$ sets, the result follows.

Example: 3.4.11 This example shows that the converse of the above theorem is not true.

From example: 3.2.5, E is $BNN_Q{}^*SO$ but not $BNN_Q{}^*RO$.

Theorem: 3.4.12 The arbitrary union of $BNN_Q{}^*RO$ sets is $BNN_Q{}^*RO$ in U.

Proof. Let $\{E_\alpha\}_{\alpha\in\Omega}$ is a collection of $BNN_Q{}^*RO$ sets in $(U,\tau_{R_{BNN^*}}(Q))$. Then for each $\alpha\in\Omega$, $E_\alpha \subseteq BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(E_\alpha))$.

$$E_1 \cup E_2 \cup ...... \subseteq BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(E_1)) \cup BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(E_2)) \cup ....$$
$$\subseteq BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(E_1) \cup BNN_Q{}^*cl(E_2) \cup .....)$$
$$\subseteq BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(E_1 \cup E_2 \cup ....))$$

Hence $\bigcup_{\alpha\in\Omega} E_\alpha$ is $BNN_Q{}^*RO$ set in U.

Theorem: 3.4.13 In a BNN^*TS $(U,\tau_{R_{BNN^*}}(Q))$, the $BNN_Q{}^*$ - open sets $0_{BNN^*}, 1_{BNN^*}$, $B_{BN}{}^*(Q), BN_1{}^*(Q)$ and $\overline{BN}{}^*(Q)$ with $BNN_Q{}^*cl(\overline{BN}{}^*(Q)) = (BN_2{}^*(Q))^C$ are the only $BNN_Q{}^*RO$ sets in U.

Proof. **Table 1.** $BNN_Q{}^*$ - interior closure of each $BNN_Q{}^*O$ sets

$BNN_Q{}^*O$ **Set (E)**	$BNN_Q{}^*cl(E)$	$BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(E))$
$\overline{BN}{}^*(Q)$	$(BN_2{}^*(Q))^C$	$\overline{BN}{}^*(Q)$
$\underline{BN}{}^*(Q)$	$(B_{BN}{}^*(Q))^C$	$BN_1{}^*(Q)$
$B_{BN}{}^*(Q)$	$(BN_1{}^*(Q))^C$	$B_{BN}{}^*(Q)$
$BN_1{}^*(Q)$	$(B_{BN}{}^*(Q))^C$	$BN_1{}^*(Q)$
$BN_2{}^*(Q)$	$(BN_1{}^*(Q))^C$	$B_{BN}{}^*(Q)$

Since $BNN_Q{}^*RO$ sets are $BNN_Q{}^*O$, then $0_{BNN^*}, 1_{BNN^*}, B_{BN}{}^*(Q), BN_1{}^*(Q), \overline{BN}{}^*(Q)$ with $BNN_Q{}^*cl(\overline{BN}{}^*(Q)) = (BN_2{}^*(Q))^C$ are the only $BNN_Q{}^*RO$ sets in U.

Theorem: 3.4.14 Finite Intersection of $BNN_Q{}^*RO$ sets is $BNN_Q{}^*RO$.

Proof. From theorem 3.4.13, we have $0_{BNN^*}, 1_{BNN^*}, B_{BN}{}^*(Q), BN_1{}^*(Q), \overline{BN}{}^*(Q)$ with $BNN_Q{}^*cl(\overline{BN}{}^*(Q)) = (BN_2{}^*(Q))^C$ are the only $BNN_Q{}^*RO$ sets in U. If E is any one of the above $BNN_Q{}^*O$ sets, then $0_{BNN^*} \cap E = 0_{BNN^*}$ and $1_{BNN^*} \cap E = E$ are $BNN_Q{}^*RO$ sets. $B_{BN}{}^*(Q) \cap BN_1{}^*(Q) = B_{BN}{}^*(Q)$, $B_{BN}{}^*(Q) \cap \overline{BN}{}^*(Q) = B_{BN}{}^*(Q)$, $BN_1{}^*(Q) \cap \overline{BN}{}^*(Q) = BN_1{}^*(Q)$. Thus finite intersection of $BNN_Q{}^*RO$ sets is $BNN_Q{}^*RO$.

Remark: 3.4.15 The intersection and union of any two $BNN_Q{}^*RC$ sets are $BNN_Q{}^*RC$.

Theorem: 3.4.16 $BNN_Q{}^*RC$ sets are $BNN_Q{}^*C$ sets.

Proof. If E is $BNN_Q{}^*RC$ in $(U,\tau_{R_{BNN^*}}(Q))$, then $E = BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E))$. Now $BNN_Q{}^*cl(E) = BNN_Q{}^*cl(BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E))) = BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E)) = E$. Hence E is $BNN_Q{}^*C$ in U.

Theorem: 3.4.17 $BNN_Q{}^*RC$ sets are $BNN_Q{}^*PC$ sets.

Proof. The proof follows from the definitions of $BNN_Q{}^*RC$ and $BNN_Q{}^*PC$ sets.

*3.5 Bipolar Neutrosophic Nano * β Open Sets*

Definition: 3.5.1 Let E be a BN set in a BNN^*TS $(U,\tau_{R_{BNN^*}}(Q))$. Then E is said to be $BNN_Q{}^*$ - β -open set ($BNN_Q{}^*\beta O$ set) of U if $E \subseteq BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(E)))$. The complement of $BNN_Q{}^*\beta O$ set is called $BNN_Q{}^*$ - β - closed set ($BNN_Q{}^*\beta C$ set) of U.

Theorem: 3.5.2 $BNN_Q{}^*O$ sets are $BNN_Q{}^*\beta O$ sets.

Proof. Let E be a $BNN_Q{}^*O$ in $(U,\tau_{R_{BNN^*}}(Q))$. Then $BNN_Q{}^*\operatorname{int}(E) = E$. We have $E \subseteq BNN_Q{}^*cl(E)$. $E \subseteq BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E)) \subseteq BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(E)))$. Hence E is $BNN_Q{}^*\beta O$ in U.

Theorem: 3.5.3 $BNN_Q{}^*SO$ sets are $BNN_Q{}^*\beta O$ sets.

Proof. Let E be a $BNN_Q{}^*SO$ in $(U,\tau_{R_{BNN^*}}(Q))$. Then $E \subseteq BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E))$. We have $E \subseteq BNN_Q{}^*cl(E)$. $E \subseteq BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E)) \subseteq BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(E)))$. Hence E is $BNN_Q{}^*\beta O$ in U.

Theorem: 3.5.4 $BNN_Q{}^*RO$ sets are $BNN_Q{}^*\beta O$ sets.

Proof. Let E be a $BNN_Q{}^*RO$ in $(U,\tau_{R_{BNN^*}}(Q))$. Then $E = BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(E))$.

$E = BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(E)) \subseteq BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(E)))$. Hence E is $BNN_Q{}^*\beta O$ in U.

Theorem: 3.5.5 $BNN_Q{}^*\alpha O$ sets are $BNN_Q{}^*\beta O$ sets.

Proof. Let E be a $BNN_Q{}^*\alpha O$ in $\left(U,\tau_{R_{BNN^*}}(Q)\right)$.

Then $E\subseteq BNN_Q{}^*\operatorname{int}\left(BNN_Q{}^*cl\left(BNN_Q{}^*\operatorname{int}(E)\right)\right)$.

$E\subseteq BNN_Q{}^*\operatorname{int}\left(BNN_Q{}^*cl\left(BNN_Q{}^*\operatorname{int}(E)\right)\right)\subseteq BNN_Q{}^*\operatorname{int}\left(BNN_Q{}^*cl(E)\right)\subseteq BNN_Q{}^*cl\left(BNN_Q{}^*\operatorname{int}\left(BNN_Q{}^*cl(E)\right)\right)$.

Hence E is $BNN_Q{}^*\beta O$ in U.

Theorem: 3.5.6 $BNN_Q{}^*PO$ sets are $BNN_Q{}^*\beta O$ sets.

Proof. Let E be a $BNN_Q{}^*PO$ in $\left(U,\tau_{R_{BNN^*}}(Q)\right)$. Then $E\subseteq BNN_Q{}^*\operatorname{int}\left(BNN_Q{}^*cl(E)\right)$. $E\subseteq BNN_Q{}^*\operatorname{int}\left(BNN_Q{}^*cl(E)\right)\subseteq BNN_Q{}^*cl\left(BNN_Q{}^*\operatorname{int}\left(BNN_Q{}^*cl(E)\right)\right)$. Hence E is $BNN_Q{}^*\beta O$ in U.

Remark: 3.5.7 The following example shows that the converses of the theorems 3.5.2, 3.5.3, 3.5.4 and 3.5.5 are not true.

Example: 3.5.8 Let $E=\begin{Bmatrix}\langle p_1,(0.4,0.4,0.5,-0.2,-0.3,-0.8)\rangle\\ \langle p_2,(0.4,0.5,0.6,-0.3,-0.3,-0.6)\rangle\\ \langle p_3,(0.1,0.2,0.9,-0.4,-0.4,-0.6)\rangle\end{Bmatrix}$.

From example 3.4.4, $BNN_Q{}^*cl\left(BNN_Q{}^*\operatorname{int}\left(BNN_Q{}^*cl(E)\right)\right)=\left(BN_2{}^*(Q)\right)^C$. Also $E\subseteq(BN_2{}^*(Q))^C$. E is $BNN_Q{}^*\beta O$

And

(i) $BNN_Q{}^*\operatorname{int}(E)\neq E$, E is not $BNN_Q{}^*O$ set.

(ii) $BNN_Q{}^*cl\left(BNN_Q{}^*\operatorname{int}(E)\right)=0_{BNN^*}$ and $E\not\subset 0_{BNN^*}$. So E is not $BNN_Q{}^*SO$ set.

(iii) $BNN_Q{}^*\operatorname{int}\left(BNN_Q{}^*cl(E)\right)=\overline{BN}^*(Q)\neq E$. So E is not $BNN_Q{}^*RO$ set.

(iv) $BNN_Q{}^*\operatorname{int}\left(BNN_Q{}^*cl\left(BNN_Q{}^*\operatorname{int}(E)\right)\right)=0_{BNN^*}$ and $E\not\subset 0_{BNN^*}$. So E is not $BNN_Q{}^*\alpha O$ set.

Example: 3.5.9 This example shows that the converse of theorem 3.5.6 is not true.

Let $E=\begin{Bmatrix}\langle p_1,(0.2,0.2,0.8,-0.3,-0.3,-0.7)\rangle\\ \langle p_2,(0.2,0.3,0.5,-0.3,-0.3,-0.6)\rangle\\ \langle p_3,(0.2,0.2,0.7,-0.3,-0.4,-0.6)\rangle\end{Bmatrix}$.

$BNN_Q{}^*\operatorname{int}\left(BNN_Q{}^*cl(E)\right)=BN_2{}^*(Q)$, $E\not\subset BN_2{}^*(Q)$.

$BNN_Q{}^*cl\left(BNN_Q{}^*\operatorname{int}\left(BNN_Q{}^*cl(E)\right)\right)=\left(\overline{BN}^*(Q)\right)^C$, $E\subseteq\left(\overline{BN}^*(Q)\right)^C$.

Hence E is $BNN_Q{}^*\beta O$ but not $BNN_Q{}^*PO$.

Theorem: 3.5.10 Arbitrary union of $BNN_Q{}^*\beta O$ sets is $BNN_Q{}^*\beta O$ set.

Proof. If $\{E_\alpha\}_{\alpha\in\Omega}$ is a collection of $BNN_Q{}^*\beta O$ sets in $\left(U,\tau_{R_{BNN^*}}(Q)\right)$, then for each $\alpha\in\Omega$, $E_\alpha\subseteq BNN_Q{}^*cl\left(BNN_Q{}^*\operatorname{int}\left(BNN_Q{}^*cl(E_\alpha)\right)\right)$.

$$\begin{aligned}E_1\cup E_2\cup\ldots\ldots&\subseteq BNN_Q{}^*cl\left(BNN_Q{}^*\operatorname{int}\left(BNN_Q{}^*cl(E_1)\right)\right)\cup BNN_Q{}^*cl\left(BNN_Q{}^*\operatorname{int}\left(BNN_Q{}^*cl(E_2)\right)\right)\cup\ldots\ldots\ldots\\&=BNN_Q{}^*cl\left(BNN_Q{}^*\operatorname{int}\left(BNN_Q{}^*cl(E_1)\right)\cup BNN_Q{}^*\operatorname{int}\left(BNN_Q{}^*cl(E_2)\right)\cup\ldots\right)\\&\subseteq BNN_Q{}^*cl\left(BNN_Q{}^*\operatorname{int}\left(BNN_Q{}^*cl(E_1)\cup BNN_Q{}^*cl(E_2)\cup\ldots\ldots\right)\right)\\&=BNN_Q{}^*cl\left(BNN_Q{}^*\operatorname{int}\left(BNN_Q{}^*cl(E_1\cup E_2\cup\ldots.)\right)\right)\end{aligned}$$

Hence $\bigcup_{\alpha\in\Omega}E_\alpha$ is $BNN_Q{}^*\beta O$ set in U.

Theorem: 3.5.11 $BNN_Q{}^*SO(U,Q)\cup BNN_Q{}^*PO(U,Q)\subset BNN_Q{}^*\beta O(U,Q)$.

Proof. The proof follows from theorems 3.5.3 and 3.5.4.

Theorem: 3.5.12 If F is BN subset of U and E is $BNN_Q{}^*PO$ in U such that $E\subseteq F\subseteq BNN_Q{}^*cl\left(BNN_Q{}^*\operatorname{int}(E)\right)$, then F is $BNN_Q{}^*\beta O$.

Proof. Since E is $BNN_Q{}^*PO$ in U, $E\subseteq BNN_Q^*\operatorname{int}\left(BNN_Q^*cl(E)\right)$.

Now $F\subseteq BNN_Q{}^*cl\left(BNN_Q{}^*\operatorname{int}(E)\right)\subseteq BNN_Q{}^*cl\left(BNN_Q{}^*\operatorname{int}\left(BNN_Q{}^*\operatorname{int}\left(BNN_Q{}^*cl(E)\right)\right)\right)$

$=BNN_Q{}^*cl\left(BNN_Q{}^*\operatorname{int}\left(BNN_Q{}^*cl(E)\right)\right)\subseteq BNN_Q{}^*cl\left(BNN_Q{}^*\operatorname{int}\left(BNN_Q{}^*cl(F)\right)\right)$.

Hence $F\subseteq BNN_Q{}^*cl\left(BNN_Q{}^*\operatorname{int}\left(BNN_Q{}^*cl(F)\right)\right)$. Then F is $BNN_Q{}^*\beta O$.

Theorem: 3.5.13 Each $BNN_Q{}^*\beta O$ set which is $BNN_Q{}^*SC$ is $BNN_Q{}^*SO$.

Proof. Let E be $BNN_Q{}^*\beta O$ set which is $BNN_Q{}^*SC$. Then $E\subseteq BNN_Q{}^*cl\left(BNN_Q{}^*\operatorname{int}\left(BNN_Q{}^*cl(E)\right)\right)$ and $BNN_Q{}^*\operatorname{int}\left(BNN_Q{}^*cl(E)\right)\subseteq E$. Hence $BNN_Q{}^*\operatorname{int}\left(BNN_Q{}^*cl(E)\right)\subseteq E\subseteq BNN_Q{}^*cl\left(BNN_Q{}^*\operatorname{int}\left(BNN_Q{}^*cl(E)\right)\right)$. Since $BNN_Q{}^*\operatorname{int}\left(BNN_Q{}^*cl(E)\right)=G$ is a $BNN_Q{}^*O$ set in U, that there exists a $BNN_Q{}^*O$ set such that $G\subseteq E\subseteq BNN_Q{}^*cl(G)$. Therefore E is $BNN_Q{}^*SO$ set.

Example: 3.5.14 The statement of the above theorem is shown in this example.

Let $U=\{p_1,p_2,p_3,p_4\}, U/R=\{\{p_1,p_3\},\{p_2,p_4\}\}$, $Q=\left\{\begin{matrix}\langle p_1,(0.2,0.3,0.6,-0.5,-0.6,-0.4)\rangle\\ \langle p_2,(0.4,0.4,0.6,-0.6,-0.5,-0.3)\rangle\\ \langle p_3,(0.3,0.3,0.7,-0.6,-0.6,-0.3)\rangle\\ \langle p_4,(0.2,0.4,0.7,-0.5,-0.5,-0.5)\rangle\end{matrix}\right\}$.

$\overline{BN}^*(Q)=\left\{\begin{matrix}\langle p_1,(0.3,0.3,0.6,-0.6,-0.6,-0.3)\rangle\\ \langle p_2,(0.4,0.4,0.6,-0.6,-0.5,-0.3)\rangle\\ \langle p_3,(0.3,0.3,0.6,-0.6,-0.6,-0.3)\rangle\\ \langle p_4,(0.4,0.4,0.6,-0.6,-0.5,-0.3)\rangle\end{matrix}\right\}$, $\underline{BN}^*(Q)=\left\{\begin{matrix}\langle p_1,(0.2,0.3,0.7,-0.5,-0.6,-0.4)\rangle\\ \langle p_2,(0.2,0.4,0.7,-0.5,-0.5,-0.5)\rangle\\ \langle p_3,(0.2,0.3,0.7,-0.5,-0.6,-0.4)\rangle\\ \langle p_4,(0.2,0.4,0.7,-0.5,-0.5,-0.5)\rangle\end{matrix}\right\}$,

$B_{BN}{}^*(Q)=\left\{\begin{matrix}\langle p_1,(0.3,0.3,0.6,-0.4,-0.4,-0.5)\rangle\\ \langle p_2,(0.4,0.4,0.6,-0.5,-0.5,-0.5)\rangle\\ \langle p_3,(0.3,0.3,0.6,-0.4,-0.4,-0.5)\rangle\\ \langle p_4,(0.4,0.4,0.6,-0.5,-0.5,-0.5)\rangle\end{matrix}\right\}$, $BN_1{}^*(Q)=\left\{\begin{matrix}\langle p_1,(0.3,0.3,0.6,-0.5,-0.6,-0.4)\rangle\\ \langle p_2,(0.4,0.4,0.6,-0.5,-0.5,-0.5)\rangle\\ \langle p_3,(0.3,0.3,0.6,-0.5,-0.6,-0.4)\rangle\\ \langle p_4,(0.4,0.4,0.6,-0.5,-0.5,-0.5)\rangle\end{matrix}\right\}$,

$BN_2{}^*(Q)=\left\{\begin{matrix}\langle p_1,(0.2,0.3,0.7,-0.4,-0.4,-0.5)\rangle\\ \langle p_2,(0.2,0.4,0.6,-0.5,-0.5,-0.5)\rangle\\ \langle p_3,(0.2,0.3,0.7,-0.4,-0.4,-0.5)\rangle\\ \langle p_4,(0.2,0.4,0.6,-0.5,-0.5,-0.5)\rangle\end{matrix}\right\}$.

$\tau_{R_{BNN^*}(Q)}=\{0_{BNN^*},\overline{BN}^*(Q),\underline{BN}^*(Q),B_{BN}{}^*(Q),BN_1{}^*(Q),BN_2{}^*(Q),1_{BNN^*}\}$.

Let $E=\left\{\begin{matrix}\langle p_1,(0.5,0.6,0.5,-0.4,-0.4,-0.5)\rangle\\ \langle p_2,(0.6,0.5,0.6,-0.5,-0.5,-0.5)\rangle\\ \langle p_3,(0.5,0.6,0.5,-0.4,-0.4,-0.5)\rangle\\ \langle p_4,(0.6,0.5,0.6,-0.5,-0.5,-0.5)\rangle\end{matrix}\right\}$, $E\subseteq BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(E)))=(BN_1{}^*(Q))^C$ and $BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(E))=B_{BN}{}^*(Q)\subseteq E$. Therefore E is both $BNN_Q{}^*\beta O$ and $BNN_Q{}^*SC$. Also $E\subseteq BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E))=(BN_1{}^*(Q))^C$. Hence E is $BNN_Q{}^*SO$.

Theorem: 3.5.15 Each $BNN_Q{}^*\beta O$ set which is $BNN_Q{}^*\alpha C$ is $BNN_Q{}^*C$.

Proof. Let E be $BNN_Q{}^*\beta O$ set which is $BNN_Q{}^*\alpha C$. Then $E\subseteq BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(E)))$ and $BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(E)))\subseteq E$.

Hence $BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(E)))\subseteq E\subseteq BNN_Q{}^*cl\ (BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(E)))$.

This implies $E=BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(E)))\Rightarrow E=BNN_Q{}^*cl(E)$. Hence E is $BNN_Q{}^*C$ set.

Theorem: 3.5.16 Arbitrary intersection of $BNN_Q{}^*\beta C$ sets is $BNN_Q{}^*\beta C$ set.

Proof. If $\{E_\alpha\}_{\alpha\in\Omega}$ is a collection of $BNN_Q{}^*\beta C$ sets in $(U,\tau_{R_{BNN^*}}(Q))$, then for each $\alpha\in\Omega$, $BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E_\alpha)))\subseteq E_\alpha$.

$$\begin{aligned}E_1\cup E_2\cup\ldots\ldots &\subseteq BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E_1)))\cap BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E_2)))\cap\ldots\ldots\ldots\\ &\subseteq BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E_1))\cap BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E_2))\cap\ldots)\\ &= BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E_1)\cap BNN_Q{}^*\operatorname{int}(E_2)\cap\ldots\ldots))\\ &\subseteq BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E_1\cap E_2\cap\ldots.)))\end{aligned}$$

Hence $\bigcap_{\alpha\in\Omega}E_\alpha$ is $BNN_Q{}^*\beta C$ set in U.

Remark: 3.5.17 Figure-1 shows the relationships among $BNN_Q{}^*O$, $BNN_Q{}^*PO$, $BNN_Q{}^*\alpha O$, $BNN_Q{}^*RO$, $BNN_Q{}^*\beta O$ and $BNN_Q{}^*SO$ in a BNN^*TS $(U,\tau_{R_{BNN^*}}(Q))$.

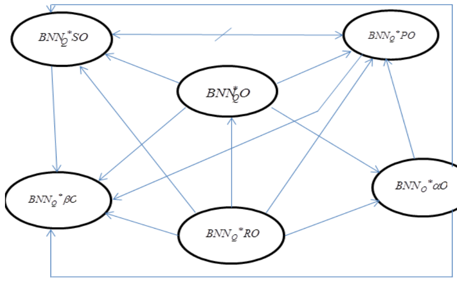

Figure 1. Relationship between the weaker forms of open sets in $BNN_Q{}^*TS$

4. Conclusion

Bipolar neutrosophic set is the base for many topological spaces. In topology, the topological structures such as closedness and openness are the important concepts. It helps to determine the continuity of a mapping between the topologies. Many researchers have proposed various types of topologies with bipolar neutrosophic set. In this paper, we introduced new family of sets namely, bipolar neutrosophic nano* preopen, semi open, $\alpha-$ open , regular open and $\beta-$ open sets in a new topology Bipolar Neutrosophic Nano* topology . Further, some important results based on the corresponding sets are derived and discussed through several examples. As we know neutrosophic sets and nano topology are the roots for many real life applications, we expect that the proposed sets will serve contributions to some future works to the new researchers in real life problems as well as in algebra, geometry and analysis of other sub-branches of mathematics. Our future work will consist of applications of the proposed sets and topology in decision making problems. There are numerous Neutrosophy based decision making algorithms available. In future, we will explore decision making scenarios and try to define novel algorithms by applying proposed concepts. Also, image processing is one of the field which uses neutrosophic logic. We will try to develop image processing algorithms based on proposed neutrosophic topology.

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Neutrosophic Algebraic Mathematical Morphology

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ABSTRACT

In this paper, we introduce and study the NeutroAlgebra structure and many of operations and properties of the mathematical morphology. This is a generalization of the operations of fuzzy and classical mathematical morphology. An explanation of the new given operations is provided through several examples and experimental results. Since mathematical morphology deals with forms and is used in image processing, we consider in this research the Indeterminate Image (i.e. image with missing, unclear, or overlapping pixels), whose basic morphological operator's dilation, erosion, opening and closing transform an indeterminate image into another indeterminate image. Therefore, in fact, we deal with neutro-dilation, neutro-erosion, neutro-opening and neutro-closing. For a determinate image (i.e. image with no indeterminacy), the classical morphological operators transform it also into a determinate image, while the neutro-morphological operators into an indeterminate image. All work from below is available for both the indeterminate and determinate image.

Keywords: Neutrosophic Fuzzy Set, Neutrosophic Crisp Sets, Mathematical Morphology, Neutrosophic Fuzzy Mathematical Morphology, Neutrosophic Crisp Mathematical Morphology, Neutro-Morphological Operators.

1. Introduction

In classical algebraic structures for mathematical morphology, all axioms are 100%, and all operations are 100% well defined, but in real life, in many cases these restrictions are too harsh, since in our world we have things that only partially verify some laws or some operations. Neutrosophy introduces a new concept, which represent indeterminacy with respect to some event, which can solve certain problems that cannot be solved by fuzzy logic and crisp logic. .In 1995, Smarandache initiated the theory of NFS as a new mathematical tool for handling problems involving imprecise indeterminacy, and inconsistent data. Several researchers dealing with the concept of NFS such as Bhowmik and Pal in [14] and Salama et al. introduced many applications in [6-13]. In [6] Salama introduced the concept of neutrosophic crisp sets, to represent any event by a triple crisp structure. A crisp structure is a structure whose all elements are characterized by the same given Relationships and Attributes. A NeutroStructure is a structure that has at least one NeutroRelation or one NeutroAttribute, and neither AntiRelation nor AntiAttribute. In 2019 and 2020, Smarandache [1, 2, 3, 4] generalized the classical Algebraic Structures to NeutroAlgebraic Structures. Neutrosophic mathematical morphology is most commonly applied to digital images, but it can be employed as well on graphs, surface meshes, solids, and many other spatial structures. Established in 1964, mathematical morphology was firstly introduced by Georges Matheron and Jean Serra, as a branch of image processing [29]. As morphology is the study of shape, mathematical morphology mostly deals with the mathematical theory of describing shapes using set theory. In image processing, the basic morphological operator's dilation, erosion, opening and closing form the fundamentals of this theory [29]. A morphological operator transforms an image into another image, using some structuring element, which can be chosen by the user. Mathematical morphology stands somewhat apart from traditional linear image processing, since the basic operations of morphology are non-linear in nature, and thus make use of a totally different type of algebra than the linear algebra. At first, the theory was purely based on set theory and operators, which defined for binary cases only. Later on the theory was extended to grayscale images also, where the theory of lattices was introduced by Petros Maragos, who also gave a representation theory for image processing as a scientific branch, Mathematical Morphology expanded worldwide during the 1990s. It is also during that period, different models based on fuzzy set theory were introduced [16, 1717]. Today, mathematical morphology remains a challenging research field, e.g. [14 - 44].

2. **Terminologies**

We recall some relevant basic preliminaries, and in particular, the work of Smarandache in [1-5], Salama et al. [6-13], and some references in [14-53].

2.1 Abbreviations

1. Crisp Mathematical Morphology (CMM)
2. Fuzzy Mathematical Morphology (FMM).
3. Neutrosophic Fuzzy Set (NFS)
4. Neutrosophic Crisp Set (NCS)
5. Neutrosophic Fuzzy Morphological (NFM)
6. Neutrosophic Fuzzy Dilation (NFD)
7. Neutrosophic Fuzzy Erosion (NFE)
8. Neutrosophic Fuzzy Opening (NFO)
9. Neutrosophic Fuzzy Closing (NFC)
10. Neutrosophic Fuzzy Filters (NFF).
11. Neutrosophic Fuzzy Gradient Boundary (NFGB)
12. Neutrosophic Fuzzy External Boundary (NFEB)
13. Neutrosophic Fuzzy Internal Boundary (NFIB)
14. Neutrosophic Fuzzy Outline Boundary (NFOB)
15. Neutrosophic Fuzzy Mathematical Relation (NFMR)

16. Neutrosophic Crisp Mathematical Morphology (NCMM)
17. Neutrosophic Crisp Dilation (NCD).
18. Neutrosophic Crisp Erosion (NCE).
19. Neutrosophic Crisp Opening (NCO)
20. Neutrosophic Crisp Closing (NCC).
21. Neutrosophic Crisp External Boundary (NCEB).

2.2 Neutrosophic Intensity Image:

To transform the Image from its Spatial (Cartesian) Domain into Neutrosophic Domain, we should investigate the necessary mathematical tools as follow:

The image as a mathematical object (Spatial Domain) is an image mathematically represented by an $m\times\ n$ matrix $I=\left[g_{ij}\right]_{m\times n}$, with entities $g\ (i,j)$ corresponding to the intensity to the given pixel located at the node (i,j) The image in the Neutrosophic Domain (ND) where each pixel of the image is represented by P_{ij} having three components $P_{ij}=(T_{ij},I_{ij},F_{ij})$; Where,

T (i, j) $=\frac{\bar{g}(i,j)-\bar{g}_{min}}{\bar{g}_{max}-\bar{g}_{min}}$, I (i, j) = $\frac{\delta(i,j)-\delta_{min}}{\delta_{max}-\delta_{min}}$, F (i, j) $=1-\text{T}\ \ (\text{i},\text{j})=\ \frac{\bar{g}_{\max}-\bar{g}(\text{i},\text{j})}{\bar{\bar{g}}_{\max}-\bar{g}_{\min}}$

$\bar{g}(i,j)$ is the mean intensity in some neighborhood w of the pixel given by:

$\bar{g}(i,j)=\frac{1}{w^2}\sum_{k=i-\frac{w}{2}}^{m=i+\frac{w}{2}}\sum_{l=j-\frac{w}{2}}^{n=j+\frac{w}{2}}g(k,l)$. Also, $\bar{g}_{max}=\ \ max\,\bar{g}(i,j),\ \bar{g}_{min}=min\,\bar{g}(i,j),$

$\delta(i,j)=abs(g(i,j)-\bar{g}(i,j)),\ \delta_{max}=\max\delta(i,j),\ \delta_{min}=\min\delta(i,j)$. Hence, the image in the neutrosophic domain becomes a 3D matrix $I_{ND}=[T_{ij}\quad I_{ij}\quad F_{ij}]$, of $(m\times n\times 3)$ dimension.

2.3 Implementation and Experimental Results:

In this section, the following suggested algorithm has been used to transform the cartesian image domain into the neutrosophic image domain.

Step 1: Read the grayscale image.

Step 2: Compute the local mean intensity for each pixel in the image.

Step 3: Compute the maximum and minimum values of the local mean intensities.

Step 4: Compute the divergence between the intensity of each pixel and its local mean intensity.

Step 5: Compute the maximum and the minimum values of the divergence induced in the previous step.

Step 6: Construct the truth, indeterminacy, and falseness matrices T, I, F for each pixel.

Most neutrosophic morphological operations can be obtained by combining theoretical operations of the neutrosophic set with two traditional and basic image operations, dilation and erosion, the following section has been dedicated to this issue.

3. NFM Operations:

In this section, we introduce and study the neutrosophic algebraic structures and many operations and properties of mathematical neutrosophic morphology. This is a generalization of the classical mathematical morphological operations. An explanation of the new given operations is provided through several examples with giving experimental results. "Lena" image has been used to investigate the effect of each of the given operators on the image. Basic definitions for neutrosophic morphological operations are extracted and a study of its algebraic properties is presented. In our work, we demonstrate that neutrosophic morphological operations inherit properties and restrictions of fuzzy mathematical morphology. The operations of NFD, NFE, NFO, and NFC of the neutrosophic image by neutrosophic structuring element, are defined in terms of their membership, indeterminacy, and non-membership functions; which are defined for the first time as far as we know.

3.1. NFD and NFE:

The two basic operations for the construction of neutrosophic fuzzy morphological operators, namely,NFD and NFE. are based on the two Minkowski set operations, the Minkowski addition and subtraction of two NFS;respectively. We may define them as follows:

The process of structuring element B on image A and moving it across the image in a way like convolution is defined as a dilation operation. The two main inputs for the dilation operator [21] are the image, which is to be dilated, and a set of coordinate points known as a structuring element, which is can be defined also as a kernel. The exact effect of the dilation on the input image is determined by this structuring element [20]. Its dilation is defined as a set operation. A is dilated by B, written as A⊕B.

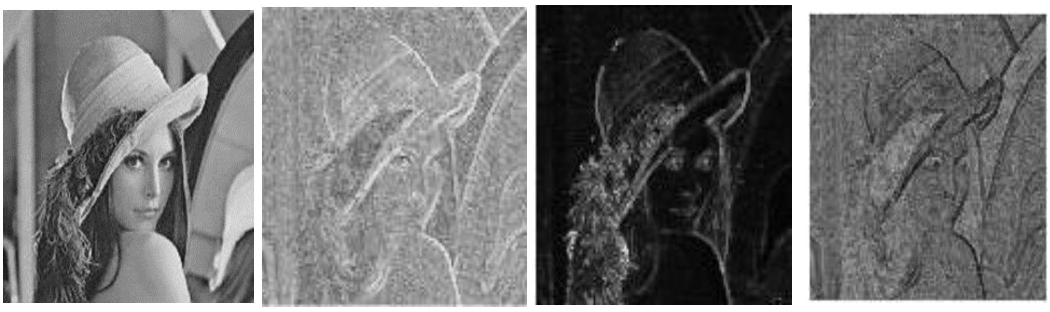
3.1.1 Definition

NFD of Type I:

Let A and B be two NFSs; then the NFD of type I is given as

$\left(A \mathbin{\widetilde{\oplus}} B\right) = \langle T_{A\widetilde{\oplus}B}, I_{A\widetilde{\oplus}B}, F_{A\widetilde{\oplus}B}\rangle$; where for each $u, v \in Z^2$

$T_{A\widetilde{\oplus}B}(v) = \sup\limits_{u\in Z^2} min(T_A(v+u), T_B(u)),\ I_{A\widetilde{\oplus}B}(v) = \sup\limits_{u\in Z^2} min(I_A(v+u), I_B(u)), F_{A\widetilde{\oplus}B}(v) = \inf\limits_{u\in Z^2} max(1 - F_A(v+u), 1 - F_B(u)).$

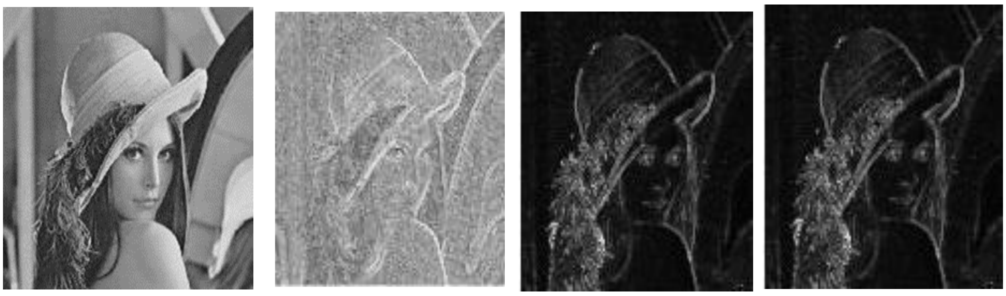


(a) **(b)** $T_{A\widetilde{\oplus}B}$ **(b)** $I_{A\widetilde{\oplus}B}$ (b) $F_{A\widetilde{\oplus}B}$

Fig 3.1.1 (I): Applying the NFD operator: (a) Original image , (b) Neutrosophic Fuzzy components of the dilated image in type I $\langle T_{A\widetilde{\oplus}B},\ I_{A\widetilde{\oplus}B}\ ,\ F_{A\widetilde{\oplus}B}\rangle$ respectively.

NFD of Type II:

$T_{A\widetilde{\oplus}B}(v) = \sup\limits_{u\in Z^2} min(T_A(v+u), T_B(u)),\ I_{A\widetilde{\oplus}B} = \inf\limits_{u\in Z^2} max(I_A(v+u), 1 - I_B(u)),\ F_{A\widetilde{\oplus}B} = \inf\limits_{u\in Z^2} max(F_A(v+u), 1 - F_B(u))$



(a) (b) $T_{A\widetilde{\oplus}B}$ (b) $I_{A\widetilde{\oplus}B}$ (b) $F_{A\widetilde{\oplus}B}$

Fig.3.1.1 (II): Applying the neutrosophic dilation operator: a) Original image b) Neutrosophic Fuzzy components of the dilated image in type II $\langle T_{A\widetilde{\oplus}B},\ I_{A\widetilde{\oplus}B},\ F_{A\widetilde{\oplus}B}\rangle$ respectively.

3.2 NFE Operation:

The erosion process is as same as dilation, but the pixels are converted to 'white', not 'black'. The two main inputs for the erosion operator are the image that is to be eroded and a set of coordinate points known as a structuring element, which is defined also as a kernel. The exact effect of the erosion on the input image is determined by this structuring element. The followings are the mathematical definitions of erosion type I and erosion type II for grey-scale images.

3.2.1 **Definition (**NFE of Type I, II):

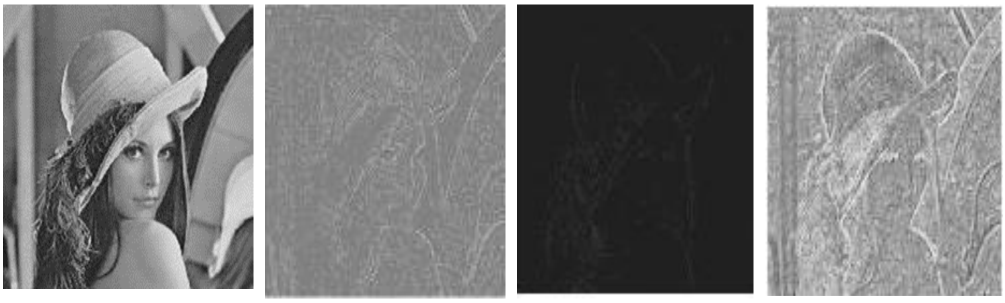
Let A and B be two neutrosophic sets, The neutrosophic fuzzy erosion of a neutrosophic set B from a neutrosophic set A is defined as $(A \widetilde{\ominus} B) = \langle T_{A\widetilde{\ominus}B}, I_{A\widetilde{\ominus}B}, F_{A\widetilde{\ominus}B}\rangle$; where for each $u, v \in Z^2$. The three components, $T_{A\widetilde{\ominus}B}, I_{A\widetilde{\ominus}B}, F_{A\widetilde{\ominus}B}$ are to be defined in different types as follows:

NFE of Type I:

Let A and B be two NFS, then the NFE is given by

$\left(\mathrm{A} \widetilde{\ominus} \mathrm{B}\right) = \left\langle \mathrm{T}_{\mathrm{A}\widetilde{\ominus}\mathrm{B}}, \mathrm{I}_{\mathrm{A}\widetilde{\ominus}\mathrm{B}}, \mathrm{F}_{\mathrm{A}\widetilde{\ominus}\mathrm{B}}\right\rangle$; where for each $u, v \in Z^2$, $\mathrm{T}_{\mathrm{A}\widetilde{\ominus}\mathrm{B}}(\mathrm{v}) = \inf\limits_{\mathrm{u}\in\mathrm{Z}^2} \max(\mathrm{T}_\mathrm{A}(\mathrm{v}+\mathrm{u}), 1 - \mathrm{T}_\mathrm{B}(\mathrm{u}))$,

$\mathrm{I}_{\mathrm{A}\widetilde{\ominus}\mathrm{B}}(\mathrm{v}) = \inf\limits_{\mathrm{u}\in\mathrm{Z}^2} \max(\mathrm{I}_\mathrm{A}(\mathrm{v}+\mathrm{u}), 1 - \mathrm{I}_\mathrm{B}(\mathrm{u}))$, $\mathrm{F}_{\mathrm{A}\widetilde{\ominus}\mathrm{B}}(\mathrm{v}) = \sup\limits_{\mathrm{u}\in\mathrm{Z}^2} \min\left(1 - \mathrm{F}_\mathrm{A}(\mathrm{v}+\mathrm{u}), \mathrm{F}_\mathrm{B}(\mathrm{u})\right)$.

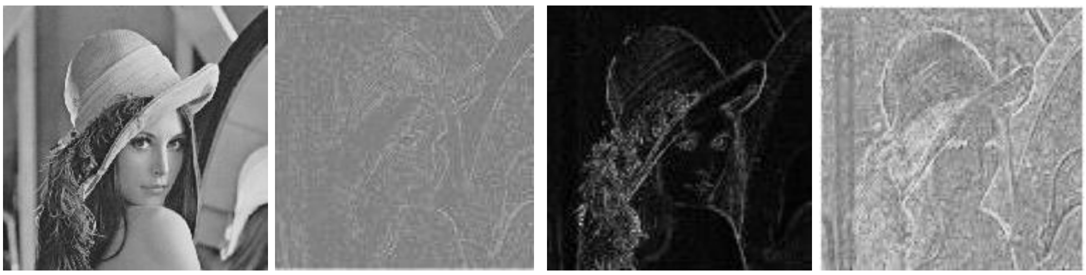


(a) (b) $T_{A\tilde{\ominus}B}$ (b) $I_{A\tilde{\ominus}B}$ (b) $F_{A\tilde{\ominus}B}$

Fig. 3.2.1 (I): Applying the NFE operator: (a) original image (b) neutrosophic components of the eroded in type I $\langle T_{A\ominus B}, I_{A\ominus B}, F_{A\ominus B}\rangle$ respectively.

NFE of Type II:

$T_{A\tilde{\ominus}B}(v) = \inf_{u\in Z^2} max(T_A(v+u), 1-T_B(u))$, $I_{A\tilde{\ominus}B}(v) = \sup_{u\in Z^2} min(I_A(v+u), I_B(u))$, $F_{A\tilde{\ominus}B}(v) = \sup_{u\in Z^2} min\big(F_A(v+u), F_B(u)\big).$



(a) (b) $T_{A\tilde{\ominus}B}$ (b) $I_{A\tilde{\ominus}B}$ (b) $F_{A\tilde{\ominus}B}$

Fig. 3.2.1 (II): Applying the neutrosophic erosion operator: (a) original image (b) neutrosophic components of the eroded in type II $\langle T_{A\tilde{\ominus}B}(v),\ I_{A\tilde{\ominus}B}(v),\ F_{A\tilde{\ominus}B}\rangle$ respectively.

3.3 NFO and NFC Operations:

The combination of the two main neutrosophic fuzzy operations, dilation and erosion, can produce more complex sequences. Opening and closing are the most useful of these for morphological filtering. An opening operation is defined as an erosion followed by dilation using the same structuring element for both operations. The basic two inputs for the opening operator are an image to be opened and a structuring element. The grey-level opening consists simply of grey-level erosion followed by grey-level dilation. The morphological opening $\circ$ and closing $\bullet$ are defined by:

$A \circ B = (A \widetilde{\ominus} B) \widetilde{\oplus} B, \quad A \bullet B = (A \widetilde{\oplus} B) \widetilde{\ominus} B.$

From a granularity perspective, opening and closing provide coarser descriptions of the neutrosophic fuzzy set A. The opening describes A as closely as possible without using the individual pixels but by fitting (possibly overlapping) copies of E within A. The closing describes the complement of A by fitting copies of E^* outside A. The actual set is always contained within these two extremes: $A \circ B \subseteq A \subseteq A \bullet B$ and the informal notion of fitting copies of E, or of E^*, within a set is made precisely in these equations:

The operator $P(E) \to P(E): A \to A \circ B$ is called the opening by B; it is the composition of the erosion $\ominus$, followed by the dilation $\oplus$. On the other hand, the operator $P(E) \to P(E): A \to A \bullet B$ is called the closing. To understand what a closing operation does: imagine the closing applied to a set; the dilation will expand object boundaries, which will be partly undone by the following erosion. Small, (i.e., smaller than the structuring element) holes and thin tube-like structures in the interior or at the boundaries of objects will be filled up by the dilation, and not reconstructed by the erosion, in as much as these structures no longer have a boundary for the erosion to act upon. In this sense, the term 'closing' is a well-chosen one, as the operation removes holes and thin cavities. In the same sense, the opening opens up holes that are near (with respect to the size of the structuring element) a boundary and removes small object protuberances.

Definition 3.3.1 (NFO of Type I, II):

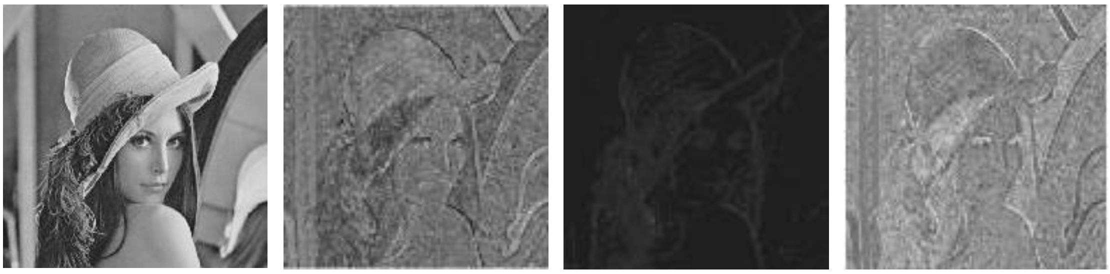
Two types of neutrosophic fuzzy opening operations NFO may defined as $NOF: (A \tilde{\circ} B) = \langle T_{A\tilde{\circ}B}, I_{A\tilde{\circ}B}, F_{A\tilde{\circ}B} \rangle$ where $u, v, w \in Z^2$.

NFO of Type I

$$T_{A\tilde{\circ}B}(v) = \sup_{w,v,u\in Z^2} \min\left[\inf_{z\in R^n} \max\big(A(v-u+w), 1-B(w)\big), B(u)\right],$$

$$I_{A\tilde{\circ}B}(v) = \sup_{w,v,u\in Z^2} \min\left[\inf_{z\in R^n} \max\big(A(v-u+w), 1-B(w)\big), B(u)\right],$$

$$F_{A\tilde{\circ}B}(v) = \inf_{w,v,u\in Z^2} \max\left[\sup_{z\in R^n} \min\big(1-A(v-u+w), B(w)\big), 1-B(u)\right].$$



(a) (b) $T_{A\tilde{\circ}B}$ (b) $I_{A\tilde{\circ}B}$ (b) $F_{A\tilde{\circ}B}$

Fig.3.3.1 (I): Applying the neutrosophic fuzzy opening operator: (a) Original image.

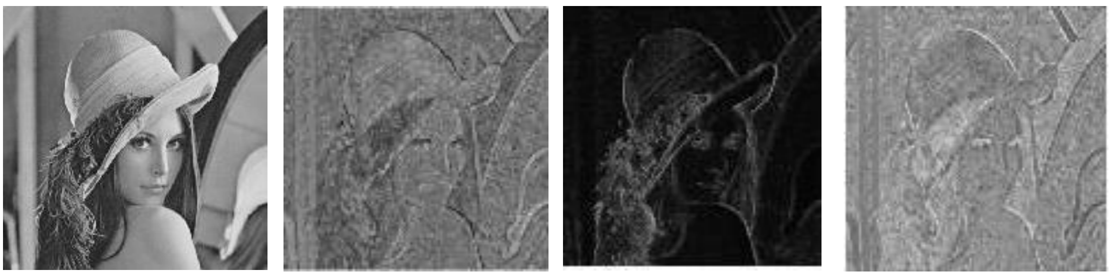
(b) Neutrosophic fuzzy opening components in type I $\langle T_{A\tilde{\circ}B}, I_{A\tilde{\circ}B}, F_{A\tilde{\circ}B}\rangle$ respectively.

NFO of Type II:

$$T_{A\tilde{\circ}B}(v) = \sup_{w,v,u\in Z^2} \min\ \left[\left(\inf \max\big(T_A(v-u+w), 1-T_B(w)\big), T_B(u)\right]\right.,$$

$$I_{A\tilde{\circ}B}(v) = \sup_{w,v,u\in Z^2} \min\ \left[\ \inf \max\big(I_A(v-u+w), I_B(w)\big), 1-I_B(u)\right],$$

$$F_{A\tilde{\circ}B}(v) = \inf_{w,v,u\in Z^2} \max\ \left[\ \sup \min\big(F_A(v-u+w), F_B(w)\big), 1-F_B(u)\right].$$



(a) (b) $T_{A\tilde{\circ}B}(v)$ (b) $I_{A\tilde{\circ}B}(v)$ (b) $F_{A\tilde{\circ}B}(v)$

Fig.3.3.1 (II): Applying the neutrosophic opening operator: (a) Original image (b) Neutrosophic opening components in type II $\langle T_{A\tilde{\circ}B}(v), I_{A\tilde{\circ}B}(v), F_{A\tilde{\circ}B}(v)\rangle$ respectively.

Definition 3.3.2 (NFC of Type I, II)

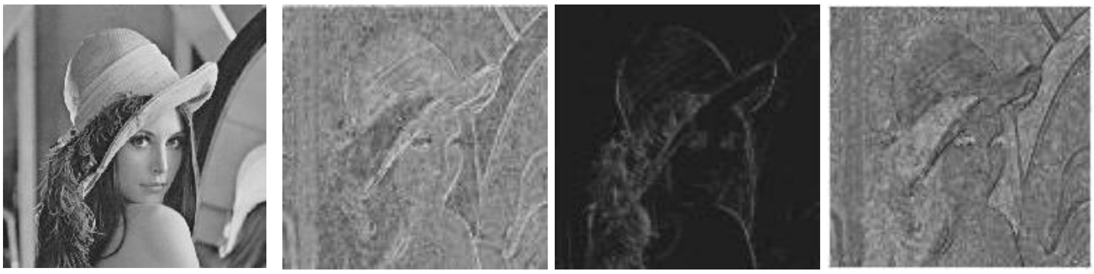
Let A and B be two types as the following may define two neutrosophic sets: $(A \tilde{\bullet} B) = \langle T_{A\tilde{\bullet}B}, I_{A\tilde{\bullet}B}, F_{A\tilde{\bullet}B}\rangle$ where

Type I

$$T_{A\tilde{\bullet}B}(v) = \inf_{w,v,u\in Z^2} max\left[sup\, min\left(T_A(v-u+w), T_B(w)\right), 1 - T_B(u)\right],$$

$$I_{A\tilde{\bullet}B}(v) = \inf_{w,v,u\in Z^2} max\left[sup\, min\left(I_A(v-u+w), I_B(w)\right), 1 - I_B(u)\right],$$

$$F_{A\tilde{\bullet}B}(v) = \sup_{w,v,u\in Z^2} min\left[inf\, max\left(1 - F_A(v-u+w), 1 - F_B(w)\right), F_B(u)\right]$$



(a) (b) $T_{A\tilde{\bullet}B}$ (b) $I_{A\tilde{\bullet}B}$ (b) $F_{A\tilde{\bullet}B}$

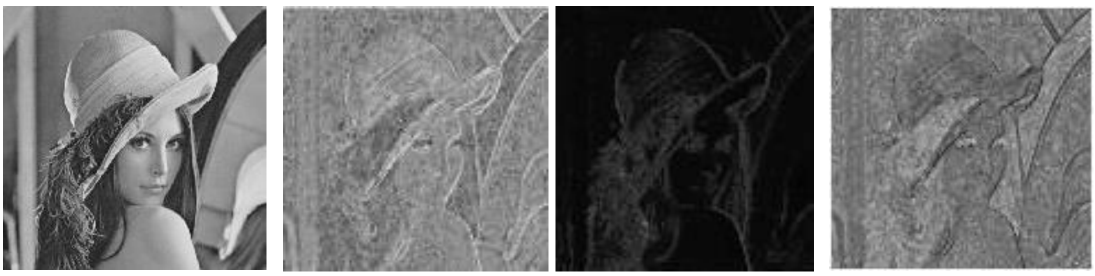
Fig. 3.3.2 (I): Applying the neutrosophic closing operator: (a) Original image (b) Neutrosophic closing components in type I $\langle T_{A\tilde{\bullet}B}, I_{A\tilde{\bullet}B}, F_{A\tilde{\bullet}B}\rangle$ respectively.

Neutrosophic Closing Type II:

$$T_{A\tilde{\bullet}B}(v) = \inf_{w,v,u\in Z^2} max\ [\sup min\left(T_A(v-u+w), T_B(w)\right), 1 - T_B(u)]\ ,$$

$$I_{A\tilde{\bullet}B}(v) = \sup_{w,v,u\in Z^2} min\ [\inf\ max\left(I_A(v-u+w), 1 - I_B(w)\right), I_B(u)],$$

$$F_{A\tilde{\bullet}B}(v) = \sup_{w,v,u\in Z^2} min\ [\inf max\left(F_A(v-u+w), 1 - F_B(w)\right), F_B(u)].$$



(a) (b) $T_{A\tilde{\bullet}B}(v)$ (b) $I_{A\tilde{\bullet}B}(v)$ (b) $F_{A\tilde{\bullet}B}(v)$

Fig.3.3.2 (II): Applying the neutrosophic closing operator: a) Original image b) Neutrosophic closing components in type II $\langle T_{A\bullet B}(v),\ I_{A\bullet B}(v),\ F_{A\bullet B}(v)\rangle$ respectively.

3.4 Algebraic Properties of Neutrosophic Fuzzy Morphological Operations:

This part has been dedicated to investigate some of the algebraic properties of the neutrosophic fuzzy morphological operations; i.e. NFD, NFE, NFO and neutrosophic fuzzy closing. The algebraic properties for neutrosophic fuzzy mathematical morphology erosion and dilation, as well as for NFO and closing operations are now considered.

3.4.1 Duality Theorem of NFD:

Let A and B be two NFSs. Then the NFE and the NFD both are dual operations, i.e. $(A^c \widetilde{\oplus} B)^c = \left\langle T_{(A^c\widetilde{\oplus}B)^c}, I_{(A^c\widetilde{\oplus}B)^c}, F_{(A^cA\widetilde{\oplus}B)^c}\right\rangle$; where for each u & $v \in Z^2$ we have:

$$T_{(A^c\widetilde{\oplus}B)^c}(v) = 1 - T_{(A^c\widetilde{\oplus}B)}(v) = 1 - \sup_{v,u\in Z^2} \min(T_{A^c}(v+u), T_B(u)) = \inf_{v,u\in Z^2}[1 - \min(T_{A^c}(v+u), T_B(u))] = \inf_{v,u\in Z^2}[\max(1 - T_{A^c}(v+u), 1 - T_B(u))] = \inf_{v,u\in Z^2}[\max(T_A(v+u), 1 - T_B(u))] = T_{A\ominus B}(v)$$

$$I_{(A^c\widetilde{\oplus}B)^c}(v) = 1 - I_{(A^c\widetilde{\oplus}B)}(v) = 1 - \sup_{v,u\in Z^2} \min(I_{A^c}(v+u), I_B(u)) = \inf_{v,u\in Z^2}[1 - \min(I_{A^c}(v+u), I_B(u))] = \inf_{v,u\in Z^2}[\max(1 - I_{A^c}(v+u), 1 - I_B(u))] = \inf_{v,u\in Z^2}[\max(I_A(v+u), 1 - I_B(u))] = I_{A\ominus B}(v)$$

$$F_{(A^c\widetilde{\oplus}B)^c}(v) = 1 - F_{(A^c\widetilde{\oplus}B)}(v)$$
$$= 1 - \inf_{v,u\in Z^2} \max(1 - F_{A^c}(v+u), 1 - F_B(u)) = \sup_{v,u\in Z^2}[1 - \max(1 - F_{A^c}(v+u), 1 - F_B(u))] = \sup_{v,u\in Z^2}[\min(1 - F_A(v+u), F_B(u))] = F_{A\ominus B}(v)$$

Suppose the set A is the image under processing and the set B is the structuring element, the NFO and the NFC are defined respectively, as define the neutrosophic fuzzy binary operation $\circ$ and $\bullet$ by setting for any A and B $\in \mathcal{N}(E)$.

3.4.2. Duality Theorem of NFC:

Let A and B are two NFSs, then the NFO and the NFC are also dual operation i.e:$T(A^c \tilde{\bullet} B)^c =$ $\langle T_{(A^c \tilde{\bullet} B)^c}\ , I_{(A^c \tilde{\bullet} B)^c}\ , F_{(A^c \tilde{\bullet} B)^c}\rangle$, where for all v, u$\in Z^2$

$$T_{(A^c \tilde{\bullet} B)^c}(v) = 1 - T_{(A^c \tilde{\bullet} B)}(v) = 1 - \inf_{v,u,w\in Z^2} \max\left[\sup_{v,u,w\in Z^2} \min\left(T_{A^c}(v-u+w), T_{B(w)}\right), 1 - T_B(u)\right]$$

$$= \sup_{v,u,w\in Z^2} \min\left[1 - \sup_{v,u,w\in Z^2} \min\left(T_{A^c}(v-u+w), T_{B(w)}\right), 1 - \left(1 - T_B(u)\right)\right] = \sup_{v,u,w\in Z^2} \min\left[\inf_{v,u,w\in Z^2} \max\left(1 - T_{A^c}(v-u+w), 1 - T_{B(w)}\right), T_B(u)\right]$$

$$= \sup_{v,u,w\in Z^2} \min\left[\inf_{v,u,w\in Z^2} \max\left(T_{A^c}(v-u+w), 1 - T_{B(w)}\right), T_B(u)\right] = T_{A\circ B}$$

$$I_{(A^c \tilde{\bullet} B)^c}(v) = 1 - I_{(A^c \tilde{\bullet} B)}(v) = 1 - \inf_{v,u,w\in Z^2} \max\left[\sup_{v,u,w\in Z^2} \min\left(I_{A^c}(v-u+w), I_{B(w)}\right), 1 - I_B(u)\right]$$

$$= \sup_{v,u,w\in Z^2} \min\left[1 - \sup_{v,u,w\in Z^2} \min\left(I_{A^c}(v-u+w), I_{B(w)}\right), 1 - \left(1 - I_B(u)\right)\right] = \sup_{v,u,w\in Z^2} \min\left[\inf_{v,u,w\in Z^2} \max\left(1 - I_{A^c}(v-u+w), 1 - I_{B(w)}\right), I_B(u)\right]$$

$$= \sup_{v,u,w\in Z^2} \min\left[\inf_{v,u,w\in Z^2} \max\left(I_{A^c}(v-u+w), 1 - I_{B(w)}\right), I_B(u)\right] = I_{A\circ B}$$

$$F_{(A^c \tilde{\bullet} B)^c}(v) = 1 - F_{(A^c \tilde{\bullet} B)}(v) = 1 - \sup_{v,u,w\in Z^2} \min\left[\inf_{v,u,w\in Z^2} \max\left(1 - F_A(v-u+w), 1 - F_{B(w)}\right), F_B(u)\right]$$

$$= \inf_{v,u,w\in Z^2} \max\left[1 - \inf_{v,u,w\in Z^2} \max\left(1 - F_A(v-u+w), 1 - F_{B(w)}\right), F_B(u)\right] = \inf_{v,u,w\in Z^2} \max\left[\sup_{v,u,w\in Z^2} \min\left(1 - F_A(v-u+w), F_{B(w)}\right), 1 - F_B(u)\right] = F_{A\circ B}$$

3.5. Neutrosophic Fuzzy Mathematical Relation

3.5.1. Definition: Let A is a NFS and R be a neutrosophic relation on nonempty crisp set X then $A \oplus R$ my be defined by two types:

$\mathbf{A \widetilde{\oplus} R} = \langle \mathbf{T_{A\widetilde{\oplus}R}}\ , \mathbf{I_{A\widetilde{\oplus}R}}\ , \mathbf{F_{A\widetilde{\oplus}R}} \rangle$

$T_{A\widetilde{\oplus}R}(v) = \sup_{v,u\in Z^2} \min(T_A(v+u), F_R(u)),\ I_{A\oplus R}(v) = \sup_{v,u\in Z^2} \min(I_A(v+u), F_R(u))$,

$F_{A\oplus R}(y) = \inf_{u\in Z^2} \max(1 - F_A(v+u), 1 - F_R(u)).$

Given two relations on X, say R and S, the places you can get to following an arrow in R and then following an arrow in S are exactly the places you can get to by following an arrow in R; S. Formally, we have the right monoid action:

Lemma 3.5.1: The operation just defined, $\oplus$, provides a right action for the monoid of relations on the non-empty crisp set X on the power set. Specifically, for all $A \in PX$ and for all $R\&S \in RX$,

$\mathbf{A} \widetilde{\oplus} \mathbf{1} = \langle \mathbf{T}_{\mathbf{A}\oplus\mathbf{1}}, \mathbf{I}_{\mathbf{A}\oplus\mathbf{1}}, \mathbf{F}_{\mathbf{A}\oplus\mathbf{1}} \rangle$

$$\mathrm{T}_{\mathrm{A}\oplus 1}(\mathrm{v}) = \sup_{\mathrm{v,u}\in \mathrm{Z}^2} \min(\mathrm{T_A}(\mathrm{v}+\mathrm{u}), 1) = \sup_{\mathrm{v,u}\in \mathrm{Z}^2} \big(\mathrm{T_A}(\mathrm{u}+\mathrm{v})\big) = \mathrm{T_A}$$

$$\mathrm{I}_{\mathrm{A}\oplus 1}(\mathrm{v}) = \sup_{\mathrm{v,u}\in \mathrm{Z}^2} \min(\mathrm{I_A}(\mathrm{v}+\mathrm{u}), 1) = \sup_{\mathrm{v,u}\in \mathrm{Z}^2} \big(\mathrm{I_A}(\mathrm{u}+\mathrm{v})\big) = \mathrm{I_A}$$

$$\mathrm{F}_{\mathrm{A}\oplus 1}(\mathrm{v}) = \inf_{\mathrm{v,u}\in \mathrm{Z}^2} \max(1-\mathrm{F_A}(\mathrm{v}+\mathrm{u}), 1-1) = \inf_{\mathrm{v,u}\in \mathrm{Z}^2} \big(1-\mathrm{F_A}(\mathrm{v}+\mathrm{u})\big) = \mathrm{F}_{\mathrm{A}^c}$$

Lemma 3.5. 2 : Let A_i are indexed set neutrosophic fuzzy subsets on the non-empty crisp set X, then $\bigvee_i \mathrm{A_i} \widetilde{\oplus} \mathrm{R} = \left\langle \mathrm{T}_{\bigvee_i \mathrm{A_i}\oplus \mathrm{R}}, \mathrm{I}_{\bigvee_i \mathrm{A_i}\oplus \mathrm{R}}, \mathrm{F}_{\left(\bigvee_i \mathrm{A_i}\oplus \mathrm{R}\right)^c} \right\rangle$, where $\mathrm{T}_{\bigvee_i \mathrm{A_i}\oplus \mathrm{R}} = \mathrm{T}_{\bigvee_i(\mathrm{A_i}\oplus \mathrm{R})}$, $\mathrm{I}_{\bigvee_i \mathrm{A_i}\oplus \mathrm{R}} = \mathrm{I}_{\bigvee_i(\mathrm{A_i}\oplus \mathrm{R})}$, $\mathrm{F}_{\left(\bigvee_i \mathrm{A_i}\oplus \mathrm{R}\right)^c} = \mathrm{F}_{\bigwedge_i(\mathrm{A_i}\oplus \mathrm{R})^c}$.

3.6. Basic Properties of the Neutrosophic Fuzzy Morphological Operations:

3.6.1 Properties of the NFD Operation:

3.6.1.1 Proposition

The neutrosophic Minkowski-addition satisfies the following properties

i. **Commutativity:** $(\forall \mathrm{A}, \mathrm{B} \in \mathcal{N}(\mathrm{Z}^2))\big(\langle \mathrm{T}_{\mathrm{A}\widetilde{\oplus}\mathrm{B}}, \mathrm{I}_{\mathrm{A}\widetilde{\oplus}\mathrm{B}}, \mathrm{F}_{\mathrm{A}\widetilde{\oplus}\mathrm{B}} \rangle = \langle \mathrm{T}_{\mathrm{B}\widetilde{\oplus}\mathrm{A}}, \mathrm{I}_{\mathrm{B}\widetilde{\oplus}\mathrm{A}}, \mathrm{F}_{\mathrm{B}\widetilde{\oplus}\mathrm{A}} \rangle\big)$;

ii. **Associativity:** $(\forall \mathrm{A}, \mathrm{B}, \mathrm{C} \in \mathcal{N}(\mathrm{Z}^2))$

$$\big(\langle \mathrm{T}_{(\mathrm{A}\widetilde{\oplus}\mathrm{B})\widetilde{\oplus}\mathrm{C}}, \mathrm{I}_{(\mathrm{A}\widetilde{\oplus}\mathrm{B})\widetilde{\oplus}\mathrm{C}}, \mathrm{F}_{(\mathrm{A}\widetilde{\oplus}\mathrm{B})\widetilde{\oplus}\mathrm{C}} \rangle = \langle \mathrm{T}_{\mathrm{A}\widetilde{\oplus}(\mathrm{B}\widetilde{\oplus}\mathrm{C})}, \mathrm{I}_{\mathrm{A}\widetilde{\oplus}(\mathrm{B}\widetilde{\oplus}\mathrm{C})}, \mathrm{F}_{\mathrm{A}\widetilde{\oplus}(\mathrm{B}\widetilde{\oplus}\mathrm{C})} \rangle\big)$$

Proof Straight forward

Notice that the property $\langle \mathrm{T}_{-(\mathrm{A}\widetilde{\oplus}\mathrm{B})}, \mathrm{I}_{-(\mathrm{A}\widetilde{\oplus}\mathrm{B})}, \mathrm{F}_{-(\mathrm{A}\widetilde{\oplus}\mathrm{B})} \rangle = \langle \mathrm{T}_{(-\mathrm{A})\widetilde{\oplus}(-\mathrm{B})}, \mathrm{I}_{(-\mathrm{A})\widetilde{\oplus}(-\mathrm{B})}, \mathrm{F}_{(-\mathrm{A})\widetilde{\oplus}(-\mathrm{B})} \rangle$

3.6.1.2. Proposition

The neutrosophic dilation satisfies the following properties

i. **Neutrosophic Monotonicity (**increasing in both arguments):

$(\forall A, B, C \in \mathcal{N}(Z^2))(A \subseteq B \Longrightarrow \langle T_{A\widetilde{\oplus}C}, I_{A\widetilde{\oplus}C}, F_{A\widetilde{\oplus}C}\rangle \subseteq \langle T_{B\widetilde{\oplus}C}, I_{B\widetilde{\oplus}C}, F_{B\widetilde{\oplus}C}\rangle)$, here we have

$T_{A\widetilde{\oplus}C} \subseteq T_{B\widetilde{\oplus}C}$, $I_{A\widetilde{\oplus}C} \subseteq I_{B\widetilde{\oplus}C}$, $F_{A\widetilde{\oplus}C} \supseteq F_{B\widetilde{\oplus}C}$

$(\forall A, B, C \in \mathcal{N}(Z^2))(A \subseteq B \Longrightarrow \langle T_{C\widetilde{\oplus}A}, I_{C\widetilde{\oplus}A}, F_{C\widetilde{\oplus}A}\rangle \subseteq \langle T_{C\widetilde{\oplus}B}, I_{C\widetilde{\oplus}B}, F_{C\widetilde{\oplus}B}\rangle)$, here we have

$T_{C\widetilde{\oplus}A} \subseteq T_{C\widetilde{\oplus}B}$, $I_{C\widetilde{\oplus}A} \subseteq I_{C\widetilde{\oplus}B}$, $F_{C\widetilde{\oplus}A} \supseteq F_{C\widetilde{\oplus}B}$

ii. **Interaction with Zadeh's intersection;**

For any family $(A_i | i \in I)$ in $\mathcal{N}(Z^2)$and $B \in \mathcal{N}(Z^2)$, we have:

$\langle T_{\bigcap_{i\in I} A_i\widetilde{\oplus}B}, I_{\bigcap_{i\in I} A_i\widetilde{\oplus}B}, F_{\bigcap_{i\in I} A_i\widetilde{\oplus}B}\rangle \subseteq \langle T_{\bigcap_{i\in I}(A_i\widetilde{\oplus}B)}, I_{\bigcap_{i\in I}(A_i\widetilde{\oplus}B)}, F_{\bigcap_{i\in I}(A_i\widetilde{\oplus}B)}\rangle$, where

$T_{\bigcap_{i\in I} A_i\widetilde{\oplus}B} \subseteq T_{\bigcap_{i\in I}(A_i\widetilde{\oplus}B)}$, $I_{\bigcap_{i\in I} A_i\widetilde{\oplus}B} \subseteq I_{\bigcap_{i\in I}(A_i\widetilde{\oplus}B)}, F_{\bigcap_{i\in I} A_i\widetilde{\oplus}B} \subseteq F_{\bigcap_{i\in I}(A_i\widetilde{\oplus}B)}$.

Also we have, $\langle T_{B\widetilde{\oplus}\bigcap_{i\in I}A_i}, I_{B\widetilde{\oplus}\bigcap_{i\in I}A_i} F_{B\widetilde{\oplus}\bigcap_{i\in I}A_i}\rangle \subseteq \langle T_{\bigcap_{i\in I}(B\widetilde{\oplus}A_i)}, I_{T_{\bigcap_{i\in I}(B\widetilde{\oplus}A_i)}}, F_{T_{\bigcap_{i\in I}(B\widetilde{\oplus}A_i)}}\rangle$, where

$T_{B\widetilde{\oplus}\bigcap_{i\in I}A_i} \subseteq T_{\bigcap_{i\in I}(B\widetilde{\oplus}A_i)}$, $I_{B\widetilde{\oplus}\bigcap_{i\in I}A_i} \subseteq I_{\bigcap_{i\in I}(B\widetilde{\oplus}A_i)}$, $F_{B\widetilde{\oplus}\bigcap_{i\in I}A_i} \subseteq F_{\bigcap_{i\in I}(B\widetilde{\oplus}A_i)}$.

iii. **Interaction with Zadeh's union:**

For any family$(A_i | i \in I)$ in $\mathcal{N}(Z^2)$ and $B \in \mathcal{N}(Z^2)$, we have

$\langle T_{\bigcup_{i\in I} A_i\widetilde{\oplus}B}, I_{\bigcup_{i\in I} A_i\widetilde{\oplus}B}, F_{\bigcup_{i\in I} A_i\widetilde{\oplus}B}\rangle \supseteq \langle T_{\bigcup_{i\in I}(A_i\widetilde{\oplus}B)}, I_{\bigcup_{i\in I}(A_i\widetilde{\oplus}B)}, F_{\bigcup_{i\in I}(A_i\widetilde{\oplus}B)}\rangle$, where

$T_{\bigcup_{i\in I} A_i\widetilde{\oplus}B} \supseteq T_{\bigcup_{i\in I}(A_i\widetilde{\oplus}B)}$, $I_{\bigcup_{i\in I} A_i\widetilde{\oplus}B} \supseteq I_{\bigcup_{i\in I}(A_i\widetilde{\oplus}B)}$, $F_{\bigcup_{i\in I} A_i\widetilde{\oplus}B} \supseteq F_{\bigcup_{i\in I}(A_i\widetilde{\oplus}B)}$.

Also we have, $\langle T_{B\widetilde{\oplus}\bigcup_{i\in I}A_i}, I_{B\widetilde{\oplus}\bigcup_{i\in I}A_i}, F_{B\widetilde{\oplus}\bigcup_{i\in I}A_i}\rangle \supseteq \langle T_{\bigcup_{i\in I}(B\widetilde{\oplus}A_i)}, I_{\bigcup_{i\in I}(B\widetilde{\oplus}A_i)}, F_{\bigcup_{i\in I}(B\widetilde{\oplus}A_i)}\rangle$, where

$T_{B\widetilde{\oplus}\bigcup_{i\in I}A_i} \supseteq T_{\bigcup_{i\in I}(B\widetilde{\oplus}A_i)}$, $I_{B\widetilde{\oplus}\bigcup_{i\in I}A_i} \supseteq I_{\bigcup_{i\in I}(B\widetilde{\oplus}A_i)}$, $F_{B\widetilde{\oplus}\bigcup_{i\in I}A_i} \supseteq F_{\bigcup_{i\in I}(B\widetilde{\oplus}A_i)}$.

Proof.

The proof of the first property is straightforward.

i. $\langle T_{A\widetilde{\oplus}C}, I_{A\widetilde{\oplus}C}, F_{A\widetilde{\oplus}C}\rangle \subseteq \langle T_{B\widetilde{\oplus}C}, I_{B\widetilde{\oplus}C}, F_{B\widetilde{\oplus}C}\rangle$.

ii. $\langle T_{\bigcap_{i\in I} A_i\widetilde{\oplus}B}, I_{\bigcap_{i\in I} A_i\widetilde{\oplus}B}, F_{\bigcap_{i\in I} A_i\widetilde{\oplus}B}\rangle \subseteq \langle T_{\bigcap_{i\in I}(A_i\widetilde{\oplus}B)}, I_{\bigcap_{i\in I}(A_i\widetilde{\oplus}B)}, F_{\bigcap_{i\in I}(A_i\widetilde{\oplus}B)}\rangle$,

$$T_{\underset{i\in I}{\cap} A_i\widetilde{\oplus}B}(v) = \sup_{u\in Z^n} \min\left(T_{\underset{i\in I}{\cap} A_i\widetilde{\oplus}B}(v+u), T_B(u)\right) = \sup_{v,u\in Z^2} \min\left(\min_{i\in I} T_{\mathbf{A_i}\widetilde{\oplus}\mathbf{B}}(v\right.$$

$$\left. + u), T_B(u)\right) = \sup_{v,u\in Z^2} \min_{i\in I}\left(\min T_{\mathbf{A_i}\widetilde{\oplus}\mathbf{B}}(v+u), T_B(u)\right) \leq \underset{\mathbf{i\in I}}{\cap} \sup_{v,u\in Z^2} \left(\min T_{\mathbf{A_i}\widetilde{\oplus}\mathbf{B}}(v\right.$$

$$\left. + u), T_B(u)\right) \leq \underset{i\in I}{\cap} T_{(A_i\widetilde{\oplus}B)}(v+u) \leq T_{\underset{i\in I}{\cap}(A_i\widetilde{\oplus}B)}$$

$$I_{\underset{i\in I}{\cap} A_i\widetilde{\oplus}B}(v) = \sup_{v,u\in Z^2} \min\left(I_{\underset{i\in I}{\cap} A_i\widetilde{\oplus}B}(v+u), I_B(u)\right) = \sup_{v,u\in Z^2} \min\left(\min_{i\in I}\ I_{\mathbf{A_i}\widetilde{\oplus}\mathbf{B}}(v\right.$$

$$\left. + u), I_B(u)\right) = \sup_{v,u\in Z^2} \min_{i\in I}\left(\min\ I_{\mathbf{A_i}\widetilde{\oplus}\mathbf{B}}(v+u), I_B(u)\right) \leq \underset{\mathbf{i\in I}}{\cap} \sup_{u\in Z^2}\left(\min\ I_{\mathbf{A_i}\widetilde{\oplus}\mathbf{B}}(v\right.$$

$$\left. + u), I_B(u)\right) \leq \underset{i\in I}{\cap} I_{(A_i\widetilde{\oplus}B)}(v+u) \leq I_{\underset{i\in I}{\cap}(A_i\widetilde{\oplus}B)}$$

$$F_{\underset{i\in I}{\cap} A_i\widetilde{\oplus}B}(v) = \inf_{v,u\in Z^n} \max\left(1 - F_{\underset{i\in I}{\cap} A_i\widetilde{\oplus}B}(v+u), 1 - F_B(u)\right) = \inf_{v,u\in Z^n} \max\left(\min_{i\in I}\ I_{A_i{}^c\widetilde{\oplus}B}(v+u), 1 - F_B(u)\right)$$

$$\leq \inf_{v,u\in Z^n} \min_{i\in I}\left(\max\ F_{\mathbf{A_i{}^c}\widetilde{\oplus}\mathbf{B}}(v+u), 1 - F_B(u)\right)$$

$$\leq \min_{i\in I} \inf_{v,u\in Z^n}\left(\max\ F_{\mathbf{A_i{}^c}\widetilde{\oplus}\mathbf{B}}(v+u), 1 - F_B(u)\right)$$

$$\leq \underset{i\in I}{\cap} \inf_{v,u\in Z^n}\left(\max\ F_{\mathbf{A_i{}^c}\widetilde{\oplus}\mathbf{B}}(v+u), 1 - F_B(u)\right) \leq F_{\underset{i\in I}{\cap}(A_i\widetilde{\oplus}B)}$$

iii. $\langle T_{\underset{i\in I}{\cup} A_i\widetilde{\oplus}B}, I_{\underset{i\in I}{\cup} A_i\widetilde{\oplus}B}, F_{\underset{i\in I}{\cup} A_i\widetilde{\oplus}B}\rangle \supseteq \langle T_{\underset{i\in I}{\cup}(A_i\widetilde{\oplus}B)}, I_{\underset{i\in I}{\cup}(A_i\widetilde{\oplus}B)}, F_{\underset{i\in I}{\cup}(A_i\widetilde{\oplus}B)}\rangle$

$$T_{\underset{i\in I}{\cup} A_i\widetilde{\oplus}B}(v) = \sup_{v,u\in Z^2} \min\left(T_{\underset{i\in I}{\cup} A_i\widetilde{\oplus}B}(v+u), T_B(u)\right) = \sup_{v,u\in Z^2} \min\left(\max_{i\in I} T_{A_i\widetilde{\oplus}B}(v\right.$$

$$\left. + u), T_B(u)\right) \geq \sup_{v,u\in Z^2} \max_{i\in I}\left(\min T_{A_i\widetilde{\oplus}B}(v+u), T_B(u)\right) \geq \underset{i\in I}{\cup} \sup_{v,u\in Z^2}\left(\min T_{A_i\widetilde{\oplus}B}(v\right.$$

$$\left. + u), T_B(u)\right) \geq \underset{i\in I}{\cup} T_{(A_i\widetilde{\oplus}B)}(v+u) \geq T_{\underset{i\in I}{\cup}(A_i\widetilde{\oplus}B)}$$

$$I_{\underset{i\in I}{\cup} A_i\widetilde{\oplus}B}(v) = \sup_{v,u\in Z^2} \min\left(I_{\underset{i\in I}{\cup} A_i\widetilde{\oplus}B}(v+u), I_B(u)\right) = \sup_{v,u\in Z^2} \min\left(\max_{i\in I}\ I_{A_i\widetilde{\oplus}B}(v\right.$$

$$\left. + u), I_B(u)\right) \geq \sup_{v,u\in Z^2} \max_{i\in I}\left(\min\ I_{\mathbf{A_i}\widetilde{\oplus}\mathbf{B}}(v+u), I_B(u)\right) \geq \underset{\mathbf{i\in I}}{\cup} \sup_{v,u\in Z^2}\left(\min I_{\mathbf{A_i}\widetilde{\oplus}\mathbf{B}}(v\right.$$

$$\left. + u), I_B(u)\right) \geq \underset{i\in I}{\cup} I_{(A_i\widetilde{\oplus}B)}(v+u) \geq I_{\underset{i\in I}{\cup}(A_i\widetilde{\oplus}B)}$$

$$F_{\underset{i\in I}{\cup} A_i \widetilde{\oplus} B}(v) = \inf_{v,u\in Z^n} \max\left(1 - F_{\underset{i\in I}{\cup} A_i \widetilde{\oplus} B}(v+u), 1 - F_B(u)\right) = \inf_{v,u\in Z^n} \max\left(\max_{i\in I}\ I_{\mathbf{A_i}^c \widetilde{\oplus} \mathbf{B}}(v+u), 1 - F_B(u)\right)$$

$$= \inf_{v,u\in Z^n} \max_{i\in I}\left(\max\ F_{\mathbf{A_i}^c \widetilde{\oplus} \mathbf{B}}(v+u), 1 - F_B(u)\right)$$

$$\geq \max_{i\in I} \inf_{v,u\in Z^n}\left(\max\ F_{\mathbf{A_i}^c \widetilde{\oplus} \mathbf{B}}(v+u), 1 - F_B(u)\right)$$

$$\geq \underset{i\in I}{\cup} \inf_{v,u\in Z^n}\left(\max\ F_{\mathbf{A_i}^c \widetilde{\oplus} \mathbf{B}}(v+u), 1 - F_B(u)\right)$$

3.6.2 Properties of the NFE Operation:

3.6.2.1 Proposition

The NFE satisfies the following properties:

i. **Monotonicity** (increasing in the first argument and decreasing in the second argument):

$\left(\forall A, B, C \ \in \mathcal{N}(Z^2)\right)\left(A \subseteq B \Longrightarrow \langle T_{A\widetilde{\ominus}C}\ , I_{A\widetilde{\ominus}C}\ , F_{A\widetilde{\ominus}C}\rangle \subseteq \langle T_{B\widetilde{\ominus}C}\ , I_{B\widetilde{\ominus}C}\ , F_{B\widetilde{\ominus}C}\rangle\right)$, where

$T_{A\widetilde{\ominus}C} \subseteq T_{B\widetilde{\ominus}C}\ , I_{A\widetilde{\ominus}C} \subseteq I_{B\widetilde{\ominus}C}\ ,\ F_{A\widetilde{\ominus}C} \supseteq F_{B\widetilde{\ominus}C}$.

$\left(\forall A, B, C \ \in \mathcal{N}(Z^2)\right)\left(A \subseteq B \Longrightarrow \langle T_{C\widetilde{\ominus}A}\ , I_{C\widetilde{\ominus}A}\ , F_{C\widetilde{\ominus}A}\rangle \supseteq \langle T_{C\widetilde{\ominus}B}\ , I_{C\widetilde{\ominus}B}\ , F_{C\widetilde{\ominus}B}\rangle\right)$, where

$T_{C\widetilde{\ominus}A} \subseteq T_{C\widetilde{\ominus}B}\ ,\ I_{C\widetilde{\ominus}A} \subseteq I_{C\widetilde{\ominus}B}\ ,\ F_{C\widetilde{\ominus}A} \subseteq F_{C\widetilde{\ominus}B}$.

ii. **Interaction with Zadeh's intersection:**

For any family $(A_i | i \in I)$ in $\mathcal{N}(Z^2)$and $B \in \mathcal{N}(Z^2)$, we have

$\langle T_{\underset{i\in I}{\cap} A_i\widetilde{\ominus}B}, I_{\underset{i\in I}{\cap} A_i\widetilde{\ominus}B}\ , F_{\underset{i\in I}{\cap} A_i\widetilde{\ominus}B}\rangle \subseteq \langle T_{\underset{i\in I}{\cap}(A_i\widetilde{\ominus}B)}, I_{\underset{i\in I}{\cap}(A_i\widetilde{\ominus}B)}\ , F_{\underset{i\in I}{\cap}(A_i\widetilde{\ominus}B)}\rangle$, where

$T_{\underset{i\in I}{\cap} A_i\widetilde{\ominus}B} \subseteq T_{\underset{i\in I}{\cap}(A_i\widetilde{\ominus}B)}\ ,\ I_{\underset{i\in I}{\cap} A_i\widetilde{\ominus}B} \subseteq I_{\underset{i\in I}{\cap}(A_i\widetilde{\ominus}B)}, F_{\underset{i\in I}{\cap} A_i\widetilde{\ominus}B} \subseteq F_{\underset{i\in I}{\cap}(A_i\widetilde{\ominus}B)}$.

$\langle T_{B\widetilde{\ominus}\underset{i\in I}{\cap}A_i}\ , I_{B\widetilde{\ominus}\underset{i\in I}{\cap}A_i} F_{B\widetilde{\ominus}\underset{i\in I}{\cap}A_i}\rangle \supseteq \langle T_{\underset{i\in I}{\cap}(B\widetilde{\ominus}A_i)}, I_{\underset{i\in I}{\cap}(B\widetilde{\ominus}A_i)}, F_{\underset{i\in I}{\cap}(B\widetilde{\ominus}A_i)}\rangle$, where

$T_{B\widetilde{\ominus}\underset{i\in I}{\cap}A_i} \supseteq T_{\underset{i\in I}{\cap}(B\widetilde{\ominus}A_i)}\ , I_{B\widetilde{\ominus}\underset{i\in I}{\cap}A_i} \supseteq I_{\underset{i\in I}{\cap}(B\widetilde{\ominus}A_i)}, F_{B\widetilde{\ominus}\underset{i\in I}{\cap}A_i} \supseteq F_{\underset{i\in I}{\cap}(B\widetilde{\ominus}A_i)}$.

iii. **Interaction with Zadeh's union.**

For any family$(A_i | i \in I)$ in $\mathcal{N}(Z^2)$ and $B \in \mathcal{N}(Z^2)$, we have

$\langle T_{\underset{i\in I}{\cup} A_i\widetilde{\ominus}B}, I_{\underset{i\in I}{\cup} A_i\widetilde{\ominus}B}\ , F_{\underset{i\in I}{\cup} A_i\widetilde{\ominus}B}\rangle \supseteq \langle T_{\underset{i\in I}{\cup}(A_i\widetilde{\ominus}B)}, I_{\underset{i\in I}{\cup}(A_i\widetilde{\ominus}B)}\ , F_{\underset{i\in I}{\cup}(A_i\widetilde{\ominus}B)}\rangle$, where

$T_{\underset{i\in I}{\cup} A_i\widetilde{\ominus}B} \supseteq T_{\underset{i\in I}{\cup}(A_i\widetilde{\ominus}B)}\ ,\ I_{\underset{i\in I}{\cup} A_i\widetilde{\ominus}B} \supseteq I_{\underset{i\in I}{\cup}(A_i\widetilde{\ominus}B)}\ , F_{\underset{i\in I}{\cup} A_i\widetilde{\ominus}B} \supseteq\ F_{\underset{i\in I}{\cup}(A_i\widetilde{\ominus}B)}$.

$\langle T_{B\widetilde{\ominus}\underset{i\in I}{\cup}A_i}, I_{B\widetilde{\ominus}\underset{i\in I}{\cup}A_i}\ , F_{B\widetilde{\ominus}\underset{i\in I}{\cup}A_i}\rangle \subseteq \langle T_{B\widetilde{\ominus}\underset{i\in I}{\cup}A_i}\ , I_{B\widetilde{\ominus}\underset{i\in I}{\cup}A_i}\ , F_{B\widetilde{\ominus}\underset{i\in I}{\cup}A_i}\rangle$, where

$T_{B\widetilde{\ominus}\underset{i\in I}{\cup}A_i} \subseteq T_{B\widetilde{\ominus}\underset{i\in I}{\cup}A_i}$, $I_{B\widetilde{\ominus}\underset{i\in I}{\cup}A_i} \subseteq I_{B\widetilde{\ominus}\underset{i\in I}{\cup}A_i}$, $F_{B\widetilde{\ominus}\underset{i\in I}{\cup}A_i} \supseteq F_{B\widetilde{\ominus}\underset{i\in I}{\cup}A_i}$.

Proof.

The proof of the first property is straightforward.

i. $\langle T_{A\widetilde{\ominus}C}, I_{A\widetilde{\ominus}C}, F_{A\widetilde{\ominus}C}\rangle \subseteq \langle T_{B\widetilde{\ominus}C}, I_{B\widetilde{\ominus}C}, F_{B\widetilde{\ominus}C}\rangle$.

ii. $\langle T_{\underset{i\in I}{\cap}A_i\widetilde{\ominus}B}, I_{\underset{i\in I}{\cap}A_i\widetilde{\ominus}B}, F_{\underset{i\in I}{\cap}A_i\widetilde{\ominus}B}\rangle \subseteq \langle T_{\underset{i\in I}{\cap}(A_i\widetilde{\ominus}B)}, I_{\underset{i\in I}{\cap}(A_i\widetilde{\ominus}B)}, F_{\underset{i\in I}{\cap}(A_i\widetilde{\ominus}B)}\rangle$

$T_{\underset{i\in I}{\cap}A_i\widetilde{\ominus}B}(v) = \inf_{v,u\in Z^2} \max\left(T_{\underset{i\in I}{\cap}A_i\widetilde{\ominus}B}(v+u), T_B(u)\right) = \inf_{v,u\in Z^2} \max\left(\min_{i\in I}\ T_{\mathbf{A_i\widetilde{\ominus}B}}(v+\right.$

$\left.u), T_B(u)\right) \leq \inf_{v,u\in Z^2} \min_{i\in I}\left(\max\ T_{\mathbf{A_i\widetilde{\ominus}B}}(v+u), T_B(u)\right) \leq \underset{\mathbf{i\in I}}{\cap} \inf_{v,u\in Z^2}\left(\max\ T_{A_i\widetilde{\ominus}B}(v+\right.$

$\left.u), T_B(u)\right) \leq \underset{i\in I}{\cap} T_{(A_i\widetilde{\ominus}B)}(v+u) \leq T_{\underset{i\in I}{\cap}(A_i\widetilde{\ominus}B)}$.

$I_{\underset{i\in I}{\cap}A_i\widetilde{\ominus}B}(v) = \inf_{v,u\in Z^2} \max\left(I_{\underset{i\in I}{\cap}A_i\widetilde{\ominus}B}(v+u), I_B(u)\right) = \inf_{v,u\in Z^2} \max\left(\min_{i\in I}\ I_{\mathbf{A_i\widetilde{\ominus}B}}(v+\right.$

$\left.u), I_B(u)\right) \leq \inf_{v,u\in Z^2} \min_{i\in I}\left(\max\ I_{\mathbf{A_i\widetilde{\ominus}B}}(v+u), I_B(u)\right) \leq \underset{\mathbf{i\in I}}{\cap} \inf_{v,u\in Z^2}\left(\max\ I_{\mathbf{A_i\widetilde{\ominus}B}}(v+\right.$

$\left.u), I_B(u)\right) \leq \underset{i\in I}{\cap} I_{(A_i\widetilde{\ominus}B)}(v+u) \leq I_{\underset{i\in I}{\cap}(A_i\widetilde{\ominus}B)}$.

$F_{\underset{i\in I}{\cap}A_i\widetilde{\ominus}B}(v) = \sup_{v,u\in Z^2} \min\left(1 - F_{\underset{i\in I}{\cap}A_i\widetilde{\ominus}B}(v+u), 1 - F_B(u)\right) = \sup_{v,u\in Z^2} \min\left(\min_{i\in I}\ I_{A_i{}^c\widetilde{\ominus}B}(v+u), 1 - F_B(u)\right) =$

$\sup_{v,u\in Z^2} \min_{i\in I}\left(\min\ F_{A_i{}^c\widetilde{\ominus}B}(v+u), 1 - F_B(u)\right) \leq \min_{i\in I} \sup_{u\in Z^2}\left(\min\ F_{A_i{}^c\widetilde{\ominus}B}(v+u), 1 - F_B(u)\right) \leq$

$\underset{i\in I}{\cap} \sup_{v,u\in Z^2}\left(\min\ F_{A_i{}^c\widetilde{\ominus}B}(v+u), 1 - F_B(u)\right) \leq F_{\underset{i\in I}{\cap}(A_i\widetilde{\ominus}B)}$.

iii. $\langle T_{\underset{i\in I}{\cup}A_i\widetilde{\ominus}B}, I_{\underset{i\in I}{\cup}A_i\widetilde{\ominus}B}, F_{\underset{i\in I}{\cup}A_i\widetilde{\ominus}B}\rangle \supseteq \langle T_{\underset{i\in I}{\cup}(A_i\widetilde{\ominus}B)}, I_{\underset{i\in I}{\cup}(A_i\widetilde{\ominus}B)}, F_{\underset{i\in I}{\cup}(A_i\widetilde{\ominus}B)}\rangle$

$T_{\underset{i\in I}{\cup}A_i\widetilde{\ominus}B}(v) = \inf_{v,u\in Z^2} \max\left(T_{\underset{i\in I}{\cup}A_i\widetilde{\ominus}B}(v+u), T_B(u)\right) = \inf_{v,u\in Z^2} \max\left(\max_{i\in I} T_{A_i\widetilde{\ominus}B}(v+\right.$

$\left.u), T_B(u)\right) = \inf_{v,u\in Z^2} \max_{i\in I}\left(\max T_{A_i\widetilde{\ominus}B}(v+u), T_B(u)\right) \geq \underset{i\in I}{\cup} \inf_{v,u\in Z^2}\left(\max T_{A_i\widetilde{\ominus}B}(v+u), T_B(u)\right) \geq \underset{i\in I}{\cup} T_{(A_i\widetilde{\ominus}B)}(v+$

$u) \geq T_{\underset{i\in I}{\cup}(A_i\widetilde{\ominus}B)}$.

$I_{\underset{i\in I}{\cup} A_i \widetilde{\ominus} B}(v) = \inf\limits_{v,u\in Z^2} \max\left(I_{\underset{i\in I}{\cup} A_i \widetilde{\ominus} B}(v+u), I_B(u)\right) = \inf\limits_{v,u\in Z^2} \max\left(\max\limits_{i\in I} I_{\mathbf{A_i}\widetilde{\ominus}\mathbf{B}}(v+\right.$

$\left.u), I_B(u)\right) = \inf\limits_{v,u\in Z^2} \max\limits_{i\in I}\left(\max I_{\mathbf{A_i}\widetilde{\ominus}\mathbf{B}}(v+u), I_B(u)\right) \geq \underset{\mathbf{i\in I}}{\cup} \inf\limits_{v,u\in Z^n}\left(\max I_{\mathbf{A_i}\widetilde{\ominus}\mathbf{B}}(v+\right.$

$\left.u), I_B(u)\right) \geq \underset{i\in I}{\cup} I_{(A_i\widetilde{\ominus}B)}(v+u) \geq I_{\underset{i\in I}{\cup}(A_i\widetilde{\ominus}B)}.$

$F_{\underset{i\in I}{\cup} A_i \widetilde{\ominus} B}(v) = \sup\limits_{v,u\in Z^2} \min\left(1 - F_{\underset{i\in I}{\cup} A_i \widetilde{\ominus} B}(v+u), 1 - F_B(u)\right) = \sup\limits_{v,u\in Z^2} \min\left(\max\limits_{i\in I} I_{A_i{}^c \widetilde{\ominus} B}(v+u), 1 -\right.$

$\left.F_B(u)\right) = \sup\limits_{v,u\in Z^2} \max\limits_{i\in I}\left(\min F_{A_i{}^c\widetilde{\ominus}B}(v+u), 1 - F_B(u)\right) \geq \max\limits_{i\in I} \sup\limits_{v,u\in Z^2}\left(\min F_{A_i{}^c\widetilde{\oplus}B}(v+u), 1 - F_B(u)\right) \geq$

$\underset{i\in I}{\cup} \sup\limits_{v,u\in Z^2}\left(\min F_{A_i{}^c\widetilde{\ominus}B}(v+u), 1 - F_B(u)\right).$

3.6.3 Properties of the NFC Operation:

3.6.3.1 Proposition

The NFC satisfies the following properties

i. Monotonicity (Increasing in the first argument):

$(\forall A, B, C \in \mathcal{N}(Z^2))(A \subseteq B \Longrightarrow \langle T_{A\tilde{\bullet}C}, I_{A\tilde{\bullet}C}, F_{A\tilde{\bullet}C}\rangle \subseteq \langle T_{B\tilde{\bullet}C}, I_{B\tilde{\bullet}C}, F_{B\tilde{\bullet}C}\rangle)$, where $T_{A\tilde{\bullet}C} \subseteq T_{B\tilde{\bullet}C}$, $I_{A\tilde{\bullet}C} \subseteq I_{B\tilde{\bullet}C}$, $F_{A\tilde{\bullet}C} \subseteq F_{B\tilde{\bullet}C}$.

ii. Interaction with Zadeh's intersection:

For any family $(A_i|i \in I)$ in $\mathcal{N}(Z^2)$and $B \in \mathcal{N}(Z^2)$, we have

$\langle T_{\underset{i\in I}{\cap} A_i\tilde{\bullet}B}, I_{\underset{i\in I}{\cap} A_i\tilde{\bullet}B}, F_{\underset{i\in I}{\cap} A_i\tilde{\bullet}B}\rangle \subseteq \langle T_{\underset{i\in I}{\cap}(A_i\tilde{\bullet}B)}, I_{\underset{i\in I}{\cap}(A_i\tilde{\bullet}B)}, F_{\underset{i\in I}{\cap}(A_i\tilde{\bullet}B)}\rangle$, where $T_{\underset{i\in I}{\cap} A_i\tilde{\bullet}B} \subseteq T_{\underset{i\in I}{\cap}(A_i\tilde{\bullet}B)}$, $I_{\underset{i\in I}{\cap} A_i\tilde{\bullet}B} \subseteq I_{\underset{i\in I}{\cap}(A_i\tilde{\bullet}B)}$, $F_{\underset{i\in I}{\cap} A_i\tilde{\bullet}B} \subseteq F_{\underset{i\in I}{\cap}(A_i\tilde{\bullet}B)}$.

iii. Interaction with Zadeh's union:

For any family$(A_i|i \in I)$ in $\mathcal{N}(Z^2)$ and $B \in \mathcal{N}(Z^2)$, we have

$\langle T_{\underset{i\in I}{\cup} A_i\tilde{\bullet}B}, I_{\underset{i\in I}{\cup} A_i\tilde{\bullet}B}, F_{\underset{i\in I}{\cup} A_i\tilde{\bullet}B}\rangle \supseteq \langle T_{\underset{i\in I}{\cup}(A_i\tilde{\bullet}B)}, I_{\underset{i\in I}{\cup}(A_i\tilde{\bullet}B)}, F_{\underset{i\in I}{\cup}(A_i\tilde{\bullet}B)}\rangle$, where

$T_{\underset{i\in I}{\cup} A_i\tilde{\bullet}B} \supseteq T_{\underset{i\in I}{\cup}(A_i\tilde{\bullet}B)}$, $I_{\underset{i\in I}{\cup} A_i\tilde{\bullet}B} \supseteq I_{\underset{i\in I}{\cup}(A_i\tilde{\bullet}B)}$, $F_{\underset{i\in I}{\cup} A_i\tilde{\bullet}B} \supseteq F_{\underset{i\in I}{\cup}(A_i\tilde{\bullet}B)}$.

Proof (i) The first property (i.e. the monotonicity properties of the neutrosophic dilation and neutrosophic erosion have been satisfied from the fact that $A \subseteq B$, it follows that $\langle T_{C\widetilde{\oplus}A}, I_{C\widetilde{\oplus}A}, F_{C\widetilde{\oplus}A}\rangle \subseteq \langle T_{C\widetilde{\oplus}B}, I_{C\widetilde{\oplus}B}, F_{C\widetilde{\oplus}B}\rangle$.

The proof of **(ii) &** (**iii**) can be inherited from the property that $(A_i|i \in I)\left(\min_{i\in I} A_i \subseteq A_i \subseteq \max_{i\in I} A_i\right)$.

3.6.4 Properties of the NFO Operation:

3.6.4.1 Proposition

The neutrosophic opening satisfies the following properties

i. Monotonicity (increasing in the first argument):

$(\forall A, B, C \in \mathcal{N}(Z^2))(A \subseteq B \Longrightarrow \langle T_{A \tilde{\circ} C}, I_{A \tilde{\circ} C}, F_{A \tilde{\circ} C}\rangle \subseteq \langle T_{B \tilde{\circ} C}, I_{B \tilde{\circ} C}, F_{B \tilde{\circ} C}\rangle)$, where

$T_{A\tilde{\circ} C} \subseteq T_{B\tilde{\circ} C}$, $I_{A\tilde{\circ} C} \subseteq I_{B\tilde{\circ} C}$, $F_{A\tilde{\circ} C} \subseteq F_{B\tilde{\circ} C}$.

ii. Interaction with Zadeh's intersection:

For any family $(A_i|i \in I)$ in $\mathcal{N}(Z^2)$ and $B \in \mathcal{N}(Z^2)$, we have

$\langle T_{\bigcap_{i\in I} A_i \tilde{\circ} B}, I_{\bigcap_{i\in I} A_i \tilde{\circ} B}, F_{\bigcap_{i\in I} A_i \tilde{\circ} B}\rangle \subseteq \langle T_{\bigcap_{i\in I}(A_i \tilde{\circ} B)}, I_{\bigcap_{i\in I}(A_i \tilde{\circ} B)}, F_{\bigcap_{i\in I}(A_i \tilde{\circ} B)}\rangle$, where

$T_{\bigcap_{i\in I} A_i \tilde{\circ} B} \subseteq T_{\bigcap_{i\in I}(A_i \tilde{\circ} B)}$, $I_{\bigcap_{i\in I} A_i \tilde{\circ} B} \subseteq I_{\bigcap_{i\in I}(A_i \tilde{\circ} B)}$, $F_{\bigcap_{i\in I} A_i \tilde{\circ} B} \subseteq F_{\bigcap_{i\in I}(A_i \tilde{\circ} B)}$.

iii. Interaction with Zadeh's union:

For any family$(A_i|i \in I)$ in $\mathcal{N}(Z^2)$ and $B \in \mathcal{N}(Z^2)$, we have

$\langle T_{\bigcup_{i\in I} A_i \tilde{\circ} B}, I_{\bigcup_{i\in I} A_i \tilde{\circ} B}, F_{\bigcup_{i\in I} A_i \tilde{\circ} B}\rangle \supseteq \langle T_{\bigcup_{i\in I}(A_i \tilde{\circ} B)}, I_{\bigcup_{i\in I}(A_i \tilde{\circ} B)}, F_{\bigcup_{i\in I}(A_i \tilde{\circ} B)}\rangle$, where

$T_{\bigcup_{i\in I} A_i \tilde{\circ} B} \supseteq T_{\bigcup_{i\in I}(A_i \tilde{\circ} B)}$, $I_{\bigcup_{i\in I} A_i \tilde{\circ} B} \supseteq I_{\bigcup_{i\in I}(A_i \tilde{\circ} B)}$, $F_{\bigcup_{i\in I} A_i \tilde{\circ} B} \supseteq F_{\bigcup_{i\in I}(A_i \tilde{\circ} B)}$.

The proofs are Similar to the proofs of the foregoing proposition**.**

3.7 Neutrosophic Fuzzy Mathematical Morphological Filters:

This section is considering the differences between two or more of the basic neutrosophic fuzzy morphological operators.

3.7.1 Some Types of Boundary Extraction Filters Using NFD and NFE:

As the neutrosophic dilation thickens the regions in the true level of the image, and the neutrosophic erosion shrinks them, the neutrosophic differences between the image and its neutrosophic dilation or its neutrosophic erosion may emphasize the boundaries between regions included in the image. Therefore, several boundary filters may be obtained as follows:

3.7.1.1 Neutrosophic Fuzzy Gradient Boundary:

To commence, we will investigate the neutrosophic fuzzy gradient filter, which is the mean value of the three components of the neutrosophic difference between the neutrosophic dilation of some images and its neutrosophic erosion. We get the neutrosophic gradient of the image by applying the mean of these boundaries. If the structure element is relatively small, the homogeneous areas will not be affected by NFD and NFE, and then the subtraction tends to eliminate them. The effect of neutrosophic morphological gradient operation is shown in Fig. 3.7.1.1, Type I, II, and to be defined in two different types as follows:

Type I:

$\tilde{\partial} gradient = (1/3)\ \ [min(TA \mathbin{\widetilde{\oplus}} B(v), 1 - TA \mathbin{\widetilde{\ominus}} B(v)), min(IA \mathbin{\widetilde{\oplus}} B(v), 1 -$
$IA \mathbin{\widetilde{\ominus}} B(v)), max(FA \mathbin{\widetilde{\oplus}} B(v), 1 - FA \mathbin{\widetilde{\ominus}} B(v))]$.

In the following figure (fig.3.7.1.1 (I)), we present the results obtained when applying neutrosophic gradient boundary filter on some grayscale image.

(a) (b)

Fig. 3.7.1.1 (I): Applying the Neutrosophic Gradient Boundary: (a) Original Image (b) Neutrosophic Gradient Boundary Filtered.

Type II:

$\tilde{\partial} gradient = (1/3)[min(TA \mathbin{\widetilde{\oplus}} B(v), 1 - TA \mathbin{\widetilde{\ominus}} B(v)), max(IA \mathbin{\widetilde{\oplus}} B(v), 1 - TA \mathbin{\widetilde{\ominus}} B(v)),$
$max(FA \mathbin{\widetilde{\oplus}} B(v), 1 - TA \mathbin{\widetilde{\ominus}} B(v))]$.

In the following figure (fig.3.7.1.1 (II)), we present the results obtained when applying neutrosophic gradient boundary filter on some grayscale images.

(a) (b)

Fig.3.7.1.1 (II): Applying the Neutrosophic Gradient Boundary: (a) Original Image (b) Neutrosophic Gradient Boundary Filtered.

3.7.1.2 Neutrosophic Fuzzy External Boundary:

In this filter, a neutrosophic dilation is firstly applied to the neutrosophic image (a) by some neutrosophic structure elements (b); hence, the output filtered image will be the neutrosophic difference between neutrosophic dilated image and the neutrosophic image (a). That is, the neutrosophic external boundary of (a) is to be defined in two different types as follows:

Type I: $\tilde{\partial} ext \ = (1/3)\ [min\ (TA \widetilde{\oplus} B(v), 1\ -\ TA(v)), min\ (IA \widetilde{\oplus} B(v), 1\ - IA(v)),$

$max\ (FA \widetilde{\oplus} B(v))\ , 1\ -\ FA(v)\ ].$

In the following figure (fig.3.7.1.2 (I)), we present the results obtained when applying neutrosophic external boundary filter on some grayscale images.

(a) (b)

Fig. 3.7.1.2 (I): Applying the Neutrosophic Fuzzy External Boundary: a) Original Image

a) Neutrosophic Fuzzy External Boundary Filtered Image

Type II:

$\tilde{\partial} ext \ = \left(\frac{1}{3}\right)[min(TA \widetilde{\oplus} B(v), 1 - TA(v))\ , max\ (IA \widetilde{\oplus} B(v), 1\ -$

$IA(v))\ , max\ (FA \widetilde{\oplus} B(v))\ , 1\ -\ FA(v)\ ]\ .$

In the following figure (fig 3.7.1.2 (II)), we present the results obtained when applying the Neutrosophic Fuzzy External Boundary Filter on Some Grayscale Images.

(a) (b)

Fig.3.7.1.2 (II): Applying the Neutrosophic Fuzzy Eternal Boundary: a) Original Image

b) Neutrosophic External Boundary Filtered Image

3.7.1.3 Neutrosophic Fuzzy Internal Boundary:

The main step of the neutrosophic internal boundary filter, is to get the neutrosophic erosion of the neutrosophic image, hence, the output filtered image will be the neutrosophic difference between the original image in the neutrosophic domain and the neutrosophic eroded image that is the neutrosophic internal boundary of the neutrosophic image (a) is to be defined in two different types as follows:

Type I:

In the following figure (fig.3.7.1.3 (I)), we present the results obtained when applying neutrosophic internal boundary filter on some grayscale images.

a) b)

Fig. 3.7.1.3 (I): Applying the Neutrosophic Internal Boundary: a) Original Image

b) Neutrosophic Internal Boundary Filtered Image

Type II:

$$\tilde{\partial} int = (1/3)[min\ (T(v), 1 - (TA \,\widetilde{\ominus}\, B(v)))\ , min\ (IA(v), 1 - IA \,\widetilde{\ominus}\, B(v))\ , max\ (FA(v), 1 - FA \,\widetilde{\ominus}\, B(v))\ ].$$

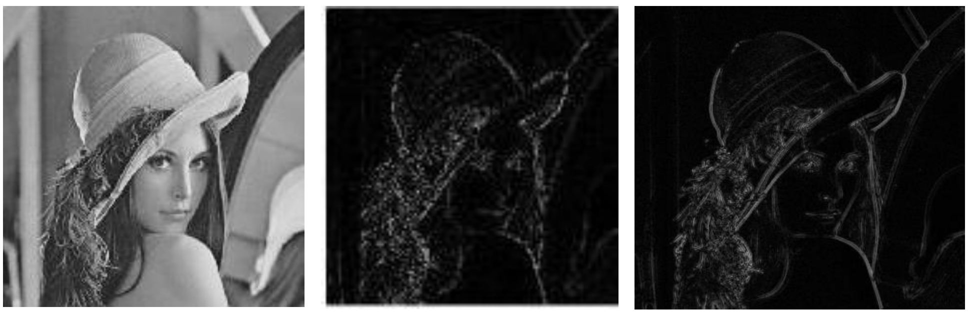
(a) **(b)**

Fig.3.7.1.3 (II): Applying the Neutrosophic Internal Boundary: a) Original Image

b) Neutrosophic Internal Boundary Filtered Image

3.7.2. Neutrosophic Fuzzy Outline Boundary:

The main step of the neutrosophic outline boundary filter, is to get the complement of the neutrosophic erosion of the neutrosophic image, hence, the output filtered image will be the neutrosophic difference between the original image in neutrosophic domain and the neutrosophic eroded image that is the neutrosophic outline boundary of the neutrosophic image A is to be defined as follows: $\tilde{\partial} outline\ (A) = (\partial 1 A1 \cup \partial 3 A3) \cap A2$, where; $\partial 1(A1) = co(A1 \ominus B1) \cap A1, \partial 3(A3) = co(A3 \oplus B3) \cup A3$. In the following figure (fig.3.7.2), we present the results obtained when applying the neutrosophic outline boundary filter on some grayscale images.

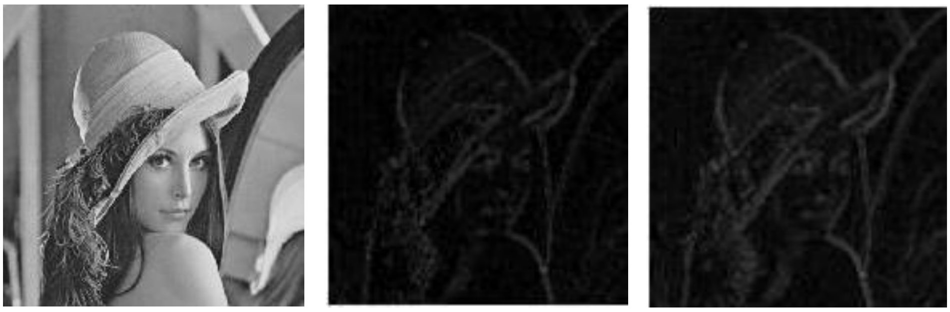


a) b) c)

Fig. 3.7.2: Neutrosophic Outline Boundary: a) Original Image, b) Neutrosophic Outline Boundary Filtered Image, c) Neutrosophic Outline Boundary Filtered Image

3.7.2.1 Some Combinations of the Neutrosophic Fuzzy External and Internal Boundary Filters

In the following figure (fig.3.7.2.1), we present the results obtained when applying the neutrosophic fuzzy sup. boundary filter on some grayscale images.



a) b) c)

Fig.3.7.2.1 Neutrosophic fuzzy sup. boundary: a) Original Image, b) Neutrosophic fuzzy sup. boundary filtered image, c) Neutrosophic fuzzy sup. boundary filtered image

4.1. Neutrosophic Crisp Mathematical Morphology:

As a generalization of the classical mathematical morphology, we present in this section the basic operations for the neutrosophic crisp mathematical morphology. To commence, we need to define the translation of a neutrosophic crisp set.

4.1.1. Definition:

Consider the space $X = R^n$ or $X = Z^n$, with origin $0 = (0, \ldots, 0)$, given that the reflection of the structuring element B mirrored in its origin is defined as:
$-B = \langle -B^1, -B^2, -B^3 \rangle$

4.1.2. Definition:

For every $p \in A$, the translation by p is the map $p: X \to X, a \to a + p$; it transforms any subset A of X into its translate by $p \in Z^2$, $A = \langle A_p^1, A_p^2, A_p^3 \rangle$. Where $A_p^1 = \{u + p: u \in A^1, p \in B^1\}$, $A_p^2 = \{u + p: u \in A^2, p \in B^2\}$, $A_p^3 = \{u + p: u \in A^3, p \in B^3\}$.

4.2. Neutrosophic Crisp Mathematical Morphological Operations:

4.2.1. Neutrosophic Crisp Dilation Operator:

Let $A, B \in \mathcal{NC}(X)$, and then we define two types of the neutrosophic crisp dilation as follows:

Type I:

$A \widetilde{\bigoplus} B = \langle A^1 \oplus B^1, A^2 \oplus B^2, A^3 \Theta B^3 \rangle$, where for each $u, v \in Z^2$, we have $A^1 \oplus B^1 = \bigcup_{b \in B^1} A_b^1$, $A^2 \oplus B^2 = \bigcup_{b \in B^2} A_b^2$, $A^3 \Theta B^3 = \bigcap_{-b \in B^3} A_{-b}^3$.

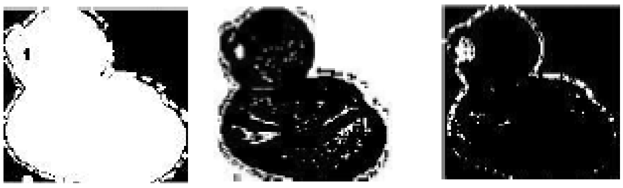


Fig. 4.2.1. (I): Neutrosophic Crisp Dilation Components in type I for $\langle A^1, A^2, A^3 \rangle$respectively.

Type II:

$A \widetilde{\bigoplus} B = \langle A^1 \oplus B^1, A^2 \Theta B^2, A^3 \Theta B^3 \rangle$, where for each $u, v \in Z^2$, we have $A^1 \oplus B^1 = \bigcup_{b \in B^1} A_b^1$, $A^2 \Theta B^2 = \bigcap_{-b \in B^2} A_{-b}^2$, $A^3 \Theta B^3 = \bigcap_{-b \in B^3} A_{-b}^3$.

Fig. 4.2.1. (II): Neutrosophic Crisp Dilation Components in type II for $\langle A^1, A^2, A^3 \rangle$ respectively.

4.2.2. Neutrosophic Crisp Erosion Operation:

Let $A, B \in \mathcal{NC}(X)$; then the neutrosophic erosion is given as two types:

Type I:

$A \widetilde{\Theta} B = \langle A^1 \Theta B^1, A^2 \Theta B^2, A^3 \oplus B^3 \rangle$, where for each $u, v \in Z^2$, we have $A^1 \Theta B^1 = \bigcap_{-b \in B^1} A_{-b}^1$, $A^2 \Theta B^2 = \bigcap_{-b \in B^2} A_{-b}^2$, $A^3 \oplus B^3 = \bigcup_{b \in B^3} A_b^3$.

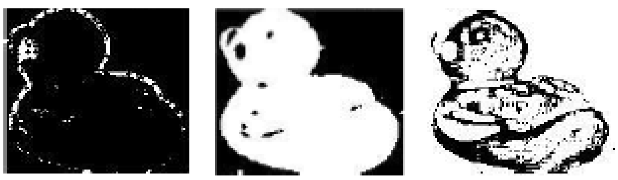


Fig. 4.2.2. (I): Neutrosophic Crisp Erosion Components in type I for $\langle A^1, A^2, A^3 \rangle$ respectively.

Type II:

$A\tilde{\Theta}B = \langle A^1 \Theta B^1, A^2 \oplus B^2, A^3 \oplus B^3 \rangle$, where for each $u, v \in Z^2$, we have $A^1 \Theta B^1 = \bigcap_{-b \in B^1} A^1_{-b}$, $A^2 \oplus B^2 = \bigcup_{b \in B^2} A^2_b$, $A^3 \oplus B^3 = \bigcup_{b \in B^3} A^3_b$.

Fig.4.2.2. (II): Neutrosophic Crisp Erosion Components in type II $\langle A^1, A^2, A^3 \rangle$ respectively.

4.2.3 Neutrosophic Crisp Opening Operation:

Let A, B$\in \mathcal{N}C(X)$; then we define two types of the neutrosophic crisp opening operator as follows:

Type I:

$A \tilde{\circ} B = \langle A^1 \circ B^1, A^2 \circ B^2, A^3 \bullet B^3 \rangle$, $A^1 \circ B^1 = (A^1 \Theta B^1) \oplus B^1$, $A^2 \circ B^2 = (A^2 \Theta B^2) \oplus B^2$, $A^3 \bullet B^3 = (A^3 \oplus B^3) \Theta B^3$.

Fig. 4.2.3. (I): Neutrosophic Crisp Opening Components in Type I $\langle A^1, A^2, A^3 \rangle$ respectively.

Type II:

$A \tilde{\circ} B = \langle A^1 \circ B^1, A^2 \bullet B^2, A^3 \bullet B^3 \rangle$, $A^1 \circ B^1 = (A^1 \Theta B^1) \oplus B^1$, $A^2 \bullet B^2 = (A^2 \oplus B^2) \Theta B^2$, $A^3 \bullet B^3 = (A^3 \oplus B^3) \Theta B^3$.

Fig. 4.2.3. (II): Neutrosophic Crisp Opening Components in type II $\langle A^1, A^2, A^3 \rangle$ respectively.

4.2.4. Neutrosophic Crisp Closing Operation:

Let A and B $\in \mathcal{N}C(X)$; then the neutrosophic closing is given as two types:

Type I:

$A \tilde{\bullet} B = \langle A^1 \bullet B^1, A^2 \bullet B^2, A^3 \circ B^3 \rangle$, $\quad A^1 \bullet B^1 = (A^1 \oplus B^1)\Theta B^1$, $\quad A^2 \bullet B^2 = (A^2 \oplus B^2)\Theta B^2$, $\quad A^3 \circ B^3 = (A^3 \Theta B^3)\oplus B^3$.

Fig.4.2.4 (I): Neutrosophic Crisp Closing Components in type I for $\langle A^1, A^2, A^3 \rangle$ respectively.

Type II:

$A \tilde{\bullet} B = \langle A^1 \bullet B^1, A^2 \circ B^2, A^3 \circ B^3 \rangle$, $\quad A^1 \bullet B^1 = (A^1 \oplus B^1)\Theta B^1$, $\quad A^2 \circ B^2 = (A^2 \Theta B^2)\oplus B^2$, $\quad A^3 \circ B^3 = (A^3 \Theta B^3)\oplus B^3$.

Fig.4.2.4. (II): Neutrosophic Crisp Closing Components in type II for $\langle A^1, A^2, A^3 \rangle$ respectively.

5. Algebraic Properties in Neutrosophic Crisp:

In this section, we investigate some of the algebraic properties of the neutrosophic crisp erosion and dilation, as well as the neutrosophic crisp opening and closing operator [15].

5. 1 Properties of the Neutrosophic Crisp Erosion Operator:

5.1.1 Proposition:

The Neutrosophic erosion satisfies the monotonicity for all A, B $\in \mathcal{NC}(Z^2)$.

Type I:

a) $A \subseteq B \Longrightarrow \langle A^1\Theta C^1, A^2\Theta C^2, A^3\Theta C^3 \rangle \subseteq \langle B^1\Theta C^1, B^2\Theta C^2, B^3\Theta C^3 \rangle$:

$A^1\Theta C^1 \subseteq B^1\Theta C^1, A^2\Theta C^2 \subseteq B^2\Theta C^2, A^3\Theta C^3 \supseteq B^3\Theta C^3$.

b) $A \subseteq B \Longrightarrow \langle C^1\Theta A^1, C^2\Theta A^2, C^3\Theta A^3 \rangle \subseteq \langle C^1\Theta B^1, C^2\Theta B^2, C^3\Theta B^3 \rangle$:

$C^1\Theta A^1 \subseteq C^1\Theta B^1, C^2\Theta A^2 \subseteq C^2\Theta B^2, C^3\Theta A^3 \supseteq C^3\Theta B^3$.

Type II:

a) $A \subseteq B \Longrightarrow \langle A^1\Theta C^1, A^2\Theta C^2, A^3\Theta C^3 \rangle \subseteq \langle B^1\Theta C^1, B^2\Theta C^2, B^3\Theta C^3 \rangle$:

$A^1\Theta C^1 \subseteq B^1\Theta C^1, A^2\Theta C^2 \supseteq B^2\Theta C^2, A^3\Theta C^3 \supseteq B^3\Theta C^3$.

b) $A \subseteq B \Longrightarrow \langle C^1 \Theta A^1, C^2 \Theta A^2, C^3 \Theta A^3 \rangle \subseteq \langle C^1 \Theta B^1, C^2 \Theta B^2, C^3 \Theta B^3 \rangle$:

$C^1 \Theta A^1 \subseteq C^1 \Theta B^1, C^2 \Theta A^2 \supseteq C^2 \Theta B^2, C^3 \Theta A^3 \supseteq C^3 \Theta B^3$.

Note that: Dislike the Neutrosophic crisp dilation operator, the Neutrosophic crisp erosion does not satisfy commutativity and the associativity properties.

5.1.2 Proposition: for any family $A_i \in \mathcal{NC}(Z^2), i \in I$, and $B \in \mathcal{NC}(Z^2)$.

Type I:

a) $\bigcap_{i\in I} A_i \widetilde{\ominus} B = \bigcap_{i\in I} (A_i \widetilde{\ominus} B) = \langle \cap A_i^1 \Theta B^1, \cap A_i^2 \Theta B^2, \cap A_i^3 \oplus B^3 \rangle = \langle \cap (A_i^1 \Theta B^1), \cap (A_i^2 \Theta B^2), \cap (A_i^3 \oplus B^3) \rangle$,

b) $B \widetilde{\ominus} \bigcap_{i\in I} A_i = \bigcap_{i\in I} (B \widetilde{\ominus} A_i) = \langle B^1 \Theta \cap A_i^1, B^2 \Theta \cap A_i^2, B^3 \oplus \cap A_i^3 \rangle = \langle \cap (B^1 \Theta A_i^1), \cap (B^2 \Theta A_i^2), \cap (B^3 \oplus A_i^3) \rangle$.

Type II:

a) $\bigcap_{i\in I} A_i \widetilde{\ominus} B = \bigcap_{i\in I} (A_i \widetilde{\ominus} B) \Longrightarrow$

$\langle \cap_{i\in I} A_i^1 \Theta B^1, \cap_{i\in I} A_i^2 \oplus B^2, \cap_{i\in I} A_i^3 \oplus B^3 \rangle = \langle \cap (A_i^1 \Theta B^1), \cap (A_i^2 \oplus B^2), \cap (A_i^3 \oplus B^3) \rangle$,

b) $B \widetilde{\ominus} \bigcap_{i\in I} A_i = \bigcap_{i\in I} (B \widetilde{\ominus} A_i) \Longrightarrow$

$\langle B^1 \Theta \cap A_i^1, B^2 \oplus \cap A_i^2, B^3 \oplus \cap A_i^3 \rangle = \langle \cap (B^1 \Theta A_i^1), \cap (B^2 \oplus A_i^2), \cap (B^3 \oplus A_i^3) \rangle$.

Proof: a) In two types:

Type I:

$\bigcap_{i\in I} A_i \widetilde{\ominus} B = \langle \cap_{i\in I} (\cap_{b\in B} A_{ib}^1), \cup_{i\in I} (\cap_{b\in B} A_{ib}^2), \cup_{i\in I} (\cap_{b\in B} A_{i(-b)}^3) \rangle =$

$\langle \cap_{i\in I} (\cap_{b\in B} A_{i(-b)}^1), \cap_{i\in I} (\cup_{b\in B} A_{i(-b)}^2), \cap_{i\in I} (\cup_{b\in B} A_{ib}^3) \rangle = \bigcap_{i\in I} (A_i \widetilde{\ominus} B)$

Type II:

Similarity, we can show that it is true in type 2,

b) The proof is similar to the (a).

5.1.3 Proposition: for any family $A_i \in \mathcal{NC}(Z^2), i \in I$, and $B \in \mathcal{NC}(Z^2)$

Type I:

a) $\cup_{i\in I} A_i \widetilde{\ominus} B = \cup_{i\in I} (A_i \widetilde{\ominus} B) \Longrightarrow$

$\langle \cup_{i\in I} A_i^1 \Theta B^1, \cup_{i\in I} A_i^2 \Theta B^2, \cup_{i\in I} A_i^3 \oplus B^3 \rangle = \langle \cup_{i\in I} (A_i^1 \Theta B^1), \cup_{i\in I} (A_i^2 \Theta B^2), \cup_{i\in I} (A_i^3 \oplus B^3) \rangle$,

b) $B \widetilde{\ominus} \cup_{i\in I} A_i = \cup_{i\in I} (B \widetilde{\ominus} A_i) \Longrightarrow$

$\langle B^1 \Theta \cup_{i\in I} A_i^1, B^2 \Theta \cup_{i\in I} A_i^2, B^3 \oplus \cup_{i\in I} A_i^3 \rangle = \langle \cup_{i\in I} (B^1 \Theta A_i^1), \cup_{i\in I} (B^2 \Theta A_i^2), \cup_{i\in I} (B^3 \oplus A_i^3) \rangle$.

Type II:

a) $\cup_{i\in I} A_i \widetilde{\ominus} B = \cup_{i\in I} (A_i \widetilde{\ominus} B) \Longrightarrow$

$\langle \cup_{i\in I} A_i^1 \Theta B^1, \cup_{i\in I} A_i^2 \oplus B^2, \cup_{i\in I} A_i^3 \oplus B^3 \rangle = \langle \cup_{i\in I} (A_i^1 \Theta B^1), \cup_{i\in I} (A_i^2 \oplus B^2), \cup_{i\in I} (A_i^3 \oplus B^3) \rangle$,

b) $B \widetilde{\ominus} \cup_{i\in I} A_i = \cup_{i\in I} (B \widetilde{\ominus} A_i) \Longrightarrow$

$\langle B^1 \Theta \cup_{i\in I} A_i^1, B^2 \oplus \cup_{i\in I} A_i^2, B^3 \oplus \cup_{i\in I} A_i^3 \rangle = \langle \cup_{i\in I} (B^1 \Theta A_i^1), \cup_{i\in I} (B^2 \oplus A_i^2), \cup_{i\in I} (B^3 \oplus A_i^3) \rangle$.

Proof: a) for both two types

Type I: $\bigcup_{i\in I} A_i \,\widetilde{\ominus}\, B = \langle \bigcap_{b\in B}\big(\bigcup_{i\in I} A^1_{i(-b)}\big), \bigcap_{b\in B}\big(\bigcup_{i\in I} A^2_{i(-b)}\big), \bigcup_{b\in B}(\bigcup_{i\in I} A^3_{ib})\rangle$

$= \langle \bigcup_{i\in I}(\bigcap_{b\in B} A^1_{i(-b)}), \bigcup_{i\in I}(\bigcap_{b\in B} A^2_{i(-b)}), \bigcup_{i\in I}(\bigcup_{b\in B} A^3_{ib})\rangle = \underset{i\in I}{\cap}(A_i \,\widetilde{\ominus}\, B).$

Type II: can be verified in a similar way as in type 1.

b) The proof is similar to the (a).

5.2 Properties of the Neutrosophic Crisp Dilation Operator):

5.2.1 Proposition:

The neutrosophic dilation satisfies the following properties: $\forall A, B \in \mathcal{NC}(Z^2)$

i) Commutativity: $A \,\widetilde{\oplus}\, B = B \,\widetilde{\oplus}\, A$.

ii) Associativity: $(A \,\widetilde{\oplus}\, B) \,\widetilde{\oplus}\, C = A \,\widetilde{\oplus}\, (B \,\widetilde{\oplus}\, C)$.

iii) Monotonicity: (increasing in both arguments):

Type I:

a) $A \subseteq B \Longrightarrow \langle A^1 \oplus C^1, A^2 \oplus C^2, A^3 \oplus C^3\rangle \subseteq \langle B^1 \oplus C^1, B^2 \oplus C^2, B^3 \oplus C^3\rangle$:

$A^1 \oplus C^1 \subseteq B^1 \oplus C^1, A^2 \oplus C^2 \subseteq B^2 \oplus C^2, A^3 \oplus C^3 \supseteq B^3 \oplus C^3$.

b) $A \subseteq B \Longrightarrow \langle C^1 \oplus A^1, C^2 \oplus A^2, C^3 \oplus A^3\rangle \subseteq \langle C^1 \oplus B^1, C^2 \oplus B^2, C^3 \oplus B^3\rangle$:

$C^1 \oplus A^1 \subseteq C^1 \oplus B^1, C^2 \oplus A^2 \subseteq C^2 \oplus B^2, C^3 \oplus A^3 \supseteq C^3 \oplus B^3$.

Type II:

a) $A \subseteq B \Longrightarrow \langle A^1 \oplus C^1, A^2 \oplus C^2, A^3 \oplus C^3\rangle \subseteq \langle B^1 \oplus C^1, B^2 \oplus C^2, B^3 \oplus C^3\rangle$:

$A^1 \oplus C^1 \subseteq B^1 \oplus C^1, A^2 \oplus C^2 \supseteq B^2 \oplus C^2, A^3 \oplus C^3 \supseteq B^3 \oplus C^3$.

b) $A \subseteq B \Longrightarrow \langle C^1 \oplus A^1, C^2 \oplus A^2, C^3 \oplus A^3\rangle \subseteq \langle C^1 \oplus B^1, C^2 \oplus B^2, C^3 \oplus B^3\rangle$:

$C^1 \oplus A^1 \subseteq C^1 \oplus B^1, C^2 \oplus A^2 \supseteq C^2 \oplus B^2, C^3 \oplus A^3 \supseteq C^3 \oplus B^3$.

Proof: i), ii), iii) are obvious in two types I and II.

5.2.2 Proposition: for any family $(A_i | i \in I)$ in $\mathcal{NC}(Z^2)$ and $B \in \mathcal{NC}(Z^2)$

Type I: $a)$ $\underset{i\in I}{\cap} \mathrm{A_i} \,\widetilde{\oplus}\, \mathrm{B} = \underset{\mathrm{i\in I}}{\cap}(\mathrm{A_i} \,\widetilde{\oplus}\, \mathrm{B}) \Longrightarrow$

$\langle \bigcap_{i\in I} A^1_i \oplus B^1, \bigcap_{i\in I} A^2_i \oplus B^2, \bigcap_{i\in I} A^3_i \Theta B^3\rangle = \langle \bigcap_{i\in I}(A^1_i \oplus B^1), \bigcap_{i\in I}(A^2_i \oplus B^2), \bigcap_{i\in I}(A^3_i \Theta B^3)\rangle.$

b) $\mathrm{B} \,\widetilde{\oplus}\, \underset{\mathrm{i\in I}}{\cap} \mathrm{A_i} = \underset{\mathrm{i\in I}}{\cap}(\mathrm{B} \,\widetilde{\oplus}\, \mathrm{A_i}) \Longrightarrow$

$\langle B^1 \oplus \bigcap_{i\in I} A^1_i, B^2 \oplus \bigcap_{i\in I} A^2_i, B^3 \Theta \bigcap_{i\in I} A^3_i\rangle = \langle \bigcap_{i\in I}(B^1 \oplus A^1_i), \bigcap_{i\in I}(B^2 \oplus A^2_i), \bigcap_{i\in I}(B^3 \Theta A^3_i)\rangle.$

Type II: a) $\underset{\mathrm{i\in I}}{\cap} \mathrm{A_i} \,\widetilde{\oplus}\, \mathrm{B} = \underset{\mathrm{i\in I}}{\cap}(\mathrm{A_i} \,\widetilde{\oplus}\, \mathrm{B}) \Longrightarrow$

$\langle \bigcap_{i\in I} A^1_i \oplus B^1, \bigcap_{i\in I} A^2_i \Theta B^2, \bigcap_{i\in I} A^3_i \Theta B^3\rangle = \langle \bigcap_{i\in I}(A^1_i \oplus B^1), \bigcap_{i\in I}(A^2_i \Theta B^2), \bigcap_{i\in I}(A^3_i \Theta B^3)\rangle.$

b) $B \widetilde{\bigoplus} \bigcap_{i \in I} A_i = \bigcap_{i \in I}(B \widetilde{\bigoplus} A_i) \Longrightarrow$

$\langle B^1 \oplus \bigcap_{i \in I} A_i^1, B^2 \Theta \bigcap_{i \in I} A_i^2, B^3 \Theta \bigcap_{i \in I} A_i^3 \rangle = \langle \bigcap_{i \in I}(B^1 \oplus A_i^1), \bigcap_{i \in I}(B^2 \Theta A_i^2), \bigcap_{i \in I}(B^3 \Theta A_i^3) \rangle$.

Proof: we will prove this property for the two types of the neutrosophic crisp intersection operator:

Type I: a) $\bigcap_{i \in I} A_i \widetilde{\bigoplus} B = \bigcap_{i \in I}(A_i \widetilde{\bigoplus} B) \Longrightarrow$

$\langle \bigcup_{b \in B}(\bigcap_{i \in I} A_{ib}^1), \bigcup_{b \in B}(\bigcap_{i \in I} A_{ib}^2), \bigcap_{b \in B}(\bigcap_{i \in I} A_{i(-b)}^3) \rangle =$

$\langle \bigcap_{i \in I}(\bigcup_{b \in B} A_{ib}^1), \bigcap_{i \in I}(\bigcup_{b \in B} A_{ib}^2), \bigcap_{i \in I}(\bigcap_{b \in B} A_{i(-b)}^3) \rangle$.

Type II: a) $\bigcap_{i \in I} A_i \widetilde{\bigoplus} B = \bigcap_{i \in I}(A_i \widetilde{\bigoplus} B) \Longrightarrow$

$\langle \bigcup_{b \in B}(\bigcap_{i \in I} A_{ib}^1), \bigcap_{b \in B}(\bigcap_{i \in I} A_{ib}^2), \bigcap_{b \in B}(\bigcap_{i \in I} A_{i(-b)}^3) \rangle =$

$\langle \bigcap_{i \in I}(\bigcup_{b \in B} A_{ib}^1), \bigcap_{i \in I}(\bigcap_{b \in B} A_{ib}^2), \bigcap_{i \in I}(\bigcap_{b \in B} A_{i(-b)}^3) \rangle$.

The proof of (b) is similar to (a).

5.2.3 Proposition: for any family of $(A_i | i \in I)$ in $\mathcal{N}C(Z^2)$**,** and $B \in \mathcal{N}C(Z^2)$

Type I:

a) $\bigcup_{i \in I} A_i \widetilde{\bigoplus} B = \bigcup_{i \in I}(A_i \widetilde{\bigoplus} B) \Longrightarrow$

$\langle \bigcup_{i \in I} A_i^1 \oplus B^1, \bigcup_{i \in I} A_i^2 \oplus B^2, \bigcup_{i \in I} A_i^3 \Theta B^3 \rangle = \langle \bigcup_{i \in I}(A_i^1 \oplus B^1), \bigcup_{i \in I}(A_i^2 \oplus B^2), \bigcup_{i \in I}(A_i^3 \Theta B^3) \rangle$.

b) $B \widetilde{\bigoplus} \bigcup_{i \in I} A_i = \bigcup_{i \in I}(B \widetilde{\bigoplus} A_i) \Longrightarrow$

$\langle B^1 \oplus \bigcup_{i \in I} A_i^1, B^2 \oplus \bigcup_{i \in I} A_i^2, B^3 \Theta \bigcup_{i \in I} A_i^3 \rangle = \langle \bigcup_{i \in I}(B^1 \oplus A_i^1), \bigcup_{i \in I}(B^2 \oplus A_i^2), \bigcup_{i \in I}(B^3 \Theta A_i^3) \rangle$.

Type II:

a) $\bigcup_{i \in I} A_i \widetilde{\bigoplus} B = \bigcup_{i \in I}(A_i \widetilde{\bigoplus} B) \Longrightarrow$

$\langle \bigcup_{i \in I} A_i^1 \oplus B^1, \bigcup_{i \in I} A_i^2 \Theta B^2, \bigcup_{i \in I} A_i^3 \Theta B^3 \rangle = \langle \bigcup_{i \in I}(A_i^1 \oplus B^1), \bigcup_{i \in I}(A_i^2 \Theta B^2), \bigcup_{i \in I}(A_i^3 \Theta B^3) \rangle$.

Proof: a) we will prove this property for the two types of the neutrosophic crisp union operator:

Type I:

$\bigcup_{i \in I} A_i \widetilde{\bigoplus} B = \langle \bigcup_{b \in B}(\bigcup_{i \in I} A_{ib}^1), \bigcup_{b \in B}(\bigcup_{i \in I} A_{ib}^2), \bigcap_{b \in B}\big(\bigcup_{i \in I} A_{i(-b)}^3\big) \rangle =$

$\bigcup_{i \in I}(A_i \widetilde{\bigoplus} B) = \langle \bigcup_{i \in I}(\bigcup_{b \in B} A_{ib}^1), \bigcup_{i \in I}(\bigcup_{b \in B} A_{ib}^2), \bigcup_{i \in I}(\bigcap_{b \in B} A_{i(-b)}^3) \rangle$

Type II:

$\bigcup_{i \in I} A_i \widetilde{\bigoplus} B = \langle \bigcup_{b \in B}(\bigcup_{i \in I} A_{ib}^1), \bigcap_{b \in B}\big(\bigcup_{i \in I} A_{i(-b)}^2\big), \bigcap_{b \in B}\big(\bigcup_{i \in I} A_{i(-b)}^3\big) \rangle =$

$\bigcup_{i \in I}(A_i \widetilde{\bigoplus} B) = \langle \bigcup_{i \in I}(\bigcup_{b \in B} A_{ib}^1), \bigcup_{i \in I}(\bigcap_{b \in B} A_{i(-b)}^2, \bigcup_{i \in I}(\bigcap_{b \in B} A_{i(-b)}^3) \rangle$.

The proof of (b) is similar to (a)

5.2.4 Proposition (Duality Theorem of Neutrosophic Crisp Dilation):

Let $A, B \in \mathcal{N}C(Z^2)$ Neutrosophic Crisp Erosion and Dilation are Dual Operations i.e.

Type I:

$co\big(co\ A \widetilde{\bigoplus} B\big) = \langle co(coA^1 \oplus B^1), co(coA^2 \oplus B^2), co(coA^3 \Theta B^3) \rangle = \langle A^1 \Theta B^1, A^2 \Theta B^2, A^3 \oplus B^3 \rangle = A \widetilde{\Theta} B$.

Type II:

$co\big(co\ A \widetilde{\bigoplus} B\big) = \langle co(coA^1 \oplus B^1), co(coA^2 \Theta B^2), co(coA^3 \Theta B^3) \rangle = \langle A^1 \Theta B^1, A^2 \oplus B^2, A^3 \oplus B^3 \rangle = A \widetilde{\Theta} B$.

5.3 Properties of the Neutrosophic Crisp Opening Operator:

5.3.1 Proposition:

The neutrosophic crisp opening satisfies the monotonicity

$\forall\ A, B \in \mathcal{N}C(Z^2)$

Type I:

$A \subseteq B \Longrightarrow \langle A^1 \circ C^1, A^2 \circ C^2, A^3 \circ C^3 \rangle \subseteq \langle B^1 \circ C^1, B^2 \circ C^2, B^3 \circ C^3 \rangle$,

$A^1 \circ C^1 \subseteq B^1 \circ C^1, A^2 \circ C^2 \subseteq B^2 \circ C^2, A^3 \circ C^3 \supseteq B^3 \circ C^3$.

Type II:

$A \subseteq B \Longrightarrow \langle A^1 \circ C^1, A^2 \circ C^2, A^3 \circ C^3 \rangle \subseteq \langle B^1 \circ C^1, B^2 \circ C^2, B^3 \circ C^3 \rangle$,

$A^1 \circ C^1 \subseteq B^1 \circ C^1, A^2 \circ C^2 \supseteq B^2 \circ C^2, A^3 \circ C^3 \supseteq B^3 \circ C^3$

5.3.2 Proposition: for any family $(A_i|i \in I)$ in $\mathcal{N}C(Z^2)$, and $B \in \mathcal{N}C(Z^2)$

Type I:

$\bigcap_{i\in I} A_i \,\tilde{\circ}\, B = \bigcap_{i\in I}(A_i \,\tilde{\circ}\, B) \Longrightarrow$
$\langle \bigcap_{i\in I} A_i^1 \circ B^1, \bigcap_{i\in I} A_i^2 \circ B^2, \bigcap_{i\in I} A_i^3 \bullet B^3 \rangle = \langle \bigcap_{i\in I}(A_i^1 \circ B^1), \bigcap_{i\in I}(A_i^2 \circ B^2), \bigcap_{i\in I}(A_i^3 \bullet B^3) \rangle$

Type II:

$\bigcap_{i\in I} A_i \,\tilde{\circ}\, B = \bigcap_{i\in I}(A_i \,\tilde{\circ}\, B) \Longrightarrow$
$\langle \bigcap_{i\in I} A_i^1 \circ B^1, \bigcap_{i\in I} A_i^2 \bullet B^2, \bigcap_{i\in I} A_i^3 \bullet B^3 \rangle = \langle \bigcap_{i\in I}(A_i^1 \circ B^1), \bigcap_{i\in I}(A_i^2 \bullet B^2), \bigcap_{i\in I}(A_i^3 \bullet B^3) \rangle$

5.3.3 Proposition: for any family $(A_i|i \in I)$ in $\mathcal{N}C(Z^2)$ and $B \in \mathcal{N}C(Z^2)$

Type I:

$\bigcup_{i\in I} A_i \,\tilde{\circ}\, B = \bigcup_{i\in I}(A_i \,\tilde{\circ}\, B) \Longrightarrow$
$\langle \bigcup_{i\in I} A_i^1 \circ B^1, \bigcup_{i\in I} A_i^2 \circ B^2, \bigcup_{i\in I} A_i^3 \bullet B^3 \rangle = \langle \bigcup_{i\in I}(A_i^1 \circ B^1), \bigcup_{i\in I}(A_i^2 \circ B^2), \bigcup_{i\in I}(A_i^3 \bullet B^3) \rangle$.

Type II:

$\bigcup_{i\in I} A_i \,\tilde{\circ}\, B = \bigcup_{i\in I}(A_i \,\tilde{\circ}\, B) \Longrightarrow$
$\langle \bigcup_{i\in I} A_i^1 \circ B^1, \bigcup_{i\in I} A_i^2 \bullet B^2, \bigcup_{i\in I} A_i^3 \bullet B^3 \rangle = \langle \bigcup_{i\in I}(A_i^1 \circ B^1), \bigcup_{i\in I}(A_i^2 \bullet B^2), \bigcup_{i\in I}(A_i^3 \bullet B^3) \rangle$.

Proof: Is like the procedure used to prove the propositions given previously.

5.4 Properties of the Neutrosophic Crisp Closing

5.4.1 Proposition:

The neutrosophic closing satisfies the monotonicity

$\forall A, B \in \mathcal{N}C(Z^2)$

Type I:

a) $A \subseteq B \Longrightarrow \langle A^1 \bullet C^1, A^2 \bullet C^2, A^3 \bullet C^3 \rangle \subseteq \langle B^1 \bullet C^1, B^2 \bullet C^2, B^3 \bullet C^3 \rangle$

$A^1 \bullet C^1 \subseteq B^1 \bullet C^1, A^2 \bullet C^2 \subseteq B^2 \bullet C^2, A^3 \bullet C^3 \supseteq B^3 \bullet C^3$

Type II:

a) $A \subseteq B \Longrightarrow \langle A^1 \bullet C^1, A^2 \bullet C^2, A^3 \bullet C^3 \rangle \subseteq \langle B^1 \bullet C^1, B^2 \bullet C^2, B^3 \bullet C^3 \rangle$

$A^1 \bullet C^1 \subseteq B^1 \bullet C^1, A^2 \bullet C^2 \supseteq B^2 \bullet C^2, A^3 \bullet C^3 \supseteq B^3 \bullet C^3$

5.4.2 Proposition: for any family $(\mathrm{A_i}|\mathrm{i} \in \mathrm{I})$, in $\mathcal{N}\mathrm{C}(\mathrm{Z}^2)$, and $B \in \mathcal{N}\mathrm{C}(\mathrm{Z}^2)$

Type I:

$\bigcap_{i\in I} A_i \,\tilde{\bullet}\, B = \bigcap_{i\in I}(A_i \,\tilde{\bullet}\, B) \Longrightarrow$

$\langle \bigcap_{i\in I} A_i^1 \bullet B^1, \bigcap_{i\in I} A_i^2 \bullet B^2, \bigcap_{i\in I} A_i^3 \bullet B^3 \rangle = \langle \bigcap_{i\in I}(A_i^1 \bullet B^1), \bigcap_{i\in I}(A_i^2 \bullet B^2), \bigcap_{i\in I}(A_i^3 \bullet B^3) \rangle$

Type II:

$\bigcap_{i\in I} A_i \,\tilde{\bullet}\, B = \bigcap_{i\in I}(A_i \,\tilde{\bullet}\, B) \Longrightarrow$

$\langle \bigcap_{i\in I} A_i^1 \bullet B^1, \bigcap_{i\in I} A_i^2 \circ B^2, \bigcap_{i\in I} A_i^3 \circ B^3 \rangle = \langle \bigcap_{i\in I}(A_i^1 \bullet B^1), \bigcap_{i\in I}(A_i^2 \circ B^2), \bigcap_{i\in I}(A_i^3 \circ B^3) \rangle$.

5.4.3 Proposition: for any family$(\mathrm{A_i}|\mathrm{i} \in \mathrm{I})$, in $\mathcal{N}\mathrm{C}(\mathrm{Z}^2)$, and $B \in \mathcal{N}\mathrm{C}(\mathrm{Z}^2)$

Type I:

$\bigcup_{i\in I} A_i \,\tilde{\bullet}\, B = \bigcup_{i\in I}(A_i \,\tilde{\bullet}\, B) \Longrightarrow$

$\langle \bigcup_{i\in I} A_i^1 \bullet B^1, \bigcup_{i\in I} A_i^2 \bullet B^2, \bigcup_{i\in I} A_i^3 \circ B^3 \rangle = \langle \bigcup_{i\in I}(A_i^1 \bullet B^1), \bigcup_{i\in I}(A_i^2 \bullet B^2), \bigcup_{i\in I}(A_i^3 \circ B^3) \rangle$.

Type II:

$\bigcup_{i\in I} A_i \,\tilde{\bullet}\, B = \bigcup_{i\in I}(A_i \,\tilde{\bullet}\, B) \Longrightarrow$

$\langle \bigcup_{i\in I} A_i^1 \bullet B^1, \bigcup_{i\in I} A_i^2 \circ B^2, \bigcup_{i\in I} A_i^3 \circ B^3 \rangle = \langle \bigcup_{i\in I}(A_i^1 \bullet B^1), \bigcup_{i\in I}(A_i^2 \circ B^2), \bigcup_{i\in I}(A_i^3 \circ B^3) \rangle$.

Proof: Is similar to the procedure used to prove the propositions given previously.

5.4.4 Proposition (Duality theorem of Neutrosophic Crisp Closing**):**

Let A, B$A, B \in \mathcal{N}\mathrm{C}(\mathrm{Z}^2)$; Neutrosophic crisp erosion and dilation are dual operations i.e.

Type I:

$co(co\ \ A \,\tilde{\bullet}\, B) = \langle co(co\ \ A^1 \bullet B^1), co(co\ \ A^2 \bullet B^2), co(co\ \ A^3 \circ B^3) \rangle = \langle A^1 \circ B^1, A^2 \circ B^2, A^3 \bullet B^3 \rangle = A \,\tilde{\circ}\, B.$

Type II:

$co(co\ \ A \,\tilde{\bullet}\, B) = \langle co(co\ \ A^1 \bullet B^1), co(co\ \ A^2 \circ B^2), co(co\ \ A^3 \circ B^3) \rangle = \langle A^1 \circ B^1, A^2 \bullet B^2, A^3 \bullet B^3 \rangle = A \,\tilde{\circ}\, B.$

6. Neutrosophic Crisp Mathematical Morphological Filters:

6.1 Neutrosophic Crisp External Boundary:

Where A^1 is the set of all pixels that belong to the foreground of the picture, A^3 contains the pixels that belong to the background while contains those pixel which do not belong to neither A^2 nor A^1.

Let $A, B \in \mathcal{N}C(Z^2)$, such that $A = \langle A^1, A^2, A^3 \rangle$, and B is some structure element of the form $B = \langle B^1, B^2, B^3 \rangle$; then the NC boundary extraction filter is defined to be:

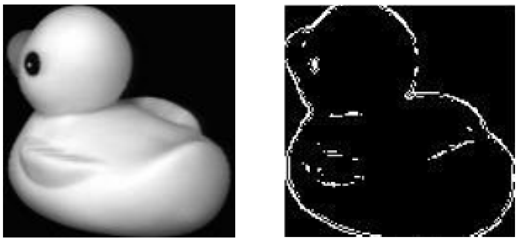
$\partial_1 A^1 = A^1 - (A^1 \Theta B^1)$,

$\partial_3 A^3 = (A^3 \Theta B^3) - A^3$,

$\partial(A) = A^2 - (\partial_1 A^1 \cup \partial_3 A^3)$,

$\partial^*(A) = A^2 - [(A^3 \oplus B^3) - (A^1 \Theta B^1)]$,

$b(A) = \partial^*(A) \cap \partial(A)$.



a) b)

Fig. 6.1: Applying the Neutrosophic Crisp External Boundary: a) the Original Image b) Neutrosophic Crisp Boundary.

6.2 Neutrosophic Crisp Top-Hat Filter:

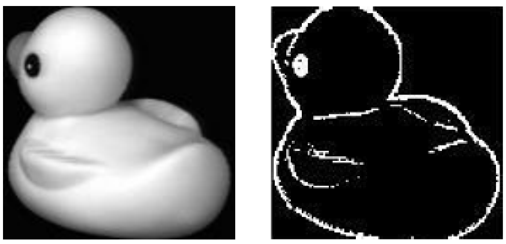
$B_1(A^1) = A^1 - (A^1 \circ B^1)$,

$B_3(A^3) = (A^3 \bullet B^3) - A^1$,

$B(A) = A^2 - (B_1(A^1) \cup B_3(A^3))$,

$B^*(A) = A^2 - [(A^1 \circ B^1) - (A^3 \bullet B^3)]$,

$T\tilde{o}p_{hat}(A) = B(A) \cap B^*(A)$.



a) b)

Fig. 6.2.: Applying the Neutrosophic Crisp Top-Hat Filter: a) Original Image b) Neutrosophic Crisp Components $\langle A^1, A^2, A^3 \rangle$ Respectively.

6.3 Bottom-Hat Filter:

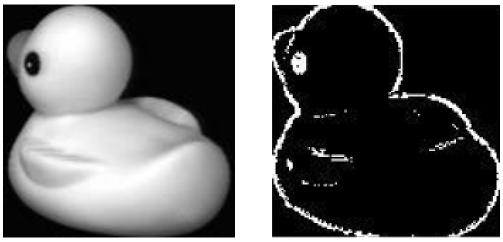
$B_1(A^1) = (A^1 \bullet B^1) - A^1,$

$B_3(A^3) = A^3 - (A^3 \circ B^3),$

$B(A) = A^2 - (B_1(A^1) \cup B_3(A^3)),$

$B^*(A) = A^2 - [(A^1 \bullet B^1) - (A^3 \circ B^3)],$

$Bo\widetilde{tt}omp_{hat}(A) = B(A) \cap B^*(A).$



a) b)

Fig. 6.3.: Applying the Neutrosophic Crisp Bottom-Hat Filter: Neutrosophic Crisp Components $\langle A^1, A^2, A^3 \rangle$ Respectively.

The following diagram represents the relationship between all types of mathematical morphology

$$\begin{array}{ccc} \textbf{Crisp Mathematical Morphology} & \longrightarrow & \textbf{Neutrosophic Crisp Mathematical Morphology} \\ \downarrow & & \downarrow \\ \textbf{Fuzzy Mathematical Morphology} & \longrightarrow & \textbf{Neutrosophic Fuzzy Mathematical Morphology} \end{array}$$

7. Conclusion:

In our work, we have proposed a new technique for analyzing and processing images; either grayscale or binary. The technique is a generalization for the fuzzy and crisp mathematical morphology; it handles the image in the neutrosophic domain.in such domain the image is analyzed into three different layers; the first layer describes how much each pixel belongs to the white set, and the third layer describes how much each pixel belongs to the non-white (black) set. In contrast, the second layer describes how much the pixel is neither white nor black. The properties of each layer were used to define the basic operations for what we called "Neutrosophic Mathematical Morphology". mainly, we introduced four basic operations; namely, the neutrosophic dilation, the neutrosophic erosion, the neutrosophic closing and the neutrosophic opening. The algebraic properties of the proposed operation were discussed. Furthermore, we introduced some advanced and generalized concepts of classical and fuzzy mathematical morphology. For this purpose, we developed serval neutrosophic crisp and fuzzy morphological operators; namely, the neutrosophic fuzzy and crisp dilation, the neutrosophic fuzzy and crisp erosion, the neutrosophic crisp opening and the neutrosophic crisp closing operators. These operators were presented in two different types, each type is determined according to the behaviour of the second component of the triple structure of the operator. Furthermore, we developed three neutrosophic crisp morphological filters; namely, the neutrosophic fuzzy and crisp boundary extraction. Some promising experimental results were presented to visualise the effect of the newly introduced operators and filters on the image in the neutrosophic fuzzy and crisp domain instead of the spatial domain.

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Multi-Attribute Group Decision-Making Based on Aggregation Operator and Score Function of Bipolar Neutrosophic Hypersoft Environment

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Abstract. Hypersoft sets (HSSs) have gained popularity because of their ability to formulate data in the form of several trait-valued disjoint sets that blend various traits. Motivated by this idea, in this study, we present a new hyper-approach referred to as bipolar neutrosophic hypersoft sets (BNHSSs) by a generalization of neutrosophic hypersoft sets (NHSSs) and bipolar fuzzy hypersoft sets (BFHSSs) or by merging and subjecting both HSSs and neutrosophic sets (NSs) to a bipolarity property of real numbers. By utilizing positive and negative neutrosophic structures, we construct different notions and operations on the basis of BNHSSs, such as an absolute BNHSS, a null BNHSS, a complement, subset-hood, a restricted and extended union, and a restricted and extended intersection, along with their related properties. Also, some operations like OR and AND on BNHSS have been initiated. In addition, some properties are displayed paired together, and some numerical hypothetical examples are given to clarify the mechanism of using these instruments. Finally, to prove the efficiency and applicability of the proposed model, we established two novel algorithms based on mathematical techniques (aggregation operator and score function) applied to our model (BNHSS). The aforementioned methods have been utilized in the resolution of a multi-attribute group decision-making (MAGDM) problem. Some discussions and comparisons between the given techniques are also presented to demonstrate their effectiveness and applicability.

Keywords: Bipolar fuzzy Set, Bipolar Neutrosophic Set, Hypersoft Set, Neutrosophic Set, Neutrosophic Hypersoft Set, Soft Set.

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1. General Introduction

In the course of our daily routines, we frequently meet a multitude of circumstances that present dual perspectives or facets of information: the first is positive and the second is negative, and here the human mind tends towards two patterns of thinking: positive thinking and negative thinking in judging such situations. Where the positive character indicates that the information to be evaluated is satisfactory or desirable, while the negative character indicates rejection or negativity of the choice.

On the other hand, multi-attribute group decision-making (MAGDM) methods seek analysis and evaluation of real-life issues that face the human mind and contain full uncertainty nature, including a positive and negative nature, in order to help the decision maker (user) in selecting the best object. In order to handle MCDM issues that contain imprecise and two sides of information (positive and negative information), Zhang introduced a new mathematical approach named bipolar fuzzy sets (BFSs) as an extension of the range of fuzzy set memberships from positive degrees to positive and negative degrees. This concept is characterized by the bifurcation of the fuzzy memberships into two poles, positive membership $\mu^+ : A \to [1, 0]$ correspond with positive preferences and desires and negative membership $\mu^- : A \to [0, -1]$ corresponds to a lack of preference and a rejection rate.

2. Literature Review

To handle the complicated MAGDM issues that contain uncertainty, indeterminacy, and consistency, Smarandache [1] proposed a new mathematical notion known as neutrosophic set (NS) by developing the ordinary fuzzy set [2] (FS) and the ordinary intuitionistic fuzzy set [3] (IFS). A neutrosophic set (NS) [4] structure is made up of three functions: truthness, indeterminacy, and falsity functions. Each element in the universal set corresponds to three membership functions, all of which lie in the closed interval [0, 1]. For decades, this novel notion has been used successfully to model uncertainty in several fields such as reasoning, control, pattern recognition, decision making, and computer vision. The NS has been extended and studied by many researchers in various fields, for example. Khalifa and Kumar [5, 6] presented novel approaches regarding trapezoidal neutrosophic numbers and linear fractional programming, respectively, under an interval-value neutrosophic environment. Sallam and Mohamed [7] utilized N-MCDM Methodology for the examination of onshore wind for electricity generation. Nishtar and Afzal [8] work on an analysis of a system for multiple combining schemes. Rodrigo and Maheswari [9] introduced properties and characterizations of a new idea of Ne-mapping namely Neu-open maps and Ne-closed maps in Ne-topological spaces. Researchers did not stop developing this concept at the real level, but rather creative works continued to the complex level, taking into account the importance and characteristics of the complex level.

Ali and Smarandache [10] have further extended the NS to the complex field by developing the notion of complex neutrosophic set (CNS), which is progressed rapidly to complex single-valued neutrosophic set (CSVNS) [11] and Q-complex neutrosophic set (Q-CNS) [12]. In addition to other studies dealing with supply chain (SC) networks [13], facing challenges for a sustainable future, and using logistic systems [14].

In 1999, Molodtsov [15] put forth the notion of a soft set (SS) as a new parametric form when he noticed a gap in the previous concepts, that is their inability to deal with real-world data in the parametric environment. The fertile hybrid environment provided by the SS provoked the attention of researchers and prompted them to create a great deal of contributions by merging the previous concepts with the properties of the SS. Maji [16] introduced and studied the basic definitions and operations of neutrosophic soft set (NSS). Deli and Broumi [17] introduced a preference relationships technique on NSSs that allows to amalgamate two NSSs. Deli [18]again developed a forecasting approach based interval-NSS. Ozturk et al. [19] introduced and studied some definitions and theorems on NS in topological spaces. Saeed et al. [20] applied similarity and distance measures on multi-polar neutrosophic soft set (mpNSS) and experimented it to handle some medical diagnosis and DM-problems. Broumi et al. [21] smelted both SS and NS to produce the idea of complex neutrosophic soft sets (CNSSs). Following them Al-Sharqi et al. [22]- [32] made a great effort to represent the idea of Bromi et al. in an interval manner. Abdel-Basset et al [33] developed a novel risk assessment framework, called RAF-CPWS, which works perfectly to estimate the risks of water and wastewater technologies. In addition, there are contributions in several fields see [34]- [43]. In some practical scenarios, traits that provide further elaboration of the choices should be separated into trait values to provide more clarity. In light of this intent , recently, Smarandache [44, 45] has suggested the HSS as an upgraded structure of the SS. Also, he clarified the mechanism of performance of this idea with FS and its extension. According to this idea, Samarandache opened the doors to develop previous models that built on SS by rehashing it into multi-trait function. At present, scholars have released several studies on HSSs. Saeed et al. [46, 47] developed fundamental HSS operations. Yolcu and Ozturk [48] prepared critical decision-making applications for fuzzy hypersoft (FHSS). Saeed et al. [49] conceptualized the notion of FHSS under interval form when they established the notion of interval-FHSS. More results were shown on IFHSS by Yolcu et. al. [50]. Some mathematical measures on neutrosophic hypersoft set (NHSS) were demonstrated by Saqlain et al. [51].

On the opposite side, the principle of bipolarity craeted to handel practical challenges encountered in everyday life, which are given by two distinct aspects. namily positive aspect and negative aspect, such as black and white, return and progress, profit and loss and et. Then, Zhang [52, 53] is the first initiaed the idea of bipolar fuzzy set (BFS) when he extension of the range of fuzzy set memberships from positive degrees to positive and negative degrees. This concept is characterized by the bifurcation of the fuzzy memberships into two poles, positive membership $\mu^+ : A \to [1, 0]$ correspond with positive preferences and desires and negative membership $\mu^- : A \to [0, -1]$ corresponds to a lack of preference and a rejection rate. Naz and Shabir [54] built some algebraic structure on fuzzy bipolar soft set (BFSS).

The idea of bipolar soft set (BSS) has been redefined by Karaaslan and Karatas [55]. Mahmood [56] improved the previous definitions of BSS by establishing the notion of T-bipolar soft sets which is more close to the concept of bipolarity as compared to the previous ones. Jana and Pal [57] applied the bipolarity information on IFSS. Deli et al. [58] elaborated on the notion of bipolar-NS (BNS). Ali et al. [59] presented bipolar-NSS (BNSS) and trailed it to decipher decision-making problems. In complex space, Mahmood and Rehman [60] first proposed an approach to bipolar complex fuzzy sets (BCFSs), which is closer to bipolarity When comparing this model with other models. Then, Aczel-Alsina aggregation operators applied by Mahmood et al. [61] on bipolar complex fuzzy information to handle MCGDM issues. Following in this direction, Al-Quran et al. [62] established the concept of complex bipolar- valued NSS as a hybrid model of BNSS and complex fuzzy set (CFS).

Recently, the concept of BSS was expanded to the bipolar hypersoft set (BHSS) by Musa and Asaad [63], and they presented some basec algebraic properties. Following this direction, Al-Quran et al. [64] extended the notion of BHSS to BFHSSs. However, BFHSSs can only handle uncertain data but not be able todeal with ambiguous, contradictory, and indeterminate information which usually results in real-life problems. To adapt to such situations, we propose a new hybrid approach, namely BNHSS, By combining the qualities that distinguish BNS and HSS from each other. BNHSS is superior to BFHSS with its three independent membership functions, which play a role in increasing the accuracy of the end decision. Therefore, the advantages and benefits of the suggested method are shown as follows. Firstly, BNHSS exhibits a high level of applicability in real-life scenarios when decision-makers seek to address dualistic or dichotomous judgemental thinking, encompassing both positive and negative perspectives. Secondly, the purpose of this study is to include the concept of bipolarity into decision-making processes through the utilization of the HSS, the HSS is equipped with a parameterization tool that enables the portrayal of sub-divided features in a more comprehensive and thorough manner. Thirdly, another advantage is the inclusion of the neutrosophic set, which possesses the capacity to simultaneously analyze and handle truth, indeterminate, and false information in order to facilitate decision-making. Finally, the suggested model incorporates all of the aforementioned components into a single framework, rendering it more suitable for addressing decision-making challenges that are not amenable to other existing decision-making models.

This article is split into the following parts: Figure 1:

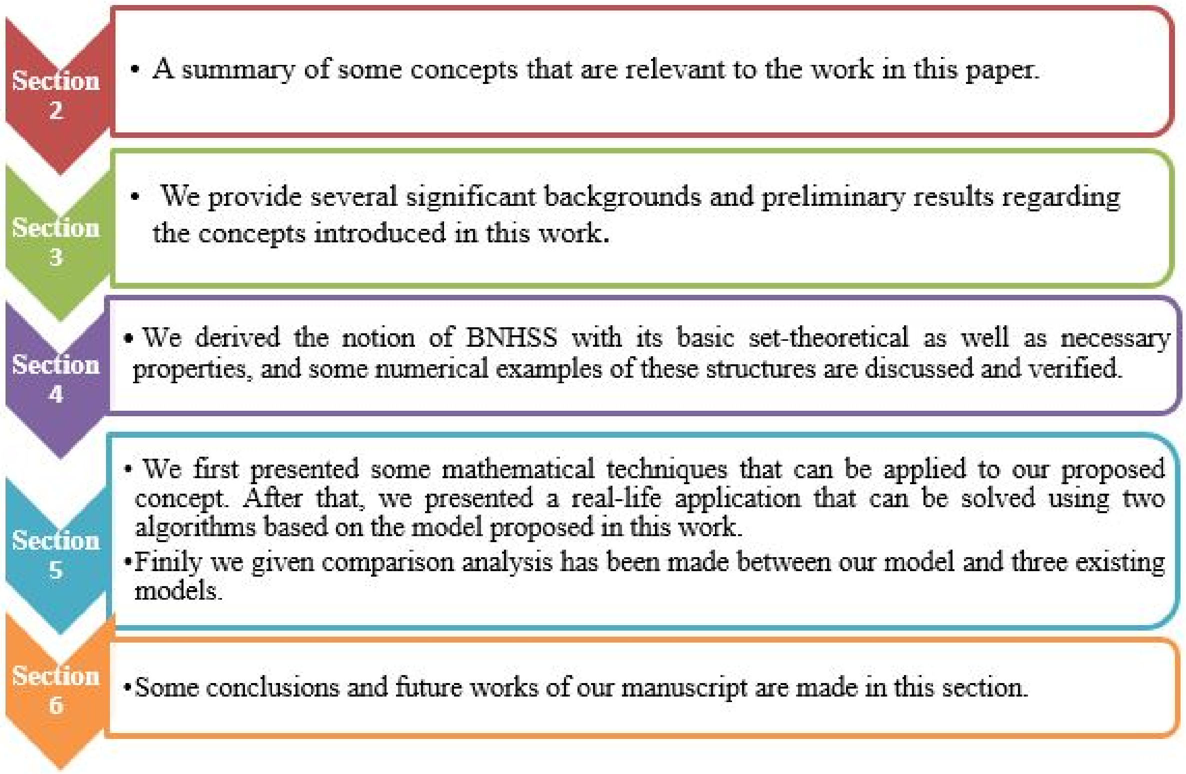

FIGURE 1. show how we organize our manuscript in a brief way.

3. Preliminaries

This part revised some ideas connected to the suggested work. We review SS, HSS, BNS, BNSS and NHSS.

Molodtsov [8] defined the idea of SS as a set-valued map that helps the user describe objects by utilizing many parameters.

Definition 3.1. [8] A SS $\left(\hat{\mathcal{G}}, \mathcal{A}\right)$ on $\hat{\mathfrak{C}}$ non-empty universal set is represented as a mapping as follows:

$$\hat{\mathcal{G}} : \mathcal{A} \to \widehat{\mathbb{P}}\left(\hat{\mathfrak{C}}\right)$$

where $\widehat{\mathbb{P}}\left(\hat{\mathfrak{C}}\right)$ is the power set of $\hat{\mathfrak{C}}$ and here both $\hat{\mathfrak{C}}$ and $\mathcal{A} \subseteq \mathfrak{M}$ refer to the non-empty universal set and the parameter family respectively.

Smarandache [44] expanded the idea of SS to HSS by modifying the function to incorporate many attributes.

Definition 3.2. [44] A HSS structures $\left(\hat{\mathcal{G}}, \mathcal{W} = \mathcal{W}_1 \times \mathcal{W}_2 \times \mathcal{W}_3 \times ... \times \mathcal{W}_n\right)$ on the non-empty universal set $\hat{\mathfrak{C}}$ portrayed as follows:
$\{(\nu, \hat{\mathcal{G}}(\nu)) : \hat{\mathcal{G}}(\nu) \subseteq \hat{\mathfrak{C}}, \forall \nu \in \mathcal{W} = \mathcal{W}_1 \times \mathcal{W}_2 \times \mathcal{W}_3 \times ... \times \mathcal{W}_n \subseteq \mathfrak{A} = \mathfrak{A}_1 \times \mathfrak{A}_2 \times \mathfrak{A}_3 \times ... \times \mathfrak{A}_n\}$, where $\mathfrak{A}_i : i = 1, 2, ..., n$ are separate sets of parameters terms and $W_i \subseteq \mathfrak{A}_i, \forall i = 1, 2, ..., n$.

Definition 3.3. [1] A NS structures

$$\hat{\mathcal{S}} = \{< \hat{\varkappa}; \mathcal{T}_{\mathcal{S}}(\hat{\varkappa}), \mathcal{I}_{\mathcal{S}}(\hat{\varkappa}), \mathcal{F}_{\mathcal{S}}(\hat{\varkappa}) >: \hat{\varkappa} \in \mathcal{X}\},$$

on non-empty universal set $\mathcal{X}$ called neutrosophic set (NS),

where $\mathcal{T}_{\mathcal{S}};\ \mathcal{I}_{\mathcal{S}};\ \mathcal{F}_{\mathcal{S}} : \mathcal{X} \to]^{-}0; 1^{+}[$ denoted to the TM,IM and FM of any object $\hat{\varkappa} \in \mathcal{X}$, respectively with $^{-}0 \leq \mathcal{T}_{\mathcal{S}} + \mathcal{I}_{\mathcal{S}} + \mathcal{F}_{\mathcal{S}} \leq 3^{+}$.

Deli et al. [59] generalized BFS by defining BNS as follows.

Definition 3.4. [59] The BNS $\mathbb{A}$ on the universe $\mathfrak{C}$ is signified as follows.

$\mathbb{A} = \{\langle \mathfrak{c}; \mathcal{T}^{+}_{\mathbb{A}}(\hat{\varkappa}), \mathcal{T}^{-}_{\mathbb{A}}(\hat{\varkappa}), \mathcal{I}^{+}_{\mathbb{A}}(\hat{\varkappa}), \mathcal{I}^{-}_{\mathbb{A}}(\hat{\varkappa}), \mathcal{F}^{+}_{\mathbb{A}}(\hat{\varkappa}), \mathcal{F}^{-}_{\mathbb{A}}(\hat{\varkappa})\rangle : \hat{\varkappa} \in \mathfrak{C}\}$, where, $\mathcal{T}^{+}, \mathcal{I}^{+}, \mathcal{F}^{+} : \mathfrak{C} \to [0, 1]$ denote, respectively the positive-TM, positive-IM and positive-FM degrees of an element $\hat{\varkappa} \in \mathfrak{C}$ to the property in line with a BNS $\mathbb{A}$, and $\mathcal{T}^{-}, \mathcal{I}^{-}, \mathcal{F}^{-} : \mathfrak{C} \to [-1, 0]$ denote, respectively the negative-TM, negative-IM and negative-FM degrees of an object $\hat{\varkappa} \in \mathfrak{C}$.

Ali et al. [59] defined BNSS and its fundamental operations as in the following two definitions.

Definition 3.5. [59] A structures $(\mathbb{F}, \mathbb{A})$ is called a BNSS over the universe $\mathfrak{C}$, where $\mathbb{F}$ is a transformation given by $\mathbb{F} : \mathbb{A} \longrightarrow BN(\mathfrak{C})$ and $BN(\mathfrak{C})$ refers to the set of all bipolar neutrosophic subsets of $\mathfrak{C}$.

Definition 3.6. [59] Suppose $(\mathbb{F}, \mathbb{A})$ and $(\mathbb{G}, \mathbb{B})$ are two BNSSs over the non-empty universal set $\mathfrak{C}$, then $(\mathbb{F}, \mathbb{A})$ is given as:
$\mathbb{F}(\mathfrak{a}) = \Big\{\langle \hat{\varkappa}, \{\mathcal{T}^{+}_{\mathbb{F}(\mathfrak{a})}(\hat{\varkappa}), \mathcal{T}^{-}_{\mathbb{F}(\mathfrak{a})}(\hat{\varkappa}), \mathcal{I}^{+}_{\mathbb{F}(\mathfrak{a})}(\hat{\varkappa}), \mathcal{I}^{-}_{\mathbb{F}(\mathfrak{a})}(\hat{\varkappa}), \mathcal{F}^{+}_{\mathbb{F}(\mathfrak{a})}(\hat{\varkappa}),$
$\mathcal{F}^{-}_{\mathbb{F}(\mathfrak{a})}(\hat{\varkappa})\}\rangle : \forall \hat{\varkappa} \in \mathfrak{C}, \mathfrak{a} \in \mathbb{A}\Big\}$ and the second BNSS $(\mathbb{G}, \mathbb{B})$ is given as $\mathbb{G}(\mathfrak{b}) = \Big\{\langle \hat{\varkappa}, \{\mathcal{T}^{+}_{\mathbb{G}(\mathfrak{b})}(\hat{\varkappa}), \mathcal{T}^{-}_{\mathbb{G}(\mathfrak{b})}(\hat{\varkappa}), \mathcal{I}^{+}_{\mathbb{G}(\mathfrak{b})}(\hat{\varkappa}), \mathcal{I}^{-}_{\mathbb{G}(\mathfrak{b})}(\hat{\varkappa}), \mathcal{F}^{+}_{\mathbb{G}(\mathfrak{b})}(\hat{\varkappa}), \mathcal{F}^{-}_{\mathbb{G}(\mathfrak{b})}(\hat{\varkappa})\}\rangle : \forall \hat{\varkappa} \in \mathfrak{C}, \mathfrak{b} \in \mathbb{B}\Big\}$. Then,

(1.) $\mathbb{F}^{c}(\mathfrak{a}) = \Big\{\langle \hat{\hat{\varkappa}}, \{\mathcal{F}^{+}_{\mathbb{F}(\mathfrak{a})}(\hat{\hat{\varkappa}}), \mathcal{F}^{-}_{\mathbb{F}(\mathfrak{a})}(\hat{\hat{\varkappa}}), 1 - \mathcal{I}^{+}_{\mathbb{F}(\mathfrak{a})}(\hat{\hat{\varkappa}}), -1 - \mathcal{I}^{-}_{\mathbb{F}(\mathfrak{a})}(\hat{\hat{\varkappa}}),$
$\mathcal{T}^{+}_{\mathbb{F}(\mathfrak{a})}(\hat{\hat{\varkappa}}), \mathcal{T}^{-}_{\mathbb{F}(\mathfrak{a})}(\hat{\hat{\varkappa}})\}\rangle : \forall \hat{\varkappa} \in \mathfrak{C}, \mathfrak{a} \in \mathbb{A}\Big\}$,

(2.) $\mathbb{F}(\mathfrak{a}) \subseteq \mathbb{G}(\mathfrak{b})$ iff:
$\mathcal{T}^{+}_{\mathbb{F}(\mathfrak{a})}(\hat{\varkappa}) \leq \mathcal{T}^{+}_{\mathbb{G}(\mathfrak{b})}(\hat{\varkappa}),\ \mathcal{T}^{-}_{\mathbb{F}(\mathfrak{a})}(\hat{\varkappa}) \geq \mathcal{T}^{-}_{\mathbb{G}(\mathfrak{b})}(\hat{\varkappa}),\ \mathcal{I}^{+}_{\mathbb{F}(\mathfrak{a})}(\hat{\varkappa}) \geq \mathcal{I}^{+}_{\mathbb{G}(\mathfrak{b})}(\hat{\varkappa}),$
$\mathcal{I}^{-}_{\mathbb{F}(\mathfrak{a})}(\hat{\varkappa}) \leq \mathcal{I}^{-}_{\mathbb{G}(\mathfrak{b})}(\hat{\varkappa}),\ \mathcal{F}^{+}_{\mathbb{F}(\mathfrak{a})}(\hat{\varkappa}) \geq \mathcal{F}^{+}_{\mathbb{G}(\mathfrak{b})}(\hat{\varkappa}), \mathcal{F}^{-}_{\mathbb{F}(\mathfrak{a})}(\hat{\varkappa}) \leq \mathcal{F}^{-}_{\mathbb{G}(\mathfrak{b})}(\hat{\varkappa}),$

(3.) $\mathbb{F}(\mathfrak{a}) \cup \mathbb{G}(\mathfrak{b}) = \{\langle \hat{\varkappa}, \mathcal{T}^{+}_{\mathbb{F}(\epsilon)}(\hat{\varkappa}) \vee \mathcal{T}^{+}_{\mathbb{G}(\epsilon)}(\hat{\varkappa}),\ \mathcal{T}^{-}_{\mathbb{F}(\epsilon)}(\hat{\varkappa}) \wedge \mathcal{T}^{-}_{\mathbb{G}(\epsilon)}(\hat{\varkappa}),$
$\mathcal{I}^{+}_{\mathbb{F}(\epsilon)}(\hat{\varkappa}) \wedge \mathcal{I}^{+}_{\mathbb{G}(\epsilon)}(\hat{\varkappa}),\ \mathcal{I}^{-}_{\mathbb{F}(\epsilon)}(\hat{\varkappa}) \vee \mathcal{I}^{-}_{\mathbb{G}(\epsilon)}(\hat{\varkappa}),\ \mathcal{F}^{+}_{\mathbb{F}(\epsilon)}(\hat{\varkappa}) \wedge \mathcal{F}^{+}_{\mathbb{G}(\epsilon)}(\hat{\varkappa}),$
$\mathcal{F}^{-}_{\mathbb{F}(\epsilon)}(\hat{\varkappa}) \vee \mathcal{F}^{-}_{\mathbb{G}(\epsilon)}(\hat{\varkappa})\rangle : \forall \hat{\varkappa} \in \mathfrak{C}, \epsilon \in \mathbb{A} \cap \mathbb{B}\},$

(4.) $\mathbb{F}(\mathfrak{a}) \cap \mathbb{G}(\mathfrak{b}) = \{\langle \hat{\varkappa}, \mathcal{T}^{+}_{\mathbb{F}(\epsilon)}(\hat{\varkappa}) \wedge \mathcal{T}^{+}_{\mathbb{G}(\epsilon)}(\hat{\varkappa}),\ \mathcal{T}^{-}_{\mathbb{F}(\epsilon)}(\hat{\varkappa}) \vee \mathcal{T}^{-}_{\mathbb{G}(\epsilon)}(\hat{\varkappa}),$
$\mathcal{I}^{+}_{\mathbb{F}(\epsilon)}(\hat{\varkappa}) \vee \mathcal{I}^{+}_{\mathbb{G}(\epsilon)}(\hat{\varkappa}),\ \mathcal{I}^{-}_{\mathbb{F}(\epsilon)}(\hat{\varkappa}) \wedge \mathcal{I}^{-}_{\mathbb{G}(\epsilon)}(\hat{\varkappa}),\ \mathcal{F}^{+}_{\mathbb{F}(\epsilon)}(\hat{\varkappa}) \vee \mathcal{F}^{+}_{\mathbb{G}(\epsilon)}(\hat{\varkappa}),$
$\mathcal{F}^{-}_{\mathbb{F}(\epsilon)}(\hat{\varkappa}) \wedge \mathcal{F}^{-}_{\mathbb{G}(\epsilon)}(\hat{\varkappa})\rangle : \forall \hat{\varkappa} \in \mathfrak{C}, \epsilon \in \mathbb{A} \cap \mathbb{B}\},$

where max $= \vee$ and min $= \wedge$.

NHSS is defined for the first time by Smarandache [50] in the following manner.

Definition 3.7. [50] A NHSS structures $\left(\hat{\mathcal{H}}, \mathcal{W} = \mathcal{W}_1 \times \mathcal{W}_2 \times \mathcal{W}_3 \times ... \times \mathcal{W}_n\right)$ on the non-empty universal set $\hat{\mathfrak{C}}$ portrayed as a mapping as follows:

$$\hat{\mathcal{H}} : \mathcal{W} \longrightarrow NH(\hat{\mathfrak{C}})$$

where the component $NH(\hat{\mathfrak{C}})$ refer to a family of all NSs over non-empty universal set $\hat{\mathfrak{C}}$ such that $\hat{\mathcal{H}}(\nu) = \{(\hat{\varkappa}, \mathfrak{T}_{\hat{\mathcal{H}}(\nu)}(\hat{\varkappa}), \mathfrak{I}_{\hat{\mathcal{H}}(\nu)}(\hat{\varkappa}), \mathfrak{F}_{\hat{\mathcal{H}}(\nu)}(\hat{\varkappa})) : \hat{\varkappa} \in \mathfrak{C}, \nu \in W = W_1 \times W_2 \times W_3 \times ... \times W_n \subseteq \mathfrak{A} = \mathfrak{A}_1 \times \mathfrak{A}_2 \times \mathfrak{A}_3 \times ... \times \mathfrak{A}_n\}$,
such that $\mathfrak{T}_{\hat{\mathcal{H}}(\nu)}(\hat{\varkappa}), \mathfrak{I}_{\hat{\mathcal{H}}(\nu)}(\hat{\varkappa})$ and $\mathfrak{F}_{\hat{\mathcal{H}}(\nu)}(\hat{\varkappa})$ are the TM, IM and FM, respectively and $\mathfrak{A}_i : i = 1, 2, ..., n$ are pairwise disjoint sets of attribute values.

Recently, Al-Quran et al. [64] have extended the notions of HSS and BFSS by introducing the notion of BFHSS as in the following definition.

Definition 3.8. [64] A BFHSS structures (Φ, Λ) on the non-empty universal set $\hat{\mathfrak{C}}$ portrayed as a mapping as follows:

$$\Phi : \Lambda \to P\left(\hat{\mathfrak{C}}\right)$$

and written as: $(\Phi, \Lambda) = \{\langle \alpha, \{(\hat{m}, \mathcal{T}^{+}_{\Phi(\alpha)}(\hat{m}), \mathcal{T}^{-}_{\Phi(\alpha)}(\hat{m})) : \forall \hat{m} \in \hat{\mathfrak{C}}\}\rangle : \alpha \in \Lambda \subseteq \Delta\}$.
where $\Lambda = \mathcal{J}_{\nu_1} \times \mathcal{J}_{\nu_1} \times ... \times \mathcal{J}_{\nu_n}$, $\Delta = \mathcal{H}_{\nu_1} \times \mathcal{H}_{\nu_1} \times ... \times \mathcal{H}_{\nu_n}$,and $P\left(\hat{\mathfrak{C}}\right)$ indicated to power of non-empty universal set $\hat{\mathfrak{C}}$.

4. Bipolar Neutrosophic Hypersoft Set

This section of our work consists of presenting the primary definition of BNHSS along with some illustrations and hypothetical examples, basic set theory operations, and some rudimentary properties.

Definition 4.1. Let $\hat{\chi}$ be a universal set and $\widehat{p}\ (\hat{\chi})$ denotes to the powerful set of $\hat{\chi}$. Let $\mu_k : k = 1, 2, ..., m$ are m-well-defined qualities that are in line with characteristics and facets values respectively , the pairwise disjoint sets $\mathcal{C}_{\mu_k}$: $k = 1, 2, ..., m$. Let $\mathcal{D}_{\mu_k}$ be the nonempty subset of $\mathcal{C}_{\mu_k}$ $\forall k = 1, 2, ..., m$.
A BNHS (Ψ, Γ) is identified by the following mapping $\Psi : \Gamma \rightarrow \widehat{p}\ (\hat{\chi})$ whose functional value is the BNS
$\Psi(\nu) = \Big\{\langle \hat{\varkappa}, \{\mathcal{T}^{+}_{\Psi(\nu)}(\hat{\varkappa}), \mathcal{T}^{-}_{\Psi(\nu)}(\hat{\varkappa}), \mathcal{I}^{+}_{\Psi(\nu)}(\hat{\varkappa}), \mathcal{I}^{-}_{\Psi(\nu)}(\hat{\varkappa}), \mathcal{F}^{+}_{\Psi(\nu)}(\hat{\varkappa}), \mathcal{F}^{-}_{\Psi(\nu)}(\hat{\varkappa})\}\rangle : \forall \hat{\varkappa} \in \hat{\chi}, \nu \in \Gamma \subseteq \Omega\Big\}$,
where $\Gamma = \mathcal{D}_{\mu_1} \times \mathcal{D}_{\mu_2} \times ... \times \mathcal{D}_{\mu_m}$ and $\Omega = \mathcal{C}_{\mu_1} \times \mathcal{C}_{\mu_2} \times ... \times \mathcal{C}_{\mu_m}$ such that $\mathcal{T}^{+}, \mathcal{I}^{+}, \mathcal{F}^{+} : \chi \rightarrow [0, 1]$ denote, respectively the positive-TM, positive-IM and positive-FM degrees of the attribute ν_* with regard to component $\hat{\varkappa}_*$ for the property in line with a BNHS (Ψ, Γ), while $\mathcal{T}^{-}, \mathcal{I}^{-}, \mathcal{F}^{-} : \chi \rightarrow [-1, 0]$ denote, respectively the negative-TM, negative-IM and negative-FM degrees of some implicit counter-property of the attribute ν_* with regard to component $\hat{\varkappa}_*$ line with a a BNHS (Ψ, Γ).

We can view the BNHS (Ψ, Γ) as follows:
$(\Psi, \Gamma) = \Big\{\Big\langle \nu, \big\{\big(\hat{\varkappa}, \mathcal{T}^{+}_{\Psi(\nu)}(\hat{\varkappa}), \mathcal{T}^{-}_{\Psi(\nu)}(\hat{\varkappa}), \mathcal{I}^{+}_{\Psi(\nu)}(\hat{\varkappa}), \mathcal{I}^{-}_{\Psi(\nu)}(\hat{\varkappa}),$
$\mathcal{F}^{+}_{\Psi(\nu)}(\hat{\varkappa}), \mathcal{F}^{-}_{\Psi(\nu)}(\hat{\varkappa})\big) : \forall \hat{\varkappa} \in \hat{\chi}\big\}\Big\rangle : \nu \in \Gamma \subseteq \Omega\Big\}$.

The following numerical example makes above definition clear.

Example 4.2. Suppose the alternatives set encompasses three mobile phones of the same brand $\hat{\chi} = \{\hat{\varkappa}_1, \hat{\varkappa}_2, \hat{\varkappa}_3\}$ and the attributes are μ_1 = Price, μ_2 = Camera resolution, μ_3 = RAM size.Suppose the attribute's values are
$\mathcal{C}_{\mu_1} = \{\alpha_1 = 1200, \alpha_2 = 1500, \alpha_3 = 2000\}$, $\mathcal{C}_{\mu_2} = \{\alpha_4 = 8MP, \alpha_5 = 12MP, \alpha_6 = 16MP\}$, $\mathcal{C}_{\mu_3} = \{\alpha_7 = 6GB, \alpha_8 = 8GB, \alpha_9 = 12GB\}$. If we take the subset $\mathcal{D}_{\mu_k}$ of $\mathcal{C}_{\mu_k}$ $\forall k = 1, 2, 3$ as follows.

$\mathcal{D}_{\mu_1} = \{\alpha_2 = 1500, \alpha_3 = 2000\}$,
$\mathcal{D}_{\mu_2} = \{\alpha_5 = 12MP\}$, $\mathcal{D}_{\mu_3} = \{\alpha_7 = 6GB, \alpha_8 = 8GB\}$. Then, we obtain the following BNHS (Ψ, Γ)

$(\Psi, \Gamma) =$
$\Big\{\Big\langle\big((\alpha_2, \alpha_5, \alpha_7), \{(\hat{\varkappa}_1, .6, -.1, .5, -.9, .8, -.1), (\hat{\varkappa}_2,$
$.7, -.4, .1, -.2, .7, -.5), (\hat{\varkappa}_3, .6, -.4, .6, -.4, .5,$
$-.7)\}\big), \big((\alpha_2, \alpha_5, \alpha_8), \{(\hat{\varkappa}_1, .5, -.2, .2, -.4, .5, -.6),$
$(\hat{\varkappa}_2, .2, -.4, .1, -.5, .3, -.6), (\hat{\varkappa}_3, .6, -.2, .1, -.3,$
$.9, -.8)\}\big), \big((\alpha_3, \alpha_5, \alpha_7), \{(\hat{\varkappa}_1, 0, -.4, .8, -.3, .4, -.5$
$), (\hat{\varkappa}_2, .6, -1, .2, -.3, .4, -.5), (\hat{\varkappa}_3, .8, -.9, .4, -.8,$

$.2, -.5)\}\big), \big((\alpha_3, \alpha_5, \alpha_8), \{(\hat{\varkappa}_1, .8, -.5, .3, -.4, .6, -.3), (\hat{\varkappa}_2, .7, -.1, .7, -.3, .8, -.9), (\hat{\varkappa}_3, 1, -.9, .2, -.5, .7, -.3)\}\big)\Big\rangle\Big\}.$

Definition 4.3. Let $\hat{\chi}$ be a non-empty universe. A BNHS denoted by $(\Psi, \Gamma)_0$, is called empty BNHS and defined as:

$(\Psi, \Gamma)_0 = \Big\{\big\langle \nu, \{\hat{\hat{\varkappa}}, 0, 0, 1, -1, 1, -1\}\big\rangle : \forall \hat{\varkappa} \in \hat{\chi},\ \forall \nu \in \Gamma \subseteq \Omega\Big\}$, where $\mathcal{T}^{+}_{\Psi(\nu)}(\hat{\hat{\varkappa}}) = \mathcal{T}^{-}_{\Psi(\nu)}(\hat{\varkappa}) = 0, \mathcal{I}^{+}_{\Psi(\nu)}(\hat{\hat{\varkappa}}) = \mathcal{F}^{+}_{\Psi(\nu)}(\hat{\hat{\varkappa}}) = 1, \mathcal{I}^{-}_{\Psi(\nu)}(\hat{\varkappa}) = \mathcal{F}^{-}_{\Psi(\nu)}(\hat{\hat{\varkappa}}) = -1,\ \forall \hat{\hat{\varkappa}} \in \hat{\chi},\ \forall \nu \in \Gamma \subseteq \Omega.$

Definition 4.4. Let $\hat{\chi}$ be a non-empty universe. A BNHS denoted by $(\Psi, \Gamma)_{\chi}$, is called absolute BNHS and defined as:

$(\Psi, \Gamma)_{\chi} = \Big\{\big\langle \nu, \{\hat{\hat{\varkappa}}, 1, -1, 0, 0, 0, 0\}\big\rangle : \forall \hat{\varkappa} \in \hat{\chi},\ \forall \nu \in \Gamma \subseteq \Omega\Big\}$, where $\mathcal{T}^{+}_{\Psi(\nu)}(\hat{\hat{\varkappa}}) = 1, \mathcal{T}^{-}_{\Psi(\nu)}(\hat{\hat{\varkappa}}) = -1$ and $\mathcal{I}^{+}_{\Psi(\nu)}(\hat{\hat{\varkappa}}) = \mathcal{F}^{+}_{\Psi(\nu)}(\hat{\hat{\varkappa}}) = \mathcal{I}^{-}_{\Psi(\nu)}(\hat{\hat{\varkappa}}) = \mathcal{F}^{-}_{\Psi(\nu)}(\hat{\hat{\varkappa}}) = 0,\ \forall \hat{\hat{\varkappa}} \in \hat{\chi},\ \forall \nu \in \Gamma \subseteq \Omega.$

The complement operator of the BNHS is defined in this part.

Definition 4.5. Let $(\Psi, \Gamma) = \Big\{\big\langle \nu, \{\big(\hat{\varkappa}, \mathcal{T}^{+}_{\Psi(\nu)}(\hat{\varkappa}), \mathcal{T}^{-}_{\Psi(\nu)}(\hat{\varkappa}), \mathcal{I}^{+}_{\Psi(\nu)}(\hat{\varkappa}), \mathcal{I}^{-}_{\Psi(\nu)}(\hat{\varkappa}), \mathcal{F}^{+}_{\Psi(\nu)}(\hat{\varkappa}), \mathcal{F}^{-}_{\Psi(\nu)}(\hat{\varkappa})\big) : \forall \hat{\varkappa} \in \hat{\chi}\}\big\rangle : \nu \in \Gamma \subseteq \Omega\Big\}$ be a BNHS. Then the complement of (Ψ, Γ) is denoted by $(\Psi, \Gamma)^c$ and is defined as:

$(\Psi, \Gamma)^c = (\Psi^c, \Gamma) = \Big\{\big\langle \nu, \{\big(\hat{\hat{\varkappa}}, \mathcal{F}^{+}_{\Psi(\nu)}(\hat{\hat{\varkappa}}), \mathcal{F}^{-}_{\Psi(\nu)}(\hat{\hat{\varkappa}}), 1 - \mathcal{I}^{+}_{\Psi(\nu)}(\hat{\hat{\varkappa}}), -1 - \mathcal{I}^{-}_{\Psi(\nu)}(\hat{\hat{\varkappa}}), \mathcal{T}^{+}_{\Psi(\nu)}(\hat{\hat{\varkappa}}), \mathcal{T}^{-}_{\Psi(\nu)}(\hat{\varkappa})\big) : \forall \hat{\hat{\varkappa}} \in \hat{\chi}\}\big\rangle : \nu \in \Gamma \subseteq \Omega\Big\}$

Now, we display the use of the complement operator through an example as follows:

Example 4.6. With reference to Example 3.2. The complement of the BNHS (Ψ, Γ) is

$(\Psi, \Gamma)^c = (\Psi^c, \Gamma) = \Big\{\Big\langle\big((\alpha_2, \alpha_5, \alpha_7), \{(\hat{\varkappa}_1, .8, -.1, .5, -.1, .6, -.1), (\hat{\varkappa}_2, .7, -.5, .9, -.8, .7, -.4), (\hat{\varkappa}_3, .5, -.7, .4, -.6, .6, -.4)\}\big), \big((\alpha_2, \alpha_5, \alpha_8), \{(\hat{\varkappa}_1, .5, -.6, .8, -.6, .5, -.2), (\hat{\varkappa}_2, .3, -.6, .9, -.5, .2, -.4), (\hat{\varkappa}_3, .9, -.8, .9, -.7, .6, -.2)\}\big), \big((\alpha_3, \alpha_5, \alpha_7), \{(\hat{\varkappa}_1, .4, -.5, .2, -.7, 0, -.4), (\hat{\varkappa}_2, .4, -.5, .8, -.7, .6, -1), (\hat{\varkappa}_3, .2, -.5, .6, -.2, .8, -.9)\}\big), \big((\alpha_3, \alpha_5, \alpha_8), \{(\hat{\varkappa}_1, .6, -.3, .7, -.6, .8, -.5), (\hat{\varkappa}_2, .8, -.9, .3, -.7, .7, -.1), (\hat{\varkappa}_3, .7, -.3, .8, -.5, 1, -.9)\}\big)\Big\rangle\Big\}.$

Proposition 4.7. *The complement of the complement of a BNHS* (Ψ, Γ) *is simply the BNHS* (Ψ, Γ) *itself. In symbols,* $((\Psi, \Gamma)^c)^c = (\Psi, \Gamma)$.

Proof: Suppose the BNHS $(\Psi,\Gamma)=\Big\{\langle\nu,\{(\hat{\varkappa},\mathcal{T}^{+}_{\Psi(\nu)}(\hat{\varkappa}),\mathcal{T}^{-}_{\Psi(\nu)}(\hat{\varkappa}),\mathcal{I}^{+}_{\Psi(\nu)}(\hat{\varkappa}),\mathcal{I}^{-}_{\Psi(\nu)}(\hat{\varkappa}),\mathcal{F}^{+}_{\Psi(\nu)}(\hat{\varkappa}),\mathcal{F}^{-}_{\Psi(\nu)}(\hat{\varkappa})) : \forall\hat{\varkappa}\in\hat{\chi}\}\rangle : \nu\in\Gamma\subseteq\Omega\Big\}$. By Definition 3.5, $(\Psi,\Gamma)^c=(\Psi^c,\Gamma)=$
$\Big\{\langle\nu,\{(\hat{\varkappa},\mathcal{F}^{+}_{\Psi(\nu)}(\hat{\varkappa}),\mathcal{F}^{-}_{\Psi(\nu)}(\hat{\varkappa}),1-\mathcal{I}^{+}_{\Psi(\nu)}(\hat{\varkappa}),-1-\mathcal{I}^{-}_{\Psi(\nu)}(\hat{\varkappa}),\mathcal{T}^{+}_{\Psi(\nu)}(\hat{\varkappa}),\mathcal{T}^{-}_{\Psi(\nu)}(\hat{\varkappa})) : \forall\hat{\varkappa}\in\hat{\chi}\}\rangle : \nu\in\Gamma\subseteq\Omega\Big\}$. Using Definition 3.5 again, we obtain.

$((\Psi,\Gamma)^c)^c=$
$\Big\{\langle\nu,\{(\hat{\varkappa},\mathcal{T}^{+}_{\Psi(\nu)}(\hat{\varkappa}),\mathcal{T}^{-}_{\Psi(\nu)}(\hat{\varkappa}),1-(1-\mathcal{I}^{+}_{\Psi(\nu)}(\hat{\varkappa})),-1-(-1-\mathcal{I}^{-}_{\Psi(\nu)}(\hat{\varkappa})),\mathcal{F}^{+}_{\Psi(\nu)}(\hat{\varkappa}),\mathcal{F}^{-}_{\Psi(\nu)}(\hat{\varkappa})) : \forall\hat{\varkappa}\in\hat{\chi}\}\rangle : \nu\in\Gamma\subseteq\Omega\Big\}$,
$=\Big\{\langle\nu,\{(\hat{\varkappa},\mathcal{T}^{+}_{\Psi(\nu)}(\hat{\varkappa}),\mathcal{T}^{-}_{\Psi(\nu)}(\hat{\varkappa}),\mathcal{I}^{+}_{\Psi(\nu)}(\hat{\varkappa}),\mathcal{I}^{-}_{\Psi(\nu)}(\hat{\varkappa}),\mathcal{F}^{+}_{\Psi(\nu)}(\hat{\varkappa}),\mathcal{F}^{-}_{\Psi(\nu)}(\hat{\varkappa})) : \forall\hat{\varkappa}\in\hat{\chi}\}\rangle : \nu\in\Gamma\subseteq\Omega\Big\}$,
$=(\Psi,\Gamma)$.

Proposition 4.8. *Assume, (Ψ,Γ) is a BNHS over $\hat{\chi}$. Then,*

1. $((\Psi,\Gamma)_0)^c=(\Psi,\Gamma)_\chi$,
2. $((\Psi,\Gamma)_\chi)^c=(\Psi,\Gamma)_0$.

1. Suppose $(\Psi,\Gamma)_0=\Big\{\langle\nu,\{\hat{\varkappa},0,0,1,-1,1,-1\}\rangle : \forall\hat{\varkappa}\in\hat{\chi},\ \forall\nu\in\Gamma\subseteq\Omega\Big\}$ is an empty BNHS. Based on Definition 11, $((\Psi,\Gamma)_0)^c=$ $\Big\{\langle\nu,\{\hat{\varkappa},1,-1,1-1,-1-(-1),0,0\}\rangle : \forall\hat{\varkappa}\in\hat{\chi},\ \forall\nu\in\Gamma\subseteq\Omega\Big\}$,
   $=\Big\{\langle\nu,\{\hat{\varkappa},1,-1,0,0,0,0\}\rangle : \forall\hat{\varkappa}\in\hat{\chi},\ \forall\nu\in\Gamma\subseteq\Omega\Big\}=(\Psi,\Gamma)_\chi$.
2. Proof of this item is similar to that of (1).

Now, we define subset-hood operator on two BNHSs.

Definition 4.9. Suppose (Ψ,Γ) and (Φ,Λ) are two BNHSs over $\hat{\chi}$. Where $(\Psi,\Gamma)=\Big\{\langle\nu,\{(\hat{\varkappa},\mathcal{T}^{+}_{\Psi(\nu)}(\hat{\varkappa}),\mathcal{T}^{-}_{\Psi(\nu)}(\hat{\varkappa}),\mathcal{I}^{+}_{\Psi(\nu)}(\hat{\varkappa}),\mathcal{I}^{-}_{\Psi(\nu)}(\hat{\varkappa}),\mathcal{F}^{+}_{\Psi(\nu)}(\hat{\varkappa}),\mathcal{F}^{-}_{\Psi(\nu)}(\hat{\varkappa})) : \forall\hat{\varkappa}\in\hat{\chi}\}\rangle : \nu\in\Gamma\subseteq\Omega\Big\}$ and $(\Phi,\Lambda)=\Big\{\langle\nu,\{(\hat{\varkappa},\mathcal{T}^{+}_{\Phi(\nu)}(\hat{\varkappa}),\mathcal{T}^{-}_{\Phi(\nu)}(\hat{\varkappa}),\mathcal{I}^{+}_{\Phi(\nu)}(\hat{\varkappa}),\mathcal{I}^{-}_{\Phi(\nu)}(\hat{\varkappa}),\mathcal{F}^{+}_{\Phi(\nu)}(\hat{\varkappa}),\mathcal{F}^{-}_{\Phi(\nu)}(\hat{\varkappa})) : \forall\hat{\varkappa}\in\hat{\chi}\}\rangle : \nu\in\Lambda\subseteq\Omega\Big\}$. We said that (Ψ,Γ) is a subset of (Φ,Λ), denoted as $(\Psi,\Gamma)\subseteq(\Phi,\Lambda)$, if:

1. $\Gamma\subseteq\Lambda$,
2. $\forall\nu\in\Gamma,\ \forall\hat{\varkappa}\in\hat{\chi},\quad \mathcal{T}^{+}_{\Psi(\nu)}(\hat{\varkappa})\le\mathcal{T}^{+}_{\Phi(\nu)}(\hat{\varkappa}),\quad \mathcal{T}^{-}_{\Psi(\nu)}(\hat{\varkappa})\ge\mathcal{T}^{-}_{\Phi(\nu)}(\hat{\varkappa}),\quad \mathcal{I}^{+}_{\Psi(\nu)}(\hat{\varkappa})\ge\mathcal{I}^{+}_{\Phi(\nu)}(\hat{\varkappa}),$ $\mathcal{I}^{-}_{\Psi(\nu)}(\hat{\varkappa})\le\mathcal{I}^{-}_{\Phi(\nu)}(\hat{\varkappa}),\quad \mathcal{F}^{+}_{\Psi(\nu)}(\hat{\varkappa})\ge\mathcal{F}^{+}_{\Phi(\nu)}(\hat{\varkappa}),\quad \mathcal{F}^{-}_{\Psi(\nu)}(\hat{\varkappa})\le\mathcal{F}^{-}_{\Phi(\nu)}(\hat{\varkappa})$.

Remark 4.10. From Definition 3.9, it is clear that $((\Psi,\Gamma)_0)\subseteq(\Psi,\Gamma)_\chi$.

The equality between two BNHSs (Ψ,Γ) and (Φ,Λ) can be defined as follows.

Definition 4.11. We said that (Ψ,Γ) is equal to (Φ,Λ), denoted as $(\Psi,\Gamma)=(\Phi,\Lambda)$, if:

1. $\Gamma = \Lambda$,
2. $\forall \nu \in \Gamma,\ \forall \hat{\hat{\varkappa}} \in \hat{\chi},\quad \mathcal{T}^{+}_{\Psi(\nu)}(\hat{\hat{\varkappa}}) = \mathcal{T}^{+}_{\Phi(\nu)}(\hat{\hat{\varkappa}}),\quad \mathcal{T}^{-}_{\Psi(\nu)}(\hat{\hat{\varkappa}}) = \mathcal{T}^{-}_{\Phi(\nu)}(\hat{\varkappa}),\quad \mathcal{I}^{+}_{\Psi(\nu)}(\hat{\hat{\varkappa}}) = \mathcal{I}^{+}_{\Phi(\nu)}(\hat{\varkappa}),$ $\mathcal{I}^{-}_{\Psi(\nu)}(\hat{\hat{\varkappa}}) = \mathcal{I}^{-}_{\Phi(\nu)}(\hat{\hat{\varkappa}}),\quad \mathcal{F}^{+}_{\Psi(\nu)}(\hat{\hat{\varkappa}}) = \mathcal{F}^{+}_{\Phi(\nu)}(\hat{\hat{\varkappa}}),\quad \mathcal{F}^{-}_{\Psi(\nu)}(\hat{\hat{\varkappa}}) = \mathcal{F}^{-}_{\Phi(\nu)}(\hat{\hat{\varkappa}}).$

The following, is a numerical example clarifies Definition 3.9.

Example 4.12. Consider Example 1 and suppose that $\mathcal{E}_{\mu_1} = \{\alpha_3 = 2000\}$, $\mathcal{E}_{\mu_2} = \{\alpha_5 = 12MP\}$, $\mathcal{E}_{\mu_3} = \{\alpha_7 = 6GB, \alpha_8 = 8GB\}$, be another subsets of $\mathcal{C}_{\mu_k}\ \forall k = 1,2,3$ and $\Lambda = \mathcal{E}_{\mu_1} \times \mathcal{E}_{\mu_2} \times \mathcal{E}_{\mu_3}$. Then, we can obtain the following BNHS (Φ,Λ), where, $(\Phi,\Lambda) =$ $\Big\{\Big\langle\big((\alpha_3,\alpha_5,\alpha_7),\{(\hat{\varkappa}_1,0,-0.2,0.9,-0.5,0.6,-0.7),(\hat{\varkappa}_2,0.3,$ $-0.8,0.3,-0.4,0.7,-0.7),(\hat{\varkappa}_3,0.6,-0.5,0.5,-0.9,0.6,-0.7)\}\big),$ $\big((\alpha_3,\alpha_5,\alpha_8),\{(\hat{\varkappa}_1,0.6,-0.3,0.5,-0.6,0.7,-0.4),(\hat{\varkappa}_2,0.6,0,$ $0.8,-0.4,0.9,-0.9),(\hat{\varkappa}_3,1,-0.7,0.3,-0.6,0.8,-0.4)\}\big)\Big\rangle\Big\}$. Based on Definition 4.9, it is clear that $(\Phi,\Lambda) \subseteq (\Psi,\Gamma)$, where $(\Psi,\Gamma) =$ $\Big\{\Big\langle\big((\alpha_2,\alpha_5,\alpha_7),\{(\hat{\hat{\varkappa}}_1,.6,-.1,.5,-.9,.8,-.1),(\hat{\varkappa}_2,$ $.7,-.4,.1,-.2,.7,-.5),(\hat{\hat{\varkappa}}_3,.6,-.4,.6,-.4,.5,$ $-.7)\}\big),\big((\alpha_2,\alpha_5,\alpha_8),\{(\hat{\varkappa}_1,.5,-.2,.2,-.4,.5,-.6),$ $(\hat{\varkappa}_2,.2,-.4,.1,-.5,.3,-.6),(\hat{\varkappa}_3,.6,-.2,.1,-.3,$ $.9,-.8)\}\big),\big((\alpha_3,\alpha_5,\alpha_7),\{(\hat{\hat{\varkappa}}_1,0,-.4,.8,-.3,.4,-.5$ $),(\hat{\hat{\varkappa}}_2,.6,-1,.2,-.3,.4,-.5),(\hat{\hat{\varkappa}}_3,.8,-.9,.4,-.8,$ $.2,-.5)\}\big),\big((\alpha_3,\alpha_5,\alpha_8),\{(\hat{\hat{\varkappa}}_1,.8,-.5,.3,-.4,.6,$ $-.3),(\hat{\hat{\varkappa}}_2,.7,-.1,.7,-.3,.8,-.9),(\hat{\varkappa}_3,1,-.9,.2,$ $-.5,.7,-.3)\}\big)\Big\rangle\Big\}$.

To combine two BNHSs into a single BNHS, we will define the following fundamental operations on BNHSs.

Definition 4.13. The restricted union of two BNHSs (Ψ,Γ) and (Φ,Λ) over the universe $\hat{\chi}$ is signified by $(\Psi,\Gamma) \Cup_R (\Phi,\Lambda)$ and stated as: $(\Pi_R,\Upsilon) = (\Psi,\Gamma) \Cup_R (\Phi,\Lambda)$, where $\Upsilon = \Gamma \cap \Lambda$ and (Π_R,Υ) is characterized as:

$$(\Pi_R,\Upsilon) = \Big\{\langle\varepsilon,\{(\hat{\hat{\varkappa}},\mathcal{T}^{+}_{\Psi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) \vee \mathcal{T}^{+}_{\Phi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right),\mathcal{T}^{-}_{\Psi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) \wedge \mathcal{T}^{-}_{\Phi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right),\mathcal{I}^{+}_{\Psi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) \wedge \mathcal{I}^{+}_{\Phi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right),\mathcal{I}^{-}_{\Psi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) \vee \mathcal{I}^{-}_{\Phi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right),\mathcal{F}^{+}_{\Psi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) \wedge \mathcal{F}^{+}_{\Phi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right),\mathcal{F}^{-}_{\Psi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) \vee \mathcal{F}^{-}_{\Phi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) : \forall\hat{\hat{\varkappa}} \in \hat{\chi}\}\rangle : \varepsilon \in \Gamma \cap \Lambda\Big\}.$$

Where $\max = \vee$ and $\min = \wedge$.

To clarify Definition 3.13, we provide the following example.

Example 4.14. Consider the BNHS (Ψ, Γ) in Example 3.2, where $(\Psi, \Gamma) =$
$\Big\{\Big\langle\big((\alpha_2, \alpha_5, \alpha_7), \{(\hat{\hat{\varkappa}}_1, .6, -.1, .5, -.9, .8, -.1), (\hat{\hat{\varkappa}}_2,$
$.7, -.4, .1, -.2, .7, -.5), (\hat{\hat{\varkappa}}_3, .6, -.4, .6, -.4, .5,$
$-.7)\}\big), \big((\alpha_2, \alpha_5, \alpha_8), \{(\hat{\hat{\varkappa}}_1, .5, -.2, .2, -.4, .5, -.6),$
$(\hat{\hat{\varkappa}}_2, .2, -.4, .1, -.5, .3, -.6), (\hat{\hat{\varkappa}}_3, .6, -.2, .1, -.3,$
$.9, -.8)\}\big), \big((\alpha_3, \alpha_5, \alpha_7), \{(\hat{\hat{\varkappa}}_1, 0, -.4, .8, -.3, .4, -.5$
$), (\hat{\hat{\varkappa}}_2, .6, -1, .2, -.3, .4, -.5), (\hat{\varkappa}_3, .8, -.9, .4, -.8,$
$.2, -.5)\}\big), \big((\alpha_3, \alpha_5, \alpha_8), \{(\hat{\hat{\varkappa}}_1, .8, -.5, .3, -.4, .6,$
$-.3), (\hat{\hat{\varkappa}}_2, .7, -.1, .7, -.3, .8, -.9), (\hat{\hat{\varkappa}}_3, 1, -.9, .2,$
$-.5, .7, -.3)\}\big)\Big\rangle\Big\}.$

Suppose that $\mathcal{H}_{\mu_1} = \{\alpha_1 = 1200, \alpha_2 = 1500\}$, $\mathcal{H}_{\mu_2} = \{\alpha_5 = 12MP, \alpha_6 = 16MP\}$, $\mathcal{H}_{\mu_3} = \{\alpha_7 = 6GB\}$, be another subsets of $\mathcal{C}_{\mu_k}$ $\forall k = 1, 2, 3$ and $\lambda = \mathcal{H}_{\mu_1} \times \mathcal{H}_{\mu_2} \times \mathcal{H}_{\mu_3}$. Then, we obtain the following BNHS (Θ, λ), where, $(\Theta, \lambda) =$
$\Big\{\Big\langle\big((\alpha_1, \alpha_5, \alpha_7), \{(\hat{\hat{\varkappa}}_1, .1, -.8, .3, -.2, .9, -.1), (\hat{\hat{\varkappa}}_2,$
$.5, -.1, .1, -.6, .3, -.2), (\hat{\hat{\varkappa}}_3, .8, -.7, .9, -.2, .9,$
$-.8)\}\big), \big((\alpha_1, \alpha_6, \alpha_7), \{(\hat{\varkappa}_1, .6, -.4, .7, -.3, .2, -.7),$
$(\hat{\hat{\varkappa}}_2, .3, -.5, .1, -.6, .4, -.7), (\hat{\hat{\varkappa}}_3, .8, -.4, .1, -.3,$
$.8, -.1)\}\big), \big((\alpha_2, \alpha_5, \alpha_7), \{(\hat{\hat{\varkappa}}_1, 0, -1, .5, -.1, .2, -.7$
$), (\hat{\hat{\varkappa}}_2, 0, -1, .8, -.5, 1, -.5), (\hat{\hat{\varkappa}}_3, .2, -1, .5, -.7,$
$.1, -.2)\}\big), \big((\alpha_2, \alpha_6, \alpha_7), \{(\hat{\hat{\varkappa}}_1, 1, -.5, .2, -.8, .5,$
$-.2), (\hat{\hat{\varkappa}}_2, 0, -.1, 1, -.3, .2, -.9), (\hat{\hat{\varkappa}}_3, 1, -.7, .5,$
$-.3, .2, -.4)\}\big)\Big\rangle\Big\}.$

The restricted union of (Ψ, Γ) and (Θ, λ) can be calculated as follows.
$(\Psi, \Gamma) \Cup_R (\Theta, \lambda) =$

$\Big\{\Big\langle\big((\alpha_2, \alpha_5, \alpha_7), \{(\hat{\hat{\varkappa}}_1, .6, -1, .5, -.1, .2, -.1), (\hat{\hat{\varkappa}}_2,$
$.7, -1, .1, -.2, .7, .5), (\hat{\hat{\varkappa}}_3, .6, -1, .5, -.4, .1,$
$-.2)\}\big)\Big\rangle\Big\}.$

The following properties hold under the BNHS union.

Proposition 4.15. *Let (Ψ, Γ), (Φ, Λ) and (Θ, λ) be three BNHSs over $\hat{\chi}$. Then,*

1. $(\Psi, \Gamma) \Cup_R (\Psi, \Gamma)_0 = (\Psi, \Gamma)$,
2. $(\Psi, \Gamma) \Cup_R (\Psi, \Gamma)_{\hat{\chi}} = (\Psi, \Gamma)_{\hat{\chi}}$,
3. $(\Psi, \Gamma) \Cup_R (\Phi, \Lambda) = (\Phi, \Lambda) \Cup_R (\Psi, \Gamma)$,
4. $\big((\Psi, \Gamma) \Cup_R (\Phi, \Lambda)\big) \Cup_R (\Theta, \lambda) = (\Psi, \Gamma) \Cup_R \big((\Phi, \Lambda) \Cup_R (\Theta, \lambda)\big)$.

Definition 4.16. The extended union of two BNHSs (Ψ,Γ) and (Φ,Λ) over the universe $\hat{\chi}$ is signified by $(\Psi,\Gamma)\Cup_E(\Phi,\Lambda)$ and stated as: $(\Pi_E,\Upsilon)=(\Psi,\Gamma)\Cup_E(\Phi,\Lambda)$, where $\Upsilon=\Gamma\cup\Lambda$ and $\forall\varepsilon\in\Upsilon,\ \ \forall\hat{\hat{\varkappa}}\in\hat{\chi}$,

$$\mathcal{T}^{+}_{\Pi_E(\varepsilon)}\left(\hat{\hat{\varkappa}}\right)=\begin{cases}\mathcal{T}^{+}_{\Psi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) & ,if\varepsilon\in\Gamma-\Lambda\\ \mathcal{T}^{+}_{\Phi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) & ,if\varepsilon\in\Lambda-\Gamma\\ \mathcal{T}^{+}_{\Psi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right)\vee\mathcal{T}^{+}_{\Phi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) & ,if\varepsilon\in\Gamma\cap\Lambda\end{cases}$$

$$\mathcal{T}^{-}_{\Pi_E(\varepsilon)}\left(\hat{\varkappa}\right)=\begin{cases}\mathcal{T}^{-}_{\Psi(\varepsilon)}\left(\hat{\varkappa}\right) & ,if\varepsilon\in\Gamma-\Lambda\\ \mathcal{T}^{-}_{\Phi(\varepsilon)}\left(\hat{\varkappa}\right) & ,if\varepsilon\in\Lambda-\Gamma\\ \mathcal{T}^{-}_{\Psi(\varepsilon)}\left(\hat{\varkappa}\right)\wedge\mathcal{T}^{-}_{\Phi(\varepsilon)}\left(\hat{\varkappa}\right) & ,if\varepsilon\in\Gamma\cap\Lambda\end{cases}$$

$$\mathcal{I}^{+}_{\Pi_E(\varepsilon)}\left(\hat{\varkappa}\right)=\begin{cases}\mathcal{I}^{+}_{\Psi(\varepsilon)}\left(\hat{\varkappa}\right) & ,if\varepsilon\in\Gamma-\Lambda\\ \mathcal{I}^{+}_{\Phi(\varepsilon)}\left(\hat{\varkappa}\right) & ,if\varepsilon\in\Lambda-\Gamma\\ \mathcal{I}^{+}_{\Psi(\varepsilon)}\left(\hat{\varkappa}\right)\wedge\mathcal{I}^{+}_{\Phi(\varepsilon)}\left(\hat{\varkappa}\right) & ,if\varepsilon\in\Gamma\cap\Lambda\end{cases}$$

$$\mathcal{I}^{-}_{\Pi_E(\varepsilon)}\left(\hat{\varkappa}\right)=\begin{cases}\mathcal{I}^{-}_{\Psi(\varepsilon)}\left(\hat{\varkappa}\right) & ,if\varepsilon\in\Gamma-\Lambda\\ \mathcal{I}^{-}_{\Phi(\varepsilon)}\left(\hat{\varkappa}\right) & ,if\varepsilon\in\Lambda-\Gamma\\ \mathcal{I}^{-}_{\Psi(\varepsilon)}\left(\hat{\varkappa}\right)\vee\mathcal{I}^{-}_{\Phi(\varepsilon)}\left(\hat{\varkappa}\right) & ,if\varepsilon\in\Gamma\cap\Lambda\end{cases}$$

$$\mathcal{F}^{+}_{\Pi_E(\varepsilon)}\left(\hat{\varkappa}\right)=\begin{cases}\mathcal{F}^{+}_{\Psi(\varepsilon)}\left(\hat{\varkappa}\right) & ,if\varepsilon\in\Gamma-\Lambda\\ \mathcal{F}^{+}_{\Phi(\varepsilon)}\left(\hat{\varkappa}\right) & ,if\varepsilon\in\Lambda-\Gamma\\ \mathcal{F}^{+}_{\Psi(\varepsilon)}\left(\hat{\varkappa}\right)\wedge\mathcal{F}^{+}_{\Phi(\varepsilon)}\left(\hat{\varkappa}\right) & ,if\varepsilon\in\Gamma\cap\Lambda\end{cases}$$

$$\mathcal{F}^{-}_{\Pi_E(\varepsilon)}\left(\hat{\varkappa}\right)=\begin{cases}\mathcal{F}^{-}_{\Psi(\varepsilon)}\left(\hat{\varkappa}\right) & ,if\varepsilon\in\Gamma-\Lambda\\ \mathcal{F}^{-}_{\Phi(\varepsilon)}\left(\hat{\varkappa}\right) & ,if\varepsilon\in\Lambda-\Gamma\\ \mathcal{F}^{-}_{\Psi(\varepsilon)}\left(\hat{\varkappa}\right)\vee\mathcal{F}^{-}_{\Phi(\varepsilon)}\left(\hat{\varkappa}\right) & ,if\varepsilon\in\Gamma\cap\Lambda\end{cases}$$

Where max $=\vee$ and min $=\wedge$.

To clarify Definition 3.16, we provide the following hypothetical example.

Example 4.17. Consider Example 3.14. The extended union of (Ψ,Γ) and (Θ,λ) can be calculated as follows.

$(\Psi,\Gamma)\Cup_E(\Theta,\lambda)=$

$\Big\{\Big\langle\big((\alpha_2,\alpha_5,\alpha_7),\{(\hat{\hat{\varkappa}}_1,.6,-1,.5,-.1,.2,-.1),(\hat{\hat{\varkappa}}_2,$
$.7,-1,.1,-.2,.7,.5),(\hat{\hat{\varkappa}}_3,.6,-1,.5,-.4,.1,$
$-.2)\}\big),\big((\alpha_2,\alpha_5,\alpha_8),\{(\hat{\hat{\varkappa}}_1,.5,-.2,.2,-.4,.5,-.6),$
$(\hat{\hat{\varkappa}}_2,.2,-.4,.1,-.5,.3,-.6),(\hat{\hat{\varkappa}}_3,.6,-.2,.1,-.3,$
$.9,-.8)\}\big),\big((\alpha_3,\alpha_5,\alpha_7),\{(\hat{\hat{\varkappa}}_1,0,-.4,.8,-.3,.4,-.5$

$), (\hat{\hat{\varkappa}}_2, .6, -1, .2, -.3, .4, -.5), (\hat{\hat{\varkappa}}_3, .8, -.9, .4, -.8,$
$.2, -.5)\}\big), \big((\alpha_3, \alpha_5, \alpha_8), \{(\hat{\hat{\varkappa}}_1, .8, -.5, .3, -.4, .6,$
$-.3), (\hat{\varkappa}_2, .7, -.1, .7, -.3, .8, -.9), (\hat{\hat{\varkappa}}_3, 1, -.9, .2,$
$-.5, .7, -.3)\}\big), \big((\alpha_1, \alpha_5, \alpha_7), \{(\hat{\hat{\varkappa}}_1, .1, -.8, .3, -.2,$
$.9, -.1), (\hat{\hat{\varkappa}}_2, .5, -.1, .1, -.6, .3, -.2), (\hat{\hat{\varkappa}}_3, .8, -.7,$
$.9, -.2, .9, -.8)\}\big), \big((\alpha_1, \alpha_6, \alpha_7), \{(\hat{\hat{\varkappa}}_1, .6, -.4, .7,$
$-.3, .2, -.7), (\hat{\hat{\varkappa}}_2, .3, -.5, .1, -.6, .4, -.7), (\hat{\hat{\varkappa}}_3, .8,$
$-.4, .1, -.3, .8, -.1)\}\big), \big((\alpha_2, \alpha_6, \alpha_7), \{(\hat{\hat{\varkappa}}_1, 1, -.5, .2,$
$-.8, .5, -.2), (\hat{\hat{\varkappa}}_2, 0, -.1, 1, -.3, .2, -.9), (\hat{\hat{\varkappa}}_3, 1, -.7,$
$.5, -.3, .2, -.4)\}\big)\Big\rangle\Big\}.$

Definition 4.18. The restricted intersection of two BNHSs (Ψ, Γ) and (Φ, Λ) over the non-empty universe $\hat{\chi}$ is signified by $(\Psi, \Gamma) \Cap_R (\Phi, \Lambda)$ and stated as: $(\Xi_R, \Upsilon) = (\Psi, \Gamma) \Cap_R (\Phi, \Lambda)$, where $\Upsilon = \Gamma \cap \Lambda$ and (Ξ_R, Υ) is characterized as:

$$(\Xi_R, \Upsilon) = \Big\{\big\langle \varepsilon, \big\{\big(\hat{\varkappa}, \mathcal{T}^{+}_{\Psi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) \wedge \mathcal{T}^{+}_{\Phi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right), \mathcal{T}^{-}_{\Psi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) \vee \mathcal{T}^{-}_{\Phi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right), \mathcal{I}^{+}_{\Psi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) \vee \mathcal{I}^{+}_{\Phi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right), \mathcal{I}^{-}_{\Psi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) \wedge \mathcal{I}^{-}_{\Phi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right), \mathcal{F}^{+}_{\Psi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) \vee \mathcal{F}^{+}_{\Phi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right), \mathcal{F}^{-}_{\Psi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) \wedge \mathcal{F}^{-}_{\Phi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) : \forall \hat{\varkappa} \in \hat{\chi}\big\}\big\rangle : \varepsilon \in \Gamma \cap \Lambda\Big\}.$$

Where max $= \vee$ and min $= \wedge$.

To clarify Definition 4.18, we provide the following hypothetical example.

Example 4.19. Consider Example 3.14. The restricted intersection of (Ψ, Γ) and (Θ, λ) can be calculated as follows.

$(\Psi, \Gamma) \Cap_R (\Theta, \lambda) =$

$$\Big\{\Big\langle\big((\alpha_2, \alpha_5, \alpha_7), \{(\hat{\varkappa}_1, 0, -.1, .5, -.9, .8, -.7), (\hat{\hat{\varkappa}}_2, 0, -.4, .8, -.5, 1, -.5), (\hat{\hat{\varkappa}}_3, .2, -.4, .6, -.7, .5, -.7)\}\big)\Big\rangle\Big\}.$$

The following properties hold under the BNHS intersection.

Proposition 4.20. *Let (Ψ, Γ), (Φ, Λ) and (Θ, λ) be three BNHSs over $\hat{\chi}$. Then,*

1. $(\Psi, \Gamma) \Cap_R (\Psi, \Gamma)_0 = (\Psi, \Gamma)_0$,
2. $(\Psi, \Gamma) \Cap_R (\Psi, \Gamma)_{\hat{\chi}} = (\Psi, \Gamma)$,
3. $(\Psi, \Gamma) \Cap_R (\Phi, \Lambda) = (\Phi, \Lambda) \Cap_R (\Psi, \Gamma)$,
4. $\big((\Psi, \Gamma) \Cap_R (\Phi, \Lambda)\big) \Cap_R (\Theta, \lambda) = (\Psi, \Gamma) \Cap_R \big((\Phi, \Lambda) \Cap_R (\Theta, \lambda)\big)$.

Definition 4.21. The extended intersection of two BNHSs (Ψ,Γ) and (Φ,Λ) over the universe $\hat{\chi}$ is signified by $(\Psi,\Gamma)\Cap_E(\Phi,\Lambda)$ and stated as: $(\Delta_E,\Upsilon)=(\Psi,\Gamma)\Cap_E(\Phi,\Lambda)$, where $\Upsilon=\Gamma\cup\Lambda$ and $\forall\varepsilon\in\Upsilon,\quad\forall\hat{\varkappa}\in\hat{\chi}$,

$$\mathcal{T}^{+}_{\Delta_E(\varepsilon)}(\hat{\varkappa})=\begin{cases}\mathcal{T}^{+}_{\Psi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) & ,if\varepsilon\in\Gamma-\Lambda\\ \mathcal{T}^{+}_{\Phi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) & ,if\varepsilon\in\Lambda-\Gamma\\ \mathcal{T}^{+}_{\Psi(\varepsilon)}(\hat{\varkappa})\wedge\mathcal{T}^{+}_{\Phi(\varepsilon)}(\hat{\varkappa}) & ,if\varepsilon\in\Gamma\cap\Lambda\end{cases}$$

$$\mathcal{T}^{-}_{\Delta_E(\varepsilon)}\left(\hat{\hat{\varkappa}}\right)=\begin{cases}\mathcal{T}^{-}_{\Psi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) & ,if\varepsilon\in\Gamma-\Lambda\\ \mathcal{T}^{-}_{\Phi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) & ,if\varepsilon\in\Lambda-\Gamma\\ \mathcal{T}^{-}_{\Psi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right)\vee\mathcal{T}^{-}_{\Phi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) & ,if\varepsilon\in\Gamma\cap\Lambda\end{cases}$$

$$\mathcal{I}^{+}_{\Delta_E(\varepsilon)}\left(\hat{\hat{\varkappa}}\right)=\begin{cases}\mathcal{I}^{+}_{\Psi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) & ,if\varepsilon\in\Gamma-\Lambda\\ \mathcal{I}^{+}_{\Phi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) & ,if\varepsilon\in\Lambda-\Gamma\\ \mathcal{I}^{+}_{\Psi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right)\vee\mathcal{I}^{+}_{\Phi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) & ,if\varepsilon\in\Gamma\cap\Lambda\end{cases}$$

$$\mathcal{I}^{-}_{\Delta_E(\varepsilon)}\left(\hat{\hat{\varkappa}}\right)=\begin{cases}\mathcal{I}^{-}_{\Psi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) & ,if\varepsilon\in\Gamma-\Lambda\\ \mathcal{I}^{-}_{\Phi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) & ,if\varepsilon\in\Lambda-\Gamma\\ \mathcal{I}^{-}_{\Psi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right)\wedge\mathcal{I}^{-}_{\Phi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) & ,if\varepsilon\in\Gamma\cap\Lambda\end{cases}$$

$$\mathcal{F}^{+}_{\Delta_E(\varepsilon)}\left(\hat{\hat{\varkappa}}\right)=\begin{cases}\mathcal{F}^{+}_{\Psi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) & ,if\varepsilon\in\Gamma-\Lambda\\ \mathcal{F}^{+}_{\Phi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) & ,if\varepsilon\in\Lambda-\Gamma\\ \mathcal{F}^{+}_{\Psi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right)\vee\mathcal{F}^{+}_{\Phi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) & ,if\varepsilon\in\Gamma\cap\Lambda\end{cases}$$

$$\mathcal{F}^{-}_{\Delta_E(\varepsilon)}\left(\hat{\hat{\varkappa}}\right)=\begin{cases}\mathcal{F}^{-}_{\Psi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) & ,if\varepsilon\in\Gamma-\Lambda\\ \mathcal{F}^{-}_{\Phi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) & ,if\varepsilon\in\Lambda-\Gamma\\ \mathcal{F}^{-}_{\Psi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right)\wedge\mathcal{F}^{-}_{\Phi(\varepsilon)}\left(\hat{\hat{\varkappa}}\right) & ,if\varepsilon\in\Gamma\cap\Lambda\end{cases}$$

Where max $=\vee$ and min $=\wedge$.

To clarify Definition 3.21, we provide the following hypothetical example.

Example 4.22. Take Example 3.14. The extended intersection of (Ψ,Γ) and (Θ,λ) can be calculated as follows.

$(\Psi,\Gamma)\Cap_E(\Theta,\lambda)=$
$\Big\{\Big\langle\big((\alpha_2,\alpha_5,\alpha_7),\{(\hat{\varkappa}_1,0,-.1,.5,-.9,.8,-.7),(\hat{\hat{\varkappa}}_2,$
$0,-.4,.8,-.5,1,-.5),(\hat{\hat{\varkappa}}_3,.2,-.4,.6,-.7,.5,$
$-.7)\}\big),\big((\alpha_2,\alpha_5,\alpha_8),\{(\hat{\hat{\varkappa}}_1,.5,-.2,.2,-.4,.5,-.6),$
$(\hat{\hat{\varkappa}}_2,.2,-.4,.1,-.5,.3,-.6),(\hat{\hat{\varkappa}}_3,.6,-.2,.1,-.3,$
$.9,-.8)\}\big),\big((\alpha_3,\alpha_5,\alpha_7),\{(\hat{\hat{\varkappa}}_1,0,-.4,.8,-.3,.4,-.5$

$), (\hat{\varkappa}_2, .6, -1, .2, -.3, .4, -.5), (\hat{\hat{\varkappa}}_3, .8, -.9, .4, -.8,$
$.2, -.5)\}\big), \big((\alpha_3, \alpha_5, \alpha_8), \{(\hat{\hat{\varkappa}}_1, .8, -.5, .3, -.4, .6,$
$-.3), (\hat{\hat{\varkappa}}_2, .7, -.1, .7, -.3, .8, -.9), (\hat{\hat{\varkappa}}_3, 1, -.9, .2,$
$-.5, .7, -.3)\}\big), \big((\alpha_1, \alpha_5, \alpha_7), \{(\hat{\hat{\varkappa}}_1, .1, -.8, .3, -.2,$
$.9, -.1), (\hat{\hat{\varkappa}}_2, .5, -.1, .1, -.6, .3, -.2), (\hat{\hat{\varkappa}}_3, .8, -.7,$
$.9, -.2, .9, -.8)\}\big), \big((\alpha_1, \alpha_6, \alpha_7), \{(\hat{\hat{\varkappa}}_1, .6, -.4, .7,$
$-.3, .2, -.7), (\hat{\hat{\varkappa}}_2, .3, -.5, .1, -.6, .4, -.7), (\hat{\hat{\varkappa}}_3, .8,$
$-.4, .1, -.3, .8, -.1)\}\big), \big((\alpha_2, \alpha_6, \alpha_7), \{(\hat{\hat{\varkappa}}_1, 1, -.5, .2,$
$-.8, .5, -.2), (\hat{\hat{\varkappa}}_2, 0, -.1, 1, -.3, .2, -.9), (\hat{\hat{\varkappa}}_3, 1, -.7,$
$.5, -.3, .2, -.4)\}\big)\Big\rangle\Big\}.$

In the following, we define AND and OR operations on BNHSs.

Definition 4.23. Let (Ψ, Γ) and (Φ, Λ) be two BNHSs over the universe $\hat{\chi}$. Then, AND operation is a BNHS over $\hat{\chi}$ and signified by

$(\Psi, \Gamma) \bigtriangledown (\Phi, \Lambda) = (\Re, \Gamma \times \Lambda)$, where, $\Re(\nu_i, \eta_j) = \Psi(\nu_i) \bar{\cap} \Phi(\eta_j)$, $\forall (\nu_i, \eta_j) \in \Gamma \times \Lambda$, where $\bar{\cap}$ is a BN-intersection.

The following is an example on AND operation.

Example 4.24. Consider Example 4, where $\nu_1 = (\alpha_2, \alpha_5, \alpha_7)$, $\nu_2 = (\alpha_2, \alpha_5, \alpha_8)$ are the hypersoft parameters(attributes) for the BNHS (Ψ, Γ) and $\eta_1 = (\alpha_1, \alpha_5, \alpha_7), \eta_2 = (\alpha_1, \alpha_6, \alpha_7)$ are the hypersoft parameters(attributes) for the BNHS (Θ, λ). Then $\Gamma \times \lambda = \{(\nu_1, \eta_1), (\nu_1, \eta_2), (\nu_2, \eta_1), (\nu_2, \eta_2)\}$. The values of $(\Psi, \Gamma) \bigtriangledown (\Theta, \lambda) = (\Re, \Gamma \times \lambda)$ is as follows.

$\{\langle((\nu_1 \times \eta_1), \{(\hat{\hat{\varkappa}}_1, .1, -.8, .5, -.2, .9, -.1), (\hat{\hat{\varkappa}}_2, .5, -.4, .1, -.2, .7, -.2),$
$(\hat{\hat{\varkappa}}_3, .6, -.7, .9, -.2, .9, -.7)\}), ((\nu_1 \times \eta_2), \{(\hat{\hat{\varkappa}}_1, .6, -.4, .7, -.3, .8, -.1),$
$(\hat{\hat{\varkappa}}_2, .2, -.5, .1, -.2, .7, -.5), (\hat{\hat{\varkappa}}_3, .6, -.4, .6, -.3, .8, -.1)\}),$
$((\nu_2 \times \eta_1), \{(\hat{\hat{\varkappa}}_1, .1, -.8, .3, -.2, .9, -.1), (\hat{\hat{\varkappa}}_2, .2, -.5, .1, -.5, .3, -.2),$
$(\hat{\hat{\varkappa}}_3, .6, -.7, .1, -.3, .9, -.8)\}), ((\nu_2 \times \eta_2), \{(\hat{\hat{\varkappa}}_1, .5, -.4, .7, -.3, .5, -.6),$
$(\hat{\hat{\varkappa}}_2, .3, -.1, .2, -.3, .4, -.5), (\hat{\hat{\varkappa}}_3, .8, -.9, .4, -.3, .8, -.1)\})\rangle\}$

Here, we provide the definition of OR operation.

Definition 4.25. Let (Ψ, Γ) and (Φ, Λ) be two BNHSs over the universe $\hat{\chi}$. Then, OR operation is a BNHS over $\hat{\chi}$ and signified by $(\Psi, \Gamma) \bigtriangleup (\Phi, \Lambda) = (\Sigma, \Gamma \times \Lambda)$, where, $\Sigma(\nu_i, \eta_j) = \Psi(\nu_i) \bar{\cup} \Phi(\eta_j), \forall (\nu_i, \eta_j) \in \Gamma \times \Lambda$, where $\bar{\cup}$ is a BN-union.

Example 4.26. Consider Example 3.24. Then, $(\Psi, \Gamma) \triangle (\Phi, \Lambda) = (\Sigma, \Gamma \times \Lambda)$ is calculated as follows.

$$= \{\langle ((\nu_1 \times \eta_1), \ \{(\hat{\hat{\varkappa}}_1, .6, -.1, .3, -.9, .8, -.1),$$
$$(\hat{\hat{\varkappa}}_2, .7, -.1, .1, -.6, .3, -.5),$$
$$(\hat{\hat{\varkappa}}_3, .8, -.4, .6, -.4, .5, -.8)\}),$$
$$((\nu_1 \times \eta_2), \ \{(\hat{\hat{\varkappa}}_1, .6, -.1, .5, -.9, .2, -.7),$$
$$(\hat{\hat{\varkappa}}_2, .7, -.4, .1, -.6, .4, -.7),$$
$$(\hat{\hat{\varkappa}}_3, .8, -.4, .1, -.4, .5, -.7)\}),$$
$$((\nu_2 \times \eta_1), \ \{(\hat{\hat{\varkappa}}_1, .5, -.2, .2, -.4, .5, -.6),$$
$$(\hat{\hat{\varkappa}}_2, .5, -.1, .1, -.6, .3, -.6),$$
$$(\hat{\hat{\varkappa}}_3, .8, -.2, .1, -.3, .9, -.8)\}),$$
$$((\nu_2 \times \eta_2), \ \{(\hat{\hat{\varkappa}}_1, .6, -.2, .2, -.4, .2, -.7),$$
$$(\hat{\hat{\varkappa}}_2, .3, -.4, .1, -.6, .3, -.7),$$
$$(\hat{\hat{\varkappa}}_3, .8, -.4, .1, -.8, .2, -.5)\})\rangle\}$$

5. Applicability of BNHSSs in MAGDM based on mathematical tools

In this part, we will demonstrate the mechanism for applying our proposed approach to dealing with real-life problems that include uncertainty data with two sides (positive and negative) by proposing two algorithms based on some mathematical tools that can be adapted to our approach, such as the score function (SF) of BNHSS and the aggregation operator (AO) of BNHS. Therefore, we will first begin by presenting the mathematical definitions for each SF of BNHSS and AO of BNHS.

Definition 5.1. For BNHSN $\Psi = \left(\mathcal{T}_\Psi^+, \mathcal{T}_\Psi^-, \mathcal{I}_\Psi^+, \mathcal{I}_\Psi^-, \mathcal{F}_\Psi^+, \mathcal{F}_\Psi^-\right)$ then the SF value defined as $S(\Psi) = \frac{\left(\mathcal{T}_\Psi^+ + 1 - \mathcal{I}_\Psi^+ + 1 - \mathcal{F}_\Psi^+ + 1 + \mathcal{T}_\Psi^- - \mathcal{I}_\Psi^- - \mathcal{F}_\Psi^-\right)}{6}$.

Definition 5.2. Assume that (Ψ, Γ) be a BNHS over $\hat{\chi}$. Where $(\Psi, \Gamma) = \left\{\langle \nu, \{(\hat{\hat{\varkappa}}, \mathcal{T}_{\Psi(\nu)}^+(\hat{\hat{\varkappa}}), \mathcal{T}_{\Psi(\nu)}^-(\hat{\varkappa}), \mathcal{I}_{\Psi(\nu)}^+(\hat{\hat{\varkappa}}), \mathcal{I}_{\Psi(\nu)}^-(\hat{\varkappa}), \mathcal{F}_{\Psi(\nu)}^+(\hat{\hat{\varkappa}}), \mathcal{F}_{\Psi(\nu)}^-(\hat{\hat{\varkappa}})) : \forall \hat{\hat{\varkappa}} \in \hat{\chi}\} \rangle : \nu \in \Gamma \subseteq \Omega\right\}$. Then AO of BNHS, denoted by $\widehat{\mathfrak{B}}_{agg}$ and defined as the following:

$$\widehat{\mathfrak{B}}_{agg} = \left\{\Xi_{\widehat{\mathfrak{B}}}\left(\hat{\hat{\varkappa}}\right) : \hat{\hat{\varkappa}} \in \hat{\chi}\right\}$$

Such that:

$$\Xi_{\widehat{\mathfrak{B}}}(\hat{\varkappa}) = \frac{1}{2\,|\Omega \times \hat{\chi}|} \sum_{\upsilon \in \Gamma \subseteq \Omega} \left(\left|1 - \mathcal{I}_{\Psi(\hat{\varkappa})}^+ \left(\mathcal{T}_{\Psi(\hat{\varkappa})}^+ - \mathcal{F}_{\Psi(\hat{\varkappa})}^+\right) + 1 - \mathcal{I}_{\Psi(\hat{\varkappa})}^- \left(\mathcal{T}_{\Psi(\hat{\varkappa})}^- - \mathcal{F}_{\Psi(\hat{\varkappa})}^-\right)\right|\right)$$

For the purpose of solving this problem, we will organize above two algorithms based on definitions 4.1 and 4.2, as shown in Figures 1 and 2.

5.1. *Numerical Example*

Choosing a professor to work at a private university: Private universities are always looking to improve their academic reputation, so they work to select teaching staff according to strict standards. Therefore, this selection process is classified as a multi-criteria selection problem. Here, in this partial section, we assume that a private university wants to choose a professor to teach genetics in the Department of Biological Sciences in the College of Science among a number of applicants according to multiple criteria, including academic qualification, scientific degree, and scientific experience. Also, these standards have sub-criteria that are compatible with HSSs. Accordingly, two committees were selected from the college deanship to undertake the task of interviewing each candidate individually in accordance with the criteria mentioned above. Based on this interview, the two committees formulate their opinions in accordance with our proposed model.

Assumptions:

(1) Let $\hat{\chi} = \{\hat{\varkappa}_1, \hat{\varkappa}_2, \hat{\varkappa}_3\}$ be the set of candidates to fill the job advertised.
(2) Let μ be a set of attributes include μ_1 = Academic Qualification, μ_2 = Scientific Degree,μ_3 =Scientific Experience : the criteria upon which selection is made.
(3) The attributes mentioned in (2) are categorized into the following:
$\mu_1 = \alpha_1$ =Phd ,α_2 = Post Doctorate
$\mu_2 = \alpha_3$ =Assistant Professor ,α_4 = Associate Professor
$\mu_3 = \alpha_5$ =3 years ,α_6 = 5 years ,α_7 = 10 years

Now, we can apply the two proposed algorithms 1 and 2 to help the committee choose suitable candidates as follows:

Algorithm 1. Using score function (SF) values $S(\Psi)$ to choose suitable candidate

Step 1. Put up BNHSSs $(\Psi,\Gamma)_{G_1}, (\Psi,\Gamma)_{G_2}$ respectively, based on expert opinions (two committees).

Step 2. Calculating the union value $(\Psi,\Gamma)_{G_1 \cup G_2}$ between two BNHSSs which given in step 1.

Step 3. Compute the value SF value of $(\Psi,\Gamma)_{G_1 \cup G_2}$ based on definition 4.1.

Step 4. Find the value $\mathcal{M}_i = \sum_{i=1}^{3} S(\Psi)_i$ for the candidate X_i,$i = 1, 2, 3$.

Step 5. Decision: Choose the highest value of $\mathcal{M}_i$.

Step 6. End algorithm 1.

In addition Figure 2 bellow representation of algorithm 1.

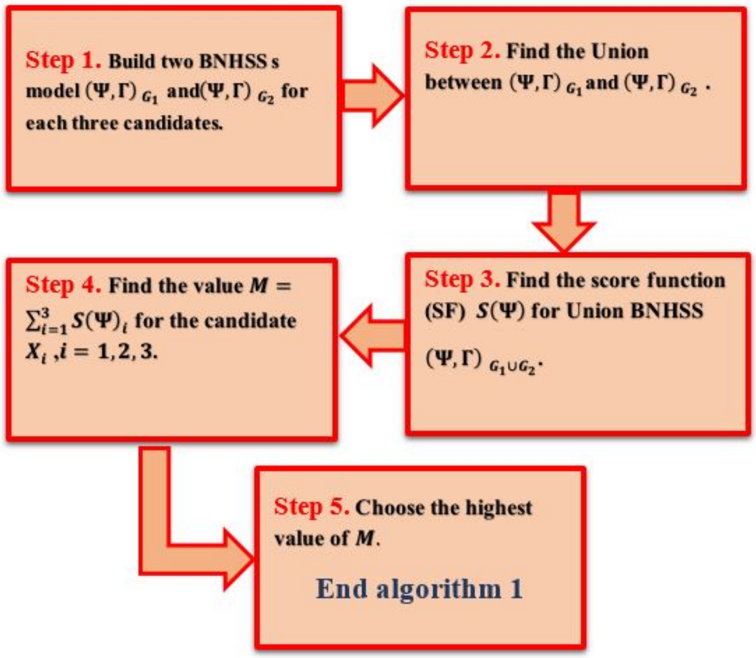

FIGURE 2. Algorithm 1. depends on score function (SF) values $S(\Psi)$

Step1. The jury members put their valuable opinions of each candidate in the form of two BNHSSs separately, as follows:

$(\Psi, \Gamma)_{G_1} =$
$\{(\alpha_1, \alpha_3, \alpha_5), [\hat{\varkappa}_1, 0.2, -0.1, 0.5, -0.9, 0.8, -0.7], [\hat{\varkappa}_2, 0.2, -0.1, 0.5, -0.9, 0.8, -0.7],$
$[\hat{\varkappa}_3, 0.2, -0.1, 0.5, -0.9, 0.8, -0.7],$
$(\alpha_1, \alpha_4, \alpha_5), [\hat{\varkappa}_1, 0.4, -0.3, 0.1, -0.4, 0.3, -0.6], [\hat{\varkappa}_2, 0.3, -0.4, 0.2, -0.6, 0.8, -0.5],$
$[\hat{\varkappa}, 0.4, -0.3, 0.7, -0.4, 0.9, -0.2],$
$(\alpha_1, \alpha_4, \alpha_7), [\hat{\varkappa}_1, 0.3, -0.2, 0.6, -0.4, 0.2, -0.9], [\hat{\varkappa}_2, 0.2, -0.6, 0.8, -0.2, 0.9, -0.1],$
$[\hat{\varkappa}_3, 0.4, -0.2, 0.9, -0.3, 0.8, -0.8],$
$(\alpha_1, \alpha_3, \alpha_6), [\hat{\varkappa}_1, 0.3, -0.6, 0.3, -0.5, 0.8, -0.3], [\hat{\varkappa}_2, 0.6, -0.8, 0.3, -0.6, 0.3, -0.9],$
$[\hat{\varkappa}_3, 0.3, -0.5, 0.7, -0.7, 0.9, -0.1],$
$(\alpha_2, \alpha_3, \alpha_5), [\hat{\varkappa}_1, 0.3, -0.8, 0.2, -0.4, 0.3, -0.9], [\hat{\varkappa}_2, 0.3, -0.1, 0.2, -0.4, 0.8, -0.3],$

$[\hat{\varkappa}_3, 0.1, -0.4, 0.3, -0.8, 0.4, -0.7]$,
$(\alpha_2, \alpha_4, \alpha_5), [\hat{\varkappa}_1, 0.2, -0.1, 0.5, -0.9, 0.8, -0.7], [\hat{\varkappa}_2, 0.6, -0.8, 0.3, -0.6, 0.3, -0.9]$,
$[\hat{\varkappa}_3, 0.2, -0.1, 0.5, -0.9, 0.8, -0.7]$,
$(\alpha_2, \alpha_4, \alpha_7), [\hat{\varkappa}_1, 0.2, -0.2, 0.4, -0.2, 0.7, -0.3], [\hat{\varkappa}_2, 0.5, -0.3, 0.9, -0.4, 0.3, -0.5]$,
$[\hat{\varkappa}_3, 0.4, -0.5, 0.8, -0.7, 0.4, -0.3]$,
$(\alpha_2, \alpha_3, \alpha_6), [\hat{\varkappa}_1, 0.5, -0.4, 0.8, -0.3, 0.8, -0.2], [\hat{\varkappa}_2, 0.3, -0.6, 0.3, -0.5, 0.8, -0.3]$,
$[\hat{\varkappa}_3, 0.3, -0.5, 0.7, -0.7, 0.9, -0.1]\}$

$(\Psi, \Gamma)_{G_2} =$
$\{(\alpha_1, \alpha_3, \alpha_5), [x_1, 0.5, -0.8, 0.1, -0.5, 0.4, -0.3], [x_2, 0.5, -0.3, 0.2, -0.5, 0.3, -0.9]$,
$[x_3, 0.3, -0.5, 0.2, -0.9, 0.4, -0.2]$,
$(\alpha_1, \alpha_4, \alpha_5), [x_1, 0.3, -0.6, 0.3, -0.5, 0.8, -0.3], [x_2, 0.6, -0.8, 0.3, -0.6, 0.3, -0.9]$,
$[x_3, 0.3, -0.5, 0.7, -0.7, 0.9, -0.1]$,
$(\alpha_1, \alpha_4, \alpha_7), [x_1, 0.2, -0.1, 0.5, -0.9, 0.8, -0.7], [x_2, 0.6, -0.8, 0.3, -0.6, 0.3, -0.9]$,
$[x_3, 0.2, -0.1, 0.5, -0.9, 0.8, -0.7]$,
$(\alpha_1, \alpha_3, \alpha_6), [x_1, 0.6, -0.6, 0.2, -0.8, 0.5, -0.4], [x_2, 0.3, -0.9, 0.2, -0.9, 0.2, -0.7]$,
$[x_3, 0.7, -0.8, 0.9, -0.2, 0.8, -0.3]$,
$(\alpha_2, \alpha_3, \alpha_5), [x_1, 0.1, -0.8, 0.2, -0.7, 0.3, -0.4], [x_2, 0.3, -0.1, 0.8, -0.4, 0.5, -0.3]$,
$[x_3, 0.2, -0.5, 0.3, -0.6, 0.3, -0.7]$,
$(\alpha_2, \alpha_4, \alpha_5), [x_1, 0.8, -0.6, 0.3, -0.6, 0.4, -0.8], [x_2, 0.5, -0.9, 0.4, -0.8, 0.7, -0.9]$,
$[x_3, 0.9, -0.1, 0.5, -0.5, 0.8, -0.7]$,
$(\alpha_2, \alpha_4, \alpha_7), [x_1, 0.2, -0.2, 0.4, -0.2, 0.7, -0.3], [x_2, 0.5, -0.3, 0.9, -0.4, 0.3, -0.5]$,
$[x_3, 0.4, -0.5, 0.8, -0.7, 0.4, -0.3]$,
$(\alpha_2, \alpha_3, \alpha_6), [x_1, 0.7, -0.2, 0.8, -0.9, 0.8, -0.2], [x_2, 0.8, -0.6, 0.8, -0.6, 0.8, -0.8]$,
$[x_3, 0.4, -0.5, 0.7, -0.7, 0.4, -0.1]\}$

Step 2. We follow the implementation of the two algorithms, precisely the second step , by calculating the union value between two BNHSSs. $(\Psi, \Gamma)_{G_1 \cup G_2}$ as follows .
$(\Psi, \Gamma)_{G_1 \cup G_2} =$
$\{(\alpha_1, \alpha_3, \alpha_5), [x_1, 0.5, -0.8, 0.1, -0.5, 0.4, -0.3], [x_2, 0.5, -0.3, 0.2, -0.5, 0.3, -0.9]$,
$[x_3, 0.3, -0.5, 0.2, -0.9, 0.8, -0.2]$,
$(\alpha_1, \alpha_4, \alpha_5), [x_1, 0.3, -0.6, 0.1, -0.4, 0.3, -0.3], [x_2, 0.6, -0.8, 0.3, -0.6, 0.3, -0.5]$,
$[x_3, 0.4, -0.5, 0.7, -0.4, 0.9, -0.1]$,
$(\alpha_1, \alpha_4, \alpha_7), [x_1, 0.3, -0.1, 0.5, -0.4, 0.2, -0.7], [x_2, 0.6, -0.8, 0.8, -0.2, 0.3, -0.1]$,
$[x_3, 0.4, -0.2, 0.5, -0.3, 0.8, -0.7]$,

$(\alpha_1, \alpha_3, \alpha_6), [x_1, 0.6, -0.6, 0.2, -0.8, 0.5, -0.3], [x_2, 0.6, -0.8, 0.2, -0.6, 0.2, -0.9],$
$[x_3, 0.7, -0.5, 0.7, -0.2, 0.8, -0.1],$
$(\alpha_2, \alpha_3, \alpha_5), [x_1, 0.3, -0.8, 0.2, -0.4, 0.3, -0.4], [x_2, 0.3, -0.1, 0.2, -0.4, 0.5, -0.3],$
$[x_3, 0.7, -0.5, 0.7, -0.2, 0.8, -0.1],$
$(\alpha_2, \alpha_4, \alpha_5), [x_1, 0.8, -0.1, 0.3, -0.9, 0.4, -0.7], [x_2, 0.6, -0.8, 0.3, -0.8, 0.3, -0.9],$
$[x_3, 0.9, -0.1, 0.5, -0.5, 0.8, -0.7],$
$(\alpha_2, \alpha_4, \alpha_7), [x_1, 0.8, -0.6, 0.4, -0.2, 0.7, -0.3], [x_2, 0.5, -0.9, 0.4, -0.4, 0.3, -0.5],$
$[x_3, 0.9, -0.5, 0.5, -0.5, 0.4, -0.3],$
$(\alpha_2, \alpha_3, \alpha_6), [x_1, 0.7, -0.2, 0.8, -0.3, 0.8, -0.2], [x_2, 0.8, -0.3, 0.3, -0.5, 0.3, -0.3],$
$[x_3, 0.4, -0.5, 0.7, -0.7, 0.4, -0.1]\}$

TABLE 1. SF values of $\hat{\varkappa}_i$ for candidates

$\mathcal{K}_\pi$	**SF for value $\hat{\varkappa}_1$**	**SF value for $\hat{\varkappa}_2$**	**SF value for $\hat{\varkappa}_3$**
$(\alpha_1, \alpha_3, \alpha_5)$	0.50	0.65	0.48
$(\alpha_1, \alpha_4, \alpha_5)$	0.50	0.55	0.30
$(\alpha_1, \alpha_4, \alpha_7)$	0.60	0.33	0.48
$(\alpha_1, \alpha_3, \alpha_6)$	0.47	0.65	0.33
$(\alpha_2, \alpha_3, \alpha_5)$	0.47	0.63	0.56
$(\alpha_2, \alpha_4, \alpha_5)$	0.43	0.50	0.55
$(\alpha_2, \alpha_4, \alpha_7)$	0.50	0.61	0.43
$(\alpha_2, \alpha_3, \alpha_6)$	0.76	0.65	0.62
Total Values of $\mathcal{M}_i$	$\mathcal{M}_1$= 2.756	$\mathcal{M}_2$= 2.527	$\mathcal{M}_3$= 2.936
Final Decision	$\mathcal{M}_1= \times$	$\mathcal{M}_2=\times$	$\mathcal{M}_3= \surd$

Step 3 . Table 1 collects the rest of the steps (3,4 and 5) mentioned in Algorithm 1, and the choice falls on the candidate $\hat{\varkappa}_3$.

Algorithm 2. Using the aggregation value $\widehat{\mathfrak{B}}_{agg}$ for candidates $\hat{\varkappa}_i$

Step 1. Put up BNHSSs $(\Psi, \Gamma)_{G_1}, (\Psi, \Gamma)_{G_2}$ respectively, based on expert opinions (two committees).

Step 2. Calculating the union value $(\Psi, \Gamma)_{G_1 \cup G_2}$ between two BNHSSs which given in step 1.

Step 3. Find the aggregation value $\widehat{\mathfrak{B}}_{agg}$ for Union BNHSS $(\Psi, \Gamma)_{G_1 \cup G_2}$ based on definition 4.2.

Step 4. Decision: Choose the highest value for the candidate X_i,$i = 1, 2, 3.$ to choose a suitable candidate.

Step 5. End algorithm 2.

In addition Figure 3 bellow representation of algorithm 2.

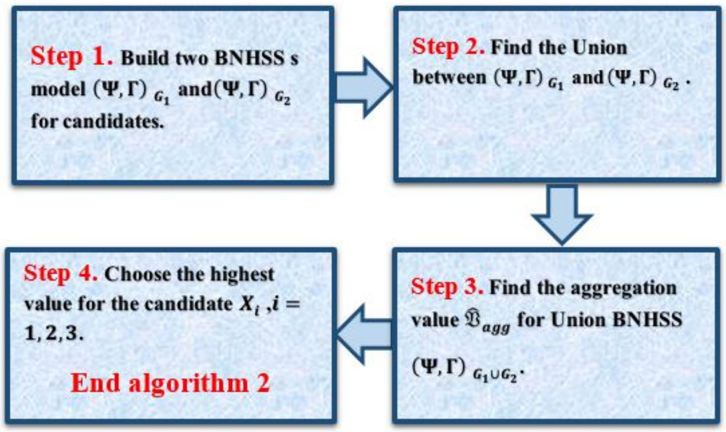

FIGURE 3. Algorithm 2. depends on aggregation values $\widehat{\mathfrak{B}}_{agg}$

Step 1 and Step 2 : These steps are the same as in steps 1 and 2 of algorithm 1. **Step 3 .**Table 2 collects the rest of the steps (3 and 4) mentioned in Algorithm 2, and the choice falls on the candidate $\hat{\varkappa}_3$.

TABLE 2. Aggregation value $\widehat{\mathfrak{B}}_{agg}$ of $\hat{\varkappa}_i$ for candidates

$\Xi_{\widehat{\mathfrak{B}}}(\hat{\varkappa})_i$	**Aggregation value** $\widehat{\mathfrak{B}}_{agg}$
$\Xi_{\widehat{\mathfrak{B}}}(\hat{\varkappa})_1$	0.870
$\Xi_{\widehat{\mathfrak{B}}}(\hat{\varkappa})_2$	0.838
$\Xi_{\widehat{\mathfrak{B}}}(\hat{\varkappa})_3$	0.896
Final Decision	$\mathcal{M}_1 = \Xi_{\widehat{\mathfrak{B}}}(\hat{\varkappa})_1 = \times$
	$\mathcal{M}_2 = \Xi_{\widehat{\mathfrak{B}}}(\hat{\varkappa})_2 = \times$
	$\mathcal{M}_3 = \Xi_{\widehat{\mathfrak{B}}}(\hat{\varkappa})_3 = \surd$

5.2. *Comparison analysis*

In this section, we prepared Table 3 and Figure 1 to compare the two algorithms presented in this part of the work. Both algorithms (algorithm 1 based on score function (SF) and algorithm 2 based on aggregation value) rely mainly on analyzing the data of the problem to be solved using our concept presented in this work.

In another instance of similar comparison, Table 4 provides another method of comparison with some of the previous works mentioned in the previous study in the first part of this

Table 3. Comparison between the values obtained from the two algorithms

Methods	$\hat{\varkappa}_1$	$\hat{\varkappa}_2$	$\hat{\varkappa}_3$	**Ranking**
SF for value $\hat{\varkappa}_i$	2.756	2.527	2.936	$\mathcal{W}_3 \succ \mathcal{W}_1 \succ \mathcal{W}_2$
Aggregation value $\widehat{\mathfrak{B}}_{agg}$	0.870	0.838	0.896	$\mathcal{W}_3 \succ \mathcal{W}_1 \succ \mathcal{W}_2$

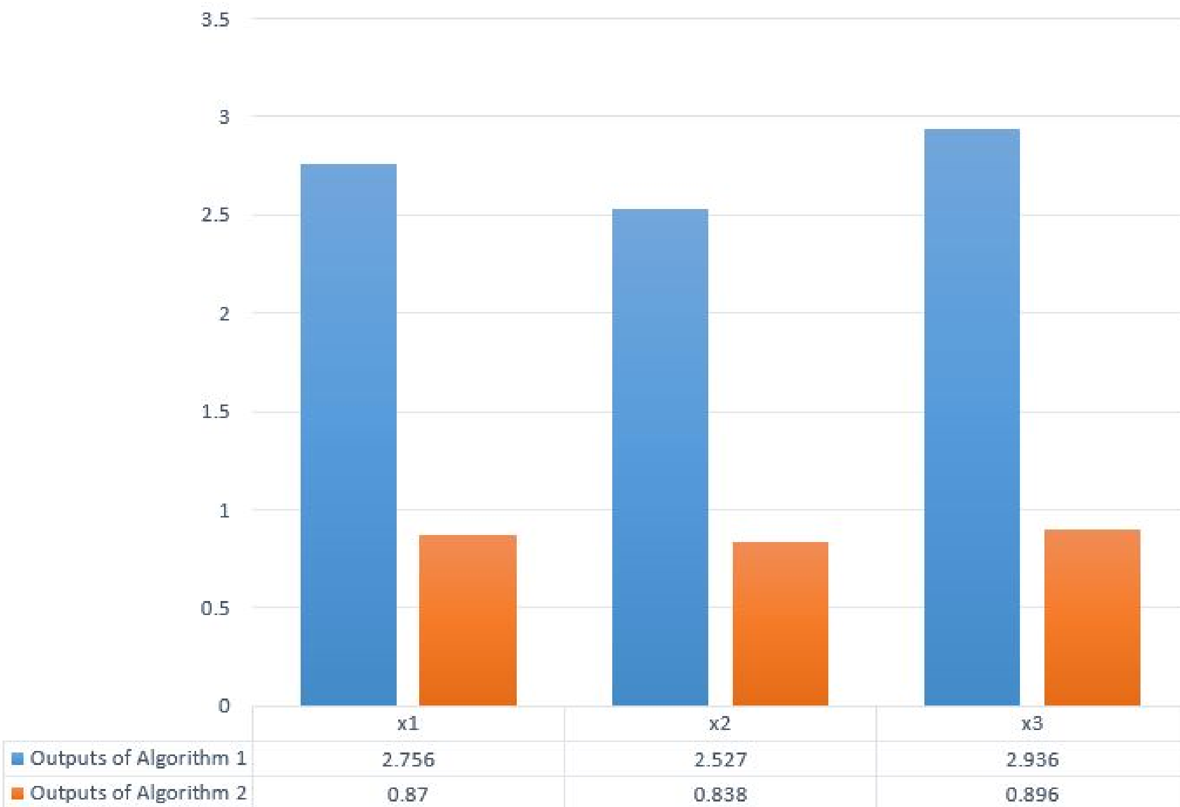

Figure 4. A statistical chart showing the vivid comparison between the numerical outputs of the two proposed algorithms

work. Our proposed concept is compared with some existing extensions of the fuzzy soft set under bipolarity, such as: bipolar fuzzy soft set(BFSS), bipolar intuitionistic fuzzy soft set (BIFSS), bipolar neutrosophic soft set (BNSS), and bipolar fuzzy hypersoft set(BFHSS) based on their structural composition, where TMD, IMD, FMD, SS, and HSS indicate to three NS memberships degree, soft set, and hypersoft set, respectively.

From Table 4, we notice that our concept is different from the previous concepts mentioned in the literature, and therefore it can be said that our proposed concept is more comprehensive than the previous concepts in covering ambiguous data of a positive and negative nature at the same time.

TABLE 4. The vivid comparison between the proposed structure and the existing structure.

Methods	**Authors**	**TMD**	**IMD**	**FMD**	**SS**	**HSS**
BFS	Zhang [52]	$\surd$	$\times$	$\times$	$\times$	$\times$
BFSS	Naz and Shabir [54]	$\surd$	$\times$	$\times$	$\surd$	$\times$
BIFSS	Jana and Pal [57]	$\surd$	$\times$	$\surd$	$\surd$	$\times$
BNSS	Ali et al. [59]	$\surd$	$\surd$	$\surd$	$\surd$	$\times$
BFHSS	Al-Quran et al. [64]	$\surd$	$\times$	$\times$	$\surd$	$\surd$
BNHSS	**Propose model**	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$

6. Conclusions

In this work, the novel idea of a new hybrid model of BNHSS by merging both neutrosophic sets (NSs) and HSSs under the bipolarity property of real numbers is provided. Furthermore, we studied its properties and necessary operations, such as complements, subset, unions, and intersections. Subsequently, we describe some operations, like "AND" and "OR," as well as their properties and some numerical examples. Two algorithms are discussed that rely on some mathematical methods (aggregation operator and score function) to deal with MAGDM in the BNHSS environment. In this study we attempt to develop more sophisticated model which has the advantages of all the previous models, however,there are still certain challenges with the work that is being suggested. . In BNHSS, we have only taken into consideration the evaluation information given in one dimension, where the time dimension does not enter into determining its fate. For the purpose of dealing with such data, we recommend that future studies combine the tools presented in this work with complex numbers. Also, the proposed model could be investigated more by proposing some aggregation operators such as Heronian mean, power mean, Hamacher, Bonferroni mean and Dombis aggregation operators to solve the existing decision making problems.

Conflicts of Interest: The authors declare no conflict of interest.

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Waste to Energy Treatment for India Using CRITIC-WASPAS Method under Interval-Valued Pythagorean Neutrosophic Fuzzy Set

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Abstract. Municipal solid waste management (MSWM) has emerged as a major issue in India due to the massive amounts of waste generated on a daily basis. Governments are focusing on establishing a proper waste management system, including a timeline for the installation of waste processing and disposal facilities to reduce waste. For this problem, waste to energy (WtE) technologies have been identified because of their ability to convert waste into green energy while minimizing associated issues. This research investigates the different kinds of WtE treatments based on various factors, namely eco-friendly, budget friendly, technical, and social aspects. The findings of this study revealed which WtE treatments are best suited for waste management systems that convert green energy. In this paper, we propose the best WtE treatment for India using the interval-valued pythagorean neutrosophic fuzzy set (PNFS). We employ the WASPAS-CRITIC method in the interval-valued pythagorean neutrosophic fuzzy environment for this WtE treatment problem. Finally, a numerical example and a comparison are provided to illustrate the reliability and efficiency of the proposed technique.

Keywords: Neutrosophic Fuzzy Set, Pythagorean Neutrosophic Fuzzy Set, Interval-Valued Pythagorean Neutrosophic Fuzzy Set, WASPAS, CRITIC, WtE Treatments.

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1. Introduction

Municipal solid waste (MSW), hazardous wastes, industrial wastes, agricultural wastes, and bio-medical wastes are some of the most common types of solid waste. Growing waste generation and unscientific waste disposal methods are leading to the release of GHG (methane, CO_2, etc.) into the environment. MSW is a complex mix of food waste, metals, glass, yard trimmings, woody waste, non-recyclable paper and plastic, construction and demolition waste, rags, and wastewater treatment sludge. When used as a raw material for power production, MSW presents several challenges: it has a low power content, high moisture, a diverse composition, and is copious. Managing waste safely is critical for the environment and the long-term goals for both the public and private sectors. Harvest trash, livestock wastes, slaughter waste, forest wastes, and other agricultural recycle waste materials are examples. WtE routes helps to converted waste into useful power forms such as bio-hydrogen, biogas, bio-alcohols, and so on, allowing for the sustainable global development. Sum of solid waste is produced annually across country as a form of by-product of industrial, municipal, agricultural, mining, and other processes [1].

MSW management has originally involved in discharging waste in open dumps and burning it to decrease waste volumes. Dangerous Industrial waste was frequently disposed of alongside municipal garbage and refuse in open dumps or landfills. Contaminated Groundwater, toxic fume and greenhouse gas emissions, land contamination, and large pest and disease vector populations, such as rats, flies, and mosquitoes, all of them have been tied into these old landfills in the past. To reduce these environmental impacts when we dispose of MSW, the treatment of waste into energy is now done within the framework of waste management regulations [23]. Most countries are focusing on WtE projects for municipal solid waste.

In this study, we propose an appropriate WtE treatment for India using a fuzzy approach MCDM model. Reducing waste and finding new environmentally friendly forms of energy will help countries solve energy demand problems and develop a hygienic society in the near future. WtE treatments would be the best way to get renewable energy. These innovative technologies can generate huge amounts of heat and energy from waste, reducing the serious environmental problems associated with MSW and reducing the use of junk fuels that emit gas. Green houses cause climate change and global water consumption. These WtE treatments can also be useful to society due to their economic and environmental benefits.

In 1965, Zadeh introduced fuzzy logic concepts [2] to address the problems of human decision-making under unreliability. Fuzzy sets (FS) have some limitations when non membership concepts are involved. To tackle this issue, Atanassov [3] transformed a FS into an intuitionistic fuzzy set (IFS) by including a non-membership function. Yager [4–6] developed the Pythagorean fuzzy set (PFS), which has a larger solution space in ambiguous and imprecise environments. In addition, when compared to FS and IFS, the Pythagorean treatment for India Using CRITIC-WASPAS Method under IVPNFS fuzzy number provided a more comprehensive computational model. Smarandache [7] introduced the concepts of neutrosophic set and neutrosophic probability and their logic, which contains of three logics: truth, indeterminacy, and falsity-membership degree. Interval-valued fuzzy sets were introduced by Zadeh [2].

An interval-valued membership function defines an interval-valued fuzzy set (IVFS). IVFs are a subset of L-fuzzy sets [8] and type-2 fuzzy sets [9].

The interval-valued Pythagorean fuzzy set (IVPFS) is a PFS extension [10]. Due to a lack of easily available information, experts may find it difficult to explain their ideas accurately with a specific number for many real-world decision-related difficulties, but they can do so by using an interval number between [0, 1]. This entails the idea of IVPFS, which permits both the degrees of a set's membership and absence to have an interval value. It should be emphasized that IVPFS turns into PFS when the interval values' upper and lower limits are identical, proving that the latter is a special case of the former [11, 12]. In order to provide a more dependable solution to the WtE treatment problem, we apply the suggested model in this research to combine two sets, such as Pythagorean and Neutrosophic fuzzy sets, in interval form, namely as an interval-valued Pythagorean Neutrosophic fuzzy set.

Some new operations and properties for IVPFS are proposed by Peng and Yang [12]. Garg [13] discussed an accuracy function for IVPFS to solve the MCDM problem. Liang et al. [14] introduced the interval-valued Pythagorean fuzzy weighted aggregating operators. Garg [15] demonstrated an improved score function for a Pythagorean fuzzy set-based TOPSIS method with interval values. Chen [16] examined the IVPF outranking algorithm for the MCDM problem. Rahman et al. [17] proposed IVPF geometric aggregation operators for the MCGDM problem. Stephy Stephen and Helen [18] discussed the IVPN set and their application using TOPSIS. Narmada devi and Sowmiya [19, 20] introduced the Octagonal neutrosophic fuzzy number in game and sequencing problem. Jansi et al. [21] examined the basic operations and correlation measure of PNS set. Abdel-Basset et al. [22] used a hybrid MCDM approach in a neutrosophic environment. Khan et al. [23] explored the effects of renewable electricity generation from waste. Van Thanh et al. [24] proposed a fuzzy MCDM model to evaluate and select a location for a solid WtE plant in Vietnam. Kurbatova and Abu-Qdais [25] used AHP to evaluate the various waste-to-energy options and chose the best technology for Moscow. Hezam et al. [26] examined the optimal selection of recycling plant site. Sleem et al. [27] investigated te product's target demographic using CRITIC model under neutrosophic set. Gamal and Mohamed [28] proposed the industrial robots selection using hybrid MCDM approach. Narmada Devi et al. [29] proposed the suitable waste to energy technology for India using MULTIMOORA method. The majority of WtE options were identified under different MCDMs using various fuzzy sets in the studies reviewed above. In this study, we identify the appropriate WtE treatment for India based on the WASPAS model under an interval-valued Pythagorean Neutrosophic fuzzy set. The pictorial representation of the algorithm is shown in figure 1.

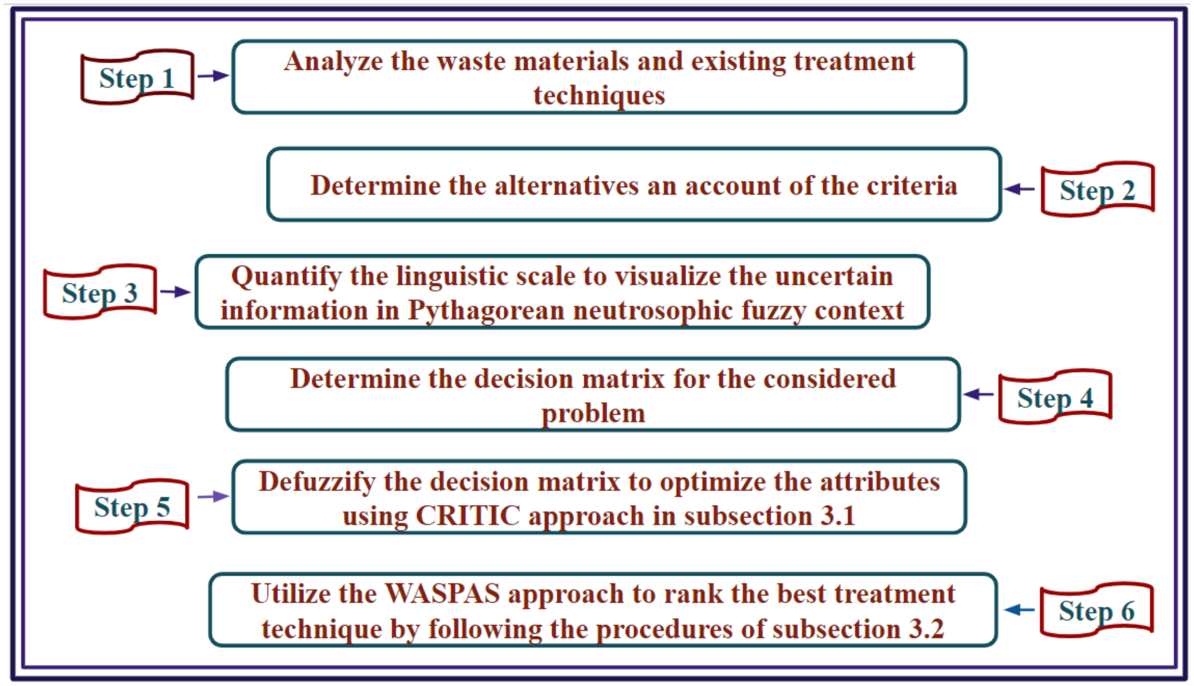

FIGURE 1. **Procedures of integrated fuzzy MCDM approach**

2. Preliminaries

Definition 2.1. [4–6] Consider Ω to be a non-empty set. Then a Pythagorean fuzzy set P over Ω, which is defined as follows:

$$P = \{(f, \alpha(f), \beta(f))|f \in \Omega\}$$

where $\alpha_P(f) : \Omega \to [0,1]$ and $\beta_P(f) : \Omega \to [0,1]$ define the membership and non-membership, of the element $f \in \Omega$ to P.

$$0 \leq (\alpha_P(f))^2 + (\beta_P(f))^2 \leq 1$$

Suppose $(\alpha_P(f))^2+(\beta_P(f))^2 \leq 1$ then there is a degree of indeterminacy of $f \in \Omega$ to P defined by $\alpha_P(f) : \sqrt{1 - [(\alpha_P(f))^2 + (\beta_P(f))^2]}$ and $\alpha_P(f) \in [0,1]$. In follows, $(\alpha_P(f))^2+(\beta_P(f))^2 = 1$. Otherwise, $\alpha_P(f) = 0$ whenever $(\alpha_P(f))^2 + (\beta_P(f))^2 = 1$.

Definition 2.2. [7]

A Neutrosophic fuzzy set N on Ω is an object of the form:

$$N = \{(f, \alpha_N(f), \gamma_N(f), \beta_N(f)) : f \in \Omega\}$$

where $\alpha_N(f), \gamma_N(f), \beta_N(f) \in [0,1], 0 \leq \alpha_N(f) + \gamma_N(f) + \beta_N(f) \leq 3$ for all $f \in \Omega$, $\alpha_N(f)$, $\gamma_N(f)$, $\beta_N(f)$ are degrees of membership, indeterminacy and non-membership, respectively.

TABLE 1. Interval-Valued Pythagorean Neutrosophic fuzzy linguistic scale

Linguistic term	membership values	indeterminacy values	non-membership values
Extremely elevated (EE)	[0.1, 0.2]	[0.5, 0.6]	[0.8, 0.9]
Average elevated (AE)	[0.2, 0.4]	[0.4, 0.5]	[0.6, 0.8]
Average (A)	[0.4, 0.6]	[0.3, 0.4]	[0.4, 0.6]
Average dropped (AD)	[0.6, 0.8]	[0.2, 0.3]	[0.2, 0.4]
Extremely dropped (ED)	[0.8, 0.9]	[0.1, 0.2]	[0.1, 0.2]

Definition 2.3. [10] A Pythagorean Neutrosophic fuzzy set (PNFS) with T and F are dependent Neutrosophic components D on Ω is in the form

$$D = \{(f, \alpha_D(f), \gamma_D(f), \beta_D(f)) : f \in \Omega\}$$

where $\alpha_D(f), \gamma_D(f), \beta_D(f) \in [0,1]$, $0 \leq (\alpha_D(f))^2 + (\gamma_D(f))^2 + (\beta_D(f))^2 \leq 2$, for all $f \in \Omega$, $\alpha_D(f)$, $\gamma_D(f)$ and $\beta_D(f)$ are degrees of membership, indeterminacy, non-membership, respectively.

Definition 2.4. [12,13] A Interval-Valued Pythagorean Neutrosophic fuzzy set (PNFS) with T and F are dependent Neutrosophic components C on Ω is in the form

$$C = \{(f, [\alpha_C^L(f), \alpha_C^U(f)], [\gamma_C^L(f), \gamma_C^U(f)], [\beta_C^L(f), \beta_C^U(f)] : f \in \Omega\}$$

where $[\alpha_C^L(f), \alpha_C^U(f)], [\gamma_C^L(f), \gamma_C^U(f)], [\beta_C^L(f), \beta_C^U(z)] \in [0,1]$,

$$0 \leq [\frac{\alpha_C^L(f) + \alpha_C^U(f)}{2}]^2, [\frac{\gamma_C^L(f) + \gamma_C^U(f)}{2}]^2, [\frac{\beta_C^L(f) + \beta_C^U(f)}{2}]^2 \leq 2$$

, for all $f \in \Omega$, $[\alpha_C^L(f), \alpha_C^U(f)]$ is the degree of membership, $[\gamma_C^L(f), \gamma_C^U(f)]$ is the degree of indeterminacy and $[\beta_C^L(f), \beta_C^U(f)]$ is the degree of non-membership.

Definition 2.5. [13] The score function of the Pythagorean Neutrosophic fuzzy sets with dependent Pythogorean Neutrospohic components I and F are defined as:

$$S_C(x) = [T_C^L + (1 - I_C^U) + (1 - F_C^U), T_C^U + (1 - I_C^L) + (1 - F_C^L)]$$

with the condition $0 \leq [\frac{\alpha_C^L(x)+\alpha_C^U(x)}{2}]^2, [\frac{\gamma_C^L(x)+\gamma_C^U(x)}{2}]^2, [\frac{\beta_C^L(x)+\beta_C^U(x)}{2}]^2 \leq 2$.

Note:

The linguistic variables with Interval-Valued Pythgorean Neutrosophic fuzzy number to evaluate the WtE treatment based on selected criteria and the linguistic scale is presented in Table 1.

3. **Mathematical methods**

3.1. *The CRITIC method*

The CRITIC approach is one of the objective weighing methods suggested by Diakoulaki et al. [34]. It employs a decision matrix explicitly to compute criterion weights objectively. There is no requirement for decision-makers opinions or pairwise comparisons, as in other weighing procedures. Based on an analysis of the evaluation matrix, it collects all of the preference information contained in the evaluation criteria. Further, the objective weight is determined by quantifying the inherent information of each criterion. The procedure for obtaining objective weight includes not only the standard deviation of the criteria but also the correlation between the other criteria.

The steps of the CRITIC method are presented below [35].
Here, the problem has m alternatives $K_i(i = 1, 2, ..., m)$ and n criteria $V_j(j = 1, 2, ..., n)$.

Step 1: Here is the DM K as it is formed. It compares the performance of various alternatives based on selected criteria.

$$K = \begin{bmatrix} [k_{11}^L, k_{11}^U] & [k_{12}^L, k_{12}^U] & \dots & [k_{1n}^L, k_{1n}^U] \\ [k_{21}^L, k_{21}^U] & [k_{22}^L, k_{22}^U] & \dots & [k_{2n}^L, k_{2n}^U] \\ \dots & \dots & \dots & \dots \\ [k_{m1}^L, k_{m1}^U] & [k_{m2}^L, k_{m2}^U] & \dots & [k_{mn}^L, k_{mn}^U] \end{bmatrix} \tag{1}$$

Step 2: The DM is normalized by applying the below equation:

$$k_{ij}^* = \frac{k_{ij} - min(k_{ij})}{max(k_{ij}) - min(k_{ij})}, i = 1, 2, ..., m, j = 1, 2, ..., n \tag{2}$$

k_{ij} is the normalized value of i^{th} alternative on j^{th} criterion.

Step 3: Both the criterion's standard deviation (SD) and its correlation with other criteria are considered when determining the criteria's weights. The weight of the j^{th} criterion (w_j) is calculated as follows:

$$w_j = \frac{H_j}{\sum_{j=1}^{n} H_j} \tag{3}$$

where H_j is the quantity of information which obtained as:

$$H_j = \Gamma_j \sum_{j=1}^{n} (1 - t_{jj'}) \tag{4}$$

where j is SD of the j^{th} criterion and $t_{jj'}$ is the correlation coefficient between the two criteria. It is possible to conclude that this method gives more weight to the criterion with a high SD and a low correlation with other criteria [36]. A significantly higher value of H_j indicates that more detail is obtained from criterion, implying that the criterion's relative importance for the decision making problem is greater.

3.2. *The WASPAS method*

The WASPAS method was created by Chakraborty and Zavadskas [30]. The WSM method computes an alternative's entirety as a weighted sum of the criteria standards, whereas the WPM technique calculates an alternative's score as a product of the scaled grading of every criteria to a power equal to the weight of the specified criterion [31]. Furthermore to these approaches, WASPAS efforts to achieve the highest precision for estimation by optimising weighted aggregated functions [30]. The combined optimality on criteria values computed based on the results of these two models for rank the alternatives. The model is actually proposed as the best MCDM method in terms of accuracy or verification of accuracy when those two methods are used together.

The algorithm of the WASPAS model are as follows [32, 33]:

Step 1: Create the initial decision matrix (DM).

$$K = \begin{bmatrix} [k_{11}^L, k_{11}^U] & [k_{12}^L, k_{12}^U] & \dots & [k_{1n}^L, k_{1n}^U] \\ [k_{21}^L, k_{21}^U] & [k_{22}^L, k_{22}^U] & \dots & [k_{2n}^L, k_{2n}^U] \\ \dots & \dots & \dots & \dots \\ [k_{m1}^L, k_{m1}^U] & [k_{m2}^L, k_{m2}^U] & \dots & [k_{mn}^L, k_{mn}^U] \end{bmatrix} \tag{5}$$

where m represents the alternatives, n represents the criteria and k_{ij} is the performance value of i^{th} alternative with respect to j^{th} criteria.

Step 2: Calculate the linear normalized decision matrix using the following equations:

For benefit criteria:

$$\bar{k}_{ij} = \frac{k_{ij}}{\max_j k_{ij}} \tag{6}$$

For non-benefit criteria:

$$\bar{k}_{ij} = \frac{\min_j k_{ij}}{k_{ij}} \tag{7}$$

Where $\bar{k}_{ij}$ is the normalized value of k_{ij}.

Step 3: Compute the measures of WSM (L_j^1) and WPM (L_j^2) for each alternative by applying the below equation:

$$L_j^1 = \sum_{i=1}^{m} w_i \bar{k}_{ij} \tag{8}$$

$$L_j^2 = \prod_{i=1}^{m} (\bar{k}_{ij})^{w_i} \tag{9}$$

Step 4: To obtain the aggregate measure of the WASPAS model for every alternative using the below expression:

$$L_j = \delta L_j^1 + (1-\delta)L_j^2 \tag{10}$$

Where δ is the parameter of the model. It can take values in $[0-1]$. When $\delta = 1$, the WASPAS model is transformed to WSM, and $\delta = 0$ into WPM model.

Step 5: Finally, according to decreasing values of L_i, rank the alternatives.

4. **Application**

A considerable quantity of waste is produced in developing countries. The primary explanations for propelling waste generation and creating distinguished social and environmental concerns are accelerating urbanisation, economic expansion, population increase, and modern technology. Waste management is increasingly focused on sophisticated waste reduction strategies, but they are still looking for the optimum response to that issue with no adverse environmental or social impact. As a result, we have to figure out the optimal WtE treatment to create green energy from MSW wastes, thus contributing to environmental sustainability. In this work, we presented the WASPAS approach using an IVPNFS to discover the optimal WtE therapy for India. Based on the parameters we identified, we picked four types of WtE procedures.

5. **Numerical example**

In this section, we discuss the WtE treatment under the interval-valued pythagorean neutrosophic fuzzy set using the CRITIC-WASPAS method. Here, the experts evaluate this problem based on four criteria. The WtE treatment are: K_1- photo-biological process; K_2- dark fermentation; K_3- microbiological fuel cells; and K_4- microbial electrolysis cells. In this paper, experts evaluate the WtE treatment using the WASPAS method under IVPNFS. The linguistic scale is used to form a decision matrix. We are now analyzing the problem under proposed method.

5.1. *CRITIC method*

Step 1: Here is the decision matrix K as it is formed which is shown in Table 2. Using the IVPNFSs score function to create the DM shown in Table 3.

Step 2: The DM is normalized by applying the equation (8) and the normalized matrix is given in Table 4.

Step 3: Finally, the weight values of the criteria are computed by using the equation (9) and (10). The weight values of the criteria are 0.2772, 0.3938, 0.2735, 0.0555.

Table 2. Initial decision matrix

	V_1	V_2	V_3	V_4
K_1	([0.2,0.4], [0.3, 0.4], [0.4, 0.6])	([0.1, 0.2], [0.3, 0.4], [0.2, 0.4])	([0.4, 0.6], [0.1, 0.2], [0.2, 0.4])	([0.1, 0.2], [0.4, 0.5], [0.2, 0.4])
K_2	([0.6, 0.8], [0.3, 0.4], [0.1, 0.2])	([0.4, 0.6], [0.1, 0.2], [0.1, 0.2])	([0.2, 0.4], [0.3, 0.4], [0.2, 0.4])	([0.1, 0.2], [0.2, 0.3], [0.6, 0.8])
K_3	([0.6, 0.8], [0.4, 0.5], [0.4, 0.6])	([0.6, 0.8], [0.4, 0.5], [0.1, 0.2])	([0.6, 0.8], [0.1, 0.2], [0.2, 0.4])	([0.1, 0.2], [0.4, 0.5], [0.6, 0.8])
K_4	([0.1, 0.2], [0.4, 0.5], [0.4, 0.6])	([0.1, 0.2], [0.2, 0.3], [0.4, 0.6])	([0.1, 0.2], [0.5 0.6], [0.2, 0.4])	([0.2, 0.4], [0.3, 0.4], [0.1, 0.2])

Table 3. Decision matrix

	V_1	V_2	V_3	V_4
K_1	-0.4752	-0.3676	1.0154	-0.5015
K_2	0.4513	0.3781	-0.2236	-0.8551
K_3	0.5820	0.7627	0.5820	-0.1076
K_4	-0.7354	-0.5307	-0.6562	-0.1219

Table 4. Normalized decision matrix

	V_1	V_2	V_3	V_4
K_1	1.0000	1.0000	0.6700	-0.0384
K_2	0.9952	1.0000	0.6255	0.0442
K_3	0.6700	0.6255	1.0000	-0.7090
K_4	-0.0384	0.0442	-0.7090	1.0000

5.2. *WASPAS method:*

The WtE treatment and the criteria are given below:

$$\begin{aligned} &K_1 - \text{Photo - biological processes} \\ &K_2 - \text{Dark fermentation} \\ &K_3 - \text{Microbiological fuel cells} \\ &K_4 - \text{Microbial electrolysis cells} \\ \\ &V_1 - \text{Ecosystem} \\ &V_2 - \text{Cost} \\ &V_3 - \text{Technical aspects} \\ &V_4 - \text{Social aspects} \end{aligned} \tag{11}$$

Step 1: The decision matrices are shown in Table 2 and 3.

Step 2: Calculate the linear normalized decision matrix using the equations (2) and (3).

Step 3: Calculated the measures of WSM (L_j^1) and WPM (L_j^2) for each alternative by using

TABLE 5. WSM and WPM values

WtE treatment	WSM (L_j^1)	WPM (L_j^2)
K_1	0.0482	0.0039
K_2	-0.1468	0.0110
K_3	0.9630	0.0012
K_4	-0.8402	-0.0014

TABLE 6. The final ranking results for proposed method

WtE	L_j	Rank
K_1	0.0299	2
K_2	-0.0569	4
K_3	0.4833	1
K_4	-0.4221	3

the equations (4) and (5). The WSM and WPM values are presented in Table 5.

Step 4: Finally, rank the alternatives according to decreasing values of L_j. The final ranking results is shown in Table 6.

From this Table 6, K_3- Microbiological fuel cells in WtE treatment is the most suitable and eco-friendly treatment, which make more green energy to keep environment clean and provide great employment to our society.

6. Comparison and sensitivity analysis

6.1. *Comparison Analysis*

To show the suggested approach's efficacy in comparison to other approaches from the literature, this section compares it against an assortment of those methods. The proposed approach was compared to two MCDM techniques: TOPSIS [33] and VIKOR [37]. These MCDM approaches employ the same weights. The results of the ranking order comparison are shown in table 7. The suggested ranking yields different outcomes from the compared models. As a result, when compared to existing MCDM approaches, the suggested methodology generates more dependable findings.

6.2. *Sensitivity analysis*

This model's sensitivity analysis compares the outcomes of four cases. Case 1 is the study's outcome, and Cases 2, 3, and 4 are the other outcomes discovered by varying the weights of the criteria, which are given in Table 8. Sensitivity analysis reveals that changing the weights

Table 7. Comparison analysis results

WtE	VIKOR	Rank	TOPSIS	Rank	Proposed method	Rank
K_1	0	1	0.6980	1	0.0299	2
K_2	0.4616	2	0.4419	3	-0.0569	4
K_3	1	4	0.3186	4	0.4833	1
K_4	0.4835	3	0.5196	2	-0.4221	3

Table 8. Weights in sensitivity analysis

WtE	Case 1	Case 2	Case 3	Case 4
V_1	0.2772	0.0555	0.2735	0.3938
V_2	0.3938	0.2772	0.0555	0.2735
V_3	0.2735	0.3938	0.2772	0.0555
V_4	0.0555	0.2735	0.3938	0.2772

Table 9. Sensitivity analysis results

WtE	Case 1	Rank	Case 2	Rank	Case 3	Rank	Case 4	Rank
K_1	0.247	3	0.286	2	0.178	4	0.289	1
K_2	0.286	2	0.178	4	0.289	1	0.247	3
K_3	0.178	4	0.289	1	0.247	3	0.286	2
K_4	0.289	1	0.247	3	0.286	2	0.178	4

of the criteria affects the ranking order. Those results of sensitivity analysis are presented in Table 9.

7. Conclusion

The normative waste disposal practices used in India, such as mass burning and dumps, have had detrimental effects on the environment and the general population. The nation, nonetheless, has identified the unexpected implications and harms of such approaches and has recommended ecologically friendly and cost-effective waste management options. Notwithstanding rising oil and other fossil fuel costs and the depletion of fossil fuels, demand for energy is increasing. If India prioritises economic and logistical planning, failures may be avoided. Furthermore, the entire country should seek to strengthen the regulatory framework, which may result in people, industry stakeholders, and shareholders fighting the process.We presented suggestions for the most effective and feasible treatment of WtE for waste management and energy generation in India, which eliminates huge quantities of greenhouse gases and carbon from the atmosphere and leads to global warming along with alterations in the climate, according to the research findings.

As a result, this research was conducted in order to present a broad, systematic framework that might aid policymakers in determining the most effective WtE treatment for constructing waste management systems in India. The IVPNFS score function and the IVPNF-WASPAS technique based on it are provided in this work. IVPNFNs are used to represent the characteristics of each WtE therapy. New trends in WtE have been recognised as the cleanest and most advantageous WtE technology in the present environment based on the suggested approach for determining the most suitable solution for the aforementioned issue. The suggested approach stated that the energy provided by microbiological fuel cells (K_3) is superior to other strategies in terms of releasing enough energy to partially cover the costs. This method contributes to waste reduction while also producing energy, which will help with future energy demand issues.

However, advancements in technical tools and techniques for updating WtE technologies are on the horizon. In addition, integrated outranking technologies with improved theoretical underpinnings will be pursued in the future.

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Some remarks on Δ^m-Cesàro summability in neutrosophic normed spaces

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Abstract. In this paper, we define the notion of a generalized summability, called Cesàro summability in neutrosophic normed spaces (briefly NNS). We obtain conditions under which ordinary summability follows from Cesàro summability. Later, we define a concept of slowly oscillating sequences in NNS and establish related Tauberian Theorems in neutrosophic normed spaces.

Keywords: Neutrosophic normed spaces, Cesàro summability, slow oscillation and Tauberian theorem.

MSC: 46S40, 11B39, 03E72 and 40G15.

1. Introduction

In Analysis, we usually face many situations where the analytic solutions to some problems seem difficult due to the divergence of an infinite series or a power series. Consequently, we look forward to a modified method of convergence that can sum up the divergence series in some sense and call it a method of summability. A well-known method is due to Cesàro for number sequences known as Cesàro summability and is defined as follows:

"A sequence $x = (x_n)$ of numbers is said to be Cesàro summable [or $(C, 1)-$ summable to x_0 if

$$\lim_{n\to\infty}\left(\frac{x_1 + x_2 + \ldots x_n}{n}\right) = x_0."$$

If $\lim_{n\to\infty}(x_n) = x_0$, then (x_n) is $(C, 1)-$summable to x_0 however, the reverse way implication may not be true. But by adding some additional conditions on sequence called "Tauberian conditions", we obtained the result in the reverse way too. These results obtained by imposing Tauberian conditions are known as Tauberian Theorems. In past years, many interesting works have been carried out in this direction and various kinds of Tauberian theorems have been proved. For some historical view on Cesàro summability and Tauberian Theorems, we refer to the reader [1], [13]-[15] and [24]-[26].

On the other side, Zadeh [28] observed first time that many real-life situations cannot be set in the framework of classical sets. Therefore, to deal with such situations, in 1965, he proposed the idea of fuzzy sets via introducing the membership function. Later, a revolutionary development on fuzzy sets has been started. Many existing ideas have been developed again by applying fuzzy logic. During this developmental phase, several intriguing generalizations of fuzzy sets have emerged in the literature. For example: intuitionistic fuzzy sets (IFS) [2], vague fuzzy sets [5], neutrosophic sets (NS) [12], interval-valued fuzzy sets [27], etc. Analogous to the classical set theory, these sets have also been employed to introduce novel spaces, including fuzzy normed spaces ([6], [11]), intuitionistic fuzzy normed spaces ([7], [8], [19], [21]), and neutrosophic normed spaces ([3], [4], [9], [10], [17], [18], [20], [22], [23]). To develop these spaces mathematically and topologically, we need to define the concept of limit as one of the fundamental concepts. Some interesting works in this direction can be found in [7] - [11], etc. Recently, Talo and Yavuz [25] studied Cesàro summability and proved some Tauberian theorems in an intuitionistic fuzzy normed space. As neutrosophic normed spaces are generalizations of intuitionistic fuzzy normed spaces so it is natural to extend Cesàro summability and related concepts in these spaces. In present paper, we define Cesàro summability, slowly oscillating sequences and prove some Tauberian theorems in neutrosophic normed spaces. We organize the paper as follows, the first and second sections are introductory and provide basic information needed in the sequel. In third section we define Δ^m-Cesàro summability in NNS and obtained certain results. Finally in last section we define Slowly oscillating sequences in NNS and establish related Tauberian Theorems in neutrosophic normed spaces.

2. **Background and Preliminaries**

This section begin with a short review on some definitions and results.

Throughout this work, I will denote the closed interval $[0, 1]$, and $\mathbb{N}$ and $\mathbb{R}^+$ denotes the set of positive integers and positive reals, respectively.

Definition 2.1 [8] "A map from $\circ : I \times I$ to I is said to be a continuous $t-$norm if, $\forall$ $f, g, h, i \in I$ we have:

(i) $f \circ g = g \circ f$;

(ii)$f \circ (g \circ h) = (f \circ g) \circ h$;

(iii) $\circ$ is continuous;

(iv) $f \circ 1 = f$ and

(v) $f \circ g \leq h \circ i$ whenever $f \leq h$ and $g \leq i$."

Definition 2.2 [8] "A map from $\diamond : I \times I$ to I is said to be a continuous triangular conorm or $t-$conorm if for all $f, g, h, i \in I$ we have:

(i) $f \diamond g = g \diamond f$;

(ii)$f \diamond (g \diamond h) = (f \diamond g) \circ h$;

(iii) $\diamond$ is continuous;

(iv) $f \diamond 0 = f$ for every $f \in [0,1]$

(v) $f \diamond g \leq h \diamond i$ whenever $f \leq h$ and $g \leq i$."

Definition 2.3 [10] "A four tuple $V = (F, N, \ \circ, \diamond, \ )$ where F be a vector space, $N = \{\langle \vartheta, \mathcal{H}(\vartheta), \mathcal{I}(\vartheta), \mathcal{J}(\vartheta)\rangle : \vartheta \in F\}$ be a normed space with $N : F \times \mathbb{R}^+ \to I$ and $\circ$, $\diamond$ respectively are continuous $t-$norm and continuous $t-$conorm, is called a neutrosophic normed spaces (NNS) if the following conditions hold: For every $u, v \in F$ and $\mathfrak{y}_1, \mathfrak{y}_2 > 0$ and for every $\alpha \neq 0$ we have (i) $0 \leq \mathcal{H}(u, \ \mathfrak{y}_1) \leq 1, 0 \leq \ \mathcal{I}(u, \ \mathfrak{y}_1) \leq 1, 0 \leq \mathcal{J}(u, \ \mathfrak{y}_1) \leq 1$ for every $\mathfrak{y}_1 \in \mathbb{R}^+$;

(ii) $\mathcal{H}(u, \ \mathfrak{y}_1) + \ \mathcal{I}(u, \ \mathfrak{y}_1) + \mathcal{J}(u, \ \mathfrak{y}_1) \leq 3$ for $\mathfrak{y}_1 \in \mathbb{R}^+$;

(iii) $\mathcal{H}(u, \ \mathfrak{y}_1) = 1 \quad (\text{for } \mathfrak{y}_1 > 0)$ if and only if $u = \theta$;

(iv) $\mathcal{H}(\alpha u, \ \mathfrak{y}_1) = \mathcal{H}\left(u, \ \frac{\mathfrak{y}_1}{|\alpha|}\right)$; (v) $\mathcal{H}(u, \ \mathfrak{y}_1) \circ \mathcal{H}(v, \ \mathfrak{y}_2) \leq \mathcal{H}(u + v, \ \mathfrak{y}_1 + \mathfrak{y}_2)$;

(vi) $\mathcal{H}(u, \ .)$ is continuous non-decreasing function;

(vii) $\lim_{\mathfrak{y}_1 \to \infty} \mathcal{H}(u, \ \mathfrak{y}_1) = 1$;

(viii) $\mathcal{I}(u, \ \mathfrak{y}_1) = 0 \quad (\text{for } \mathfrak{y}_1 > 0)$ if and only if $u = \theta$;

(ix) $\mathcal{I}(\alpha u, \ \mathfrak{y}_1) = \mathcal{I}\left(u, \ \frac{\mathfrak{y}_1}{|\alpha|}\right)$;

(x) $\mathcal{I}(u, \ \mathfrak{y}_1) \diamond \mathcal{I}(v, \ \mathfrak{y}_2) \geq \mathcal{I}(u + v, \ \mathfrak{y}_1 + \mathfrak{y}_2)$;

(xi) $\mathcal{I}(u, \ .)$ is continuous non-decreasing function;

(xii) $\lim_{\mathfrak{y}_1 \to \infty} \mathcal{I}(u, \ \mathfrak{y}_1) = 0$;

(xiii) $\mathcal{J}(u, \ \mathfrak{y}_1) = 0 \quad (\text{for } \mathfrak{y}_1 > 0)$ if and only if $u = \theta$;

(xiv) $\mathcal{J}(\alpha u, \ \mathfrak{y}_1) = \mathcal{J}\left(u, \ \frac{\mathfrak{y}_1}{|\alpha|}\right)$;

(xv) $\mathcal{J}(u, \ \mathfrak{y}_1) \diamond \mathcal{J}(v, \ \mathfrak{y}_2) \geq \mathcal{J}(u + v, \ \mathfrak{y}_1 + \mathfrak{y}_2)$;

(xvi) $\mathcal{J}(u, \ .)$ is continuous non-decreasing function;

(xvii) $\lim_{\mathfrak{y}_1 \to \infty} \mathcal{J}(u, \ \mathfrak{y}_1) = 0$;

(xviii) If $\mathfrak{y}_1 \leq 0$, then $\mathcal{H}(u, \ \mathfrak{y}_1) = 0, \quad \mathcal{I}(u, \ \mathfrak{y}_1) = 1 \quad$ and $\quad \mathcal{J}(u, \ \mathfrak{y}_1) = 1$.

We call $N = (\mathcal{H}, \mathcal{I}, \mathcal{J})$, the neutrosophic norm and $V = (F, \mathcal{H}, \mathcal{I}, \mathcal{J}, \circ, \diamond)$, the neutrosophic normed space."

For some examples on these spaces we refer [10].

"A sequence (u_n) in a neutrosophic normed spaces V is said to convergent if for each $\varepsilon > 0$ and $\mathfrak{y} > 0$, there exists a positive integer m and $u_0 \in F$ such that $\mathcal{H}(u_n - u_0, \mathfrak{y}) > 1 - \varepsilon$, $\mathcal{I}(u_n - u_0, \mathfrak{y}) < \varepsilon$ and $\mathcal{J}(u_n - u_0, \mathfrak{y}) < \varepsilon$ for all $n \geq m$. This is equivalent to say that

$\lim_{n\to\infty}\mathcal{H}(u_n-u_0,\mathfrak{y})=1$, $\lim_{n\to\infty}\mathcal{I}(u_n-u_0,\mathfrak{y})=0$ and $\lim_{n\to\infty}\mathcal{J}(u_n-u_0,\mathfrak{y})=0$ and we write $N-\lim_{n\to\infty}u_n=u_0$."

"A sequence (u_n) is said to be Cauchy if for each $\varepsilon>0$ and $\mathfrak{y}>0$, there exists a positive integer p such that $\mathcal{H}(u_k-u_n,\ \mathfrak{y})>1-\varepsilon$, $\mathcal{I}(u_k-u_n,\ \mathfrak{y})<\varepsilon$ and $\mathcal{J}(u_k-u_n,\ \mathfrak{y})<\varepsilon$ for all $k,\ n\geq p$."

"Let w denotes the set of all sequences in the neutrosophic normed space $V=(F,\mathcal{H},\mathcal{I},\mathcal{J},\circ,\diamond)$. Define $\Delta^m: w\to w$ by

$$\Delta^0\ a_k=a_k;$$

$$\Delta^1\ a_k=a_k-a_{k+1};$$

$$\Delta^m\ a_k=\Delta^{m-1}(a_k-a_{k+1})m\geq 2 \text{ and } \forall\ k\in\mathbb{N}.$$

We now demonstrate two important Lemmas of [24].

For $\mu>0$ and $n\in\mathbb{N}$, let $\mu_n=\lfloor\mu n\rfloor$ i.e, the sequence of integral parts of the product μn."

If we define $\langle\mu\rangle=\mu-\lfloor\mu\rfloor$, then we have the following Lemmas.

Lemma 2.1 [24] "(i) If $\mu>1$, then $\mu_n>n,\ \forall\ n\in\mathbb{N}-\{0\}$ along with $n>\langle\mu\rangle^{-1}$.

(ii) If $0<\mu<1$, then $\mu_n<n,\ \forall\ n\in\mathbb{N}-\{0\}$."

Lemma 2.2[24] "(i) If $\mu>1$, then $\forall\ n\in\mathbb{N}-\{0\}$ along with $n\geq\frac{3\mu-1}{\mu(\mu-1)}$, we have

$$\frac{\mu}{(\mu-1)}<\frac{\mu_n+1}{\mu_n-n}<\frac{2\mu}{\mu-1}.$$

(ii) If $0<\mu<1$, then $\forall\ n\in\mathbb{N}-\{0\}$ along with $n=\mu^{-1}$ we have

$$0<\frac{\mu_n+1}{n-\mu_n}<\frac{2\mu}{1-\mu}.$$

We now turn towards our main section. Throughout the work, V denotes a neutrosophic normed space with neutrosophic norm N unless otherwise stated and θ, the $0-$th element in V."

3. Δ^m-Cesàro summability in NNS

Definition 3.1 A sequence $u=(u_n)$ in V is called Δ^m-Cesàro summable [or $(C,\Delta^m,1)-$summable w.r.t. N] to u_0 if $N-\lim_{n\to\infty}\sigma_n=u_0$ where the sequence (σ_n) is precisely defined by

$$\sigma_n=\frac{v_1+v_2+\ldots v_n}{n}=\frac{\sum_{k=1}^n v_k}{n}.\quad (n\in\mathbb{N}) \text{ and}$$

$$v_n=\Delta^m u_n=\sum_{p=0}^m(-1)^p\binom{m}{p}u_{n+p}.$$

This is similar to say, for $\epsilon > 0$ and $\mathfrak{y} > 0$ there exist $n_0 \in \mathbb{N}$ satisfying

$$\mathcal{H}(\sigma_n - u_0, \mathfrak{y}) > 1 - \epsilon \text{ and } \mathcal{I}(\sigma_n - u_0, \mathfrak{y}) < \epsilon, \mathcal{J}(\sigma_n - u_0, \mathfrak{y}) < \epsilon.$$

In this case, we abbreviate it as $N(C, \Delta^m, 1) - \lim_{n\to\infty} u_n = u_0$.

Next Theorem gives the relationship between $N-$convergence and $N(C, \Delta^m, 1)-$summability.

Theorem 3.1 For any sequence $u = (u_n)$ in V, if $N - \lim_{n\to\infty} \Delta^m u_n = u_0$, then $N(C, \Delta^m, 1) - \lim_{n\to\infty} u_n = u_0$.

Proof. Assume that $N - \lim_{n\to\infty} \Delta^m u_n = u_0$. We wish to prove that $N(C, \Delta^m, 1) - \lim_{n\to\infty} u_n = u_0$. Let $\epsilon > 0$ be given and take $\mathfrak{y} > 0$. As $N - \lim_{n\to\infty} \Delta^m u_n = u_0$ so $\exists$ $n_1 \in \mathbb{N}$ satisfying, for all $n \geq n_1$

$$\mathcal{H}\left(\Delta^m u_n - u_0, \frac{\mathfrak{y}}{2}\right) > 1 - \epsilon \text{ and } \mathcal{I}\left(\Delta^m u_n - u_0, \frac{\mathfrak{y}}{2}\right) < \epsilon, \mathcal{J}\left(\Delta^m u_n - u_0, \frac{\mathfrak{y}}{2}\right) < \epsilon;$$

Moreover,

$$\lim_{n\to\infty} \mathcal{H}\left(\sum_{k=1}^{n_1} \Delta^m u_k - u_0, \frac{n\mathfrak{y}}{2}\right) = 1 \text{ and } \lim_{n\to\infty} \mathcal{I}\left(\sum_{k=1}^{n_1} \Delta^m u_k - u_0, \frac{n\mathfrak{y}}{2}\right) = 0,$$
$$\lim_{n\to\infty} \mathcal{J}\left(\sum_{k=1}^{n_1} \Delta^m u_k - u_0, \frac{n\mathfrak{y}}{2}\right) = 0;$$

gives another $n_2 \in \mathbb{N}$ with $n \geq n_2$ such that

$$\mathcal{H}\left(\sum_{k=1}^{n_1} \Delta^m u_k - u_0, \frac{n\mathfrak{y}}{2}\right) > 1 - \epsilon \text{ and } \mathcal{I}\left(\sum_{k=1}^{n_1} \Delta^m u_k - u_0, \frac{n\mathfrak{y}}{2}\right) < \epsilon,$$
$$\mathcal{J}\left(\sum_{k=1}^{n_1} \Delta^m u_k - u_0, \frac{n\mathfrak{y}}{2}\right) < \epsilon.$$

Now, for $n > \max\{n_1, n_2\}$ we have

$$\begin{aligned}
\mathcal{H}\left(\frac{1}{n}\sum_{k=1}^{n} \Delta^m u_k - u_0, \mathfrak{y}\right) &= \mathcal{H}\left(\frac{1}{n}\sum_{k=1}^{n} (\Delta^m u_k - u_0), \mathfrak{y}\right) = \mathcal{H}\left(\sum_{k=1}^{n} (\Delta^m u_k - u_0), n\mathfrak{y}\right) \\
&\geq \min\left\{\mathcal{H}\left(\sum_{k=1}^{n_1} (\Delta^m u_k - u_0), n\frac{\mathfrak{y}}{2}\right), \mathcal{H}\left(\sum_{k=n_1+1}^{n} (\Delta^m u_k - u_0), n\frac{\mathfrak{y}}{2}\right)\right\} \\
&\geq \min\left\{\mathcal{H}\left(\sum_{k=1}^{n_1} (\Delta^m u_k - u_0), n\frac{\mathfrak{y}}{2}\right), \mathcal{H}\left(\sum_{k=n_1+1}^{n} (\Delta^m u_k - u_0), (n - n_1).\frac{\mathfrak{y}}{2}\right)\right\} \\
&\geq \min\left\{\mathcal{H}\left(\sum_{k=1}^{n_1} (\Delta^m u_k - u_0), n\frac{\mathfrak{y}}{2}\right), \mathcal{H}\left((\Delta^m u_{n_1+1} - u_0), \frac{\mathfrak{y}}{2}\right), \mathcal{H}\left((\Delta^m u_{n_1+2} - u_0), \frac{\mathfrak{y}}{2}\right), \cdots \right. \\
&\qquad \left. \mathcal{H}\left((\Delta^m u_n - u_0), \frac{\mathfrak{y}}{2}\right)\right\} \\
&> (1 - \epsilon) \text{ and}
\end{aligned}$$

$$\mathcal{I}\bigg(\frac{1}{n}\sum_{k=1}^{n}\Delta^m u_k-u_0,\mathfrak{y}\bigg)=\mathcal{I}\bigg(\frac{1}{n}\sum_{k=1}^{n}(\Delta^m u_k-u_0),\mathfrak{y}\bigg)=\mathcal{I}\bigg(\sum_{k=1}^{n}(\Delta^m u_k-u_0),n\mathfrak{y}\bigg)$$
$$<\max\bigg\{\mathcal{I}\bigg(\sum_{k=1}^{n_1}(\Delta^m u_k-u_0),n\frac{\mathfrak{y}}{2}\bigg),\mathcal{I}\bigg(\sum_{k=n_1+1}^{n}(\Delta^m u_k-u_0),n\frac{\mathfrak{y}}{2}\bigg)\bigg\}$$
$$<\max\bigg\{\mathcal{I}\bigg(\sum_{k=1}^{n_1}(\Delta^m u_k-u_0),n\frac{\mathfrak{y}}{2}\bigg),\mathcal{I}\bigg(\sum_{k=n_1+1}^{n}(\Delta^m u_k-u_0),(n-n_1)\frac{\mathfrak{y}}{2}\bigg)\bigg\}$$
$$<\max\bigg\{\mathcal{I}\bigg(\sum_{k=1}^{n_1}(\Delta^m u_k-u_0),n\frac{\mathfrak{y}}{2}\bigg),$$
$$\mathcal{I}\bigg((\Delta^m u_{n_1+1}-u_0),\frac{\mathfrak{y}}{2}\bigg),\mathcal{I}\bigg((\Delta^m u_{n_1+2}-u_0),\frac{\mathfrak{y}}{2}\bigg),\cdots\mathcal{I}\bigg((\Delta^m u_n-u_0),\frac{\mathfrak{y}}{2}\bigg)\bigg\}<\epsilon.$$

Similarly one can show

$$\mathcal{J}\bigg(\frac{1}{n}\sum_{k=1}^{n}\Delta^m u_k-u_0,\mathfrak{y}\bigg)<\epsilon.$$

This implies that $N(C,\Delta^m,1)-\lim_{n\to\infty}u_n=u_0$, which completes the proof of the Theorem.□

Example 3.1 Let $(\mathbb{R},|.|)$ denote the space of reals with the usual norm. For $a,b\in[0,1]$, let the $t-$ norm and $t-$conorm are defined by

$$a\circ b=ab \text{ and } a\diamond b=a+b-ab$$

Let, $u\in\mathbb{R}$ and $\mathfrak{y}>0$ with $\mathfrak{y}>|u|$. Define $\mathcal{H},\mathcal{I}$ and $\mathcal{J}$ as follows:

$$\mathcal{H}(u,\mathfrak{y})=\frac{\mathfrak{y}}{\mathfrak{y}+|u|},\mathcal{I}(u,\mathfrak{y})=\frac{|u|}{\mathfrak{y}+|u|} \text{ and } \mathcal{J}(u,\mathfrak{y})=\frac{|u|}{\mathfrak{y}},$$

then $N(\mathcal{H},\mathcal{I},\mathcal{J})$ is a neutrosophic norm and $(\mathbb{R},\circ,\diamond,\mathcal{H},\mathcal{I},\mathcal{J})$ is a NNS.

Define a sequence (u_n) by $u_n=(-1)^n$, then for $m=1$, $\Delta^1 u_n=2(-1)^n$ and therefore the sequence σ_n is given by

$$\sigma_n=\frac{2(-1)^1+2(-1)^2+\cdots+2(-1)^n}{n}=0 \text{ or } \frac{-2}{n},$$

according as n is even or odd respectively.

Case-I: If n is even, then $\sigma_n=0$, and therefore we have

$$\lim_{n\to\infty}\mathcal{H}(0,\mathfrak{y})=1 \text{ and } \lim_{n\to\infty}\mathcal{I}(0,\mathfrak{y})=\lim_{n\to\infty}\mathcal{J}(0,\mathfrak{y})=0.$$

(by Definition NNS)

Case-II: If n is odd, then

$$\mathcal{H}(\sigma_n - 0, \mathfrak{y}) = \mathcal{H}(\sigma_n, \mathfrak{y}) = \frac{\mathfrak{y}}{\mathfrak{y} + |\sigma_n|} = \frac{\mathfrak{y}}{\mathfrak{y} + |\frac{-2}{n}|}$$

$$\text{so,} \quad \lim_{n\to\infty} \mathcal{H}(\sigma_n - 0, \mathfrak{y}) = \lim_{n\to\infty} \frac{\mathfrak{y}}{\mathfrak{y} + |\frac{-2}{n}|} = 1;$$

and

$$\mathcal{I}(\sigma_n - 0, \mathfrak{y}) = \mathcal{I}(\sigma_n, \mathfrak{y}) = \frac{|\sigma_n|}{\mathfrak{y} + |\sigma_n|} = \frac{|\frac{-2}{n}|}{\mathfrak{y} + |\frac{-2}{n}|} \text{ gives}$$

$$\lim_{n\to\infty} \mathcal{I}(\sigma_n - 0, \mathfrak{y}) = \lim_{n\to\infty} \frac{|\frac{-2}{n}|}{\mathfrak{y} + |\frac{-2}{n}|} = 0;$$

$$\mathcal{J}(\sigma_n - 0, \mathfrak{y}) = \mathcal{J}(\sigma_n, \mathfrak{y}) = \frac{\|\sigma_n\|}{\mathfrak{y}} = \frac{|\frac{-2}{n}|}{\mathfrak{y}} \text{ will imply}$$

$$\lim_{n\to\infty} \mathcal{J}(\sigma_n - \theta, \mathfrak{y}) = \lim_{n\to\infty} \frac{|\frac{-2}{n}|}{\mathfrak{y}} = 0.$$

Hence, in both cases,

$$\lim_{n\to\infty} \mathcal{H}(\sigma_n - 0, \mathfrak{y}) = 1, \lim_{n\to\infty} \mathcal{I}(\sigma_n - 0, \mathfrak{y}) = \lim_{n\to\infty} \mathcal{J}(\sigma_n - 0, \mathfrak{y}) = 0,$$

and therefore, $N - \lim_{n\to\infty} \sigma_n = 0$ i.e., $N(C, \Delta, 1) - \lim_{n\to\infty} u_n = 0$.

But clearly the sequence $(u_n) = 2(-1)^n$ is not $N-$convergent as

$$\mathcal{H}(u_n - u_0, \mathfrak{y}) = \frac{\mathfrak{y}}{\mathfrak{y} + \|u_n - u_0\|} = \frac{\mathfrak{y}}{\mathfrak{y} + |2(-1)^n - u_0|} = \begin{cases} \frac{\mathfrak{y}}{\mathfrak{y}+|-2-u_0|} & \text{if } n \text{ is odd;} \\ (\frac{\mathfrak{y}}{\mathfrak{y}+|2-u_0|}) & \text{if } n \text{ is even.} \end{cases}$$

Thus, if we choose $u_0 = -2$ when n is odd and $u_0 = 2$ when n is even, then we have

$$\lim_{n\to\infty} \mathcal{H}(u_n - u_0, \mathfrak{y}) = \begin{cases} 1 & \text{if } n \text{ is odd;} \\ 1 & \text{if } n \text{ is even.} \end{cases}$$

Similarly one can show

$$\lim_{n\to\infty} \mathcal{I}(u_n - u_0, \mathfrak{y}) = \lim_{n\to\infty} \mathcal{J}(u_n - u_0, \mathfrak{y}) \begin{cases} 0 & \text{if } n \text{ is odd;} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

In this way we obtain two subsequences of the sequence $u_n = (-1)^n$ corresponding to sets of even and odd integers and which are $N-$convergent to different limits. This shows that $(u_n) = (-1)^n$ is not $N-$convergent. □

The following Theorem gives the reverse way of Theorem 3.1 via applying some additional conditions.

Theorem 3.2 For any sequence $u = (u_n)$ in V, if $N(C,\Delta^m,1) - \lim_{n\to\infty} u_n = u_0$, then $N - \lim_{n\to\infty} \Delta^m u_n = u_0$ if and only if

$$(i) \sup_{\mu>1}\left[\liminf_{n\to\infty} \mathcal{H}\left(\frac{1}{\mu_n - n}\sum_{k=n+1}^{\mu_n}(\Delta^m u_k - \Delta^m u_n), \mathfrak{y}\right)\right] = 1;$$

$$(ii) \inf_{\mu>1}\left[\limsup_{n\to\infty} \mathcal{I}\left(\frac{1}{\mu_n - n}\sum_{k=n+1}^{\mu_n}(\Delta^m u_k - \Delta^m u_n), \mathfrak{y}\right)\right] = 0;$$

$$(iii) \inf_{\mu>1}\left[\limsup_{n\to\infty} \mathcal{J}\left(\frac{1}{\mu_n - n}\sum_{k=n+1}^{\mu_n}(\Delta^m u_k - \Delta^m u_n), \mathfrak{y}\right)\right] = 0.$$

(Here for $\mu > 0, \mu_n = \lfloor \mu n \rfloor$i.e., the integral part of μn)

Proof. Necessity: Let, $u = (u_n)$ be any sequence in V with $N(C,\Delta^m,1) - \lim_{n\to\infty} u_n = u_0$. We first assume that $N - \lim_{n\to\infty} \Delta^m u_n = u_0$ and obtain conditions $(i),(ii)$ and (iii). Let $\mathfrak{y} > 0$ and take $\mu > 1$. Then, by Lemma 2.1, for each $n \in \mathbb{N} - \{0\}$ we have $\mu_n > n$ and $n \geq \frac{1}{\langle\mu\rangle}$ where $\langle\mu\rangle = \mu - \lfloor\mu\rfloor$. Moreover, by Lemma 2.1, we can write the difference of $(\Delta^m u_n - \Delta^m \sigma_n)$ as

$$\Delta^m u_n - \sigma_n = \frac{\mu_n + 1}{\mu_n - n}\left[\sigma_{\mu_n} - \sigma_n\right] - \frac{1}{\mu_n - n}\sum_{k=n+1}^{\mu_n}\left(\Delta^m u_k - \Delta^m u_n\right);$$

and therefore by Lemma 2.2, we have for $n \geq \frac{3\mu-1}{\mu(\mu-1)}$,

$$\mathcal{H}\left(\frac{\mu_n+1}{\mu_n-n}\left[\sigma_{\mu_n} - \sigma_n\right], \mathfrak{y}\right) = \mathcal{H}\left(\left[\sigma_{\mu_n} - \sigma_n\right], \frac{\mathfrak{y}}{\frac{\mu_n+1}{\mu_n-n}}\right)$$
$$\geq \mathcal{H}\left(\left[\sigma_{\mu_n} - \sigma_n\right], \frac{\mathfrak{y}}{\frac{2\mu}{\mu-1}}\right)$$

$$\text{so we have, } \lim_{n\to\infty} \mathcal{H}\left(\frac{\mu_n+1}{\mu_n-n}\left[\sigma_{\mu_n} - \sigma_n\right], \mathfrak{y}\right) = \mathcal{H}\left(0, \frac{\mathfrak{y}}{\frac{2\mu}{\mu-1}}\right) = 1.$$

as (σ_n) is a Cauchy sequence.

Now,

$$\mathcal{I}\left(\frac{\mu_n+1}{\mu_n-n}\left[\sigma_{\mu_n}-\sigma_n\right],\mathfrak{y}\right)=\mathcal{I}\left(\left[\sigma_{\mu_n}-\sigma_n\right],\frac{\mathfrak{y}}{\frac{\mu_n+1}{\mu_n-n}}\right)$$
$$\leq \mathcal{I}\left(\left[\sigma_{\mu_n}-\sigma_n\right],\frac{\mathfrak{y}}{\frac{2\mu}{\mu-1}}\right)$$

so we have, $\lim_{n\to\infty}\mathcal{I}\left(\frac{\mu_n+1}{\mu_n-n}\left[\sigma_{\mu_n}-\sigma_n\right],\mathfrak{y}\right)=\mathcal{I}\left(0,\frac{\mathfrak{y}}{\frac{2\mu}{\mu-1}}\right)=0;$

similarly,

$$\mathcal{J}\left(\frac{\mu_n+1}{\mu_n-n}\left[\sigma_{\mu_n}-\sigma_n\right],\mathfrak{y}\right)=\mathcal{J}\left(\left[\sigma_{\mu_n}-\sigma_n\right],\frac{\mathfrak{y}}{\frac{\mu_n+1}{\mu_n-n}}\right)$$
$$\leq \mathcal{J}\left(\left[\sigma_{\mu_n}-\sigma_n\right],\frac{\mathfrak{y}}{\frac{2\mu}{\mu-1}}\right)$$

and therefore, $\lim_{n\to\infty}\mathcal{J}\left(\frac{\mu_n+1}{\mu_n-n}\left[\sigma_{\mu_n}-\sigma_n\right],\mathfrak{y}\right)=\mathcal{J}\left(0,\frac{\mathfrak{y}}{\frac{2\mu}{\mu-1}}\right)=0.$

Hence, we obtain

$$\lim_{n\to\infty}\mathcal{H}\left(\frac{\mu_n+1}{\mu_n-n}\left[\sigma_{\mu_n}-\sigma_n\right],\mathfrak{y}\right)=1,\lim_{n\to\infty}\mathcal{I}\left(\frac{\mu_n+1}{\mu_n-n}\left[\sigma_{\mu_n}-\sigma_n\right],\mathfrak{y}\right)$$
$$=\lim_{n\to\infty}\mathcal{I}\left(\frac{\mu_n+1}{\mu_n-n}\left[\sigma_{\mu_n}-\sigma_n\right],\mathfrak{y}\right)=0,$$

which immediately imply (i), (ii) and (iii).

Sufficiency: Suppose (i), (ii) and (iii) holds. We shall show that $N-\lim_{n\to\infty}\Delta^m u_n=u_0$. For this, let $\epsilon>0$ be given and take $\mathfrak{y}>0$. By hypothesis, there exists a $\mu>1$ and a $m_1\in\mathbb{N}$ such that for $n>m_1$, we have

$$\mathcal{H}\left(\frac{1}{\mu_n-n}\sum_{k=n+1}^{\mu_n}(\Delta^m u_k-\Delta^m u_n),\frac{\mathfrak{y}}{3}\right)>1-\epsilon \text{ and } \mathcal{I}\left(\frac{1}{\mu_n-n}\sum_{k=n+1}^{\mu_n}(\Delta^m u_k-\Delta^m u_n),\frac{\mathfrak{y}}{3}\right)<\epsilon,$$
$$\mathcal{J}\left(\frac{1}{\mu_n-n}\sum_{k=n+1}^{\mu_n}(\Delta^m u_k-\Delta^m u_n),\frac{\mathfrak{y}}{3}\right)<\epsilon;$$

Since, $N(C,\Delta^m,1)-\lim_{n\to\infty}u_n=u_0$, so we have another $m_2\in\mathbb{N}$ such that for all $n>m_2$ we have

$$\mathcal{H}\left(\sigma_n-u_0,\frac{\mathfrak{y}}{3}\right)>1-\epsilon \text{ and } \mathcal{I}\left(\sigma_n-u_0,\frac{\mathfrak{y}}{3}\right)<\epsilon,\mathcal{J}\left(\sigma_n-u_0,\frac{\mathfrak{y}}{3}\right)<\epsilon.$$

Moreover, as $N-\lim_{n\to\infty}\frac{\mu_n+1}{\mu_n-n}\left[\sigma_{\mu_n}-\sigma_n\right]=0$, so there is $m_3\in\mathbb{N}$ such that for all $n>m_3$ we have

$$\mathcal{H}\left(\frac{\mu_n+1}{\mu_n-n}\left[\sigma_{\mu_n}-\sigma_n\right],\frac{\mathfrak{y}}{3}\right)>1-\epsilon \text{ and } \mathcal{I}\left(\frac{\mu_n+1}{\mu_n-n}\left[\sigma_{\mu_n}-\sigma_n\right],\frac{\mathfrak{y}}{3}\right)<\epsilon,$$
$$\mathcal{J}\left(\frac{\mu_n+1}{\mu_n-n}\left[\sigma_{\mu_n}-\sigma_n\right],\frac{\mathfrak{y}}{3}\right)<\epsilon.$$

Now,

$$\begin{aligned}\mathcal{H}(\Delta^m u_n-u_0,\mathfrak{y})&=\mathcal{H}(\Delta^m u_n-\sigma_n+\sigma_n-u_0,\mathfrak{y})\\&=\mathcal{H}\left(\frac{\mu_n+1}{\mu_n-n}\left[\sigma_{\mu_n}-\sigma_n\right]-\frac{1}{\mu_n-n}\sum_{k=n+1}^{\mu_n}\left(\Delta^m u_k-\Delta^m u_n\right)+\sigma_n-u_0,\mathfrak{y}\right)\end{aligned}$$
$$\geq\min\left\{\mathcal{H}\left(\frac{\mu_n+1}{\mu_n-n}\left[\sigma_{\mu_n}-\sigma_n\right],\frac{\mathfrak{y}}{3}\right),\mathcal{H}\left(\frac{1}{\mu_n-n}\sum_{k=n+1}^{\mu_n}\left(\Delta^m u_k-\Delta^m u_n\right),\frac{\mathfrak{y}}{3}\right),\mathcal{H}\left(\sigma_n-u_0,\frac{\mathfrak{y}}{3}\right)\right\};$$

and

$$\begin{aligned}\mathcal{I}(\Delta^m u_n-u_0,\mathfrak{y})&=\mathcal{I}(\Delta^m u_n-\sigma_n+\sigma_n-u_0,\mathfrak{y})\\&=\mathcal{I}\left(\frac{\mu_n+1}{\mu_n-n}\left[\sigma_{\mu_n}-\sigma_n\right]-\frac{1}{\mu_n-n}\sum_{k=n+1}^{\mu_n}\left(\Delta^m u_k-\Delta^m u_n\right)+\sigma_n-u_0,\mathfrak{y}\right)\end{aligned}$$
$$\leq\max\left\{\mathcal{I}\left(\frac{\mu_n+1}{\mu_n-n}\left[\sigma_{\mu_n}-\sigma_n\right],\frac{\mathfrak{y}}{3}\right),\mathcal{I}\left(\frac{1}{\mu_n-n}\sum_{k=n+1}^{\mu_n}\left(\Delta^m u_k-\Delta^m u_n\right),\frac{\mathfrak{y}}{3}\right),\mathcal{I}\left(\sigma_n-u_0,\frac{\mathfrak{y}}{3}\right)\right\},$$
$$\begin{aligned}\mathcal{J}(\Delta^m u_n-u_0,\mathfrak{y})&=\mathcal{J}(\Delta^m u_n-\sigma_n+\sigma_n-u_0,\mathfrak{y})\\&=\mathcal{J}\left(\frac{\mu_n+1}{\mu_n-n}\left[\sigma_{\mu_n}-\sigma_n\right]-\frac{1}{\mu_n-n}\sum_{k=n+1}^{\mu_n}\left(\Delta^m u_k-\Delta^m u_n\right)+\sigma_n-u_0,\mathfrak{y}\right)\end{aligned}$$
$$\leq\max\left\{\mathcal{J}\left(\frac{\mu_n+1}{\mu_n-n}\left[\sigma_{\mu_n}-\sigma_n\right],\frac{\mathfrak{y}}{3}\right),\mathcal{J}\left(\frac{1}{\mu_n-n}\sum_{k=n+1}^{\mu_n}\left(\Delta^m u_k-\Delta^m u_n\right),\frac{\mathfrak{y}}{3}\right),\mathcal{J}\left(\sigma_n-u_0,\frac{\mathfrak{y}}{3}\right)\right\},$$

Thus, if we select $m=\max\{m_1,m_2,m_3\}$, then we have $\mathcal{H}(\Delta^m u_n-u_0,\mathfrak{y})>1-\epsilon$ and $\mathcal{I}(\Delta^m u_n-u_0,\mathfrak{y})<\epsilon$, $\mathcal{J}(\Delta^m u_n-u_0,\mathfrak{y})<\epsilon$ and therefore $N-\lim_{n\to\infty}\Delta^m u_n=u_0$. $\square$

The case for $0<\mu<1$ follows similarly by using the expression

$$\Delta^m u_n-\sigma_n=\frac{\mu_n+1}{n-\mu_n}\left[\sigma_n-\sigma_{\mu_n}\right]-\frac{1}{n-\mu_n}\sum_{k=n+1}^{\mu_n}\left(\Delta^m u_n-\Delta^m u_k\right).$$

Another similar result related to Cesàro summability and $N-$convergence is as follows.

Theorem 3.3 For any sequence $u=(u_n)$ in V, if $N(C,1)-\lim_{n\to\infty}u_n=u_0$, then $N-$

$\lim_{n\to\infty} u_n = u_0$ if and only if

$$(i)\ \sup_{0<\mu<1}\left[\liminf_{n\to\infty}\mathcal{H}\left(\frac{1}{n-\mu_n}\sum_{k=\mu_n+1}^{n}(\Delta^m u_n-\Delta^m u_k),\mathfrak{y}\right)\right]=1;$$

$$(ii)\ \inf_{0<\mu<1}\left[\limsup_{n\to\infty}\mathcal{I}\left(\frac{1}{n-\mu_n}\sum_{k=\mu_n+1}^{n}(\Delta^m u_n-\Delta^m u_k),\mathfrak{y}\right)\right]=0;$$

$$(iii)\ \inf_{0<\mu<1}\left[\limsup_{n\to\infty}\mathcal{J}\left(\frac{1}{n-\mu_n}\sum_{k=\mu_n+1}^{n}(\Delta^m u_n-\Delta^m u_k),\mathfrak{y}\right)\right]=0.\square$$

4. Slowly oscillating sequences in NNS

For μ_n, the sequence of integer part of μn, the concept of Δ^m-slowly oscillating sequences in neutrosophic normed spaces is defined as follow.

Definition 4.1 A sequence $u=(u_n)$ in V is called slowly oscillating if for all $\mathfrak{y}>0$

$$(i)\sup_{\mu>1}\left[\liminf_{n\to\infty}\left\{\min_{n<k\leq\ \mu_n}\mathcal{H}(\Delta^m u_k-\Delta^m u_n,\mathfrak{y})\right\}\right]=1 \text{ and}$$

$$(ii)\ \inf_{\mu>1}\left[\limsup_{n\to\infty}\left\{\max_{n<k\leq\mu_n}\mathcal{I}(\Delta^m u_k-\Delta^m u_n,\mathfrak{y})\right\}\right]=0,$$

$$(iii)\ \inf_{\mu>1}\left[\limsup_{n\to\infty}\left\{\max_{n<k\leq\mu_n}\mathcal{J}(\Delta^m u_k-\Delta^m u_n,\mathfrak{y})\right\}\right]=0.$$

Above definition immediately gives the following remarks.

Remark 4.1 In Definition 3.2, $\sup_{\mu>1}$ and $\inf_{\mu>1}$ is equivalent to say $\lim_{\mu\to1^+}$.

Remark 4.2 A sequence $u=(u_n)$ in V is Δ^m-slowly oscillating, if and only if, for all $0<\epsilon<1$ and $\mathfrak{y}>0$ there exists $\mu>1$ and $n_0(\epsilon,\mathfrak{y})\in\mathbb{N}$ such that

$$\mathcal{H}(\Delta^m u_k-\Delta^m u_n,\mathfrak{y})>1-\epsilon \text{ and } \mathcal{I}(\Delta^m u_k-\Delta^m u_n,\mathfrak{y})<\epsilon, \mathcal{J}(\Delta^m u_k-\Delta^m u_n,\mathfrak{y})<\epsilon.$$

holds for every $n_0\leq n<k\leq\mu_n$.

Example 4.1 Let $(\mathbb{R},|\cdot|)$ be a normed space. Let $a\circ b=ab$ and $a\diamond b=a+b-ab\ \forall\ a,b\in[0,1]$. For all $u\in\mathbb{R}$ and every $\mathfrak{y}>0$, we consider $\mathcal{H}(u,\mathfrak{y})=\frac{\mathfrak{y}}{\mathfrak{y}+|u|}, \mathcal{I}(u,\mathfrak{y})=\frac{|u|}{\mathfrak{y}+|u|}, \mathcal{J}(u,\mathfrak{y})=\frac{|u|}{\mathfrak{y}}$, then $(\mathbb{R},\circ,\diamond,\mathcal{H},\mathcal{I},\mathcal{J})$ is a NNS. Define a sequence (u_n) as follows:

$u_1=1$,

$u_2=u_3=1+\frac{1}{2}$,

$u_4=u_5=u_6=u_7=1+\frac{1}{2}+\frac{1}{3}$,

...

$u_{2^n} = u_{2^n+1} = \ldots = u_{2^{n+1}-1} = \sum_{j=1}^{n+1} \frac{1}{j}$.

Given $\epsilon > 0$, let $\delta = 1$ and $m = 0$. Choose $n_0 \in \mathbb{N}$ s.t $\frac{1}{n_0} < \epsilon$. Then if $n > n_0$ and $n \leq k \leq 2n$, we have

$\mathcal{H}(u_k - u_n, \mathfrak{y}) = \frac{\mathfrak{y}}{\mathfrak{y}+|u_k-u_n|} > 1-\epsilon$ and $\mathcal{I}(u_k - u_n, \mathfrak{y}) = \frac{|u_k-u_n|}{\mathfrak{y}+|u_k-u_n|} < \epsilon$, $\mathcal{J}(u_k - u_n, \mathfrak{y}) = \frac{|u_k-u_n|}{\mathfrak{y}} < \epsilon$.

This shows that (u_n) is slowly oscillating sequence in $(\mathbb{R}, \circ, \diamond, \mathcal{H}, \mathcal{I}, \mathcal{J})$

Theorem 4.1 Let $u = (u_n)$ in V be a Δ^m-slowly oscillating sequence. Then for every $\mathfrak{y} > 0$, the conditions (i), (ii) and (iii) in Definition 4.1 are respectively equivalent to

(i) $\sup_{0<\mu<1} \left[\liminf_{n\to\infty} \left\{ \min_{\mu n < k \leq n} \mathcal{H}(\Delta^m u_k - \Delta^m u_n, \mathfrak{y}) \right\} \right] = 1$ and

(ii) $\inf_{0<\mu<1} \left[\limsup_{n\to\infty} \left\{ \max_{\mu n < k \leq n} \mathcal{I}(\Delta^m u_k - \Delta^m u_n, \mathfrak{y}) \right\} \right] = 0,$

(iii) $\inf_{0<\mu<1} \left[\limsup_{n\to\infty} \left\{ \max_{\mu n < k \leq n} \mathcal{J}(\Delta^m u_k - \Delta^m u_n, \mathfrak{y}) \right\} \right] = 0.$

Proof. We first prove that the following conditions are equivalent:

$$\sup_{\mu>1} \left[\liminf_{n\to\infty} \left\{ \min_{n < k \leq \mu n} \mathcal{H}(\Delta^m u_k - \Delta^m u_n, \mathfrak{y}) \right\} \right] = 1,$$
$$\sup_{0<\mu<1} \left[\liminf_{n\to\infty} \left\{ \min_{\mu n < k \leq n} \mathcal{H}(\Delta^m u_k - \Delta^m u_n, \mathfrak{y}) \right\} \right] = 1.$$

Let $\mathfrak{y} > 0$ be given and for $\mu > 1$, we define

$$f_1(\mu) = \liminf_{n\to\infty} \left\{ \min_{n < k \leq \lfloor \mu n \rfloor} \mathcal{H}(\Delta^m u_k - \Delta^m u_n, \mathfrak{y}) \right\} \text{ and}$$
$$f_2\left(\frac{1}{\mu}\right) = \liminf_{k\to\infty} \left\{ \min_{\lfloor \frac{k}{\mu} \rfloor < n \leq k} \mathcal{H}(\Delta^m u_k - \Delta^m u_n, \mathfrak{y}) \right\}$$

By definition of $\liminf$ in f_1, we have a subsequence (n_r) with

$$f_1(\mu) = \lim_{r\to\infty} \left\{ \min_{n_r < k \leq \lfloor \mu n_r \rfloor} \mathcal{H}(\Delta^m u_k - \Delta^m u_{n_r}, \mathfrak{y}) \right\}.$$

This gives rise another subsequence (k_r) satisfying $n_r < k_r \leq \lfloor \mu n_r \rfloor$ with

$$\min_{n_r < k \leq \lfloor \mu n_r \rfloor} \mathcal{H}(\Delta^m u_k - \Delta^m u_{n_r}, \mathfrak{y}) = \mathcal{H}(\Delta^m u_{k_r} - \Delta^m u_{n_r}, \mathfrak{y}).$$

Since, $n_r < k_r \leq \lfloor \mu n_r \rfloor$, so by Remark 3 [16], $n_r \in \left(\lfloor (\frac{k_r}{\mu} \rfloor, k_r \right)$, and therefore we have

$$\begin{aligned} f_2\left(\frac{1}{\mu}\right) = \liminf_{k\to\infty} \left\{ \min_{\lfloor \frac{k}{\mu} \rfloor < n \leq k} \mathcal{H}(\Delta^m u_k - \Delta^m u_n, \mathfrak{y}) \right\} &\leq \lim_{r\to\infty} \left\{ \min_{\lfloor \frac{k_r}{\mu} \rfloor < n \leq k_r} \mathcal{H}(\Delta^m u_{k_r} - \Delta^m u_n, \mathfrak{y}) \right\} \\ &\leq \lim_{r\to\infty} \mathcal{H}(\Delta^m u_{k_r} - \Delta^m u_{n_r}, \mathfrak{y}) \\ &= \lim_{r\to\infty} \left\{ \min_{n_r < k \leq \lfloor \mu n_r \rfloor} \mathcal{H}(\Delta^m u_k - \Delta^m u_{n_r}, \mathfrak{y}) \right\} \\ &= f_1\left(\frac{1}{\mu}\right). \end{aligned}$$

Similarly, we can have $f_2\left(\frac{1}{\mu}\right) \geq f_1\left(\frac{1}{\mu}\right)$ by changing their roles and therefore we have $f_1\left(\frac{1}{\mu}\right) = f_2\left(\frac{1}{\mu}\right)$. This shows that both expressions

$$\sup_{\mu>1} \left[\liminf_{n\to\infty} \left\{ \min_{n<k\leq\ \mu_n} \mathcal{H}(\Delta^m u_k - \Delta^m u_n, \mathfrak{y}) \right\} \right] = 1,$$

$$\sup_{0<\mu<1} \left[\liminf_{n\to\infty} \left\{ \min_{\mu_n<k\leq\ n} \mathcal{H}(\Delta^m u_k - \Delta^m u_n, \mathfrak{y}) \right\} \right] = 1.$$

are equivalent.

Following the same line of proof, one can easily obtain the equivalence of other pairs of expressions. □

Example 4.2 Consider the neutrosophic normed space $((\mathbb{R}, \circ, \diamond, \mathcal{H}, \mathcal{I}, \mathcal{J})$ as defined in Example 3.1.

Define a sequence (u_n) by $u_n = \sum_{i=1}^{n} \left(\frac{1}{i}\right)$ and take $\mathfrak{y} > 0$.

Let $0 < \epsilon < 1$ be given and select $\mu = \frac{\mathfrak{y}\epsilon}{1-\epsilon} + 1$.

Now, for all n satisfying $1 < n < k < \mu_n$, we have

$$\begin{aligned} \|u_k - u_n\| &= \left\| \sum_{i=1}^{k} \left(\frac{1}{i}\right) - \sum_{i=1}^{n} \left(\frac{1}{i}\right) \right\| \\ &= \left\| \sum_{i=n+1}^{k} \left(\frac{1}{i}\right) \right\| \leq \sum_{i=n+1}^{k} \left(\frac{1}{i}\right) \\ &< \frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n} \\ &= \frac{k-n}{n} = \frac{k}{n} - 1 < \mu - 1 = \frac{\mathfrak{y}\epsilon}{1-\epsilon} (\text{by selection of } n,\ k \text{ and } \mu); \end{aligned}$$

and therefore

$$\mathcal{H}\left(u_k - u_n, \mathfrak{y}\right) = \frac{\mathfrak{y}}{\mathfrak{y} + \|u_k - u_n\|} > \frac{\mathfrak{y}}{\mathfrak{y} + \frac{\mathfrak{y}\epsilon}{1-\epsilon}} = 1 - \epsilon \quad \text{and}$$

$$\mathcal{I}\left(u_k - u_n, \mathfrak{y}\right) = \frac{\|u_k - u_n\|}{\mathfrak{y} + \|u_k - u_n\|} < \frac{\left(\frac{\mathfrak{y}\epsilon}{1-\epsilon}\right)}{\mathfrak{y} + \left(\frac{\mathfrak{y}\epsilon}{1-\epsilon}\right)} = \epsilon.$$

Similarly, one can have $\mathcal{J}\left(u_k - u_n, \mathfrak{y}\right) < \epsilon$.

This shows that (u_n) is slowly oscillating in $(\mathbb{R}, \circ, \diamond, \mathcal{H}, \mathcal{I}, \mathcal{J})$. $\square$

Theorem 4.2 Let V be a normed space with norm $\|.\|$ and $(\mathbb{R}, \circ, \diamond, \mathcal{H}, \mathcal{I}, \mathcal{J})$ be the neutrosophic normed space as in Example 3.1. Then, a sequence $u = (u_n)$ is Δ^m-slowly oscillating in V if and only if it is so in $(\mathbb{R}, \circ, \diamond, \mathcal{H}, \mathcal{I}, \mathcal{J})$.

Proof. We first assume that $u = (u_n)$ is Δ^m-slowly oscillating in V. Let, $\mathfrak{y} > 0$ and $0 < \epsilon < 1$. Select $\epsilon' = \frac{\mathfrak{y}\epsilon}{1-\epsilon}$, then by Remark 4.2 there exists $\mu > 1$ and $n_0(\epsilon, \mathfrak{y}) \in \mathbb{N}$ such that

$$\mathcal{H}(\Delta^m u_k - \Delta^m u_n, \mathfrak{y}) > 1 - \epsilon \text{ and } \mathcal{I}(\Delta^m u_k - \Delta^m u_n, \mathfrak{y}) < \epsilon, \mathcal{J}(\Delta^m u_k - \Delta^m u_n, \mathfrak{y}) < \epsilon.$$

holds for every $n_0 \leq n < k \leq \mu_n$.
This proves that $u = (u_n)$ is Δ^m-slowly oscillating in $(\mathbb{R}, \circ, \diamond, \mathcal{H}, \mathcal{I}, \mathcal{J})$.

Conversely, assume that $u = (u_n)$ is Δ^m-slowly oscillating in $(\mathbb{R}, \circ, \diamond, \mathcal{H}, \mathcal{I}, \mathcal{J})$. Then for $0 < \epsilon < \frac{1}{2}$ and $\mathfrak{y} = 1 > 0$, then there exists $\mu > 1$ and $n_0(\epsilon, 1) \in \mathbb{N}$ such that

$$\mathcal{H}(\Delta^m u_k - \Delta^m u_n, 1) > 1 - \epsilon \text{ and } \mathcal{I}(\Delta^m u_k - \Delta^m u_n, 1) < \epsilon, \mathcal{J}(\Delta^m u_k - \Delta^m u_n, 1) < \epsilon.$$

holds for every $n_0 \leq n < k \leq \mu_n$.

Now, for $n_0 \leq n < k \leq \mu_n$, $\mathcal{H}(\Delta^m u_k - \Delta^m u_n, 1) > 1 - \epsilon$ will immediately gives

$$1 - \epsilon < \frac{1}{1 + \|\Delta^m u_k - \Delta^m u_n\|} \quad \text{or} \quad \|\Delta^m u_k - \Delta^m u_n\| < \frac{\epsilon}{1+\epsilon} < 2\epsilon = \epsilon',$$

and therefore $u = (u_n)$ is Δ^m-slowly oscillating in V. $\square$

Theorem 4.3 If $u = (u_n)$ is any Δ^m-slowly oscillating sequence in V, then

$$(i) \sup_{\mu>1} \left[\liminf_{n\to\infty} \mathcal{H}\left(\frac{1}{\mu_n - n} \sum_{k=n+1}^{\mu_n} (\Delta^m u_k - \Delta^m u_n), \mathfrak{y}\right)\right] = 1;$$

$$(ii) \inf_{\mu>1} \left[\limsup_{n\to\infty} \mathcal{I}\left(\frac{1}{\mu_n - n} \sum_{k=n+1}^{\mu_n} (\Delta^m u_k - \Delta^m u_n), \mathfrak{y}\right)\right] = 0;$$

$$(iii) \inf_{\mu>1} \left[\limsup_{n\to\infty} \mathcal{J}\left(\frac{1}{\mu_n - n} \sum_{k=n+1}^{\mu_n} (\Delta^m u_k - \Delta^m u_n), \mathfrak{y}\right)\right] = 0.$$

Proof. Suppose that $u = (u_n)$ is any Δ^m-slowly oscillating sequence in V. Then for $\mathfrak{y} > 0$ and $0 < \epsilon < 1$ there exists $\mu > 1$ and $n_0(\epsilon, \mathfrak{y}) \in \mathbb{N}$ such that

$$\mathcal{H}(\Delta^m u_k - \Delta^m u_n, \mathfrak{y}) > 1 - \epsilon \text{ and } \mathcal{I}(\Delta^m u_k - \Delta^m x_n, \mathfrak{y}) < \epsilon, \mathcal{J}(\Delta^m u_k - \Delta^m u_n, \mathfrak{y}) < \epsilon.$$

holds for every $n_0 \leq n < k \leq \mu_n$. Now,

$$\begin{aligned} \mathcal{H}\left(\frac{1}{\mu_n - n}\sum_{k=n+1}^{\mu_n}(\Delta^m u_k - \Delta^m u_n), \mathfrak{y}\right) &= \mathcal{H}\left(\sum_{k=n+1}^{\mu_n}(\Delta^m u_k - \Delta^m u_n), (\mu_n - n)\mathfrak{y}\right) \\ &\geq \min\{\mathcal{H}(\Delta^m u_{n+1} - \Delta^m u_n), \mathcal{H}(\Delta^m u_{n+2} - \Delta^m u_n), \cdots \mathcal{H}(\Delta^m u_{\mu_n} - \Delta^m u_n)\} \\ &> 1 - \epsilon \end{aligned}$$

and

$$\begin{aligned} \mathcal{I}\left(\frac{1}{\mu_n - n}\sum_{k=n+1}^{\mu_n}(\Delta^m u_k - \Delta^m u_n), \mathfrak{y}\right) &= \mathcal{I}\left(\sum_{k=n+1}^{\mu_n}(\Delta^m u_k - \Delta^m u_n), (\mu_n - n)\mathfrak{y}\right) \\ &\leq \max\{\mathcal{I}(\Delta^m u_{n+1} - \Delta^m u_n), \mathcal{I}(\Delta^m u_{n+2} - \Delta^m u_n), \cdots \mathcal{I}(\Delta^m u_{\mu_n} - \Delta^m u_n)\} \\ &< \epsilon, \\ \mathcal{J}\left(\frac{1}{\mu_n - n}\sum_{k=n+1}^{\mu_n}(\Delta^m u_k - \Delta^m u_n), \mathfrak{y}\right) &= \mathcal{J}\left(\sum_{k=n+1}^{\mu_n}(\Delta^m u_k - \Delta^m u_n), (\mu_n - n)\mathfrak{y}\right) \\ &\leq \max\{\mathcal{J}(\Delta^m u_{n+1} - \Delta^m u_n), Y(\Delta^m u_{n+2} - \Delta^m u_n), \cdots \mathcal{J}(\Delta^m u_{\mu_n} - \Delta^m u_n)\} \\ &< \epsilon. \end{aligned}$$

This proves the Theorem. □

Theorem 4.4 If $u = (u_n)$ is any Δ^m-slowly oscillating sequence in V which is $N(C, \Delta^m, 1)-$summable with $N(C, \Delta^m, 1) - \lim_{n\to\infty} u_n = u_0$, then $N - \lim_{n\to\infty} \Delta^m u_n = u_0$.
Proof. The proof is omitted as it can be obtain with the help of Theorem 3.2 and Theorem 4.3. □.

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Solving Neutrosophic Fuzzy Transportation Problem Of Type-II

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Abstract. Transportation problems offer a structured approach to optimize the allocation of resources, minimize transportation costs, and improve overall efficiency in supply chain and logistics management, leading to several advantages for businesses and organizations. Fuzzy transportation problems are particularly relevant in supply chain and logistics management when dealing with uncertain demand, fluctuating costs, or imprecise data and the intuitionistic fuzzy transportation problem is a more advanced modeling technique that takes into account the nuanced handling of uncertainty and imprecision using intuitionistic fuzzy sets(IFS). It provides a more realistic approach to decision-making in situations where classical or fuzzy models may not capture the subtleties of uncertainty in data. In this article, we demonstrate a novel approach to resolving transportation problems in a neutrosophic atmosphere. Neutrosophic set is an extension of fuzzy and IFS and it is classified by three independent membership grades: truth, indeterminacy, and falsity membership grade. These sets are better suited to handle imprecise parameters. Transportation cost and demand are taken as neutrosophic numbers. Vogel's approximation method is used to get the optimum solution of this neutrosophic transportation problem. Also, we performed a numerical instance to figure out the successful outcome of our suggested technique.

Keywords: Neutrosophic Sets; Neutrosophic Triangle Fuzzy Numbers; Ordering of Triangle Fuzzy Numbers; Neutrosophic Minimum Total Cost; Vogel's Approximation Method.

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1. Introduction

In the current landscape of intense market competition, several firms are actively seeking more effective strategies to enhance their ability to generate and provide value to their consumers, therefore fortifying their overall position. The task of efficiently and securely delivering items to clients while minimizing costs has grown more complex. In order to address this formidable task, transportation models provide a robust foundation. The optimization issue discussed is well recognized within the field of operational research and was first formulated by Hitchcock in 1941.The primary objective of the transportation problem is to ascertain the optimal shipment schedule that reduces the overall shipping cost, while simultaneously meeting the constraints of supply limitations and demand needs. The classical transportation problem pertains to a distinct category of linear programming problems.

In [1-8], many authors developed the concept of transportation in fuzzy, intuitionistic fuzzy and neutrosophic environments. According to these findings, in this paper, alternative simple methods are proposed for solving neutrosophic fuzzy transportation promblems and for solving neutrosophic fully fuzzy transportation problems. Vogel's approximation method is used to find initial basic feasible solution for neutrosophic fuzzy and neutrosophic fully fuzzy transportation problems.

The transportation problem, while the cost for shipping a single unit of a good from a particular source to a target is quantified by neutrosophic numbers, however the availability and demand can be illustrated with real numbers, is usually referred to as the neutrosophic transportation problem.

The transportation problem, when the characteristics such as the cost of transmitting a unit amount of a product from a particular source to a specific destination, the availability, and the demand, are presented as neutrosophic numbers, is commonly known as the neutrosophic fully fuzzy transportation problem. Problem pertaining to the distribution of goods and services, wherein the cost involved in transporting a singular unit of a particular item from a designated origin to a specified destination is quantified using neutrosophic fuzzy numbers, while the accessibility and requirement is indicated using real numbers, is referred to as the neutrosophic transportation problem.

The only difference between the classical methods and the neutrosophic fuzzy methods is that in the neutrosophic fuzzy numbers, the arithmetic operations of neutrosophic fuzzy numbers are used instead of arithmetic operations of real numbers.

2. Preliminaries

Definition:1 [9]

Let x be any element belongs to the universal set X. A neutrosophic set A in X is demonstrated by truth T_A, indeterminacy I_A and falsity-membership function F_A. Here, $T_A(x)$, $I_A(x)$ and $F_A(x)$ are nothing but the real standard or non-standard elements of [0,1]. i.e.,

$$T_A : X \to [0,1]$$
$$I_A : X \to [0,1]$$
$$F_A : X \to [0,1]$$

and no restriction on the sum of $T_A(X)$,$I_A(X)$ and $F_A(X)$, and also
$0 \leq supT_A(X) + supI_A(X) + supF_A(X) \leq 3$.

Definition:2 [9]

A single valued triangular neutrosophic number $\langle (a,b,c); w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}} \rangle$, is a unique neutrosophic set on the real number **R**, of which the truth, indeterminacy and falsity-membership functions are given as follows:

$$\mu_{\tilde{a}}(x) = \begin{cases} \frac{(x-a)w_{\tilde{a}}}{(b-a)}, & (a \leq x < b) \\ \frac{(c-x)w_{\tilde{a}}}{(c-b)}, & (b \leq x \leq c) \\ 0, & otherwise \end{cases}$$

$$\nu_{\tilde{a}}(x) = \begin{cases} \frac{(b-x+u_{\tilde{a}}(x-a))}{(b-a)}, & (a \leq x < b) \\ \frac{(x-b+u_{\tilde{a}}(c-x))}{(c-b)}, & (b \leq x \leq c) \\ 0, & otherwise \end{cases}$$

$$\lambda_{\tilde{a}}(x) = \begin{cases} \frac{(b-x+y_{\tilde{a}}(x-a))}{(b-a)}, & (a \leq x < b) \\ \frac{(x-b+y_{\tilde{a}}(c-x))}{(c-b)}, & (b \leq x \leq c) \\ 0, & otherwise \end{cases}$$

Definition:3 [9]

Let $w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}} \in [0,1]$ and $a_1, a_2, a_3, a_4 \in \mathbb{R}$ such that $a_1 \leq a_2 \leq a_3 \leq a_4$. Then a single valued trapezoidal neutrosophic number, $\tilde{a} = \langle (a_1, a_2, a_3, a_4); w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}} \rangle$ is a unique neutrosophic set on the real line $\mathbb{R}$, of which the truth, indeterminacy, and falsity-membership functions are given as follows:

$$\mu_{\tilde{a}}(x) = \begin{cases} w_{\tilde{a}}(\frac{x-a_1}{a_2-a_1}), & for\ a_1 \leq x \leq a_2 \\ w_{\tilde{a}}, & for\ a_2 \leq x \leq a_3 \\ w_{\tilde{a}}(\frac{a_4-x}{a_4-a_3}), & for\ a_3 \leq x \leq a_4 \\ 0, & otherwise, \end{cases}$$

$$\nu_{\tilde{a}}(x) = \begin{cases} \frac{a_2-x+u_{\tilde{a}}(x-a_1)}{a_2-a_1}, & for\ a_1 \le x \le a_2 \\ u_{\tilde{a}}, & for\ a_2 \le x \le a_3 \\ \frac{x-a_3+u_{\tilde{a}}(a_4-x)}{a_4-a_3}, & for\ a_3 \le x \le a_4 \\ 1, & otherwise, \end{cases}$$

$$\lambda_{\tilde{a}}(x) = \begin{cases} \frac{a_2-x+y_{\tilde{a}}(x-a_1)}{a_2-a_1}, & for\ a_1 \le x \le a_2 \\ y_{\tilde{a}}, & for\ a_2 \le x \le a_3 \\ \frac{x-a_3+y_{\tilde{a}}(a_4-x)}{a_4-a_3}, & for\ a_3 \le x \le a_4 \\ 1, & otherwise, \end{cases}$$

where $w_{\tilde{a}}, u_{\tilde{a}}$, and $y_{\tilde{a}}$ implies the maximum truth, minimum indeterminacy and minimum falsity membership degree, respectively. A single valued trapezoidal neutrosophic number $\tilde{a} = \langle (a_1, a_2, a_3, a_4); w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}} \rangle$ may approximately identical to $[a_2, a_3]$ and it is denoted to be an ill-defined quantity about a.

2.1. *Arithmetic Operations on Triangular Neutrosophic Fuzzy Numbers*

Let $\tilde{A}^N = (a_1, a_2, a_3; a_1', a_2, a_3'; a_1'', a_2, a_3'')$ and $\tilde{B}^N = (b_1, b_2, b_3; b_1', b_2, b_3'; b_1'', b_2, b_3'')$ be two triangular neutrosophic fuzzy numbers.Then

i) $\tilde{A}^N \oplus \tilde{B}^N =$
$(a_1 + b_1, a_2 + b_2, a_3 + b_3; a_1' + b_1', a_2 + b_2, a_3' + b_3'; a_1'' + b_1'', a_2 + b_2, a_3'' + b_3'')$

ii)$\tilde{A}^N \ominus \tilde{B}^N =$
$(a_1 - b_3, a_2 - b_2, a_3 - b_1; a_1' - b_3', a_2 - b_2, a_3' - b_1'; a_1'' - b_3'', a_2 - b_2, a_3'' - b_1'')$

iii)$\tilde{A}^N \otimes \tilde{B}^N = (m_1, m_2, m_3; m_1', m_2, m_3'; m_1'', m_{2,}, m_3'')$, where
$m_1 = min\{a_1b_1, a_1b_3, a_3b_1, a_3b_3\}$, $m_2(= m_2' = m_2'') = a_2b_2$,
$m_3 = max\{a_1b_1, a_1b_3, a_3b_1, a_3b_3\}$, $m_1' = min\{a_1'b_1', a_1'b_3', a_3'b_1', a_3'b_3'\}$,
$m_3' = max\{a_1'b_1', a_1'b_3', a_3'b_1', a_3'b_3'\}$, $m_1'' = min\{a_1''b_1'', a_1''b_3'', a_3''b_1'', a_3''b_3''\}$,
$m_3'' = max\{a_1''b_1'', a_1''b_3'', a_3''b_1'', a_3''b_3''\}$,

iv)$\lambda\tilde{A}^N = \begin{cases} (\lambda a_1, \lambda a_2, \lambda a_3; \lambda a_1', \lambda a_2, \lambda a_3'; \lambda a_1'', \lambda a_2, \lambda a_3''); \lambda \ge 0, \\ (\lambda a_3, \lambda a_2, \lambda a_1; \lambda a_3', \lambda a_2, \lambda a_1'; \lambda a_3'', \lambda a_2, \lambda a_1''); \lambda < 0, \end{cases}$

2.2. *Ordering of Triangular Neutrosophic Fuzzy Numbers*

Let $\tilde{A}^N = (a_1, a_2, a_3; a_1', a_2, a_3'; a_1'', a_2, a_3'')$ and $\tilde{B}^N = (b_1, b_2, b_3; b_1', b_2, b_3'; b_1'', b_2, b_3'')$ be two triangular neutrosophic fuzzy numbers.Then

i)$\tilde{A}^N \succeq \tilde{B}^N if R(\tilde{A}^N) \ge R(\tilde{B}^N)$
ii)$\tilde{A}^N \approx \tilde{B}^N if R(\tilde{A}^N) = R(\tilde{B}^N)$
where $R(\tilde{A}^N) = \frac{[a_1+2a_2+a_3+a_1'+2a_2+a_3'+a_1''+2a_2+a_3'']}{12}$
and $R(\tilde{B}^N) = \frac{[b_1+2b_2+b_3+b_1'+2b_2+b_3'+b_1''+2b_2+b_3'']}{12}$

2.3. *Arithmetic Operations on Trapezoidal Neutrosophic Fuzzy Numbers*

Let $\tilde{A}^N = (a_1, a_2, a_3, a_4; a_1', a_2', a_3', a_4'; a_1'', a_2'', a_3'', a_4'')$ and $\tilde{B}^N = (b_1, b_2, b_3, b_4; b_1', b_2', b_3', b_4'; b_1'', b_2'', b_3'', b_4'')$ be two triangular neutrosophic fuzzy numbers.Then

i) $\tilde{A}^N \oplus \tilde{B}^N = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4; a_1' + b_1', a_2' + b_2', a_3' + b_3', a_4' + b_4';$
$a_1'' + b_1'', a_2'' + b_2'', a_3'' + b_3'', a_4'' + b_4'')$

ii)$\tilde{A}^N \ominus \tilde{B}^N = (a_1 - b_4, a_2 - b_2, a_3 - b_2, a_4 - b_1; a_1' - b_4', a_2' - b_2', a_3' - b_2', a_4' - b_1';$
$a_1'' - b_4'', a_2'' - b_2'', a_3'' - b_2'', a_4'' - b_1'')$

iii)$\tilde{A}^N \otimes \tilde{B}^N = (m_1, m_2, m_3, m_4; m_1', m_2', m_3', m_4'; m_1'', m_{2,}'', m_3'', m_4'')$, where

$m_1 = min\{a_1b_1, a_1b_4, a_4b_1, a_4b_4\}$, $m_2 = min\{a_2b_2, a_2b_3, a_3b_2, a_3b_3\}$

$m_3 = max\{a_2b_2, a_2b_3, a_3b_2, a_3b_3\}$, $m_4 = max\{a_1b_1, a_1b_4, a_4b_1, a_4b_4\}$,

$m_1' = min\{a_1'b_1', a_1'b_4', a_4'b_1', a_4'b_4'\}$, $m_2' = min\{a_2'b_2', a_2'b_3', a_3'b_2', a_3'b_3'\}$,

$m_3' = max\{a_2'b_2', a_2'b_3', a_3'b_2', a_3'b_3'\}$, $m_4' = max\{a_1'b_1', a_1'b_4', a_4'b_1', a_4'b_4'\}$,

$m_1'' = min\{a_1''b_1'', a_1''b_4'', a_4''b_1'', a_4'b_4'\}$, $m_2'' = min\{a_2''b_2'', a_2''b_3'', a_3''b_2'', a_3''b_3''\}$,

$m_3'' = max\{a_2''b_2'', a_2''b_3'', a_3''b_2'', a_3''b_3''\}$, $m_4'' = max\{a_1''b_1'', a_1''b_4'', a_4''b_1'', a_4''b_4''\}$,

$$\text{iv)}\lambda\tilde{A}^N = \begin{cases} (\lambda a_1, \lambda a_2, \lambda a_3, \lambda a_4; \lambda a_1', \lambda a_2', \lambda a_3', \lambda a_4'; \lambda a_1'', \lambda a_2'', \lambda a_3'', \lambda a_4''); \lambda \geq 0, \\ (\lambda a_4 \lambda a_3, \lambda a_2, \lambda a_1; \lambda a_4' \lambda a_3', \lambda a_2', \lambda a_1'; \lambda a_4'' \lambda a_3'', \lambda a_2, \lambda a_1''); \lambda < 0, \end{cases}$$

2.4. *Ordering of Trapezoidal Neutrosophic Fuzzy Numbers*

Let $\tilde{A}^N = (a_1, a_2, a_3, a_4; a_1', a_2', a_3', a_4'; a_1'', a_2'', a_3'', a_4'')$ and $\tilde{B}^N = (b_1, b_2, b_3, b_4; b_1', b_2', b_3', b_4'; b_1'', b_2'', b_3'', b_4'')$ be two triangular neutrosophic fuzzy numbers.Then

i)$\tilde{A}^N \succeq \tilde{B}^{(N)} if R(\tilde{A}^{(N)}) \geq R(\tilde{B}^{(N)})$

ii)$\tilde{A}^N \approx \tilde{B}^{(N)} if R(\tilde{A}^{(N)}) = R(\tilde{B}^{(N)})$

where $R(\tilde{A}^N) = \frac{[a_1+a_2+a_3+a_4+a_1'+a_2'+a_3'+a_4'+a_1''+a_2''+a_3''+a_4'']}{12}$

and $R(\tilde{B}^N) = \frac{[b_1+b_2+b_3+b_4+b_1'+b_2'+b_3'+b_4'+b_1''+b_2''+b_3''+b_4'']}{12}$

3. A New Method For Solving Neutrosophic Fuzzy Transportation Problem Of Type-II

3.1. *Notations:*

c_{ij}=Unit neutrosophic transportation cost;

a_i=Neutrosophic availability:

d_j=Neutrosophic demand:

x_{ij}=Neutrosophic quantity.

3.2. *Algorithm for proposed method*

The stepwise procedure of proposed method is carried out as follows.

Step (1): Construct a neutrosophic fuzzy balanced transportation problem as in below table.

Sources	**Destination D1**	**Destination D2**	**...**	**Destination Dn**	**Availabilities**
S_1	$a_{11}, b_{11}, c_{11}; a'_{11}, b_{11}, c'_{11}; a''_{11}, b_{11}, c''_{11}$	$a_{12}, b_{12}, c_{12}; a'_{12}, b_{12}, c'_{12}; a''_{12}, b_{12}, c''_{12}$	...	$a_{1n}, b_{1n}, c_{1n}; a'_{1n}, b_{1n}, c'_{1n}; a''_{1n}, b_{1n}, c''_{1n}$	a
S_2	$a_{21}, b_{21}, c_{21}; a'_{21}, b_{21}, c'_{21}; a''_{21}, b_{21}, c''_{21}$	$a_{22}, b_{22}, c_{22}; a'_{22}, b_{22}, c'_{22}; a''_{22}, b_{22}, c''_{22}$	...	$a_{2n}, b_{2n}, c_{2n}; a'_{2n}, b_{2n}, c'_{2n}; a''_{2n}, b_{2n}, c''_{2n}$	b
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
S_n	$a_{n1}, b_{n1}, c_{n1}; a'_{n1}, b_{n1}; c'_{n1}; a''_{n1}, b_{n1}; c''_{n1}$	$a_{n2}, b_{n2}, c_{n2}; a'_{n2}, b_{n2}, c'_{n2}; a''_{n2}, b_{n2}, c''_{n2}$	...	$a_{nn}, b_{nn}, c_{nn}; a'_{nn}, b_{nn}, c'_{nn}; a''_{nn}, b_{nn}, c''_{nn}$	z
Demand	A	B	.....	Z	

Step (2): In general, The above table may be expressed as follows:

Minimize

$$R[\sum_{i=1}^{m}\sum_{j=1}^{n}(c_{ij1}, c_{ij2}, c_{ij3}; c'_{ij1}, c_{ij2}, c'_{ij3}; c''_{ij1}, c_{ij2}, c''_{ij3})x_{ij}]$$

subject to the constraints

$$\sum_{j=1}^{n} x_{ij} = a_i; i = 1, 2, ..., m,$$

$$\sum_{i=1}^{m} x_{ij} = b_j; j = 1, 2, ...,$$

$$x_{ij} \geq 0; i = 1, 2, ..., m; j = 1, 2, ..., n. \tag{1}$$

Step (3): Using the relation,

$$R[\sum_{i=1}^{m}\sum_{j=1}^{n}(a_{ij}, b_{ij}, c_{ij}; a'_{ij}, b_{ij}, c'_{ij}; a''_{ij}, b_{ij}, c''_{ij})] = \sum_{i=1}^{m}\sum_{j=1}^{n} R(a_{ij}, b_{ij}, c_{ij}; a'_{ij}, b_{ij}, c'_{ij}; a''_{ij}, b_{ij}, c''_{ij})$$

,the above problem can be stated as

Minimize

$$\sum_{i=1}^{m}\sum_{j=1}^{n} R[(c_{ij1}, c_{ij2}, c_{ij3}; c'_{ij1}, c_{ij2}, c'_{ij3}; c''_{ij1}, c_{ij2}, c''_{ij3})x_{ij})]$$

subject to the constraints

$$\sum_{j=1}^{n} x_{ij} = a_i; i = 1, 2, ..., m,$$

$$\sum_{i=1}^{m} x_{ij} = b_j; j = 1, 2, ..., n,$$

$$x_{ij} \geq 0; i = 1, 2, ..., m; j = 1, 2, ..., n. \tag{2}$$

Step (4): The expression

$R(\lambda \times (a, b, c; a', b, c'; a'', b, c'')) = \lambda \times R(a, b, c; a', b, c'; a'', b, c'')$ can be used to rewrite the above problem as

Minimize

$$\sum_{i=1}^{m}\sum_{j=1}^{n} R(c_{ij1}, c_{ij2}, c_{ij3}; c'_{ij1}, c_{ij2}, c'_{ij3}; c''_{ij1}, c_{ij2}, c''_{ij3}) \times x_{ij})$$

subject to the constraints

$$\sum_{j=1}^{n} x_{ij} = a_i; i = 1, 2, ..., m,$$

$$\sum_{i=1}^{m} x_{ij} = b_j; j = 1, 2, ..., n,$$

$$x_{ij} \geq 0; i = 1, 2, ..., m; j = 1, 2, ..., n. \qquad (3)$$

Step(5) : With the help of $R(a, b, c; a', b, c'; a'', b, c'') = \frac{a+2b+c+a'+2b+c'+a''+b+c''}{12}$, rewrite the above problem

Minimize

$$\sum_{i=1}^{m}\sum_{j=1}^{n} \frac{[c_{ij1} + 2c_{ij2} + c_{ij3} + c'_{ij1} + 2c_{ij2} + c'_{ij3} + c''_{ij1} + 2c_{ij2} + c''_{ij3}]}{12} \times x_{ij}$$

subject to the constraints

$$\sum_{j=1}^{n} x_{ij} = a_i; i = 1, 2, ..., m,$$

$$\sum_{i=1}^{m} x_{ij} = b_j; j = 1, 2, ..., n,$$

$$x_{ij} \geq 0; i = 1, 2, ..., m; j = 1, 2, ..., n. \qquad (4)$$

Step(6) : Find the optimal solution by using Vogel's approximation method.

Step(7) :The minimum neutrosophic fuzzy transportation cost is

$$\sum_{i=1}^{m}\sum_{j=1}^{n} (c_{ij1}, c_{ij2}, c_{ij3}; c'_{ij1}, c_{ij2}, c'_{ij3}; c''_{ij1}, c_{ij2}, c''_{ij3}) \times x_{ij}$$

3.3. *Numerical example*

Step (1): The existing neutrosophic fuzzy balanced transportation problem can be given below.

Sources	**Destination D1**	**Destination D2**	**Destination D3**	**Destination D4**	**Availabilities**
S 1	2,4,5;1,4,6;0.1,4,6.1	2,5,7;1,5,8;0.1,5,8.1	4,6,8;3,6,9;2.1,6,9.1	4,7,8;3,7,9;2.1,7,9.1	11
S 2	4,6,8;3,6,9;2.1,6,9.2	3,7,12;2,7,13;1.2,7,13.2	10,15,20;8,15,22;7.2,15,22.1	11,12,13;10,12,14;9.2,12,14.2	11
S 3	3,4,6;1,4,8;0.2,4,8.5	8,10,13;5,10,16;4.1,10,16.2	2,3,5;1,3,6;0.2,3,6.2	6,10,14;5,10,15;4.2,10,15.1	11
S 4	2,4,6;1,4,7;0.1,4,7.2	3,9,10;2,9,12;0.2,9,12.1	3,6,10;2,6,12;0.1,6,12.3	3,4,5;2,4,8;0.1,4,8.2	12
Demand	16	10	8	11	

Step(2): The above problem can be transformed into the neutrosophic fuzzy linear programming problem.

Minimize

$[(2, 4, 5; 1, 4, 6; 0.1, 4, 6.1)x_{11} \oplus (2, 5, 7; 1, 5, 8; 0.1, 5, 8.1)x_{12} \oplus$

$(4,6,8;3,6,9;2.1,6,9.1)x_{13} \oplus (4,7,8;3,7,9;2.1,7,9.1)x_{14} \oplus$
$(4,6,8;3,6,9;2.1,6,9.2)x_{21} \oplus (3,7,12;2,7,13;1.2,7,13.2)x_{22} \oplus$
$(10,15,20;8,15,22;7.2,15,22.1)x_{23} \oplus (11,12,13;10,12,14;9.2,12,14.2)x_{24} \oplus$
$(3,4,6;1,4,8;0.2,4,8.5)x_{31} \oplus (8,10,13;5,10,16;4.1,10,16.2)x_{32} \oplus$
$(2,3,5;1,3,6;0.2,3,6.2)x_{33} \oplus (6,10,14;5,10,15;4.2,10,15.1)x_{34} \oplus$
$(2,4,6;1,4,7;0.1,4,7.2)x_{41} \oplus (3,9,10;2,9,12;0.2,9,12.1)x_{42} \oplus$
$(3,6,10;2,6,12;0.1,6,12.3)x_{43} \oplus (3,4,5;2,4,8;0.1,4,8.2)x_{44}]$
subject to the constraints
$x_{11}+x_{12}+x_{13}+x_{14}=11,$
$x_{21}+x_{22}+x_{23}+x_{24}=11,$
$x_{31}+x_{32}+x_{33}+x_{34}=11,$
$x_{41}+x_{42}+x_{43}+x_{44}=12,$
$x_{11}+x_{21}+x_{31}+x_{41}=16,$
$x_{12}+x_{22}+x_{32}+x_{42}=10,$
$x_{13}+x_{23}+x_{33}+x_{43}=8,$
$x_{14}+x_{24}+x_{34}+x_{44}=11,$
$x_{ij} \geq 0; i=1,2,3,4; j=1,2,3,4.$

Step(3): By step (3) in the algorithm, we have

Minimize
R $[(2,4,5;1,4,6;0.1,4,6.1)x_{11} \oplus (2,5,7;1,5,8;0.1,5,8.1)x_{12} \oplus$
$(4,6,8;3,6,9;2.1,6,9.1)x_{13} \oplus (4,7,8;3,7,9;2.1,7,9.1)x_{14} \oplus$
$(4,6,8;3,6,9;2.1,6,9.2)x_{21} \oplus (3,7,12;2,7,13;1.2,7,13.2)x_{22} \oplus$
$(10,15,20;8,15,22;7.2,15,22.1)x_{23} \oplus (11,12,13;10,12,14;9.2,12,14.2)x_{24} \oplus$
$(3,4,6;1,4,8;0.2,4,8.5)x_{31} \oplus (8,10,13;5,10,16;4.1,10,16.2)x_{32} \oplus$
$(2,3,5;1,3,6;0.2,3,6.2)x_{33} \oplus (6,10,14;5,10,15;4.2,10,15.1)x_{34} \oplus$
$(2,4,6;1,4,7;0.1,4,7.2)x_{41} \oplus (3,9,10;2,9,12;0.2,9,12.1)x_{42} \oplus$
$(3,6,10;2,6,12;0.1,6,12.3)x_{43} \oplus (3,4,5;2,4,8;0.1,4,8.2)x_{44}]$
subject to the constraints
$x_{11}+x_{12}+x_{13}+x_{14}=11,$
$x_{21}+x_{22}+x_{23}+x_{24}=11,$
$x_{31}+x_{32}+x_{33}+x_{34}=11,$
$x_{41}+x_{42}+x_{43}+x_{44}=12,$
$x_{11}+x_{21}+x_{31}+x_{41}=16,$
$x_{12}+x_{22}+x_{32}+x_{42}=10,$
$x_{13}+x_{23}+x_{33}+x_{43}=8,$
$x_{14}+x_{24}+x_{34}+x_{44}=11,$
$x_{ij} \geq 0; i=1,2,3,4; j=1,2,3,4.$

Step(4): Using step(4), the above problem becomes

Minimize

$[R([(2,4,5;1,4,6;0.1,4,6.1)x_{11}) \oplus R((2,5,7;1,5,8;0.1,5,8.1)x_{12}) \oplus$
$R((4,6,8;3,6,9;2.1,6,9.1)x_{13}) \oplus R((4,7,8;3,7,9;2.1,7,9.1)x_{14}) \oplus$
$R((4,6,8;3,6,9;2.1,6,9.2)x_{21}) \oplus R((3,7,12;2,7,13;1.2,7,13.2)x_{22}) \oplus$
$R((10,15,20;8,15,22;7.2,15,22.1)x_{23}) \oplus R((11,12,13;10,12,14;9.2,12,14.2)x_{24}) \oplus$
$R((3,4,6;1,4,8;0.2,4,8.5)x_{31}) \oplus R((8,10,13;5,10,16;4.1,10,16.2)x_{32}) \oplus$
$R((2,3,5;1,3,6;0.2,3,6.2)x_{33}) \oplus R((6,10,14;5,10,15;4.2,10,15.1)x_{34}) \oplus$
$R((2,4,6;1,4,7;0.1,4,7.2)x_{41}) \oplus R((3,9,10;2,9,12;0.2,9,12.1)x_{42}) \oplus$
$R(3,6,10;2,6,12;0.1,6,12.3)x_{43} \oplus R((3,4,5;2,4,8;0.1,4,8.2)x_{44})]$
subject to the constraints
$x_{11}+x_{12}+x_{13}+x_{14}=11,$
$x_{21}+x_{22}+x_{23}+x_{24}=11,$
$x_{31}+x_{32}+x_{33}+x_{34}=11,$
$x_{41}+x_{42}+x_{43}+x_{44}=12,$
$x_{11}+x_{21}+x_{31}+x_{41}=16,$
$x_{12}+x_{22}+x_{32}+x_{42}=10,$
$x_{13}+x_{23}+x_{33}+x_{43}=8,$
$x_{14}+x_{24}+x_{34}+x_{44}=11,$
$x_{ij} \geq 0; i=1,2,3,4; j=1,2,3,4.$
Step(5): The relation in step(5) connect the later problem into below one
Minimize
$[R((2,4,5;1,4,6;0.1,4,6.1)x_{11} \oplus R(2,5,7;1,5,8;0.1,5,8.1)x_{12} \oplus$
$R(4,6,8;3,6,9;2.1,6,9.1)x_{13} \oplus R(4,7,8;3,7,9;2.1,7,9.1)x_{14} \oplus$
$R(4,6,8;3,6,9;2.1,6,9.2)x_{21} \oplus R(3,7,12;2,7,13;1.2,7,13.2)x_{22} \oplus$
$R(10,15,20;8,15,22;7.2,15,22.1)x_{23} \oplus R(11,12,13;10,12,14;9.2,12,14.2)x_{24} \oplus$
$R(3,4,6;1,4,8;0.2,4,8.5)x_{31} \oplus R(8,10,13;5,10,16;4.1,10,16.2)x_{32} \oplus$
$R(2,3,5;1,3,6;0.2,3,6.2)x_{33} \oplus R(6,10,14;5,10,15;4.2,10,15.1)x_{34} \oplus$
$R(2,4,6;1,4,7;0.1,4,7.2)x_{41} \oplus R(3,9,10;2,9,12;0.2,9,12.1)x_{42} \oplus$
$R(3,6,10;2,6,12;0.1,6,12.3)x_{43} \oplus R(3,4,5;2,4,8;0.1,4,8.2)x_{44}]$
subject to the constraints
$x_{11}+x_{12}+x_{13}+x_{14}=11,$
$x_{21}+x_{22}+x_{23}+x_{24}=11,$
$x_{31}+x_{32}+x_{33}+x_{34}=11,$
$x_{41}+x_{42}+x_{43}+x_{44}=12,$
$x_{11}+x_{21}+x_{31}+x_{41}=16,$
$x_{12}+x_{22}+x_{32}+x_{42}=10,$
$x_{13}+x_{23}+x_{33}+x_{43}=8,$
$x_{14}+x_{24}+x_{34}+x_{44}=11,$
$x_{ij} \geq 0; i=1,2,3,4; j=1,2,3,4.$
Step(6): Using the expression $R(a,b,c;a',b,c';a'',b,c'')=\frac{a+2b+c+a'+2b+c'+a''+b+c''}{12}$, rewrite the

above problem as
Minimize
$(3.68x_{11} + 4.68x_{12} + 5.93x_{13} + 6.43x_{14} + 5.94x_{21} + 7.2x_{22} + 14.9x_{23} + 11.95x_{24} + 4.22x_{31} + 10.19x_{32} + 3.2x_{33} + 9.94x_{34} + 3.94x_{41} + 7.775x_{42} + 6.28x_{43} + 4.19x_{44})$
subject to the constraints
$x_{11} + x_{12} + x_{13} + x_{14} = 11,$
$x_{21} + x_{22} + x_{23} + x_{24} = 11,$
$x_{31} + x_{32} + x_{33} + x_{34} = 11,$
$x_{41} + x_{42} + x_{43} + x_{44} = 12,$
$x_{11} + x_{21} + x_{31} + x_{41} = 16,$
$x_{12} + x_{22} + x_{32} + x_{42} = 10,$
$x_{13} + x_{23} + x_{33} + x_{43} = 8,$
$x_{14} + x_{24} + x_{34} + x_{44} = 11,$
$x_{ij} \geq 0; i = 1, 2, 3, 4; j = 1, 2, 3, 4.$
Step(7): Solving the crisp linear programming problem by Vogel's approximation method, the obtained optimal solution is
$x_{11} = 1, x_{12} = 10, x_{13} = 0, x_{14} = 0, x_{21} = 11, x_{22} = 0, x_{23} = 0, x_{24} = 0,$
$x_{31} = 3, x_{32} = 0, x_{33} = 8, x_{41} = 1, x_{42} = 0, x_{43} = 0, x_{44} = 11.$
Step(8): Using the optimal solution, the minimum neutrosophic fuzzy transportation cost is
$(2, 4, 5; 1, 4, 6; 0.1, 4, 6.1) \times 1 \oplus (2, 5, 7; 1, 5, 8; 0.1, 5, 8.1) \times 10 \oplus$
$(4, 6, 8; 3, 6, 9; 2.1, 6, 9.2) \times 11 \oplus (3, 4, 6; 1, 4, 8; 0.2, 4, 8.5) \times 3 \oplus$
$(2, 3, 5; 1, 3, 6; 0.2, 3, 6.2) \times 8 \oplus (2, 4, 6; 1, 4, 7; 0.1, 4, 7.2) \times 1 \oplus$
$(3, 4, 5; 2, 4, 8; 0.1, 4, 8.2) \times 11 = (126, 204, 282; 78, 204, 352; 26.5, 204, 359.7)$

Conclusion:

In the proposed method, the new algorithm for finding optimal solution for the transportation problem under neutrosophic environment by Vogel's approximation method is established. The final results of the stated approach are investigated through a numerical example. Using this concept, the comparision between existing methods and proposed method and various applications in neutrosophic transportaion problems will be carried out in future.

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Neutrosophic Estimators in Two-Phase Survey Sampling

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Abstract. Point estimates in survey sampling only provide a single value for the parameter being studied and are consequently vulnerable to changes caused by sampling error. In order to cope with ambiguity, indeterminacy, and uncertainty in data, Florentin Smarandache's neutrosophic technique, which generates interval estimates with high probability, offers a helpful solution. To estimate the neutrosophic population mean of the studied variable, this research provides new neutrosophic factor type exponential estimators using well-known neutrosophic auxiliary parameters. For the first-degree of approximation, the study derives the bias and Mean Squared Error (MSE) of the proposed estimators. Characterising constants have neutrosophic optimal values, and for these optimum values, the least value of the neutrosophic MSE is obtained. Notably, the proposed neutrosophic estimators outperform the corresponding adapted classical estimators since their estimated interval falls under the minimal MSE and lies within the estimated interval of the proposed neutrosophic estimators. The theoretical results are supported by empirical data from real data sets acquired by the "Ministry of Earth Sciences" and the "India Meteorological Department (IMD), Pune, India," as well as simulated data sets produced via Neutrosophic Normal Distribution. The estimator with the lowest MSE is suggested for practical applications across many domains, providing greater accuracy and reliability in parameter estimation when utilising the neutrosophic methodology.

Keywords: Neutrosophic Statistics, Bias, Mean Square Error, Auxiliary information, Exponential Estimator, Factor-type Estimator, Two-Phase Sampling, Relative Efficiency (%).

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1. **Introduction**

Sampling becomes a crucial method in scientific study when dealing with big populations and having time and resource constraints. In these situations, we use statistical techniques and estimators to estimate the relevant parameters of interest. The sample mean ($\bar{y}$), which approximates the population mean ($\bar{Y}$) of the study variable Y, is one of the most often used estimators. The sample mean is a fair estimator of the population mean, but because of the sample mean's potential for significant estimation variability, the sampling distribution may not be highly representative of the real population mean $\bar{Y}$. Therefore, even if adding a little amount of bias, researchers look for ways to increase the precision and accuracy of estimators. Incorporating data from auxiliary variables (X) that show significant positive or negative relationships with the study variable Y is one efficient method to do this. We may improve the effectiveness of the estimators by taking use of the correlation between the study variable and these auxiliary variables. The ratio as well as product methods of estimate are two common approaches for using the auxiliary variable information. The ratio between the study variable and the auxiliary variable is calculated in the ratio method to determine estimators. Estimators are created using the product consider by multiplying the study variable by the auxiliary variable. These methods work especially well when the line of regression crosses the origin. The regression technique of estimation, however, is better suited when the regression line does not cross the origin. It entails fitting a regression line to the study variable and any auxiliary variables, then estimating the population parameter using the regression equation. Due to its adaptability and extensive applicability, the ratio approach is frequently used in practical applications. It is used in a variety of sectors including agriculture to estimate crop yields, economics to evaluate revenue and investment, and healthcare to examine hospital facilities and health indicators. As scientific investigation advances, one current area of emphasis is the estimate of population parameters utilising known auxiliary variables with positive relations. With more precise and trustworthy insights into the underlying population features, this study intends to expand and improve estimating approaches.

In classical sampling theory, estimating methods for the population parameter Y employ a variety of methodologies, including ratio, product, and regression type estimators, when the data consists of precise numerical values. Several researchers in the discipline of classical statistics are devoting their efforts to developing and refining various estimators for Y, especially when information on the auxiliary variable X is available. The conventional ratio estimator, which makes use of a positively correlated auxiliary variable called X, was one of the groundbreaking developments in classical sampling theory introduced by Cochran [1]. As an auxiliary parameter, we are using the known population mean ($\bar{X}$) of X. Based on this, later scholars investigated how to improve the estimation of Y by incorporating widely used auxiliary factors including the coefficient of variation (CV), coefficient of skewness, coefficient of kurtosis, standard deviation, quartiles, and others. As an illustration, Sisodia and Dwivedi [2] developed a modified ratio estimator for $\bar{Y}$ based on the known CV of X, and Bahl and Tuteja [3] offered an exponential ratio estimator using the known CV of X to achieve enhanced estimate of $\bar{Y}$. Upadhyaya and Singh [4] provided two ratio estimators that both sought to more accurately estimate $\bar{Y}$ using the given coefficient of kurtosis and CV of X. Similar to Upadhyaya and Singh [4], Singh and Tailor [5] concentrated on using the established population correlation coefficient between Y and X to enhance $\bar{Y}$ estimations, demonstrating superior outcomes to other estimators. The ratio estimator has to be changed in order to progress the estimation of $\bar{Y}$. To boost the estimate of $\bar{Y}$, Singh et al. [6] provided modifications based on the well-known kurtosis coefficient of X, and Kadilar and Cingi [7] proposed a number of modified ratio estimators depending on well-known data regarding well-known auxiliary parameters. Using the given skewness and kurtosis of X, Yan and Tian [8] proposed two ratio type estimators for $\bar{Y}$, which outperformed competing estimators. A method for improved estimate of the population mean $\bar{Y}$ employing auxiliary parameters associated with the characteristic was provided in Singh and Solanki [9]. Yadav and kadilar [10] worked on improving a family of ratio and product estimations for Y with known parameters of X, While Grover and Kaur [11] concentrated on a general family of estimators for $\bar{Y}$ using transformed X. Vishwakarma and Kumar [12] proposed a generalised family of known auxiliary parameters-based dual to ratio-cum-product estimators for $\bar{Y}$. Cekim and Cingi [13] used the lowest and maximum values of linear adjustments of X to create a unique ratio estimate for $\bar{Y}$. Both Subzar et al. [14] and Yadav et al. [15] offered new families of $\bar{Y}$ estimators in accordance with the known population median of the study variable, demonstrating receives over competing estimators. Subzar et al. [14] produced numerous effective estimators for $\bar{Y}$ using auxiliary parameters. The next step forward came from Zaman and Dunder [16], who suggested an entirely novel modified ratio type estimator built on the exponential parameter of an auxiliary variable. To increase the effectiveness of the estimators, Yadav et al. [17] proposed an improved family of $\bar{Y}$ estimators employing known Y and X parameters. In their estimation technique, Yadav and Zaman [18] used well-known traditional as well as non-traditional auxiliary variables. These efforts are only a fraction of the numerous attempts made by numerous authors to improve $\bar{Y}$ estimate within the framework of classical sampling theory utilising known data regarding both traditional along with non-traditional, robust and non-robust auxiliary variables. The hunt for more precise and effective estimators is still an active and developing subject of study in this discipline.

1.1. *Research Gap*

The standard presumption of classical sampling theory is that the data are deterministic and that there is no uncertainty in the measurements of the observable features. However, in actual settings, we frequently come across data for the properties that are being investigated that are not properly specified. This happens across a number of industries, namely information technology, systems for decision-making, financial data analysis, much more. Fuzzy logic, developed by Prof. Lofti A. Zadeh [19] in 1965, gives a method for addressing situations when precise measurements are not available. Dealing with such indeterminate data demands alternative ways. Fuzzy logic can handle confusing, murky, or inaccurate data, but it does not completely take into account measures that are not known in advance. Contrarily, neurosophic logic expands fuzzy logic to take into consideration both the determinate along with indeterminate aspects of observations, which is especially useful when working with uncertain or ambiguous data. Fuzzy and neutrosophic logic have been created and extensively used in numerous applications for decision-making and other operations. When there is any degree of indeterminacy in the data, neutrosophic statistics, a derivative of classical statistics, takes into action. It is used when observations made about the population or sample are hazy, ambiguous, or imprecise. In systems with uncertainty, neutrosophic statistics are especially helpful because they enable the interpretation of neutrosophic data in situations where the sample size might not be ideal. Numerous applications of neutrosophic statistics have been used by researchers. It has been applied to analyse impacts, make group decisions, analyse medical data, estimate variables, track traffic accidents, create goodness-of-fit tests, research wind speed distributions, and make judgements in challenging situations with unknowns. Ultimately, fuzzy and neutrosophic logic along with statistics provide useful tools for handling ambiguous and imprecise data in real-world situations, enabling researchers and decision-makers to conduct more thorough and accurate studies and make better choices in challenging situations. In comparison to the fuzzy set, the neutrosophic set has performed better when handling uncertainty in practical settings. Neutosophic parameterized hypersoft set theory has been studied for its potential as a useful tool for applications involving decision-making. They have created cutting-edge decision-making techniques that can successfully manage uncertainty in complicated situations by introducing and examining the neutrosophic parameterized hypersoft set along with its basic properties and functions. Traditional approaches in statistical analysis may be inadequate when working with interval-valued data and uncertain situations. Modified Sign tests that take into account both the real form of the observations and the ambiguous nature of the data have been presented as a solution to this problem. These enhancements' appropriateness for nonparametric decision-making with interval-valued data has been tested using real-world data sets. Particular difficulties have been encountered in the diagnostic and decision-making processes in the medical area. Researchers have developed methods based on the generalisation of multipolar neutrosophic soft sets to overcome these challenges. These methods include informational measurements like distance, similarities, and correlation coefficient to offer a thorough framework for making decisions in the face of ambiguity. Single-valued brittle estimates may produce inaccurate and skewed results in conventional survey sample studies when data is presumed to be certain and clear. Since neutrosophic data frequently appears in real-world circumstances, Neutrosophic statistics becomes a useful alternative to conventional methods. It is a useful option in a range of scenarios where standard approaches would not be enough due to its capacity to manage indeterminacy and uncertainty. Neutosophic data, where data from experiments or populations may be expressed as interval-valued neutrosophic numbers, is characterised by ambiguous and contradictory values, non-clear contentions, and inaccurate interval values. In real-world situations, ambiguous data predominates over definite data. As a result, statistical methods that can properly handle neutrosophic data are becoming more and more important.

1.2. *Scope of the Neutrosophic Study*

The act of acquiring data for various variables in research can be expensive, especially when working with unclear or confusing data. Traditional techniques can be costly and dangerous when attempting to estimate genuine parameter values in the context of uncertain data. However, neutrosophic statistical computation provides a way to investigate data that is uncertain or for which there is inadequate information, taking into account competing viewpoints. Traditional statistics are unable to do an accurate analysis of the data due to the problem of indeterminacy, where some observations fall within a range of uncertain values. Neutosophic statistics, which are adaptable and all-encompassing, replace traditional statistics in such unsettling situations. While several research have been conducted in sample surveys to investigate neutrosophy, the particular use of ratio estimation with neutrosophic data is pretty new and requires substantial attention to meet the issues given by uncertain data systems. Numerous situations in the actual world find use for neutrosophic estimators. Neutosophic statistics may be superior to conventional techniques, for instance, in analysing machine product measurements with small errors or evaluating health parameters through various testing processes. When observations of the research variable are not deterministic but rather non-deterministic, reflecting the intrinsic uncertainties present, the use of neutrosophic estimators enables improved estimate of population means.

Although fuzzy statistics solves the issues raised by ambiguous, confusing, or imprecise data, indeterminate measures are not taken into account. Neutosophic logic, on the other hand, extends fuzzy logic more broadly by include both the determinate and indeterminate parts of observations. Analysis of situations involving ambiguous or inaccurate observations makes use of neutrosophic logic Aslam ([20] & [21]). Bellman and Zadeh [22] employed this strategy to improve decision-making precision. Different approaches based on fuzzy logic then started to appear, and they now play a big part in decision-making across many different areas. Similar to Liu and Mahmood [23], who proposed the idea of advanced fuzzy sets and showed how they might be expanded to create complicated neutrosophic sets. Interval-valued neutrosophic sets were used in a framework described by Li et al. [24] that displayed fuzzy sets together with their generalisations and aided decision-making. Aslam [25] included neutrosophic statistics in the investigation of skewness and kurtosis estimators for wind speed distributions under uncertainty. Chinnadurai and Bharathivelan [26] developed a paradigm for making decisions that favours badly damaged machines when assessing damages in a neutrosophic environment. Mohanta and Pal [27] proposed a number of single-valued neutrosophic graph (SVNG)-related ideas while emphasising the significance of fuzzy and neutrosophic sets in reducing uncertainty in real-world contexts. Zulqarnain et al. [28] concentrated on algorithms for generalised multi-polar neutrosophic soft sets with information measures to address issues in medical diagnosis and decision-making. They developed the idea of multipolar neutrosophic soft sets by introducing several informational metrics for hypothetical decision-making situations. Tahir et al. [29] emphasised the drawbacks of conventional survey sample studies that rely on precise and definite data, and it promoted the use of neutrosophic statistics in situations where the data exhibits such features. Neutrosophic data includes ambiguous and uncertain variable values, unclear statements, and erratic interval values. Data of this kind can be represented as interval-valued neutrosophic numbers, where initially uncertain observed values are assumed to fall within predetermined ranges. Neutosophic statistical methods must be developed and used since uncertain data are common in real-world situations. It can be expensive to collect data for various study variables, especially when dealing with unclear data. Therefore, using conventional techniques to determine unknown actual parameter values from uncertain data can be expensive and risky.

These factors were taken into account when Tahir et al. [29] first developed a neutrosophic ratio-type estimate technique. For analysing data with uncertainty, limited information, and opposing beliefs, neutral statistical analysis is useful. When observations fall inside an undefined value range, traditional statistics have trouble. In such circumstances, neutrosophic statistics act as a versatile and all-encompassing replacement for classical statistics. The ratio estimate approach is still relatively new in the field of sample surveys under Neutrosophy and needs additional consideration when dealing with unreliable data systems. For instance, neutrosophic statistics may be preferable to conventional methods for measurements of machine goods with small faults or health metrics acquired from various testing techniques. When dealing with nondeterministic observations of study variables, Neutrosophic estimators frequently outperform classical estimators.

1.3. *Flow Chart of the Proposed Study*

The provided flowchart (Figure - 1) presents demonstrations of the suggested factor type exponential estimators that fall within the aforementioned group of estimators. Divergent neutrosophic exponential estimators are obtained depending on the different values of d. With the use of neutrosophic statistics and by expanding on the concepts offered by Yadav and Smarandache [30], this study develops a unique method for factor type exponential estimators. We may modify the estimators to fit certain circumstances and gain greater performance in a number of applications by carefully selecting the values of d. As a result of the study's use of neutrosophic statistics, the estimate procedure gains a distinctive component that enhances its adaptability to ambiguous or indeterminate data. A similar strategy may have a big influence on a number of fields, including economics, engineering, and social sciences. To evaluate these neutrosophic estimators' performance to more established techniques, empirical assessments are essential. Under specified circumstances, the suggested factor type neutrosophic exponential estimators demonstrate promise in terms of providing accurate and dependable estimates, making a significant addition to the area of statistics. It is necessary to do further study in this field to fully explore the possibilities of factor type exponential estimators and neutrosophic statistics, opening the way for improvements in statistical estimation and analysis.

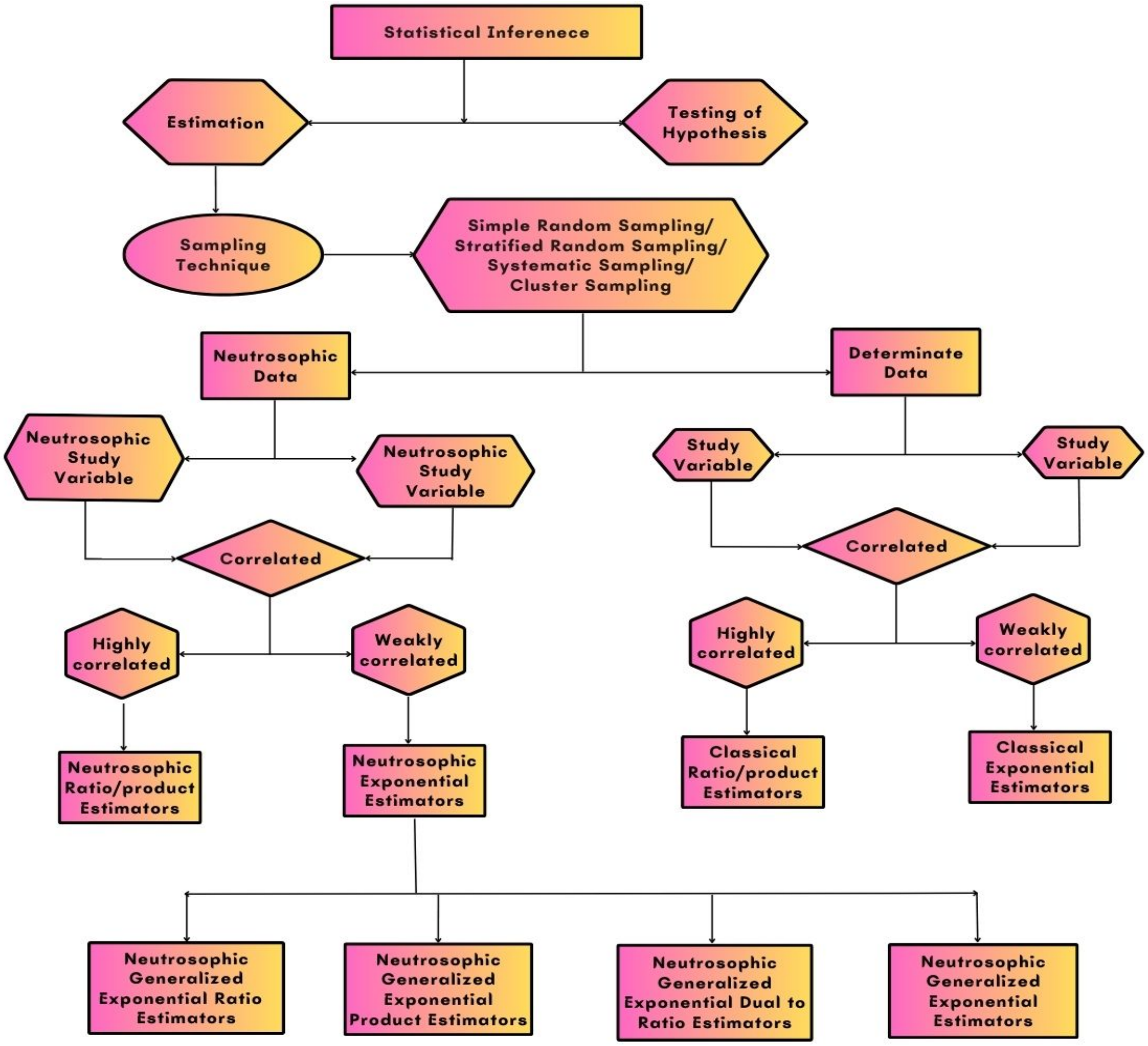

FIGURE 1. Flow Chart of the Proposed Study In Neutrosophic Framework.

1.4. *Observations and Terminology in Neutrosophic Statistics*

One of the main observations inside the neutrosophic environment is the use of quantitative neutrosophic data, where a number may live within an unknown interval $[N\Phi_L, N\Phi_U]$. Interval values of neutrosophic numbers can be stated in numerous ways. Neutosophic interval values are specifically specified in Yadav and Smarandache [30] as $Z_{(N\Phi)} = Z_{(N\Phi_L)} + Z_{(N\Phi_U)} I_{(N\Phi)}$, where $I_{(N\Phi)} \in [I_{(N\Phi_L)}, I_{(N\Phi_U)}]$. For the considered neutrosophic data, we use the same notations as Yadav and Smarandache [30], which take the interval form as $Z_{(N\Phi)} \in [Z_{(N\Phi_L)}, Z_{(N\Phi_U)}]$, where $Z_{(N\Phi_L)}$ and $Z_{(N\Phi_U)}$ indicate the respective Lower and Upper values of the neutrosophic variable $Z_{(N\Phi_L, N\Phi_U)}$. We use the simple random sampling without replacement (SRSWOR) approach to extract a neutrosophic random sample of size $n_{(N\Phi)} \in [n_{(N\Phi_L)}, n_{(N\Phi_U)}]$ from the aforementioned population under the assumption that the neutrosophic population consists of $N_{(N\Phi)} \in [N_{(N\Phi_L)}, N_{(N\Phi_U)}]$ unique units. For the neutrosophic data under discussion, each observation on the i^{th} unit of the sample for the study variable is designated as $Y_{(N\Phi)} \in [Y_{(N\Phi_L)}, Y_{(N\Phi_U)}]$, and the secondary variable $X_{(N\Phi)} \in [X_{(N\Phi_L)}, X_{(N\Phi_U)}]$. we define $\bar{Y}_{(N\Phi)} = \frac{1}{N}\sum_{i=1}^{N_{(N\Phi)}} Y_{i_{(N\Phi)}}$ and $\bar{X}_{(N\Phi)} = \frac{1}{N}\sum_{i=1}^{N_{(N\Phi)}} X_{i_{(N\Phi)}}$ as the population means for the neutrosophic variables $Y_{(N\Phi)}$ and $X_{(N\Phi)}$, respectively, acting as the overall averages of the neutrosophic data set. The neutrosophic study variable $Y_{(N\Phi)}$ has a sample mean of $\bar{y}_{(N\Phi)} = \frac{1}{n}\sum_{i=1}^{n_{(N\Phi)}} y_{i_{(N\Phi)}}$ and $X_{(N\Phi)}$, has a sample mean of $\bar{x}_{(N\Phi)} = \frac{1}{N}\sum_{i=1}^{n_{(N\Phi)}} x_{i_{(N\Phi)}}$. Additionally, The neutrosophic coefficients of variation for $Y_{(N\Phi)}$ and $X_{(N\Phi)}$ are given as $C_{y_{(N\Phi)}}$ and $C_{x_{(N\Phi)}}$, respectively. Additionally, the correlation coefficient between the neutrosophic variables $Y_{(N\Phi)}$ and $X_{(N\Phi)}$ denoted as $\rho_{y_{(N\Phi)}x_{(N\Phi)}}$. The neutrosophic coefficients of skewness and kurtosis for the neutrosophic variable $X_{(N\Phi)}$ are calculated as $\beta_{1(x_{(N\Phi)})}$ and $\beta_{2(x_{(N\Phi)})}$, respectively.

When the value of $\bar{X}_{(N\Phi)}$ is unavailable or unknown, the technique of a two-phase sampling is used in neutrosophic framework to estimate the population mean, denoted as $y_{(N\Phi)}$. To choose the required sample in the neutrosophic double sampling technique, the following steps are taken:

Case I: A large sample, designated as S', is drawn over the population employing SRSWOR, having a size of $n'_{(N\Phi)}$, ($n'_{(N\Phi)}$ being less than $N'_{(N\Phi)}$). This sample is used to collect observations that are entirely connected to the auxiliary variable $x_{(N\Phi)}$, with the goal of estimating the population mean $\bar{X}_{(N\Phi)}$ associated with this auxiliary variable.

Case II: A sample with the symbol S and a size of $n_{(N\Phi)}$ is chosen, where $(n_{(N\Phi)} < N_{(N\Phi)})$. This sample is taken directly from the population, which has the size $N_{(N\Phi)}$, or from the set of S' characters. This sample's goal is to collect data on both the primary neutrosophic study variable and the secondary neutrosophic auxiliary variable.

Employing the neutrosophic framework, we are adopting this approach to research. The corresponding sample mean, particularly is provided by, is the best estimate for the population mean.

$$t_{0_{(N\Phi)}} = \bar{y}_{(N\Phi)} \tag{1}$$

We are introducing the subsequent expression within the framework of neutrosophic statistics. The variance of $t_{0_{(N\Phi)}}$ is obtained as follows:

$$V(t_{0_{(N\Phi)}}) = \gamma_{(N\Phi)}\bar{Y}^2_{(N\Phi)}C^2_{y_{(N\Phi)}} \tag{2}$$

where,

$\gamma_{(N\Phi)} = \frac{1}{n_{(N\Phi)}} - \frac{1}{N_{(N\Phi)}}$, $C_{y_{(N\Phi)}} = \frac{S_{y_{(N\Phi)}}}{\bar{Y}_{(N\Phi)}}$ and $S^2_{y_{(N\Phi)}} = \frac{1}{N_{(N\Phi)}-1}\sum_{i=1}^{N_{(N\Phi)}}(y_{i_{(N\Phi)}} - \bar{Y}_{(N\Phi)})^2$.

We are incorporating the following idea into the framework of neutrosophic statistics. In the setting of simple random sampling, Cochran [2] proposed a conventional estimate of the population mean using auxiliary data. To improve our understanding, this strategy is being included into the design of neutrosophic statistics.

$$t_{R_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left(\frac{\bar{X}_{(N\Phi)}}{\bar{x}_{(N\Phi)}} \right) \tag{3}$$

In this study, a new neutrosophic factor type exponential estimator is introduced for improving the estimate of the parameter $Y_{N\phi}$ utilising known $X_{N\phi}$ parameter values. In the first degree of approximation, the sample characteristics of the suggested estimator are investigated.

- This article introduces Florentin Smarandache's neutrosophic technique in Two Phase survey sampling.
- Addresses ambiguity, indeterminacy, and uncertainty in data.
- Proposes new estimators for neutrosophic population mean.
- Utilizes well-known neutrosophic auxiliary parameters.
- Derives bias and MSE for proposed estimators in the first-degree of approximation.
- Identifies optimal values for characterizing constants.
- Validates theoretical findings with real datasets from Earth Sciences and Meteorological Departments.
- Includes simulated data sets from Neutrosophic Normal Distribution.

Advantages:

- Handles ambiguity and uncertainty in data.
- Optimizes Mean Squared Error for reliability.
- Supported by empirical data from real and simulated sets.

Disadvantages:

- May introduce complexity and pose interpretability challenges.
- Computational burden not discussed.
- Generalizability to various domains requires further investigation.

While previous studies (Including more study for the reference Uma & Nandhitha [31], Jdid & Smarandache [32], Abdel-Basset et al. [33], Gamal et al. [34,35]) have made strides in neutrosophic systems, our study uniquely focuses on two phase survey sampling using Neutrosophic statistics—an area largely unexplored in survey sampling.

The full paper is divided into sections that include the introduction, some existing adapted estimators, proposed estimators, numerical study, simulation study, conclusssion and references. By adding to the ongoing study of neutrosophic statistical methods and their applications, the article is divided into these components.

2. Some Existing Estimators

As far as we are aware, this study is a fresh and ground-breaking use of Neutroshopic statistics that incorporates two-phase sample estimators. To the best of our knowledge, this is the first thorough investigation of the use of these estimators within the framework of two phase neutroshopic sampling. The use of two-phase sample approaches in the context of neutroshopic statistics, a relatively new and specialised discipline, gives up intriguing opportunities for investigation and evaluation. This work presents a cutting-edge strategy to dealing with complicated data gathering scenarios by integrating the distinctive elements of Neutroshopic statistics with the complexities of two-phase sampling. The use of these estimators in Neutroshopic statistics indicates a forward-thinking and innovative approach to statistical analysis, paving the way for future advances and discoveries in this rapidly growing subject. We hope that this new study will benefit the larger scientific community by fostering a better knowledge of statistical approaches in the context of Neutroshopic sampling. We are using previously suggested estimators that have been adjusted for neutrosophic two-phase sampling in this part. This novel method bridges the gap between traditional two-phase sampling and Neutrosophic statistics, improving population mean estimate in the presence of uncertainties and missing data.

2.1. *Adapted Kumar and Bahl Estimators*

We are integrating the conventional ratio estimator for two-phase sampling suggested by Kumar and Bahl [36] into the Neutroshopic statistical framework. This modification attempts to boost the accuracy of statistical analysis in this specialised sector by improving population mean estimate under uncertainty.

$$t^d_{R_{(N\Phi)}} = \bar{y}_{(N\Phi)}\left(\frac{\bar{x}_{1_{(N\Phi)}}}{\bar{X}_{(N\Phi)}}\right) \tag{4}$$

where

$\bar{x}_{1_{(N\Phi)}} = \frac{1}{n_{(N\Phi)}'}\sum_{i=1}^{n'_{(N\Phi)}} x_{i_{(N\Phi)}}$.

MSE of $t^d_{R_{(N\Phi)}}$, for Case-I and Case-II are given as,

$$MSE(t^d_{R_{(N\Phi)}})_I = \bar{Y}^2_{(N\Phi)}\left[\gamma_{(N\Phi)}C^2_{y_{(N\Phi)}} + \gamma^{**}_{(N\Phi)}C^2_{x_{(N\Phi)}}(1-2C_{(N\Phi)})\right] \tag{5}$$

$$MSE(t^d_{R_{(N\Phi)}})_{II} = \bar{Y}^2_{(N\Phi)}\left[\gamma_{(N\Phi)}C^2_{y_{(N\Phi)}} + \gamma^{***}_{(N\Phi)}C^2_{x_{(N\Phi)}} - 2\gamma_{(N\Phi)}C_{(N\Phi)}C^2_{x_{(N\Phi)}}\right] \tag{6}$$

where,

$\gamma^*_{(N\Phi)} = \left(\frac{1}{n'_{(N\Phi)}} - \frac{1}{N_{(N\Phi)}}\right)$, $\gamma^{**}_{(N\Phi)} = \left(\frac{1}{n_{(N\Phi)}} - \frac{1}{n'_{(N\Phi)}}\right)$, $\gamma^{***}_{(N\Phi)} = \left(\gamma_{(N\Phi)} + \gamma^*_{(N\Phi)}\right)$, $C_{x_{(N\Phi)}} = \frac{S_{x_{(N\Phi)}}}{\bar{X}_{(N\Phi)}}$, $C_{(N\Phi)} = \rho_{y_{(N\Phi)}x_{(N\Phi)}}\left(\frac{C_{y_{(N\Phi)}}}{C_{x_{(N\Phi)}}}\right)$,

$S^2_{x_{(N\Phi)}} = \frac{1}{N_{(N\Phi)}-1}\sum_{i=1}^{N_{(N\Phi)}}(x_{i_{(N\Phi)}} - \bar{X}_{(N\Phi)})^2$, and

$\rho_{y_{(N\Phi)}x_{(N\Phi)}} = \frac{1}{N_{(N\Phi)}}\sum_{i=1}^{N_{(N\Phi)}}(y_{i_{(N\Phi)}} - \bar{Y}_{(N\Phi)})(x_{i_{(N\Phi)}} - \bar{X}_{(N\Phi)})$

2.2. *Adapted Singh and Choudhury Estimator*

In the context of Neutroshopic statistics, we are using the dual to product estimator of population mean suggested by Singh and Choudhury [37] for two-phase sampling. This specialised estimator improves population mean estimate in Neutroshopic data and is designed to manage uncertainties, making statistical studies more trustworthy.

$$t^d_{P_{(N\Phi)}} = \bar{y}_{(N\Phi)}\left(\frac{\bar{x}_{(N\Phi)}}{\bar{x}_{1_{(N\Phi)}}}\right) \tag{7}$$

MSE for Case-I and Case-II are given as,

$$MSE(t^d_{P_{(N\Phi)}})_I = \bar{Y}^2_{(N\Phi)}\left[\gamma_{(N\Phi)}C^2_{y_{(N\Phi)}} + \gamma^{**}_{(N\Phi)}C^2_{x_{(N\Phi)}}(1 + 2C_{(N\Phi)})\right] \tag{8}$$

$$MSE(t^d_{P_{(N\Phi)}})_{II} = \bar{Y}^2_{(N\Phi)}\left[\gamma_{(N\Phi)}C^2_{y_{(N\Phi)}} + \gamma^{***}_{(N\Phi)}C^2_{x_{(N\Phi)}} + 2\gamma_{(N\Phi)}C_{(N\Phi)}C^2_{x_{(N\Phi)}}\right] \tag{9}$$

2.3. *Adapted Singh and Vishwakarma Estimators*

In the setting of Neutroshopic statistics, we are adopting Singh and Vishwakarma's [38] suggested exponential type ratio and product estimators. In two-phase sampling circumstances, these estimators are tailored to manage uncertainties and enhance population mean estimation, increasing the precision and efficacy of statistical studies in the Neutroshopic setting.

$$t^d_{Re_{(N\Phi)}} = \bar{y}_{(N\Phi)}exp\left(\frac{\bar{x}_{1_{(N\Phi)}} - \bar{x}_{(N\Phi)}}{\bar{x}_{1_{(N\Phi)}} + \bar{x}_{(N\Phi)}}\right) \tag{10}$$

$$t^d_{Pe_{(N\Phi)}} = \bar{y}_{(N\Phi)}exp\left(\frac{\bar{x}_{(N\Phi)} - \bar{x}_{1_{(N\Phi)}}}{\bar{x}_{(N\Phi)} + \bar{x}_{1_{(N\Phi)}}}\right) \tag{11}$$

MSE of both the estimators for Case-I and Case-II are given as,

$$MSE(t^d_{Re_{(N\Phi)}})_I = \bar{Y}^2_{(N\Phi)}\left[\gamma_{(N\Phi)}C^2_{y_{(N\Phi)}} + \gamma^{**}_{(N\Phi)}C^2_{x_{(N\Phi)}}(\frac{1}{4} - C_{(N\Phi)})\right] \tag{12}$$

$$MSE(t^d_{Re_{(N\Phi)}})_{II} = \bar{Y}^2_{(N\Phi)}\left[\gamma_{(N\Phi)}C^2_{y_{(N\Phi)}} + \frac{1}{4}\gamma^{***}_{(N\Phi)}C^2_{x_{(N\Phi)}} - \gamma_{(N\Phi)}C^2_{x_{(N\Phi)}}C_{(N\Phi)}\right] \tag{13}$$

$$MSE(t^d_{Pe_{(N\Phi)}})_I = \bar{Y}^2_{(N\Phi)}\left[\gamma_{(N\Phi)}C^2_{y_{(N\Phi)}} + \gamma^{**}_{(N\Phi)}C^2_{x_{(N\Phi)}}(\frac{1}{4} + C_{(N\Phi)})\right] \tag{14}$$

$$MSE(t^d_{Pe_{(N\Phi)}})_{II} = \bar{Y}^2_{(N\Phi)}\left[\gamma_{(N\Phi)}C^2_{y_{(N\Phi)}} + \frac{1}{4}\gamma^{***}_{(N\Phi)}C^2_{x_{(N\Phi)}} + \gamma_{(N\Phi)}C^2_{x_{(N\Phi)}}C_{(N\Phi)}\right] \tag{15}$$

2.4. *Adapted Kumar and Bahl Estimator*

We are using Kumar and Bahl's [36] dual to ratio estimator for the population mean in two-phase sampling in the context of Neutroshopic statistics. By addressing uncertainties in Neutroshopic data, this specialised estimator improves the precision of population mean estimate and makes it possible to make well-informed decisions in challenging sampling situations.

$$t^{*d}_{R_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left(\frac{\bar{x}^{*d}_{(N\Phi)}}{\bar{x}_{1_{(N\Phi)}}} \right) \tag{16}$$

MSE for Case-I and Case-II are given as,

$$MSE(t^{*d}_{R_{(N\Phi)}})_I = \bar{Y}^2_{(N\Phi)} \left[\gamma_{(N\Phi)} C^2_{y_{(N\Phi)}} + g_{(N\Phi)} \gamma^{**}_{(N\Phi)} C^2_{x_{(N\Phi)}} (g_{(N\Phi)} - 2C_{(N\Phi)}) \right] \tag{17}$$

$$MSE(t^{*d}_{R_{(N\Phi)}})_{II} = \bar{Y}^2_{(N\Phi)} \left[\gamma_{(N\Phi)} C^2_{y_{(N\Phi)}} + g_{(N\Phi)} C^2_{x_{(N\Phi)}} (g_{(N\Phi)} \gamma^{***}_{(N\Phi)} - 2\gamma_{(N\Phi)} C_{(N\Phi)}) \right] \tag{18}$$

where, $g_{(N\Phi)} = \frac{n_{(N\Phi)}}{n_{1_{(N\Phi)}} - n_{(N\Phi)}}$

2.5. *Adapted Singh and Choudhury Estimator*

For two-phase sampling in Neutroshopic statistics, we use Singh and Choudhury's [37] dual to product estimator. This estimator is intended to deal with uncertainties in Neutroshopic data, boosting population mean estimation accuracy and allowing for successful statistical analysis in complicated sampling settings.

$$t^{*d}_{P_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left(\frac{\bar{x}_{1_{(N\Phi)}}}{\bar{x}^{*d}_{(N\Phi)}} \right) \tag{19}$$

MSE for Case-I and Case-II are given as,

$$MSE(t^{*d}_{P_{(N\Phi)}})_I = \bar{Y}^2_{(N\Phi)} \left[\gamma_{(N\Phi)} C^2_{y_{(N\Phi)}} + g_{(N\Phi)} \gamma^{**}_{(N\Phi)} C^2_{x_{(N\Phi)}} (g_{(N\Phi)} + 2C_{(N\Phi)}) \right] \tag{20}$$

$$MSE(t^{*d}_{P_{(N\Phi)}})_{II} = \bar{Y}^2_{(N\Phi)} \left[\gamma_{(N\Phi)} C^2_{y_{(N\Phi)}} + g_{(N\Phi)} C^2_{x_{(N\Phi)}} (g_{(N\Phi)} \gamma^{***}_{(N\Phi)} + 2\gamma_{(N\Phi)} C_{(N\Phi)}) \right] \tag{21}$$

2.6. *Adapted Kalita and Singh Estimators*

In Neutroshopic statistics, we are employing Kalita and Singh's exponential dual to ratio and exponential dual to product estimators [39] in two-phase sampling. These estimators manage uncertainty in Neutroshopic data, improving accuracy and efficacy in challenging sampling circumstances.

$$t^{*d}_{Re_{(N\Phi)}} = \bar{y}_{(N\Phi)} exp \left(\frac{\bar{x}^{*d}_{(N\Phi)} - \bar{x}_{1_{(N\Phi)}}}{\bar{x}^{*d}_{(N\Phi)} + \bar{x}_{1_{(N\Phi)}}} \right) \tag{22}$$

$$t^{*d}_{Pe_{(N\Phi)}} = \bar{y}_{(N\Phi)} exp \left(\frac{\bar{x}_{1_{(N\Phi)}} - \bar{x}^{*d}_{(N\Phi)}}{\bar{x}_{1_{(N\Phi)}} + \bar{x}^{*d}_{(N\Phi)}} \right) \tag{23}$$

MSE of both the estimators for Case-I and Case-II are given as,

$$MSE(t^{*d}_{Re_{(N\Phi)}})_I = \bar{Y}^2_{(N\Phi)}\left[\gamma_{(N\Phi)}C^2_{y_{(N\Phi)}} + g_{(N\Phi)}\gamma^{**}_{(N\Phi)}C^2_{x_{(N\Phi)}}(\frac{1}{4}g_{(N\Phi)} - C_{(N\Phi)})\right] \tag{24}$$

$$\begin{aligned} MSE(t^{*d}_{Re_{(N\Phi)}})_{II} =& \bar{Y}^2_{(N\Phi)}\Big[\gamma_{(N\Phi)}C^2_{y_{(N\Phi)}} + \frac{1}{4}g^2_{(N\Phi)}\gamma^{***}_{(N\Phi)}C^2_{x_{(N\Phi)}} \\ &- \gamma_{(N\Phi)}g_{(N\Phi)}C^2_{x_{(N\Phi)}}C_{(N\Phi)}\Big] \end{aligned} \tag{25}$$

$$MSE(t^{*d}_{Pe_{(N\Phi)}})_I = \bar{Y}^2_{(N\Phi)}\left[\gamma_{(N\Phi)}C^2_{y_{(N\Phi)}} + g_{(N\Phi)}\gamma^{**}_{(N\Phi)}C^2_{x_{(N\Phi)}}(\frac{1}{4}g_{(N\Phi)} + C_{(N\Phi)})\right] \tag{26}$$

$$\begin{aligned} MSE(t^{*d}_{Pe_{(N\Phi)}})_{II} =& \bar{Y}^2_{(N\Phi)}\Big[\gamma_{(N\Phi)}C^2_{y_{(N\Phi)}} + \frac{1}{4}g^2_{(N\Phi)}\gamma^{***}_{(N\Phi)}C^2_{x_{(N\Phi)}} \\ &+ \gamma_{(N\Phi)}g_{(N\Phi)}C^2_{x_{(N\Phi)}}C_{(N\Phi)}\Big] \end{aligned} \tag{27}$$

2.7. *Adapted Subhash et al. Estimators*

In Neutroshopic statistics, we use Subhash et al. [40] modified ratio and product estimators, which are based on Kalita and Singh's work [39]. These estimators deal with uncertainty, improving population mean estimate in two-phase sampling and so leading to more robust statistical studies.

$$\jmath^{*d}_{Re_{(N\Phi)}} = \alpha\bar{y}_{(N\Phi)} + (1-\alpha)t^{*d}_{Re_{(N\Phi)}} \tag{28}$$

$$\jmath^{*d}_{Pe_{(N\Phi)}} = \delta\bar{y}_{(N\Phi)} + (1-\delta)t^{*d}_{Pe_{(N\Phi)}} \tag{29}$$

where, α and δ are the scalars constant.

MSE of both the estimators for Case-I and Case-II are given as,

$$\begin{aligned} MSE(\jmath^{*d}_{Re_{(N\Phi)}})_I =& \bar{Y}^2_{(N\Phi)}\Big[\gamma_{(N\Phi)}C^2_{y_{(N\Phi)}} + g_{(N\Phi)}\gamma^{**}_{(N\Phi)}C^2_{x_{(N\Phi)}}\Big(\frac{1}{4}g_{(N\Phi)} - C_{(N\Phi)}\Big) \\ &- \gamma^{**}_{(N\Phi)}\frac{A^2_{(N\Phi)}}{4B_{(N\Phi)}}\Big] \end{aligned} \tag{30}$$

$$\begin{aligned} MSE(\jmath^{*d}_{Re_{(N\Phi)}})_{II} =& \bar{Y}^2_{(N\Phi)}\Big[\gamma_{(N\Phi)}C^2_{y_{(N\Phi)}} + \frac{g^2_{(N\Phi)}}{4}\gamma^{***}_{(N\Phi)}C^2_{x_{(N\Phi)}} \\ &- g_{(N\Phi)}\gamma_{(N\Phi)}C^2_{x_{(N\Phi)}}C_{(N\Phi)} - \frac{A^{*2}_{(N\Phi)}}{4B^*_{(N\Phi)}}\Big] \end{aligned} \tag{31}$$

$$\begin{aligned} MSE(\jmath^{*d}_{Pe_{(N\Phi)}})_I =& \bar{Y}^2_{(N\Phi)}\Big[\gamma_{(N\Phi)}C^2_{y_{(N\Phi)}} + g_{(N\Phi)}\gamma^{**}_{(N\Phi)}C^2_{x_{(N\Phi)}}(\frac{g_{(N\Phi)}}{4} \\ &+ C_{(N\Phi)}) - \gamma^{**}_{(N\Phi)}\frac{D^2_{(N\Phi)}}{4B_{(N\Phi)}}\Big] \end{aligned} \tag{32}$$

$$MSE(\jmath^{*d}_{Pe_{(N\Phi)}})_{II} = \bar{Y}^2_{(N\Phi)}\Big[\gamma_{(N\Phi)}C^2_{y_{(N\Phi)}} + \frac{g^2_{(N\Phi)}}{4}\gamma^{***}_{(N\Phi)}C^2_{x_{(N\Phi)}} + g_{(N\Phi)}\gamma_{(N\Phi)}C^2_{x_{(N\Phi)}}C_{(N\Phi)} - \frac{D^{*2}_{(N\Phi)}}{4B^*_{(N\Phi)}}\Big] \quad (33)$$

Where

$A_{(N\Phi)} = C^2_{x_{(N\Phi)}}(g_{(N\Phi)} - 2C_{(N\Phi)})$, $B_{(N\Phi)} = C^2_{x_{(N\Phi)}}$, $D_{(N\Phi)} = C^2_{x_{(N\Phi)}}(g_{(N\Phi)} + 2C_{(N\Phi)})$,
$A^*_{(N\Phi)} = g_{(N\Phi)}\gamma^{***}_{(N\Phi)}C^2_{x_{(N\Phi)}} - 2\gamma_{(N\Phi)}C^2_{x_{(N\Phi)}}C_{(N\Phi)}$, $B^*_{(N\Phi)} = \gamma^{***}_{(N\Phi)}C^2_{x_{(N\Phi)}}$,
$D^*_{(N\Phi)} = g_{(N\Phi)}\gamma^{***}_{(N\Phi)}C^2_{x_{(N\Phi)}} + 2\gamma_{(N\Phi)}C^2_{x_{(N\Phi)}}C_{(N\Phi)}$ and $g_{(N\Phi)} = \frac{n_{(N\Phi)}}{n_{1_{(N\Phi)}} - n_{(N\Phi)}}$

3. Proposed Estimators

We have presented a generalised class of factor-type estimators for two-phase sampling in the setting of Neutroshopic statistics, building on previous work and driven by the generic character of exponential and factor type estimators. This new family of estimators is designed particularly to deal with the uncertainties and difficulties inherent with Neutroshopic. We want to improve the accuracy and reliability of population mean estimate and progress statistical studies in the field of Neutroshopic statistics by introducing this novel technique. This suggestion contributes significantly to the knowledge and use of statistical approaches in the field of Neutroshopic two-phase sampling.

$$\Im^{yp}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)}\left[\frac{(A+c)\bar{x}_{1_{(N\Phi)}} + f_{(N\Phi)}B\bar{x}_{(N\Phi)}}{(A+f_{(N\Phi)}B)\bar{x}_{1_{(N\Phi)}} + C\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{a(\bar{x}_{1_{(N\Phi)}} - \bar{x}_{(N\Phi)})}{a(\bar{x}_{1_{(N\Phi)}} + \bar{x}_{(N\Phi)}) + 2b}\right\} \quad (34)$$

where, $A = (d-1)(d-2)$; $B = (d-1)(d-4)$; $C = (d-2)(d-3)(d-4)$, α and d are the characterizing scalars, (a≠0, b) are real constants or functions of populations parameters of the known auxiliary variable. For the fixed value of α, we get some generalized exponential estimators for different values of d, which is dicuss later on in this Section as a particular cases. Members of this class of proposed neutrosophic estimators are given in Table 5.

Note:

(i) For $\alpha = 0$, in eqaution (34) our proposed estimator $\Im^{yp}_{ft_{(N\Phi)}}$ reduces to modified exponential estimators for suitable values of (a≠0, b) members of this class of neutrosophic estimators are given in Tables [9 & 14].

$$\Im^{yp}_{exp_{(N\Phi)}} = \bar{y}_{(N\Phi)} exp\left\{\frac{a(\bar{x}_{1_{(N\Phi)}} - \bar{x}_{(N\Phi)})}{a(\bar{x}_{1_{(N\Phi)}} + \bar{x}_{(N\Phi)}) + 2b}\right\} \quad (35)$$

(ii) For $\alpha = 1$, in equation (34) our proposed estimator $\Im^{yp}_{ft_{(N\Phi)}}$ reduces to genralized factor type ratio exponential estimators for suitable values of (a≠0, b) members of this class of neutrosophic estimators are given in Tables [10].

$$\Im^{yp}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)}\left[\frac{(A+c)\bar{x}_{1_{(N\Phi)}} + f_{(N\Phi)}B\bar{x}_{(N\Phi)}}{(A+f_{(N\Phi)}B)\bar{x}_{1_{(N\Phi)}} + C\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{a(\bar{x}_{1_{(N\Phi)}} - \bar{x}_{(N\Phi)})}{a(\bar{x}_{1_{(N\Phi)}} + \bar{x}_{(N\Phi)}) + 2b}\right\} \quad (36)$$

Members of this class of proposed neutrosophic estimators are given in Table 10. **Particular Cases of Proposed Estimator:**

(1) When $d = 1$ in the values of A, B and C in equation (34) estimator $\Im^{yp}_{ft_{(N\Phi)}}$ takes the form as

$$\Im^{Re}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{1_{(N\Phi)}}}{\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{a(\bar{x}_{1_{(N\Phi)}} - \bar{x}_{(N\Phi)})}{a(\bar{x}_{1_{(N\Phi)}} + \bar{x}_{(N\Phi)}) + 2b}\right\}$$

which is generalised ratio type exponetial estimators in two phase sampling. For suitable values of (a≠0, b), members of proposed neutrosophic estimators $\Im^{Re}_{ft_{(N\Phi)}}$ are given in Table [6].

(2) When $d = 2$ in the values of A, B and C in equation (34) estimator $\Im^{yp}_{ft_{(N\Phi)}}$ takes the form as

$$\Im^{Pe}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{(N\Phi)}}{\bar{x}_{1_{(N\Phi)}}}\right]^{\alpha} exp\left\{\frac{a(\bar{x}_{1_{(N\Phi)}} - \bar{x}_{(N\Phi)})}{a(\bar{x}_{1_{(N\Phi)}} + \bar{x}_{(N\Phi)}) + 2b}\right\}$$

which is generalised product type exponetial estimators in two phase sampling. For suitable values of (a≠0, b), members of proposed neutrosophic estimators $\Im^{Pe}_{ft_{(N\Phi)}}$ are given in Table [7].

(3) When $d = 3$ in the values of A, B and C in equation (34) estimator $\Im^{yp}_{ft_{(N\Phi)}}$ takes the form as

$$\Im^{DR}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{1_{(N\Phi)}} - f\bar{x}_{(N\Phi)}}{\bar{x}_{1_{(N\Phi)}} - \bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{a(\bar{x}_{1_{(N\Phi)}} - \bar{x}_{(N\Phi)})}{a(\bar{x}_{1_{(N\Phi)}} + \bar{x}_{(N\Phi)}) + 2b}\right\}$$

which is generalised dual to ratio type exponetial estimators in two phase sampling. For suitable values of (a≠0, b), members of proposed neutrosophic estimators $\Im^{DR}_{ft_{(N\Phi)}}$ are given in Table [8].

(4) When $d = 4$ in the values of A, B and C in equation (34) estimator $\Im^{yp}_{ft_{(N\Phi)}}$ takes the form as

$$\Im^{exp}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} exp\left\{\frac{a(\bar{x}_{1_{(N\Phi)}} - \bar{x}_{(N\Phi)})}{a(\bar{x}_{1_{(N\Phi)}} + \bar{x}_{(N\Phi)}) + 2b}\right\}$$

which is generalised ratio type exponetial estimators in two phase sampling. For suitable values of (a≠0, b), members of proposed neutrosophic estimators $\Im^{exp}_{ft_{(N\Phi)}}$ are given in Table [9].

Similarly, Particular Cases of Proposed Estimator when $\alpha = 1$:

(1) When $d = 1$ in the values of A, B and C in equation (36) estimator $\Im^{yp}_{ft_{(N\Phi)}}$ takes the form as

$$\Im^{Re}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{1_{(N\Phi)}}}{\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{a(\bar{x}_{1_{(N\Phi)}} - \bar{x}_{(N\Phi)})}{a(\bar{x}_{1_{(N\Phi)}} + \bar{x}_{(N\Phi)}) + 2b}\right\}$$

which is generalised ratio type exponetial estimators in two phase sampling. For suitable values of (a≠0, b), members of proposed neutrosophic estimators $\Im^{Re}_{ft_{(N\Phi)}}$ are given in Table [11].

(2) When $d = 2$ in the values of A, B and C in equation (36) estimator $\Im^{yp}_{ft_{(N\Phi)}}$ takes the form as

$$\Im^{Pe}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{(N\Phi)}}{\bar{x}_{1_{(N\Phi)}}}\right] exp\left\{\frac{a(\bar{x}_{1_{(N\Phi)}} - \bar{x}_{(N\Phi)})}{a(\bar{x}_{1_{(N\Phi)}} + \bar{x}_{(N\Phi)}) + 2b}\right\}$$

which is generalised product type exponetial estimators in two phase sampling. For suitable values of (a≠0, b), members of proposed neutrosophic estimators $\Im^{Pe}_{ft_{(N\Phi)}}$ are given in Table [12].

(3) When $d = 3$ in the values of A, B and C in equation (36) estimator $\Im^{yp}_{ft_{(N\Phi)}}$ takes the form as

$$\Im^{DR}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{1_{(N\Phi)}} - f\bar{x}_{(N\Phi)}}{\bar{x}_{1_{(N\Phi)}} - \bar{x}_{(N\Phi)}}\right] exp\left\{\frac{a(\bar{x}_{1_{(N\Phi)}} - \bar{x}_{(N\Phi)})}{a(\bar{x}_{1_{(N\Phi)}} + \bar{x}_{(N\Phi)}) + 2b}\right\}$$

which is generalised dual to ratio type exponetial estimators in two phase sampling. For suitable values of (a≠0, b), members of proposed neutrosophic estimators $\Im^{DR}_{ft_{(N\Phi)}}$ are given in Table [13].

(4) When $d = 4$ in the values of A, B and C in equation (36) estimator $\Im^{yp}_{ft_{(N\Phi)}}$ takes the form as

$$\Im^{exp}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} exp\left\{\frac{a(\bar{x}_{1_{(N\Phi)}} - \bar{x}_{(N\Phi)})}{a(\bar{x}_{1_{(N\Phi)}} + \bar{x}_{(N\Phi)}) + 2b}\right\}$$

which is generalised ratio type exponetial ratio estimators in two phase sampling. For suitable values of (a≠0, b), members of proposed neutrosophic estimators $\Im^{exp}_{ft_{(N\Phi)}}$ are given in Table [14].

4. **Bias and MSE**

To derive the expressions of Bias and MSE we have following two cases for the proposed class of estimators.

Case I: A large sample, designated as S', is drawn over the population employing SRSWOR, having a size of $n'_{(N\Phi)}$, ($n'_{(N\Phi)}$ being less than $N'_{(N\Phi)}$). This sample is used to collect observations that are entirely connected to the auxiliary variable $x_{(N\Phi)}$, with the goal of estimating the population mean $\bar{X}_{(N\Phi)}$ associated with this auxiliary variable.

Case II: A sample with the symbol S and a size of $n_{(N\Phi)}$ is chosen, where ($n_{(N\Phi)} < N_{(N\Phi)}$). This sample is taken directly from the population, which has the size $N_{(N\Phi)}$, or from the set of S' characters. This sample's goal is to collect data on both the primary neutrosophic study variable and the secondary neutrosophic auxiliary variable.

4.1. *Case I*

To derive the expression for Bias and MSE for Case I, consider the following transformations as follows

$\bar{y}_{(N\Phi)} = \bar{Y}_{(N\Phi)}(1 + e_{0_{(N\Phi)}})$, $\bar{x}_{(N\Phi)} = \bar{X}_{(N\Phi)}(1 + e_{1_{(N\Phi)}})$, and $\bar{x}_{1_{(N\Phi)}} = \bar{X}_{(N\Phi)}(1 + e_{2_{(N\Phi)}})$
such that $E(e_{0_{(N\Phi)}}) = E(e_{1_{(N\Phi)}}) = E(e_{2_{(N\Phi)}}) = 0$ and $E(e^2_{0_{(N\Phi)}}) = \gamma_{(N\Phi)} C^2_{y_{(N\Phi)}}$,
$E(e^2_{1_{(N\Phi)}}) = \gamma_{(N\Phi)} C^2_{x_{(N\Phi)}}$, $E(e^2_{2_{(N\Phi)}}) = \gamma^*_{(N\Phi)} C^2_{x_{(N\Phi)}}$, $E(e_{0_{(N\Phi)}} e_{1_{(N\Phi)}}) = \gamma_{(N\Phi)} C_{(N\Phi)} C^2_{x_{(N\Phi)}}$,
$E(e_{0_{(N\Phi)}} e_{2_{(N\Phi)}}) = \gamma^*_{(N\Phi)} C_{(N\Phi)} C^2_{x_{(N\Phi)}}$, $E(e_{1_{(N\Phi)}} e_{2_{(N\Phi)}}) = \gamma^*_{(N\Phi)} C^2_{x_{(N\Phi)}}$,

$\gamma_{(N\Phi)} = \left(\frac{1}{n_{(N\Phi)}} - \frac{1}{N_{(N\Phi)}}\right)$, $\gamma^{*}_{(N\Phi)} = \left(\frac{1}{n1_{(N\Phi)}} - \frac{1}{N_{(N\Phi)}}\right)$, $\gamma^{**}_{(N\Phi)} = \gamma_{(N\Phi)} - \gamma^{*}_{(N\Phi)} = \left(\frac{1}{n_{(N\Phi)}} - \frac{1}{n1_{(N\Phi)}}\right)$, $C_{(N\Phi)} = \rho_{y_{(N\Phi)}x_{(N\Phi)}}\frac{C_{y_{(N\Phi)}}}{C_{x_{(N\Phi)}}}$ and $g_{(N\Phi)} = \frac{n_{(N\Phi)}}{n1_{(N\Phi)} - n_{(N\Phi)}}$
under the above transformations and from equation (34) expressing estimators in terms of e's, we get

$$\begin{aligned}\Im^{yp}_{ft_{(N\Phi)}} =& \bar{Y}_{(N\Phi)}(1+e_{0_{(N\Phi)}})\left[\frac{(A+c)\bar{X}_{(N\Phi)}(1+e_{2_{(N\Phi)}}) + f_{(N\Phi)}B\bar{X}_{(N\Phi)}(1+e_{1_{(N\Phi)}})}{(A+f_{(N\Phi)}B)\bar{X}_{(N\Phi)}(1+e_{2_{(N\Phi)}}) + C\bar{X}_{(N\Phi)}(1+e_{1_{(N\Phi)}})}\right]^{\alpha} \\ & exp\left\{\frac{a(\bar{X}_{(N\Phi)}(1+e_{2_{(N\Phi)}}) - \bar{X}_{(N\Phi)}(1+e_{1_{(N\Phi)}}))}{a(\bar{X}_{(N\Phi)}(1+e_{2_{(N\Phi)}}) + \bar{X}_{(N\Phi)}(1+e_{1_{(N\Phi)}})) + 2b}\right\}\end{aligned} \tag{37}$$

on simplyfying we get

$$\begin{aligned}\Im^{yp}_{ft_{(N\Phi)}} - \bar{Y}_{(N\Phi)} =& \bar{Y}_{(N\Phi)}\Big[e_{0_{(N\Phi)}} + e_{1_{(N\Phi)}}\left(\alpha\xi_{(N\Phi)} - \frac{k_{(N\Phi)}}{2}\right) - e_{2_{(N\Phi)}}\left(\alpha\xi_{(N\Phi)} - \frac{k_{(N\Phi)}}{2}\right) \\ & + e^2_{1_{(N\Phi)}}\left(-\alpha\xi_{(N\Phi)}\phi_{2_{(N\Phi)}} + \frac{\alpha(\alpha-1)}{2}\xi^2_{(N\Phi)} - \frac{\alpha\xi_{(N\Phi)}k_{(N\Phi)}}{2} + \frac{3}{8}k^2_{(N\Phi)}\right) \\ & + e^2_{2_{(N\Phi)}}\left(\alpha\xi_{(N\Phi)}\phi_{4_{(N\Phi)}} + \frac{\alpha(\alpha-1)}{2}\xi^2_{(N\Phi)} - \frac{\alpha\xi_{(N\Phi)}k_{(N\Phi)}}{2} - \frac{k^2_{(N\Phi)}}{8}\right) \\ & + e_{0_{(N\Phi)}}e_{1_{(N\Phi)}}\left(\alpha\xi_{(N\Phi)} - \frac{k_{(N\Phi)}}{2}\right) - e_{0_{(N\Phi)}}e_{2_{(N\Phi)}}\left(\alpha\xi_{(N\Phi)} - \frac{k_{(N\Phi)}}{2}\right) \\ & + e_{1_{(N\Phi)}}e_{2_{(N\Phi)}}\Big(\alpha\xi_{(N\Phi)}\phi_{4_{(N\Phi)}} - \alpha\xi_{(N\Phi)}\phi_{4_{(N\Phi)}} - \alpha(\alpha-1)\xi^2_{(N\Phi)} \\ & + \alpha\xi_{(N\Phi)}k_{(N\Phi)} - \frac{k^2_{(N\Phi)}}{4}\Big)\Big]\end{aligned} \tag{38}$$

where $\phi_{1_{(N\Phi)}} = \frac{f_{(N\Phi)}B}{A+f_{(N\Phi)}B+c}$, $\phi_{2_{(N\Phi)}} = \frac{C}{A+f_{(N\Phi)}B+c}$, $\phi_{3_{(N\Phi)}} = \frac{A+C}{A+f_{(N\Phi)}B+c}$, $\phi_{4_{(N\Phi)}} = \frac{A+f_{(N\Phi)}B}{A+f_{(N\Phi)}B+c}$, $\xi_{(N\Phi)} = \phi_{1_{(N\Phi)}} - \phi_{2_{(N\Phi)}} = \phi_{4_{(N\Phi)}} - \phi_{3_{(N\Phi)}}$ and $k_{(N\Phi)} = \left(\frac{a\bar{X}_{(N\Phi)}}{a\bar{X}_{(N\Phi)}+b}\right)$

To obtain Bias of the estimators we will take expectation of equation (38) and then by substituting the value of the considered transformations, we get

$$\begin{aligned} Bias[\Im^{yp}_{ft_{(N\Phi)}}] =& \bar{Y}_{(N\Phi)}\Big[\Big(\frac{1}{n_{(N\Phi)}}-\frac{1}{N_{(N\Phi)}}\Big)C^2_{x_{(N\Phi)}}\Big(-\alpha\xi_{(N\Phi)}\phi_{2_{(N\Phi)}}-\frac{\alpha(\alpha-1)}{2}\xi^2_{(N\Phi)} \\ &-\frac{\alpha\xi_{(N\Phi)}k_{(N\Phi)}}{2}+\frac{3}{8}k^2_{(N\Phi)}\Big)+\Big(\frac{1}{n_{1_{(N\Phi)}}}-\frac{1}{N_{(N\Phi)}}\Big)C^2_{x_{(N\Phi)}}\Big(\alpha\xi_{(N\Phi)}\phi_{4_{(N\Phi)}} \\ &+\frac{\alpha(\alpha-1)}{2}\xi^2_{(N\Phi)}-\frac{\alpha\xi_{(N\Phi)}k_{(N\Phi)}}{2}-\frac{k^2_{(N\Phi)}}{8}\Big) \\ &+\Big(\frac{1}{n_{(N\Phi)}}-\frac{1}{N_{(N\Phi)}}\Big)\rho C_{y_{(N\Phi)}}C_{x_{(N\Phi)}}\Big(\alpha\xi_{(N\Phi)}-\frac{k_{(N\Phi)}}{2}\Big) \\ &-\Big(\frac{1}{n_{1_{(N\Phi)}}}-\frac{1}{N_{(N\Phi)}}\Big)\rho C_{y_{(N\Phi)}}C_{x_{(N\Phi)}}\Big(\alpha\xi_{(N\Phi)}-\frac{k_{(N\Phi)}}{2}\Big) \\ &+\Big(\frac{1}{n_{1_{(N\Phi)}}}-\frac{1}{N_{(N\Phi)}}\Big)C^2_{x_{(N\Phi)}}\Big(-\alpha(\alpha-1)\xi^2_{(N\Phi)}+\alpha\xi_{(N\Phi)}k_{(N\Phi)} \\ &-\frac{k^2_{(N\Phi)}}{4}\Big)\Big] \end{aligned} \tag{39}$$

Squaring both sides of the equation (38) and then taking expectation on both sides, the MSE will take the structure as

$$\begin{aligned} MSE[\Im^{yp}_{ft_{(N\Phi)}}] =& \bar{Y}^2_{(N\Phi)}\Big[\Big(\frac{1}{n_{(N\Phi)}}-\frac{1}{N_{(N\Phi)}}\Big)C^2_{y_{(N\Phi)}}+\Big(\alpha\xi_{(N\Phi)}-\frac{k_{(N\Phi)}}{2}\Big)^2\Big(\frac{1}{n_{(N\Phi)}} \\ &-\frac{1}{n_{1_{(N\Phi)}}}\Big)C^2_{x_{(N\Phi)}}+2\big(\alpha\xi_{(N\Phi)}-\frac{k_{(N\Phi)}}{2}\big)\Big(\frac{1}{n_{(N\Phi)}} \\ &-\frac{1}{n_{1_{(N\Phi)}}}\Big)\rho_{(N\Phi)}C_{y_{(N\Phi)}}C_{x_{(N\Phi)}}\Big] \end{aligned} \tag{40}$$

Now , we can obtain the optimal value of α by differentiating equation (40) with respect to α and equating its to zero we will get

$$\alpha = \frac{1}{\xi_{(N\Phi)}}\left\{\frac{k_{(N\Phi)}}{2}-\rho_{(N\Phi)}\frac{C_{y_{(N\Phi)}}}{C_{x_{(N\Phi)}}}\right\} \tag{41}$$

we can get the minimum MSE of $\Im^{yp}_{ft_{(N\Phi)}}$ by substituting the value of α in equation (40)

$$MSE[\Im^{yp}_{ft}]_{min_{(N\Phi)}} = \bar{Y}^2_{(N\Phi)}\Big[C^2_{y_{(N\Phi)}}(\gamma_{(N\Phi)}-\gamma^{**}_{(N\Phi)}\rho^2_{(N\Phi)})\Big] \tag{42}$$

4.1.1. *Properties of the particular cases of the proposed estimators*

(i) When $d = 1$ in the values of A, B and C, $\Im^{yp}_{ft_{(N\Phi)}}$ becomes $\Im^{Re}_{ft_{(N\Phi)}}$

Bias of the estimator $\Im^{Re}_{ft_{(N\Phi)}}$:

$$\begin{aligned}Bias[\Im^{Re}_{ft_{(N\Phi)}}] =&\bar{Y}_{(N\Phi)}\Big[\Big(\frac{1}{n_{(N\Phi)}}-\frac{1}{N_{(N\Phi)}}\Big)C^2_{x_{(N\Phi)}}\Big(\alpha-\frac{\alpha(\alpha-1)}{2}+\frac{\alpha k_{(N\Phi)}}{2}+\frac{3}{8}k^2_{(N\Phi)}\Big)\\&+\Big(\frac{1}{n_{1_{(N\Phi)}}}-\frac{1}{N_{(N\Phi)}}\Big)C^2_{x_{(N\Phi)}}\Big(\frac{\alpha(\alpha-1)}{2}+\frac{\alpha k_{(N\Phi)}}{2}-\frac{k^2_{(N\Phi)}}{8}\Big)\\&+\Big(\frac{1}{n_{(N\Phi)}}-\frac{1}{N_{(N\Phi)}}\Big)\rho_{(N\Phi)}C_{y_{(N\Phi)}}C_{x_{(N\Phi)}}\Big(-\alpha-\frac{k_{(N\Phi)}}{2}\Big)\\&-\Big(\frac{1}{n_{1_{(N\Phi)}}}-\frac{1}{N_{(N\Phi)}}\Big)\rho_{(N\Phi)}C_{y_{(N\Phi)}}C_{x_{(N\Phi)}}\Big(-\alpha-\frac{k_{(N\Phi)}}{2}\Big)\\&+\Big(\frac{1}{n_{1_{(N\Phi)}}}-\frac{1}{N_{(N\Phi)}}\Big)C^2_{x_{(N\Phi)}}\Big(-\alpha(\alpha-1)-\alpha k_{(N\Phi)}-\frac{k^2_{(N\Phi)}}{4}\Big)\Big]\end{aligned} \tag{43}$$

MSE of the estimator $\Im^{Re}_{ft_{(N\Phi)}}$:

$$\begin{aligned}MSE[\Im^{Re}_{ft_{(N\Phi)}}] =&\bar{Y}^2_{(N\Phi)}\Big[\Big(\frac{1}{n_{(N\Phi)}}-\frac{1}{N_{(N\Phi)}}\Big)C^2_{y_{(N\Phi)}}+\Big(-\alpha-\frac{k_{(N\Phi)}}{2}\Big)^2\Big(\frac{1}{n_{(N\Phi)}}-\frac{1}{n_{1_{(N\Phi)}}}\Big)C^2_{x_{(N\Phi)}}\\&+2\Big(-\alpha-\frac{k_{(N\Phi)}}{2}\Big)\Big(\frac{1}{n_{(N\Phi)}}-\frac{1}{n_{1_{(N\Phi)}}}\Big)\rho_{(N\Phi)}C_{y_{(N\Phi)}}C_{x_{(N\Phi)}}\Big]\end{aligned} \tag{44}$$

Optimal values of α

$$\alpha=-\left\{\frac{k_{(N\Phi)}}{2}-\rho_{(N\Phi)}\frac{C_{y_{(N\Phi)}}}{C_{x_{(N\Phi)}}}\right\} \tag{45}$$

Minimum MSE of the estimator $\Im^{Re}_{ft_{(N\Phi)}}$:

$$MSE[\Im^{Re}_{ft}]_{min_{(N\Phi)}}=\bar{Y}^2_{(N\Phi)}\Big[C^2_{y_{(N\Phi)}}(\gamma_{(N\Phi)}-\gamma^{**}_{(N\Phi)}\rho^2_{(N\Phi)})\Big] \tag{46}$$

(ii) When $d = 2$ in the values of A, B and C, $\Im^{yp}_{ft_{(N\Phi)}}$ becomes $\Im^{Pe}_{ft_{(N\Phi)}}$

Bias of the estimator $\Im^{Pe}_{ft_{(N\Phi)}}$:

$$\begin{aligned}Bias[\Im^{Pe}_{ft_{(N\Phi)}}] =&\bar{Y}_{(N\Phi)}\Big[\Big(\frac{1}{n_{(N\Phi)}}-\frac{1}{N_{(N\Phi)}}\Big)C^2_{x_{(N\Phi)}}\Big(-\frac{\alpha(\alpha-1)}{2}-\frac{\alpha k_{(N\Phi)}}{2}+\frac{3}{8}k^2_{(N\Phi)}\Big)\\&+\Big(\frac{1}{n_{1_{(N\Phi)}}}-\frac{1}{N_{(N\Phi)}}\Big)C^2_{x_{(N\Phi)}}\Big(\alpha+\frac{\alpha(\alpha-1)}{2}-\frac{\alpha k}{2}-\frac{k^2_{(N\Phi)}}{8}\Big)\\&+\Big(\frac{1}{n_{(N\Phi)}}-\frac{1}{N_{(N\Phi)}}\Big)\rho_{(N\Phi)}C_{y_{(N\Phi)}}C_{x_{(N\Phi)}}\Big(\alpha-\frac{k_{(N\Phi)}}{2}\Big)\\&-\Big(\frac{1}{n_{1_{(N\Phi)}}}-\frac{1}{N_{(N\Phi)}}\Big)\rho_{(N\Phi)}C_{y_{(N\Phi)}}C_{x_{(N\Phi)}}\Big(\alpha-\frac{k_{(N\Phi)}}{2}\Big)\\&+\Big(\frac{1}{n_{1_{(N\Phi)}}}-\frac{1}{N_{(N\Phi)}}\Big)C^2_{x_{(N\Phi)}}\Big(-\alpha(\alpha-1)+\alpha k_{(N\Phi)}-\frac{k^2_{(N\Phi)}}{4}\Big)\Big]\end{aligned} \tag{47}$$

MSE of the estimator $\Im^{Pe}_{ft_{(N\Phi)}}$:

$$MSE[\Im^{Pe}_{ft_{(N\Phi)}}] = \bar{Y}^2_{(N\Phi)}\Big[\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{N_{(N\Phi)}}\Big)C^2_{y_{(N\Phi)}} + \Big(\alpha - \frac{k_{(N\Phi)}}{2}\Big)^2\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{n_{1_{(N\Phi)}}}\Big)C^2_{x_{(N\Phi)}} + 2\big(\alpha - \frac{k_{(N\Phi)}}{2}\big)\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{n_{1_{(N\Phi)}}}\Big)\rho_{(N\Phi)}C_{y_{(N\Phi)}}C_{x_{(N\Phi)}}\Big] \tag{48}$$

Optimal values of α

$$\alpha = \left\{\frac{k_{(N\Phi)}}{2} - \rho_{(N\Phi)}\frac{C_{y_{(N\Phi)}}}{C_{x_{(N\Phi)}}}\right\} \tag{49}$$

Minimum MSE of the estimator $\Im^{Pe}_{ft_{(N\Phi)}}$:

$$MSE[\Im^{Pe}_{ft}]_{min_{(N\Phi)}} = \bar{Y}^2_{(N\Phi)}\Big[C^2_{y_{(N\Phi)}}(\gamma_{(N\Phi)} - \gamma^{**}_{(N\Phi)}\rho^2_{(N\Phi)})\Big] \tag{50}$$

(iii) When $d = 3$ in the values of A, B and C, $\Im^{yp}_{ft_{(N\Phi)}}$ becomes $\Im^{DR}_{ft_{(N\Phi)}}$

Bias of the estimator $\Im^{DR}_{ft_{(N\Phi)}}$:

$$\begin{aligned} Bias[\Im^{DR}_{ft_{(N\Phi)}}] =& \bar{Y}_{(N\Phi)}\Big[\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{N_{(N\Phi)}}\Big)C^2_{x_{(N\Phi)}}\Big(-\frac{\alpha(\alpha-1)}{2}\aleph^2 - \frac{\alpha\aleph k_{(N\Phi)}}{2} + \frac{3}{8}k^2_{(N\Phi)}\Big) \\ &+ \Big(\frac{1}{n_{1_{(N\Phi)}}} - \frac{1}{N_{(N\Phi)}}\Big)C^2_{x_{(N\Phi)}}\Big(\alpha\aleph + \frac{\alpha(\alpha-1)}{2}\aleph^2 - \frac{\alpha\aleph k_{(N\Phi)}}{2} - \frac{k^2_{(N\Phi)}}{8}\Big) \\ &+ \Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{N_{(N\Phi)}}\Big)\rho_{(N\Phi)}C_{y_{(N\Phi)}}C_{x_{(N\Phi)}}\Big(\alpha\aleph - \frac{k_{(N\Phi)}}{2}\Big) - \Big(\frac{1}{n_{1_{(N\Phi)}}} \\ &- \frac{1}{N_{(N\Phi)}}\Big)\rho_{(N\Phi)}C_{y_{(N\Phi)}}C_{x_{(N\Phi)}}\Big(\alpha\aleph - \frac{k_{(N\Phi)}}{2}\Big) \\ &+ \Big(\frac{1}{n_{1_{(N\Phi)}}} - \frac{1}{N_{(N\Phi)}}\Big)C^2_{x_{(N\Phi)}}\Big(-\alpha(\alpha-1)\aleph^2 + \alpha\aleph k - \frac{k^2_{(N\Phi)}}{4}\Big)\Big] \end{aligned} \tag{51}$$

Where $\aleph = \frac{-n_{(N\Phi)}}{n_{1_{(N\Phi)}} - n_{(N\Phi)}}$.

MSE of the estimator $\Im^{DR}_{ft_{(N\Phi)}}$:

$$MSE[\Im^{DR}_{ft_{(N\Phi)}}] = \bar{Y}^2_{(N\Phi)}\Big[\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{N_{(N\Phi)}}\Big)C^2_{y_{(N\Phi)}} + \Big(\alpha\aleph - \frac{k_{(N\Phi)}}{2}\Big)^2\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{n_{1_{(N\Phi)}}}\Big)C^2_{x_{(N\Phi)}} + 2\big(\alpha\aleph - \frac{k_{(N\Phi)}}{2}\big)\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{n_{1_{(N\Phi)}}}\Big)\rho_{(N\Phi)}C_{y_{(N\Phi)}}C_{x_{(N\Phi)}}\Big] \tag{52}$$

Optimal values of α

$$\alpha = \frac{1}{\aleph}\left\{\frac{k_{(N\Phi)}}{2} - \rho_{(N\Phi)}\frac{C_{y_{(N\Phi)}}}{C_{x_{(N\Phi)}}}\right\} \tag{53}$$

Minimum MSE of the estimator $\Im^{DR}_{ft_{(N\Phi)}}$:

$$MSE[\Im^{DR}_{ft}]_{min_{(N\Phi)}} = \bar{Y}^2_{(N\Phi)}\Big[C^2_{y_{(N\Phi)}}(\gamma_{(N\Phi)} - \gamma^{**}_{(N\Phi)}\rho^2_{(N\Phi)})\Big] \tag{54}$$

(iv) When $d = 4$ in the values of A, B and C, $\Im^{yp}_{ft_{(N\Phi)}}$ becomes $\Im^{exp}_{ft_{(N\Phi)}}$

Bias of the estimator $\Im^{exp}_{ft_{(N\Phi)}}$:

$$Bias[\Im^{exp}_{ft_{(N\Phi)}}] = \bar{Y}_{(N\Phi)}\Big[\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{N_{(N\Phi)}}\Big)C^2_{x_{(N\Phi)}}\Big(\frac{3}{8}k^2_{(N\Phi)}\Big) + \Big(\frac{1}{n_{1_{(N\Phi)}}} - \frac{1}{N_{(N\Phi)}}\Big)C^2_{x_{(N\Phi)}}\Big(- \frac{k^2_{(N\Phi)}}{8}\Big) + \Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{N_{(N\Phi)}}\Big)\rho_{(N\Phi)}C_{y_{(N\Phi)}}C_{x_{(N\Phi)}}\Big(- \frac{k_{(N\Phi)}}{2}\Big) - \Big(\frac{1}{n_{1_{(N\Phi)}}} - \frac{1}{N_{(N\Phi)}}\Big)\rho_{(N\Phi)}C_{y_{(N\Phi)}}C_{x_{(N\Phi)}}\Big(- \frac{k_{(N\Phi)}}{2}\Big) + \Big(\frac{1}{n_{1_{(N\Phi)}}} - \frac{1}{N_{(N\Phi)}}\Big)C^2_{x_{(N\Phi)}}\Big(- \frac{k^2_{(N\Phi)}}{4}\Big)\Big] \tag{55}$$

MSE of the estimator $\Im^{exp}_{ft_{(N\Phi)}}$:

$$MSE[\Im^{yp}_{ft_{(N\Phi)}}] = \bar{Y}^2_{(N\Phi)}\Big[\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{N_{(N\Phi)}}\Big)C^2_{y_{(N\Phi)}} + \Big(- \frac{k_{(N\Phi)}}{2}\Big)^2\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{n_{1_{(N\Phi)}}}\Big)C^2_{x_{(N\Phi)}} + 2\Big(- \frac{k_{(N\Phi)}}{2}\Big)\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{n_{1_{(N\Phi)}}}\Big)\rho_{(N\Phi)}C_{y_{(N\Phi)}}C_{x_{(N\Phi)}}\Big] \tag{56}$$

Remarks: Similarly, For the proposed Estimator when $\alpha = 1$

(i) When $d = 1$ in the values of A, B and C, $\Im^{yp}_{vk_{(N\Phi)}}$ becomes $\Im^{Re}_{vk_{(N\Phi)}}$

Bias of the estimator $\Im^{Re}_{vk_{(N\Phi)}}$:

$$Bias[\Im^{Re}_{vk_{(N\Phi)}}] = \bar{Y}_{(N\Phi)}\Big[\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{N_{(N\Phi)}}\Big)C^2_{x_{(N\Phi)}}\Big(1 + \frac{k_{(N\Phi)}}{2} + \frac{3}{8}k^2_{(N\Phi)}\Big) + \Big(\frac{1}{n_{1_{(N\Phi)}}} - \frac{1}{N_{(N\Phi)}}\Big)C^2_{x_{(N\Phi)}}\Big(\frac{k_{(N\Phi)}}{2} - \frac{k^2_{(N\Phi)}}{8}\Big) + \Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{N_{(N\Phi)}}\Big)\rho_{(N\Phi)}C_{y_{(N\Phi)}}C_{x_{(N\Phi)}}\Big(- 1 - \frac{k_{(N\Phi)}}{2}\Big) - \Big(\frac{1}{n_{1_{(N\Phi)}}} - \frac{1}{N_{(N\Phi)}}\Big)\rho_{(N\Phi)}C_{y_{(N\Phi)}}C_{x_{(N\Phi)}}\Big(- 1 - \frac{k_{(N\Phi)}}{2}\Big) + \Big(\frac{1}{n_{1_{(N\Phi)}}} - \frac{1}{N_{(N\Phi)}}\Big)C^2_{x_{(N\Phi)}}\Big(- k_{(N\Phi)} - \frac{k^2_{(N\Phi)}}{4}\Big)\Big] \tag{57}$$

MSE of the estimator $\Im^{Re}_{vk_{(N\Phi)}}$:

$$MSE[\Im^{Re}_{vk_{(N\Phi)}}] = \bar{Y}^2_{(N\Phi)}\Big[\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{N_{(N\Phi)}}\Big)C^2_{y_{(N\Phi)}} + \Big(- 1 - \frac{k_{(N\Phi)}}{2}\Big)^2\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{n_{1_{(N\Phi)}}}\Big)C^2_{x_{(N\Phi)}} + 2\Big(- 1 - \frac{k_{(N\Phi)}}{2}\Big)\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{n_{1_{(N\Phi)}}}\Big)\rho_{(N\Phi)}C_{y_{(N\Phi)}}C_{x_{(N\Phi)}}\Big] \tag{58}$$

(ii) When $d = 2$ in the values of A, B and C, $\Im^{yp}_{vk_{(N\Phi)}}$ becomes $\Im^{Pe}_{vk_{(N\Phi)}}$

Bias of the estimator $\Im^{Pe}_{vk_{(N\Phi)}}$:

$$\begin{aligned} Bias[\Im^{Pe}_{vk_{(N\Phi)}}] =& \bar{Y}_{(N\Phi)}\Big[\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{N_{(N\Phi)}}\Big)C^2_{x_{(N\Phi)}}\Big(-\frac{k_{(N\Phi)}}{2} + \frac{3}{8}k^2_{(N\Phi)}\Big) \\ &+ \Big(\frac{1}{n_{1_{(N\Phi)}}} - \frac{1}{N_{(N\Phi)}}\Big)C^2_{x_{(N\Phi)}}\Big(1 - \frac{k_{(N\Phi)}}{2} - \frac{k^2_{(N\Phi)}}{8}\Big) \\ &+ \Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{N_{(N\Phi)}}\Big)\rho_{(N\Phi)}C_{y_{(N\Phi)}}C_{x_{(N\Phi)}}\Big(1 - \frac{k_{(N\Phi)}}{2}\Big) \\ &- \Big(\frac{1}{n_{1_{(N\Phi)}}} - \frac{1}{N_{(N\Phi)}}\Big)\rho_{(N\Phi)}C_{y_{(N\Phi)}}C_{x_{(N\Phi)}}\Big(1 - \frac{k_{(N\Phi)}}{2}\Big) \\ &+ \Big(\frac{1}{n_{1_{(N\Phi)}}} - \frac{1}{N_{(N\Phi)}}\Big)C^2_{x_{(N\Phi)}}\Big(k_{(N\Phi)} - \frac{k^2_{(N\Phi)}}{4}\Big)\Big] \end{aligned} \tag{59}$$

MSE of the estimator $\Im^{Pe}_{vk_{(N\Phi)}}$:

$$\begin{aligned} MSE[\Im^{Pe}_{vk_{(N\Phi)}}] =& \bar{Y}^2_{(N\Phi)}\Big[\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{N_{(N\Phi)}}\Big)C^2_{y_{(N\Phi)}} + \Big(1 - \frac{k_{(N\Phi)}}{2}\Big)^2\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{n_{1_{(N\Phi)}}}\Big)C^2_{x_{(N\Phi)}} \\ &+ 2\big(1 - \frac{k_{(N\Phi)}}{2}\big)\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{n_{1_{(N\Phi)}}}\Big)\rho_{(N\Phi)}C_{y_{(N\Phi)}}C_{x_{(N\Phi)}}\Big] \end{aligned} \tag{60}$$

(iii) When $d = 3$ in the values of A, B and C, $\Im^{yp}_{vk_{(N\Phi)}}$ becomes $\Im^{DR}_{vk_{(N\Phi)}}$

Bias of the estimator $\Im^{DR}_{vk_{(N\Phi)}}$:

$$\begin{aligned} Bias[\Im^{DR}_{vk_{(N\Phi)}}] =& \bar{Y}_{(N\Phi)}\Big[\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{N_{(N\Phi)}}\Big)C^2_{x_{(N\Phi)}}\Big(-\frac{\aleph k_{(N\Phi)}}{2} + \frac{3}{8}k^2_{(N\Phi)}\Big) \\ &+ \Big(\frac{1}{n_{1_{(N\Phi)}}} - \frac{1}{N_{(N\Phi)}}\Big)C^2_{x_{(N\Phi)}}\Big(\aleph - \frac{\aleph k_{(N\Phi)}}{2} - \frac{k^2_{(N\Phi)}}{8}\Big) \\ &+ \Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{N_{(N\Phi)}}\Big)\rho_{(N\Phi)}C_{y_{(N\Phi)}}C_{x_{(N\Phi)}}\Big(\aleph - \frac{k_{(N\Phi)}}{2}\Big) \\ &- \Big(\frac{1}{n_{1_{(N\Phi)}}} - \frac{1}{N_{(N\Phi)}}\Big)\rho_{(N\Phi)}C_{y_{(N\Phi)}}C_{x_{(N\Phi)}}\Big(\aleph - \frac{k_{(N\Phi)}}{2}\Big) \\ &+ \Big(\frac{1}{n_{1_{(N\Phi)}}} - \frac{1}{N_{(N\Phi)}}\Big)C^2_{x_{(N\Phi)}}\Big(\aleph k_{(N\Phi)} - \frac{k^2_{(N\Phi)}}{4}\Big)\Big] \end{aligned} \tag{61}$$

Where $\aleph = \frac{-n_{(N\Phi)}}{n_{1_{(N\Phi)}} - n_{(N\Phi)}}$.

MSE of the estimator $\Im^{DR}_{vk_{(N\Phi)}}$:

$$\begin{aligned} MSE[\Im^{DR}_{vk_{(N\Phi)}}] =& \bar{Y}^2_{(N\Phi)}\Big[\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{N_{(N\Phi)}}\Big)C^2_{y_{(N\Phi)}} + \Big(\aleph - \frac{k_{(N\Phi)}}{2}\Big)^2\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{n_{1_{(N\Phi)}}}\Big)C^2_{x_{(N\Phi)}} \\ &+ 2\big(\aleph - \frac{k_{(N\Phi)}}{2}\big)\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{n_{1_{(N\Phi)}}}\Big)\rho_{(N\Phi)}C_{y_{(N\Phi)}}C_{x_{(N\Phi)}}\Big] \end{aligned} \tag{62}$$

(iv) When $d = 4$ in the values of A, B and C, $\Im^{yp}_{vk_{(N\Phi)}}$ becomes $\Im^{exp}_{vk_{(N\Phi)}}$

Bias of the estimator $\Im^{exp}_{vk_{(N\Phi)}}$:

$$Bias[\Im^{exp}_{vk_{(N\Phi)}}] = \bar{Y}_{(N\Phi)}\Big[\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{N_{(N\Phi)}}\Big)C^2_{x_{(N\Phi)}}\Big(\frac{3}{8}k^2_{(N\Phi)}\Big) + \Big(\frac{1}{n_{1_{(N\Phi)}}} - \frac{1}{N_{(N\Phi)}}\Big)C^2_{x_{(N\Phi)}}\Big(\tag{63}$$

$$- \frac{k^2_{(N\Phi)}}{8}\Big) + \Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{N_{(N\Phi)}}\Big)\rho_{(N\Phi)}C_{y_{(N\Phi)}}C_{x_{(N\Phi)}}\Big(-\frac{k_{(N\Phi)}}{2}\Big)$$

$$- \Big(\frac{1}{n_{1_{(N\Phi)}}} - \frac{1}{N_{(N\Phi)}}\Big)\rho_{(N\Phi)}C_{y_{(N\Phi)}}C_{x_{(N\Phi)}}\Big(-\frac{k_{(N\Phi)}}{2}\Big)$$

$$+ \Big(\frac{1}{n_{1_{(N\Phi)}}} - \frac{1}{N_{(N\Phi)}}\Big)C^2_{x_{(N\Phi)}}\Big(-\frac{k^2_{(N\Phi)}}{4}\Big)\Big]$$

MSE of the estimator $\Im^{exp}_{vk_{(N\Phi)}}$:

$$MSE[\Im^{yp}_{vk_{(N\Phi)}}] = \bar{Y}^2_{(N\Phi)}\Big[\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{N_{(N\Phi)}}\Big)C^2_{y_{(N\Phi)}} + \Big(-\frac{k_{(N\Phi)}}{2}\Big)^2\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{n_{1_{(N\Phi)}}}\Big)C^2_{x_{(N\Phi)}}$$

$$+ 2\Big(-\frac{k_{(N\Phi)}}{2}\Big)\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{n_{1_{(N\Phi)}}}\Big)\rho_{(N\Phi)}C_{y_{(N\Phi)}}C_{x_{(N\Phi)}}\Big] \tag{64}$$

4.2. *Case II*

To derive the expression for Bias and MSE for Case I, consider the following transformations as follows

$\bar{y}_{(N\Phi)} = \bar{Y}_{(N\Phi)}(1 + e_{0_{(N\Phi)}})$, $\bar{x}_{(N\Phi)} = \bar{X}_{(N\Phi)}(1 + e_{1_{(N\Phi)}})$, and $\bar{x}_{1_{(N\Phi)}} = \bar{X}_{(N\Phi)}(1 + e_{2_{(N\Phi)}})$
such that $E(e_{0_{(N\Phi)}}) = E(e_{1_{(N\Phi)}}) = E(e_{2_{(N\Phi)}}) = 0$ and $E(e^2_{0_{(N\Phi)}}) = \gamma_{(N\Phi)}C^2_{y_{(N\Phi)}}$,
$E(e^2_{1_{(N\Phi)}}) = \gamma_{(N\Phi)}C^2_{x_{(N\Phi)}}$, $E(e^2_{2_{(N\Phi)}}) = \gamma^*_{(N\Phi)}C^2_{x_{(N\Phi)}}$,
$E(e_{0_{(N\Phi)}}e_{1_{(N\Phi)}}) = \gamma_{(N\Phi)}C_{(N\Phi)}C^2_{x_{(N\Phi)}}$, $E(e_{0_{(N\Phi)}}e_{2_{(N\Phi)}}) = 0$,
$E(e_{1_{(N\Phi)}}e_{2_{(N\Phi)}}) = 0$, $\gamma_{(N\Phi)} = \left(\frac{1}{n_{(N\Phi)}} - \frac{1}{N_{(N\Phi)}}\right)$, $\gamma^*_{(N\Phi)} = \left(\frac{1}{n_{1_{(N\Phi)}}} - \frac{1}{N_{(N\Phi)}}\right)$,
$\gamma^{**}_{(N\Phi)} = \gamma_{(N\Phi)} - \gamma^*_{(N\Phi)} = \left(\frac{1}{n_{(N\Phi)}} - \frac{1}{n_{1_{(N\Phi)}}}\right)$,
$\gamma^{***}_{(N\Phi)} = \gamma_{(N\Phi)} + \gamma^*_{(N\Phi)}$, $C_{(N\Phi)} = \rho_{y_{(N\Phi)}x_{(N\Phi)}}\frac{C_{y_{(N\Phi)}}}{C_{x_{(N\Phi)}}}$ and $g_{(N\Phi)} = \frac{n_{(N\Phi)}}{n_{1_{(N\Phi)}} - n_{(N\Phi)}}$
under the above transformations and from equation (34) expressing estimators in terms of e's, we get

$$\Im^{yp}_{ft_{(N\Phi)}} = \bar{Y}_{(N\Phi)}(1 + e_{0_{(N\Phi)}})\left[\frac{(A + c)\bar{X}(1 + e_{2_{(N\Phi)}}) + f_{(N\Phi)}B\bar{X}_{(N\Phi)}(1 + e_{1_{(N\Phi)}})}{(A + f_{(N\Phi)}B)\bar{X}_{(N\Phi)}(1 + e_{2_{(N\Phi)}}) + C\bar{X}_{(N\Phi)}(1 + e_1)}\right]^{\alpha} \tag{65}$$

$$exp\left\{\frac{a(\bar{X}_{(N\Phi)}(1 + e_{2_{(N\Phi)}}) - \bar{X}_{(N\Phi)}(1 + e_{1_{(N\Phi)}}))}{a(\bar{X}_{(N\Phi)}(1 + e_{2_{(N\Phi)}}) + \bar{X}_{(N\Phi)}(1 + e_{1_{(N\Phi)}})) + 2b}\right\}$$

on simplyfying we get

$$\begin{aligned}\Im^{yp}_{ft_{(N\Phi)}} - \bar{Y}_{(N\Phi)} =& \bar{Y}_{(N\Phi)}\Big[e_{0_{(N\Phi)}} + e_{1_{(N\Phi)}}\Big(\alpha\xi_{(N\Phi)} - \frac{k_{(N\Phi)}}{2}\Big) - e_{2_{(N\Phi)}}\Big(\alpha\xi_{(N\Phi)} - \frac{k_{(N\Phi)}}{2}\Big)\\ &+ e^2_{1_{(N\Phi)}}\Big(-\alpha\xi_{(N\Phi)}\phi_{2_{(N\Phi)}} + \frac{\alpha(\alpha-1)}{2}\xi^2_{(N\Phi)} - \frac{\alpha\xi_{(N\Phi)}k_{(N\Phi)}}{2} + \frac{3}{8}k^2_{(N\Phi)}\Big)\\ &+ e^2_{2_{(N\Phi)}}\Big(\alpha\xi_{(N\Phi)}\phi_{4_{(N\Phi)}} + \frac{\alpha(\alpha-1)}{2}\xi^2_{(N\Phi)} - \frac{\alpha\xi_{(N\Phi)}k_{(N\Phi)}}{2} - \frac{k^2_{(N\Phi)}}{8}\Big)\\ &+ e_{0_{(N\Phi)}}e_{1_{(N\Phi)}}\Big(\alpha\xi_{(N\Phi)} - \frac{k_{(N\Phi)}}{2}\Big) - e_{0_{(N\Phi)}}e_{2_{(N\Phi)}}\Big(\alpha\xi_{(N\Phi)} - \frac{k_{(N\Phi)}}{2}\Big)\\ &+ e_{1_{(N\Phi)}}e_{2_{(N\Phi)}}\Big(\alpha\xi_{(N\Phi)}\phi_{4_{(N\Phi)}} - \alpha\xi_{(N\Phi)}\phi_{4_{(N\Phi)}} - \alpha(\alpha-1)\xi^2_{(N\Phi)}\\ &+ \alpha\xi_{(N\Phi)}k_{(N\Phi)} - \frac{k^2_{(N\Phi)}}{4}\Big)\Big] \end{aligned} \tag{66}$$

To obtain Bias of the estimators we will take expectation of equation (66) and then by substituting the value of the considered transformations, we get

$$\begin{aligned}Bias[\Im^{yp}_{ft_{(N\Phi)}}] =& \bar{Y}_{(N\Phi)}\Big[\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{N_{(N\Phi)}}\Big)C_{x_{(N\Phi)}}\Big\{C_{x_{(N\Phi)}}\Big(-\alpha\xi_{(N\Phi)}\phi_{2_{(N\Phi)}} + \frac{\alpha(\alpha-1)}{2}\xi^2_{(N\Phi)}\\ &- \frac{\alpha\xi_{(N\Phi)}k_{(N\Phi)}}{2} + \frac{3}{8}k^2_{(N\Phi)}\Big) + \rho_{(N\Phi)}C_{y_{(N\Phi)}}\Big(\alpha\xi_{(N\Phi)} - \frac{k_{(N\Phi)}}{2}\Big)\Big\}\\ &+ \Big(\frac{1}{n_{1_{(N\Phi)}}} - \frac{1}{N_{(N\Phi)}}\Big)C^2_{x_{(N\Phi)}}\Big(\alpha\xi_{(N\Phi)}\phi_{4_{(N\Phi)}} + \frac{\alpha(\alpha-1)}{2}\xi^2_{(N\Phi)} - \frac{\alpha\xi_{(N\Phi)}k_{(N\Phi)}}{2}\\ &- \frac{k^2_{(N\Phi)}}{8}\Big)\Big] \end{aligned} \tag{67}$$

Squaring both sides of the equation (66) and then taking expectation on both sides, the MSE will take the structure as

$$\begin{aligned}MSE[\Im^{yp}_{ft_{(N\Phi)}}] =& \bar{Y}^2_{(N\Phi)}\Big[\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{N_{(N\Phi)}}\Big)C^2_{y_{(N\Phi)}} + \Big(\alpha\xi_{(N\Phi)} - \frac{k_{(N\Phi)}}{2}\Big)^2\Big(\frac{1}{n_{(N\Phi)}}\\ &- \frac{1}{n_{1_{(N\Phi)}}}\Big)C^2_{x_{(N\Phi)}} + 2\big(\alpha\xi_{(N\Phi)} - \frac{k_{(N\Phi)}}{2}\big)\Big(\frac{1}{n_{(N\Phi)}}\\ &- \frac{1}{N_{(N\Phi)}}\Big)\rho_{(N\Phi)}C_{y_{(N\Phi)}}C_{x_{(N\Phi)}}\Big] \end{aligned} \tag{68}$$

Now , we can obtain the optimal value of α by differentiating equation (68) with respect to α and equating its to zero we will get

$$\alpha = \frac{1}{\xi_{(N\Phi)}}\left\{\frac{k_{(N\Phi)}}{2} - \rho_{(N\Phi)}\frac{C_{y_{(N\Phi)}}}{C_{x_{(N\Phi)}}}\right\} \tag{69}$$

we can get the minimum MSE of $\Im^{yp}_{ft_{(N\Phi)}}$ by substituting the value of α in equation (68)

$$MSE[\Im^{yp}_{ft_{(N\Phi)}}]_{min} = \bar{Y}^2_{(N\Phi)}C^2_{y_{(N\Phi)}}\gamma_{(N\Phi)}\Big[1 - \frac{\gamma_{(N\Phi)}\rho^2}{\gamma^{**}_{(N\Phi)}}\Big] \tag{70}$$

4.2.1. *Properties of the particular cases of the proposed estimators*

(i) When $d = 1$ in the values of A, B and C, $\Im^{yp}_{ft_{(N\Phi)}}$ becomes $\Im^{Re}_{ft_{(N\Phi)}}$

Bias of the estimator $\Im^{Re}_{ft_{(N\Phi)}}$:

$$\begin{aligned} Bias[\Im^{yp}_{ft_{(N\Phi)}}] =& \bar{Y}_{(N\Phi)}\Big[\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{N_{(N\Phi)}}\Big)C_{x_{(N\Phi)}}\Big\{C_{x_{(N\Phi)}}\Big(\alpha + \frac{\alpha(\alpha-1)}{2} + \frac{\alpha k_{(N\Phi)}}{2} + \frac{3}{8}k^2_{(N\Phi)}\Big) \\ & + \rho_{(N\Phi)}C_{y_{(N\Phi)}}\Big(-\alpha - \frac{k_{(N\Phi)}}{2}\Big)\Big\} + \Big(\frac{1}{n_{1_{(N\Phi)}}} - \frac{1}{N_{(N\Phi)}}\Big)C^2_{x_{(N\Phi)}}\Big(\\ & + \frac{\alpha(\alpha-1)}{2} + \frac{\alpha k_{(N\Phi)}}{2} - \frac{k^2_{(N\Phi)}}{8}\Big)\Big] \end{aligned} \tag{71}$$

MSE of the estimator $\Im^{Re}_{ft}$:

$$\begin{aligned} MSE[\Im^{Re}_{ft_{(N\Phi)}}] =& \bar{Y}^2_{(N\Phi)}\Big[\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{N_{(N\Phi)}}\Big)C^2_{y_{(N\Phi)}} + \Big(-\alpha - \frac{k_{(N\Phi)}}{2}\Big)^2\Big(\frac{1}{n_{(N\Phi)}} \\ & - \frac{1}{n_{1_{(N\Phi)}}}\Big)C^2_{x_{(N\Phi)}} + 2\big(-\alpha - \frac{k_{(N\Phi)}}{2}\big)\Big(\frac{1}{n_{(N\Phi)}} \\ & - \frac{1}{N_{(N\Phi)}}\Big)\rho_{(N\Phi)}C_{y_{(N\Phi)}}C_{x_{(N\Phi)}}\Big] \end{aligned} \tag{72}$$

Optimal values of α

$$\alpha = -\left\{\frac{k_{(N\Phi)}}{2} - \rho_{(N\Phi)}\frac{C_{y_{(N\Phi)}}}{C_{x_{(N\Phi)}}}\right\} \tag{73}$$

Minimum MSE of the estimator $\Im^{Re}_{ft_{(N\Phi)}}$:

$$MSE[\Im^{Re}_{ft}]_{min_{(N\Phi)}} = \bar{Y}^2_{(N\Phi)}C^2_{y_{(N\Phi)}}\gamma_{(N\Phi)}\Big[1 - \frac{\gamma_{(N\Phi)}\rho^2}{\gamma^{**}_{(N\Phi)}}\Big] \tag{74}$$

(ii) When $d = 2$ in the values of A, B and C, $\Im^{yp}_{ft_{(N\Phi)}}$ becomes $\Im^{Pe}_{ft_{(N\Phi)}}$

Bias of the estimator $\Im^{Pe}_{ft_{(N\Phi)}}$:

$$\begin{aligned} Bias[\Im^{Pe}_{ft_{(N\Phi)}}] =& \bar{Y}_{(N\Phi)}\Big[\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{N_{(N\Phi)}}\Big)C_{x_{(N\Phi)}}\Big\{C_{x_{(N\Phi)}}\Big(\frac{\alpha(\alpha-1)}{2} - \frac{\alpha k_{(N\Phi)}}{2} + \frac{3}{8}k^2_{(N\Phi)}\Big) \\ & + \rho_{(N\Phi)}C_{y_{(N\Phi)}}\Big(\alpha - \frac{k_{(N\Phi)}}{2}\Big)\Big\} + \Big(\frac{1}{n_{1_{(N\Phi)}}} - \frac{1}{N_{(N\Phi)}}\Big)C^2_{x_{(N\Phi)}}\Big(\alpha + \frac{\alpha(\alpha-1)}{2} \\ & - \frac{\alpha k_{(N\Phi)}}{2} - \frac{k^2_{(N\Phi)}}{8}\Big)\Big] \end{aligned} \tag{75}$$

MSE of the estimator $\Im^{Pe}_{ft_{(N\Phi)}}$:

$$\begin{aligned} MSE[\Im^{Pe}_{ft_{(N\Phi)}}] =& \bar{Y}^2_{(N\Phi)}\Big[\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{N_{(N\Phi)}}\Big)C^2_{y_{(N\Phi)}} + \Big(\alpha - \frac{k_{(N\Phi)}}{2}\Big)^2\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{n_{1_{(N\Phi)}}}\Big)C^2_{x_{(N\Phi)}} \\ & + 2\big(\alpha - \frac{k_{(N\Phi)}}{2}\big)\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{N_{(N\Phi)}}\Big)\rho_{(N\Phi)}C_{y_{(N\Phi)}}C_{x_{(N\Phi)}}\Big] \end{aligned} \tag{76}$$

Optimal values of α

$$\alpha = \left\{ \frac{k_{(N\Phi)}}{2} - \rho_{(N\Phi)} \frac{C_{y_{(N\Phi)}}}{C_{x_{(N\Phi)}}} \right\} \tag{77}$$

Minimum MSE of the estimator $\Im^{Pe}_{ft_{(N\Phi)}}$:

$$MSE[\Im^{Pe}_{ft}]_{min_{(N\Phi)}} = \bar{Y}^2_{(N\Phi)} C^2_{y_{(N\Phi)}} \gamma_{(N\Phi)} \Big[1 - \frac{\gamma_{(N\Phi)}\rho^2}{\gamma^{**}_{(N\Phi)}}\Big] \tag{78}$$

(iii) When $d = 3$ in the values of A, B and C, $\Im^{yp}_{ft_{(N\Phi)}}$ becomes $\Im^{DR}_{ft_{(N\Phi)}}$

Bias of the estimator $\Im^{DR}_{ft_{(N\Phi)}}$:

$$\begin{aligned} Bias[\Im^{DR}_{ft_{(N\Phi)}}] =& \bar{Y}_{(N\Phi)} \Big[\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{N_{(N\Phi)}} \Big) C_{x_{(N\Phi)}} \Big\{ C_{x_{(N\Phi)}} \Big(\frac{\alpha(\alpha-1)}{2}\aleph^2 - \frac{\alpha\aleph k_{(N\Phi)}}{2} \\ &+ \frac{3}{8}k^2_{(N\Phi)} \Big) + \rho_{(N\Phi)} C_{y_{(N\Phi)}} \Big(\alpha\aleph - \frac{k_{(N\Phi)}}{2} \Big) \Big\} + \Big(\frac{1}{n_{1_{(N\Phi)}}} \\ &- \frac{1}{N_{(N\Phi)}} \Big) C^2_{x_{(N\Phi)}} \Big(\alpha\aleph + \frac{\alpha(\alpha-1)}{2}\aleph^2 - \frac{\alpha\aleph k_{(N\Phi)}}{2} - \frac{k^2_{(N\Phi)}}{8} \Big) \Big] \end{aligned} \tag{79}$$

Where $\aleph = \frac{-n_{(N\Phi)}}{n_{1_{(N\Phi)}} - n_{(N\Phi)}}$.

MSE of the estimator $\Im^{DR}_{ft_{(N\Phi)}}$:

$$\begin{aligned} MSE[\Im^{DR}_{ft_{(N\Phi)}}] =& \bar{Y}^2_{(N\Phi)} \Big[\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{N_{(N\Phi)}} \Big) C^2_{y_{(N\Phi)}} + \Big(\alpha\aleph - \frac{k_{(N\Phi)}}{2} \Big)^2 \Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{n_{1_{(N\Phi)}}} \Big) C^2_{x_{(N\Phi)}} \\ &+ 2\big(\alpha\aleph - \frac{k_{(N\Phi)}}{2} \big) \Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{N_{(N\Phi)}} \Big) \rho_{(N\Phi)} C_{y_{(N\Phi)}} C_{x_{(N\Phi)}} \Big] \end{aligned} \tag{80}$$

Optimal values of α

$$\alpha = \frac{1}{\aleph} \left\{ \frac{k_{(N\Phi)}}{2} - \rho_{(N\Phi)} \frac{C_{y_{(N\Phi)}}}{C_{x_{(N\Phi)}}} \right\} \tag{81}$$

Minimum MSE of the estimator $\Im^{DR}_{ft_{(N\Phi)}}$:

$$MSE[\Im^{yp}_{ft}]_{min_{(N\Phi)}} = \bar{Y}^2_{(N\Phi)} C^2_{y_{(N\Phi)}} \gamma_{(N\Phi)} \Big[1 - \frac{\gamma_{(N\Phi)}\rho^2}{\gamma^{**}_{(N\Phi)}}\Big] \tag{82}$$

(iv) When $d = 4$ in the values of A, B and C, $\Im^{yp}_{ft_{(N\Phi)}}$ becomes $\Im^{exp}_{ft_{(N\Phi)}}$

Bias of the estimator $\Im^{exp}_{ft_{(N\Phi)}}$:

$$\begin{aligned} Bias[\Im^{exp}_{ft_{(N\Phi)}}] =& \bar{Y}_{(N\Phi)} \Big[\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{N_{(N\Phi)}} \Big) C_{x_{(N\Phi)}} \Big\{ C_{x_{(N\Phi)}} \Big(\frac{3}{8}k^2_{(N\Phi)} \Big) + \rho_{(N\Phi)} C_{y_{(N\Phi)}} \Big(\\ &- \frac{k_{(N\Phi)}}{2} \Big) \Big\} + \Big(\frac{1}{n_{1_{(N\Phi)}}} - \frac{1}{N_{(N\Phi)}} \Big) C^2_{x_{(N\Phi)}} \Big(- \frac{k^2_{(N\Phi)}}{8} \Big) \Big] \end{aligned} \tag{83}$$

MSE of the estimator $\Im^{exp}_{ft_{(N\Phi)}}$:

$$MSE[\Im^{exp}_{ft_{(N\Phi)}}] = \bar{Y}^2_{(N\Phi)}\Big[\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{N_{(N\Phi)}}\Big)C^2_{y_{(N\Phi)}} + \Big(-\frac{k_{(N\Phi)}}{2}\Big)^2\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{n_{1_{(N\Phi)}}}\Big)C^2_{x_{(N\Phi)}} + 2\Big(-\frac{k_{(N\Phi)}}{2}\Big)\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{N_{(N\Phi)}}\Big)\rho_{(N\Phi)}C_{y_{(N\Phi)}}C_{x_{(N\Phi)}}\Big] \tag{84}$$

Remarks: Similarly, For the proposed Estimator when $\alpha = 1$

(i) When $d = 1$ in the values of A, B and C, $\Im^{yp}_{vk_{(N\Phi)}}$ becomes $\Im^{Re}_{vk_{(N\Phi)}}$

Bias of the estimator $\Im^{Re}_{vk_{(N\Phi)}}$:

$$Bias[\Im^{Re}_{vk_{(N\Phi)}}] = \bar{Y}_{(N\Phi)}\Big[\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{N_{(N\Phi)}}\Big)C_{x_{(N\Phi)}}\Big\{C_{x_{(N\Phi)}}\Big(1 + \frac{k_{(N\Phi)}}{2} + \frac{3}{8}k^2_{(N\Phi)}\Big) + \rho_{(N\Phi)}C_{y_{(N\Phi)}}\Big(-1 - \frac{k_{(N\Phi)}}{2}\Big)\Big\} + \Big(\frac{1}{n_{1_{(N\Phi)}}} - \frac{1}{N_{(N\Phi)}}\Big)C^2_{x_{(N\Phi)}}\Big(\frac{k_{(N\Phi)}}{2} - \frac{k^2_{(N\Phi)}}{8}\Big)\Big] \tag{85}$$

MSE of the estimator $\Im^{Re}_{vk_{(N\Phi)}}$:

$$MSE[\Im^{Re}_{vk_{(N\Phi)}}] = \bar{Y}^2_{(N\Phi)}\Big[\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{N_{(N\Phi)}}\Big)C^2_{y_{(N\Phi)}} + \Big(-1 - \frac{k_{(N\Phi)}}{2}\Big)^2\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{n_{1_{(N\Phi)}}}\Big)C^2_{x_{(N\Phi)}} + 2\Big(-1 - \frac{k_{(N\Phi)}}{2}\Big)\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{N_{(N\Phi)}}\Big)\rho_{(N\Phi)}C_{y_{(N\Phi)}}C_{x_{(N\Phi)}}\Big] \tag{86}$$

(ii) When $d = 2$ in the values of A, B and C, $\Im^{yp}_{vk_{(N\Phi)}}$ becomes $\Im^{Pe}_{vk_{(N\Phi)}}$

Bias of the estimator $\Im^{Pe}_{vk_{(N\Phi)}}$:

$$Bias[\Im^{Pe}_{vk_{(N\Phi)}}] = \bar{Y}_{(N\Phi)}\Big[\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{N_{(N\Phi)}}\Big)C_{x_{(N\Phi)}}\Big\{C_{x_{(N\Phi)}}\Big(-\frac{k_{(N\Phi)}}{2} + \frac{3}{8}k^2_{(N\Phi)}\Big) + \rho_{(N\Phi)}C_{y_{(N\Phi)}}\Big(1 - \frac{k_{(N\Phi)}}{2}\Big)\Big\} + \Big(\frac{1}{n_{1_{(N\Phi)}}} - \frac{1}{N_{(N\Phi)}}\Big)C^2_{x_{(N\Phi)}}\Big(1 + \frac{k_{(N\Phi)}}{2} - \frac{k^2_{(N\Phi)}}{8}\Big)\Big] \tag{87}$$

MSE of the estimator $\Im^{Pe}_{vk_{(N\Phi)}}$:

$$MSE[\Im^{Pe}_{vk_{(N\Phi)}}] = \bar{Y}^2_{(N\Phi)}\Big[\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{N_{(N\Phi)}}\Big)C^2_{y_{(N\Phi)}} + \Big(1 - \frac{k_{(N\Phi)}}{2}\Big)^2\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{n_{1_{(N\Phi)}}}\Big)C^2_{x_{(N\Phi)}} + 2\Big(1 - \frac{k_{(N\Phi)}}{2}\Big)\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{N_{(N\Phi)}}\Big)\rho_{(N\Phi)}C_{y_{(N\Phi)}}C_{x_{(N\Phi)}}\Big] \tag{88}$$

(iii) When $d = 3$ in the values of A, B and C, $\Im^{yp}_{vk_{(N\Phi)}}$ becomes $\Im^{DR}_{vk_{(N\Phi)}}$

Bias of the estimator $\Im^{DR}_{vk_{(N\Phi)}}$:

$$Bias[\Im^{DR}_{vk_{(N\Phi)}}] = \bar{Y}_{(N\Phi)}\Big[\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{N_{(N\Phi)}}\Big)C_{x_{(N\Phi)}}\Big\{C_{x_{(N\Phi)}}\Big(-\frac{\aleph k_{(N\Phi)}}{2} + \frac{3}{8}k^2_{(N\Phi)}\Big) + \rho_{(N\Phi)}C_{y_{(N\Phi)}}\Big(\aleph - \frac{k_{(N\Phi)}}{2}\Big)\Big\} + \Big(\frac{1}{n_{1_{(N\Phi)}}} - \frac{1}{N_{(N\Phi)}}\Big)C^2_{x_{(N\Phi)}}\Big(\aleph + \frac{\aleph k_{(N\Phi)}}{2} - \frac{k^2_{(N\Phi)}}{8}\Big)\Big] \tag{89}$$

Where $\aleph = \frac{-n_{(N\Phi)}}{n_{1_{(N\Phi)}} - n_{(N\Phi)}}$.

MSE of the estimator $\Im^{DR}_{vk_{(N\Phi)}}$:

$$MSE[\Im^{DR}_{vk_{(N\Phi)}}] = \bar{Y}^2_{(N\Phi)}\Big[\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{N_{(N\Phi)}}\Big)C^2_{y_{(N\Phi)}} + \Big(\aleph - \frac{k_{(N\Phi)}}{2}\Big)^2\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{n_{1_{(N\Phi)}}}\Big)C^2_{x_{(N\Phi)}} + 2\big(\aleph - \frac{k_{(N\Phi)}}{2}\big)\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{N_{(N\Phi)}}\Big)\rho_{(N\Phi)}C_{y_{(N\Phi)}}C_{x_{(N\Phi)}}\Big] \tag{90}$$

(iv) When $d = 4$ in the values of A, B and C, $\Im^{yp}_{vk_{(N\Phi)}}$ becomes $\Im^{exp}_{vk_{(N\Phi)}}$

Bias of the estimator $\Im^{exp}_{vk_{(N\Phi)}}$:

$$Bias[\Im^{exp}_{vk_{(N\Phi)}}] = \bar{Y}_{(N\Phi)}\Big[\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{N_{(N\Phi)}}\Big)C_{x_{(N\Phi)}}\Big\{C_{x_{(N\Phi)}}\Big(\frac{3}{8}k^2_{(N\Phi)}\Big) + \rho_{(N\Phi)}C_{y_{(N\Phi)}}\Big(-\frac{k_{(N\Phi)}}{2}\Big)\Big\} + \Big(\frac{1}{n_{1_{(N\Phi)}}} - \frac{1}{N_{(N\Phi)}}\Big)C^2_{x_{(N\Phi)}}\Big(-\frac{k^2_{(N\Phi)}}{8}\Big)\Big] \tag{91}$$

MSE of the estimator $\Im^{exp}_{vk_{(N\Phi)}}$:

$$MSE[\Im^{exp}_{vk_{(N\Phi)}}] = \bar{Y}^2_{(N\Phi)}\Big[\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{N_{(N\Phi)}}\Big)C^2_{y_{(N\Phi)}} + \Big(-\frac{k_{(N\Phi)}}{2}\Big)^2\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{n_{1_{(N\Phi)}}}\Big)C^2_{x_{(N\Phi)}} + 2\big(-\frac{k_{(N\Phi)}}{2}\big)\Big(\frac{1}{n_{(N\Phi)}} - \frac{1}{N_{(N\Phi)}}\Big)\rho_{(N\Phi)}C_{y_{(N\Phi)}}C_{x_{(N\Phi)}}\Big] \tag{92}$$

5. Numerical Study

The study goal of this in-depth investigation study is to investigate and compare several Neutrosophic estimators for estimating the population mean within the context of Neutrosophic two-phase sampling. To accomplish this goal, we chose real-world data from the open public website “https://data.gov.in/resource/seasonal-and-annual-minimum-maximum-temperature-series-1901-2019.” The dataset includes complete data on All India Seasonal and Annual Temperature the series of circuits including minimum, maximum, and mean temperatures over a long period of time.

The information was made available under the National Data Sharing and Accessibility Policy (NDSAP) by prestigious organisations such as the Ministry of Earth Sciences and the

India Meteorological Department (IMD), Pune. The diversity of temperature data enables us to obtain significant insights into temperature variations and patterns throughout seasons and years.

Our major goal is to compare the performance of the proposed Neutrosophic estimators to those existing estimators in the unique situation of two-phase sampling. We aim to test the efficiency, accuracy, and robustness of these estimators in calculating population means by doing this research using real data from India.

The consequences of our research go beyond the specific dataset, since the findings may have broader applicability in a variety of domains that use two-phase sampling and Neutrosophic statistics. We think that by adding to the knowledge base in this field, we will improve understanding and practical use of Neutrosophic estimators in real-world settings, allowing for better informed decision-making and developing statistical approaches.

Discriptions of Datasets are given in Table [1]. The auxiliary variable $X_{(N\Phi)}$ for Population A reflects the minimum and highest temperatures reported in January and February. The study variable $Y_{(N\Phi)}$, on the other hand, reflects the lowest and highest temperatures reported from March to May.

The auxiliary variable $X_{(N\Phi)}$ represents the minimum and maximum temperatures measured from June to September for Population B, whereas the study variable $Y_{(N\Phi)}$ reflects the minimum and maximum temperatures reported from October to December.

6. Simulation Study

The fundamental goal of this advanced scientific research was to validate and assess the effectiveness of proposed Neutrosophic esitimators and up against adapted Neutrosophic estimators for the study variable $Y_{(N\Phi)}$. To achieve this purpose, the researchers employed Neutrosophic data with known auxiliary parameters and the Neutrosophic normal distribution in a rigorous simulation workouts. Neutrosophic normal distributions were used to create the study Neutrosophic variable, designated as $Y_{(N\Phi)}$ as neutrosophic study variable, and the auxiliary variable, labelled as $X_{(N\Phi)}$.

The parameters for $Y_{(N\Phi)} \sim \text{NN}(\mu_{y_{(N\Phi)}}, \sigma^2_{y_{(N\Phi)}})$, where $\mu_{y_{(N\Phi)}}$ stood for the mean, and $\sigma_{y_{(N\Phi)}}$ for the standard deviation. Similar to this, the parameters for $X_{(N\Phi)} \sim \text{NN}(\mu_{x_{(N\Phi)}}, \sigma^2_{x_{(N\Phi)}})$, where $\mu_{x_{(N\Phi)}}$ stood for mean, and $\sigma_{x_{(N\Phi)}}$ for standard deviation. Specific parameter values for $Y_{(N\Phi)}$ and $X_{(N\Phi)}$ were chosen in order to aid numerical demonstration, creating a simulated dataset with 100 normally distributed observations for each variable. The parameters were set for $Y_{(N\Phi)} \sim \text{NN}([76.0, 54.9], [(12.9)^2, (17.2)^2])$, and for $X_{(N\Phi)} \sim \text{NN}([17.2, 18.4], [(5.8)^2, (6.7)^2])$.

For the simulated Neutrosophic data, the researchers produced descriptive statistics, giving a thorough breakdown of the dataset's features. The researchers intended to carefully assess the efficiency of the suggested Neutrosophic estimators and compare them to other estimators, so they carried out this complex simulation study. The paper offers important new understandings on the use of Neutrosophic statistics to handle uncertainties and indeterminacies in challenging real-world data analysis. In scenarios involving neutrosophic data and two-phase sampling, the results of this study have the potential to expand statistical approaches and improve knowledge of population parameter estimate.

7. Conclusions

We introduced a novel family of neutrosophic factor-type exponential estimators in two phase sampling for estimating neutrosophic population mean ($\bar{Y}_{(N\Phi)}$) using known neutrosophic auxiliary parameters in this thorough study. We thoroughly explored the neutrosophic sampling characteristics, specifically the bias and mean squared error (MSE), with an emphasis on degree one approximation.

We determined the neutrosophic lowest MSE by doing thorough investigations to determine the characterising scalars' neutrosophic optimal values for the proposed estimator. Several adapted neutrosophic competing estimators, such as $t_{0_{(N\Phi)}}$ $t^{d}_{R_{(N\Phi)}}$, $t^{d}_{P_{(N\Phi)}}$, $t^{d}_{Re_{(N\Phi)}}$, $t^{d}_{Pe_{(N\Phi)}}$, $t^{*d}_{R_{(N\Phi)}}$, $t^{*d}_{P_{(N\Phi)}}$, $t^{*d}_{Re_{(N\Phi)}}$, $t^{*d}_{Pe_{(N\Phi)}}$, $\jmath^{*d}_{Re_{(N\Phi)}}$, $\jmath^{*d}_{Pe_{(N\Phi)}}$ estimators were used to compare the performance of our proposed estimators, $\Im^{yp}_{ft_{(N\Phi)}}$.

Our investigations indisputably showed that our suggested estimators $\Im^{yp}_{ft_{(N\Phi)}}$, as shown by their reduced bias and MSE values, exhibited superior efficiency than the current estimators. Tables [5, 6, 7, 8, 9, 10, 11, 12, 13, and 14] provide specific examples of our suggested estimators for various values of (a, b). Further evidence that our proposed estimator performed better than any other neutrosophic competing estimators for Neutosophic proposed estimators can be seen in Tables [2, 3, 4].

We strongly advise their use for the estimate of neutrosophic population mean $\bar{Y}_{(N\Phi)}$ in numerous domains of application based on the excellent performance and efficiency proven by our introduced class of estimators. It is significant to highlight that neutrosophic estimators offer better population mean estimate in situations when study variable data are nondeterministic. It is recognised that neutrosophic estimators may still be better to classical estimators in circumstances when study variable data are indeterministic.

Table 1. Descriptive statistics of the Population - 1, Population - 2 and Simulated Data

Parameter	Population-1 Neutrosophic Value	Population-2 Neutrosophic Value	Simulated Data Neutrosophic Value
$N_{(N\Phi)}$	[119, 119]	[119, 119]	[100, 100]
$n_{(N\Phi)}$	[24, 24]	[24, 24]	[24, 24]
$n_{1_{(N\Phi)}}$	[44, 44]	[44, 44]	[44, 44]
$\bar{Y}_{(N\Phi)}$	[20.6937, 3055933]	[16.58437, 27.23613]	[20.6937, 31.55933]
$\bar{X}_{(N\Phi)}$	[13.90807, 24.6737]	[23.30966, 31.21807]	[13.90807, 24.6737]
$C_{y_{(N\Phi)}}$	[0.02657039, 0.02539073]	[0.03440212, 0.02580739]	[0.02657039, 0.02539073]
$C_{x_{(N\Phi)}}$	[0.04133301, 0.03918572]	[0.01435458, 0.01423702]	[0.04133301, 0.03918572]
$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	[0.6273783, 0.7201808]	[0.6737446, 0.7504361]	[0.01838219, -0.05842247]

Table 2. Mean Square Error of the Neutrosophic Estimators based on Population - 1.

Estimators	Case-I MSE	Case-II MSE
$t_{0_{(N\Phi)}}$	[0.01005628, 0.02135852]	[0.01005628, 0.02135852]
$t^{d}_{R_{(N\Phi)}}$	[0.01273596, 0.02677414]	[0.02524190, 0.05277647]
$t^{d}_{P_{(N\Phi)}}$	[0.03508849, 0.07387339]	[0.0644997, 0.1354970]
$t^{d}_{Re_{(N\Phi)}}$	[0.007932134, 0.016825020]	[0.008945461, 0.018872943]
$t^{d}_{Pe_{(N\Phi)}}$	[0.01910840, 0.04037465]	[0.02857436, 0.06023321]
$t^{*d}_{R_{(N\Phi)}}$	[0.01659733, 0.03480892]	[0.03663452, 0.07652684]
$t^{*d}_{P_{(N\Phi)}}$	[0.04342036, 0.09132803]	[0.08374387, 0.17579148]
$t^{*d}_{Re_{(N\Phi)}}$	[0.008338662, 0.017656234]	[0.01081217, 0.02274252]
$t^{*d}_{Pe_{(N\Phi)}}$	[0.02175018, 0.04591579]	[0.03436685, 0.07237484]
$\jmath^{*d}_{Re_{(N\Phi)}}$	[0.007802574, 0.016571881]	[0.007289519, 0.015482204]
$\jmath^{*d}_{Pe_{(N\Phi)}}$	[0.007802574, 0.016571881]	[0.007289519, 0.015482204]
$\Im^{yp}_{ft_{(N\Phi)}}$	**[0.007802574, 0.015051057]**	**[0.003104505, 0.001902523]**

TABLE 3. Mean Square Error of the Neutrosophic Estimators based on Population - 2.

	Case-I	Case-II
Estimators	MSE	MSE
$t_{0_{(N\Phi)}}$	[0.01082764, 0.01643405]	[0.01082764, 0.01643405]
$t^{d}_{R_{(N\Phi)}}$	[0.008434704, 0.012325983]	[0.007436709, 0.011372767]
$t^{d}_{P_{(N\Phi)}}$	[0.01536729, 0.02623753]	[0.01961242, 0.03580566]
$t^{d}_{Re_{(N\Phi)}}$	[0.009362831, 0.013668091]	[0.00845794, 0.01211462]
$t^{d}_{Pe_{(N\Phi)}}$	[0.01282912, 0.02062387]	[0.01454580, 0.02433106]
$t^{*d}_{R_{(N\Phi)}}$	[0.008213724, 0.012187818]	[0.007405786, 0.012077748]
$t^{*d}_{P_{(N\Phi)}}$	[0.01653282, 0.02888168]	[0.02201664, 0.04139722]
$t^{*d}_{Re_{(N\Phi)}}$	[0.009134271, 0.013285761]	[0.008145817, 0.011680043]
$t^{*d}_{Pe_{(N\Phi)}}$	[0.01329382, 0.02163269]	[0.01545124, 0.02633978]
$\jmath^{*d}_{Re_{(N\Phi)}}$	[0.008029139, 0.012186527]	[0.007392062, 0.011219580]
$\jmath^{*d}_{Pe_{(N\Phi)}}$	[0.008029139, 0.012186527]	[0.007392062, 0.011219580]
$\Im^{yp}_{ft_{(N\Phi)}}$	**[0.008029139, 0.011164511]**	**[0.0021953943, 0.0001796326]**

TABLE 4. Mean Square Error of the Neutrosophic Estimators based on Simulated Data.

	Case-I	Case-II
Estimators	MSE	MSE
$t_{0_{(N\Phi)}}$	[5.636972, 9.212937]	[5.636972, 9.212937]
$t^{d}_{R_{(N\Phi)}}$	[14.86116, 16.35115]	[27.39777, 26.10238]
$t^{d}_{P_{(N\Phi)}}$	[15.27578, 16.81980]	[28.09102, 26.88596]
$t^{d}_{Re_{(N\Phi)}}$	[7.89119, 10.93891]	[10.99052, 13.33735]
$t^{d}_{Pe_{(N\Phi)}}$	[8.098501, 11.173233]	[11.33714, 13.72914]
$t^{*d}_{R_{(N\Phi)}}$	[18.96955, 19.54820]	[37.05571, 33.62776]
$t^{*d}_{P_{(N\Phi)}}$	[19.46710, 20.11058]	[37.88761, 34.56805]
$t^{*d}_{Re_{(N\Phi)}}$	[8.907923, 11.726456]	[13.38767, 15.19911]
$t^{*d}_{Pe_{(N\Phi)}}$	[9.156696, 12.007645]	[5.635613, 9.210716]
$\jmath^{*d}_{Re_{(N\Phi)}}$	[5.635832, 9.211075]	[5.635613, 9.210716]
$\jmath^{*d}_{Pe_{(N\Phi)}}$	[5.635832, 9.211075]	[5.635613, 9.210716]
$\Im^{yp}_{ft_{(N\Phi)}}$	**[5.635832, 9.194130]**	**[5.633787, 9.160360]**

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Conflicts of Interest: The authors declare no conflict of interest.

Appendix A

TABLE 5. Members of the Neutrosophic Proposed Class of Estimators $\Im^{yp}_{ft_{(N\Phi)}}$.

S.No.	a	b	Estimator
1.	1	0	$\Im^{yp(1)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)}\left[\frac{(A+c)\bar{x}_{1(N\Phi)}+f_{(N\Phi)}B\bar{x}_{(N\Phi)}}{(A+f_{(N\Phi)}B)\bar{x}_{1(N\Phi)}+C\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})}\right\}$
2.	1	1	$\Im^{yp(2)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)}\left[\frac{(A+c)\bar{x}_{1(N\Phi)}+f_{(N\Phi)}B\bar{x}_{(N\Phi)}}{(A+f_{(N\Phi)}B)\bar{x}_{1(N\Phi)}+C\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2}\right\}$
3.	1	$S_{x_{(N\Phi)}}$	$\Im^{yp(3)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)}\left[\frac{(A+c)\bar{x}_{1(N\Phi)}+f_{(N\Phi)}B\bar{x}_{(N\Phi)}}{(A+f_{(N\Phi)}B)\bar{x}_{1(N\Phi)}+C\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2S_{x_{(N\Phi)}}}\right\}$
4.	1	$\beta_{2(x_{(N\Phi)})}$	$\Im^{yp(4)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)}\left[\frac{(A+c)\bar{x}_{1(N\Phi)}+f_{(N\Phi)}B\bar{x}_{(N\Phi)}}{(A+f_{(N\Phi)}B)\bar{x}_{1(N\Phi)}+C\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2\beta_{2(x_{(N\Phi)})}}\right\}$
5.	1	$C_{x_{(N\Phi)}}$	$\Im^{yp(5)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)}\left[\frac{(A+c)\bar{x}_{1(N\Phi)}+f_{(N\Phi)}B\bar{x}_{(N\Phi)}}{(A+f_{(N\Phi)}B)\bar{x}_{1(N\Phi)}+C\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2C_{x_{(N\Phi)}}}\right\}$
6.	1	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$\Im^{yp(6)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)}\left[\frac{(A+c)\bar{x}_{1(N\Phi)}+f_{(N\Phi)}B\bar{x}_{(N\Phi)}}{(A+f_{(N\Phi)}B)\bar{x}_{1(N\Phi)}+C\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2\rho_{y_{(N\Phi)}x_{(N\Phi)}}}\right\}$
7.	$S_{x_{(N\Phi)}}$	1	$\Im^{yp(7)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)}\left[\frac{(A+c)\bar{x}_{1(N\Phi)}+f_{(N\Phi)}B\bar{x}_{(N\Phi)}}{(A+f_{(N\Phi)}B)\bar{x}_{1(N\Phi)}+C\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{S_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{S_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2}\right\}$
8.	$S_{x_{(N\Phi)}}$	$\beta_{2(x_{(N\Phi)})}$	$\Im^{yp(8)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)}\left[\frac{(A+c)\bar{x}_{1(N\Phi)}+f_{(N\Phi)}B\bar{x}_{(N\Phi)}}{(A+f_{(N\Phi)}B)\bar{x}_{1(N\Phi)}+C\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{S_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{S_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2\beta_{2(x_{(N\Phi)})}}\right\}$
9.	$S_{x_{(N\Phi)}}$	$C_{x_{(N\Phi)}}$	$\Im^{yp(9)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)}\left[\frac{(A+c)\bar{x}_{1(N\Phi)}+f_{(N\Phi)}B\bar{x}_{(N\Phi)}}{(A+f_{(N\Phi)}B)\bar{x}_{1(N\Phi)}+C\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{S_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{S_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2C_{x_{(N\Phi)}}}\right\}$
10.	$S_{x_{(N\Phi)}}$	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$\Im^{yp(10)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)}\left[\frac{(A+c)\bar{x}_{1(N\Phi)}+f_{(N\Phi)}B\bar{x}_{(N\Phi)}}{(A+f_{(N\Phi)}B)\bar{x}_{1(N\Phi)}+C\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{S_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{S_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2\rho_{y_{(N\Phi)}x_{(N\Phi)}}}\right\}$
11.	$\beta_{2(x_{(N\Phi)})}$	1	$\Im^{yp(11)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)}\left[\frac{(A+c)\bar{x}_{1(N\Phi)}+f_{(N\Phi)}B\bar{x}_{(N\Phi)}}{(A+f_{(N\Phi)}B)\bar{x}_{1(N\Phi)}+C\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{\beta_{2(x_{(N\Phi)})}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{\beta_{2(x_{(N\Phi)})}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2}\right\}$
12.	$\beta_{2(x_{(N\Phi)})}$	$S_{x_{(N\Phi)}}$	$\Im^{yp(12)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)}\left[\frac{(A+c)\bar{x}_{1(N\Phi)}+f_{(N\Phi)}B\bar{x}_{(N\Phi)}}{(A+f_{(N\Phi)}B)\bar{x}_{1(N\Phi)}+C\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{\beta_{2(x_{(N\Phi)})}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{\beta_{2(x_{(N\Phi)})}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2S_{x_{(N\Phi)}}}\right\}$
13.	$\beta_{2(x_{(N\Phi)})}$	$C_{x_{(N\Phi)}}$	$\Im^{yp(13)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)}\left[\frac{(A+c)\bar{x}_{1(N\Phi)}+f_{(N\Phi)}B\bar{x}_{(N\Phi)}}{(A+f_{(N\Phi)}B)\bar{x}_{1(N\Phi)}+C\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{\beta_{2(x_{(N\Phi)})}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{\beta_{2(x_{(N\Phi)})}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2C_{x_{(N\Phi)}}}\right\}$
14.	$\beta_{2(x_{(N\Phi)})}$	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$\Im^{yp(14)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)}\left[\frac{(A+c)\bar{x}_{1(N\Phi)}+f_{(N\Phi)}B\bar{x}_{(N\Phi)}}{(A+f_{(N\Phi)}B)\bar{x}_{1(N\Phi)}+C\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{\beta_{2(x_{(N\Phi)})}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{\beta_{2(x_{(N\Phi)})}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2\rho_{y_{(N\Phi)}x_{(N\Phi)}}}\right\}$
15.	$C_{x_{(N\Phi)}}$	1	$\Im^{yp(15)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)}\left[\frac{(A+c)\bar{x}_{1(N\Phi)}+f_{(N\Phi)}B\bar{x}_{(N\Phi)}}{(A+f_{(N\Phi)}B)\bar{x}_{1(N\Phi)}+C\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{C_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{C_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2}\right\}$
16.	$C_{x_{(N\Phi)}}$	$S_{x_{(N\Phi)}}$	$\Im^{yp(16)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)}\left[\frac{(A+c)\bar{x}_{1(N\Phi)}+f_{(N\Phi)}B\bar{x}_{(N\Phi)}}{(A+f_{(N\Phi)}B)\bar{x}_{1(N\Phi)}+C\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{C_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{C_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2S_{x_{(N\Phi)}}}\right\}$
17.	$C_{x_{(N\Phi)}}$	$\beta_{2(x_{(N\Phi)})}$	$\Im^{yp(17)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)}\left[\frac{(A+c)\bar{x}_{1(N\Phi)}+f_{(N\Phi)}B\bar{x}_{(N\Phi)}}{(A+f_{(N\Phi)}B)\bar{x}_{1(N\Phi)}+C\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{C_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{C_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2\beta_{2(x_{(N\Phi)})}}\right\}$
18.	$C_{x_{(N\Phi)}}$	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$\Im^{yp(18)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)}\left[\frac{(A+c)\bar{x}_{1(N\Phi)}+f_{(N\Phi)}B\bar{x}_{(N\Phi)}}{(A+f_{(N\Phi)}B)\bar{x}_{1(N\Phi)}+C\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{C_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{C_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2\rho_{y_{(N\Phi)}x_{(N\Phi)}}}\right\}$
19.	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	1	$\Im^{yp(19)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)}\left[\frac{(A+c)\bar{x}_{1(N\Phi)}+f_{(N\Phi)}B\bar{x}_{(N\Phi)}}{(A+f_{(N\Phi)}B)\bar{x}_{1(N\Phi)}+C\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2}\right\}$
20.	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$S_{x_{(N\Phi)}}$	$\Im^{yp(20)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)}\left[\frac{(A+c)\bar{x}_{1(N\Phi)}+f_{(N\Phi)}B\bar{x}_{(N\Phi)}}{(A+f_{(N\Phi)}B)\bar{x}_{1(N\Phi)}+C\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2S_{x_{(N\Phi)}}}\right\}$
21.	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$\beta_{2(x_{(N\Phi)})}$	$\Im^{yp(21)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)}\left[\frac{(A+c)\bar{x}_{1(N\Phi)}+f_{(N\Phi)}B\bar{x}_{(N\Phi)}}{(A+f_{(N\Phi)}B)\bar{x}_{1(N\Phi)}+C\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2\beta_{2(x_{(N\Phi)})}}\right\}$
22.	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$C_{x_{(N\Phi)}}$	$\Im^{yp(22)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)}\left[\frac{(A+c)\bar{x}_{1(N\Phi)}+f_{(N\Phi)}B\bar{x}_{(N\Phi)}}{(A+f_{(N\Phi)}B)\bar{x}_{1(N\Phi)}+C\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2C_{x_{(N\Phi)}}}\right\}$

TABLE 6. Members of the Neutrosophic Proposed Class of Estimators $\Im^{yp}_{ft_{(N\Phi)}}$ for $d = 1$.

S.No.	a	b	**Estimator**
1.	1	0	$\Im^{Re(1)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{1(N\Phi)}}{\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})}\right\}$
2.	1	1	$\Im^{Re(2)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{1(N\Phi)}}{\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2}\right\}$
3.	1	$S_{x_{(N\Phi)}}$	$\Im^{Re(3)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{1(N\Phi)}}{\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2S_{x_{(N\Phi)}}}\right\}$
4.	1	$\beta_{2(x_{(N\Phi)})}$	$\Im^{Re(4)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{1(N\Phi)}}{\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2\beta_{2(x_{(N\Phi)})}}\right\}$
5.	1	$C_{x_{(N\Phi)}}$	$\Im^{Re(5)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{1(N\Phi)}}{\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2C_{x_{(N\Phi)}}}\right\}$
6.	1	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$\Im^{Re(6)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{1(N\Phi)}}{\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2\rho_{y_{(N\Phi)}x_{(N\Phi)}}}\right\}$
7.	$S_{x_{(N\Phi)}}$	1	$\Im^{Re(7)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{1(N\Phi)}}{\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{S_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{S_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2}\right\}$
8.	$S_{x_{(N\Phi)}}$	$\beta_{2(x_{(N\Phi)})}$	$\Im^{Re(8)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{1(N\Phi)}}{\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{S_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{S_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2\beta_{2(x)_{(N\Phi)}}}\right\}$
9.	$S_{x_{(N\Phi)}}$	$C_{x_{(N\Phi)}}$	$\Im^{Re(9)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{1(N\Phi)}}{\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{S_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{S_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2C_{x_{(N\Phi)}}}\right\}$
10.	$S_{x_{(N\Phi)}}$	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$\Im^{Re(10)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{1(N\Phi)}}{\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{S_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{S_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2\rho_{y_{(N\Phi)}x_{(N\Phi)}}}\right\}$
11.	$\beta_{2(x_{(N\Phi)})}$	1	$\Im^{Re(11)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{1(N\Phi)}}{\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{\beta_{2(x_{(N\Phi)})}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{\beta_{2(x_{(N\Phi)})}(\bar{x}_1+\bar{x}_{(N\Phi)})+2}\right\}$
12.	$\beta_{2(x_{(N\Phi)})}$	$S_{x_{(N\Phi)}}$	$\Im^{Re(12)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{1(N\Phi)}}{\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{\beta_{2(x_{(N\Phi)})}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{\beta_{2(x_{(N\Phi)})}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2S_{x_{(N\Phi)}}}\right\}$
13.	$\beta_{2(x_{(N\Phi)})}$	$C_{x_{(N\Phi)}}$	$\Im^{Re(13)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{1(N\Phi)}}{\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{\beta_{2(x_{(N\Phi)})}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{\beta_{2(x_{(N\Phi)})}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2C_{x_{(N\Phi)}}}\right\}$
14.	$\beta_{2(x_{(N\Phi)})}$	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$\Im^{Re(14)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{1(N\Phi)}}{\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{\beta_{2(x_{(N\Phi)})}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{\beta_{2(x_{(N\Phi)})}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2\rho_{y_{(N\Phi)}x_{(N\Phi)}}}\right\}$
15.	$C_{x_{(N\Phi)}}$	1	$\Im^{Re(15)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{1(N\Phi)}}{\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{C_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{C_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2}\right\}$
16.	$C_{x_{(N\Phi)}}$	$S_{x_{(N\Phi)}}$	$\Im^{Re(16)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{1(N\Phi)}}{\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{C_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{C_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2S_{x_{(N\Phi)}}}\right\}$
17.	$C_{x_{(N\Phi)}}$	$\beta_{2(x_{(N\Phi)})}$	$\Im^{Re(17)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{1(N\Phi)}}{\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{C_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{C_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2\beta_{2(x_{(N\Phi)})}}\right\}$
18.	$C_{x_{(N\Phi)}}$	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$\Im^{Re(18)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{1(N\Phi)}}{\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{C_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{C_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2\rho_{y_{(N\Phi)}x_{(N\Phi)}}}\right\}$
19.	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	1	$\Im^{Re(19)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{1(N\Phi)}}{\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}_1+\bar{x}_{(N\Phi)})+2}\right\}$
20.	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$S_{x_{(N\Phi)}}$	$\Im^{Re(20)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{1(N\Phi)}}{\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2S_{x_{(N\Phi)}}}\right\}$
21.	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$\beta_{2(x_{(N\Phi)})}$	$\Im^{Re(21)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{1(N\Phi)}}{\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2\beta_{2(x_{(N\Phi)})}}\right\}$
22.	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	C_x	$\Im^{Re(22)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{1(N\Phi)}}{\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2C_{x_{(N\Phi)}}}\right\}$

TABLE 7. Members of the Neutrosophic Proposed Class of Estimators $\Im^{yp}_{ft_{(N\Phi)}}$ for $d = 2$.

S.No.	a	b	Estimator
1.	1	0	$\Im^{Pe(1)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{(N\Phi)}}{\bar{x}_{1(N\Phi)}}\right]^{\alpha} exp\left\{\frac{(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})}\right\}$
2.	1	1	$\Im^{Pe(2)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{(N\Phi)}}{\bar{x}_{1(N\Phi)}}\right]^{\alpha} exp\left\{\frac{(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2}\right\}$
3.	1	$S_{x_{(N\Phi)}}$	$\Im^{Pe(3)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{(N\Phi)}}{\bar{x}_{1(N\Phi)}}\right]^{\alpha} exp\left\{\frac{(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2S_{x_{(N\Phi)}}}\right\}$
4.	1	$\beta_{2(x_{(N\Phi)})}$	$\Im^{Pe(4)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{(N\Phi)}}{\bar{x}_{1(N\Phi)}}\right]^{\alpha} exp\left\{\frac{(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2\beta_{2(x_{(N\Phi)})}}\right\}$
5.	1	$C_{x_{(N\Phi)}}$	$\Im^{Pe(5)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{(N\Phi)}}{\bar{x}_{1(N\Phi)}}\right]^{\alpha} exp\left\{\frac{(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2C_{x_{(N\Phi)}}}\right\}$
6.	1	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$\Im^{Pe(6)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{(N\Phi)}}{\bar{x}_{1(N\Phi)}}\right]^{\alpha} exp\left\{\frac{(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2\rho_{y_{(N\Phi)}x_{(N\Phi)}}}\right\}$
7.	$S_{x_{(N\Phi)}}$	1	$\Im^{Pe(7)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{(N\Phi)}}{\bar{x}_{1(N\Phi)}}\right]^{\alpha} exp\left\{\frac{S_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{S_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2}\right\}$
8.	$S_{x_{(N\Phi)}}$	$\beta_{2(x_{(N\Phi)})}$	$\Im^{Pe(8)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{(N\Phi)}}{\bar{x}_{1(N\Phi)}}\right]^{\alpha} exp\left\{\frac{S_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{S_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x})+2\beta_{2(x_{(N\Phi)})}}\right\}$
9.	$S_{x_{(N\Phi)}}$	$C_{x_{(N\Phi)}}$	$\Im^{Pe(9)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{(N\Phi)}}{\bar{x}_{1(N\Phi)}}\right]^{\alpha} exp\left\{\frac{S_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{S_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2C_{x_{(N\Phi)}}}\right\}$
10.	$S_{x_{(N\Phi)}}$	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$\Im^{Pe(10)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{(N\Phi)}}{\bar{x}_{1(N\Phi)}}\right]^{\alpha} exp\left\{\frac{S_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{S_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2\rho_{y_{(N\Phi)}x_{(N\Phi)}}}\right\}$
11.	$\beta_{2(x_{(N\Phi)})}$	1	$\Im^{Pe(11)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{(N\Phi)}}{\bar{x}_{1(N\Phi)}}\right]^{\alpha} exp\left\{\frac{\beta_{2(x_{(N\Phi)})}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{\beta_{2(x_{(N\Phi)})}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2}\right\}$
12.	$\beta_{2(x_{(N\Phi)})}$	$S_{x_{(N\Phi)}}$	$\Im^{Pe(12)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{(N\Phi)}}{\bar{x}_{1(N\Phi)}}\right]^{\alpha} exp\left\{\frac{\beta_{2(x_{(N\Phi)})}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{\beta_{2(x_{(N\Phi)})}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2S_{x_{(N\Phi)}}}\right\}$
13.	$\beta_{2(x_{(N\Phi)})}$	$C_{x_{(N\Phi)}}$	$\Im^{Pe(13)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{(N\Phi)}}{\bar{x}_{1(N\Phi)}}\right]^{\alpha} exp\left\{\frac{\beta_{2(x_{(N\Phi)})}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{\beta_{2(x_{(N\Phi)})}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2C_{x_{(N\Phi)}}}\right\}$
14.	$\beta_{2(x_{(N\Phi)})}$	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$\Im^{Pe(14)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{(N\Phi)}}{\bar{x}_{1(N\Phi)}}\right]^{\alpha} exp\left\{\frac{\beta_{2(x_{(N\Phi)})}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{\beta_{2(x_{(N\Phi)})}(\bar{x}_1+\bar{x}_{(N\Phi)})+2\rho_{y_{(N\Phi)}x_{(N\Phi)}}}\right\}$
15.	$C_{x_{(N\Phi)}}$	1	$\Im^{Pe(15)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{(N\Phi)}}{\bar{x}_{1(N\Phi)}}\right]^{\alpha} exp\left\{\frac{C_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{C_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2}\right\}$
16.	$C_{x_{(N\Phi)}}$	$S_{x_{(N\Phi)}}$	$\Im^{Pe(16)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{(N\Phi)}}{\bar{x}_{1(N\Phi)}}\right]^{\alpha} exp\left\{\frac{C_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x})}{C_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2S_{x_{(N\Phi)}}}\right\}$
17.	$C_{x_{(N\Phi)}}$	$\beta_{2(x_{(N\Phi)})}$	$\Im^{Pe(17)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{(N\Phi)}}{\bar{x}_{1(N\Phi)}}\right]^{\alpha} exp\left\{\frac{C_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{C_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2\beta_{2(x_{(N\Phi)})}}\right\}$
18.	$C_{x_{(N\Phi)}}$	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$\Im^{Pe(18)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{(N\Phi)}}{\bar{x}_{1(N\Phi)}}\right]^{\alpha} exp\left\{\frac{C_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{C_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2\rho_{y_{(N\Phi)}x_{(N\Phi)}}}\right\}$
19.	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	1	$\Im^{Pe(19)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{(N\Phi)}}{\bar{x}_{1(N\Phi)}}\right]^{\alpha} exp\left\{\frac{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2}\right\}$
20.	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$S_{x_{(N\Phi)}}$	$\Im^{Pe(20)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{(N\Phi)}}{\bar{x}_{1(N\Phi)}}\right]^{\alpha} exp\left\{\frac{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2S_{x_{(N\Phi)}}}\right\}$
21.	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$\beta_{2(x_{(N\Phi)})}$	$\Im^{Pe(21)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{(N\Phi)}}{\bar{x}_{1(N\Phi)}}\right]^{\alpha} exp\left\{\frac{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2\beta_{2(x_{(N\Phi)})}}\right\}$
22.	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$C_{x_{(N\Phi)}}$	$\Im^{Pe(22)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{(N\Phi)}}{\bar{x}_{1(N\Phi)}}\right]^{\alpha} exp\left\{\frac{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2C_{x_{(N\Phi)}}}\right\}$

Table 8. Members of the Neutrosophic Proposed Class of Estimators $\Im^{yp}_{ft_{(N\Phi)}}$ for $d = 3$.

S.No.	a	b	Estimator
1.	1	0	$\Im^{DR(1)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}1_{(N\Phi)}-f_{(N\Phi)}\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})}\right\}$
2.	1	1	$\Im^{DR(2)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}1_{(N\Phi)}-f_{(N\Phi)}\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2}\right\}$
3.	1	$S_{x_{(N\Phi)}}$	$\Im^{DR(3)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}1_{(N\Phi)}-f_{(N\Phi)}\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2S_{x_{(N\Phi)}}}\right\}$
4.	1	$\beta_{2(x_{(N\Phi)})}$	$\Im^{DR(4)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}1_{(N\Phi)}-f_{(N\Phi)}\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2\beta_{2(x_{(N\Phi)})}}\right\}$
5.	1	$C_{x_{(N\Phi)}}$	$\Im^{DR(5)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}1_{(N\Phi)}-f_{(N\Phi)}\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2C_{x_{(N\Phi)}}}\right\}$
6.	1	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$\Im^{DR(6)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}1_{(N\Phi)}-f_{(N\Phi)}\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2\rho_{y_{(N\Phi)}x_{(N\Phi)}}}\right\}$
7.	$S_{x_{(N\Phi)}}$	1	$\Im^{DR(7)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}1_{(N\Phi)}-f_{(N\Phi)}\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{S_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{S_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2}\right\}$
8.	$S_{x_{(N\Phi)}}$	$\beta_{2(x_{(N\Phi)})}$	$\Im^{DR(8)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}1_{(N\Phi)}-f_{(N\Phi)}\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{S_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{S_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2\beta_{2(x_{(N\Phi)})}}\right\}$
9.	$S_{x_{(N\Phi)}}$	$C_{x_{(N\Phi)}}$	$\Im^{DR(9)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}1_{(N\Phi)}-f_{(N\Phi)}\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{S_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{S_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2C_{x_{(N\Phi)}}}\right\}$
10.	$S_{x_{(N\Phi)}}$	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$\Im^{DR(10)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}1_{(N\Phi)}-f_{(N\Phi)}\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{S_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{S_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2\rho_{y_{(N\Phi)}x_{(N\Phi)}}}\right\}$
11.	$\beta_{2(x_{(N\Phi)})}$	1	$\Im^{DR(11)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}1_{(N\Phi)}-f_{(N\Phi)}\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{\beta_{2(x_{(N\Phi)})}(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{\beta_{2(x_{(N\Phi)})}(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2}\right\}$
12.	$\beta_{2(x_{(N\Phi)})}$	$S_{x_{(N\Phi)}}$	$\Im^{DR(12)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}1_{(N\Phi)}-f_{(N\Phi)}\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{\beta_{2(x_{(N\Phi)})}(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{\beta_{2(x_{(N\Phi)})}(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2S_{x_{(N\Phi)}}}\right\}$
13.	$\beta_{2(x_{(N\Phi)})}$	$C_{x_{(N\Phi)}}$	$\Im^{DR(13)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}1_{(N\Phi)}-f_{(N\Phi)}\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{\beta_{2(x_{(N\Phi)})}(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{\beta_{2(x_{(N\Phi)})}(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2C_{x_{(N\Phi)}}}\right\}$
14.	$\beta_{2(x_{(N\Phi)})}$	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$\Im^{DR(14)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}1_{(N\Phi)}-f_{(N\Phi)}\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{\beta_{2(x_{(N\Phi)})}(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{\beta_{2(x_{(N\Phi)})}(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2\rho_{y_{(N\Phi)}x_{(N\Phi)}}}\right\}$
15.	$C_{x_{(N\Phi)}}$	1	$\Im^{DR(15)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}1_{(N\Phi)}-f_{(N\Phi)}\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{C_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{C_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2}\right\}$
16.	$C_{x_{(N\Phi)}}$	$S_{x_{(N\Phi)}}$	$\Im^{DR(16)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}1_{(N\Phi)}-f_{(N\Phi)}\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{C_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}-\bar{x})}{C_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2S_{x_{(N\Phi)}}}\right\}$
17.	$C_{x_{(N\Phi)}}$	$\beta_{2(x_{(N\Phi)})}$	$\Im^{DR(17)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}1_{(N\Phi)}-f_{(N\Phi)}\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{C_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{C_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2\beta_{2(x_{(N\Phi)})}}\right\}$
18.	$C_{x_{(N\Phi)}}$	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$\Im^{DR(18)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}1_{(N\Phi)}-f_{(N\Phi)}\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{C_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{C_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2\rho_{y_{(N\Phi)}x_{(N\Phi)}}}\right\}$
19.	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	1	$\Im^{DR(19)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}1_{(N\Phi)}-f_{(N\Phi)}\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2}\right\}$
20.	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$S_{x_{(N\Phi)}}$	$\Im^{DR(20)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}1_{(N\Phi)}-f_{(N\Phi)}\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2S_{x_{(N\Phi)}}}\right\}$
21.	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$\beta_{2(x_{(N\Phi)})}$	$\Im^{DR(21)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}1_{(N\Phi)}-f_{(N\Phi)}\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2\beta_{2(x_{(N\Phi)})}}\right\}$
22.	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$C_{x_{(N\Phi)}}$	$\Im^{DR(22)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}1_{(N\Phi)}-f_{(N\Phi)}\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)}}\right]^{\alpha} exp\left\{\frac{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2C_{x_{(N\Phi)}}}\right\}$

TABLE 9. Members of the Neutrosophic Proposed Class of Estimators $\Im^{yp}_{ft_{(N\Phi)}}$ for $d = 4$.

S.No.	a	b	**Estimator**
1.	1	0	$\Im^{exp(1)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} exp\left\{\frac{(\bar{x}1_{(N\Phi)} - \bar{x}_{(N\Phi)})}{(\bar{x}1_{(N\Phi)} + \bar{x}_{(N\Phi)})}\right\}$
2.	1	1	$\Im^{exp(2)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} exp\left\{\frac{(\bar{x}1_{(N\Phi)} - \bar{x}_{(N\Phi)})}{(\bar{x}1_{(N\Phi)} + \bar{x}_{(N\Phi)}) + 2}\right\}$
3.	1	$S_{x_{(N\Phi)}}$	$\Im^{exp(3)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} exp\left\{\frac{(\bar{x}1_{(N\Phi)} - \bar{x}_{(N\Phi)})}{(\bar{x}1_{(N\Phi)} + \bar{x}_{(N\Phi)}) + 2S_{x_{(N\Phi)}}}\right\}$
4.	1	$\beta_{2(x_{(N\Phi)})}$	$\Im^{exp(4)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} exp\left\{\frac{(\bar{x}1_{(N\Phi)} - \bar{x}_{(N\Phi)})}{(\bar{x}1_{(N\Phi)} + \bar{x}_{(N\Phi)}) + 2\beta_{2(x_{(N\Phi)})}}\right\}$
5.	1	$C_{x_{(N\Phi)}}$	$\Im^{exp(5)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} exp\left\{\frac{(\bar{x}1_{(N\Phi)} - \bar{x}_{(N\Phi)})}{(\bar{x}1_{(N\Phi)} + \bar{x}_{(N\Phi)}) + 2C_{x_{(N\Phi)}}}\right\}$
6.	1	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$\Im^{exp(6)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} exp\left\{\frac{(\bar{x}1_{(N\Phi)} - \bar{x}_{(N\Phi)})}{(\bar{x}1_{(N\Phi)} + \bar{x}_{(N\Phi)}) + 2\rho_{y_{(N\Phi)}x_{(N\Phi)}}}\right\}$
7.	$S_{x_{(N\Phi)}}$	1	$\Im^{exp(7)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} exp\left\{\frac{S_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)} - \bar{x}_{(N\Phi)})}{S_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)} + \bar{x}_{(N\Phi)}) + 2}\right\}$
8.	$S_{x_{(N\Phi)}}$	$\beta_{2(x_{(N\Phi)})}$	$\Im^{exp(8)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} exp\left\{\frac{S_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)} - \bar{x}_{(N\Phi)})}{S_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)} + \bar{x}_{(N\Phi)}) + 2\beta_{2(x_{(N\Phi)})}}\right\}$
9.	$S_{x_{(N\Phi)}}$	$C_{x_{(N\Phi)}}$	$\Im^{exp(9)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} exp\left\{\frac{S_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)} - \bar{x})}{S_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)} + \bar{x}_{(N\Phi)}) + 2C_{x_{(N\Phi)}}}\right\}$
10.	$S_{x_{(N\Phi)}}$	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$\Im^{exp(10)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} exp\left\{\frac{S_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)} - \bar{x}_{(N\Phi)})}{S_{x_{(N\Phi)}}(\bar{x}1 + \bar{x}_{(N\Phi)}) + 2\rho_{y_{(N\Phi)}x_{(N\Phi)}}}\right\}$
11.	$\beta_{2(x_{(N\Phi)})}$	1	$\Im^{exp(11)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} exp\left\{\frac{\beta_{2(x_{(N\Phi)})}(\bar{x}1_{(N\Phi)} - \bar{x}_{(N\Phi)})}{\beta_{2(x_{(N\Phi)})}(\bar{x}1_{(N\Phi)} + \bar{x}_{(N\Phi)}) + 2}\right\}$
12.	$\beta_{2(x_{(N\Phi)})}$	$S_{x_{(N\Phi)}}$	$\Im^{exp(12)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} exp\left\{\frac{\beta_{2(x_{(N\Phi)})}(\bar{x}1_{(N\Phi)} - \bar{x}_{(N\Phi)})}{\beta_{2(x_{(N\Phi)})}(\bar{x}1_{(N\Phi)} + \bar{x}_{(N\Phi)}) + 2S_{x_{(N\Phi)}}}\right\}$
13.	$\beta_{2(x)_{(N\Phi)}}$	$C_{x_{(N\Phi)}}$	$\Im^{exp(13)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} exp\left\{\frac{\beta_{2(x_{(N\Phi)})}(\bar{x}1_{(N\Phi)} - \bar{x}_{(N\Phi)})}{\beta_{2(x_{(N\Phi)})}(\bar{x}1_{(N\Phi)} + \bar{x}_{(N\Phi)}) + 2C_{x_{(N\Phi)}}}\right\}$
14.	$\beta_{2(x_{(N\Phi)})}$	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$\Im^{exp(14)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} exp\left\{\frac{\beta_{2(x_{(N\Phi)})}(\bar{x}1_{(N\Phi)} - \bar{x}_{(N\Phi)})}{\beta_{2(x_{(N\Phi)})}(\bar{x}1_{(N\Phi)} + \bar{x}_{(N\Phi)}) + 2\rho_{y_{(N\Phi)}x_{(N\Phi)}}}\right\}$
15.	$C_{x_{(N\Phi)}}$	1	$\Im^{exp(15)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} exp\left\{\frac{C_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)} - \bar{x}_{(N\Phi)})}{C_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)} + \bar{x}_{(N\Phi)}) + 2}\right\}$
16.	$C_{x_{(N\Phi)}}$	$S_{x_{(N\Phi)}}$	$\Im^{exp(16)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} exp\left\{\frac{C_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)} - \bar{x}_{(N\Phi)})}{C_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)} + \bar{x}_{(N\Phi)}) + 2S_{x_{(N\Phi)}}}\right\}$
17.	$C_{x_{(N\Phi)}}$	$\beta_{2(x_{(N\Phi)})}$	$\Im^{exp(17)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} exp\left\{\frac{C_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)} - \bar{x}_{(N\Phi)})}{C_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)} + \bar{x}_{(N\Phi)}) + 2\beta_{2(x_{(N\Phi)})}}\right\}$
18.	$C_{x_{(N\Phi)}}$	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$\Im^{exp(18)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} exp\left\{\frac{C_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)} - \bar{x}_{(N\Phi)})}{C_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)} + \bar{x}_{(N\Phi)}) + 2\rho_{y_{(N\Phi)}x_{(N\Phi)}}}\right\}$
19.	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	1	$\Im^{exp(19)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} exp\left\{\frac{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}1_{(N\Phi)} - \bar{x}_{(N\Phi)})}{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}1_{(N\Phi)} + \bar{x}_{(N\Phi)}) + 2}\right\}$
20.	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$S_{x_{(N\Phi)}}$	$\Im^{exp(20)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} exp\left\{\frac{\rho_{yx_{(N\Phi)}}(\bar{x}1_{(N\Phi)} - \bar{x}_{(N\Phi)})}{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}1_{(N\Phi)} + \bar{x}_{(N\Phi)}) + 2S_{x_{(N\Phi)}}}\right\}$
21.	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$\beta_{2(x_{(N\Phi)})}$	$\Im^{exp(21)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} exp\left\{\frac{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}1_{(N\Phi)} - \bar{x}_{(N\Phi)})}{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}1_{(N\Phi)} + \bar{x}_{(N\Phi)}) + 2\beta_{2(x_{(N\Phi)})}}\right\}$
22.	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$C_{x_{(N\Phi)}}$	$\Im^{exp(22)}_{ft_{(N\Phi)}} = \bar{y}_{(N\Phi)} exp\left\{\frac{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}1_{(N\Phi)} - \bar{x}_{(N\Phi)})}{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}1_{(N\Phi)} + \bar{x}_{(N\Phi)}) + 2C_{x_{(N\Phi)}}}\right\}$

TABLE 10. Members of the Neutrosophic Proposed Class of Estimators $\Im^{yp}_{vk_{(N\Phi)}}$ for $\alpha = 1$.

S.No.	a	b	**Estimator**
1.	1	0	$\Im^{yp(1)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{(A+c)\bar{x}_{1(N\Phi)}+f_{(N\Phi)}B\bar{x}_{(N\Phi)}}{(A+f_{(N\Phi)}B)\bar{x}_{1(N\Phi)}+C\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})}\right\}$
2.	1	1	$\Im^{yp(2)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{(A+c)\bar{x}_{1(N\Phi)}+f_{(N\Phi)}B\bar{x}_{(N\Phi)}}{(A+f_{(N\Phi)}B)\bar{x}_{1(N\Phi)}+C\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2}\right\}$
3.	1	$S_{x_{(N\Phi)}}$	$\Im^{yp(3)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{(A+c)\bar{x}_{1(N\Phi)}+f_{(N\Phi)}B\bar{x}_{(N\Phi)}}{(A+f_{(N\Phi)}B)\bar{x}_{1(N\Phi)}+C\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2S_{x_{(N\Phi)}}}\right\}$
4.	1	$\beta_{2(x_{(N\Phi)})}$	$\Im^{yp(4)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{(A+c)\bar{x}_{1(N\Phi)}+f_{(N\Phi)}B\bar{x}_{(N\Phi)}}{(A+f_{(N\Phi)}B)\bar{x}_{1(N\Phi)}+C\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2\beta_{2(x_{(N\Phi)})}}\right\}$
5.	1	$C_{x_{(N\Phi)}}$	$\Im^{yp(5)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{(A+c)\bar{x}_{1(N\Phi)}+f_{(N\Phi)}B\bar{x}_{(N\Phi)}}{(A+f_{(N\Phi)}B)\bar{x}_{1(N\Phi)}+C\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2C_{x_{(N\Phi)}}}\right\}$
6.	1	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$\Im^{yp(6)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{(A+c)\bar{x}_{1(N\Phi)}+f_{(N\Phi)}B\bar{x}_{(N\Phi)}}{(A+f_{(N\Phi)}B)\bar{x}_{1(N\Phi)}+C\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2\rho_{y_{(N\Phi)}x_{(N\Phi)}}}\right\}$
7.	$S_{x_{(N\Phi)}}$	1	$\Im^{yp(7)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{(A+c)\bar{x}_{1(N\Phi)}+f_{(N\Phi)}B\bar{x}_{(N\Phi)}}{(A+f_{(N\Phi)}B)\bar{x}_{1(N\Phi)}+C\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{S_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{S_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2}\right\}$
8.	$S_{x_{(N\Phi)}}$	$\beta_{2(x_{(N\Phi)})}$	$\Im^{yp(8)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{(A+c)\bar{x}_{1(N\Phi)}+f_{(N\Phi)}B\bar{x}_{(N\Phi)}}{(A+f_{(N\Phi)}B)\bar{x}_{1(N\Phi)}+C\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{S_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{S_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2\beta_{2(x_{(N\Phi)})}}\right\}$
9.	$S_{x_{(N\Phi)}}$	$C_{x_{(N\Phi)}}$	$\Im^{yp(9)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{(A+c)\bar{x}_{1(N\Phi)}+f_{(N\Phi)}B\bar{x}_{(N\Phi)}}{(A+f_{(N\Phi)}B)\bar{x}_{1(N\Phi)}+C\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{S_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{S_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2C_{x_{(N\Phi)}}}\right\}$
10.	$S_{x_{(N\Phi)}}$	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$\Im^{yp(10)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{(A+c)\bar{x}_{1(N\Phi)}+f_{(N\Phi)}B\bar{x}_{(N\Phi)}}{(A+f_{(N\Phi)}B)\bar{x}_{1(N\Phi)}+C\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{S_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{S_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2\rho_{y_{(N\Phi)}x_{(N\Phi)}}}\right\}$
11.	$\beta_{2(x_{(N\Phi)})}$	1	$\Im^{yp(11)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{(A+c)\bar{x}_{1(N\Phi)}+f_{(N\Phi)}B\bar{x}_{(N\Phi)}}{(A+f_{(N\Phi)}B)\bar{x}_{1(N\Phi)}+C\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{\beta_{2(x_{(N\Phi)})}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{\beta_{2(x_{(N\Phi)})}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2}\right\}$
12.	$\beta_{2(x_{(N\Phi)})}$	$S_{x_{(N\Phi)}}$	$\Im^{yp(12)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{(A+c)\bar{x}_{1(N\Phi)}+f_{(N\Phi)}B\bar{x}_{(N\Phi)}}{(A+f_{(N\Phi)}B)\bar{x}_{1(N\Phi)}+C\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{\beta_{2(x_{(N\Phi)})}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{\beta_{2(x_{(N\Phi)})}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2S_{x_{(N\Phi)}}}\right\}$
13.	$\beta_{2(x_{(N\Phi)})}$	$C_{x_{(N\Phi)}}$	$\Im^{yp(13)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{(A+c)\bar{x}_{1(N\Phi)}+f_{(N\Phi)}B\bar{x}_{(N\Phi)}}{(A+f_{(N\Phi)}B)\bar{x}_{1(N\Phi)}+C\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{\beta_{2(x_{(N\Phi)})}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{\beta_{2(x_{(N\Phi)})}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2C_{x_{(N\Phi)}}}\right\}$
14.	$\beta_{2(x_{(N\Phi)})}$	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$\Im^{yp(14)}_{vk} = \bar{y}_{(N\Phi)} \left[\frac{(A+c)\bar{x}_{1(N\Phi)}+f_{(N\Phi)}B\bar{x}_{(N\Phi)}}{(A+f_{(N\Phi)}B)\bar{x}_{1(N\Phi)}+C\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{\beta_{2(x_{(N\Phi)})}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{\beta_{2(x_{(N\Phi)})}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2\rho_{y_{(N\Phi)}x_{(N\Phi)}}}\right\}$
15.	$C_{x_{(N\Phi)}}$	1	$\Im^{yp(15)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{(A+c)\bar{x}_{1(N\Phi)}+f_{(N\Phi)}B\bar{x}_{(N\Phi)}}{(A+f_{(N\Phi)}B)\bar{x}_{1(N\Phi)}+C\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{C_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{C_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2}\right\}$
16.	$C_{x_{(N\Phi)}}$	$S_{x_{(N\Phi)}}$	$\Im^{yp(16)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{(A+c)\bar{x}_{1(N\Phi)}+f_{(N\Phi)}B\bar{x}_{(N\Phi)}}{(A+f_{(N\Phi)}B)\bar{x}_{1(N\Phi)}+C\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{C_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{C_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2S_{x_{(N\Phi)}}}\right\}$
17.	$C_{x_{(N\Phi)}}$	$\beta_{2(x_{(N\Phi)})}$	$\Im^{yp(17)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{(A+c)\bar{x}_{1(N\Phi)}+f_{(N\Phi)}B\bar{x}_{(N\Phi)}}{(A+f_{(N\Phi)}B)\bar{x}_{1(N\Phi)}+C\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{C_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{C_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2\beta_{2(x_{(N\Phi)})}}\right\}$
18.	$C_{x_{(N\Phi)}}$	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$\Im^{yp(18)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{(A+c)\bar{x}_{1(N\Phi)}+f_{(N\Phi)}B\bar{x}_{(N\Phi)}}{(A+f_{(N\Phi)}B)\bar{x}_{1(N\Phi)}+C\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{C_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{C_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2\rho_{y_{(N\Phi)}x_{(N\Phi)}}}\right\}$
19.	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	1	$\Im^{yp(19)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{(A+c)\bar{x}_{1(N\Phi)}+f_{(N\Phi)}B\bar{x}_{(N\Phi)}}{(A+f_{(N\Phi)}B)\bar{x}_{1(N\Phi)}+C\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2}\right\}$
20.	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$S_{x_{(N\Phi)}}$	$\Im^{yp(20)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{(A+c)\bar{x}_{1(N\Phi)}+f_{(N\Phi)}B\bar{x}_{(N\Phi)}}{(A+f_{(N\Phi)}B)\bar{x}_{1(N\Phi)}+C\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x})}{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2S_{x_{(N\Phi)}}}\right\}$
21.	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$\beta_{2(x)}$	$\Im^{yp(21)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{(A+c)\bar{x}_{1(N\Phi)}+f_{(N\Phi)}B\bar{x}_{(N\Phi)}}{(A+f_{(N\Phi)}B)\bar{x}_{1(N\Phi)}+C\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2\beta_{2(x_{(N\Phi)})}}\right\}$
22.	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$C_{x_{(N\Phi)}}$	$\Im^{yp(22)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{(A+c)\bar{x}_{1(N\Phi)}+f_{(N\Phi)}B\bar{x}_{(N\Phi)}}{(A+f_{(N\Phi)}B)\bar{x}_{1(N\Phi)}+C\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2C_{x_{(N\Phi)}}}\right\}$

TABLE 11. Members of the Neutrosophic Proposed Class of Estimators $\Im_{vk_{(N\Phi)}}^{yp}$ for $\alpha = 1$ and $d = 1$.

S.No.	a	b	**Estimator**
1.	1	0	$\Im_{vk_{(N\Phi)}}^{Re(1)} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{1(N\Phi)}}{\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})}\right\}$
2.	1	1	$\Im_{vk_{(N\Phi)}}^{Re(2)} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{1(N\Phi)}}{\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2}\right\}$
3.	1	$S_{x_{(N\Phi)}}$	$\Im_{vk_{(N\Phi)}}^{Re(3)} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{1(N\Phi)}}{\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{(\bar{x}_{1(N\Phi)}+\bar{x})+2S_{x_{(N\Phi)}}}\right\}$
4.	1	$\beta_{2(x_{(N\Phi)})}$	$\Im_{vk_{(N\Phi)}}^{Re(4)} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{1(N\Phi)}}{\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2\beta_{2(x_{(N\Phi)})}}\right\}$
5.	1	$C_{x_{(N\Phi)}}$	$\Im_{vk_{(N\Phi)}}^{Re(5)} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{1(N\Phi)}}{\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2C_{x_{(N\Phi)}}}\right\}$
6.	1	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$\Im_{vk_{(N\Phi)}}^{Re(6)} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{1(N\Phi)}}{\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2\rho_{y_{(N\Phi)}x_{(N\Phi)}}}\right\}$
7.	$S_{x_{(N\Phi)}}$	1	$\Im_{vk_{(N\Phi)}}^{Re(7)} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{1(N\Phi)}}{\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{S_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{S_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2}\right\}$
8.	$S_{x_{(N\Phi)}}$	$\beta_{2(x_{(N\Phi)})}$	$\Im_{vk_{(N\Phi)}}^{Re(8)} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{1(N\Phi)}}{\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{S_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x})}{S_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2\beta_{2(x_{(N\Phi)})}}\right\}$
9.	$S_{x_{(N\Phi)}}$	$C_{x_{(N\Phi)}}$	$\Im_{vk_{(N\Phi)}}^{Re(9)} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{1(N\Phi)}}{\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{S_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{S_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2C_{x_{(N\Phi)}}}\right\}$
10.	$S_{x_{(N\Phi)}}$	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$\Im_{vk_{(N\Phi)}}^{Re(10)} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{1(N\Phi)}}{\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{S_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{S_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2\rho_{y_{(N\Phi)}x_{(N\Phi)}}}\right\}$
11.	$\beta_{2(x_{(N\Phi)})}$	1	$\Im_{vk_{(N\Phi)}}^{Re(11)} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{1(N\Phi)}}{\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{\beta_{2(x_{(N\Phi)})}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{\beta_{2(x_{(N\Phi)})}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2}\right\}$
12.	$\beta_{2(x_{(N\Phi)})}$	$S_{x_{(N\Phi)}}$	$\Im_{vk}^{Re(12)} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{1(N\Phi)}}{\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{\beta_{2(x_{(N\Phi)})}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{\beta_{2(x_{(N\Phi)})}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2S_{x_{(N\Phi)}}}\right\}$
13.	$\beta_{2(x_{(N\Phi)})}$	$C_{x_{(N\Phi)}}$	$\Im_{vk_{(N\Phi)}}^{Re(13)} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{1(N\Phi)}}{\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{\beta_{2(x_{(N\Phi)})}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{\beta_{2(x_{(N\Phi)})}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2C_{x_{(N\Phi)}}}\right\}$
14.	$\beta_{2(x_{(N\Phi)})}$	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$\Im_{vk_{(N\Phi)}}^{Re(14)} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{1(N\Phi)}}{\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{\beta_{2(x_{(N\Phi)})}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{\beta_{2(x_{(N\Phi)})}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2\rho_{y_{(N\Phi)}x_{(N\Phi)}}}\right\}$
15.	$C_{x_{(N\Phi)}}$	1	$\Im_{vk_{(N\Phi)}}^{Re(15)} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{1(N\Phi)}}{\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{C_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{C_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2}\right\}$
16.	$C_{x_{(N\Phi)}}$	$S_{x_{(N\Phi)}}$	$\Im_{vk}^{Re(16)} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{1(N\Phi)}}{\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{C_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{C_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2S_{x_{(N\Phi)}}}\right\}$
17.	$C_{x_{(N\Phi)}}$	$\beta_{2(x_{(N\Phi)})}$	$\Im_{vk_{(N\Phi)}}^{Re(17)} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{1(N\Phi)}}{\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{C_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{C_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2\beta_{2(x_{(N\Phi)})}}\right\}$
18.	$C_{x_{(N\Phi)}}$	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$\Im_{vk_{(N\Phi)}}^{Re(18)} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{1(N\Phi)}}{\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{C_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{C_{x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2\rho_{y_{(N\Phi)}x_{(N\Phi)}}}\right\}$
19.	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	1	$\Im_{vk_{(N\Phi)}}^{Re(19)} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{1(N\Phi)}}{\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2}\right\}$
20.	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$S_{x_{(N\Phi)}}$	$\Im_{vk_{(N\Phi)}}^{Re(20)} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{1(N\Phi)}}{\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2S_{x_{(N\Phi)}}}\right\}$
21.	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$\beta_{2(x_{(N\Phi)})}$	$\Im_{vk_{(N\Phi)}}^{Re(21)} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{1(N\Phi)}}{\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2\beta_{2(x_{(N\Phi)})}}\right\}$
22.	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$C_{x_{(N\Phi)}}$	$\Im_{vk_{(N\Phi)}}^{Re(22)} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{1(N\Phi)}}{\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}-\bar{x}_{(N\Phi)})}{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}_{1(N\Phi)}+\bar{x}_{(N\Phi)})+2C_{x_{(N\Phi)}}}\right\}$

TABLE 12. Members of the Neutrosophic Proposed Class of Estimators $\Im^{yp}_{vk_{(N\Phi)}}$ for $\alpha = 1$ and $d = 2$.

S.No.	a	b	**Estimator**
1.	1	0	$\Im^{Pe(1)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}}\right] exp\left\{\frac{(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})}\right\}$
2.	1	1	$\Im^{Pe(2)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}}\right] exp\left\{\frac{(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2}\right\}$
3.	1	$S_{x_{(N\Phi)}}$	$\Im^{Pe(3)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}}\right] exp\left\{\frac{(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2S_{x_{(N\Phi)}}}\right\}$
4.	1	$\beta_{2(x_{(N\Phi)})}$	$\Im^{Pe(4)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}}\right] exp\left\{\frac{(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2\beta_{2(x_{(N\Phi)})}}\right\}$
5.	1	$C_{x_{(N\Phi)}}$	$\Im^{Pe(5)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}}\right] exp\left\{\frac{(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2C_{x_{(N\Phi)}}}\right\}$
6.	1	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$\Im^{Pe(6)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}}\right] exp\left\{\frac{(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2\rho_{y_{(N\Phi)}x_{(N\Phi)}}}\right\}$
7.	$S_{x_{(N\Phi)}}$	1	$\Im^{Pe(7)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}}\right] exp\left\{\frac{S_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{S_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2}\right\}$
8.	$S_{x_{(N\Phi)}}$	$\beta_{2(x_{(N\Phi)})}$	$\Im^{Pe(8)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}}\right] exp\left\{\frac{S_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}-\bar{x})}{S_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2\beta_{2(x_{(N\Phi)})}}\right\}$
9.	$S_{x_{(N\Phi)}}$	$C_{x_{(N\Phi)}}$	$\Im^{Pe(9)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}}\right] exp\left\{\frac{S_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{S_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2C_{x_{(N\Phi)}}}\right\}$
10.	$S_{x_{(N\Phi)}}$	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$\Im^{Pe(10)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}}\right] exp\left\{\frac{S_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{S_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2\rho_{y_{(N\Phi)}x_{(N\Phi)}}}\right\}$
11.	$\beta_{2(x_{(N\Phi)})}$	1	$\Im^{Pe(11)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}}\right] exp\left\{\frac{\beta_{2(x_{(N\Phi)})}(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{\beta_{2(x_{(N\Phi)})}(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2}\right\}$
12.	$\beta_{2(x_{(N\Phi)})}$	$S_{x_{(N\Phi)}}$	$\Im^{Pe(12)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}}\right] exp\left\{\frac{\beta_{2(x_{(N\Phi)})}(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{\beta_{2(x_{(N\Phi)})}(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2S_{x_{(N\Phi)}}}\right\}$
13.	$\beta_{2(x_{(N\Phi)})}$	$C_{x_{(N\Phi)}}$	$\Im^{Pe(13)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}}\right] exp\left\{\frac{\beta_{2(x_{(N\Phi)})}(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{\beta_{2(x_{(N\Phi)})}(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2C_{x_{(N\Phi)}}}\right\}$
14.	$\beta_{2(x_{(N\Phi)})}$	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$\Im^{Pe(14)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}}\right] exp\left\{\frac{\beta_{2(x_{(N\Phi)})}(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{\beta_{2(x_{(N\Phi)})}(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2\rho_{y_{(N\Phi)}x_{(N\Phi)}}}\right\}$
15.	$C_{x_{(N\Phi)}}$	1	$\Im^{Pe(15)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}}\right] exp\left\{\frac{C_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{C_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2}\right\}$
16.	$C_{x_{(N\Phi)}}$	$S_{x_{(N\Phi)}}$	$\Im^{Pe(16)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}}\right] exp\left\{\frac{C_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{C_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2S_{x_{(N\Phi)}}}\right\}$
17.	$C_{x_{(N\Phi)}}$	$\beta_{2(x_{(N\Phi)})}$	$\Im^{Pe(17)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}}\right] exp\left\{\frac{C_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{C_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2\beta_{2(x_{(N\Phi)})}}\right\}$
18.	$C_{x_{(N\Phi)}}$	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$\Im^{Pe(18)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}}\right] exp\left\{\frac{C_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{C_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2\rho_{y_{(N\Phi)}x_{(N\Phi)}}}\right\}$
19.	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	1	$\Im^{Pe(19)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}}\right] exp\left\{\frac{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2}\right\}$
20.	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	S_x	$\Im^{Pe(20)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}}\right] exp\left\{\frac{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2S_{x_{(N\Phi)}}}\right\}$
21.	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$\beta_{2(x_{(N\Phi)})}$	$\Im^{Pe(21)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}}\right] exp\left\{\frac{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2\beta_{2(x_{(N\Phi)})}}\right\}$
22.	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$C_{x_{(N\Phi)}}$	$\Im^{Pe(22)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}}\right] exp\left\{\frac{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2C_{x_{(N\Phi)}}}\right\}$

TABLE 13. Members of the Neutrosophic Proposed Class of Estimators $\Im^{yp}_{vk_{(N\Phi)}}$ for $\alpha = 1$ and $d = 3$.

S.No.	a	b	Estimator
1.	1	0	$\Im^{DR(1)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}1_{(N\Phi)}-f_{(N\Phi)}\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})}\right\}$
2.	1	1	$\Im^{DR(2)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}1_{(N\Phi)}-f_{(N\Phi)}\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2}\right\}$
3.	1	$S_{x_{(N\Phi)}}$	$\Im^{DR(3)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}1_{(N\Phi)}-f_{(N\Phi)}\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2S_{x_{(N\Phi)}}}\right\}$
4.	1	$\beta_{2(x_{(N\Phi)})}$	$\Im^{DR(4)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}1_{(N\Phi)}-f_{(N\Phi)}\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2\beta_{2(x_{(N\Phi)})}}\right\}$
5.	1	$C_{x_{(N\Phi)}}$	$\Im^{DR(5)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}1_{(N\Phi)}-f_{(N\Phi)}\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{(\bar{x}1_{(N\Phi)}+\bar{x})+2C_{x_{(N\Phi)}}}\right\}$
6.	1	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$\Im^{DR(6)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}1_{(N\Phi)}-f_{(N\Phi)}\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2\rho_{y_{(N\Phi)}x_{(N\Phi)}}}\right\}$
7.	$S_{x_{(N\Phi)}}$	1	$\Im^{DR(7)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}1_{(N\Phi)}-f_{(N\Phi)}\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{S_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{S_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2}\right\}$
8.	$S_{x_{(N\Phi)}}$	$\beta_{2(x_{(N\Phi)})}$	$\Im^{DR(8)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}1_{(N\Phi)}-f_{(N\Phi)}\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{S_{x_{(N\Phi)}}(\bar{x}1-\bar{x}_{(N\Phi)})}{S_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2\beta_{2(x_{(N\Phi)})}}\right\}$
9.	$S_{x_{(N\Phi)}}$	$C_{x_{(N\Phi)}}$	$\Im^{DR(9)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}1_{(N\Phi)}-f_{(N\Phi)}\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{S_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{S_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2C_{x_{(N\Phi)}}}\right\}$
10.	$S_{x_{(N\Phi)}}$	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$\Im^{DR(10)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}1_{(N\Phi)}-f_{(N\Phi)}\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{S_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{S_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2\rho_{y_{(N\Phi)}x_{(N\Phi)}}}\right\}$
11.	$\beta_{2(x_{(N\Phi)})}$	1	$\Im^{DR(11)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}1_{(N\Phi)}-f_{(N\Phi)}\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{\beta_{2(x_{(N\Phi)})}(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{\beta_{2(x_{(N\Phi)})}(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2}\right\}$
12.	$\beta_{2(x_{(N\Phi)})}$	$S_{x_{(N\Phi)}}$	$\Im^{DR(12)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}1_{(N\Phi)}-f_{(N\Phi)}\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{\beta_{2(x_{(N\Phi)})}(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{\beta_{2(x_{(N\Phi)})}(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2S_{x_{(N\Phi)}}}\right\}$
13.	$\beta_{2(x_{(N\Phi)})}$	$C_{x_{(N\Phi)}}$	$\Im^{DR(13)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}1_{(N\Phi)}-f_{(N\Phi)}\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{\beta_{2(x_{(N\Phi)})}(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{\beta_{2(x_{(N\Phi)})}(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2C_{x_{(N\Phi)}}}\right\}$
14.	$\beta_{2(x_{(N\Phi)})}$	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$\Im^{DR(14)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}1_{(N\Phi)}-f_{(N\Phi)}\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{\beta_{2(x_{(N\Phi)})}(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{\beta_{2(x_{(N\Phi)})}(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2\rho_{y_{(N\Phi)}x_{(N\Phi)}}}\right\}$
15.	$C_{x_{(N\Phi)}}$	1	$\Im^{DR(15)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}1_{(N\Phi)}-f_{(N\Phi)}\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{C_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{C_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2}\right\}$
16.	$C_{x_{(N\Phi)}}$	$S_{x_{(N\Phi)}}$	$\Im^{DR(16)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}1_{(N\Phi)}-f_{(N\Phi)}\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{C_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{C_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2S_{x_{(N\Phi)}}}\right\}$
17.	$C_{x_{(N\Phi)}}$	$\beta_{2(x_{(N\Phi)})}$	$\Im^{DR(17)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}1_{(N\Phi)}-f_{(N\Phi)}\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{C_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}-\bar{x})}{C_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2\beta_{2(x_{(N\Phi)})}}\right\}$
18.	$C_{x_{(N\Phi)}}$	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$\Im^{DR(18)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}1_{(N\Phi)}-f_{(N\Phi)}\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{C_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{C_{x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}+x_{(\bar{N}\Phi)})+2\rho_{y_{(N\Phi)}x_{(N\Phi)}}}\right\}$
19.	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	1	$\Im^{DR(19)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}1_{(N\Phi)}-f_{(N\Phi)}\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2}\right\}$
20.	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$S_{x_{(N\Phi)}}$	$\Im^{DR(20)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}1_{(N\Phi)}-f_{(N\Phi)}\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2S_{x_{(N\Phi)}}}\right\}$
21.	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$\beta_{2(x_{(N\Phi)})}$	$\Im^{DR(21)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}1_{(N\Phi)}-f_{(N\Phi)}\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2\beta_{2(x_{(N\Phi)})}}\right\}$
22.	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$C_{x_{(N\Phi)}}$	$\Im^{DR(22)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} \left[\frac{\bar{x}1_{(N\Phi)}-f_{(N\Phi)}\bar{x}_{(N\Phi)}}{\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)}}\right] exp\left\{\frac{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}-\bar{x}_{(N\Phi)})}{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}1_{(N\Phi)}+\bar{x}_{(N\Phi)})+2C_{x_{(N\Phi)}}}\right\}$

TABLE 14. Members of the Neutrosophic Proposed Class of Estimators $\Im^{yp}_{vk_{(N\Phi)}}$ for $\alpha = 1$ and $d = 4$.

S.No.	a	b	Estimator
1.	1	0	$\Im^{exp(1)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} exp \left\{ \frac{(\bar{x}_{1_{(N\Phi)}} - \bar{x}_{(N\Phi)})}{(\bar{x}_{1_{(N\Phi)}} + \bar{x}_{(N\Phi)})} \right\}$
2.	1	1	$\Im^{exp(2)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} exp \left\{ \frac{(\bar{x}_{1_{(N\Phi)}} - \bar{x}_{(N\Phi)})}{(\bar{x}_{1_{(N\Phi)}} + \bar{x}_{(N\Phi)}) + 2} \right\}$
3.	1	$S_{x_{(N\Phi)}}$	$\Im^{exp(3)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} exp \left\{ \frac{(\bar{x}_{1_{(N\Phi)}} - \bar{x}_{(N\Phi)})}{(\bar{x}_{1_{(N\Phi)}} + \bar{x}_{(N\Phi)}) + 2S_{x_{(N\Phi)}}} \right\}$
4.	1	$\beta_{2(x_{(N\Phi)})}$	$\Im^{exp(4)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} exp \left\{ \frac{(\bar{x}_{1_{(N\Phi)}} - \bar{x}_{(N\Phi)})}{(\bar{x}_{1_{(N\Phi)}} + \bar{x}_{(N\Phi)}) + 2\beta_{2(x_{(N\Phi)})}} \right\}$
5.	1	$C_{x_{(N\Phi)}}$	$\Im^{exp(5)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} exp \left\{ \frac{(\bar{x}_{1_{(N\Phi)}} - \bar{x}_{(N\Phi)})}{(\bar{x}_{1_{(N\Phi)}} + \bar{x}_{(N\Phi)}) + 2C_{x_{(N\Phi)}}} \right\}$
6.	1	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$\Im^{exp(6)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} exp \left\{ \frac{(\bar{x}_{1_{(N\Phi)}} - \bar{x}_{(N\Phi)})}{(\bar{x}_{1_{(N\Phi)}} + \bar{x}_{(N\Phi)}) + 2\rho_{y_{(N\Phi)}x_{(N\Phi)}}} \right\}$
7.	$S_{x_{(N\Phi)}}$	1	$\Im^{exp(7)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} exp \left\{ \frac{S_{x_{(N\Phi)}}(\bar{x}_{1_{(N\Phi)}} - \bar{x}_{(N\Phi)})}{S_{x_{(N\Phi)}}(\bar{x}_{1_{(N\Phi)}} + \bar{x}_{(N\Phi)}) + 2} \right\}$
8.	$S_{x_{(N\Phi)}}$	$\beta_{2(x_{(N\Phi)})}$	$\Im^{exp(8)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} exp \left\{ \frac{S_{x_{(N\Phi)}}(\bar{x}_{1_{(N\Phi)}} - \bar{x}_{(N\Phi)})}{S_{x_{(N\Phi)}}(\bar{x}_{1_{(N\Phi)}} + \bar{x}_{(N\Phi)}) + 2\beta_{2(x_{(N\Phi)})}} \right\}$
9.	$S_{x_{(N\Phi)}}$	$C_{x_{(N\Phi)}}$	$\Im^{exp(9)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} exp \left\{ \frac{S_{x_{(N\Phi)}}(\bar{x}_{1_{(N\Phi)}} - \bar{x})}{S_{x_{(N\Phi)}}(\bar{x}_{1_{(N\Phi)}} + \bar{x}_{(N\Phi)}) + 2C_{x_{(N\Phi)}}} \right\}$
10.	$S_{x_{(N\Phi)}}$	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$\Im^{exp(10)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} exp \left\{ \frac{S_{x_{(N\Phi)}}(\bar{x}_{1_{(N\Phi)}} - \bar{x}_{(N\Phi)})}{S_{x_{(N\Phi)}}(\bar{x}_1 + \bar{x}_{(N\Phi)}) + 2\rho_{y_{(N\Phi)}x_{(N\Phi)}}} \right\}$
11.	$\beta_{2(x_{(N\Phi)})}$	1	$\Im^{exp(11)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} exp \left\{ \frac{\beta_{2(x_{(N\Phi)})}(\bar{x}_{1_{(N\Phi)}} - \bar{x}_{(N\Phi)})}{\beta_{2(x_{(N\Phi)})}(\bar{x}_{1_{(N\Phi)}} + \bar{x}_{(N\Phi)}) + 2} \right\}$
12.	$\beta_{2(x_{(N\Phi)})}$	$S_{x_{(N\Phi)}}$	$\Im^{exp(12)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} exp \left\{ \frac{\beta_{2(x_{(N\Phi)})}(\bar{x}_{1_{(N\Phi)}} - \bar{x}_{(N\Phi)})}{\beta_{2(x_{(N\Phi)})}(\bar{x}_{1_{(N\Phi)}} + \bar{x}_{(N\Phi)}) + 2S_{x_{(N\Phi)}}} \right\}$
13.	$\beta_{2(x)_{(N\Phi)}}$	$C_{x_{(N\Phi)}}$	$\Im^{exp(13)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} exp \left\{ \frac{\beta_{2(x_{(N\Phi)})}(\bar{x}_{1_{(N\Phi)}} - \bar{x}_{(N\Phi)})}{\beta_{2(x_{(N\Phi)})}(\bar{x}_{1_{(N\Phi)}} + \bar{x}_{(N\Phi)}) + 2C_{x_{(N\Phi)}}} \right\}$
14.	$\beta_{2(x_{(N\Phi)})}$	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$\Im^{exp(14)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} exp \left\{ \frac{\beta_{2(x_{(N\Phi)})}(\bar{x}_{1_{(N\Phi)}} - \bar{x}_{(N\Phi)})}{\beta_{2(x_{(N\Phi)})}(\bar{x}_{1_{(N\Phi)}} + \bar{x}_{(N\Phi)}) + 2\rho_{y_{(N\Phi)}x_{(N\Phi)}}} \right\}$
15.	$C_{x_{(N\Phi)}}$	1	$\Im^{exp(15)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} exp \left\{ \frac{C_{x_{(N\Phi)}}(\bar{x}_{1_{(N\Phi)}} - \bar{x}_{(N\Phi)})}{C_{x_{(N\Phi)}}(\bar{x}_{1_{(N\Phi)}} + \bar{x}_{(N\Phi)}) + 2} \right\}$
16.	$C_{x_{(N\Phi)}}$	$S_{x_{(N\Phi)}}$	$\Im^{exp(16)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} exp \left\{ \frac{C_{x_{(N\Phi)}}(\bar{x}_{1_{(N\Phi)}} - \bar{x}_{(N\Phi)})}{C_{x_{(N\Phi)}}(\bar{x}_{1_{(N\Phi)}} + \bar{x}_{(N\Phi)}) + 2S_{x_{(N\Phi)}}} \right\}$
17.	$C_{x_{(N\Phi)}}$	$\beta_{2(x_{(N\Phi)})}$	$\Im^{exp(17)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} exp \left\{ \frac{C_{x_{(N\Phi)}}(\bar{x}_{1_{(N\Phi)}} - \bar{x}_{(N\Phi)})}{C_{x_{(N\Phi)}}(\bar{x}_{1_{(N\Phi)}} + \bar{x}_{(N\Phi)}) + 2\beta_{2(x_{(N\Phi)})}} \right\}$
18.	$C_{x_{(N\Phi)}}$	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$\Im^{exp(18)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} exp \left\{ \frac{C_{x_{(N\Phi)}}(\bar{x}_{1_{(N\Phi)}} - \bar{x}_{(N\Phi)})}{C_{x_{(N\Phi)}}(\bar{x}_{1_{(N\Phi)}} + \bar{x}_{(N\Phi)}) + 2\rho_{y_{(N\Phi)}x_{(N\Phi)}}} \right\}$
19.	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	1	$\Im^{exp(19)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} exp \left\{ \frac{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}_{1_{(N\Phi)}} - \bar{x}_{(N\Phi)})}{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}_{1_{(N\Phi)}} + \bar{x}_{(N\Phi)}) + 2} \right\}$
20.	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$S_{x_{(N\Phi)}}$	$\Im^{exp(20)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} exp \left\{ \frac{\rho_{yx_{(N\Phi)}}(\bar{x}_{1_{(N\Phi)}} - \bar{x}_{(N\Phi)})}{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}_{1_{(N\Phi)}} + \bar{x}_{(N\Phi)}) + 2S_{x_{(N\Phi)}}} \right\}$
21.	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$\beta_{2(x_{(N\Phi)})}$	$\Im^{exp(21)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} exp \left\{ \frac{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}_{1_{(N\Phi)}} - \bar{x}_{(N\Phi)})}{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}_{1_{(N\Phi)}} + \bar{x}_{(N\Phi)}) + 2\beta_{2(x_{(N\Phi)})}} \right\}$
22.	$\rho_{y_{(N\Phi)}x_{(N\Phi)}}$	$C_{x_{(N\Phi)}}$	$\Im^{exp(22)}_{vk_{(N\Phi)}} = \bar{y}_{(N\Phi)} exp \left\{ \frac{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}_{1_{(N\Phi)}} - \bar{x}_{(N\Phi)})}{\rho_{y_{(N\Phi)}x_{(N\Phi)}}(\bar{x}_{1_{(N\Phi)}} + \bar{x}_{(N\Phi)}) + 2C_{x_{(N\Phi)}}} \right\}$

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Harnessing Tree Soft Set and Soft Computing Techniques' Capabilities in Bioinformatics: Analysis, Improvements, and Applications

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Abstract

A crucial aspect of bioinformatics is sequence comparison, which entails matching recently discovered biological sequences with previously identified sequences kept in databases. To find similarities between two or more nucleotide or amino acid sequences, sequence alignment organizes the sequences. Understanding the functional, structural, and evolutionary links between the sequences is made easier by looking at these areas of commonality. This study highlighted types of alignment. Also, proposed an effective methodology for deciding which algorithm can be utilized and satisfying the objective. Hence, Multi-Criteria Decision-Making (MCDM) techniques have been harnessed with Neutrosophic theory as a supporter in uncertain situations. Herein Single Value Neutrosophic Sets (SVNSs) as a type of uncertainty theory-Neutrosophic. This process requires a set of criteria leveraged in judgment. Also, Tree Soft Sets (TrSS) are applied for the first time to model the required criteria to facilitate the decision process. The hybrid techniques are applied to support stakeholders in making optimal decisions for optimal alignment algorithms among various algorithms such as pairwise and sequence algorithms. The results of the implementation of this decision technique indicated that multiple sequence alignment is the best compared with pairwise algorithms. Thus, we implemented multiple sequences in our study and employed logic programming to perform sequence matching. To ensure optimal alignment, the approach is tested on different sets of 16S rRNA gene of Actinobacteria (Streptomyces) sequences taken from NCBI. Then, the results are compared with MEGA.

Keywords: Bioinformatics; Multiple sequence alignment; Logic Programming; Multi-Criteria Decision-Making (MCDM); Single Value Neutrosophic Sets (SVNSs); Tree Soft Sets (TrSS)

1. Introduction

All living organism cells are composed of genetic codes that are passed from one generation to another. This is the reason for some living organisms are biologically similar and some are distinct. The genetic code can be represented as a sequence of alphabets, such as four base pairs of DNA and RNA, or twenty amino acids of protein [1]. These sequences are called biological sequences and over time a lot of changes called mutations occur in these sequences.

The field of Bioinformatics aims to align many biological sequences to derive their evolutionary relationships through comparative sequence analysis.

Bioinformatics applies computations to biological sequences to analyze and manipulate them. Sequence alignment (SA) is the most basic and essential module of computational bioinformatics and has varied applications in sequence assembly, sequence annotation, structural and functional prediction, and evolutionary or phylogeny relationship analysis.

SA is a field of research that focuses on the development of tools for comparing and finding similar sequences of (RNA, DNA, or amino acids) base pairs with the help of computers. The degree of similarity is used to measure gene and protein homology, classify genes and proteins, predict biological function, secondary and tertiary protein structure, detect point mutations, construct evolutionary trees, etc.

This study works into two phases. The first one is analyzing and examining existing alignment algorithms for deciding and utilizing optimal ones. In this phase, we are volunteering a combination of effective techniques to achieve the phase's objective. These techniques are utilized for preferencing and prioritizing SA alternatives based on a set of criteria. Hence, MCDM techniques are one of the utilized techniques in our study. Due to the ability of MCDM to treat this circumstance. TrSS model is volunteered for modeling the determined criteria and clarifying the relationship between these criteria. MCDM has been boosted by SVNSs in opacity circumstances Second phase: the results of phase one received by phase two to apply as optimal alternative for alignment. According to the results of the first phase, multiple sequences are the optimal alternative which applies for alignment.

Accordingly, we developed an algorithm that applied a logic program to align multiple biological sequences. SWI-Prolog (*http://www.swi-prolog.org*) is used to implement our proposed algorithm. Furthermore, we apply our implemented algorithm on eight different sets of 16S rRNA gene of Actinobacteria (Streptomyces) sequences: Seq1, Seq2, Seq3, Seq4, Seq5, Seq6, Seq7, and Seq8, were collected from GenBank at National Center for Biotechnology Information (NCBI). Also, we will use MEGA (Molecular Evolutionary Genetic Analysis Software for microcomputer), available at (*http://www. megasoftware.net*) to align the selected eight sequences. Each sequence set will be aligned using both methods fifty times and the execution times for all the fifty runs will be averaged.

Based on the average execution time, we compare the two methods to see which method reduces the execution time, speeds the performance, and decreases the memory location used to make the sequence alignment.

The objective of this study is summarized into several points:

1. Conducting surveys for prior studies and perspectives related to our scope.
2. Next, the results of the previous step entailed determining the effective and popular algorithms for alignment and we treated them as alternatives (Alts).

3. Leveraging decision techniques such as MCDM, SVNSs, and TrSS model to analyze the alternatives based on determined aspects and recommend the optimal.
4. We employ the recommended alternative to implement in our study.
5. We are observing the results of implementing the recommended algorithm and discussing it in the results and discussion section.

The outline of this study is as follows: Section 2 reviews the literature related to sequence alignment. The methodology used for sequence alignment of two methods is discussed in Section 3. Experimental results and their discussions are presented in Section 4. Finally, Section 5 discusses the obtained results. Finally, our conclusion of the study is represented in Section 6.

2. Prior Perspectives: Theoretical background related to our scope.

In this section, we conducted surveys for prior studies that embraced our notion. Firstly, we exhibited the principles for the concept alignment by showcasing its types and branches. Secondly, we collected the previous perspectives and studies from other scholars.

2.1 Comprehensive Visions for Sequence Alignment

A biological sequence is a sequence of characters from an alphabet. For DNA sequence, the character alphabet is {A, C, G, T}, for RNA sequence, the alphabet is {A, C, G, U}, and for RNA sequence is composed of A, C, G, U. For protein sequence, character set is {A, R, N, D, C, Q, E, G, H, I, L, K, M, F, P, S, T, W, Y, V}. Sequence alignment is the process of identifying one-to-one correspondence among subunits of sequences to measure the similarities among them. These similar regions provide functional, structural, and evolutionary information about the sequences under study. Aligned sequences are generally represented as rows within a matrix. Gaps ('-') are inserted between the characters so that identical or similar characters are aligned in successive columns. Gaps represent the insertion of a character in or a deletion of a character from a biological sequence. Sequence alignment of two biological sequences is called pair-wise sequence alignment, and in case more than two biological sequences are involved, it is called multiple sequence alignment [2]. The sequence alignment is divided into:

2.1.1 Global Alignment

Closely related sequences which are of the same length are very much appropriate for global alignment. Here, the alignment is carried out from the beginning till the end of the sequence to find out the best possible alignment as in Figure 1

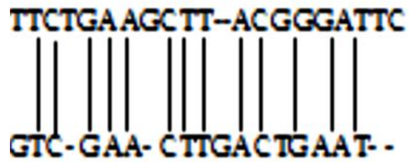

Fig. 1: Global Alignment of two biological sequences

2.1.2 Local Alignment

Sequences that are suspected to have similar or even dissimilar sequences can be compared with the local alignment method. It finds local regions with a high level of similarity as in Figure 2.

Pairwise Sequence Alignment is used to identify regions of similarity that may indicate functional, structural, and/or evolutionary relationships between two biological sequences (protein or nucleic acid). This type of alignment is based on numbers. Multiple sequence alignment (MSA) is the alignment of three or more biological sequences of similar length and therefore it is included in the alignment based on numbers. From the output of MSA applications, homology can be inferred and the evolutionary relationship between the sequences can be studied.

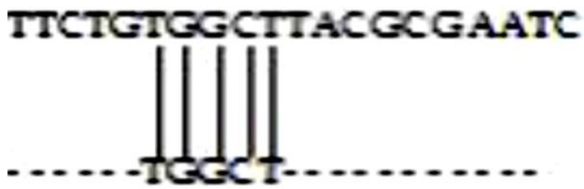

Fig. 2: Local Alignment of two biological sequences

2.2 Comprehensive Related Studies

Biological sequences databases are growing exponentially resulting in extensive demands on the implementation of new fast and efficient sequence alignment algorithms. Most of the work in the sequence alignment field has been primarily intended to provide new fast and efficient alignment methods.

The Needleman-Wunsch algorithm [3] employs a global alignment on two query sequences and is used widely in bioinformatics to align protein or nucleotide sequences. It uses a dynamic programming method to ensure the alignment is optimum by exploring all possible alignments and choosing the best.

While, the Smith–Waterman algorithm is a well-known algorithm for performing local sequence alignment that is for determining similar regions between two nucleotide or protein sequences

[4],[5]. Instead of looking at the total sequence, the Smith–Waterman algorithm compares segments of all possible lengths and optimizes the similarity measure.

In all the algorithms that had been proposed, the main objective of the researchers had been to apply different techniques to provide efficient alignment algorithms in terms of time and memory requirements.

Logic programming has been applied to develop logical databases to retrieve information about metabolic pathways, to identify and model genome structure [6] and to model protein interaction networks [7], [8]

3. Methodology: TrSS for Modelling Sequence Alignment Algorithms and Selection Procedures

Herein, we are leveraging the soft set notion represented in TrSS which was introduced by Smarandache. [9]. In TrSS we are clarifying and modeling various algorithms of sequence alignment (SA) into nodes at some levels. The purpose of modelling the determined algorithms into TrSS for make optimal decisions for selecting optimal and appropriate algorithms in our study. Hence, we are taking advantage of MCDM techniques and utilizing these techniques in the constructed tree to bolster us in making optimal decisions as clarified in the following steps:

Step 1: Construct a Tree and determine its nodes.

- ✓ At level 1: this level includes main aspects of sequence alignment {Matching Efficiency Node 1(N_1), Producing Phylogenetic Trees =Node2 (N_2), Prediction Efficiently= Node3 (N_3)}.
- ✓ At level 2: this level is divided into various branches based on previous branches of N_1, N_2 , and N_3 . Thereby, {Identify unknown sequence =$N_{1.1}$, Accuracy=$N_{1.2}$ } are considering sub-node of N_1. Also, {Finding out the relationship between the matched sequences = $N_{2.1}$, Easy of analyzing= $N_{2.2}$}are considering sub-node of N_2 . Finally, {Predicting protein efficiently= $N_{3.1}$, Predicting gene locations efficiently= $N_{3.2}$ } are considering sub-node of N_3.

Step 2: Determining Influential Aspects.

- ✓ in this step, the crucial factor in decision-making is determining the influential factors which impact the decision process. In this study, the decision process is conducted on three main aspects and six sub-aspects.
- ✓ The role of MCDM techniques is starting to work. Herein, we are employing entropy as a technique of MCDM to analyze determined sequence alignment's aspects. For boosting entropy, we are merging Neutrosophic theory for bolstering entropy in ambiguous situations. This theory is proposed by Smarandache [10]. Due to the ability of neutrosophic to apply in indeterminacy situations as mentioned

in [11] through measuring possible degrees of membership as truth, false, also indeterminancy a. Hence, we are implementing SVNSs as in [12] as a type of Neutrosophic theory. The aspects' weights are derived from entropy analysis and these weights have been obtained through the following several steps.

Step 2.1: We had an encounter with three specialists in this field to prioritize the determined alternative algorithms through determined aspects in Figure 3.

Step 2.2: Resulted from the encounter with three Neutrosophic decision matrices for three specialists. These matrices formed as in Eq. (1)

$$X^n = \begin{pmatrix} Asp_{11}^n & Asp_{12}^n & \cdots & Asp_{1n}^n \\ \vdots & & \ddots & \vdots \\ Asp_{m1}^n & Asp_{m2}^n & \cdots & Asp_{mn}^n \end{pmatrix} \quad (1)$$

Where:

X^n indicated to prioritize each specialist $-$ based decision matrix.

Step 2.3: Eq. (2) is employed for transforming neutrosophic matrices into crisp matrices.

$$\mathbf{s}(Q_{ij}) = \frac{(2 + Tr - Fl - In)}{3} \quad (2)$$

Where:

Tr, Fl, In refer to truth, false, and indeterminacy respectively.

Step 2.4: Crisp matrices are amalgamated based on Eq.(3) into a single matrix so-called an aggregated decision matrix.

$$\partial_{ij} = \frac{(\sum_{j=1}^{N} Q_{ij})}{S} \quad (3)$$

Where:

Q_{ij} refers to the value of the criterion in the matrix, and S refers to the number of specialists.

Step 2.5: Eq. (4) is normalizing an aggregated matrix.

$$Nor_{ij} = \frac{\partial_{ij}}{\sum_{j=1}^{n} \partial_{ij}} \quad (4)$$

Where:

$\sum_{j=1}^{n} \partial_{ij}$ represents the sum of each aspect in an aggregated matrix per column.

Step 2.6: Entropy of the normalized matrix is computed through Eq. (5).

$$\mathrm{En}_{j=-h\sum_{i=1}^{n}\mathrm{Nor}_{ij}}\ \ln \mathrm{Nor}_{ij} \tag{5}$$

Where:

$$h = \frac{1}{\ln(\mathrm{Alts})} \tag{6}$$

Step 3: Reaching the optimal Decision for sequence alignment algorithm.

- ✓ This is the final step in the decision-making process, selecting the optimal algorithm between two SA algorithms. Alternative 1(Alt_1) = Pairwise alignment; Alternative 2(Alt_2) = Multiple SA algorithms.
- ✓ COPRAS is employed in this study as a technique of MCDM with hybridization of SVNs for ranking and prioritizing two Alts based on aspects and sub-aspects of SA. This process facilitates decision-making for optimal Alt. The hybridization process is implemented as follows:

  Step 3.1: Leveraging normalized matrix produced from previous step two and aspects' weights generated from entropy based on SVNSs to produce a weighted decision matrix through following Eq. (7).

$$\mathscr{B}_{ij} = w_i * \mathrm{Nor}_{ij} \tag{7}$$

Step 3.2: Eqs (8) and (9) are employed for computing the Sum of the weighted decision matrix.

$$S_{+i} = \sum_{j=1}^{n} \mathcal{B}_{+ij}\,,\ \text{for beneficial criteria} \tag{8}$$

$$S_{-i} = \sum_{j=1}^{n} \mathscr{B}_{-ij}\,,\ \text{for nonbeneficial criteria} \tag{9}$$

Step 3.3: the relative importance of alternatives is calculated based on Eq. (10).

$$Q_i = s_{+i} + \frac{s_{-\min}\ \sum_{i=1}^{m} s_{-i}}{s_{-i}\ \sum_{i=1}^{m}(s_{-m/}s_{-i})} \tag{10}$$

where I = 1, 2,…,m, and $s_{-m} = s_{-i}$ all aspects and sub-aspects are beneficial.

Step 3.4: quantity utility U_i for each Alt is based on Eq. (11) to rank Alts.

$$U_i = \left[\frac{Q_i}{Q_{\max}}\right] \times 100\% \tag{11}$$

4. Comprehensive Analysis

Herein, this section is divided into two sub-sections, each one responsible for exhibiting results and Consequent to each other. The first sub-section involving the results of the application of the methodology has been exhibited. The second sub-section is prepared based on the results of the first sub-section.

4.1 Analysis of Implementing Proposed Methodology

Herein, we discuss the results of implementing entropy-COPRAS under SVNSs based on TrSS. The resulting Alt as optimal SA is applied in this study.

4.1.1 Encounter with specialists: three specialists contributed to rating and prioritizing two Alts based on aspects and sub-aspects of SA which were modelled in the TrSS model.

4.1.2 Analyzing and obtaining weights for aspects and sub-aspects: this step involves two dimensions. First dimension, we obtain the main aspects' weights. The second dimension is obtaining sub-aspects' weights.

- **First dimension: Extracting the main aspects' weights Procedures**.

1. Three constructed neutrosophic decision matrices based on the SVNS scale which applied in [13] are transformed into crisp matrices based on Eq.(2).
2. These crisp matrices are amalgamated into the aggregated matrix by Eq. (3) as in Table 1.
3. Table 2 represents a normalized matrix based on Eq. (4).
4. Entropy for normalized matrix is calculated by Eq. (5) as in Table 3.
5. Final Aspects' weights are exhibited in Figure 3 through Eq. (6). This Figure indicates that main Aspect 1 outperforms main Aspect 2 and main Aspect 2.
6.

Table 1. An aggregated matrix of Aspects at level 1 for N1-N2

	ASP_1	ASP_2	ASP_3
Alt_1	0.226666667	0.594444444	0.777777778
Alt_2	0.494444444	0.36	0.66

Table 2. Normalized matrix of Aspects at level 1 for N1-N2

	ASP_1	ASP_2	ASP_3
Alt_1	0.314329738	0.622817229	0.540958269
Alt_2	0.685670262	0.377182771	0.459041731

Table 3. Entropy of Normalized matrix of Aspects at level 1 for N1-N2

	ASP_1	ASP_2	ASP_3
Alt_1	-0.363777805	-0.29490531	-0.3323719
Alt_2	-0.258743457	-0.36776278	-0.3574164
$\sum_{i=1}^{m} X_{ij}$	-0.622521262	-0.662668097	-0.689788258
$-h\sum_{i=1}^{m} X_{ij} \ln X_{ij}$	0.568592766	0.540771009	0.521976737

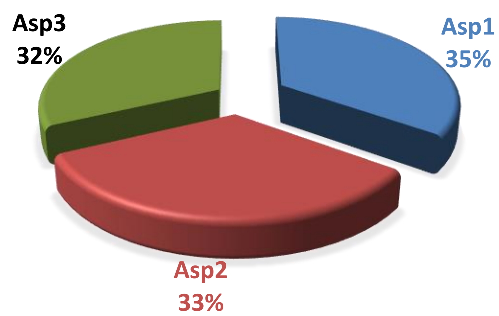

Fig 3. Weights of Main Aspects in Level 1 for N1-N2

- **Second dimension: Extracting Sub- aspects' weights Procedures**.

1. Three Neutrosophic decision matrices are constructed for {Sub-Asp $_{1.1}$, Sub-Asp $_{1.2}$}; { Sub-Asp $_{2.1}$, Sub-Asp $_{2.2}$ }; { Sub-Asp $_{3,1}$, Sub-Asp $_{3.2}$ } and transformed into crisp matrices based on Eq.(2).
2. Eq.(3) is exploited for aggregating each pair of sub_Aspects into an aggregated matrix belonging to the main node (Aspect) at level 1.
3. Figure 4 indicates that sub_Aspect 1.1 outperforms sub_Aspect 1.2.
4. Figure 5 indicates that sub_Aspect 2.1 outperforms sub_Aspect 2.2.
5. Figure 6 indicates that sub_Aspect 3.1 outperforms sub_Aspect 3.2.

4.1.3 Ranking and prioritizing SA algorithms

1. Eq. (7) plays a critical role in the normalized matrix to generate a weighted decision matrix as in Table 4.
2. Eq. (8) is applied to obtain a sum weighted where all Aspects are beneficial
3. through Eq. (11), Quantity utility U_i for each alternative is calculated to rank the alternatives and results illustrated in Figure 7. Alt $_2$ (Multiple Alignment algorithm) is an optimal algorithm.

Table 4. Weighted decision matrix

	ASP$_1$	**ASP$_2$**	**ASP$_3$**
Alt$_1$	0.109557517	0.206456898	0.173089327
Alt $_2$	0.238985759	0.125031841	0.146878658

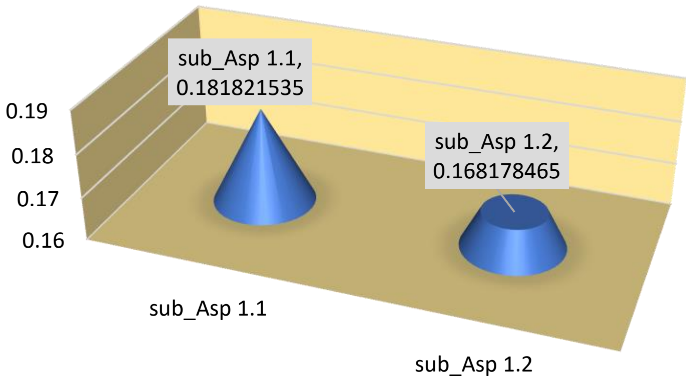

Fig.4. Final Weights of Sub Aspects 1.1 to 1.2 in Level 2

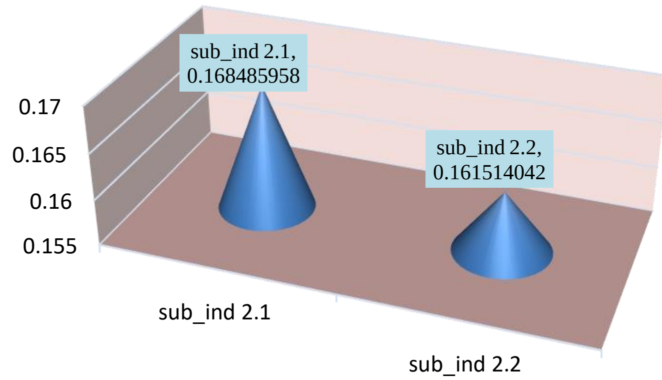

Fig.5. Final Weights of Sub Aspects 2.1 to 2.2 in Level 2

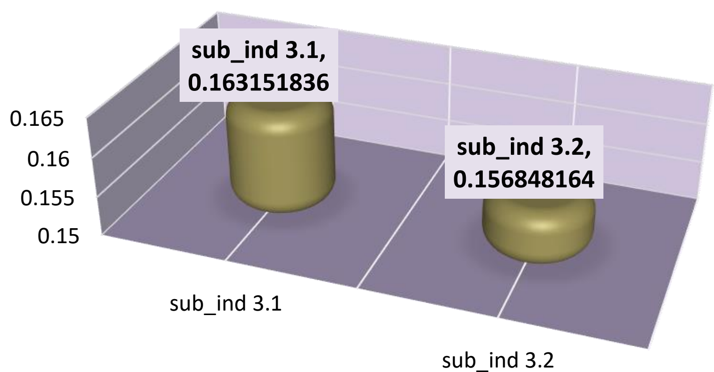

Fig.6. Final Weights of Sub Aspects 3.1 to 3.2 in Level 2

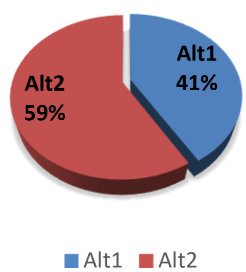

Fig. 7. Ranking two sequence algorithms

4.2 Analysis of Implementing Multiple Sequence Algorithm

Based on the results of the implementation of MCDM techniques under SVNS based on TrSS, the multiple sequence algorithm outperforms another algorithm. Hence, we used multiple sequences for aligning to determine the similarity between the sequences, and then based on their degree of similarity the sequences were aligned.

In this study, we applied the developed algorithm given in [14]. The algorithm described an application for the logic programming paradigm for large-scale comparison of complete microbial genomes. We used SWI-Prolog language to implement our proposed algorithm. Where Prolog is a general-purpose logic programming language associated with artificial intelligence and computational linguistics.

4.2.1 Implementing the Algorithm

We have divided the implementation of the algorithm into three stages, the First stage, extracting genome information from GenBank, the Second stage, identifying homologous genes using BLAST [15], and the Third stage, alignment of homologous gene pairs using the Smith-Waterman software. The Smith-Waterman algorithm[16],[17] is a matrix-based dynamic programming technique to align two sequences. Smith–Waterman algorithm is a local sequence alignment; that is, for determining similar regions between two strings or nucleotide or protein sequences. Instead of looking at all the sequences, the Smith–Waterman algorithm only compares segments of all possible lengths and then optimizes the similarity measure.

4.2.2 Obtained Sequences from Genbank

Eight different sets of 16S rRNA gene of Actinobacteria (Streptomyces) sequences: Seq1, Seq2, Seq3, Seq4, Seq5, Seq6, Seq7, and Seq8, were collected from Genbank at NCBI (see Appendix). Identification of bacteria by using the molecular method (16S rDNA sequence) is more accurate than the traditional biochemical methods. The use of 16S rRNA gene sequences to study bacterial phylogeny and taxonomy has been by far the most common housekeeping genetic marker used for some reasons. These reasons include:

(i) It is present in almost all bacteria, often existing as a multigene family or operons.
(ii) The function of the 16S rRNA gene over time has not changed, suggesting that random sequence changes are a more accurate measure of time (evolution); and
(iii) The 16S rRNA gene (1,500 bp) is large enough for informatics purposes [18].

Details of the obtained sequence sets are listed in Table 5.

Table 5: Identification of Streptomyces.

Sequences	Streptomyces isolates	GenBank number	Base pair (bp)
Seq1	S. albidofuscus	Later name is S. pyridomyceticus BankIt1507621 JQ625331	900
Seq2	S. ambofaciens	BankIt1507642 JQ625332	703
Seq3	S. canarius	BankIt1507650 JQ625337	849
Seq4	S. chibaensis	Later name is S. corchorusii BankIt1507649 JQ625336	851
Seq5	S. coelicolor	BankIt1507648 JQ625335	944
Seq6	S. corchorusii	BankIt1507647 JQ625334	834
Seq7	S. nigrifaciens	Later name is S. flavovirens BankIt1507149 JQ625330	716
Seq8	S. parvullus	BankIt1507645 JQ625333	787

In this study, we will use MEGA to align the selected sequences. MEGA software is an integrated suite of tools for statistics-based comparative analysis of molecular sequence data based on evolutionary principles [19], [20]. MEGA is being used by biologists in a large number of laboratories for reconstructing the evolutionary histories of species and inferring the extent and nature of selective forces shaping the evolution of genes and species. Additionally, MEGA is used in many classrooms as a tool for teaching the methods used in evolutionary bioinformatics.

4.2.3 Results of Multiple Sequence

We have been extracting an algorithm that employs logic programming to measure the similarity of sequences. To guarantee the optimal alignment of the sequences we are using prolog language.

The algorithm is tested on various sets of real genome sequences taken from NCBI, and the processing time for the alignment process on these data sets has been computed.

To evaluate the performance of this approach, eight sets (Seq1, Seq2, Seq3, Seq4, Seq5, Seq6, Seq7, and Seq8) of 16S rRNA gene of Actinobacteria (Streptomyces) sequences have been used.

Data sets are used to find out the effect of varying the number of sequences being aligned on the processing time. The alignment of eight sequences by using MEGA is shown in Fig. 8.

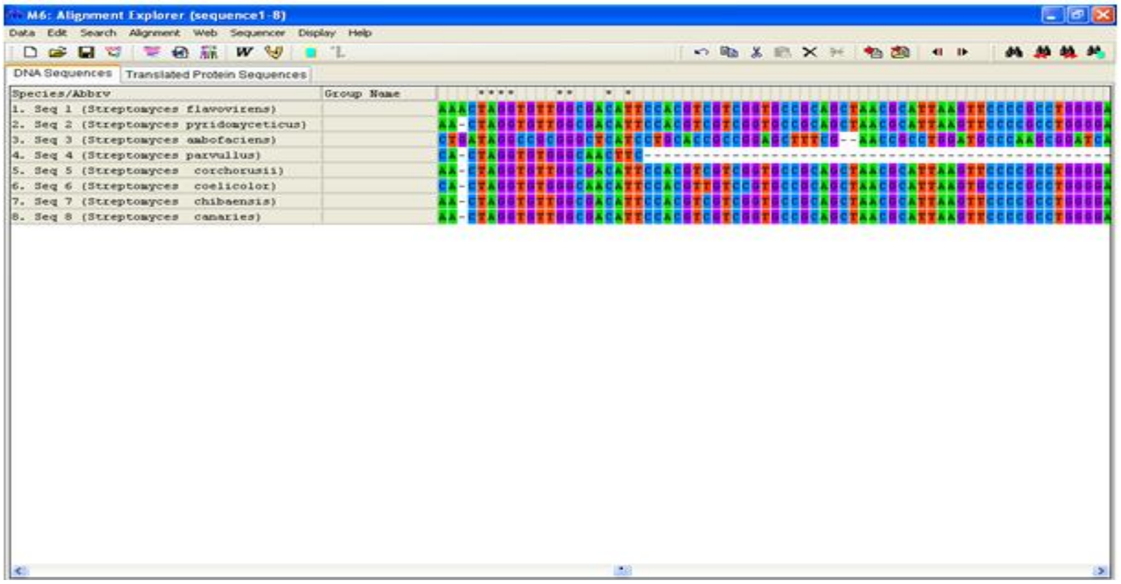

Fig. 8: Alignment of Eight Sequences.

To compare the amount of time needed to process the two methods of alignment being discussed, the processing time has been calculated. Each sequence set has been aligned using both methods fifty times and the execution times for all the fifty runs have been averaged. This average execution time has been used for the comparison. The average processing time for eight sets (Seq1, Seq2, Seq3, Seq4, Seq5, Seq6, Seq7 and Seq8) of 16S rRNA gene of Actinobacteria (Streptomyces) sequences are tabulated in Table 6 and Table 7 respectively.

Table 6: Average processing time (in seconds) for sequences Seq1, Seq2, Seq3, and Seq4.

Number of Sequences	Seq1 (Length:900 bp)		Seq2 (Length: 703 bp)		Seq3 (Length: 849 bp)		Seq4 (Length: 851 bp)	
	Logic Programming Method	MEGA Method	Logic Programming Method	MEGA Method	Logic Programming Method	MEGA Method	Logic Programming Method	MEGA Method
10	20.25	15.64	3.98	4.30	15.12	13.95	16.86	14.98
20	35.36	29.40	6.95	7.59	28.26	27.12	29.52	28.10
50	50.21	45.43	26.62	27.98	43.20	41.56	45.34	43.87
70	66.52	58.23	37.80	39.13	60.53	59.96	62.20	59.93
100	109.32	96.30	69.76	71.05	89.82	87.16	91.25	89.75
120	123.31	116.54	85.34	86.98	115.34	112.19	117.52	111.63
150	226.62	207.14	130.65	132.12	207.20	199.92	209.89	197.23

Table 7: The average processing time (in seconds) for sequences Seq5, Seq6, Seq7, and Seq8.

Number of Sequences	Seq5 (Length:944 bp)		Seq6(Length: 834 bp)		Seq7 (Length: 716 bp)		Seq8 (Length: 787 bp)	
	Logic Programming Method	MEGA Method	Logic Programming Method	MEGA Method	Logic Programming Method	MEGA Method	Logic Programming Method	MEGA Method
10	24.36	19.15	12.38	10.23	4.65	5.75	8.5	7.15
20	39.65	32.76	26.50	23.45	7.38	8.93	19.23	17.33
50	56.37	51.78	41.82	36.89	28.65	29.95	35.82	33.56
70	71.44	66.16	57.15	51.14	40.10	41.87	49.5	47.12
100	125.82	119.24	87.28	80.55	71.28	72.89	79.83	77.15
120	130.12	124.92	112.23	108.89	87.51	88.98	95.89	93.25
150	236.22	229.19	205.81	198.46	132.72	134.12	143.29	141.65

In the following, we give the line graph for the average processing time over fifty runs of both the methods on the eight sequences of (Seq1, Seq2, Seq3, Seq4, Seq5, Seq6, Seq7 and Seq8) of 16S rRNA gene of Actinobacteria (Streptomyces) in Fig. 9, Fig. 10, Fig. 11, Fig. 12, Fig. 13, Fig. 14, Fig. 15 and Fig. 16 respectively.

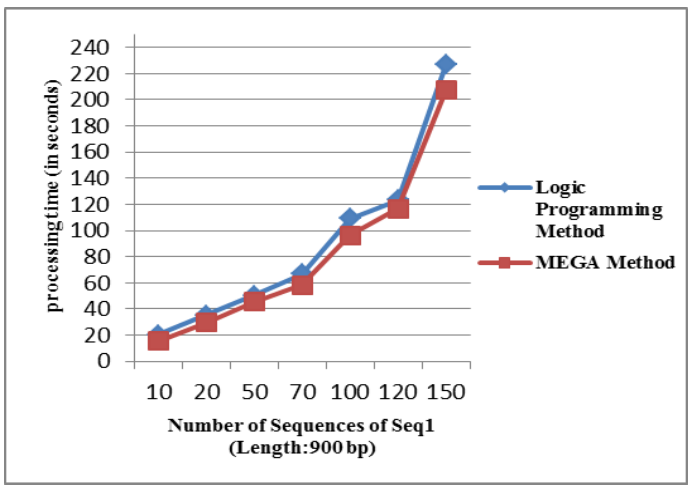

Fig. 9: Average processing time's line graph for Logic Programming and MEGA methods on Seq1.

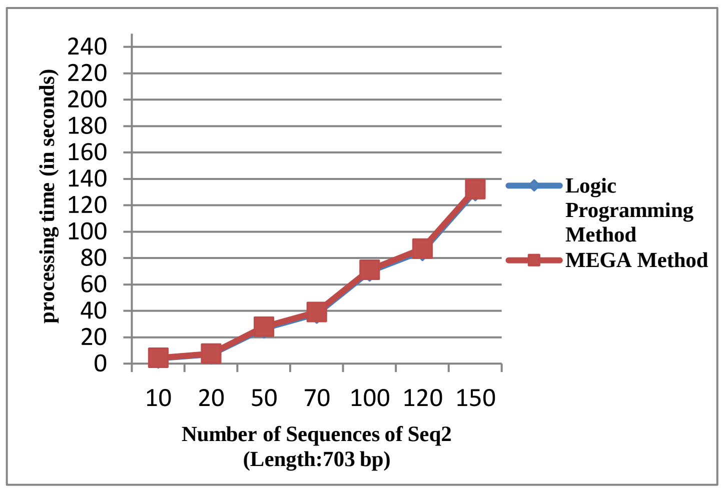

Fig. 10: Average processing time's line graph for Logic Programming and MEGA methods on Seq2.

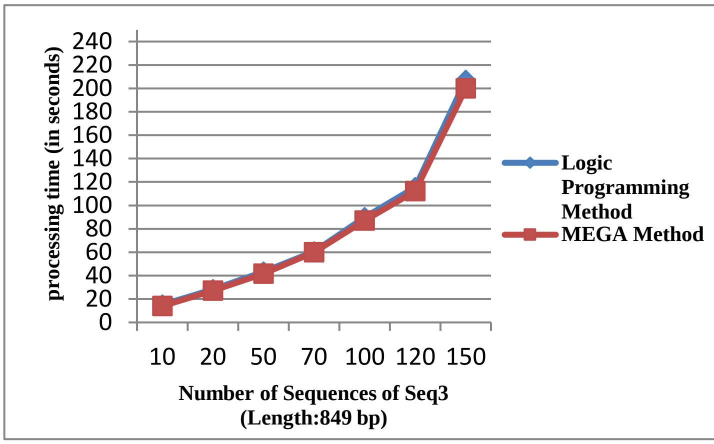

Fig. 11: Average processing time's line graph for Logic Programming and MEGA methods on Seq3

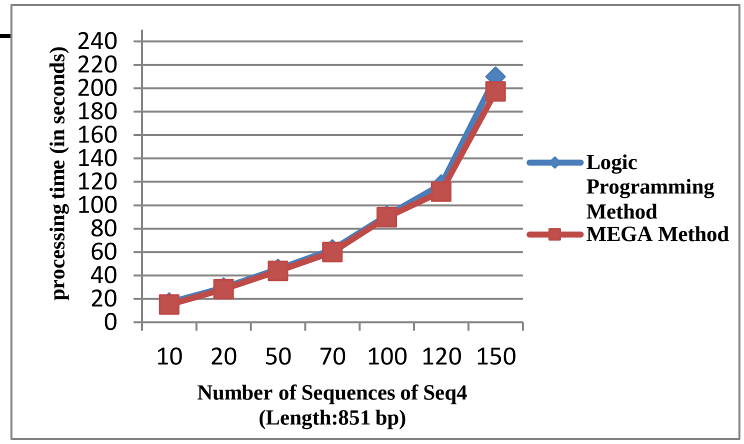

Fig. 12: Average processing time's line graph for Logic Programming and MEGA methods on Seq4

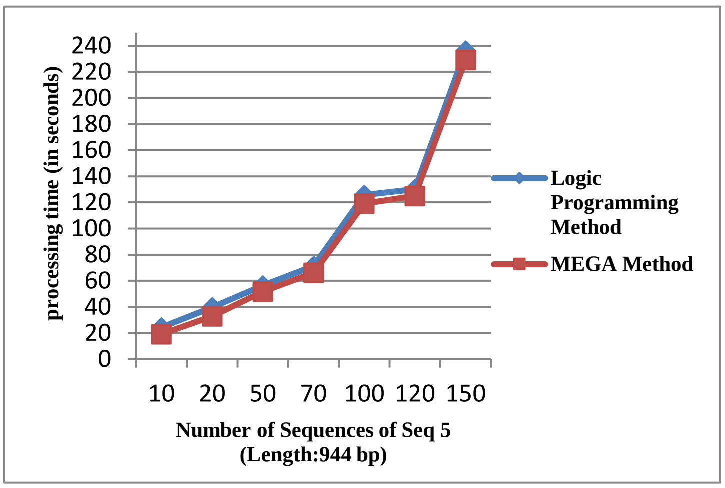

Fig. 13: Average processing time's line graph for Logic Programming and MEGA methods on Seq 5.

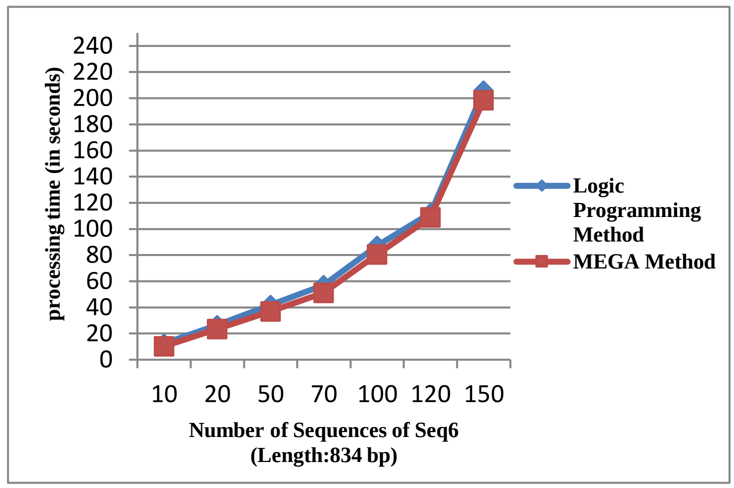

Fig. 14: Average processing time's line graph for Logic Programming and MEGA methods on Seq6

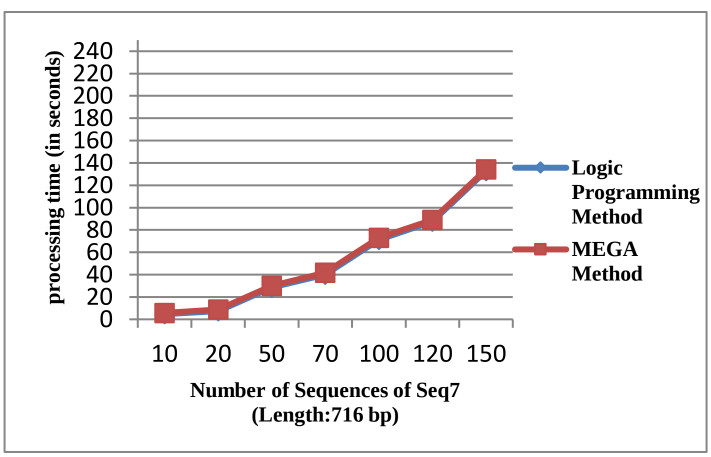

Fig. 15: Average processing time's line graph for Logic Programming and MEGA methods on Seq7.

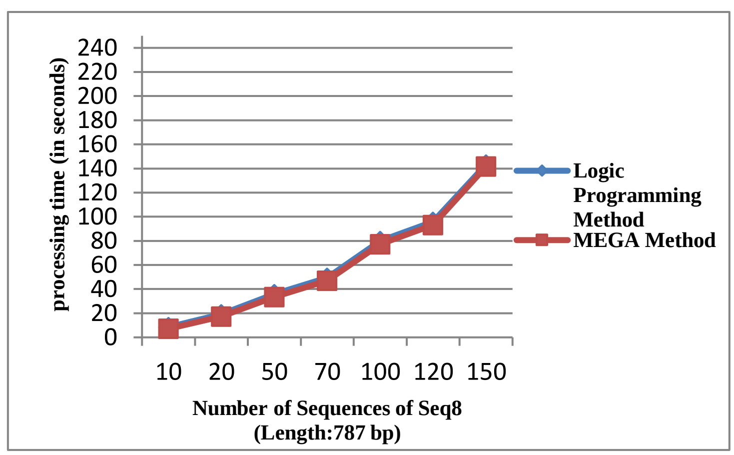

Fig. 16: Average processing time's line graph for Logic Programming and MEGA methods on Seq8.

5. Discussion

In this study, we leveraged the ability of MCDM techniques (i.e. entropy -COPRAS) under the authority of SVNSs for supporting MCDM in indeterminacy situations the objective of implementing these techniques is to recommend the optimal algorithm that we can utilize in our problem. The recommendation occurs based on a prioritizing process for a set of criteria/aspects and sub-aspects. Hence, we are modeling the decision-making process by using TrSS to express relationships between main aspects and sub-aspects. The results from the implementation of these techniques in the decision process indicated that multiple sequence algorithms in contrast to pairwise algorithms. Thus, we are implementing multiple sequences in our study. The experiments of applying multiple sequences for data sets in Table 2 and Table 3 show the effect of variation in the number of sequences on the processing time of the two alignment methods. From the processing times of sequences Seq2 and Seq7 in Table 5 and Table 6, we obtain that the processing time of the Logic Programming method takes less time as compared to the MEGA method for the sequences of length in the range 703-716 bp.

From the processing times of sequences Seq1, Seq3, Seq4, Seq5, Seq6, and Seq8 in Table 2 and Table 3, we obtain that the processing time of the Logic Programming method is higher than the MEGA method of sequences of length 787 – 944 bp.

From the obtained experimental results, we conclude that if the number and length of involved sequences are large, the Logic Programming method is very inefficient. Furthermore, we have that the Logic Programming method outperforms the MEGA method if the length of involved sequences is in the range 703-716 bp.

6. Conclusion

One of the most important steps in comparing biological sequences is thought to be sequence alignment. To find similarities between two or more nucleotide or amino acid sequences, sequence alignment organizes the sequences. Understanding the functional, structural, and evolutionary links between the sequences is made easier by looking at these areas of commonality.

Hence, utilizing suitable SA is critical. Herein, we discussed the methodology for selecting an optimal algorithm to perform the task of alignment. We utilized TrSS for the first time for modeling the aspects which contributed to the selection process. Also, MCDM worked with SVNSs to serve our objective. These techniques recommended multiple alignments for the alignment process.

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Appendix

The Eight sets sequences Seq1, Seq2, Seq3, Seq4, Seq5, Seq6, Seq7 and Seq8, are given as follows:

Seq 1

GTTGGTGGGGTGATGGCCTACCAAGGCGACGACGGGTAGCCGGCCTGAGAGGGCGACCGGCCACACTGG
GACTGAACACGGCCCAGACTCCTACGGGAGGCAGCAGTGGGGAATATTGCACAATGGGCGAAAGCCTGA
TGCAGCGACGCCGCGTGAGGGATGACGGCCTTCGGGTTGTAAACCTCTTTCAGCAGGGAAGAAGCGAAA
GGGACGGTACCTGCAGAAGAAGCCTTCTTGAATAACTACGTGCCAGCAGCCGCGGTAATACGTAGGGCG
CAAGCGTTGTCCGGAATTATTGGGCGTAAAGAGCTCGTAGGCGGCTTGTCACGTCGGATGTGAAAGCCC
GGGGCTTAACCCCGGGTCTGCATTCGATACGGGCTAGCTAGAGTGTGGTAGGGGAGATCGGAATTCCTG
GTGTAGCGGTGAAATGCGCAGATATCAGGAGGAACACCGGTGGCGAAGGCGGATCTCTGGGCCATTACT
GACGCTGAGGAGCGAAAGCGTGGGGAGCGAACAGGATTAGATACCCTGGTAGTCCACGCCGTAAACGTT
GGGAAACTAGGTGTTGGCGACATTCCACGTCGTCGGTGCCGCAGCTAACGCATTAAGTTCCCCGCCTGG
GGAGTACGGCCGCAAGGCTAAAACTCAAAGGAATTGACGGGGGCCCGCACAAGCAGCGGAGCATGTGGC
TTAATTCGACGCAACGCGAAGAACCT

Seq 2

CGCATGGGGGTTGGTGTAAAGCTCCGGCGGTGCAGGATGAGCCCGCGGCCTATCAGCTTGTTGGTGGGG
TAATGGCCTACCAAGGCGACGACGGGTAGCCGGCCTGAGAGGGCGACCGGCCACACTGGGACTGAGACA
CGGCCCAGACTCCTACGGGAGGCAGCAGTGGGGAATATTGCACAATGGGCGAAAGCCTGATGCAGCGAC
GCCGCGTGAGGGATGACGGCCTTCGGGTTGTAAACCTCTTTCAGCAGGGAAGAAGCGCAAGTGACGGTA
CCTGCAGAAGAAGCACCGGCTAACTACGTGCCAGCAGCCGCGGTAATACGTAGGGTGCGAGCGTTGTCC
GGAATTATTGGGCGTAAAGAGCTCGTAGGCGGCCTGTCGCGTCGGATGTGAAAGCCCGGGGCTTAACCC
CGGGTCTGCATTCGATACGGGCAGGCTAGAGTGTGGTAGGGGAGATCGGAATTCCTGGTGTAGCGGTGA
AATGCGCAGATATCAGGAGGAACACCGGTGGCGAAGGCGGATCTCTGGGCCATTACTGACGCTGAGGAG
CGAAAGCGTGGGGAGCGAACAGGATTAGATACCCTGGTAGTCCACGCCGTAAACGTTGGGAACTAGGTG
TTGGCGACATTCCACGTCGTCGGTGCCGCAGCTAACGCATTAAGTTCCCCGCCTGGGGAGTACGGCCGC
AAGGCTAAAACTCAAAGGAATTGACGGGGGCCCGCACAAGCAGCGGAGCATGTGGCTTAATTCGACAGA
CCAACGCGAAGAACCTTACCAAGGCTTGACATATACCGGAAACGGCTAGAGATAGTCGCCCCCTTGTGG
TCGGTATACAGGTGGTGCATGGTTGTCGTCAGCTCGTGTCGTGAGATGTTGGGTTAAGTCCCGCAACGA
GCG

Seq 3

GCTCCTCAGCGTCAGTATCGGCCCAGAGATCCGCCTTCGCCACCGGTGTTCCTCCTGATATCTGCGCAT
TTCACCGCTACACCAGGAATTCCGATCTCCCCTACCGAACTCTAGCCTGCCCGTATCGACTGCAGACCC
GGGGTTAAGCCCCGGGCTTTCACAACCGACGCGACAAGCCGCCTACGAGCTCTTTACGCCCAATAATTC
CGGACAACGCTCGCGCCCTACGTATTACCGCGGCTGCTGGCACGTAGTTAGCCGGCGCTTCTTCTGCAG
GTACCGTCACTTGCGCTTCTTCCCTGCTGAAAGAGGTTTACAACCCGAAGGCCGTCATCCCTCACGCGG
CGTCGCTGCATCAGGCTTGCGCCCATTGTGCAATATTCCCCACTGCTGCCTCCCGTAGGAGTCTGGGCC
GTGTCTCAGTCCCAGTGTGGCCGGTCGCCCTCTCAGGCCGGCTACCCGTCGTCGCCTTGGTGAGCCGTT
ACCTCACCAACAAGCTGATAGGCCGCGGGCTCATCCTGCACCGCCGGAGCTTTCGAACCGCCTGGATGC
CCAAGCGGATCAGTATCCGGTATTAGACCCCGTTTCCAGGGCTTGTCCCAGAGTGCAGGGCAGATTGCC
CACGTGTTACTCACCCGTTCGCCACTAATCCCCACCGAAGTGGTTCATCGTTCGACTTGCATGTGTTAA
GCACGCCGCCAGC

Seq 4

ACGAACGCTGGCGGCGTGCTTAACACATGCAAGTCGAACGATGAACCACTTCGGTGGGGATTAGTGGCG
AACGGGTGAGTAACACGTGGGCAATCTGCCCTGCACTCTGGGACAAGCCCTGGAAACGGGGTCTAATAC
CGGATACTGACCTTCACGGGCATCTGTGAAGGTCGAAAGCTCCGGCGGTGCAGGATGAGCCCGCGGCCT
ATCAGCTTGTTGGTGAGGTAATGGCTCACCAAGGCGACGACGGGTAGCCGGCCTGAGAGGGCGACCGGC
CACACTGGGACTGAGACACGGCCCAGACTCCTACGGGAGGCAGCAGTGGGGAATATTGCACAATGGGCG
AAAGCCTGATGCAGCGACGCCGCGTGAGGGATGACGGCCTTCGGGTTGTAAACCTCTTTCAGCAGGGAA
GAAGCGAAAGTGACGGTACCTGCAGAAGAAGCGCCGGCTAACTACGTGCCAGCACCGCGGTAATACGTA
GGGCGCAAGCGTTGTCCGGAATTATTGGGCGTAAAGAGCTCGTAGGCGGCTTGTCACGTCGGTTGTGAA
AGCCCGGGGCTTAACCCCGGGTCTGCAGTCGATACGGGCAGGCTAGAGTTCGGTAGGGGAGATCGGAAT
TCCTGGTGTAGCGGTGAAATGCGCAGATATCAGGAGGAACACCGGTGGCGAAGGCGGATCTCTGGGCCG
ATACTGACGCTGAGGAGCGAAAGCGTGGGGAGCGAACAGGATTAGATACCCTGGTAGTCCACGCCGTAA
ACGGTGGGCACTAGGTGTGGGCAACTTC

Seq 5

AGTGGCGGACGGGTGAGGAATACATCGGAATCTACCTTGTCGTGGGGGATAACGTCTGGAAACGGGGTC
TAATACCGGATACCACTCTCGCAGGCATCTGTGAGGGTTGAAAGCTCCGGCGGTGAAGGATGAGCCCGC
GGCCTATCAGCTTGTTGGTGAGGTAATGGCTCACCAAGGCGACGACGGGTAGCCGGCCTGAGAGGGCGA

CCGGCCACACTGGGACTGAGACACGGCCCAGACTCCTACGGGAGGCAGCAGTGGGGAATATTGCACAAT
GGGCGAAAGCCTGATGCACGACGCCGCGTGAGGGATGACGGCCTTCGGGTTGTAAACCTCTTTCAGCAG
GGAAGAAGCGAAAGTGACGGTACCTGCAGAAGAAGCGCCGGCTAACTACGTGCCAGCAGCCGCGGTAAT
ACGTAGGGCGCAAGCGTTGTCCGGAATTATTGGGCGTAAAGAGCTCGTAGGCGGCTTGTCACGTCGGGT
GTGAAAGCCCGGGGCTTAACCCCGGGTCTGCATTCGATACGGGCTAGCTAGAGTGTGGTAGGGGAGATC
GGAATTCCTGGTGTAGCGGTGAAATGCGCAGATATCAGGAGGAACACCGGTGGCGAAGGCGGATCTCTG
GGCCATTACTGACGCTGAGGAGCGAAAGCGTGGGGAGCGAACAGGATTAGATACCCTGGTAGTCCACGC
CGTAAACGGTGGGAACTAGGTGTTGGCGACATTCCACGTCGTCGGTGCCGCAGCTAACGCATTAAGTTC
CCCGCCTGGGGAGTACGGCCGCAAGGCTAAAACTCAAAGGAATTGACGGGGGCCCGCACAAGCAGCGGA
GCATGT

Seq 6

TATTGGGCGTAAAGAGCTCGTAGGCGGCTTGTCAGCGTCGGTTGTGAAAGCCCGGGGCTTAACCCCGGG
TCTGCAGTCGATACGGGCAGGCTAGAGTTCGGTAGGGGAGATCGGAATTCCTGGTGTAGCGGTGAAATG
CGCAGATATCAGGAGGAACACCGGTGGCGAAGGCGGATCTCTGGGCCGATACTGACGCTGAGGAGCGAA
AGCGTGGGGAGCGAACAGGATTAGATACCCTGGTAGTCCACGCCGTAAACGGTGGGCACTAGGTGTGGG
CAACATTCCACGTTGTCCGTGCCGCAGCTAACGCATTAAGTGCCCCGCCTGGGGAGTACGGCCGCAAGG
CTAAAACTCAAAGGAATTGACGGGGGCCCGCACAAGCGGCGGAGCATGTGGCTTAATTCGACGCAACGC
GAAGAACCTTACCAAGGCTTGACATACACCGGAAACGTCTGGAGACAGGCGCCCCCTTGTGGTCGGTGT
ACAGGTGGTGCATGGCTGTCGTCAGCTCGTGTCGTGAGATGTTGGGTTAAGTCCCGCAACGAGCGCAAC
CCTTGTCCCGTGTTGCCAGCAAGCCCTTCGGGGTGTTGGGGACTCACGGGAGACCGCCGGGGTCAACTC
GGAGGAAGGTGGGGACGACGTCAAGTCATCATGCCCCTTATGTCTTGGGCTGCACACGTGCTACAATGG
CCGGTACAATGAGCTGCGATACCGCGAGGTGGAGCGAATCTCAAAAAGCCGGTCTCAGTTCGGATTGGG
GTCTGCAACTCGACCCCATGAAGTCGGAGTCGCTAGTAATCGCAGATCAGCATTGCTGCGGTGAATACG
TTCCCGGGCCTTGTACACACCGCCCGTCACGTCACGAAAGTCGGTAACACCCGAAGCCGGTGGCCCAAC
CCCTTGTGGGAGGGAGCTGTCGAAGTGGGACTGGCGATGGACGAGTC

Seq 7:

GGATGAGCCCGCGGCCTATCAGCTTGTTGGTGAGGTAACGGCTCACCAAGGCGACGACGGGTAGCCGGC
CTGAGAGGGCGACCGGCCACACTGGGACTGAGACACGGCCCAGACTCCTACGGGAGGCAGCAGTGGGGA
ATATTGCACAATGGGCGNAAGCCTGATGCAGCGACGCCGCGTGAGGGATGACGGCCTTCGGGTTGTAAA
CCTCTTTCAGCAGGGAAGAAGCGAAAGTGACGGTACCTGCAGAAGAAGCGCCGGCTAACTACGTGCCAG
CAGCCGCGGTAATACGTAGGGCGCAAGCGTTGTCCGGAATTATTGGGCGTAAAGAGCTCGTAGGCGGCT
TGTCACGTCGGGTGTGAAAGCCCGGGGCTTAACCCCGGGTCTGCATTCGATACGGGCTAGCTAGAGTGT
GGTAGGGGAGATCGGAATTCCTGGTGTAGCGGTGAAATGCGCAGATATCAGGAGGAACACCGGTGGCGA
AGGCGGATCTCTGGGCCATTACTGACGCTGAGGAGCGAAAGCGTGGGGAGCGAACAGGATTAGATACCC
TGGTAGTCCACGCCGTAAACGGTGGGAACTAGGTGTTGGCGACATTCCACGTCGTCGGTGCCGCAGCTA
ACGCATTAAGTTCCCCGCCTGGGGAGTACGGCCGCAAGGCTAAAACTCAAAGGAATTGACGGGGGCCCG
CACAAGCAGCGGAGCATGTGGCTTAATTCGACGCAACGCGAAGAACCTTACCAAGGCTTGACATACACC
GGAAAACCCTGGAGACAGGGTCCCCCTTGTGGTCGGTGTACAGGTGGTGCATGGCTGTCGTCAGCTCGT
GTCGTGAGATGTTGGGTTAAGTC

Seq 8:

AGCTTGTTGGTGAGGTAACGGCTCACCAAGGCGACGACGGGTAGCCGGCCTGAGAGGGCGACCGGCCAC
ACTGGGACTGAGACACGGCCCAGACTCCTACGGGAGGCAGCAGTGGGGAATATTGCACAATGGGCGAAA
GCCTGATGCAGCGACGCCGCGTGAGGGATGACGGCCTTCGGGTTGTAAACCTCTTTCAGCAGGGAAGAA
GCGAAAGTGACGGTACCTGCAGAAGAAGCGCCGGCTAACTACGTGCCAGCAGCCGCGGTAATACGTAGG
GCGCAAGCGTTGTCCGGAATTATTGGGCGTAAAGAGCTCGTAGGCGGCTTTCACGTCGGGTGTGAAAGC
CCGGGGCTTAACCCCGGGTCTGCATTCGATACGGGCTAGCTAGAGTGTGGTAGGGGAGATCGGAATTCC
TGGTGTAGCGGTGAAATGCGCAGATATCAGGAGGAACACCGGTGGCGAAGGCGGATCTCTGGGCCATTA
CTGACGCTGAGGAGCGAAAGCGTGGGGAGCGAACAGGATTAGATACCCTGGTAGTCCACGCCGTAAACG
GTGGGAACTAGGTGTTGGCGACATTCCACGTCGTCGGTGCCGCAGCTAACGCATTAAGTTCCCCGCCTG
GGGAGTACGGCCGCAAGGCTAAAACTCAAAGGAATTGACGGGGGCCCGCACAAGCAGCGGAGCATGTGG
CTTAATTCGACGCAACGCGAAGAACCTTACCAAGGCTTGACATACACCGGAAAACCCTGGAGACAGGGT
CCCCCTTGTGGTCGGTGTACAGGTGGTGCATGGCTGTCGTCAGCTCGTGTCGTGAGATGTTGGGTTAAG
TCCCGCAACGAGCGCAACCCT

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The MultiAlist System of Thought

(philosophical essay)

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Abstract: The goal of this short note is to expand the concepts of 'pluralism', 'neutrosophy', 'refined neutrosophy', 'refined neutrosophic set', 'multineutrosophic set', and 'plithogeny' (Smarandache 2002, 2013, 2017, 2019, 2021, 2023a, 2023b, 2023c), into a larger category that I will refer to as **MultiAlism** (or **MultiPolar**). As a straightforward generalization, I propose the conceptualization of a *MultiPolar System* (different from a *PluriPolar System*), which is formed not only by multiple elements that might be random, or contradictory, or adjuvant, but also by accepting features from more than one basic system (UniPolar, BiPolar, TriPolar, or PluriPolar systems). PluriAlism is a closed dynamic system without neutralities nor indeterminacies, while MultiAlism is an open dynamic system with neutralities and indeterminacies. PluriAlism is a uni-system (formed by elements from a single system), while MultiAlism is a MultiSystem (formed by elements from many systems).

Keywords: Monism; Dualism; Trialism; Pluralism; Neutrosophy; Refined Neutrosophy; MultiNeutrosophy; Refined Neutrosophic Set, MultiNeutrosophic Set; Plithogeny; Multialism; Zoroastrianism; Neutral Monism; neo-Vedanta.

1. Introduction

Many casual interactions with non-Western peers from academics have opened my eyes during the past two decades to themes that – except for a few committed and non-biased specialists – are still approached superficially in what we still call *The Occident*. In our Western World, some Eastern ideas, principles, and actions remain misunderstood or wrongly judged, because we still have an obstinacy to fit them without nuances into our unique methods of thoughts. The frequent visits I made to the Non-Western World, to international conferences and scientific seminars, or post-doctoral in applied mathematical and technological research, provided me with an unmediated contact with these diverse cultures, allowing me to improve the understanding of their systems of thinking, and resulting in many traveling memories I wrote about their custom, religion, philosophy, history, geography, and life.

In this regard, *Zoroastrianism* serves as an illustration. Its somehow paradoxical aspects bedazzle most Western observers, making them confused when they try to categorize the religion among monotheistic, dualist, or pluralist systems. However, imposing concepts whose meanings have been referenced to other doctrines will not succeed in an attempt to fully define this religion, and rather than pointing out monotheistic or polytheistic features, or even prompt neutrosophic features – as I did myself in one of my scilogs (Smarandache, 2023, 84 et ss.) – would it not be more beneficial for thinking to broaden the current categories?

Alternatively, we may look in the Western philosophy at the *neutral monism*, which – to put it simplistically – holds that the mind and body are not two distinct entities, but are rather *composed of* *the same neutral "stuff"*, or as a fluid (indeterminate) margin between non-physical and physical (Smarandache 2023c). In this respect, David Hume proposed "impressions" or "perceptions" as primary realities of experience, while William James thought that the *neutral core material* is a "booming, buzzing confusion" called "pure experience", and Bertrand Russell, more towards our times, referred to the *neutral entities* as "sensibilia". *Neutral monism* is actually pluralist[1] in that it recognizes the existence of multiple such elements (as opposed to metaphysical monism), but it is monist in that it holds that the fundamental components of the universe are all of the same kind (against mind-body dualism). Since we do not fall strictly into the category of monism anymore, by accepting neutralities or indeterminacies – would it not be more beneficial for thinking to broaden the current categories?

In what follow, I will provide a few more examples of this kind; however, I have no doubt the readers can add their own examples to complete the picture. The examples are not limited, but the question persists: would it not be more beneficial for thinking to broaden the current categories?

It happened that I was reading a very recent study by Ethan Brauer once the sketch of an answer to the above question has settled on its own on the paper. Brauer's extensive paper addresses a completely different and narrow topic, but which can be expanded from its limited sphere – modal analysis of potential infinity. Brauer extended a theory of classical second-order arithmetic to include intrinsically well-motivated axioms for lawless sequences.[2] Free choice sequences are central to the intuitionistic theory of the continuum, but since intuitionistic analysis theorems defy the classical analysis, many mathematicians reject the concept. (Brauer 2023)

Mutatis mutandis, our quest is similar.

2. UniPolar, BiPolar, TriPolar, PluriPolar, and more general MultiPoar Systems. Definitions and examples

In this section, I will scrutinize definitions and meanings of the basic Western systems (of organization) of thoughts, and exemplify them, including scenarios from Eastern doctrines.

2.1. Monism: all is one

Monism is a philosophy and metaphysical doctrine that postulates a single, ultimate, cohesive reality. The universe is composed of a single, overarching 'idea' or 'substance', or only one ultimate deity, <A>. Everything else is just a manifestation of this one reality/substance/deity. This is a **UniPolar System**,

i.e. $\underline{<A> = \infty}$, where <A> is an 'idea', a 'substance', *et caetera*, and ∞ is 'world', 'reality', 'all'.

The monist schools of philosophy claim that either everything is material (materialism) or everything is mental (idealism), and abolish the distinction between the body and the mind in favor of explaining all phenomena as expressions of a single unifying principle.[3]

Christian Wolff coined the term 'monism' in the eighteenth century in his work "Rational Thoughts" [German Logic] (1728): "we must admit of one necessary, self-existent Being" (Wolff, 1770). Wolff delves further into the systems of mind-body connection in the "Psychologia Rationalis" (1734). He believes in the validity of Leibnizian monadology, but only applied to ideas, refuting the monistic panpsychism that is central to Leibniz's metaphysics.[4]

[1] Griffin, N. (1998). 'Neutral monism'. In *The Routledge Encyclopedia of Philosophy*. Taylor and Francis. Retrieved 23 Dec. 2023, from https://www.rep.routledge.com/articles/thematic/neutral-monism/v-1.

[2] Which leaded Brauer to a theory that is called MC_{LS}.

[3] O'Conaill, D.(2019). 'Monism.' In The Routledge Encyclopedia of Philosophy. Taylor and Francis. Retrieved 21 Dec. 2023, from https://www.rep.routledge.com/articles/thematic/monism/v-2.

[4] Hettche, Matt and Corey Dyck, "Christian Wolff", *The Stanford Encyclopedia of Philosophy* (2019), Edward N. Zalta (ed.). Retrieved 17 Dec. 2023, from https://plato.stanford.edu/archives/win2019/entries/wolff-christian.

Looking back in time and towards the East, monism has been widely discussed in connection with the Indian philosophy, particularly in "Uttara Mīmāṃsā" (also known as "Vedānta"). Many schools of thought have emerged out there, all basing their doctrines on the authority of the same corpus known as "Prasthānatrayī".

In Hinduism, the idea of Brahman —the ultimate reality or supreme cosmic power —is frequently connected to monism. Most Hindus follow monastic principles and hold that Brahman is everything and everything is Brahman.[5] The philosophy of Advaita Vedānta, which is frequently referred to as a type of absolute nondualism, also reflects this viewpoint.

In one accessible simplification, one can reduce the monism to two types: a substantive monism, in religions like Buddhism and Hinduism in the East, or philosophers like Spinoza in the West, and attributive monism, with sub-tyes as idealism, physicalism, or neutral monism. The first reduces the reality to a single substance, or states that the world is only varied because this one substance exists in plural forms, while the second asserts that there is a one category of being that encompasses a wide plurality of distinct objects or substances.

Despite being essentially monistic, attributive monism appears to be rather pluralistic, but substantival monism is strongly hostile to pluralism.

In that it reduces the physical cosmos to a single principle, pantheism is similar to monism: "Pantheists are monists" (Owen, 1971, 65), even though the pantheist deity is imperfect, expanding and continuously creating, or also extending beyond space and time in panentheism —a conceptions of God present as well in some Christian confessions —therefore surpassing the simplification of monistic attribution.

2.2. Dualism: all is two

Dualism explains the world (or reality) by two fundamental, diametrically opposed, and irreducible principles. In religion, it generally refers to the conviction that the universe was created by two ultimate antagonistic forces, gods, or groups of angelic or demonic creatures. Since dualism is a system formed by two contrasted parts, this is a **BiPolar System**:

i.e. <u><*A*> + <*antiA*> = ∞</u>.

where <A> is an 'idea', a 'substance', *et caetera*, <antiA> is its opposite or negation, and ∞ is 'world', 'reality', 'all'.

I would probably not be wrong if I affirmed that this system is for ages a dominant worldview in Western way of thinking, with Descartes and Hegel being the first two figures that spring to mind, completed by Kant's cognitive dualism, which distinguished between the faculties of sensibility and understanding, . Examples of epistemological dualism include being and thought, subject and object; and, on the other hand, examples of metaphysical dualism being matter and spirit, body and mind, good and evil.

Glancing eastward, we observe that most historians of religion use the ancient Iranian religion Zoroastrianism as a clear case of eschatological dualism, advocating that it is based on two conflicting principles: Ahura Mazda, the deity of light and truth, and Angra Mainyu, the destroying enemy.

An ongoing conflict exists between the good, spiritual realm of light and the bad, material realm of darkness, according to the ancient Iranian religion of Manichaeism.

Furthermore, as its name says by ityself, *dvaita* — the Sanskrit word *dvaita* actually means 'dualism' (Flood, 1996, 245) — is a dualist school of *Vedanta*, asserting that there is an everlasting separation between the particular self and the ultimate, in opposition to the *advaita* (non-dualist) philosophy. Although *dvaita* was dualist in that sense, it proposed an autonomous God named Vishnu as the ruler of the independent and separate entities of matter and soul. More specifically, *dvaita* recognized three absolute and eternally existing entities: God, souls (*atman*), and primordial substance (*prakriti*).

[5] Leeming, D.A. (2014). 'Brahman.' In: Leeming, D.A. (eds) Encyclopedia of Psychology and Religion. Springer, Boston, MA. https://doi.org/10.1007/978-1-4614-6086-2_9052.

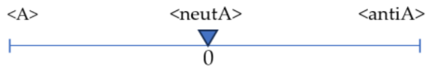
2.3. Trialism: all is three

Trialism was introduced in philosophy by John Cottingham as “a grouping of three notions” (Cottingham, 1985, 219), an alternative viewpoint to Descartes' dualism, with the addition of sensation next to mind and body: “It turns out that there are features that belong to the mind alone, features that belong to the body alone, and what may be called hybrid features - features that belong to man *qua* embodied being” (*Ibidem*; see also Cottingham, 2021).

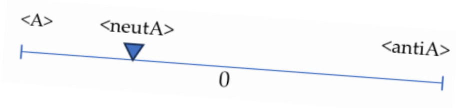
Trialism is thus a system formed by three contrasted or entirely different parts, and similarly, *trichotomy* is a division of three opposites (or entirely different) two by two things.

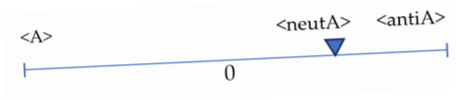
A three-poles system was also proposed by *neutrosophy* (Smarandache 1995, 2013), which operates with three independent opposites, found in equilibrium: <A>, <neutA>, and <antiA>, called *Neutrosophic Triad*. All ‘ideas’ <A> are considered in conjunction with their opposites or negations <antiA> and the range of neutralities <neutA> between them, while <nonA> is the collective term for the ideas <antiA> and <neutA>. In neutrosophy, the three poles may be fluid two by two.

The balance between <A> and <antiA> rests on <neutA>. In other words, <neutA> is imagined as a buffer zone between <A> and <antiA>:



Moving <neutA> to the left, or to the right, i.e. if the neutral/indeterminacy part is pushed towards <A>, or <antiA> (the indeterminacy degree increases), then one of them gets stronger (having less indeterminacy), and the balance gets in disequilibrium:





Based on neutrosophy, the associated **TriPolar System** can be described as:

<A> + <neutA> + <antiA> = ∞,

where <A> is an ‘idea’, a ‘substance’, *et caetera*, <antiA> is its opposite or negation, <neutA> is the range of neutralities between them, and ∞ is ‘world’, ‘reality’, ‘all’.

I point out here no more than that the neutrosophy is an extension of both the ancient Chinese Yin-Yang philosophy and dialectics (Smarandache 2013), and also remind the reader that the trialism was associated with Christianity as well, e.g. for holding that human beings are composed of three separate essences: a body, a soul, and a spirit.[6]

2.4. Pluralism: all is plurality

Pluralism is a wordview of plurality, used in philosophy to contrast with monism (the idea that everything is one), with dualism (the idea that everything is two), and arguably with trialism (the idea that everything is three). Pluralism can be defined as a system in which more than two (arguably three) groups, principles, states, ideas, *et caterea*, coexist. This is a **PluriPolar System**:

<pluriA> = ∞,

where <pluriA> means more than two (arguably three) ‘ideas’, *et caetera*, and ∞ is ‘world’, ‘reality’, ‘all’.

In metaphysics, pluralism is the idea that reality is actually made up of a variety of substances found in nature, while in ontology the concept describes various forms, kinds, or modes of existence.

[6] This understanding stems from taking 1 Thessalonians 5:231 literally: “And the very God of peace sanctify you wholly; and I pray God your whole spirit and soul and body be preserved blameless unto the coming of our Lord Jesus Christ.”

Buddhism is given as an example of a pluralistic religion. Many Buddhist traditions do not declare a single ultimate truth and recognize the validity of multiple paths to enlightenment, advocating conversation and understanding with people of other faiths.

Another example might be the Bahá' Faith, which holds that all major faiths have the same spiritual basis, are descended from the same divine source (God), and differ only in their social teachings in accordance with the necessities of the eras in which they were revealed.

3. MultiAlism: all is open

We observed in the short, quick and without going into depth evaluation of the basic systems that we previously discussed that it is challenging to strictly include some non-Western doctrines (or even Western!) in one group or another. Certain doctrines/ideologies/ideas acknowledge several components from various systems. Some beliefs are classified as monistic, yet they clearly contain components of pluralism as well; others, on the other hand, are classified as nondualistic but cannot be classified as either strictly UniPolar, or PluriPolar systems. Nor the concept of nonduality, a common thread in Taoism, Mahayana Buddhism, or Advaita Vedanta (Loy, 1998), does suffice, being a rather fuzzy concept, which might finally include anything that does not fall into a BiPolar System of thought, regardless of distinctions, or mutations.

Numerous schools of thought have extensively examined the dynamics between the opposites <A> and <antiA>. These concepts are known by various names, including dialectics, Yin-Yang, Manichaeism, dualism, Dharma-Adharma, and many others. However, the neutral (or indeterminacy) part (<neutA>) between these opposites has rather either been ignored or retracted. The neutral or indeterminate, as I emphasized in my studies on neutrosophic theory (Smarandache 2002, 2013), usually intervenes in the dynamics (or conflicts) from one side or the other, tipping the balance in one direction or the other. The boundaries between the opposites can be either fluid (when there is some overlapping or indeterminate/neutral part between the opposites) or rigid (when <A> and <antiA> are clearly separated).

In Occasionalism, for example, the God is a neutral ($\langle neutA_1 \rangle$) between mind ($\langle A_1 \rangle$) and body ($\langle antiA_1 \rangle$), as a particular case, i.e. where one has only one dynamic, between $\langle A_1 \rangle$ and $\langle antiA_1 \rangle$ (one neutrosophic triad). In MultiAlism, one has dynamics between many neutrosophic triads:

($\langle A_1 \rangle$, $\langle neutA_1 \rangle$, $\langle antiA_1 \rangle$), ($\langle A_2 \rangle$, $\langle neutA_2 \rangle$, $\langle antiA_2 \rangle$), ...

And so forth.

By convention let's use the prefix "*pluri*" when talking about the elements of a single system, and "*multi*" when talking about the elements of many systems.

Therefore, the **PluriPolar System** accepts and deals with the dynamicity of opposites, but not with the neutralities or indeterminacies between them:

$\underline{\langle (pluri)A \rangle + \langle (pluri)antiA \rangle = \infty}$.

This simple observation instigated the idea of a generalizing and integrative construct into which to accommodate theories that mix parts from many systems. I unpretentiously call this construct **multialism**, and clearly differentiate it from **pluralism**, and consequently call the related system the **multialist system**, conceiving it as a **MultiPolar System** that accepts and is open to combinations of opposites and neutrals (indeterminacies), e.g.:

$\underline{\langle (multi)A \rangle + \langle (multi)neutA \rangle + \langle (multi)antiA \rangle = \infty}$.

The MultiPolar System accepts and deals with neutralities and indeterminacies between the opposites, but it is not necessarily to contain them. As such, the MultiPolar System is an extension of the PluriPolar System.

Let us test out two examples from religion before returning with more in-depth studies in later papers.

3.1. Zoroastrianism

Zoroastrianism offers a perplexing picture of a religion (about the state and prospects of the study of this religion, a must read is Stausberg, 2008) whose followers worship several sacred beings, called *yazatas*, in addition to a single deity, *Ahura Mazda* (or *Ohrmazd* in Middle Persian).[7]

These *yazatas*[8] —somehow remembering us the Roman tutelary deities *Lares*[9]— include natural objects or phenomena (earth, water, wind, sun, moon, etc.). Other individual deities manifest their presence, among which *Anahita* (fertility), *Armaiti* (right-mindedness), *Ai* (reward), or *Rasnu* ('justice'). Furthermore, *Ahura Mazda*'s faces strong opposition from the personification of evil, *Ahreman* in Middle Persian (or *Angra Mainyu* in Avestan). Its only goal is to ruin *Ohrmazd*'s good world.

This makes the **Zoroastrianism** to Hintze to be both dualistic, polytheistic, and monotheistic (a "mixture of seemingly monotheistic, polytheistic, and dualistic features", Hintze, 2014, 225 et ss.), in an attempt to put an end to the debates in literature which went from defining **Zoroastrianism** as a "dualistic monotheism" (Gnoli, 1994, 480) to a "monotheistic dualism" (Schwartz, 2002, 64). Added to this are the interpretations of existence of a *Divine Triad*, or a *dialogical triad* in **Zoroastrianism**: "The Deity is also not a monadic one, but a dialogical triad (and there may be other aspects) who exists in relationship" (Louchakova-Schwartz, 2018, 481).

Furthermore, I observe the obvious neutrosophic features of *yazatas*: the balance between good and evil tilts according to their (neutrosophic) actions (*vedi supra*, 2.3).

In our approach, these characteristics makes the **Zoroastrianism** a ***multialist*** religion, including elements from all basic systems:

<A> [*Ohrmazd*] + <(multi)A> [deities] + <neutA> [actions of *yazatas*] + <antiA> [*Ahreman*] = ∞.

3.2. Vedanta and neo-Vedanta schools

Other instances of **multialism** are generated by the different interpretations of Vedanta. Independently, the Vedanta schools may appear utterly distinct due to significant discrepancies in ontology, soteriology, and epistemology.

Let us remind the main schools of Vedanta, and their interpretations: Advaita (non-dualism), Dvaitadvaita (difference and non-difference), Vishishtadvaita (qualified non-dualism), Dvaita (dualism), Suddhadvaita (pure non-dualism), Achintya-Bheda-Abheda (inconceivable difference and non-difference) (Isaeva, 1992; Clooney, 1993).

Coming closer to our days, modern developments (so-called neo-Vedanta) propagated the idea that the divine, the absolute, exists within all human beings. Acceptance of many kinds of worship is a key component of Swami Vivekananda's philosophy, an exponent of neo-Vedanta, emphasizing the idea of *acceptance* rather than *tolerance*. This neo-Vedanta school holds that no other types of worship are incorrect. Life is a quest trip from one truth to another, from a lesser truth to a greater one. The truth is not anyone's property, and the nature of all souls is truth. Actually, Vivekananda "reconciles Dvaita or dualism and Advaita or non-dualism" (Sooklal, 1993, 48).

According to Vivekananda, the perfect man possesses all the components of philosophy, mysticism, passion, action in right measure to create a harmoniously balanced whole (*Ibidem*, 42). To my understanding, the components are supposed to exist in a balanced (and hence *neutrosophic*) manner rather than just in their *plurality*, and yet being monistical manifestations of *one*,

i.e. <A> + <(multi)A> + <neutA>,

which makes me consider it a *multialist doctrine*.

[7] Duchesne-Guillemin, Jacques. "Zoroastrianism". Encyclopedia Britannica, 8 Nov. 2023, https://www.britannica.com/topic/Zoroastrianism. Accessed 11 December 2023.

[8] Britannica, The Editors of Encyclopaedia. "yazata". Encyclopedia Britannica, 3 Apr. 2014, https://www.britannica.com/topic/yazata. Accessed 11 December 2023.

[9] Britannica, The Editors of Encyclopaedia. "Lar". Encyclopedia Britannica, 14 Feb. 2018, https://www.britannica.com/topic/Lar-Roman-deities. Accessed 11 December 2023.

Conclusions

As an extension of the concepts of 'pluralism', 'neutrosophy', 'refined neutrosophy', 'refined neutrosophic set', 'multineutrosophic set', and 'plithogeny' (Smarandache 2002, 2013, 2017, 2019, 2021, 2023a, 2023b, 2023c), I introduced in this short note the concept of MultiAlism, to which corresponds a MultiPolar system of thought. A possible advantage of this system could free from ambiguities the other systems, especially the PluriPolar system, where *plural elements* – more or less equal – coexist or are tolerated to exist and contains their opposites, but not their neutralities or indeterminacies between them; while the MultiPolar system is open to accept in various combinations and mutations, the opposites and their neutralities or indeterminacies between them, from *more than one system*. In other words, the UniPolar, BiPolar, TriPolar, and PluriPolar systems are uni-valent systems (one excludes the other), whilst the MultiPolar System is a multi-valent system (it includes more than one system) and accepts neutralities and indeterminacies between opposites.

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