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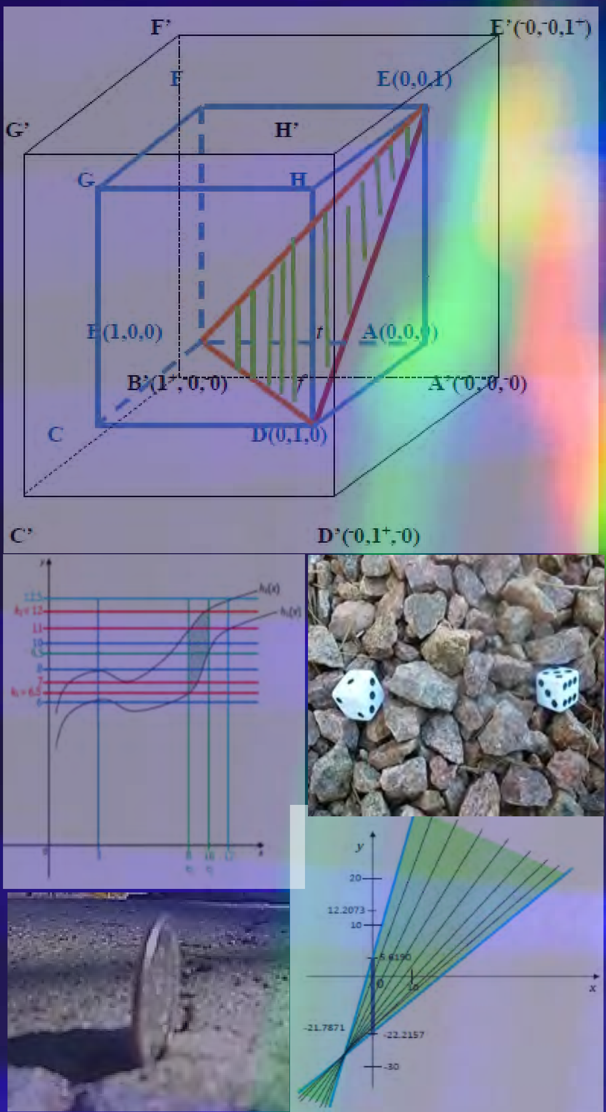

<A> <neutA> <antiA>

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Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]^-0, 1^+[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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Google Dictionaries have translated the neologisms "**neutrosophy**" (1) and"**neutrosophic**" (2), coined in 1995 for the first time, into about 100 languages.
FOLDOC Dictionary of Computing (**1, 2**), Webster Dictionary (**1, 2**), Wordnik (**1**),Dictionary.com, The Free Dictionary (**1**), Wiktionary (**2**), YourDictionary (**1, 2**),OneLook Dictionary (**1, 2**), Dictionary / Thesaurus (**1**), Online Medical Dictionary (**1,2**), Encyclopedia (**1, 2**), Chinese Fanyi Baidu Dictionary (**2**), Chinese Youdao Dictionary (**2**) etc. have included these scientific neologisms.

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An Application of Neutrosophic Sets to Decision Making

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Abstract: Frequently in real life situations decision making takes place under fuzzy conditions, because the corresponding goals and/or the existing constraints are not clearly defined. Maji et al. introduced in 2002 a method of parametric decision making using soft sets as tools and representing their tabular form as a binary matrix. As we explain here, however, in cases where some or all of the parameters used for the characterization of the elements of the universal set are of fuzzy texture, their method does not give always the best decision making solution. In order to tackle this problem, we modified in earlier works the method of Maji et al. by replacing the binary elements in the tabular form of the corresponding soft set either by grey numbers or by triangular fuzzy numbers. In this work, in order to tackle more efficiently cases in which the decision maker has doubts even about the correctness of the fuzzy/qualitative characterizations assigned to some or all of the elements of the universal set, we replace the binary elements of the tabular form by neutrosophic triplets. Our new, neutrosophic decision making method is illustrated by an application concerning the choice of a new player by a soccer club.

Keywords: decision making (DM); fuzzy set (FS); neutrosophic set (NS); soft set (SS); grey number (GN)

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1. Introduction

Decision Making (DM) is a fundamental process in a great spectrum of human activities and many books have been written about it, helping decision makers to make smarter choices easier and quicker; e.g. see [1, 2], etc. Frequently in real life situations, however, DM takes place under fuzzy conditions, since the corresponding goals and/or the existing constraints are not clearly defined. Several methods have been also proposed for successful DM in such cases, e.g. [3-5], etc.

Maji et al. introduced in 2002 a method of parametric DM using *soft sets (SS)* as tools and representing their tabular form as a binary matrix [6]. When some or all of the parameters used for the characterization of the elements of the set of the discourse (houses in their example) are of fuzzy texture (beautiful and cheap in their example), however, their method does not always give the best solution to the corresponding DM problem. This happens, because they replace the parameters in the tabular form of the corresponding SS with the binary elements 0, 1. In other words, their method, although it starts with a fuzzy framework (SS), it continues by using bivalent logic for obtaining the required decision (beautiful or not beautiful and cheap or not cheap in their example)! In order to tackle this problem, we modified in earlier works the method of Maji et al. by using either *triangular fuzzy numbers (TFNs)* [7] or *grey numbers (GNs)* – see [8], or [9] (section 5.3) - instead of the binary elements in the tabular form of the corresponding SS.

In reality, however, cases appear in which the decision maker has doubts about the correctness of the fuzzy/qualitative characterizations assigned to some or all of the elements of the set of the discourse (e.g. very beautiful, rather beautiful, etc.). In order to study more efficiently the DM process in the previous cases, we introduce here *neutrosophic sets (NSs)* and we replace the binary elements of the method of Maji et al. by *neutrosophic triplets.* The rest of the paper is organized as follows: Section 2 contains the necessary background about NSs, SSs and GNs needed for the understanding of the paper. The DM method of Maji et al., our modification using GNs, as well as the new neutrosophic DM method are developed in section 3, illustrated by examples concerning the choice of a new player by a soccer club. The article closes with a discussion on the results obtained in it, including some hints for future research, and the final conclusions, presented in section 4.

2. Mathematical Background

2.1 Fuzzy Sets and Fuzzy Logic

Zadeh extended in 1965 the concept of crisp set to the concept of FS [10] on the purpose of dealing with the existing in real life partial truths (e.g. rather good, almost true, etc.) and of expressing mathematically definitions with no clear boundaries (e.g. high mountains, clever people, etc.) This was succeeded by replacing the characteristic function by the *membership function*, which maps each element of the universal set U to the unit interval [0, 1]. In fact, if A is the corresponding FS in U and m: U → [0, 1] is its membership function, m(x) is called the *membership degree* of x in A, for all elements x in U. The closer m(x) to 1, the better x satisfies the characteristic property of A. And, although the FS A is typically defined as the set of all ordered pairs of the form (x. m(x)), ∀ x∈U, for reasons of simplicity many authors identify A with its membership function.

Most of notions and operations on crisp sets are extended in a natural way to FSs (e.g. see [11]). Based on the concept of FS, Zadeh introduced the infinite-valued *fuzzy logic (FL)*, in which the truth values are modelled by numbers in the unit interval [12]. FL does not oppose the traditional bivalent logic of Aristole (384-322 BC), which used to be for many centuries the basic tool of human reasoning being "responsible" for the growth of science and human civilization all this time; on the contrary it completes and extends it [13]

In a later stage, when membership functions were reinterpreted as *possibility* distributions, FSs and FL were used to embrace *uncertainty* modelling [14, 15]. The frequently appearing in real life uncertainty is due to several reasons, like randomness, imprecise or incomplete data, vague information, etc. *Probability* theory has been proved sufficient to tackle only the cases of uncertainty which are due to randomness [16]. Starting from Zadeh's FSs, however, several theories have been developed during the last years on the purpose of tackling more effectively all the forms of the existing uncertainty. The main among those theories are briefly reviewed in [17]. In the present paper we are going to use elements from the theories of NSs, SSs and GNs needed for its understanding, which are exposed below.

2.2 Neutrosophic Sets

Atanassov in 1986, considered in addition to Zadeh's membership degree the degree of *non-membership* and extended FS to the notion of *intuitionistic FS (IFS)* [18]. Smarandache in 1995, inspired by the frequently appearing in real life neutralities - like <friend, neutral, enemy>, <win, draw, defeat>, <high, medium, short>, etc. - generalized IFS to the concept of NS by adding the degree of *indeterminacy* or *neutrality* [19]. The word "neutrosophy" is a synthesis of the word "neutral΄ and the Greek word "sophia" (wisdom) and means "the knowledge of neutral thought". The simplest form of a NS is defined as follows:

Definition 1: A *single valued NS (SVNS)* A in the universe U is of the form

$$A = \{(x,T(x),I(x),F(x)): x \in U,\ T(x),I(x),F(x) \in [0,1],\ 0 \leq T(x)+I(x)+F(x) \leq 3\} \quad (1)$$

In equation (1) T(x), I(x), F(x) are the degrees of *truth* (or membership), *indeterminacy* (or neutrality) *and falsity* (or non-membership) of x in A respectively, called the *neutrosophic components* of x. For simplicity, we write A<T, I, F>.

Indeterminacy is defined to be in general everything that exists between the opposites of truth and falsity [20].

Example 1: Let U be the set of the players of a soccer club and let A be the SVNS of the good players of the club. Then each player x is characterized by a *neutrosophic triplet* (t, i, f) with respect to A, with t, i, f in [0, 1]. For example, x(0.7, 0.1, 0.4) ∈ A means that the coach of the club is 70% sure that x is a good player, but at the same time he has a 10% doubt about it and a 40% belief that x is not a good player. In particular, x (0, 1, 0) ∈ A means that the coach does not know absolutely nothing about player x (new player).

If the sum T(x) + I(x) + F(x) < 1, then it leaves room for incomplete information about x, if it is equal to 1 for complete information and if it is >1 for *inconsistent* (i.e. contradiction tolerant) information about x. A SVNS may contain simultaneously elements leaving room to all the previous types of information. All notions and operations defined on FSs are naturally extended to SVNSs [21].

Summation of neutrosophic triplets is equivalent to the union of NSs. That is why the neutrosophic summation and implicitly its extension to neutrosophic scalar multiplication can be defined in many ways, equivalently to the known in the literature neutrosophic union operators [22]. For the needs of the present work, writing the elements of a SVNS A in the form of neutrosophic triplets and considering them simply as <u>ordered triplets</u> we define addition and scalar product as follows:

Definition 2: Let (t_1, i_1, f_1), (t_2, i_2, f_2) be in A and let k be a positive number. Then:

- The *sum* $(t_1, i_1, f_1) + (t_2, i_2, f_2) = (t_1+ t_2,\ i_1+ i_2, f_1+ f_2)$ (2)
- The *scalar product* $k(t_1, i_1, f_1) = (kt_1, k\, i_1, kf_1)$ (3)

Remark 1: The summation and scalar product of the elements of a SVNS A with respect to Definition 2 <u>need not</u> be a closed operation in A, since it may happen that $(t_1+ t_2)+(i_1+ i_2)+(f_1+ f_2)>3$ or $kt_1+k\, i_1+kf_1>3$. With the help of Definition 2, however, one can define <u>in A</u> the *mean value* of a finite number of elements of A as follows:

Definition 3: Let A be a SVNS and let (t_1, i_1, f_1), (t_2, i_2, f_2), …., (t_k, i_k, f_k) be a finite number of elements of A. Assume that (t_i, i_i, f_i) appears n_i times in an application, i = 1,2,…., k. Set $n = n_1+n_2+\ldots+n_k$. Then the *mean value* of all these elements of A is defined to be <u>the element of A</u>

$$(t_m, i_m, f_m) = \frac{1}{n}[n_1(t_1, i_1, f_1)+n_2(t_2, i_2, f_2)+\ldots+n_k(t_k, i_k, f_k)] \quad (4)$$

2.3 Soft Sets

A disadvantage of FSs and of all their extensions involving membership degrees (like IFSs, NSs, etc.), is that there is not any exact rule for defining properly the corresponding membership functions. The methods used for this are usually statistical or intuitive/empirical. Moreover, the definition of the membership function is not unique depending on the "signals" that each observer receives from the environment. For example, defining the FS of "young people" one could consider as young all persons aged less than 30 years and another one all persons aged less than 40 years. As a result the second observer will assign membership degree 1 to all people aged between 30 and 40 years, whereas the first one will assign to them membership degrees less than 1. Analogous differences are logically expected to appear to the membership degrees of all the other ages. In other words, the only restriction for the definition of the membership function is that it must be compatible to the common sense; otherwise the resulting FS does not give a creditable description of the corresponding real situation. This could happen, for instance, if in the previous example people aged more than 70 years possessed membership degrees ≥0.5.

For overpassing this problem, the concept of *interval-valued FS (IVFS)* was introduced in 1975. An IVFS is defined by mapping the universe U to the set of the closed subintervals of [0, 1] [23]. Other related to FS theories were also developed in which the definition of a membership function is not necessary (*grey systems and numbers* [24]), or it is overpassed by using a pair of crisp sets giving the lower and upper bound of the original set (*rough sets [25]*). Molodstov, introduced in 1999 the concept of SS for tackling the uncertainty in a parametric manner, not needing, therefore, the definition of a membership function [26]. Namely, a SS is defined as follows:

Definition 4: Let E be a set of parameters, let A be a subset of E, and let f be a map from A into the power set P(U) of the universe U. Then the SS (f, A) in U is defined as the set of the ordered pairs

$$(f, A) = \{(e, f(e)): e \in A\} \qquad (5)$$

In other words, a SS is a parametrized family of subsets of U. The name "soft" was given due to the fact that the form of (f, A) depends on the parameters of A. For each $e \in A$, its image f(e) in P(U) is called the *value set* of e in (f, A), while f is called the *approximation function* of (f, A).

Example 2: Let U= $\{H_1, H_2, H_3\}$ be a set of houses and let E = $\{e_1, e_2, e_3\}$ be the set of parameters e_1=cheap, e_2=beautiful and e_3= expensive. Let us further assume that H_1, H_2 are cheap, H_3 is expensive and H_2, H_3 are beautiful houses. Then, a map f: E $\rightarrow$ P(U) is defined by $f(e_1)=\{H_1, H_2\}$, $f(e_2)=\{H_2, H_3\}$ and $f(e_3)=\{H_3\}$. Therefore, the SS (f, E) in U is the set of the ordered pairs

$$(f, E) = \{(e_1, \{H_1, H_2\}), (e_2, \{H_2, H_3\}, (e_3, \{H_3\}\} \qquad (6)$$

Maji et al. [6] introduced a *tabular representation* of SSs in the form of a binary matrix in order to be stored easily in a computer's memory. For instance, the tabular representation of the soft set (f, E) of the previous example is given by Table 1.

Table 1. Tabular representation of the SS of Example 2

	e_1	**e_2**	**e_3**
H_1	1	0	0
H_2	1	1	0
H_3	0	1	1

A FS in U with membership function y = m(x) is a SS in U of the form (f, [0, 1]), where $f(\alpha)=\{x \in U: m(x) \geq \alpha\}$ is the corresponding *a-cut* of the FS, for each α in [0, 1]. Consequently the concept of SS is a generalization of the concept of FS. All notions and operations defined on FSs are extended in a natural way to SSs [27].

2.4 Grey Numbers

The theory of *grey systems* [24] introduces an alternative way for managing the uncertainty in case of approximate data. A grey system is understood to be any system which lacks information, such as structure message, operation mechanism or/and behavior document.

Closed real intervals, are used for performing the necessary calculations in grey systems. In fact, a closed real interval [x, y] could be considered as representing a real number T, termed as a *grey number (GN)*, whose exact value in [x, y] is unknown. We write then T ∈ [x, y]. A GN T, however, is frequently accompanied by a *whitenization function* f: [x, y] → [0, 1], such that, if f(a) approaches 1, then a in [x, y] approaches the unknown value of T. If no whitenization function is defined, it is logical to consider as a representative crisp approximation of the GN A the real number

$$V(A) = \frac{x \circledcirc y}{2} \qquad (7)$$

The arithmetic operations on GNs are introduced with the help of the known arithmetic of the real intervals [28]. In this work we are going to make use only of the addition of GNs and of the scalar multiplication of a GN with a positive number, which are defined as follows:

Definition 5: Let $A \in [x_1, y_1]$, $B \in [x_2, y_2]$ be two GNs and let k be a positive number. Then:

- The sum: A+B is the GN $A+B \in [x_1+y_1, x_2+y_2]$ (8)
- The scalar product kA is the GN $kA \in [kx_1, ky_1]$ (9)

3. The Parametric Assessment Method

The parametric assessment method of Maji et al. [6], our modification using GNs [8, 9] and our new method using NSs are developed in this section through suitable examples concerning the choice of a new player by a soccer club.

3.1 The Method of Maji et al.

Let us consider the following example:

Example 3: A soccer club wants to choose a new player among 6 candidates, say P_1, P_2, P_3, P_4, P_5 and P_6. The desired qualifications of the new player are to be fast, younger than 30 years, higher than 1.70 m and experienced. Assume that P_1, P_2, P_6 are the fast players, P_2, P_3, P_5,P_6 are the players being younger than 30 years, P_3, P_5 are the players with heights greater than 1.70 m and P_4 is the unique experienced player. Which is the best choice for the club?

Solution: Let U be the set of the 6 candidate players. Consider the parameters e_1=fast, e_2=younger than 30 years, e_3=higher than 1.70 m, e_4=experienced and set E = {e_1, e_2, e_3, e_4}. Define the map f: E → P(U)by f(e_1) = {P_1, P_2, P_6}, f(e_2) = {P_2, P_3, P_5, P_6}, f(e_3) = {P_3, P_5} and f(e_4) = {P_4}. Then the tabular form of the SS (f, E) is shown in Table 2.

Table 2. Tabular representation of the SS of Example 3

	e_1	e_2	e_3	e_4
P_1	1	0	0	0
P_2	1	1	0	0
P_3	0	1	1	0
P_4	0	0	0	1
P_5	0	1	1	0
P_6	1	1	0	0

The *choice value* of each player is calculated by adding the binary elements of the row of Table 2 in which he belongs. The players P_1 and P_4, therefore, have choice value 1 and all the others have choice value 2. Consequently, the right decision is to choose one of the players P_2, P_3, P_5, and P_6.

3.2 Our Method Using GNs

The previous decision, obtained by the method of Maji et al., is obviously not very helpful for the soccer club. Observe, however, that the parameters e_1 and e_4, in contrast to e_2 and e_3, have not a bivalent texture. A player, for example, could not be very fast, but quite fast, or rather experienced, etc. This inspired us to characterize the qualifications of the players with respect to the parameters e_1 and e_4 in the tabular matrix of the previous SS (f, E) by using the linguistic grades A=excellent, B=very good, C=good, D=mediocre and F= not satisfactory instead of the binary element 0. To show how one works in this case for making the right decision, let us modify Example 3 as follows:

Example 4: Reconsider Example 3 and assume that the technical manager of the soccer club, after a more careful inspection of the qualifications of the 6 candidate players, decided to use the following Table 3 instead of Table 2 for the DM process. Which is the best choice for the club in this case?

Table 3. Tabular representation of the SS of Example 4

	e_1	e_2	e_3	e_4
P_1	1	0	0	C
P_2	1	1	0	F
P_3	C	1	1	C
P_4	D	0	0	1
P_5	D	1	1	C
P_6	1	1	0	D

Solution: Assign to each of the qualitative grades A, B, C, D, F a GN, denoted for simplicity by the same letter, as follows: A = [0.85, 1], B = [0.75, 0.84], C= [0.6, 0.74], D = [0.5, 0.59], F = [0, 0.49]. From Table 3 then, one calculates, with the help of formulas (7), (8) and (9), the choice value of each player in the following way. P_1: 1+V(C) = $1+\frac{0.6+0.74}{2} \approx 1.67$, P_2: 2+V(F) = 2.245, P_3: 2+V(2C) = 3.34, P_4: 1+V(D)= 1.545, P_5: 2+V(D+C) = 3.215, P_6: 2+V(D)=2.545. The right decision, therefore, is to choose the player P_3.

Remark 2: The choices of the qualitative grades A, B, C, D, F, as well as of the intervals for translating them in the numerical scale 0-1, correspond to generally accepted standards. The decision maker, however, could use, with respect to his/her goals, more or less qualitative grades (e.g. by adding E between D and F, etc.) and could also change the corresponding intervals (e.g. by setting A= [0.9, 1], B = [0.8, 0.89], C= [0.7, 0.79], D = [0.6, 0.69], F = [0, 0.59], or otherwise). Such changes, however, does not affect the generality of our method

3.3. The Neutrosophic DM Method

As it was already mentioned in our Introduction, DM situations appear frequently in reality, in which the decision maker has doubts about the correctness of the fuzzy/qualitative grades assigned to some or all of the elements of the set of the discourse. In such cases, the best way to perform the DM process is to use NSs. In Example 4, for instance, considering the set U of the 6 candidate players as the universal set, the decision maker could define in U the NSs of the fast and of the experienced players by assigning the suitable neutrosophic triplets to each player of U. In order to have complete information, we should have t+i+f = 1, for each triplet (t, i, f). Then, he/she could continue the DM process by replacing in Table 3 the qualitative grades by the corresponding neutrosophic triplets and the binary elements 0, 1 by the neutrosophic triplets (0, 0, 1) and (1, 0, 0) respectively. This process will be illustrated by the following Example 5.

Example 5: Reconsider Example 4 and assume that the technical manager of the soccer club, being not sure about the qualitative grades assigned to each of the 6 candidate players, he decided to proceed by replacing them by neutrosophic triplets, in the way that we have previously described. As a result, the tabular matrix of the DM process took the form of the following Table 4. Which is the best decision for the club in this case?

Table4. Tabular representation of the SS of Example 5

	e_1	e_2	e_3	e_4
P_1	(1, 0, 0)	(0, 0, 1)	(0, 0, 1)	(0.6, 0.3, 0.1)
P_2	(1, 0, 0)	(1, 0, 0)	(0, 0, 1)	(0.2, 0.2, 0.6)
P_3	(0.5, 0.4, 0.1)	(1, 0, 0)	(1, 0, 0)	(0.6, 0.2, 0.2)
P_4	(0.5, 0.2, 0.3)	(0, 0, 1)	(0, 0, 1)	(1, 0, 0)
P_5	(0.5, 0.1, 0.4)	(1, 0, 0)	(1, 0, 0)	(0.6, 0.3, 0.1)
P_6	(1, 0, 0)	(1, 0, 0)	(0, 0, 1)	(0.4, 0.4, 0.2)

Solution: The choice value of each player in this case is defined to be the mean value of the neutrosophic triplets of the line of Table 4 in which he belongs. Thus, by equation (4), the choice value of P_1 is equal to $\frac{1}{4}[(1, 0, 0)+2(0, 0, 1)+(0.6, 0.3, 0.1)] = \frac{1}{4}(1.6, 0.3, 2.1) = (0.4, 0.075, 0.525)$. In the same way one finds that the choice values of P_2, P_3, P_4, P_5 and P_6 are (0.55, 0.005, 0.4), (0.775, 0.15, 0.075), (0.375, 0.05, 0.575), (0.775, 0.1, 0.125) and (0.6, 0.1, 0.3) respectively.

In this case the club's technical manager could use either an optimistic criterion by choosing the player with the greatest truth degree, or a conservative criterion by choosing the player with the lower falsity degree. Consequently, using the optimistic criterion he must choose one of the players P_3 and P_5, whereas using the conservative criterion he must choose the player P_3. A combination of the two criteria leads to the final choice of player P_3. Observe, however, that, since the indeterminacy degree of P_3 is 0.15 and of P_5 is 0.1, there is a slightly greater doubt about the qualifications of P_3 with respect to the qualifications of P_5. In other words, the choice of P_3 is connected with a slightly greater risk. In final analysis, therefore, all the neutrosophic components assigned to each player give useful information about his qualifications.

4. Discussion and Conclusions

In this paper a parametric DM method of hybrid character was presented illustrated by suitable examples about the choice of a new player from a football club. The whole discussion performed leads to the following conclusions:

- The parametric DM method is based on the introduction of a set E of parameters characterizing the elements of the set U of all possible solutions of the corresponding DM problem, the definition with respect to E of a suitable SS in U and the use of its tabular representation T as a tool for the DM process.
- When all the parameters of E are of bivalent texture (yes or no), then T takes the form of a binary matrix, wherefrom the decision maker calculates the choice value of each element of U by adding the binary elements of the row of T in which this element appears.
- When some or all of the parameters of E are of fuzzy texture and the decision maker has no doubts about the qualitative characterizations assigned to the elements of U with respect to these parameters, then the binary elements of T corresponding to the fuzzy parameters are replaced by suitable GNs and the choice value of each element of U is calculated by adding the remaining in the corresponding row of T binary elements and the representative real values of the GNs appearing in it.
- When some or all of the parameters of E are of fuzzy texture and the decision maker do has doubts about the qualitative characterizations assigned to the elements of U with respect to these parameters, then each parameter of E is expressed in the form of a NS in U and the binary elements of T are replaced by the corresponding neutrosophic triplets. In this case, the choice value of each element of U is obtained by calculating the mean value of the neutrosophic triplets of the row of T in which this element appears.

As it has been already mentioned in section 2.1 of this paper, several theories extending / generalizing Zadeh's FSs have been developed during the last years on the purpose of tackling more effectively the existing in real life uncertainty. None of these theories, however, can tackle efficiently alone all the forms of the existing uncertainty, but the combination of all of them provides an adequate framework towards this target.

The results obtained in this and earlier works of the present author give strong indications that hybrid methods applied to several situations in fuzzy environments could give better results, not only for DM, as it happened here, but also for assessment - e.g. see [9] (section 5.2) – and probably for many other topics. This is, therefore, a promising area for further research.

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Topological structures via MBJ-neutrosophic sets

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Abstract. In this paper, we introduce the concept of quasi-coincidences in MBJ-neutrosophic sets and obtain some of its properties. Also by defining the pre-image and the image of an MBJ-neutrosophic set under a mapping, we confirm that their properties are naturally extensions of the classical case. Next, we define two types of MBJ-neutrosophic neighborhood system and discuss their various properties and introduce the notion of ∘-[resp. ∗-]MBJ-neutrosophic bases and deal with some of their properties. Finally, we define $\circ$-C_I, $*$-C_I, $\circ$-C_{II} and $*$-C_{II} corresponding to the first countability and the second countability in classical topological spaces, and we obtain the relationships between them.

Keywords: MBJ-neutrosophic set; MBJ-neutrosophic topology; MBJ-neutrosophic neighborhood; MBJ-neutrosophic base and subbase; MBJ-neutrosophic local base.

1. Introduction

In the real world, we are faced with a complex system that includes various types of uncertainty to solve epidemics such as covid, conflicts between countries, and international energy and food problems. As a tool to solve such complex systems, Zadeh [1] first proposed the concept of fuzzy sets which generalises crisp sets. Smarandache [2, 3] introduced the notion of neutrosophic sets that is a triple $\langle T, I, F\rangle$ of three fuzzy sets (T, I and F are called the *truth*, the *indeterminate* and *false membership functions*) and can extend the concepts of classical sets, fuzzy sets, interval-valued fuzzy sets [4] and intuitionistic fuzzy sets [5]. We can see that neutrosophic set is applied to a variety of fields (See the site http://fs.gallup.unm.edu/neutrosophy.htm). Recently, Takallo et al. [6] defined an MBJ-neutrosophic set that generalized the neutrosophic set by replacing the interval-valued fuzzy set with the fuzzy set I of a neutrosophic set, and applied it to BCK/BCI-algebra. After that time, it can be seen from the literature that several researchers [7–10] have applied the MBJ-netrosophic set to BCK/BCI-algebras. In particular, Khalid et al. [11] studied MBJ-neutrosophic T-ideals on B-algebras. Manivasan and Kalidass [12] discussed MBJ-neutrosophic ideals on KU-algebras.

Topology has been intended in a natural way as background for geometry and modern analysis. It is not only a nice and powerful tool in many branches in Mathematics but also has had a beauty of its own. However, topology studies based on MBJ-neutrosophic sets could not be found in the literature. Then it is our aim to study basic properties for topology via MBJ-neutrosophic sets. Thus, before conducting our study, we would like to review topology studies based on fuzzy sets, intuitionistic fuzzy sets, interval-valued fuzzy sets and neutrosophic sets respectively. Chang [13] first applied fuzzy sets to topologies (See [14–20] for further researches). Coker [21] introduced the concept of intuitionistic fuzzy topologies and studied its some properties. After then, El-Latif and Khalaf [22], Singh and Srivastava [23], and Saleh [24] discussed connectedness and separation axioms in intuitionistic fuzzy topological spaces. Mondal and Samanta [25] defined an interval-valued fuzzy topology and dealt with some of its properties. After that time, Hongmei and Xuehai [26], and Kandil et al. [27] investigated separation axioms and in interval-valued fuzzy topological spaces. Smarandache [28], Lupia'n$\widetilde{e}$z [29, 30], and Salama and Alblowi [31] studied basic properties of neutrosophic topologies respectively. Kim et al. [32] defined an ordinary single valued topology by considering the single valued neutrosophic degree of openness for ordinary subsets and dealt with some of its properties. Recently, Lee et al. [33] studied topological structures based on cubic sets introduced by Jun et al. [34].

So to do this, we proceed with our research in the following order. First, we recall the concepts of fuzzy sets, interval-valued fuzzy sets and neutrosophic sets needed in the next sections. Second, we defined the inclusion, the union, the intersection and the compliment for MBJ-neutrosophic sets and obtain some of their properties. Moreover, we introduce the notion of MBJ-neutrosophic quasi-coincidences and discuss some of its properties. Third, We define two types of topologies and neighborhoods based on MBJ-neutrosophic sets and study their respective properties. Finally, by MBJ-neutrosophic set, the concepts of two types of base, subbase and local base are introduced and their respective properties are studied. Also we extend the concepts of first coutabilities and second countabilities in classical topology to the MBJ-neutrosophic sets and find the relationships between them. Furthermore, we present an example which the converse of a proposition does not hold.

Throughout this paper, let $I = [0, 1]$ and let J denote a index set.

2. Preliminaries

We list the notions of fuzzy sets, interval-valued fuzzy sets and neutrosophic sets needed in the next sections.

For a nonempty set X, a mapping $A : X \to I$ is called a *fuzzy set* in X. The special fuzzy set $\mathbf{0}$ [resp. 1] defined by:

$$\mathbf{0}(x) = 0 \text{ [resp. } \mathbf{1}(x) = 1]$$

is called the *fuzzy empty set* [resp. the *fuzzy whole set*] in X (See [1]). We denote the collection of all fuzzy sets in X by I^X.

For a nonempty set X, a mapping $\bar{A} = (A^{\in}, A^{\notin}) : X \to I \times I$ satisfying the following condition: for each $x \in X$,

$$0 \leq A^{\in}(x) + A^{\notin}(x) \leq 1$$

is called an *intuitionistic fuzzy set* in X (See [5]). The intuitionistic fuzzy sets $\bar{\mathbf{0}}$ and $\bar{\mathbf{1}}$ defined as follows: for each $x \in X$,

$$\bar{\mathbf{0}}(x) = (0, 1) \text{ and } \bar{\mathbf{1}}(x) = (1, 0)$$

are called the *intuitionistic fuzzy empty set* and the *intuitionistic fuzzy whole set* in X. $IFS(X)$ denotes the set of all intuitionistic fuzzy sets.

Let $[I]$ be the set of all closed subintervals of I and members of $[I]$ are called *interval numbers* and are denoted by $\widetilde{a}$, $\widetilde{b}$, $\widetilde{c}$, etc., where $\widetilde{a} = [a^-, a^+]$ and $0 \leq a^- \leq a^+ \leq 1$ (See [34] for the definitions of the order between two interval-valued numbers, the infimum and the supremum of an arbitrary interval-valued numbers).

For a nonempty set X, a mapping $\widetilde{A} = [A^-, A^+] : X \to [I]$ is called an *interval-valued fuzzy set* (briefly, IVFS) in X. The special interval-valued fuzzy set $\widetilde{\mathbf{0}}$ [resp. $\widetilde{\mathbf{1}}$] defined by: for each $x \in X$,

$$\widetilde{\mathbf{0}}(x) = [0, 0] \text{ [resp. } \widetilde{\mathbf{1}}(x) = [1, 1]]$$

is called the *interval-valued fuzzy empty set* [resp. the *interval-valued fuzzy whole set* in X (See [4]). $IVFS(X)$ denotes the set of all IVFSs in X.

For a nonempty set X, the form $A = \left\langle A^T, A^I, A^F \right\rangle$ is called a *neutrosophic set* in X, where $A^T : X \to I$ represents a *truth membership function*, $A^I : X \to I$ represents an *indeterminate membership* and $A^F : X \to I$ represents a *false membership function* (See [2]). We will denote the set of all neutrosophic sets in X as $NS(X)$.

3. Basic properties of MBJ-neutrosophic sets

By modifying some concepts related to fuzzy sets, interval-valued fuzzy sets, intuitionistic fuzzy sets and neutrosophic fuzzy sets, we give some basic definitions based on MBJ-neutrosophic sets.

Definition 3.1 ([6]). Let X be a nonempty set. Then the form

$$\mathcal{A} = \left\langle M_A, \widetilde{B}_A, J_A \right\rangle$$

is called an *MBJ-neutrosophic set* in X, where $M_A,\ J_A \in I^X$ which are called a *truth membership function* and a *false membership function* respectively, and $\widetilde{B}_A \in IVFS(X)$ which is called an *indeterminate interval-valued membership function.*

It is clear that for any subset A of a nonempty set X, $\langle \chi_A, [\chi_A, \chi_A], \chi_{A^c} \rangle \in MBJN(X)$, where χ_A denotes the characteristic function of A. Then we can consider an MBJ-neutrosophic set as a generalization of classical sets.

We can consider special MBJ-neutrosophic sets:

$$\ddot{\emptyset} = \left\langle \mathbf{0}, \widetilde{\mathbf{0}}, \mathbf{1} \right\rangle,\ \dot{\emptyset} = \left\langle \mathbf{0}, \widetilde{\mathbf{1}}, \mathbf{1} \right\rangle,\ \breve{\emptyset} = \left\langle \mathbf{0}, \widetilde{\mathbf{1}}, \mathbf{0} \right\rangle,\ \hat{\emptyset} = \left\langle \mathbf{0}, \widetilde{\mathbf{0}}, \mathbf{0} \right\rangle,$$

$$\ddot{X} = \left\langle \mathbf{1}, \widetilde{\mathbf{1}}, \mathbf{0} \right\rangle,\ \dot{X} = \left\langle \mathbf{1}, \widetilde{\mathbf{0}}, \mathbf{0} \right\rangle,\ \breve{X} = \left\langle \mathbf{1}, \widetilde{\mathbf{0}}, \mathbf{1} \right\rangle,\ \hat{X} = \left\langle \mathbf{1}, \widetilde{\mathbf{1}}, \mathbf{1} \right\rangle.$$

We will denote the set of all MBJ-neutrosophic sets in X as $MBJNS(X)$.

Definition 3.2. Let X be a nonempty set and let $\mathcal{A} \in MBJNS(X)$. Then the *complement* of $\mathcal{A}$, denoted by $\mathcal{A}^{c,1}$ resp. $\mathcal{A}^{c,2}$ and $\mathcal{A}^{c,3}$], is an MBJ-neutrosophic set in X defined as follows:

$$\mathcal{A}^{c,1} = \left\langle M^c_A, \widetilde{B}^c_A, J^c_A \right\rangle \text{ [resp. } \mathcal{A}^{c,2} = \left\langle J_A, \widetilde{B}_A, M_A \right\rangle \text{ and } \mathcal{A}^{c,3} = \left\langle J_A, \widetilde{B}^c_A, M_A \right\rangle].$$

Definition 3.3. Let X be a nonempty set and let $\mathcal{A},\ \mathcal{B} \in MBJNS(X)$. Then two type's inclusion relations between $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A} \sqsubset \mathcal{B}$ (called the $\circ$-*inclusion*) and $\mathcal{A} \Subset \mathcal{B}$ ($*$-*inclusion*), are defined as follows: for each $x \in X$,

(i) $\mathcal{A} \sqsubset \mathcal{B}$ if and only if $M_A(x) \leq M_A(x)$, $\widetilde{B}_A(x) \leq \widetilde{B}_B(x)$, $J_A(x) \geq J_A(x)$,

(ii) $\mathcal{A} \Subset \mathcal{B}$ if and only if $M_A(x) \leq M_A(x)$, $\widetilde{B}_A(x) \geq \widetilde{B}_B(x)$, $J_A(x) \geq J_A(x)$.

The following is an immediate consequence of Definitions 3.1 and 3.3.

Proposition 3.4. *Let X be a nonempty set and let $\mathcal{A} \in MBJNS(X)$. Then the followings are hold:*

(1) $\ddot{\emptyset} \sqsubset \mathcal{A} \sqsubset \ddot{X}$, $\ddot{\emptyset} \sqsubset \dot{\emptyset} \sqsubset \breve{\emptyset} \sqsubset \ddot{X}$, $\breve{X} \sqsubset \dot{X} \sqsubset \ddot{X}$, $\hat{X} \sqsubset \ddot{X}$,

(2) $\dot{\emptyset} \Subset \mathcal{A} \Subset \dot{X}$, $\dot{\emptyset} \Subset \hat{X} \Subset \ddot{X} \Subset \dot{X}$, $\dot{\emptyset} \Subset \ddot{\emptyset} \Subset \hat{\emptyset}$, $\dot{\emptyset} \Subset \breve{\emptyset} \Subset \hat{\emptyset}$.

Definition 3.5. Let X be a nonempty set and let $\mathcal{A},\ \mathcal{B} \in MBJNS(X)$.

(i) The *intersection* of $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A} \sqcap \mathcal{B}$ (called the $\circ$-*intersection*) and $\mathcal{A} \Cap \mathcal{B}$ (called the $*$-*intersection*), is a MBJ-neutrosophic set in X defined as follows: for each $x \in X$,

$$(\mathcal{A} \sqcap \mathcal{B})(x) = \left\langle M_A(x) \wedge M_B(x), \widetilde{B}_A(x) \wedge \widetilde{B}_B(x), J_A(x) \vee J_B(x) \right\rangle,$$

$$(\mathcal{A} \Cap \mathcal{B})(x) = \left\langle M_A(x) \wedge M_B(x), \widetilde{B}_A(x) \vee \widetilde{B}_B(x), J_A(x) \vee J_B(x) \right\rangle.$$

(ii) The *union* of $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A} \sqcup \mathcal{B}$ (called the $\circ$-*union*) and $\mathcal{A} \Cup \mathcal{B}$ (called the $*$-*union*), is an MBJ-neutrosophic set in X defined as follows: for each $x \in X$,

$$(\mathcal{A} \sqcup \mathcal{B})(x) = \left\langle M_A(x) \vee M_B(x), \widetilde{B}_A(x) \vee \widetilde{B}_B(x), J_A(x) \wedge J_B(x) \right\rangle,$$

$$(\mathcal{A} \Cup \mathcal{B})(x) = \left\langle M_A(x) \vee M_B(x), \widetilde{B}_A(x) \wedge \widetilde{B}_B(x), J_A(x) \wedge J_B(x) \right\rangle.$$

Definition 3.6. Let X be a nonempty set and let $(\mathcal{A}_j)_{j\in J} \subset MBJNS(X)$.

(i) The *intersection* of $(\mathcal{A}_j)_{j\in J}$, denoted by $\sqcap_{j\in J}\mathcal{A}_j$ and $\Cap_{j\in J}\mathcal{A}_j$, is a MBJ-neutrosophic set in X defined as follows: for each $x \in X$,

$$(\sqcap j \in J\mathcal{A}_j)(x) = \left\langle \bigwedge_{j\in J} M_{A_j}(x), \bigwedge_{j\in J} \widetilde{B}_{A_j}(x), \bigvee_{j\in J} J_{A_j}(x) \right\rangle,$$

$$(\Cap_{j\in J}\mathcal{A}_j)(x) = \left\langle \bigwedge_{j\in J} M_{A_j}(x), \bigvee_{j\in J} \widetilde{B}_{A_j}(x), \bigvee_{j\in J} J_{A_j}(x) \right\rangle.$$

(ii) The *union* of $(\mathcal{A}_j)_{j\in J}$, denoted by $\sqcup_{j\in J}\mathcal{A}_j$ and $\Cup_{j\in J}\mathcal{A}_j$, is a MBJ-neutrosophic set in X defined as follows: for each $x \in X$,

$$(\sqcup j \in J\mathcal{A}_j)(x) = \left\langle \bigvee_{j\in J} M_{A_j}(x), \bigvee_{j\in J} \widetilde{B}_{A_j}(x), \bigwedge_{j\in J} J_{A_j}(x) \right\rangle,$$

$$(\Cup_{j\in J}\mathcal{A}_j)(x) = \left\langle \bigvee_{j\in J} M_{A_j}(x), \bigwedge_{j\in J} \widetilde{B}_{A_j}(x), \bigwedge_{j\in J} J_{A_j}(x) \right\rangle.$$

From Definitions 3.1, 3.2, 3.3, 3.5 and 3.6, we obtain a similar consequence of (Theorem 1, [25]), (Corollary 2.8, [21]) and (Theorem 1.2, [35]).

Proposition 3.7. *Let X be a nonempty set, let $\mathcal{A},\ \mathcal{B},\ \mathcal{C} \in MBJNS(X)$ and let $(\mathcal{A}_j)_{j\in J} \subset MBJNS(X)$.*

(1) $\mathcal{A} \sqsubset \mathcal{A} \sqcap \mathcal{B},\ \mathcal{B} \sqsubset \mathcal{A} \sqcap \mathcal{B},\ \mathcal{A} \Subset \mathcal{A} \Cap \mathcal{B},\ \mathcal{B} \Subset \mathcal{A} \Cap \mathcal{B}$.

(2) $\mathcal{A} \sqcup \mathcal{B} \sqsubset \mathcal{A},\ \mathcal{A} \sqcup \mathcal{B} \sqsubset \mathcal{B},\ \mathcal{A} \Cup \mathcal{B} \Subset \mathcal{A},\ \mathcal{A} \Cup \mathcal{B} \Subset \mathcal{B}$.

(3) *If $\mathcal{A} \sqsubset \mathcal{B}$, then $\mathcal{A} \sqcap \mathcal{C} \sqsubset \mathcal{B} \sqcap \mathcal{C},\ \mathcal{A} \sqcup \mathcal{C} \sqsubset \mathcal{B} \sqcup \mathcal{C}$.*

(4) *If $\mathcal{A} \Subset \mathcal{B}$, then $\mathcal{A} \Cap \mathcal{C} \Subset \mathcal{B} \Cap \mathcal{C},\ \mathcal{A} \Cup \mathcal{C} \Subset \mathcal{B} \Cup \mathcal{C}$.*

(5) *If $\mathcal{A} \sqsubset \mathcal{B}$ and $\mathcal{B} \sqsubset \mathcal{C}$, then $\mathcal{A} \sqsubset \mathcal{C}$.*

(6) *If $\mathcal{A} \Subset \mathcal{B}$ and $\mathcal{B} \Subset \mathcal{C}$, then $\mathcal{A} \Subset \mathcal{C}$.*

(7) $\mathcal{A} \sqcup \mathcal{A} = \mathcal{A},\ \mathcal{A} \sqcap \mathcal{A} = \mathcal{A},\ \mathcal{A} \Cup \mathcal{A} = \mathcal{A},\ \mathcal{A} \Cap \mathcal{A} = \mathcal{A}.$

(8) $\mathcal{A} \sqcup \mathcal{B} = \mathcal{B} \sqcup \mathcal{A},\ \mathcal{A} \sqcap \mathcal{B} = \mathcal{B} \sqcap \mathcal{A},\ \mathcal{A} \Cup \mathcal{B} = \mathcal{B} \Cup \mathcal{A},\ \mathcal{A} \Cap \mathcal{B} = \mathcal{B} \Cap \mathcal{A}.$

(9) $\mathcal{A} \sqcup (\mathcal{B} \sqcup \mathcal{C}) = (\mathcal{A} \sqcup \mathcal{B}) \sqcup \mathcal{C},\ \mathcal{A} \sqcap (\mathcal{B} \sqcap \mathcal{C}) = (\mathcal{A} \sqcap \mathcal{B}) \sqcap \mathcal{C},$
$\mathcal{A} \Cup (\mathcal{B} \Cup \mathcal{C}) = (\mathcal{A} \Cup \mathcal{B}) \Cup \mathcal{C},\ \mathcal{A} \Cap (\mathcal{B} \Cap \mathcal{C}) = (\mathcal{A} \Cap \mathcal{B}) \Cap \mathcal{C}.$

(10) $\mathcal{A} \sqcup (\mathcal{B} \sqcap \mathcal{C}) = (\mathcal{A} \sqcup \mathcal{B}) \sqcap (\mathcal{A} \sqcup \mathcal{C}),\ \mathcal{A} \sqcap (\mathcal{B} \sqcup \mathcal{C}) = (\mathcal{A} \sqcap \mathcal{B}) \sqcup (\mathcal{A} \sqcap \mathcal{C}),$
$\mathcal{A} \Cup (\mathcal{B} \Cap \mathcal{C}) = (\mathcal{A} \Cup \mathcal{B}) \Cap (\mathcal{A} \Cup \mathcal{C}),\ \mathcal{A} \Cap (\mathcal{B} \Cup \mathcal{C}) = (\mathcal{A} \Cap \mathcal{B}) \Cup (\mathcal{A} \Cap \mathcal{C}).$

$(10)'$ $\mathcal{A} \sqcup (\sqcap_{j\in J}\mathcal{A}_j) = \sqcap_{j\in J}(\mathcal{A} \sqcup \mathcal{A}_j),\ \mathcal{A} \sqcap (\sqcup_{j\in J}\mathcal{A}_j) = \sqcup_{j\in J}(\mathcal{A} \sqcap \mathcal{A}_j),$
$\mathcal{A} \Cup (\Cap_{j\in J}\mathcal{A}_j) = \Cap_{j\in J}(\mathcal{A} \Cup \mathcal{A}_j),\ \mathcal{A} \Cap (\Cup_{j\in J}\mathcal{A}_j) = \Cup_{j\in J}(\mathcal{A} \Cap \mathcal{A}_j).$

(11) $(\mathcal{A} \sqcup \mathcal{B})^{c,2} = \mathcal{A}^{c,2} \Cap \mathcal{B}^{c,2},\ (\mathcal{A} \sqcup \mathcal{B})^{c,i} = \mathcal{A}^{c,i} \sqcap \mathcal{B}^{c,i},$
$(\mathcal{A} \sqcap \mathcal{B})^{c,2} = \mathcal{A}^{c,2} \Cup \mathcal{B}^{c,2},\ (\mathcal{A} \sqcap \mathcal{B})^{c,i} = \mathcal{A}^{c,i} \sqcup \mathcal{B}^{c,i},$
$(\mathcal{A} \Cup \mathcal{B})^{c,2} = \mathcal{A}^{c,2} \sqcap \mathcal{B}^{c,2},\ (\mathcal{A} \Cup \mathcal{B})^{c,i} = \mathcal{A}^{c,i} \Cap \mathcal{B}^{c,i},$
$(\mathcal{A} \Cap \mathcal{B})^{c,2} = \mathcal{A}^{c,2} \sqcup \mathcal{B}^{c,2},\ (\mathcal{A} \Cap \mathcal{B})^{c,i} = \mathcal{A}^{c,i} \Cup \mathcal{B}^{c,i}$ *for* $i = 1,\ 3.$

(12) $\ddot{\emptyset}^{c,2} = \dot{X},\ \ddot{X}^{c,2} = \dot{\emptyset},\ \ddot{\emptyset}^{c,i} = \ddot{X},\ \ddot{X}^{c,i} = \ddot{\emptyset},$
$\dot{\emptyset}^{c,2} = \ddot{X},\ \dot{X}^{c,2} = \ddot{\emptyset},\ \dot{\emptyset}^{c,i} = \dot{X},\ \dot{X}^{c,i} = \dot{\emptyset}$ *for* $i = 1,\ 3,$
$\breve{\emptyset}^{c,1} = \breve{X},\ \breve{X}^{c,1} = \breve{\emptyset},\ \breve{\emptyset}^{c,2} = \breve{\emptyset},\ \breve{X}^{c,2} = \breve{X},\ \breve{\emptyset}^{c,3} = \hat{\emptyset},\ \breve{X}^{c,3} = \hat{X},$
$\hat{\emptyset}^{c,1} = \hat{X},\ \hat{X}^{c,1} = \hat{\emptyset},\ \hat{\emptyset}^{c,2} = \hat{\emptyset},\ \hat{X}^{c,2} = \hat{X},\ \hat{\emptyset}^{c,3} = \breve{\emptyset},\ \hat{X}^{c,3} = \breve{X}.$

(13) $\mathcal{A} \sqcap \mathcal{A}^{c,i} \neq \ddot{\emptyset},\ \mathcal{A} \sqcup \mathcal{A}^{c,i} \neq \ddot{X}$ *and* $\mathcal{A} \Cap \mathcal{A}^{c,i} \neq \dot{\emptyset},\ \mathcal{A} \Cup \mathcal{A}^{c,i} \neq \dot{X}$ *in general for* $i = 1,\ 3$ (See Example 3.8).

Example 3.8. Let $X = \{a, b, c\}$ and let $\mathcal{A}$ be the MBJ-neutrosophic set in X given by: for each $x \in X$,

$$\mathcal{A} = \langle 0.5, [0.5, 0.5], 0, 5\rangle\,.$$

Then we can easily check that $\mathcal{A} \sqcap \mathcal{A}^{c,i} \neq \ddot{\emptyset},\ \mathcal{A} \sqcup \mathcal{A}^{c,i} \neq \ddot{X}$ and $\mathcal{A} \Cap \mathcal{A}^{c,i} \neq \dot{\emptyset},\ \mathcal{A} \Cup \mathcal{A}^{c,i} \neq \dot{X}$.

Remark 3.9. From Propositions 3.4 and 3.7, we can see that

$$(MBJNS(X), \sqcup, \sqcap, ^{c,i}, \ddot{\emptyset}, \ddot{X}) \text{ and } (MBJNS(X), \Cup, \Cap, ^{c,i}, \dot{\emptyset}, \dot{X})$$

form Boolean algebras except the condition (13) of Proposition 3.7.

Let $M_a,\ J_a \in I$ and let $\widetilde{B}_a \in [I]$. Then the form

$$\widetilde{\overline{a}} = \left\langle M_a, \widetilde{B}_a, J_a \right\rangle = \langle a, \widetilde{a}, \overline{a}\rangle$$

is called an *MBJ-neutrosophic number* (briefly, MBJNN). We can consider the following special MBJNNs:

$$\ddot{0} = \left\langle 0, \widetilde{0}, 1 \right\rangle,\ \dot{0} = \left\langle 0, \widetilde{1}, 1 \right\rangle,\ \breve{0} = \left\langle 0, \widetilde{1}, 0 \right\rangle,\ \hat{0} = \left\langle 0, \widetilde{0}, 0 \right\rangle,$$

$$\ddot{1} = \left\langle 1, \widetilde{1}, 0 \right\rangle,\ \dot{1} = \left\langle 1, \widetilde{0}, 0 \right\rangle,\ \breve{1} = \left\langle 1, \widetilde{0}, 1 \right\rangle,\ \hat{1} = \left\langle 1, \widetilde{1}, 1 \right\rangle.$$

We will denote the set of all MBJNNs as $I \times [I] \times I$.

Definition 3.10. Let $\widetilde{\overline{a}}$, $\widetilde{\overline{b}}$ be two MBJNNs and let $(\widetilde{\overline{a}}_j)_{j\in J}$ be a family of MBJNNs.

(i) The *order* between $\widetilde{\overline{a}}$ and $\widetilde{\overline{b}}$, denoted by $\widetilde{\overline{a}} \leq^\circ \widetilde{\overline{b}}$ [resp. $\widetilde{\overline{a}} \leq^* \widetilde{\overline{b}}$], is defined as follows:

$$\widetilde{\overline{a}} \leq^\circ \widetilde{\overline{b}} \iff a \leq b,\ \widetilde{a} \leq \widetilde{a},\ \overline{a} \geq \overline{b} \text{ [resp. } \widetilde{\overline{a}} \leq^* \widetilde{\overline{b}} \iff a \leq b,\ \widetilde{a} \geq \widetilde{a},\ \overline{a} \geq \overline{b}].$$

(ii) The *equality* of $\widetilde{\overline{a}}$ and $\widetilde{\overline{b}}$, denoted by $\widetilde{\overline{a}} = \widetilde{\overline{b}}$, is defined as follows:

$$\widetilde{\overline{a}} = \widetilde{\overline{b}} \iff \widetilde{\overline{a}} \leq^\circ \widetilde{\overline{b}},\ \widetilde{\overline{b}} \leq^\circ \widetilde{\overline{a}} \text{ or } \widetilde{\overline{a}} \leq^* \widetilde{\overline{b}},\ \widetilde{\overline{b}} \leq^* \widetilde{\overline{a}}.$$

(iii) The *infimum* of $(\widetilde{\overline{a}}_j)_{j\in J}$, denoted by $\bigwedge^\circ{}_{j\in J}\widetilde{\overline{a}}_j$ [resp. $\bigwedge^*_{j\in J}\widetilde{\overline{a}}_j$], is an MBJ-neutrosophic number defined as follows:

$$\overset{\circ}{\bigwedge}{}_{j\in J}\widetilde{\overline{a}}_j = \left\langle \bigwedge_{j\in J} a_j, \bigwedge_{j\in J} \widetilde{a}_j, \bigvee_{j\in J} \overline{a}_j \right\rangle \text{ [resp. } \overset{*}{\bigwedge_{j\in J}} \widetilde{\overline{a}}_j = \left\langle \bigwedge_{j\in J} a_j, \bigvee_{j\in J} \widetilde{a}_j, \bigvee_{j\in J} \overline{a}_j \right\rangle].$$

and

(iv) The *supremum* of $(\widetilde{\overline{a}}_j)_{j\in J}$, denoted by $\bigvee^\circ{}_{j\in J}\widetilde{\overline{a}}_j$ [resp. $\bigvee^*_{j\in J}\widetilde{\overline{a}}_j$], is an MBJ-neutrosophic number defined as follows:

$$\overset{\circ}{\bigvee}{}_{j\in J}\widetilde{\overline{a}}_j = \left\langle \bigvee_{j\in J} a_j, \bigvee_{j\in J} \widetilde{a}_j, \bigwedge_{j\in J} \overline{a}_j \right\rangle \text{ [resp. } \overset{*}{\bigvee_{j\in J}} \widetilde{\overline{a}}_j = \left\langle \bigvee_{j\in J} a_j, \bigwedge_{j\in J} \widetilde{a}_j, \bigwedge_{j\in J} \overline{a}_j \right\rangle].$$

(v) The *complement* of $\widetilde{\overline{a}}$, denoted by $\widetilde{\overline{a}}^{c,1}$ [resp. $\widetilde{\overline{a}}^{c,2}$ and $\widetilde{\overline{a}}^{c,3}$], is an MBJ-neutrosophic number defined as follows:

$$\widetilde{\overline{a}}^{c,1} = \langle 1-a, \widetilde{a}^c, 1-\overline{a}\rangle \text{ [resp. } \widetilde{\overline{a}}^{c,2} = \langle \overline{a}, \widetilde{a}, a\rangle \text{ and } \widetilde{\overline{a}}^{c,2} = \langle \overline{a}, \widetilde{a}^c, a\rangle].$$

Remark 3.11. (1) Definitions 3.3, 3.5 and 3.6 are redefined by Definition 3.10 as follows. Let X be a nonempty set, $\mathcal{A}$, $\mathcal{B} \in MBJNS(X)$ and let $(\mathcal{A}_j) \subset MBJN(X)$. Then

- $\mathcal{A} \sqsubset \mathcal{B}$ if and only if $\mathcal{A}(x) \leq^\circ \mathcal{B}(x)$ for each $x \in X$,
- $\mathcal{A} \Subset \mathcal{B}$ if and only if $\mathcal{A}(x) \leq^* \mathcal{B}(x)$ for each $x \in X$,
- $(\mathcal{A} \sqcap \mathcal{B})(x) = \mathcal{A}(x) \wedge^\circ \mathcal{B}(x)$, $(\mathcal{A} \sqcup \mathcal{B})(x) = \mathcal{A}(x) \vee^\circ \mathcal{B}(x)$ for each $x \in X$,
- $(\mathcal{A} \Cap \mathcal{B})(x) = \mathcal{A}(x) \wedge^* \mathcal{B}(x)$, $(\mathcal{A} \Cup \mathcal{B})(x) = \mathcal{A}(x) \vee^* \mathcal{B}(x)$ for each $x \in X$,
- $(\sqcap_{j\in J}\mathcal{A}_j)(x) = \bigwedge^\circ \mathcal{A}_j(x)$, $(\sqcup_{j\in J}\mathcal{A}_j)(x) = \bigvee^\circ \mathcal{A}_j(x)$ for each $x \in X$,
- $(\Cap_{j\in J}\mathcal{A}_j)(x) = \bigwedge^* \mathcal{A}_j(x)$, $(\Cup_{j\in J}\mathcal{A}_j)(x) = \bigvee^* \mathcal{A}_j(x)$ for each $x \in X$.

(2) We can easily see that MBJNNs have similar properties to Proposition 3.7, and then

$$(I \times [I] \times I, \wedge^\circ, \vee^\circ, {}^{c,i}, \ddot{0}, \ddot{1}) \text{ and } (I \times [I] \times I, \wedge^*, \vee^*, {}^{c,i}, \dot{0}, \dot{1})$$

form Boolean algebras except the property corresponding the condition (13) of Proposition 3.7.

Definition 3.12. Let X be a nonempty set, let $\widetilde{\overline{a}} \in I \times [I] \times I$ and let $\mathcal{A} \in MBJNS(X)$. Then we define two type's MBJ-neutrosophic points as followings"

(i) A is called a $\circ$-*MBJ-neutrosophic point* (briefly, $\circ$-MBJNP) with the support $x \in X$ and the value $\widetilde{\overline{a}}$ with $a > 0,\ a^- > 0,\ \overline{a} < 1$, denoted by $A = x^{\circ}_{\widetilde{\overline{a}}}$, if for each $y \in X$,

$$x^{\circ}_{\widetilde{\overline{a}}} = \begin{cases} \widetilde{\overline{a}} & \text{if } y = x \\ \ddot{0} & \text{otherwise,} \end{cases}$$

(ii) A is called a $*$- *MBJ-neutrosophic point* (briefly, $*$-MBJNP) with the support $x \in X$ and the value $\widetilde{\overline{a}}$ with $a > 0,\ a^+ < 1,\ \overline{a} < 1$, denoted by $A = x^{\circ}_{\widetilde{\overline{a}}}$, if for each $y \in X$,

$$x^{*}_{\widetilde{\overline{a}}} = \begin{cases} \widetilde{\overline{a}} & \text{if } y = x \\ \dot{0} & \text{otherwise.} \end{cases}$$

We denote the set of all fuzzy points in X by $MBJN_P(X)$.

For a nonempty set X, let x_a [resp. $x_{\widetilde{a}}$] denotes the fuzzy point [resp. interval-valued fuzzy point] in X with the support $x \in X$ and the value $a \in I$ [resp. $\widetilde{a} \in [I]$] (See [16] [resp. [25]]). We denote the set of all fuzzy points [resp. interval-valued fuzzy points] in X as $F_P(X)$ [resp. $IVF_P(X)$]. It is well-known that $A = \bigcup_{x_a \in A} x_a$ for each $A \in I^X$ (See [18]) and $\widetilde{A} = \bigcup_{x_{\widetilde{a}} \in \widetilde{A}} x_{\widetilde{a}}$ for each $\widetilde{A} \in IVFS(X)$ (See [25]).

Definition 3.13. Let X be a nonempty set, let $x^{\circ}_{\widetilde{\overline{a}}},\ x^{*}_{\widetilde{\overline{a}}} \in MBJN_P(X)$ and let $\mathcal{A} \in MBJNS(X)$. Then

(i) $x^{\circ}_{\widetilde{\overline{a}}}$ is said to *belong to* $\mathcal{A}$, denoted by $x^{\circ}_{\widetilde{\overline{a}}} \in \mathcal{A}$, if $\widetilde{\overline{a}} \leq^{\circ} \mathcal{A}(x)$,

(ii) $x^{*}_{\widetilde{\overline{a}}}$ is said to *belong to* $\mathcal{A}$, denoted by $x^{*}_{\widetilde{\overline{a}}} \in \mathcal{A}$, if $\widetilde{\overline{a}} \leq^{*} \mathcal{A}(x)$,

(iii) $x^{\circ}_{\widetilde{\overline{a}}}$ is said to $\circ$-*quasi-coincident with* $\mathcal{A}$, denoted by $x^{\circ}_{\widetilde{\overline{a}}} q^i \mathcal{A}$, if $\mathcal{A}(x) >^{\circ} \mathcal{A}^i(x)$ $(i = 1,\ 2,\ 3)$,

(iv) $x^{*}_{\widetilde{\overline{a}}}$ is said to $*$-*quasi-coincident with* $\mathcal{A}$, denoted by $x^{*}_{\widetilde{\overline{a}}} q^i \mathcal{A}$, if $\mathcal{A}(x) >^{*} \mathcal{A}^{c,i}(x)$ $(i = 1,\ 2,\ 3)$,

(v) $\mathcal{A}$ is said to be $\circ$-*quasi-coincident with* $\mathcal{B}$, denoted by $\mathcal{A} q^{\circ,i} \mathcal{B}$, if there is $x \in X$ such that $\mathcal{A}(x) >^{\circ} \mathcal{B}^i(x)$ $(i = 1,\ 2,\ 3)$,

(vi) $\mathcal{A}$ is said to be $*$-*quasi-coincident with* $\mathcal{B}$, denoted by $\mathcal{A} q^{*.i} \mathcal{B}$, if there is $x \in X$ such that $\mathcal{A}(x) >^{*} \mathcal{B}^i(x)$ $(i = 1,\ 2,\ 3)$.

It is obvious that $\mathcal{A} = \sqcup_{x^{\circ}_{\widetilde{\overline{a}}} \in \mathcal{A}} x^{\circ}_{\widetilde{\overline{a}}}$ and $\mathcal{A} = \Cup_{x^{*}_{\widetilde{\overline{a}}} \in \mathcal{A}} x^{*}_{\widetilde{\overline{a}}}$ for each $\mathcal{A} \in MBJN(X)$.

For a fuzzy point x_a and two fuzzy sets $A,\ B$, $x_a q A$ [resp. AqB] means that x_a is quasi-coincident with A [resp. A is quasi-coincident with B] (See [16]). Also, an interval-valued fuzzy point $x_{\widetilde{a}}$ and two interval-valued fuzzy sets $\widetilde{A},\ \widetilde{B}$, $x_{\widetilde{a}} q \widetilde{A}$ [resp. $\widetilde{A} q \widetilde{B}$] means that $x_{\widetilde{a}}$ is quasi-coincident with $\widetilde{A}$ [resp. $\widetilde{A}$ is quasi-coincident with $\widetilde{B}$] (See [33]).

Throughout this paper, for any fuzzy set A [resp. interval-valued fuzzy set $\widetilde{A}$] in X, if $a \geq A(x)$ [resp. $\widetilde{a} \geq \widetilde{A}(x)$], then we say that x_a $\circ$-*belongs to* A [resp. $x_{\widetilde{a}}$ $*$-*belongs to* $\widetilde{A}$] and denoted by $x^{\circ}_a \in A$ [resp. $x^{*}_{\widetilde{a}} \in \widetilde{A}$]. Moreover, if $a < A^c(x)$ [resp. $\widetilde{a} < \widetilde{A}^c(x)$], then we say that

x_a *is* $\circ$*-quasi-coincident with* A [resp. $x_{\tilde{a}}$ *is* $*$*-quasi-coincident with* $\widetilde{A}$] and denoted by $x_a q^\circ A$ [resp. $x_{\tilde{a}} q^* \widetilde{A}$].

Remark 3.14. From Definition 3.13, we can easily see that the followings hold.

(1) $x^\circ_{\tilde{\bar{a}}} \in \mathcal{A}$ if and only if $x_a \in M_A$, $x_{\tilde{a}} \in \widetilde{B}_A$, $x^\circ_{\bar{a}} \in J_A$.

(2) $x^*_{\tilde{\bar{a}}} \in \mathcal{A}$ if and only if $x_a \in M_A$, $x^*_{\tilde{a}} \in \widetilde{B}_A$, $x^\circ_{\bar{a}} \in J_A$.

(3) $x^\circ_{\tilde{\bar{a}}} q^1 \mathcal{A}$ if and only if $x_a q M_A$, $x_{\tilde{a}} q \widetilde{B}_A$, $x_{\bar{a}} q^\circ J_A$,
$x^\circ_{\tilde{\bar{a}}} q^2 \mathcal{A}$ if and only if $a > J_A(x)$, $\tilde{a} > \widetilde{B}_A(x)$, $\bar{a} < M_A(x)$,
$x^\circ_{\tilde{\bar{a}}} q^3 \mathcal{A}$ if and only if $a > J_A(x)$, $x_{\tilde{a}} q \widetilde{B}_A$, $\bar{a} < M_A(x)$.

(4) $x^*_{\tilde{\bar{a}}} q^1 \mathcal{A}$ if and only if $x_a q M_A$, $x_{\tilde{a}} q^* \widetilde{B}_A$, $x_{\bar{a}} q^\circ J_A$,
$x^*_{\tilde{\bar{a}}} q^2 \mathcal{A}$ if and only if $a > J_A(x)$, $\tilde{a} < \widetilde{B}_A(x)$, $\bar{a} < M_A(x)$,
$x^*_{\tilde{\bar{a}}} q^3 \mathcal{A}$ if and only if $a > J_A(x)$, $\tilde{a} < \widetilde{B}^c_A(x)$, $\bar{a} < M_A(x)$.

From now on, we will use only $\mathcal{A}^{c,1}$ as the complement of an MBJNS $\mathcal{A}$ in X and write $\mathcal{A}^{c,1} = \mathcal{A}^c$. Also, we use $x^\circ_{\tilde{\bar{a}}} q\mathcal{A}$, $x^*_{\tilde{\bar{a}}} q\mathcal{A}$, $\mathcal{A} q^\circ \mathcal{B}$ and $\mathcal{A} q^* \mathcal{B}$ instead of $x^\circ_{\tilde{\bar{a}}} q^1 \mathcal{A}$, $x^*_{\tilde{\bar{a}}} q^1 \mathcal{A}$, $\mathcal{A} q^{\circ,1} \mathcal{B}$ and $\mathcal{A} q^{*,1} \mathcal{B}$ respectively.

If there is $x \in X$ such that $\mathcal{A}(x) >^\circ \mathcal{B}^c(x)$ [resp. $\mathcal{A}(x) >^* \mathcal{B}^c(x)$], then we say that $\mathcal{A}$ and $\mathcal{B}$ are $\circ$- [resp. $*$-]*quasi-coincident* (*with each other*) *at* x. We say that $\mathcal{A}$ *is not* $\circ$-[resp. $*$-]*quasi-coincident with* $\mathcal{B}$, denoted by $\mathcal{A} \neg q^\circ \mathcal{B}$ [resp. $\mathcal{A} \neg q^* \mathcal{B}$], if the following conditions hold:

$$M_A \neg q M_B,\ \widetilde{B}_A \neg q \widetilde{B}_B,\ J_A \neg q^\circ J_B \text{ [resp. } M_A \neg q M_B,\ \widetilde{B}_A \neg q^* \widetilde{B}_B,\ J_A \neg q^\circ J_B],\ \text{i.e.,}$$

$$M_A(x) \leq M^c_B(x),\ \widetilde{B}_A(x) \leq \widetilde{B}^c_B(x),\ J_A(x) \geq J^c_B(x),$$

$$M_A(x) \leq M^c_B(x),\ \widetilde{B}_A(x) \geq \widetilde{B}^c_B(x),\ J_A(x) \geq J^c_B(x)$$

for each $x \in X$.

Definition 3.15. Let $\mathcal{A},\ \mathcal{B} \in MBJNS(X)$. Then

(i) $\mathcal{A}$ and $\mathcal{B}$ are said to be $\circ$-*intersecting*, if there is $x \in X$ such that

$$(M_A \cap M_B)(x) \neq 0,\ (\widetilde{B}_A \cap \widetilde{B}_B)(x) \neq [0,0],\ (J_A \cup J_B)(x) \neq 1,$$

(ii) $\mathcal{A}$ and $\mathcal{B}$ are said to be $*$-*intersecting*, if there is $x \in X$ such that

$$(M_A \cap M_B)(x) \neq 0,\ (\widetilde{B}_A \cup \widetilde{B}_B)(x) \neq [1,1],\ (J_A \cup J_B)(x) \neq 1.$$

In either case, we say that $\mathcal{A}$ and $\mathcal{B}$ $\circ$-*intersect at* x [resp. $*$-*intersect at* x].

It is obvious that if $\mathcal{A}$ and $\mathcal{B}$ are $\circ$- [resp. $*$-]quasi-coincident at x, then they are $\circ$- [resp. $*$-]intersect at x.

The following is an immediate consequence of Definition 3.13 and Remark 3.14.

Lemma 3.16. *Let $x^{\circ}_{\tilde{\tilde{a}}}$, $x^{*}_{\tilde{\tilde{a}}} \in MBJN_P(X)$ and let $\mathcal{A} \in MBJNS(X)$. Then*

(1) $x^{\circ}_{\tilde{\tilde{a}}} \in \mathcal{A}$ *if and only if* $x^{\circ}_{\tilde{\tilde{a}}} \neg q \mathcal{A}^c$,

(2) $x^{*}_{\tilde{\tilde{a}}} \in \mathcal{A}$ *if and only if* $x^{*}_{\tilde{\tilde{a}}} \neg q \mathcal{A}^c$.

The following is an immediate consequence of Definitions 3.3 and 3.13, and Lemma 3.5 (1).

Lemma 3.17. *Let $\mathcal{A}$, $\mathcal{B} \in MBJNS(X)$. Then the followings are equivalent:*

(1) $\mathcal{A} \sqsubset \mathcal{B}$,

(2) $x^{\circ}_{\tilde{\tilde{a}}} \in \mathcal{B}$ *for each* $x^{\circ}_{\tilde{\tilde{a}}} \in \mathcal{A}$,

(3) $\mathcal{A} \neg q \mathcal{B}^c$.

Also, the following is an immediate consequence of Definitions 3.3 and 3.13, and Lemma 3.5 (2).

Lemma 3.18. *Let $\mathcal{A}$, $\mathcal{B} \in MBJNS(X)$. Then the followings are equivalent:*

(1) $\mathcal{A} \Subset \mathcal{B}$,

(2) $x^{*}_{\tilde{\tilde{a}}} \in \mathcal{B}$ *for each* $x^{*}_{\tilde{\tilde{a}}} \in \mathcal{A}$,

(3) $\mathcal{A} \neg q \mathcal{B}^c$.

Lemma 3.19. *Let $\mathcal{A}$, $\mathcal{B} \in MBJNS(X)$, let $(\mathcal{A})_{j \in J} \subset MBJNS(X)$ and let $x^{\circ}_{\tilde{\tilde{a}}} \in MBJN_P(X)$.*

(1) $x^{\circ}_{\tilde{\tilde{a}}} q(\sqcup_{j \in J} \mathcal{A}_j)$ *if and only if there is* $j_0 \in J$ *such that* $x^{\circ}_{\tilde{\tilde{a}}} q \mathcal{A}_{j_0}$.

(2) $x^{\circ}_{\tilde{\tilde{a}}} q(\mathcal{A} \sqcap \mathcal{B})$ *if and only if* $x^{\circ}_{\tilde{\tilde{a}}} q \mathcal{A}$ *and* $x^{\circ}_{\tilde{\tilde{a}}} q \mathcal{B}$.

Proof. The proof is easy. □

Lemma 3.20. *Let $\mathcal{A}$, $\mathcal{B} \in MBJNS(X)$, let $(\mathcal{A})_{j \in J} \subset MBJNS(X)$ and let $x^{*}_{\tilde{\tilde{a}}} \in MBJN_P(X)$.*

(1) $x^{*}_{\tilde{\tilde{a}}} q(\Cup_{j \in J} \mathcal{A}_j)$ *if and only if there is* $j_0 \in J$ *such that* $x^{*}_{\tilde{\tilde{a}}} q \mathcal{A}_{j_0}$.

(2) $x^{*}_{\tilde{\tilde{a}}} q(\mathcal{A} \Cap \mathcal{B})$ *if and only if* $x^{*}_{\tilde{\tilde{a}}} q \mathcal{A}$ *and* $x^{*}_{\tilde{\tilde{a}}} q \mathcal{B}$.

Proof. The proof is similar to Proposition 3.19. □

Lemma 3.21. *Let X be a nonempty set and let $A \in I^X$ such that $A(x) \neq 1$ for each $x \in X$. Then $x_{a^c} q^{\circ} A$ for each $a \in I$ such that $A(x) < a < 1$.*

Proof. The proof is straightforward. □

Lemma 3.22. *Let X be a nonempty set and let $\widetilde{A} \in IVFS(X)$ such that $\widetilde{A}(x) \neq [1, 1]$ for each $x \in X$. Then $x_{\tilde{a}^c} q^{*} \widetilde{A}$ for each $\tilde{a} \in [I]$ such that $\widetilde{A}(x) < \tilde{a} < [1, 1]$.*

Proof. The proof is straightforward. □

Definition 3.23. Let X, Y be nonempty sets and let $f : X \to Y$ and let $\mathcal{A} \in MBJNS(X)$, $\mathcal{B} \in MBJNS(Y)$.

(i) The *pre-image* of $\mathcal{B}$ under f, denoted by denoted by

$$f^{-1}(\mathcal{B}) = \left\langle f^{-1}(M_B), f^{-1}(\widetilde{B}_B), f^{-1}(J_B) \right\rangle,$$

is an MBJ-neutrosophic set in X defined as follows: for each $x \in X$,

$$f^{-1}(M_B)(x) = M_B(f(x)),\ f^{-1}(\widetilde{B}_B)(x) = \widetilde{B}_B(f(x)),\ f^{-1}(J_B)(x) = J_B(f(x)).$$

(ii) The $\circ$-*image* and the $*$-*image* of $\mathcal{A}$ under f, denoted by $f^\circ(\mathcal{A})$ and $f^*(\mathcal{A})$, are cubic sets in Y respectively defined as follows: for each $y \in Y$,

$$f^\circ(\mathcal{A})(y) = \begin{cases} \left\langle \bigvee_{x\in f^{-1}(y)} M_A(x), \bigvee_{x\in f^{-1}(y)} \widetilde{B}_A(x), \bigwedge_{x\in f^{-1}(y)} J_A(x) \right\rangle & \text{if } f^{-1}(y) \neq \emptyset \\ \ddot{0} & \text{otherwise,} \end{cases}$$

$$f^*(\mathcal{A})(y) = \begin{cases} \left\langle \bigvee_{x\in f^{-1}(y)} M_A(x), \bigwedge_{x\in f^{-1}(y)} \widetilde{B}_A(x), \bigwedge_{x\in f^{-1}(y)} J_A(x) \right\rangle & \text{if } f^{-1}(y) \neq \emptyset \\ \dot{0} & \text{otherwise.} \end{cases}$$

Remark 3.24. Let us denote $\bigwedge_{x\in f^{-1}(y)} J_A(x)$ and $\bigwedge_{x\in f^{-1}(y)} \widetilde{B}_A(x)$ as $f^\circ(J_A)$ and $f^*(\widetilde{B}_A)$ respectively. Then we can see that

$$f^\circ(\mathcal{A}) = \left\langle f(M_A, f(\widetilde{B}_A), f^\circ(J_A) \right\rangle,\ f^*(\mathcal{A}) = \left\langle f(M_A, f^*(\widetilde{B}_A), f^\circ(J_A) \right\rangle.$$

We have a similar consequence of (Lemma 1.1, [17]), (Theorem 2, [25]), (Corollary 2.10, [21]) and (Theorem 1.10, [35]).

Proposition 3.25. *Let X, Y be nonempty sets, let $f : X \to Y$ be a mapping, let $\mathcal{A}$, $\mathcal{A}_1, \mathcal{A}_2 \in MBJNS(X)$, $\mathcal{B}$, $\mathcal{B}_1, \mathcal{B}_2 \in MBJNS(Y)$, let $(\mathcal{A}_j)_{j\in J} \subset MBJNS(X)$ and let $(\mathcal{B}_j)_{j\in J} \subset MBJNS(Y)$.*

(1) $f^{-1}(\mathcal{B}^c) = [f^{-1}(\mathcal{B})]^c$.

(2) $f^{-1}(\ddot{\emptyset}) = \ddot{\emptyset},\ f^{-1}(\ddot{Y}) = \ddot{Y}, f^{-1}(\dot{\emptyset}) = \dot{\emptyset},\ f^{-1}(\dot{Y}) = \dot{Y}$.

(3) $f^\circ(\mathcal{A}^c) \sqsupset [f^\circ(\mathcal{A})]^c$ *and* $f^*(\mathcal{A}^c) \Supset [f^*(\mathcal{A})]^c$, *if f is injective, then* $f^\circ(\mathcal{A}^c) = [f^\circ(\mathcal{A})]^c$ *and* $f^*(\mathcal{A}^c) = [f^*(\mathcal{A})]^c$.

(4) *If* $\mathcal{B}_1 \sqsubset \mathcal{B}_2$, *then* $f^{-1}(\mathcal{B}_1) \sqsubset f^{-1}(\mathcal{B}_2)$.

(5) *If* $\mathcal{B}_1 \Subset \mathcal{B}_2$, *then* $f^{-1}(\mathcal{B}_1) \Subset f^{-1}(\mathcal{B}_2)$.

(6) *If* $\mathcal{A}_1 \sqsubset \mathcal{A}_2$, *then* $f^\circ(\mathcal{A}_1) \sqsubset f^\circ(\mathcal{A}_2)$.

(7) *If* $\mathcal{A}_1 \Subset \mathcal{A}_2$, *then* $f^*(\mathcal{A}_1) \Subset f^*(\mathcal{A}_2)$.

(8) $f^\circ(f^{-1}(\mathcal{B})) \sqsubset \mathcal{B}$. *In particular, if f is surjective, then* $f^\circ(f^{-1}(\mathcal{B})) = \mathcal{B}$.

(9) $\mathcal{A} \sqsubset f^{-1}(f^\circ(\mathcal{A}))$. *In particular, if f is injective, then* $\mathcal{A} = f^{-1}(f^\circ(\mathcal{A}))$.

(10) $f^*(f^{-1}(\mathcal{B})) \Subset \mathcal{B}$. *In particular, if* f *is surjective, then* $f^*(f^{-1}(\mathcal{B})) = \mathcal{B}$.

(11) $\mathcal{A} \Subset f^{-1}(f^*(\mathcal{A}))$. *In particular, if* f *is injective, then* $\mathcal{A} = f^{-1}(f^*(\mathcal{A}))$.

(12) *If* $f^\circ(\mathcal{A}) \sqsubset \mathcal{B}$, *then* $\mathcal{A} \sqsubset f^{-1}(\mathcal{B})$.

(13) *If* $f^*(\mathcal{A}) \Subset \mathcal{B}$, *then* $\mathcal{A} \Subset f^{-1}(\mathcal{B})$.

(14) *For each* $x^\circ_{\tilde{\tilde{a}}} \in MBJN_P(X)$, $f^\circ(x^\circ_{\tilde{\tilde{a}}}) \in MBJN_P(Y)$ *and* $f^\circ(x^\circ_{\tilde{\tilde{a}}}) = [f(x)]^\circ_{\tilde{\tilde{a}}}$.

(15) *For each* $x^\circ_{\tilde{\tilde{a}}} \in MBJN_P(X)$, *if* $x^\circ_{\tilde{\tilde{a}}} q\mathcal{A}$, *then* $f^\circ(x^\circ_{\tilde{\tilde{a}}}) q f^\circ(\mathcal{A})$.

(16) *For each* $x^*_{\tilde{\tilde{a}}} \in MBJN_P(X)$, $f^*(x^*_{\tilde{\tilde{a}}}) \in MBJN_P(Y)$ *and* $f^*(x^*_{\tilde{\tilde{a}}}) = [f(x)]^*_{\tilde{\tilde{a}}}$.

(17) *For each* $x^*_{\tilde{\tilde{a}}} \in MBJN_P(X)$, *if* $x^*_{\tilde{\tilde{a}}} q\mathcal{A}$, *then* $f^*(x^*_{\tilde{\tilde{a}}}) q f^*(\mathcal{A})$.

(18) $f^\circ(\sqcup_{j\in J}\mathcal{A}_j) = \sqcup_{j\in J} f^\circ(\mathcal{A}_j)$.

(19) $f^*(\Cup_{j\in J}\mathcal{A}_j) = \Cup_{j\in J} f^*(\mathcal{A}_j)$.

(20) $f^{-1}(\sqcup_{j\in J}\mathcal{B}_j) = \sqcup_{j\in J} f^{-1}(\mathcal{B}_j)$ *and* $f^{-1}(\Cup_{j\in J}\mathcal{B}_j) = \Cup_{j\in J} f^{-1}(\mathcal{B}_j)$.

(21) $f^{-1}(\sqcap_{j\in J}\mathcal{B}_j) = \sqcap_{j\in J} f^{-1}(\mathcal{B}_j)$ *and* $f^{-1}(\Cap_{j\in J}\mathcal{B}_j) = \Cap_{j\in J} f^{-1}(\mathcal{B}_j)$.

(22) *If* $g : Y \to Z$ *is a mapping, then* $(g \circ f)^{-1}(\mathcal{C}) = f^{-1}(g^{-1}(\mathcal{C}))$ *for each* $\mathcal{C} \in MBJNS(Z)$, *where* $g \circ f$ *denotes the composition of* f *and* g.

Definition 3.26. Let X be a nonempty set and let $\mathcal{A} \in MBJNS(X)$. Then

(i) the $\circ$-[resp. $*$-]*support* of $\mathcal{A}$, denoted by $supp^\circ(\mathcal{A})$ [resp. $supp^*(\mathcal{A})$], is a subset of X defined as follows:

$$supp^\circ(\mathcal{A}) = \{x \in X : \mathcal{A}(x) >^\circ \ddot{0}\} \text{ [resp. } supp^*(\mathcal{A}) = \{x \in X : \mathcal{A}(x) >^* \dot{0}\}].$$

(ii) $\mathcal{A}$ is said to be $\circ$-[resp. $*$-] *finite*, if $supp^\circ(\mathcal{A})$ [resp. $supp^*(\mathcal{A})$] is finite.

Proposition 3.27. *Let* X *be a nonempty set and let* $\mathcal{A}$, $\mathcal{B} \in MBJNS(X)$. *Then*

(1) *the* $supp^\circ((\mathcal{A} \sqcup \mathcal{B})^c) = supp^\circ(\mathcal{A}^c) \sqcap supp^*(\mathcal{B}_c)$,

(2) *the* $supp^*((\mathcal{A} \sqcup \mathcal{B})^c) = supp^*(\mathcal{A}^c) \sqcap supp^*(\mathcal{B}_c)$.

Proof. The proof is straightforward. □

4. MBJ-neutrosophic neighborhoods

We define a $\circ$-[resp. $*$]-MBJ-neutrosophic neighborhood of a $\circ$-[resp. a $*$]-MBJ-neutrosophic point with respect to a $\circ$-[resp. $*$]-MBJ-neutrosophic topology and obtain its various properties.

Definition 4.1. Let X be a nonempty set and let $\mathcal{A} \in MBJNS(X)$. Then $\mathcal{A}$ is called a *constant MBJ-neutrosophic set* in X, denoted by $\mathcal{A} = C_{\tilde{\tilde{a}}}$, if there is $\tilde{\tilde{a}} \in I \times [I] \times I$ such that $\mathcal{A}(x) = \tilde{\tilde{a}}$ for each $x \in X$.

Definition 4.2. Let τ be a family of cubic sets in a nonempty set X. Consider the following conditions:

(MBJNO$_0$) $C_{\widetilde{\widetilde{a}}} \in \tau$ for each $\widetilde{\widetilde{a}} \in I \times [I] \times I$,

($\circ$-MBJNO$_1$) $\ddot{\emptyset},\ \ddot{X} \in \tau$,

($\circ$-MBJNO$_2$) $\mathcal{A} \sqcap \mathcal{B} \in \tau$ for any $\mathcal{A},\ \mathcal{B} \in \tau$,

($\circ$-MBJNO$_3$) $\sqcup_{j\in J}\mathcal{A}_j \in \tau$ for each $(\mathcal{A}_j)_{j\in J} \subset \tau$,

($*$-MBJNO$_1$) $\dot{\emptyset},\ \dot{X} \in \tau$,

($*$-MBJNO$_2$) $\mathcal{A} \Cap \mathcal{B} \in \tau$ for any $\mathcal{A},\ \mathcal{B} \in \tau$,

($*$-MBJNO$_3$) $\Cup_{j\in J}\mathcal{A}_j \in \tau$ for each $(\mathcal{A}_j)_{j\in J} \subset \tau$.

(i) τ is called a $\circ$-*MBJ-neutrosophic topology* (briefly, $\circ$-MBJNT) on X in Chang's sense, if it satisfies the conditions ($\circ$-MBJNO$_1$), ($\circ$-MBJNO$_2$) and ($\circ$-MBJNO$_3$).

(ii) τ is called a $*$-*MBJ-neutrosophic topology* (briefly, $*$-MBJNT) on X in Chang's sense, if it satisfies the conditions ($*$-MBJNO$_1$), ($*$-MBJNO$_2$) and ($*$-MBJNO$_3$).

(iii) τ is called a $\circ$-*MBJ-neutrosophic topology* (briefly, $*$-MBJNT) on X in Lowen's sense, if it satisfies the conditions (MBJNO$_0$), ($\circ$-MBJNO$_2$) and ($\circ$-MBJNO$_3$).

(iv) τ is called a $*$-*cubic topology* on X in Lowen's sense, if it satisfies the conditions (MBJNO$_0$), ($*$-MBJNO$_2$) and ($*$-MBJNO$_3$).

In either case, the pair (X, τ) is called a $\circ$-*MBJ-neutrosophic topological space* [resp. $*$-*MBJ-neutrosophic topological space*] and each member of τ is called a $\circ$-*MBJ-neutrosophic open set* (briefly, $\circ$-MBJNOS) [resp. $*$-*MBJ-neutrosophic open set* (briefly, $*$-MBJNOS)]. We will denote the set of all $\circ$-MBJNTs in Chang's sense [resp. Lowen's sense] on X as $MBJNT^\circ(X)$ [resp. $MBJNT^\circ_L(X)$]. Also, we will denote the set of all $*$-MBJNTs in Chang's sense [resp. Lowen's sense] on X as $MBJNT^*(X)$ [resp. $MBJNT^*_L(X)$]. An MBJ-neutrosophic set $\mathcal{A}$ is called a $\circ$-*MBJ-neutrosophic closed set* (briefly, $\circ$-MBJNCS) [resp. $*$-*MBJ-neutrosophic closed set* (briefly, $*$-MBJNCS)] in X, if $\mathcal{A}^c \in \tau$. For a $\circ$-MBJ-neutrosophic topological space X, we denote the set of all $\circ$-MBJNOs [resp. $\circ$-MBJNCSs] in X as $MBJNO^\circ(X)$ [resp. $MBJNC^\circ(X)$]. Also, for a $*$-MBJ-neutrosophic topological space X, we denote the set of all $*$-MBJNOSs [resp. $*$-M BJNCSs] in X as $MBJNO^*(X)$ [resp. $MBJNC^*(X)$].

Example 4.3. (1) Let $X = \{x, y\}$ and let $\mathcal{A}_j \in MBJNS(X)$ $(j = 1,\ 2,\ 3,\ 4,\ 5,\ 6)$ defined as follows:

$$\mathcal{A}_1(x) = \langle 0.4, [0.6, 0.8], 0.8\rangle\,,\ \mathcal{A}_1(y) = \langle 0.6, [0.5, 0.9], 0.7\rangle\,,$$

$$\mathcal{A}_2(x) = \langle 0.5, [0.4, 0.7], 0.4\rangle\,,\ \mathcal{A}_2(y) = \langle 0.3, [0.7, 0.8], 0.9\rangle\,,$$

$$\mathcal{A}_3(x) = \langle 0.5, [0.6, 0.8], 0.4\rangle\,,\ \mathcal{A}_3(y) = \langle 0.6, [0.7, 0.9], 0.7\rangle\,,$$

$$\mathcal{A}_4(x) = \langle 0.4, [0.4, 0.7], 0.8\rangle\,,\ \mathcal{A}_4(y) = \langle 0.3, [0.5, 0.8], 0.9\rangle\,,$$

$$\mathcal{A}_5(x) = \langle 0.5, [0.4, 0.7], 0.4\rangle\,,\ \mathcal{A}_5(y) = \langle 0.6, [0.5, 0.8], 0.7\rangle\,,$$

$$\mathcal{A}_6(x) = \langle 0.4, [0.6, 0.8], 0.8\rangle\,,\ \mathcal{A}_6(y) = \langle 0.3, [0.7, 0.9], 0.9\rangle\,.$$

Let us consider the following two families:

$$\tau = \{\ddot{\emptyset}, \ddot{X}, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4\},\ \eta = \{\dot{\emptyset}, \dot{X}, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_5, \mathcal{A}_6\}.$$

Then we can easily check that $\tau \in MBJNT^{\circ}(X)$ and $\eta \in MBJNT^{*}(X)$. Furthermore, we can easily see that

$$\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4\} \bigcup \{C_{\widetilde{\widetilde{a}}} : \widetilde{\widetilde{a}} \in I \times [I] \times I\} \in MBJNT^{\circ}_L(X)$$

and

$$\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_5, \mathcal{A}_6\} \bigcup \{C_{\widetilde{\widetilde{a}}} : \widetilde{\widetilde{a}} \in I \times [I] \times I\} \in MBJNT^{*}_L(X).$$

(2) Let X be a nonempty set and let τ be the family of MBJ-neutrosophic sets in X defined as follows:

$$\tau = \{\mathcal{A} \in MBJNS(X) : \mathcal{A} = \ddot{X} \text{ or } supp^{\circ}(\mathcal{A}^c) \text{ is } \circ\text{-finite}\}$$

$$[\text{resp. } \tau = \{\mathcal{A} \in MBJNS(X) : \mathcal{A} = \dot{X} \text{ or } supp^{*}(\mathcal{A}^c) \text{ is } *\text{-finite}\}].$$

Then by Proposition 3.27, it is obvious that $\tau \in MBJNT^{\circ}(X)$ [resp. $\tau \in MBJNT^{*}(X)$]. In this case, we will call τ as *MBJ-neutrosophic* $\circ$-[resp. $*$-]*cofinite topology* on X.

Remark 4.4. (1) From Definition 4.2, it is obvious that $\{\ddot{\emptyset},\ \ddot{X}\} \in MBJNT^{\circ}(X)$, $\{\dot{\emptyset},\ \dot{X}\} \in MBJNT^{*}(X)$ and $MBJNS(X)$ are both $\circ$-MBJNT and $*$-MBJNT on X. In this case, $\{\ddot{\emptyset},\ \ddot{X}\}$ [resp. $\{\dot{\emptyset},\ \dot{X}\}$ and $MBJNS(X)$] is called the $\circ$-*MBJ-neutrosophic indiscrete topology* [resp. $*$-*MBJ-neutrosophic indiscrete topology* and *MBJ-neutrosophic discrete topology*] on X and will be denoted by $\mathcal{I}^{\circ}$ [resp. $\mathcal{I}^{*}$ and $\mathcal{D}$]. The pair $(X, \mathcal{I}^{\circ})$ [resp. $(X, \mathcal{I}^{*})$ and $(X, \mathcal{D})$] is called a $\circ$-*MBJ-neutrosophic indiscrete space* [resp. $*$-*MBJ-neutrosophic indiscrete space* and *MBJ-neutrosophic discrete space*]. It is clear that that $\mathcal{I}^{\circ} \subset \tau \subset \mathcal{D}$ for each $\tau \in MBJNT^{\circ}(X)$ and $\mathcal{I}^{*} \subset \tau \subset \mathcal{D}$ for each $\tau \in MBJNT^{*}(X)$. Moreover, by Proposition 3.4, we can easily check that for each $\tau \in MBJNT^{\circ}(X)$ [resp. $\tau \in MBJNT^{*}(X)$], τ have the least element $\ddot{\emptyset}$ [resp. $\dot{\emptyset}$] and greatest element $\ddot{X}$ [resp. $\dot{X}$].

(2) Let T be a classical topology on a nonempty set X. Then clearly,

$$\chi^{\circ}_{\mathcal{T}} = \{\langle \chi_A, [\chi_A, \chi_A], \chi_{A^c}\rangle \in MBJN(X) : A \in T\} \in MBJNT^{\circ}(X),$$

$$\chi^{*}_{\mathcal{T}} = \{\langle \chi_A, [\chi_{A^c}, \chi_{A^c}], \chi_{A^c}\rangle \in MBJN(X) : A \in T\} \in MBJNT^{*}(X).$$

(3) We denote the set of all fuzzy topologies (See [13, 20]) on a nonempty set X as FT(X) and let $\tau^c = \{A^c \in I^X : A \in \tau\}$ for each $\tau \in FT(X)$. Then it is obvious that for each $\tau \in FT(X)$,

$$\{\langle A, [A, A], A^c\rangle \in MBJN(X) : A \in \tau\} \in MBJNT^{\circ}(X),$$

$$\{\langle A, [A^c, A^c], A^c\rangle \in MBJN(X) : A \in \tau\} \in MBJNT^{*}(X).$$

(4) Let us denote the set of all interval-valued fuzzy topologies (See [25]) on a nonempty set X as $IVFT(X)$. Then we can easily check that for each $\tau \in IVFT(X)$,

$$\{\left\langle A^-, \widetilde{A}, A^{-c}\right\rangle \in MBJN(X) : \widetilde{A} \in \tau\} \in MBJNT^\circ(X),$$

$$\{\left\langle A^-, \widetilde{A}, A^{+c}\right\rangle \in MBJN(X) : \widetilde{A} \in \tau\} \in MBJNT^\circ(X),$$

$$\{\left\langle A^+, \widetilde{A}, A^{-c}\right\rangle \in MBJN(X) : \widetilde{A} \in \tau\} \in MBJNT^\circ(X),$$

$$\{\left\langle A^+, \widetilde{A}, A^{+c}\right\rangle \in MBJN(X) : \widetilde{A} \in \tau\} \in MBJNT^\circ(X),$$

$$\{\left\langle A^-, \widetilde{A}^c, A^{-c}\right\rangle \in MBJN(X) : \widetilde{A} \in \tau\} \in MBJNT^*(X),$$

$$\{\left\langle A^-, \widetilde{A}^c, A^{+c}\right\rangle \in MBJN(X) : \widetilde{A} \in \tau\} \in MBJNT^*(X),$$

$$\{\left\langle A^+, \widetilde{A}^c, A^{-c}\right\rangle \in MBJN(X) : \widetilde{A} \in \tau\} \in MBJNT^*(X),$$

$$\{\left\langle A^+, \widetilde{A}^c, A^{+c}\right\rangle \in MBJN(X) : \widetilde{A} \in \tau\} \in MBJNT^*(X).$$

(5) We denote the set of all interval-valued fuzzy topologies [resp. cotopologies] (See [25]) and all fuzzy topologies [resp. cotopologies] (See [13, 20]) on a set X as $IVFT(X)$ [resp. $IVFCT(X)$] and $FT(X)$ [resp. $FCT(X)$] respectively, where the term "cotoplogy" means the dual of "topology". For each $\tau \in MBJNT^\circ(X)$ [resp. $\tau \in MBJNT^*(X)$], let us consider the following families:

$$\tau_{_M} = \{M_A \in I^X : \mathcal{A} \in \tau\},\ \tau_{_{\widetilde{B}}} = \{\widetilde{B}_A \in IVFSX : \mathcal{A} \in \tau\},\ \tau_{_J} = \{J_A \in I^X : \mathcal{A} \in \tau\}.$$

Then we can easily see that the followings hold:

$$\tau \in MBJNT^\circ(X) \Longleftrightarrow \tau_{_M} \in FT(X),\ \tau_{_{\widetilde{B}}} \in IVFT(X),\ \tau_{_J} \in FCT(X)$$

and

$$\tau \in MBJNT^*(X) \Longleftrightarrow \tau_{_M} \in FT(X),\ \tau_{_{\widetilde{B}}} \in IVFCT(X),\ \tau_{_J} \in FCT(X).$$

(6) Let (X, τ) be a neutrosophic topological space proposed by Salama and Alblowi [31] and consider two families τ° and τ^* defined by:

$$\tau^\circ = \{\langle A^T, [A^I, A^I], A^F\rangle \in MBJNS(X) : \mathcal{A} = \langle A^T, A^I, A^F\rangle \in \tau\}$$

and

$$\tau^* = \{\langle A^T, [A^{I,c}, A^{I,c}], A^F\rangle \in MBJNS(X) : \mathcal{A} \in \tau\},$$

where $A^{I,c}$ denotes the complement of the fuzzy set A^I. Then clearly $\tau^\circ \in MBJNT^\circ(X)$ and $\tau^* \in MBJNT^\circ(X)$. Moreover, it is well-known (Example 4.1, [31]) that every fuzzy topology is a neutrosophic topolog.

(7) Let (X, τ) be an intuitionistic fuzzy topological space introduced by Coker [21] and consider the family $\tau_{_{I,N}}$ of neutrosophic sets in X defined by:

$$\tau_{_{I,N}} = \{\left\langle A^\in, [A^\in, A^\in], A^\notin\right\rangle \in NS(X) : \bar{A} = (A^\in, A^\notin) \in \tau\}.$$

Then clearly $\tau_{I,N}$ is a neutrosophic topology on X. On the other hand, let $\tau_{I,IV}$ be the family of interval-valued fuzzy sets in X given by:

$$\tau_{I,IV} = \{[A^{\in}, A^{\notin,c}] \in IVFS(X) : \bar{A} \in \tau\},$$

where $A^{\notin,c}$ denotes the complement of the fuzzy set $A^{\notin}$. Then we can easily check that $\tau_{I,IV}$ is an interval-valued fuzzy topology on X.

(8) Let (X, τ) be an interval-valued fuzzy topological space and consider the family $\tau_{IV,N}$ defined by:

$$\tau_{IV,N} = \{\left\langle A^-, \widetilde{A}, A^{-,c}\right\rangle \in NS(X) : \bar{A} = (A^{\in}, A^{\notin}) \in \tau\},$$

where $A^{-,c}$ denotes the complement of the fuzzy set A^-. Then clearly $\tau_{IV,N}$ is a neutrosophic topology on X. On the other hand, let $\tau_{IV,I}$ be the family of intuitionistic fuzzy sets in X given by:

$$\tau_{IV,I} = \{(A^-, A^{+,c}) \in IFS(X) : \widetilde{A} \in \tau\}.$$

Then it is clear that $\tau_{I,IV}$ is an intuitionistic fuzzy topology on X.

Let T [resp. FT, IF, $IVFT$, NT and $MBJNT$] be a classical [resp. a fuzzy, an intuitionistic fuzzy, an interval-valued fuzzy, a neutrosophic and an MBJ-neutrosophic] topology on a set X. Then from (2)–(8) and Proposition 4.5, we have the following among T, FT, IFT, $IVFT$, NT and MBJNT:

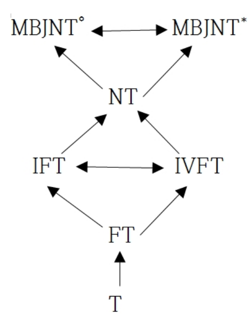

FIGURE 1. Implications among the above topologies

The following is an immediate consequence of Definition 4.2 and Proposition 3.7 (12).

Proposition 4.5. *Let X be a nonempty set. If $\tau \in MBJNT^{\circ}(X)$, then the family*

$$\tau^* = \{\left\langle M_A, \widetilde{B}_A^c, J_A\right\rangle \in MBJNS(X) : \mathcal{A} \in \tau\} \in MBJNT^*(X).$$

Also the converse holds.

The following is an immediate consequence of Definition 4.2.

Proposition 4.6. (1) *Let* $\tau \in MBJNT^{\circ}(X)$ *or* $\tau \in MBJNT^{\circ}_L(X)$ *and let* $\tau^c = \{\mathcal{A}^c : \mathcal{A} \in \tau\}$. *Then* τ^c *satisfies the following conditions:*

(MBJNC$_0$) *for each* $\widetilde{\widetilde{a}} \in I \times [I] \times I$, $C_{\widetilde{\widetilde{a}}} \in \tau^c$,

($\circ$-MBJNC$_1$) $\ddot{\emptyset},\ \ddot{X}$,

($\circ$-MBJNC$_2$) $\mathcal{A} \sqcup \mathcal{B} \in \tau^c$ *for any* $\mathcal{A},\ \mathcal{B} \in \tau^c$,

($\circ$-MBJNC$_3$) $\sqcap_{j\in J}\mathcal{A}_j \in \tau^c$ *for each* $(\mathcal{A}_j)_{j\in J} \subset \tau^c$.

(2) *Let* $\tau \in MBJNT^{*}(X)$ *or* $\tau \in MBJNT^{*}_L(X)$ *and let* $\tau^c = \{\mathcal{A}^c : \mathcal{A} \in \tau\}$. *Then* τ^c *satisfies the following conditions:*

(MBJNC$_0$) *for each* $\widetilde{\widetilde{a}} \in I \times [I] \times I$, $C_{\widetilde{\widetilde{a}}} \in \tau^c$,

($*$-MBJNC$_1$) $\dot{\emptyset},\ \dot{X}$,

($*$-MBJNC$_2$) $\mathcal{A} \Cup \mathcal{B} \in \tau^c$ *for any* $\mathcal{A},\ \mathcal{B} \in \tau^c$,

($*$-MBJNC$_3$) $\Cap_{j\in J}\mathcal{A}_j \in \tau^c$ *for each* $(\mathcal{A}_j)_{j\in J} \subset \tau^c$.

In this case, τ^c will be called a $\circ$-*MBJ-neutrosophic cotopology* [resp. $*$-*MBJ-neutrosophic cotopology*] on X.

Now we will deal with neighborhood structures based on MBJ-neutrosophic sets.

Definition 4.7. Let (X, τ) be a $\circ$-MBJ-neutrosophic topological space or a $*$-MBJ-neutrosophic topological space, let $\mathcal{A} \in MBJN(X)$ and let $x^{\circ}_{\widetilde{\widetilde{a}}},\ x^{*}_{\widetilde{\widetilde{a}}} \in MBJN_P(X)$.

(i) $\mathcal{A}$ is called a $\circ$-*MBJ-neutrosophic neighborhood* (briefly, $\circ$-MBJNN) of $x^{\circ}_{\widetilde{\widetilde{a}}}$, if there is $\mathcal{B} \in \tau$ such that $x^{\circ}_{\widetilde{\widetilde{a}}} \in \mathcal{B} \sqsubset \mathcal{A}$. A $\circ$-MBJNN $\mathcal{A}$ is said to be $\circ$-*MBJ-neutrosophic open*, if $\mathcal{A} \in \tau$. The collection of all $\circ$-MBJNNs of $x^{\circ}_{\widetilde{\widetilde{a}}}$ is called the *system of* $\circ$-*MBJ-neutrosophic neighborhoods of* $x^{\circ}_{\widetilde{\widetilde{a}}}$ and will be denoted by $\mathcal{N}(x^{\circ}_{\widetilde{\widetilde{a}}})$.

(ii) $\mathcal{A}$ is called a $\circ$-*MBJ-neutrosophic Q-neighborhood* (briefly, $\circ$-MBJNQN) of $x^{\circ}_{\widetilde{\widetilde{a}}}$, if there is $\mathcal{B} \in \tau$ such that $x^{\circ}_{\widetilde{\widetilde{a}}} q\mathcal{B} \sqsubset \mathcal{A}$. The family of all $\circ$-MBJNQNs of $x^{\circ}_{\widetilde{\widetilde{a}}}$ is called the *system of* $\circ$-*MBJ-neutrosophic Q-neighborhoods of* $x^{\circ}_{\widetilde{\widetilde{a}}}$ and will be denoted by $\mathcal{N}_Q(x^{\circ}_{\widetilde{\widetilde{a}}})$.

(iii) $\mathcal{A}$ is called a $*$-*MBJ-neutrosophic neighborhood* (briefly, $*$-MBJNN) of $x^{*}_{\widetilde{\widetilde{a}}}$, if there is $\mathcal{B} \in \tau$ such that $x^{*}_{\widetilde{\widetilde{a}}} \in \mathcal{B} \Subset \mathcal{A}$. A $*$-MBJNN $\mathcal{A}$ is said to be $*$-*MBJ-neutrosophic open*, if $\mathcal{A} \in \tau$. The collection of all $*$-MBJNNs of $x^{*}_{\widetilde{\widetilde{a}}}$ is called the *system of* $*$-*MBJ-neutrosophic neighborhoods of* $x^{*}_{\widetilde{\widetilde{a}}}$ and will be denoted by $\mathcal{N}(x^{*}_{\widetilde{\widetilde{a}}})$.

(iv) $\mathcal{A}$ is called a $*$-*MBJ-neutrosophic Q-neighborhood* (briefly, $*$-MBJNQN) of $x^{*}_{\widetilde{\widetilde{a}}}$, if there is $\mathcal{B} \in \tau$ such that $x^{*}_{\widetilde{\widetilde{a}}} q\mathcal{B} \Subset \mathcal{A}$. The family of all $*$-MBJNQNs of $x^{*}_{\widetilde{\widetilde{a}}}$ is called the *system of* $*$-*MBJ-neutrosophic Q-neighborhoods of* $x^{*}_{\widetilde{\widetilde{a}}}$ and will be denoted by $\mathcal{N}_Q(x^{*}_{\widetilde{\widetilde{a}}})$.

Example 4.8. Let (X, τ) and (X, η) be the $\circ$-MBJ-neutrosophic topological space and $*$-MBJ-neutrosophic topological space given Example 4.3. Consider four MBJ-neutrosophic points and four MBJ-neutrosophic sets in X given by:

$$x^{\circ}_{\langle 0.3,[0.4,0.6],0.5\rangle},\ x^{*}_{\langle 0.4,[0.5,0.8],0.6\rangle},\ C_{\langle 0.4,[0.5,0.7],0.3\rangle},\ C_{\langle 0.6,[0.6,0.9],0.7\rangle}$$

and

$$x^{\circ}_{\langle 0.7,[0.9,0.9],0.5\rangle},\ x^{*}_{\langle 0.6,[0.2,0.4],0.3\rangle},\ C_{\langle 0.6,[0.3,0.5],0.7\rangle},\ C_{\langle 0.6,[0.3,0.7],0.5\rangle}.$$

Then we can easily check that

$$C_{\langle 0.4,[0.5,0.7],0.3\rangle} \in \mathcal{N}(x^{\circ}_{\langle 0.3,[0.4,0.6],0.5\rangle}),\ C_{\langle 0.6,[0.6,0.9],0.7\rangle} \in \mathcal{N}(x^{*}_{\langle 0.4,[0.5,0.8],0.6\rangle})$$

and

$$C_{\langle 0.6,[0.3,0.5],0.7\rangle} \in \mathcal{N}_Q(x^{\circ}_{\langle 0.7,[0.9,0.9],0.5\rangle}),\ C_{\langle 0.6,[0.3,0.7],0.5\rangle} \in \mathcal{N}_Q(x^{*}_{\langle 0.6,[0.2,0.4],0.3\rangle}).$$

Let $\widetilde{N}(x_{\tilde{a}})$ [resp. $\widetilde{N}_Q(x_{\tilde{a}})$] be the set of all interval-valued fuzzy neighborhoods [resp. Q-neighborhoods] of an interval-valued fuzzy point $x_{\tilde{a}}$ (See [25] [resp. [33]) and let $N(x_a)$ [resp. $N_Q(x_a)$] denote the set of all fuzzy neighborhoods [resp. Q-neighborhoods] of a fuzzy point x_a (See [16]).

Remark 4.9. From Remarks 3.14 and 4.4 (5), Definitions 3.13 and 4.7, we can easily check that the following holds:

$$\mathcal{A} \in \mathcal{N}(x^{\circ}_{\tilde{\tilde{a}}}) \Longleftrightarrow M_A \in N_{\tau_M}(x_a),\ \widetilde{B}_A \in \widetilde{N}_{\tau_{\widetilde{B}}}(x_{\tilde{a}}),\ J_A \in N^{\circ}_{\tau_J}(x^{\circ}_{\bar{a}}),$$

$$\mathcal{A} \in \mathcal{N}(x^{*}_{\tilde{\tilde{a}}}) \Longleftrightarrow M_A \in N_{\tau_M}(x_a),\ \widetilde{B}_A \in \widetilde{N}^{*}_{\tau_{\widetilde{B}}}(x^{*}_{\tilde{a}}),\ J_A \in N^{\circ}_{\tau_J}(x^{\circ}_{\bar{a}}),$$

$$\mathcal{A} \in \mathcal{N}_Q(x^{\circ}_{\tilde{\tilde{a}}}) \Longleftrightarrow M_A \in N_{\tau_M,Q}(x_a),\ \widetilde{B}_A \in \widetilde{N}_{\tau_{\widetilde{B}},Q}(x_{\tilde{a}}),\ J_A \in N^{\circ}_{\tau_J,Q}(x^{\circ}_{\bar{a}}),$$

$$\mathcal{A} \in \mathcal{N}_Q(x^{*}_{\tilde{\tilde{a}}}) \Longleftrightarrow M_A \in N_{\tau_M,Q}(x_a),\ \widetilde{B}_A \in \widetilde{N}^{*}_{\tau_{\widetilde{B}},Q}(x^{*}_{\tilde{a}}),\ J_A \in N^{\circ}_{\tau_J,Q}(x^{\circ}_{\bar{a}}),$$

where $J_A \in N^{\circ}_{\tau_J}(x^{\circ}_{\bar{a}})$ if and only if there is $J_B \in \tau_J$ such that $x^{\circ}_{\bar{a}} \in J_B \supset J_A$, $\widetilde{B}_A \in \widetilde{N}^{*}_{\tau_{\widetilde{B}}}(x^{*}_{\tilde{a}})$ if and only if there is $\widetilde{B}_B \in \tau_{\widetilde{B}}$ such that $x^{*}_{\tilde{a}} \in \widetilde{B}_B \supset \widetilde{B}_A$, $J_A \in N^{\circ}_{\tau_J,Q}(x^{\circ}_{\bar{a}})$ if and only if there is $J_B \in \tau_J$ such that $x_{\bar{a}} q^{\circ} J_B \supset J_A$ and $\widetilde{B}_A \in \widetilde{N}^{*}_{\tau_{\widetilde{B}},Q}(x^{*}_{\tilde{a}})$ if and only if there is $\widetilde{B}_B \in \tau_{\widetilde{B}}$ such that $x_{\tilde{a}} q^{*} \widetilde{B}_B \supset \widetilde{B}_A$.

Theorem 4.10. *Let (X,τ) be a $\circ$-MBJ-neutrosophic topological space or a $*$-MBJ-neutrosophic topological space and let $\mathcal{A} \in MBJN(X)$.*

(1) *$\mathcal{A} \in \tau$ if and only if $\mathcal{A} \in \mathcal{N}(x^{\circ}_{\tilde{\tilde{a}}})$ for each $x^{\circ}_{\tilde{\tilde{a}}} \in \mathcal{A}$.*

(2) *$\mathcal{A} \in \tau$ if and only if $\mathcal{A} \in \mathcal{N}(x^{*}_{\tilde{\tilde{a}}})$ for each $x^{*}_{\tilde{\tilde{a}}} \in \mathcal{A}$.*

Proof. (1) By Remark 3.14 (1), it is clear that $x^{\circ}_{\tilde{\tilde{a}}} \in \mathcal{A}$ if and only if $x_a \in M_A$, $x_{\tilde{a}} \in \widetilde{B}_A$, $x^{\circ}_{\bar{a}} \in J_A$. From Proposition 1.8 in [18] and Theorem 7 in [25], we have

$$M_A \in \tau_M \iff M_A \in N_{\tau_M}(x_a) \text{ for each } x_a \in M_A$$

and

$$\widetilde{B}_A \in \tau_{\widetilde{B}} \iff \widetilde{B}_A \in N_{\tau_{\widetilde{B}}}(x_{\tilde{a}}) \text{ for each } x_{\tilde{a}} \in \widetilde{B}_A.$$

It is sufficient to prove that $J_A \in \tau_J \iff J_A \in N^{\circ}_{\tau_J}(x^{\circ}_{\bar{a}})$ for each $x^{\circ}_{\bar{a}} \in J_A$.

Suppose $J_A \in \tau_J$ and let $x^{\circ}_{\bar{a}} \in J_A$. Then clearly, $J_A \in N^{\circ}_{\tau_J}(x^{\circ}_{\bar{a}})$. Conversely, suppose the necessary condition holds. Then there is $(J_B)_{x^{\circ}_{\bar{a}}} \in \tau_J$ such that

$$x^{\circ}_{\bar{a}} \in (J_B)_{x^{\circ}_{\bar{a}}} \in \tau_J \supset J_A.$$

Thus $J_A = \bigcap_{x^{\circ}_{\bar{a}} \in J_A} (J_B)_{x^{\circ}_{\bar{a}}}$. By Remark 4.4 (5), $(J_B)_{x^{\circ}_{\bar{a}}} \in \tau_J \in FCT(X)$ for each $x^{\circ}_{\bar{a}} \in J_A$. So $J_A \in \tau_J$. Hence the result holds.

(2) From the procedure of the proof of (1), we get

$$M_A \in \tau_M \iff M_A \in N_{\tau_M}(x_a) \text{ for each } x_a \in M_A$$

and

$$J_A \in \tau_J \iff J_A \in N^{\circ}_{\tau_J}(x^{\circ}_{\bar{a}}) \text{ for each } x^{\circ}_{\bar{a}} \in J_A.$$

It is sufficient to prove that $\widetilde{B}_A \in \tau_{\widetilde{B}} \iff \widetilde{B}_A \in N_{\tau^*_{\widetilde{B}}}(x^*_{\tilde{a}})$ for each $x^*_{\tilde{a}} \in \widetilde{B}_A$.

Suppose $\widetilde{B}_A \in \tau_{\widetilde{B}}$ and let $x^*_{\tilde{a}} \in \widetilde{B}_A$. Then clearly, $\widetilde{B}_A \in N^*_{\tau_{\widetilde{B}}}(x^*_{\tilde{a}})$. Conversely, suppose the necessary condition holds. Then there is $(\widetilde{B}_B)_{x^*_{\tilde{a}}} \in \tau_{\widetilde{B}}$ such that

$$x^*_{\tilde{a}} \in (\widetilde{B}_B)_{x^*_{\tilde{a}}} \supset \widetilde{B}_A.$$

Thus $\widetilde{B}_A = \bigcap_{x^*_{\tilde{a}} \in \widetilde{B}_A} (\widetilde{B}_B)_{x^*_{\tilde{a}}}$. By Remark 4.4 (5), $(\widetilde{B}_B)_{x^*_{\tilde{a}}} \in \tau_{\widetilde{B}} \in IVFCT(X)$ for each $x^*_{\tilde{a}} \in \widetilde{B}_A$. So $\widetilde{B}_A \in \tau_{\widetilde{B}}$. Hence the result holds. □

Theorem 4.11. *Let (X, τ) be a $\circ$-MBJ-neutrosophic topological space or a $*$-MBJ-neutrosophic topological space and let $\mathcal{A} \in MBJNS(X)$.*

(1) $\mathcal{A} \in \tau$ if and only if $\mathcal{A} \in \mathcal{N}_Q(x^{\circ}_{\tilde{\bar{a}}})$ for each $x^{\circ}_{\tilde{\bar{a}}} \in MBJN_P(X)$ such that $0 < a < M_A(x)$, $[0,0] < \tilde{a} < \widetilde{B}_A(x)$, $J_A(x) < \bar{a} < 1$ and $x^{\circ}_{\tilde{\bar{a}}} q \mathcal{A}$.

(2) $\mathcal{A} \in \tau$ if and only if $\mathcal{A} \in \mathcal{N}_Q(x^*_{\tilde{\bar{a}}})$ for each $x^{\circ}_{\tilde{\bar{a}}} \in MBJN_P(X)$ such that $0 < a < M_A(x)$, $\widetilde{B}_A(x) < \tilde{a} < [1,1]$, $J_A(x) < \bar{a} < 1$ and $x^*_{\tilde{\bar{a}}} q \mathcal{A}$.

Proof. (1) From Remark 4.9, we have

$$\mathcal{A} \in \mathcal{N}_Q(x^{\circ}_{\tilde{\bar{a}}}) \Longleftrightarrow M_A \in N_{\tau_M,Q}(x_a),\ \widetilde{B}_A \in \widetilde{N}_{\tau_{\widetilde{B}},Q}(x_{\tilde{a}}),\ J_A \in N^{\circ}_{\tau_J,Q}(x_{\bar{a}}).$$

From Theorem 3.2 [19] and Lemma 4.12 [33], it is obvious that

$$M_A \in \tau_M \Longleftrightarrow M_A \in N_{\tau_M,Q}(x_a)$$

for each $x_a \in F_P(X)$ such $0 < a < M_A(x)$ and $x_a q M_A$ and

$$\widetilde{B}_A \in \tau_{\widetilde{B}} \Longleftrightarrow \widetilde{B}_A \in \widetilde{N}_{\tau_{\widetilde{B}},Q}(x_a)$$

for each $x_{\tilde{a}} \in IVF_P(X)$ such $[0,0] < \tilde{a} < \widetilde{M}_A(x)$ and $x_{\tilde{a}} q \widetilde{M}_A$. It is sufficient to show that

$$J_A \in \tau_J \Longleftrightarrow J_A \in N^{\circ}_{\tau_J,Q}(x_{\bar{a}}) \quad (1)$$

for each $x_{\overline{a}} \in F_P(X)$ such that $J_A(x) < \overline{a} < 1$ and $x_{\overline{a}} q^\circ J_A$.

Suppose $J_A \in \tau_J$ and let $x_{\overline{a}} \in F_P(X)$ such that $J_A(x) < \overline{a} < 1$ and $x_{\overline{a}} q^\circ J_A$. Then clearly, $J_A \in N^\circ_{\tau_J,Q}(x_{\overline{a}})$. Conversely, suppose the necessary condition holds. Since $J_A(x) < \overline{a} < 1$, by Lemma 3.21, $x_{\overline{a}^c} q^\circ J_A$. Then by the hypothesis, $J_A \in N^\circ_{\tau_J,Q}(x_{\overline{a}^c})$. Thus there is $U_{x_{\overline{a}^c}} \in \tau_J$ such that $x_{\overline{a}^c} q^\circ U_{x_{\overline{a}^c}} \supset J_A$. Since $x_{\overline{a}} q^\circ J_A$, $x_{\overline{a}^c} \in J_A$. So $J_A = \bigcap_{x_{\overline{a}^c} \in J_A} U_{x_{\overline{a}^c}}$. Since τ_J is a fuzzy cotopology on X and $U_{x_{\overline{a}^c}} \in \tau_J$, $J_A \in \tau_J$. Hence the result holds.

(2) From Remark 4.9, we get

$$\mathcal{A} \in \mathcal{N}_Q(x^*_{\tilde{\tilde{a}}}) \Longleftrightarrow M_A \in N_{\tau_M,Q}(x_a),\ \widetilde{B}_A \in \widetilde{N}^*_{\tau_{\widetilde{B}},Q}(x_{\widetilde{a}}),\ J_A \in N^\circ_{\tau_J,Q}(x_{\overline{a}}).$$

From (1), it is clear that

$$M_A \in \tau_M \Longleftrightarrow M_A \in N_{\tau_M,Q}(x_a)$$

for each $x_a \in F_P(X)$ such $0 < a < M_A(x)$ and $x_a q M_A$ and

$$J_A \in \tau_J \Longleftrightarrow J_A \in N^\circ_{\tau_J,Q}(x_{\overline{a}})$$

for each $x_{\overline{a}} \in F_P(X)$ such that $J_A(x) < \overline{a} < 1$ and $x_{\overline{a}} q^\circ J_A$. It is sufficient to show that

$$\widetilde{B}_A \in \tau_{\widetilde{B}} \Longleftrightarrow \widetilde{B}_A \in \widetilde{N}^*_{\tau_{\widetilde{B}},Q}(x_{\widetilde{a}}) \tag{2}$$

for each $x_{\widetilde{a}} \in IVF_P(X)$ such that $\widetilde{B}_A(x) < \widetilde{a} < [1,1]$ and $x_{\widetilde{a}} q^\circ \widetilde{B}_A$.

Suppose $\widetilde{B}_A \in \tau_{\widetilde{B}}$ and let $x_{\widetilde{a}} \in IVF_P(X)$ such that $\widetilde{B}_A(x) < \widetilde{a} < [1,1]$ and $x_{\widetilde{a}} q^* \widetilde{B}_A$. Then clearly, $\widetilde{B}_A \in N^*_{\tau_{\widetilde{B}},Q}(x_{\widetilde{a}})$. Conversely, suppose the necessary condition holds. Since $\widetilde{B}_A(x) < \widetilde{a} < [1,1]$, by Lemma 3.22, $x_{\widetilde{a}^c} q^* \widetilde{B}_A$. Then by the hypothesis, $\widetilde{B}_A \in N^*_{\tau_{\widetilde{B}},Q}(x_{\widetilde{a}^c})$. Thus there is $\widetilde{U}_{x_{\widetilde{a}^c}} \in \tau_{\widetilde{B}}$ such that $x_{\widetilde{a}^c} q^* \widetilde{U}_{x_{\widetilde{a}^c}} \supset \widetilde{B}_A$. Since $x_{\widetilde{a}} q^* \widetilde{B}_A$, $x_{\widetilde{a}^c} \in \widetilde{B}_A$. So $\widetilde{B}_A = \bigcap_{x_{\widetilde{a}^c} \in \widetilde{B}_A} \widetilde{U}_{x_{\widetilde{a}^c}}$. Since $\tau_{\widetilde{B}}$ is an interval-valued fuzzy cotopology on X and $\widetilde{U}_{x_{\widetilde{a}^c}} \in \tau_{\widetilde{B}}$, $\widetilde{B}_A \in \tau_{\widetilde{B}}$. Hence the result holds. □

Lemma 4.12. *Let (X, τ) be an interval-valued fuzzy topological space and let $x_{\widetilde{a}} \in IVF_P(X)$.*

(1) If $\widetilde{A} \in \widetilde{N}(x_{\widetilde{a}})$, then $x_{\widetilde{a}} \in \widetilde{A}$.

(2) If $\widetilde{A}$, $\widetilde{B} \in \widetilde{N}(x_{\widetilde{a}})$, then $\widetilde{A} \cap \widetilde{B} \in \widetilde{N}(x_{\widetilde{a}})$.

(3) If $\widetilde{A} \in \widetilde{N}(x_{\widetilde{a}})$ and $\widetilde{A} \subset \widetilde{B}$, then $\widetilde{B} \in \widetilde{N}(x_{\widetilde{a}})$.

(4) If $\widetilde{A} \in \widetilde{N}(x_{\widetilde{a}})$, then there is $\widetilde{B} \in \widetilde{N}(x_{\widetilde{a}})$ such that $\widetilde{B} \subset \widetilde{A}$ and $\widetilde{B} \in \widetilde{N}(y_{\widetilde{b}})$ for $y_{\widetilde{b}} \in \widetilde{B}$.

Conversely, if for each $x_{\widetilde{a}} \in IVF_P(X)$, $\widetilde{N}_{x_{\widetilde{a}}}$ satisfies the conditions (1), (2) and (3), then the family τ of IVFSs in X given by:

$$\tau = \{\widetilde{A} \in IVFS(X) : \widetilde{A} \in \widetilde{N}_{x_{\widetilde{a}}} \text{ for each } x_{\widetilde{a}} \in \widetilde{A}\}$$

is an interval-valued fuzzy topology on X. Furthermore, if $\widetilde{N}_{x_{\widetilde{a}}}$ satisfies the condition (4), then $\widetilde{N}_{x_{\widetilde{a}}}$ is exactly the system of interval-valued fuzzy neidhborhood of $x_{\widetilde{a}}$ with respect to τ, i.e., $\widetilde{N}_{x_{\widetilde{a}}} = \widetilde{N}(x_{\widetilde{a}})$.

Proof. The proof is almost similar to a classical case (See [36]). □

From Remark 4.9, Proposition 2.2 [16] and Lemma 4.12, we have the followings.

Theorem 4.13. *Let (X, τ) be a ○-MBJ-neutrosophic topological space and let $x^{\circ}_{\tilde{\tilde{a}}} \in MBJN_P(X)$.*

(1) *If $\mathcal{A} \in \mathcal{N}(x^{\circ}_{\tilde{\tilde{a}}})$, then $x^{\circ}_{\tilde{\tilde{a}}} \in \mathcal{A}$.*

(2) *If $\mathcal{A},\ \mathcal{B} \in \mathcal{N}(x^{\circ}_{\tilde{\tilde{a}}})$, then $\mathcal{A} \sqcap \mathcal{B} \in \mathcal{N}(x^{\circ}_{\tilde{\tilde{a}}})$.*

(3) *If $\mathcal{A} \in \mathcal{N}(x^{\circ}_{\tilde{\tilde{a}}})$ and $\mathcal{A} \sqsubset \mathcal{B}$, then $\mathcal{B} \in \mathcal{N}(x^{\circ}_{\tilde{\tilde{a}}})$.*

(4) *If $\mathcal{A} \in \mathcal{N}(x^{\circ}_{\tilde{\tilde{a}}})$, then there is $\mathcal{B} \in \mathcal{N}(x^{\circ}_{\tilde{\tilde{a}}})$ such that $\mathcal{B} \sqsubset \mathcal{A}$ and $\mathcal{B} \in \mathcal{N}(y^{\circ}_{\tilde{\tilde{a}}})$ for $y^{\circ}_{\tilde{\tilde{a}}} \in \mathcal{B}$.*

Conversely, if for each $x^{\circ}_{\tilde{\tilde{a}}} \in MBJN_P(X)$, $\mathcal{N}_{x^{\circ}_{\tilde{\tilde{a}}}}$ satisfies the conditions (1), (2) and (3), then the family τ of cubic sets in X given by:

$$\tau = \{\mathcal{A} \in MBJN(X) : \mathcal{A} \in \mathcal{N}_{x^{\circ}_{\tilde{\tilde{a}}}} \text{ for each } x^{\circ}_{\tilde{\tilde{a}}} \in \mathcal{A}\}$$

is a ○-MBJ-neutrosophic topology on X. Furthermore, if $\mathcal{N}^{\circ}_{x_{\tilde{\tilde{a}}}}$ satisfies the condition (4), then $\mathcal{N}_{x^{\circ}_{\tilde{\tilde{a}}}}$ is exactly the system of ○-MBJ-neutrosophic neidhborhood of $x_{\overset{\circ}{\tilde{\tilde{a}}}}$ with respect to τ, i.e., $\mathcal{N}_{x^{\circ}_{\tilde{\tilde{a}}}} = \mathcal{N}(x^{\circ}_{\tilde{\tilde{a}}})$.

Theorem 4.14. *Let (X, τ) be a ∗-MBJ-neutrosophic topological space and let $x^{*}_{\tilde{\tilde{a}}} \in MBJN_P(X)$.*

(1) *If $\mathcal{A} \in \mathcal{N}(x^{*}_{\tilde{\tilde{a}}})$, then $x^{*}_{\tilde{\tilde{a}}} \in \mathcal{A}$.*

(2) *If $\mathcal{A},\ \mathcal{B} \in \mathcal{N}(x^{*}_{\tilde{\tilde{a}}})$, then $\mathcal{A} \Cap \mathcal{B} \in \mathcal{N}(x^{*}_{\tilde{\tilde{a}}})$.*

(3) *If $\mathcal{A} \in \mathcal{N}(x^{*}_{\tilde{\tilde{a}}})$ and $\mathcal{A} \Subset \mathcal{B}$, then $\mathcal{B} \in \mathcal{N}(x^{*}_{\tilde{\tilde{a}}})$.*

(4) *If $\mathcal{A} \in \mathcal{N}(x^{*}_{\tilde{\tilde{a}}})$, then there is $\mathcal{B} \in \mathcal{N}(x^{*}_{\tilde{\tilde{a}}})$ such that $\mathcal{B} \Subset \mathcal{A}$ and $\mathcal{B} \in \mathcal{N}(y^{*}_{\tilde{\tilde{a}}})$ for $y^{*}_{\tilde{\tilde{a}}} \in \mathcal{B}$.*

Conversely, if for each $x^{*}_{\tilde{\tilde{a}}} \in MBJN_P(X)$, $\mathcal{N}_{x^{*}_{\tilde{\tilde{a}}}}$ satisfies the conditions (1), (2) and (3), then the family τ of cubic sets in X given by:

$$\tau = \{\mathcal{A} \in MBJN(X) : \mathcal{A} \in \mathcal{N}_{x^{*}_{\tilde{\tilde{a}}}} \text{ for each } x^{*}_{\tilde{\tilde{a}}} \in \mathcal{A}\}$$

is a ∗-MBJ-neutrosophic topology on X. Furthermore, if $\mathcal{N}^{*}_{x_{\tilde{\tilde{a}}}}$ satisfies the condition (4), then $\mathcal{N}_{x^{*}_{\tilde{\tilde{a}}}}$ is exactly the system of ∗-MBJ-neutrosophic neidhborhood of $x_{\overset{*}{\tilde{\tilde{a}}}}$ with respect to τ, i.e., $\mathcal{N}_{x^{*}_{\tilde{\tilde{a}}}} = \mathcal{N}(x^{*}_{\tilde{\tilde{a}}})$.

Lemma 4.15. *Let (X, τ) be a fuzzy cotopological space, let $x_a \in F_P(X)$ and let $N^{\circ}_{\tau,Q}(x_a)$ be the family of fuzzy sets in X defined as follows: for each $A \in I^X$,*

$$A \in N^{\circ}_{\tau,Q}(x_a) \ if\ and\ only\ if\ there\ is\ B \in \tau\ such\ that\ x_a q^{\circ} B \supset A.$$

(1) *If $A \in N^{\circ}_{\tau,Q}(x_a)$, then $x_a q^{\circ} A$.*

(2) *If $A,\ B \in N^{\circ}_{\tau,Q}(x_a)$, then $A \cup B \in N^{\circ}_{\tau,Q}(x_a)$.*

(3) *If $A \in N^{\circ}_{\tau,Q}(x_a)$ and $B \subset A$, then $B \in N^{\circ}_{\tau,Q}(x_a)$.*

(4) *If* $A \in N^{\circ}_{\tau,Q}(x_a)$*, then there is* $B \in N^{\circ}_{\tau,Q}(x_a)$ *such that* $A \subset B$ *and* $B \in N^{\circ}_{\tau,Q}(y_b)$ *for* $y_b q^{\circ} B$*.*

Conversely, if for each $x_a \in F_P(X)$*,* N°_{Q,x_a} *satisfies the conditions (1), (2) and (3), then the family* τ *of fuzzy sets in* X *given by:*

$$\tau = \{A \in I^X : A \in N^{\circ}_{Q,x_a} \text{ for each } x_a q^{\circ} A\}$$

is a fuzzy cotopology on X*. Furthermore, if* N°_{Q,x_a} *satisfies the condition (4), then* N°_{Q,x_a} *is exactly the system of fuzzy* $\circ$*-Q-neighborhoods of* x_a *with respect to* τ*, i.e.,* $N^{\circ}_{Q,x_a} = N^{\circ}_{\tau,Q}(x_a)$*.*

Proof. (1) Suppose $A \in N^{\circ}_{\tau,Q}(x_a)$. Then there is $U \in \tau$ such that $x_a q^{\circ} U \supset A$. Thus $a < U^c(x) \leq A^c(x)$. So $x_a q^{\circ} A$.

(2) Suppose $A,\ B \in N^{\circ}_{\tau,Q}(x_a)$. Then there are $U,\ V \in \tau$ such that $x_a q^{\circ} U \supset A$ and $x_a q^{\circ} V \supset B$. Thus $a < U^c(x) \leq A^c(x)$ and $a < V^c(x) \leq B^c(x)$. So we get

$$a < U^c(x) \wedge V^c(x) \leq A^c(x) \wedge B^c(x) = (U \cup V)^c(x) \leq (A \cup B)^c(x).$$

Hence $x_a q^{\circ} U \cup V \supset A \cup B$ and $U \cup V \in \tau$. Therefore $A \cup B \in N^{\circ}_{\tau,Q}(x_a)$.

(3) $A \in N^{\circ}_{\tau,Q}(x_a)$ and $B \subset A$. Then there is $U \in \tau$ such that $x_a q^{\circ} U \supset A$. Thus $a < U^c(x) \leq A^c(x) \leq B^c(x)$. So $x_a q^{\circ} U \supset B$. Hence $B \in N^{\circ}_{\tau,Q}(x_a)$.

(4) Suppose $A \in N^{\circ}_{\tau,Q}(x_a)$. Then there is $B \in \tau$ such that $x_a q^{\circ} B \supset A$. Since $B \supset B$, $B \in N^{\circ}_{\tau,Q}(x_a)$ and moreover, $B \in N^{\circ}_{\tau,Q}(y_b)$ for each $y_b q^{\circ} B$.

Conversely, suppose N_{Q,x_a} satisfies the conditions (1), (2) and (3) for each $x_a \in F_P(X)$. From the definition of τ, it is clear that $\mathbf{0},\ \mathbf{1} \in \tau$. Now let $A,\ B \in \tau$ and let $x_a q^{\circ}(A \cup B)$. Then by Lemma 3.19 (2), $x_a q^{\circ} A$ and $x_a q_{\circ} B$. So by the definition of τ, $A \in N^{\circ}_{Q,x_a}$ and $B \in N^{\circ}_{Q,x_a}$. By the condition (2), $A \cup B \in N_{Q,x_a}$. Hence $A \cup B \in \tau$. Finally, let $(A_j)_{j \in J} \subset \tau$, let $A = \bigcap_{j \in J} A_j$ and let $x_a q^{\circ} A$. By Lemma 3.19 (1), there is $j \in J$ such that $x_a q^{\circ} A_j$. Since $A_j \in \tau$, $A_j \in N^{\circ}_{Q,x_a}$. Since $A_j \supset A$, by the condition (3), $A \in \tau$, i.e., $\bigcap_{j \in J} A_j \in \tau$. Therefore τ is a fuzzy cotopology on X.

Now suppose N°_{Q,x_a} satisfies the conditions (4). Then we can easily prove similarly to a classical case that $N^{\circ}_{Q,x_a} = N^{\circ}_{Q}(x_a)$. □

From Remarks 4.4 (5) and 4.9, Proposition 2.2 [16], Lemma 4.18 [33] and Lemma 4.15, we obtain the following.

Theorem 4.16. *Let* (X, τ) *be a* $\circ$*-MBJ-neutrosophic topological space and let* $x^{\circ}_{\tilde{\tilde{a}}} \in MBJN_P(X)$*.*

(1) *If* $\mathcal{A} \in \mathcal{N}_Q(x^{\circ}_{\tilde{\tilde{a}}})$*, then* $x^{\circ}_{\tilde{\tilde{a}}} q \mathcal{A}$*.*

(2) *If* $\mathcal{A},\ \mathcal{B} \in \mathcal{N}_Q(x^{\circ}_{\tilde{\tilde{a}}})$*, then* $\mathcal{A} \sqcap \mathcal{B} \in \mathcal{N}_Q(x^{\circ}_{\tilde{\tilde{a}}})$*.*

(3) *If* $\mathcal{A} \in \mathcal{N}_Q(x^{\circ}_{\tilde{\tilde{a}}})$ *and* $\mathcal{A} \sqsubset \mathcal{B}$*, then* $\mathcal{B} \in \mathcal{N}_Q(x^{\circ}_{\tilde{\tilde{a}}})$*.*

(4) *If* $\mathcal{A} \in \mathcal{N}_Q(x^{\circ}_{\tilde{\tilde{a}}})$*, then there is* $\mathcal{B} \in \mathcal{N}_Q(x^{\circ}_{\tilde{\tilde{a}}})$ *such that* $\mathcal{B} \sqsubset \mathcal{A}$ *and* $\mathcal{B} \in \mathcal{N}_Q(y^{\circ}_{\tilde{\tilde{a}}})$ *for* $y^{\circ}_{\tilde{\tilde{a}}} q\mathcal{B}$*.*

Conversely, if for each $x^{\circ}_{\tilde{\tilde{a}}} \in MBJN_P(X)$*,* $\mathcal{N}_{Q,x^{\circ}_{\tilde{\tilde{a}}}}$ *satisfies the conditions (1), (2) and (3), then the family* τ *of MBJ-neutrosophic sets in* X *given by:*

$$\tau = \{\mathcal{A} \in MBJN(X) : \mathcal{A} \in \mathcal{N}_{Q,x^{\circ}_{\tilde{\tilde{a}}}} \text{ for each } x^{\circ}_{\tilde{\tilde{a}}} \in \mathcal{A}\}$$

is a $\circ$*-MBJ-neutrosophic topology on* X*. Furthermore, if* $\mathcal{N}^{\circ}_{Q,x_{\tilde{\tilde{a}}}}$ *satisfies the condition (4), then* $\mathcal{N}_{Q,x^{\circ}_{\tilde{\tilde{a}}}}$ *is exactly the system of* $\circ$*-MBJ-neutrosophic neidhborhood of* $x^{\circ}_{\tilde{\tilde{a}}}$ *with respect to* τ*, i.e.,* $\mathcal{N}_{Q,x^{\circ}_{\tilde{\tilde{a}}}} = \mathcal{N}_Q(x^{\circ}_{\tilde{\tilde{a}}})$*.*

Lemma 4.17. *Let* (X, τ) *be an interval-valued fuzzy cotopological space, let* $x_{\tilde{a}} \in IVF_P(X)$ *and let* $\widetilde{N}^*_{\tau,Q}(x_{\tilde{a}})$ *be the family of interval-valued fuzzy sets in* X *defined as follows: for each* $\widetilde{A} \in IVFS(X)$*,*

$$\widetilde{A} \in \widetilde{N}^*_{\tau,Q}(x_{\tilde{a}}) \text{ if and only if there is } \widetilde{B} \in \tau \text{ such that } x_{\tilde{a}} q^* \widetilde{B} \supset \widetilde{A}.$$

(1) *If* $\widetilde{A} \in \widetilde{N}^*_{\tau,Q}(x_{\tilde{a}})$*, then* $x_{\tilde{a}} q^* \widetilde{A}$*.*

(2) *If* $\widetilde{A},\ \widetilde{B} \in \widetilde{N}^*_{\tau,Q}(x_{\tilde{a}})$*, then* $\widetilde{A} \cup \widetilde{B} \in \widetilde{N}^*_{\tau,Q}(x_{\tilde{a}})$*.*

(3) *If* $\widetilde{A} \in \widetilde{N}^*_{\tau,Q}(x_{\tilde{a}})$ *and* $\widetilde{B} \subset \widetilde{A}$*, then* $\widetilde{B} \in \widetilde{N}^*_{\tau,Q}(x_{\tilde{a}})$*.*

(4) *If* $\widetilde{A} \in \widetilde{N}^*_{\tau,Q}(x_{\tilde{a}})$*, then there is* $\widetilde{B} \in \widetilde{N}^*_{\tau,Q}(x_{\tilde{a}})$ *such that* $\widetilde{A} \subset \widetilde{B}$ *and* $\widetilde{B} \in \widetilde{N}^*_{\tau,Q}(y_{\tilde{b}})$ *for* $y_{\tilde{b}} q^* \widetilde{B}$*.*

Conversely, if for each $x_{\tilde{a}} \in IVF_P(X)$*,* $\widetilde{N}^*_{Q,x_{\tilde{a}}}$ *satisfies the conditions (1), (2) and (3), then the family* τ *of interval-valued fuzzy sets in* X *given by:*

$$\tau = \{\widetilde{A} \in IVFS(X) : \widetilde{A} \in \widetilde{N}^*_{Q,x_{\tilde{a}}} \text{ for each } x_{\tilde{a}} q^* \widetilde{A}\}$$

is an interval-valued fuzzy cotopology on X*. Furthermore, if* $\widetilde{N}^*_{Q,x_{\tilde{a}}}$ *satisfies the condition (4), then* $\widetilde{N}^*_{Q,x_{\tilde{a}}}$ *is exactly the system of interval-valued fuzzy* $*$*-Q-neighborhoods of* $x_{\tilde{a}}$ *with respect to* τ*, i.e.,* $\widetilde{N}^*_{Q,x_{\tilde{a}}} = \widetilde{N}^*_{\tau,Q}(x_{\tilde{a}})$*.*

Proof. The proof is similar to Lemma 3.15. □

From Remarks 4.4 (5) and 4.9, Proposition 2.2 [16], Lemmas 4.15 and Lemma 4.17, we get the following.

Theorem 4.18. *Let* (X, τ) *be a* $*$*-MBJ-neutrosophic topological space and let* $x^*_{\tilde{\tilde{a}}} \in MBJN_P(X)$*.*

(1) *If* $\mathcal{A} \in \mathcal{N}_Q(x^*_{\tilde{\tilde{a}}})$*, then* $x^*_{\tilde{\tilde{a}}} q\mathcal{A}$*.*

(2) *If* $\mathcal{A},\ \mathcal{B} \in \mathcal{N}_Q(x^*_{\tilde{\tilde{a}}})$*, then* $\mathcal{A} \Cap \mathcal{B} \in \mathcal{N}_Q(x^*_{\tilde{\tilde{a}}})$*.*

(3) *If* $\mathcal{A} \in \mathcal{N}_Q(x^*_{\tilde{\tilde{a}}})$ *and* $\mathcal{A} \Subset \mathcal{B}$*, then* $\mathcal{B} \in \mathcal{N}_Q(x^*_{\tilde{\tilde{a}}})$*.*

(4) *If* $\mathcal{A} \in \mathcal{N}_Q(x^*_{\tilde{\tilde{a}}})$*, then there is* $\mathcal{B} \in \mathcal{N}_Q(x^*_{\tilde{\tilde{a}}})$ *such that* $\mathcal{B} \Subset \mathcal{A}$ *and* $\mathcal{B} \in \mathcal{N}_Q(y^*_{\tilde{\tilde{a}}})$ *for* $y^*_{\tilde{\tilde{a}}} q\mathcal{B}$*.*

Conversely, if for each $x^*_{\tilde{\tilde{a}}} \in MBJN_P(X)$, $\mathcal{N}_{Q,x^*_{\tilde{\tilde{a}}}}$ *satisfies the conditions (1), (2) and (3), then the family* τ *of MBJ-neutrosophic sets in* X *given by:*

$$\tau = \{\mathcal{A} \in MBJN(X) : \mathcal{A} \in \mathcal{N}_{Q,x^*_{\tilde{\tilde{a}}}} \text{ for each } x^*_{\tilde{\tilde{a}}} \in \mathcal{A}\}$$

is a $*$*-MBJ-neutrosophic topology on* X*. Furthermore, if* $\mathcal{N}^*_{Q,x_{\tilde{\tilde{a}}}}$ *satisfies the condition (4), then* $\mathcal{N}_{Q,x^*_{\tilde{\tilde{a}}}}$ *is exactly the system of* $*$*-MBJ-neutrosophic neidhborhood of* $x^*_{\tilde{\tilde{a}}}$ *with respect to* τ*, i.e.,* $\mathcal{N}_{Q,x^*_{\tilde{\tilde{a}}}} = \mathcal{N}_Q(x^*_{\tilde{\tilde{a}}})$.

5. MBJ-neutrosophic bases and MBJ-neutrosophic local bases

We introduce the concept of $\circ$-[resp. $*$-]MBJ-neutrosophic bases and $\circ$-[resp. $*$-]MBJ-neutrosophic local bases, and discuss some of their properties. Also we define $\circ$-C_I, $*$-C_I, $\circ$-C_{II} and $*$-C_{II}, and we obtain the relationships between them. Moreover, we give an Example that the converse of Proposition 5.23 does not hold.

Definition 5.1. Let (X, τ) be a $\circ$-[resp. $*$-]MBJ-neutrosophic topological space and let $\beta \subset \tau$, $\sigma \subset \tau$.

(i) β is called a $\circ$-*MBJ-neutrosophic base* (briefly, $\circ$-MBJNB) [resp. $*$-*MBJ-neutrosophic base* (briefly, $*$-MBJNB)] for τ, if for each $\mathcal{A} \in \tau, \mathcal{A} = \ddot{\emptyset}$ [resp. $\mathcal{A} = \dot{\emptyset}$] or there is $\beta' \subset \beta$ such that $\mathcal{A} = \sqcup\beta'$ [resp. $\mathcal{A} = \Cup\beta'$].

(ii) σ is called a $\circ$-*MBJ-neutrosophic subbase* (briefly, $\circ$-MBJNSB) [resp. $*$-*MBJ-neutrosophic subbase* (briefly, $*$-MBJNSB)] for τ, if the family $\beta = \{\sqcap\eta : \eta \text{ is a finite subset of } \sigma\}$ [resp. $\beta = \{\Cap\eta : \eta \text{ is a finite subset of } \sigma\}$] is a $\circ$-MBJNB [resp. $*$-MBJNB] for τ.

Now we will introduce the concepts of bases and subbases for a fuzzy cotopology and an interval-valued fuzzy cotopology.

Definition 5.2. Let τ be a fuzzy cotopology on a nonempty set X and let $\beta,\ \sigma \subset \tau$.

(i) β is called a $\circ$-*fuzzy base for* τ, if for each $A \in \tau$, $A = \mathbf{1}$ or there is $\beta' \subset \beta$ such that $A = \bigcap \beta'$.

(ii) σ is called a $\circ$-*fuzzy subbase for* τ, if the family $\beta = \{\bigcup \eta : \eta \text{ is a finite subset of } \sigma\}$ is a $\circ$-fuzzy base for τ.

Definition 5.3. Let τ be an interval-valued fuzzy cotopology on a nonempty set X and let $\beta,\ \sigma \subset \tau$.

(i) β is called a $*$-*interval-valued fuzzy base for* τ, if for each $\widetilde{A} \in \tau$, $\widetilde{A} = \widetilde{\mathbf{1}}$ or there is $\beta' \subset \beta$ such that $\widetilde{A} = \bigcap \beta'$.

(ii) σ is called a $*$-*interval-valued fuzzy subbase for* τ, if the family $\beta = \{\bigcup \eta : \eta \text{ is a finite subset of } \sigma\}$ is a $*$-fuzzy base for τ.

Example 5.4. (1) Let $X = \{x, y\}$ and consider the fuzzy sets A_i $(i = 1, 2, \cdots, 9)$ in X given by:

$$A_1(x) = 0.3,\ A_1(y) = 0.7,\ A_2(x) = 0.6,\ A_2(y) = 0.4,\ A_3(x) = 0.5,\ A_3(y) = 0.6,$$

$$A_4(x) = 0.6,\ A_4(y) = 0.7,\ A_5(x) = 0.5,\ A_5(y) = 0.7,\ A_6(x) = 0.6,\ A_6(y) = 0.6,$$

$$A_7(x) = 0.3,\ A_7(y) = 0.4,\ A_8(x) = 0.3,\ A_8(y) = 0.6,\ A_9(x) = 0.5,\ A_9(y) = 0.4.$$

Then we can easily see that the family $\tau = \{\mathbf{0}, \mathbf{1}, A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9\}$ is a fuzzy cotopology on X. Now let us two subfamilies β and σ of τ given by:

$$\beta = \{\mathbf{0}, A_1, A_2, A_3, A_4, A_5, A_6\},\ \sigma = \{\mathbf{0}, A_1, A_2, A_3\}.$$

Then we can easily check that β and σ are $\circ$-fuzzy base and $\circ$-fuzzy subbase for τ respectively.

(2) Let $X = \{x, y\}$ and consider the interval-valued fuzzy sets $\widetilde{A}_i$ $(i = 1, 2, \cdots, 9)$ in X given by:

$$\widetilde{A}_1(x) = [0.3, 0.5],\ A_1(y) = [0.7, 0.8],\ \widetilde{A}_2(x) = [0.6, 0.7],\ \widetilde{A}_2(y) = [0.4, 0.6],$$

$$\widetilde{A}_3(x) = [0.5, 0.6],\ \widetilde{A}_3(y) = [0.6, 0.7], \widetilde{A}_4(x) = [0.6, 0.7],\ \widetilde{A}_4(y) = [0.7, 0.8],$$

$$\widetilde{A}_5(x) = [0.5, 0.6],\ \widetilde{A}_5(y) = [0.7, 0.8],\ \widetilde{A}_6(x) = [0.6, 0.7],\ \widetilde{A}_6(y) = [0.6, 0.7],$$

$$\widetilde{A}_7(x) = [0.3, 0.5],\ \widetilde{A}_7(y) = [0.4, 0.6],\ \widetilde{A}_8(x) = [0.3, 0.5],\ \widetilde{A}_8(y) = [0.6, 0.7],$$

$$\widetilde{A}_9(x) = [0.5, 0.6],\ \widetilde{A}_9(y) = [0.4, 0.6].$$

Then we can easily see that the family $\tau = \{\widetilde{\mathbf{0}}, \widetilde{\mathbf{1}}, \widetilde{A}_1, \widetilde{A}_2, \widetilde{A}_3, \widetilde{A}_4, \widetilde{A}_5, \widetilde{A}_6, \widetilde{A}_7, \widetilde{A}_8, \widetilde{A}_9\}$ is a fuzzy cotopology on X. Now let us two subfamilies β and σ of τ given by:

$$\beta = \{\widetilde{\mathbf{0}}, \widetilde{A}_1, \widetilde{A}_2, \widetilde{A}_3, \widetilde{A}_4, \widetilde{A}_5, \widetilde{A}_6\},\ \sigma = \{\widetilde{\mathbf{0}}, \widetilde{A}_1, \widetilde{A}_2, \widetilde{A}_3\}.$$

Then we can easily check that β and σ are $*$-interval-valued fuzzy base and $*$-interval-valued fuzzy subbase for τ respectively.

(3) Let $X = \{x, y\}$ and consider the MBJ-neutrosophic sets $\mathcal{A}_i$ $(i = 1, 2, \cdots, 9)$ in X given by:

$$\mathcal{A}_1(x) = \langle 0.5, [0.5, 0.6], 0.3 \rangle,\ \mathcal{A}_1(y) = \langle 0.6, [0.6, 0.7], 0.7 \rangle,$$

$$\mathcal{A}_2(x) = \langle 0.6, [0.6, 0.8], 0.6 \rangle,\ \mathcal{A}_2(y) = \langle 0.3, [0.3, 0.5], 0.4 \rangle,$$

$$\mathcal{A}_3(x) = \langle 0.8, [0.8, 0.9], 0.5 \rangle,\ \mathcal{A}_3(y) = \langle 0.2, [0.2, 0.4], 0.6 \rangle,$$

$$\mathcal{A}_4(x) =< \langle 0.5, [0.5, 0.6], 0.5 \rangle,\ \mathcal{A}_4(y) = \langle 0.3, [0.3, 0.5], 0.7 \rangle,$$

$$\mathcal{A}_5(x) = \langle 0.5, [0.5, 0.6], 0.5 \rangle,\ \mathcal{A}_5(y) = \langle 0.2, [0.2, 0.4], 0.7 \rangle,$$

$$\mathcal{A}_6(x) = \langle 0.6, [0.6, 0.8], 0.6 \rangle,\ \mathcal{A}_6(y) = \langle 0.2, [0.2, 0.4], 0.6 \rangle,$$

$$\mathcal{A}_7(x) = \langle 0.6, [0.6, 0.8], 0.3 \rangle,\ \mathcal{A}_7(y) = \langle 0.6, [0.6, 0.7], 0.4 \rangle,$$

$$\mathcal{A}_8(x) = \langle 0.8, [0.8, 0.9], 0.3 \rangle,\ \mathcal{A}_8(y) = \langle 0.6, [0.6, 0.7], 0.6 \rangle,$$

$$\mathcal{A}_9(x) = \langle 0.8, [0.8, 0.9], 0.5\rangle\,,\ \mathcal{A}_9(y) = \langle 0.3, [0.3, 0.5], 0.4\rangle\,.$$

Then we can easily see that the family $\tau = \{\ddot{\emptyset}, \ddot{X}, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6, \mathcal{A}_7, \widetilde{A}_8, \mathcal{A}_9\}$ is a $\circ$-MBJ-neutrosophic topology on X. Now let us two subfamilies β and σ of τ given by:

$$\beta = \{\ddot{X}, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6\},\ \sigma = \{\ddot{X}, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}.$$

Then we can easily check that β and σ are $\circ$-MBJNB and $*$-MBJNSB for τ respectively.

(4) Let $X = \{x, y\}$ and consider the MBJ-neutrosophic sets $\mathcal{A}_i$ $(i = 1, 2, \cdots, 9)$ in X given by:

$$\mathcal{A}_1(x) = \langle 0.5, [0.3, 0.5], 0.3\rangle\,,\ \mathcal{A}_1(y) = \langle 0.6, [0.7, 0.8], 0.7\rangle\,,$$

$$\mathcal{A}_2(x) = \langle 0.6, [0.6, 0.7], 0.6\rangle\,,\ \mathcal{A}_2(y) = \langle 0.3, [0.4, 0.6], 0.4\rangle\,,$$

$$\mathcal{A}_3(x) = \langle 0.8, [0.5, 0.6], 0.5\rangle\,,\ \mathcal{A}_3(y) = \langle 0.2, [0.6, 0.7], 0.6\rangle\,,$$

$$\mathcal{A}_4(x) =< \langle 0.5, [0.6, 0.7], 0.5\rangle\,,\ \mathcal{A}_4(y) = \langle 0.3, [0.7, 0.8], 0.7\rangle\,,$$

$$\mathcal{A}_5(x) = \langle 0.5, [0.5, 0.6], 0.5\rangle\,,\ \mathcal{A}_5(y) = \langle 0.2, [0.7, 0.8], 0.7\rangle\,,$$

$$\mathcal{A}_6(x) = \langle 0.6, [0.6, 0.7], 0.6\rangle\,,\ \mathcal{A}_6(y) = \langle 0.2, [0.6, 0.7], 0.6\rangle\,,$$

$$\mathcal{A}_7(x) = \langle 0.6, [0.3, 0.5], 0.3\rangle\,,\ \mathcal{A}_7(y) = \langle 0.6, [0.4, 0.6], 0.4\rangle\,,$$

$$\mathcal{A}_8(x) = \langle 0.8, [0.3, 0.5], 0.3\rangle\,,\ \mathcal{A}_8(y) = \langle 0.6, [0.6, 0.7], 0.6\rangle\,,$$

$$\mathcal{A}_9(x) = \langle 0.8, [0.5, 0.6], 0.5\rangle\,,\ \mathcal{A}_9(y) = \langle 0.3, [0.4, 0.6], 0.4\rangle\,.$$

Then we can easily see that the family $\tau = \{\dot{\emptyset}, \dot{X}, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6, \mathcal{A}_7, \widetilde{A}_8, \mathcal{A}_9\}$ is a $*$-MBJ-neutrosophic topology on X. Now let us two subfamilies β and σ of τ given by:

$$\beta = \{\dot{X}, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6\},\ \sigma = \{\dot{X}, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}.$$

Then we can easily check that β and σ are $*$-MBJNB and $*$-MBJNSB for τ respectively.

Remark 5.5. (1) Let (X, τ) be a $\circ$-MBJ-neutrosophic topological space or $*$-MBJ-neutrosophic topological space and let $\beta \subset \tau$. Consider the following families:

$$\beta_M = \{M_A \in I^X : \mathcal{A} \in \beta\},\ \beta_{\widetilde{B}} = \{\widetilde{B}_A \in IVFS(X) : \mathcal{A} \in \beta\},\ \beta_J = \{J_A \in I^X : \mathcal{A} \in \beta\}.$$

Then from Remark 4.4 (5), Definitions 5.1 and 5.2, we can easily check that

(a) β is a $\circ$-MBJNB for τ if and only if β_M is a fuzzy base for τ_M, $\beta_{\widetilde{B}}$ is an interval-valued fuzzy base for $\tau_{\widetilde{B}}$ and β_J is a $\circ$-fuzzy base for τ_J,

(b) β is a $*$-MBJNB for τ if and only if β_M is a fuzzy base for τ_M, $\beta_{\widetilde{B}}$ is a $*$-interval-valued fuzzy base for $\tau_{\widetilde{B}}$ and β_J is a $\circ$-fuzzy base for τ_J.

(2) Let (X, τ) be a $\circ$-MBJ-neutrosophic topological space or $*$-MBJ-neutrosophic topological space and let $\sigma \subset \tau$. Consider the following families:

$$\sigma_M = \{M_A \in I^X : \mathcal{A} \in \sigma\},\ \sigma_{\widetilde{B}} = \{\widetilde{B}_A \in IVFS(X) : \mathcal{A} \in \sigma\},\ \sigma_J = \{J_A \in I^X : \mathcal{A} \in \sigma\}.$$

Then from Remark 4.4 (5), Definitions 5.1 and 5.2, we can easily see that

(a) σ is a $\circ$-MBJNSB for τ if and only if σ_M is a fuzzy subbase for τ_M, $\sigma_{\widetilde{B}}$ is an interval-valued fuzzy subbase for $\tau_{\widetilde{B}}$ and σ_J is a $\circ$-fuzzy subbase for τ_J,

(b) σ is a $*$-MBJNSB for τ if and only if σ_M is a fuzzy subbase for τ_M, $\sigma_{\widetilde{B}}$ is a $*$-interval-valued fuzzy subbase for $\tau_{\widetilde{B}}$ and σ_J is a $\circ$-fuzzy subbase for τ_J.

Lemma 5.6. *Let (X, τ) be a fuzzy cotopological space and let $\beta \subset \tau$. Then β is a $\circ$-fuzzy base for τ if and only if for each $x_a \in F_P(X)$ and for each fuzzy closed $\circ$-Q-neighborhood A of x_a, there is $B \in \beta$ such that $x_a q^\circ B \supset A$.*

Proof. $(\Rightarrow)$ The proof is straightforward from the definition of a $\circ$-fuzzy base and the necessary condition of Lemma 3.20 (1).

$(\Leftarrow)$ Suppose the necessary condition holds. Assume that β is not a $\circ$-fuzzy base for τ. Then there is $A \in \tau$ such that $U = \bigcap\{B \in \beta : B \supset A\} \neq A$. Thus there is $x \in X$ such that $U(x) > A(x)$. Let $a = U^c(x)$. Then clearly, $A(x) + a < U(x) + a < 1$. Thus $x_a q^\circ A$. On the other hand, let $B \in \beta$ such that $B \supset A$. Then clearly, $B \supset U$. Thus $B(x) + a \geq U(x) + a = 1$. So $x_a \neg q^\circ \widetilde{B}$. This contradicts the hypothesis. □

From Proposition 2.4 in [16], Lemma 7 in [33], Lemmas 3.19 (1) and 5.6, we have the following.

Theorem 5.7. *Let (X, τ) be a $\circ$-MBJ-neutrosophic topological space and let $\beta \subset \tau$. Then β is a $\circ$-MBJNB for τ if and only if for each $x^\circ_{\tilde{\tilde{a}}} \in MBJN_P(X)$ and for each $\circ$-MBJ-neutrosophic open Q-neighborhood $\mathcal{A}$ of $x^\circ_{\tilde{\tilde{a}}}$, there is $\mathcal{B} \in \beta$ such that $x^\circ_{\tilde{\tilde{a}}} q^\circ \mathcal{B} \sqsubset \mathcal{A}$.*

Lemma 5.8. *Let (X, τ) be an interval-valued fuzzy cotopological space and let $\beta \subset \tau$. Then β is a $*$-interval-valued fuzzy base for τ if and only if for each $x_{\tilde{a}} \in IVF_P(X)$ and for each fuzzy closed $*$-Q-neighborhood $\widetilde{A}$ of $x_{\tilde{a}}$, there is $\widetilde{B} \in \beta$ such that $x_{\tilde{a}} q^* B \supset A$.*

Proof. The proof is similar to Lemma 5.6. □

Theorem 5.9. *Let (X, τ) be a $*$-MBJ-neutrosophic topological space and let $\beta \subset \tau$. Then β is a $*$-MBJNB for τ if and only if for each $x^*_{\tilde{\tilde{a}}} \in MBJN_P(X)$ and for each $*$-MBJ-neutrosophic open Q-neighborhood $\mathcal{A}$ of $x^*_{\tilde{\tilde{a}}}$, there is $\mathcal{B} \in \beta$ such that $x^*_{\tilde{\tilde{a}}} q^* \mathcal{B} \Subset \mathcal{A}$.*

Proof. The proof is straightforward from Proposition 2.4 in [16], Lemmas 5.8 and 5.6. □

The following gives a necessary and sufficient condition for a subset of $MBJNS(X)$ to be a $\circ$-MBJNB for a $\circ$-MBJ-neutrosophic topology on a set X.

Theorem 5.10. *Let X be a set and let $\beta \subset MBJNS(X)$. Then β is a $\circ$-MBJNB for some $\circ$-MBJ-neutrosophic topology τ if and only if the followings hold:*

(1) $\ddot{X} = \sqcup\beta$,

(2) *if* $\mathcal{B}_1,\ \widetilde{B}_2 \in \beta$ *and* $x^{\circ}_{\tilde{\tilde{a}}} \in \mathcal{B}_1 \sqcup \mathcal{B}_2$, *then there is* $\mathcal{B} \in \beta$ *such that*

$$x^{\circ}_{\tilde{\tilde{a}}} \in \mathcal{B} \sqsubset \mathcal{B}_1 \sqcap \mathcal{B}_2.$$

In this case, τ is called the *$\circ$-MBJ-neutrosophic topology on X generated by β.*

Proof. ($\Rightarrow$) Suppose β is a $\circ$-MBJNB for a $\circ$-MBJ-neutrosophic topology τ. Since $\ddot{X} \in \tau$, $\ddot{X} = \sqcap\beta$. Then the condition (1) holds. Now suppose $\mathcal{B}_1,\ \widetilde{B}_2 \in \beta$ and $x^{\circ}_{\tilde{\tilde{a}}} \in \mathcal{B}_1 \sqcap \mathcal{B}_2$. Since $\beta \subset \tau$, $\mathcal{B}_1,\ \widetilde{B}_2 \in \tau$. Then $\mathcal{B}_1 \sqcap \mathcal{B}_2 \in \tau$. Since $x^{\circ}_{\tilde{\tilde{a}}} \in \mathcal{B}_1 \sqcap \mathcal{B}_2$, $\mathcal{B}_1 \sqcap \mathcal{B}_2 \neq \ddot{\emptyset}$. By the definition of a $\circ$-MBJNB, there is $\beta' \subset \beta$ such that $\mathcal{B}_1 \sqcap \mathcal{B}_2 = \sqcup\beta'$. Thus there is $\mathcal{B} \in \beta$ such that $x^{\circ}_{\tilde{\tilde{a}}} \in \mathcal{B} \sqsubset \mathcal{B}_1 \sqcap \mathcal{B}_2$. So the condition (2) holds.

($\Leftarrow$) Suppose the conditions (1) and (2) hold and let

$$\tau = \{\mathcal{U} \in MBJNS(X) : \mathcal{U} = \ddot{\emptyset} \text{ or there is } \beta' \subset \beta \text{ such that } \mathcal{U} = \sqcup\beta'\}.$$

Then clearly, $\ddot{\emptyset},\ \ddot{X} \in \tau$. Thus the condition ($\circ$-MBJNO$_1$) holds. Now suppose $\mathcal{U}_1,\ \mathcal{U}_2 \in \tau$ and $x^{\circ}_{\tilde{\tilde{a}}} \in \mathcal{U}_1 \sqcap \mathcal{U}_2$. Then there are $\mathcal{B}_1,\ \mathcal{B}_2 \in \beta$ such that $x^{\circ}_{\tilde{\tilde{a}}} \in \mathcal{B}_1 \sqsubset \mathcal{U}_1$ and $x_{\tilde{\tilde{a}}} \in \mathcal{B}_2 \sqsubset \mathcal{U}_2$. Thus $x^{\circ}_{\tilde{\tilde{a}}} \in \mathcal{B}_1 \sqcap \mathcal{B}_2 \sqsubset \mathcal{U}_1 \sqcap \mathcal{U}_2$. By the condition (2), there is $\mathcal{B} \in \beta$ such that $x^{\circ}_{\tilde{\tilde{a}}} \in \mathcal{B} \sqsubset \mathcal{U}_1 \sqcap \mathcal{U}_2$. So $\mathcal{U}_1 \sqcap \mathcal{U}_2 \in \tau$. Hence the condition ($\circ$-MBJNO$_2$) holds. Since τ consists of all $\circ$-MBJ unions of members of β, the $\circ$-MBJ union of any family of members of τ is also a member of τ. Then ($\circ$-MBJNO$_3$) holds. This completes the proof. □

Also, we have a necessary and sufficient condition for a subset of $MBJNS(X)$ to be a $*$-MBJNB for a $*$-MBJ-neutrosophic topology on a set X.

Theorem 5.11. *Let X be a set and let $\beta \subset MBJNS(X)$. Then β is a $*$-MBJNB for some $*$-MBJ-neutrosophic topology τ if and only if the followings hold:*

(1) $\ddot{X} = \Cup\beta$,

(2) *if* $\mathcal{B}_1,\ \widetilde{B}_2 \in \beta$ *and* $x^{*}_{\tilde{\tilde{a}}} \in \mathcal{B}_1 \Cap \mathcal{B}_2$, *then there is* $\mathcal{B} \in \beta$ *such that*

$$x^{*}_{\tilde{\tilde{a}}} \in \mathcal{B} \Subset \mathcal{B}_1 \Cap \mathcal{B}_2.$$

In this case, τ is called the *$*$-MBJ-neutrosophic topology on X generated by β.*

Proof. The proof is similar to Theorem 5.10. □

The following provides a sufficient condition for a subset of $MBJNS(X)$ to be a $\circ$-MBJNB for a $\circ$-MBJ-neutrosophic topology on a set X.

Proposition 5.12. *Let X be a set and let $\sigma \subset MBJNS(X)$ such that $\ddot{X} = \sqcup\sigma$. Then there is a unique $\circ$-MBJ-neutrosophc topology τ on X such that σ is a $\circ$-MBJNSB for τ.* In this case, τ is called the *$\circ$-MBJ-neutrosophc topology on X generated by σ.*

Proof. Let $\beta = \{\sqcap\eta : \eta$ is a finite subset of $\sigma\}$ and let

$$\tau = \{\mathcal{U} \in MBJNS(X) : \mathcal{U} = \ddot{\emptyset} \text{ or there is } \beta^{'} \subset \beta \text{ such that } \mathcal{U} = \sqcup\beta^{'}\}.$$

Then clearly, $\ddot{X}$, $\ddot{\emptyset} \in \tau$ by the definition of τ. Thus τ satisfies the condition ($\circ$-MBJNO$_1$). Let $\mathcal{U}_j \in \tau$ for each $j \in J$. Then there is $\beta_j \subset \beta$ such that $\mathcal{U}_j = \sqcup\{\mathcal{B} \in MBJNS(X) : \mathcal{B} \in \beta_j\}$. Thus $\sqcup_{j\in J}\mathcal{U}_j = \sqcup_{j\in J}(\sqcup_{\mathcal{B}\in\beta_j}\mathcal{B}$. So $\sqcup_{j\in J}\mathcal{U}_j \in \tau$. Hence the condition ($\circ$-MBJNO$_3$) holds. Finally, suppose $\mathcal{U}_1$, $\mathcal{U}_2 \in \tau$ and $x^{\circ}_{\tilde{\tilde{a}}} \in \mathcal{U}_1 \sqcap \mathcal{U}_2$. Then by Theorem 5.10, there are $\mathcal{B}_1$, $\mathcal{B}_2 \in \beta$ such that $x^{\circ}_{\tilde{\tilde{a}}} \in \mathcal{B}_1 \sqcap \mathcal{B}_2$, $\mathcal{B}_1 \sqsubset \mathcal{U}_1$ and $\mathcal{B}_2 \sqsubset \mathcal{U}_2$. Since each of $\mathcal{B}_1$ and $\mathcal{B}_2$ is the $\circ$-intersection of a finite number of members of σ, $\mathcal{B}_1 \sqcap \mathcal{B}_2 \in \beta$. So there is $\beta^{'} \subset \beta$ such that $\mathcal{U}_1 \sqcap \mathcal{U}_2 = \sqcup_{\mathcal{B}\in\beta^{'}}\mathcal{B}$. Hence $\mathcal{U}_1 \sqcap \mathcal{U}_2 \in \tau$, i.e., the condition ($\circ$-MBJNO$_2$) holds. Therefore $\tau \in PCT(X)$. It is obvious that τ is the unique $\circ$-MBJ-neutrosophic topology on X having σ as a $\circ$-MBJNSB. □

Also, we get a sufficient condition for a subset of $MBJNS(X)$ to be a $\circ$-MBJNB for a $\circ$-MBJ-neutrosophic topology on a set X.

Proposition 5.13. *Let X be a set and let $\sigma \subset MBJNS(X)$ such that $\dot{X} = \Cup\sigma$. Then there is a unique $*$-MBJ-neutrosophc topology τ on X such that σ is a $*$-MBJNSB for τ.* In this case, τ is called the *$*$-MBJ-neutrosophc topology on X generated by σ.*

Proof. The proof is similar to Proposition 5.12. □

Definition 5.14. Let (X, τ) be a $\circ$-MBJ-neutrosophc topological space or $*$-MBJ-neutrosophc topological space, let $x^{\circ}_{\tilde{\tilde{a}}}$, $x^{*}_{\tilde{\tilde{a}}} \in MBJN_P(X)$ and let $\beta(x^{\circ}_{\tilde{\tilde{a}}}) \subset \mathcal{N}(x^{\circ}_{\tilde{\tilde{a}}})$, $\beta(x^{*}_{\tilde{\tilde{a}}}) \subset \mathcal{N}(x^{*}_{\tilde{\tilde{a}}})$.

(i) $\beta(x^{\circ}_{\tilde{\tilde{a}}})$ is called a *$\circ$-MBJ-neutrosophic neighborhood base* (briefly, $\circ$-MBJNNB) for $\mathcal{N}(x^{\circ}_{\tilde{\tilde{a}}})$, if for each $\mathcal{A} \in \mathcal{N}(x^{\circ}_{\tilde{\tilde{a}}})$, there is $\mathcal{A} \in \beta(x^{\circ}_{\tilde{\tilde{a}}})$ such that $\mathcal{B} \sqsubset \mathcal{A}$.

(ii) $\beta(x^{*}_{\tilde{\tilde{a}}})$ is called a *$*$-MBJ-neutrosophic neighborhood base* (briefly, $*$-MBJNNB) for $\mathcal{N}(x^{*}_{\tilde{\tilde{a}}})$, if for each $\mathcal{A} \in \mathcal{N}(x^{*}_{\tilde{\tilde{a}}})$, there is $\mathcal{A} \in \beta(x^{*}_{\tilde{\tilde{a}}})$ such that $\mathcal{B} \Subset \mathcal{A}$.

(iii) (X, τ) is said to *satisfy the $\circ$-first axiom of countability* or to be $\circ$-C_I, if each $x^{\circ}_{\tilde{\tilde{a}}} \in MBJN_P(X)$ has a countable $\circ$-MBJNNB.

(iv) (X, τ) is said to *satisfy the $*$-first axiom of countability* or to be $*$-C_I, if each $x^{*}_{\tilde{\tilde{a}}} \in MBJN_P(X)$ has a countable $*$-MBJNNB.

From Remark 4.9 and 4.4 (5), we can rewrite Definition 5.14 as followings.

Remark 5.15. Let $\beta(x_a) \subset N_{\tau_M}(x_a)$, $\beta(x_{\tilde{a}}) \subset \widetilde{N}_{\tau_{\tilde{B}}}(x_{\tilde{a}})$, $\beta(x^*_{\tilde{a}}) \subset \widetilde{N}^*_{\tau_{\tilde{B}}}(x^*_{\tilde{a}})$, $\beta(x^\circ_{\bar{a}}) \subset N^\circ_{\tau_J}(x^\circ_{\bar{a}})$. Then we have

(1) $\beta(x^\circ_{\tilde{\tilde{a}}})$ is a $\circ$-MBJNNB for $\mathcal{N}(x^\circ_{\tilde{\tilde{a}}}) \Longleftrightarrow$

(i) $\beta(x_a)$ is a fuzzy neighborhood base for $N_{\tau_M}(x_a)$,

(ii) $\beta(x_{\tilde{a}})$ is an interval-valued fuzzy neighborhood base for $\widetilde{N}_{\tau_{\tilde{B}}}(x_{\tilde{a}})$,

(iii) $\beta(x^\circ_{\bar{a}})$ is a $\circ$-fuzzy neighborhood base for $N^\circ_{\tau_J}(x^\circ_{\bar{a}})$.

(2) $\beta(x^*_{\tilde{\tilde{a}}})$ is a $*$-MBJNNB for $\mathcal{N}(x^*_{\tilde{\tilde{a}}}) \Longleftrightarrow$

(i) $\beta(x_a)$ is a fuzzy neighborhood base for $N_{\tau_M}(x_a)$,

(ii) $\beta(x^*_{\tilde{a}})$ is an $*$-interval-valued fuzzy neighborhood base for $\widetilde{N}^*_{\tau_{\tilde{B}}}(x^*_{\tilde{a}})$,

(iii) $\beta(x^\circ_{\bar{a}})$ is a $\circ$-fuzzy neighborhood base for $N^\circ_{\tau_J}(x^\circ_{\bar{a}})$.

(3) (X, τ) is a $\circ$-$C_I \Longleftrightarrow$

(i) (X, τ_M) is a fuzzy C_I (See [?]),

(ii) $(X, \tau_{\tilde{B}})$ is an interval-valued fuzzy C_I, i.e., each $x_{\tilde{a}} \in IVF_P(X)$ has a countable interval-valued fuzzy neighborhood base for $\widetilde{N}_{\tau_{\tilde{B}}}(x_{\tilde{a}})$,

(iii) (X, τ_J) is a fuzzy $\circ$-C_I, i.e., each $x^\circ_{\bar{a}} \in F_P(X)$ has a countable $\circ$-fuzzy neighborhood base for $N^\circ_{\tau_J}(x^\circ_{\bar{a}})$.

(4) (X, τ) is a $*$-$C_I \Longleftrightarrow$

(i) (X, τ_M) is a fuzzy C_I,

(ii) $(X, \tau_{\tilde{B}})$ is an interval-valued fuzzy $*$-C_I, i.e., each $x_{\tilde{a}} \in IVF_P(X)$ has a countable $*$-interval-valued fuzzy neighborhood base for $\widetilde{N}^*_{\tau_{\tilde{B}}}(x^*_{\tilde{a}})$,

(iii) (X, τ_J) is a fuzzy $\circ$-C_I.

Definition 5.16. Let (X, τ) be a $\circ$-MBJ-neutrosophc topological space or $*$-MBJ-neutrosophc topological space, let $x^\circ_{\tilde{\tilde{a}}},\ x^*_{\tilde{\tilde{a}}} \in MBJN_P(X)$ and let $\beta_Q(x^\circ_{\tilde{\tilde{a}}}) \subset \mathcal{N}_Q(x^\circ_{\tilde{\tilde{a}}})$, $\beta_Q(x^*_{\tilde{\tilde{a}}}) \subset \mathcal{N}_Q(x^*_{\tilde{\tilde{a}}})$.

(i) $\beta_Q(x^\circ_{\tilde{\tilde{a}}})$ is called a $\circ$-*MBJ-neutrosophic Q- neighborhood base* (briefly, $\circ$-MBJNQNB) for $\mathcal{N}_Q(x^\circ_{\tilde{\tilde{a}}})$, if for each $\mathcal{A} \in \mathcal{N}_Q(x^\circ_{\tilde{\tilde{a}}})$, there is $\mathcal{B} \in \beta_Q(x^\circ_{\tilde{\tilde{a}}})$ such that $\mathcal{B} \sqsubset \mathcal{A}$.

(ii) $\beta_Q(x^*_{\tilde{\tilde{a}}})$ is called a $*$-*MBJ-neutrosophic Q-neighborhood base* (briefly, $*$-MBJNQNB) for $\mathcal{N}_Q(x^*_{\tilde{\tilde{a}}})$, if for each $\mathcal{A} \in \mathcal{N}_Q(x^*_{\tilde{\tilde{a}}})$, there is $\mathcal{B} \in \beta_Q(x^*_{\tilde{\tilde{a}}})$ such that $\mathcal{B} \Subset \mathcal{A}$.

(iii) (X, τ) is said to *satisfy the* $\circ$-*Q-first axiom of countability* or to be $\circ$-Q-C_I, if each $x^\circ_{\tilde{\tilde{a}}} \in MBJN_P(X)$ has a countable $\circ$-MBJNQNB.

(iv) (X, τ) is said to *satisfy the* $*$-*first axiom of countability* or to be $*$-Q-C_I, if each $x^*_{\tilde{\tilde{a}}} \in MBJN_P(X)$ has a countable $*$-MBJNQNB.

From Remark 4.9, we can rewrite Definition 5.16 as followings.

Remark 5.17. Let $\beta(x_a) \subset N_{\tau_M}(x_a)$, $\beta(x_{\tilde{a}}) \subset \widetilde{N}_{\tau_{\tilde{B}}}(x_{\tilde{a}})$, $\beta(x^*_{\tilde{a}}) \subset \widetilde{N}^*_{\tau_{\tilde{B}}}(x^*_{\tilde{a}})$, $\beta(x^\circ_{\bar{a}}) \subset N^\circ_{\tau_J}(x^\circ_{\bar{a}})$. Then we have

(1) $\beta_Q(x^{\circ}_{\tilde{\tilde{a}}})$ is a $\circ$-MBJNQNB for $\mathcal{N}(x^{\circ}_{\tilde{\tilde{a}}}) \Longleftrightarrow$

(i) $\beta_Q(x_a)$ is a fuzzy Q-neighborhood base for $N_{\tau_M,Q}(x_a)$,

(ii) $\beta_Q(x_{\tilde{a}})$ is an interval-valued fuzzy Q-neighborhood base for $\widetilde{N}_{\tau_{\widetilde{B}},Q}(x_{\tilde{a}})$,

(iii) $\beta_Q(x^{\circ}_{\bar{a}})$ is a $\circ$-fuzzy Q-neighborhood base for $N^{\circ}_{\tau_J,Q}(x^{\circ}_{\bar{a}})$.

(2) $\beta_Q(x^{*}_{\tilde{\tilde{a}}})$ is a $*$-MBJNQNB for $\mathcal{N}_Q(x^{*}_{\tilde{\tilde{a}}}) \Longleftrightarrow$

(i) $\beta_Q(x_a)$ is a fuzzy Q-neighborhood base for $N_{\tau_M,Q}(x_a)$,

(ii) $\beta_Q(x^{*}_{\tilde{a}})$ is an $*$-interval-valued fuzzy Q-neighborhood base for $\widetilde{N}^{*}_{\tau_{\widetilde{B}},Q}(x^{*}_{\tilde{a}})$,

(iii) $\beta_Q(x^{\circ}_{\bar{a}})$ is a $\circ$-fuzzy Q-neighborhood base for $N^{\circ}_{\tau_J,Q}(x^{\circ}_{\bar{a}})$.

(3) (X, τ) is a $\circ$-Q-$C_I \Longleftrightarrow$

(i) (X, τ_M) is a fuzzy Q-C_I (See [16]),

(ii) $(X, \tau_{\widetilde{B}})$ is an interval-valued fuzzy Q-C_I, i.e., each $x_{\tilde{a}} \in IVF_P(X)$ has a countable interval-valued fuzzy Q-neighborhood base for $\widetilde{N}_{\tau_{\widetilde{B}},Q}(x_{\tilde{a}})$,

(iii) (X, τ_J) is a fuzzy $\circ$-Q-C_I, i.e., each $x^{\circ}_{\bar{a}} \in F_P(X)$ has a countable $\circ$-Q-fuzzy neighborhood base for $N^{\circ}_{\tau_J,Q}(x^{\circ}_{\bar{a}})$.

(4) (X, τ) is a $*$-Q-$C_I \Longleftrightarrow$

(i) (X, τ_M) is a fuzzy Q-C_I,

(ii) $(X, \tau_{\widetilde{B}})$ is an interval-valued fuzzy $*$-Q-C_I, i.e., each $x_{\tilde{a}} \in IVF_P(X)$ has a countable $*$-interval-valued fuzzy Q-neighborhood base for $\widetilde{N}^{*}_{\tau_{\widetilde{B}},Q}(x^{*}_{\tilde{a}})$,

(iii) (X, τ_J) is a fuzzy $\circ$-Q-C_I.

Example 5.18. (1) Let $X = \{x, y\}$ and let $\tau = \{\ddot{\emptyset}, \ddot{X}, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6, \mathcal{A}_7, \widetilde{A}_8, \mathcal{A}_9\}$ be the $\circ$-MBJ-neutrosophic topology on X given in Example 5.4 (3). Consider two families $\mathcal{N}_Q(x^{\circ}_{\tilde{\tilde{a}}})$ and $\mathcal{N}_Q(y^{\circ}_{\tilde{\tilde{b}}})$of MBJ-neutrosophic sets in X defined by:

$$\mathcal{N}_Q(x^{\circ}_{\tilde{\tilde{a}}}) = \{\mathcal{A} \in MBJNS(X) : \mathcal{A}_1 \sqsubset \mathcal{A}\}$$

and

$$\mathcal{N}_Q(y^{\circ}_{\tilde{\tilde{b}}}) = \{\mathcal{A} \in MBJNS(X) : \mathcal{A}_1 \sqsubset \mathcal{A}\},$$

where $\widetilde{\bar{a}} >^{\circ} \mathcal{A}^c_1(x) = \langle 0.5, [0.4, 0.5], 0.7\rangle$, i.e., $a > 0.5$, $\tilde{a} > [0.4, 0.5]$, $\bar{a} < 0.7$ and $\widetilde{\bar{b}} >^{\circ} \mathcal{A}^c_1(y) = \langle 0.4, [0.3, 0.4], 0.3\rangle$, i.e., $a > 0.4$, $\tilde{a} > [0.3, 0.4]$, $\bar{a} < 0.3$. Then we can easily see that $\mathcal{N}_Q(x^{\circ}_{\tilde{\tilde{a}}})$ and $\mathcal{N}_Q(y^{\circ}_{\tilde{\tilde{b}}})$ are MBJ-neutrosophic neighborhood system of $x^{\circ}_{\tilde{\tilde{a}}}$ and $y^{\circ}_{\tilde{\tilde{b}}}$ with respect to τ respectively. Now consider the subfamily $\beta_Q(x^{\circ}_{\tilde{\tilde{a}}})$ [resp. $\beta_Q(y^{\circ}_{\tilde{\tilde{b}}})$] of $\mathcal{N}_Q(x^{\circ}_{\tilde{\tilde{a}}})$ [resp. $\mathcal{N}_Q(y^{\circ}_{\tilde{\tilde{b}}})$] given by: for each $n \in \mathbb{N}$,

$$\beta_Q(x^{\circ}_{\tilde{\tilde{a}}}) = \{\mathcal{A} \in \mathcal{N}_Q(x^{\circ}_{\tilde{\tilde{a}}}) : \widetilde{\bar{a}} = \left\langle 0.5 + \frac{1}{n}, [0.4 + \frac{1}{n}, 0.5 + \frac{1}{n}], 0.7 - \frac{1}{n} \right\rangle\}$$

$$[\text{resp. } \beta_Q(y^{\circ}_{\tilde{\tilde{b}}}) = \{\mathcal{A} \in \mathcal{N}_Q(y^{\circ}_{\tilde{\tilde{b}}}) : \widetilde{\bar{b}} = \left\langle 0.4 + \frac{1}{n}, [0.3 + \frac{1}{n}, 0.4 + \frac{1}{n}], 0.3 - \frac{1}{n} \right\rangle\}].$$

Then it is obvious that $\beta_Q(x^{\circ}_{\widetilde{\widetilde{a}}})$ [resp. $\beta_Q(y^{\circ}_{\widetilde{\widetilde{b}}})$] is a $\circ$-MBJNQNB for $\mathcal{N}_Q(x^{\circ}_{\widetilde{\widetilde{a}}})$ [resp. $\mathcal{N}_Q(y^{\circ}_{\widetilde{\widetilde{b}}})$]. Moreover, $\beta_Q(x^{\circ}_{\widetilde{\widetilde{a}}})$ [resp. $\beta_Q(y^{\circ}_{\widetilde{\widetilde{b}}})$] is a countable $\circ$-MBJNQNB. Note that if there is no condition for MBJNN $\widetilde{\widetilde{a}}$, then $\mathcal{N}_Q(x^{\circ}_{\widetilde{\widetilde{a}}}) = \{\ddot{X}\}$ for each $x_{\widetilde{\widetilde{a}}} \in MBJN_P(X)$. Thus (X, τ) is $\circ$-Q-C_I. Note that by placing constraints on MBJNN $\widetilde{\widetilde{a}}$, we can make $\mathcal{N}_Q(x^{\circ}_{\widetilde{\widetilde{a}}})$ have more members of τ.

(2) Let $X = \{x, y\}$ and let $\tau = \{\ddot{\emptyset}, \ddot{X}, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6, \mathcal{A}_7, \widetilde{A}_8, \mathcal{A}_9\}$ be the $*$-MBJ-neutrosophic topology on X given in Example 5.4 (4). Consider the family $\mathcal{N}_Q(x^{*}_{\widetilde{\widetilde{a}}})$ of MBJ-neutrosophic sets in X defined by:

$$\mathcal{N}_Q(x^{*}_{\widetilde{\widetilde{a}}}) = \{\mathcal{A} \in MBJNS(X) : \mathcal{A}_2 \Subset \mathcal{A}\}$$

and

$$\mathcal{N}_Q(y^{*}_{\widetilde{\widetilde{b}}}) = \{\mathcal{A} \in MBJNS(X) : \mathcal{A}_2 \Subset \mathcal{A}\},$$

where $\widetilde{\widetilde{a}} >^* \mathcal{A}^c_2(x) = \langle 0.4, [0.3, 0.4], 0.4\rangle$, i.e., $a > 0.4$, $\widetilde{a} < [0.3, 0.4]$, $\bar{a} < 0.4$ and $\widetilde{\widetilde{b}} >^* \mathcal{A}^c_2(y) = \langle 0.7, [0.4, 0.6], 0.6\rangle$, i.e., $a > 0.7$, $\widetilde{a} < [0.4, 0.6]$, $\bar{a} < 0.6$. Then we can easily check that $\mathcal{N}_Q(x^{*}_{\widetilde{\widetilde{a}}})$ and $\mathcal{N}_Q(y^{*}_{\widetilde{\widetilde{b}}})$ are MBJ-neutrosophic neighborhood system of $x^{*}_{\widetilde{\widetilde{a}}}$ and $y^{*}_{\widetilde{\widetilde{b}}}$ with respect to τ respectively. Now consider the subfamily $\beta_Q(x^{*}_{\widetilde{\widetilde{a}}})$ [resp. $\beta_Q(y^{*}_{\widetilde{\widetilde{b}}})$] of $\mathcal{N}_Q(x^{*}_{\widetilde{\widetilde{a}}})$ [resp. $\mathcal{N}_Q(y^{*}_{\widetilde{\widetilde{b}}})$] given by: for each $n \in \mathbb{N}$,

$$\beta_Q(x^{*}_{\widetilde{\widetilde{a}}}) = \{\mathcal{A} \in \mathcal{N}_Q(x^{*}_{\widetilde{\widetilde{a}}}) : \widetilde{\widetilde{a}} = \left\langle 0.4 + \frac{1}{n}, [0.3 - \frac{1}{n}, 0.4 - \frac{1}{n}], 0.4 - \frac{1}{n} \right\rangle\}$$

$$\text{[resp. } \beta_Q(y^{*}_{\widetilde{\widetilde{b}}}) = \{\mathcal{A} \in \mathcal{N}_Q(y^{*}_{\widetilde{\widetilde{b}}}) : \widetilde{\widetilde{b}} = \left\langle 0.7 + \frac{1}{n}, [0.4 - \frac{1}{n}, 0.6 - \frac{1}{n}], 0.6 - \frac{1}{n} \right\rangle\}.$$

Then it is clear that $\beta_Q(x^{*}_{\widetilde{\widetilde{a}}})$ [resp. $\beta_Q(y^{*}_{\widetilde{\widetilde{b}}})$] is a $*$-MBJNQNB for $\mathcal{N}_Q(x^{*}_{\widetilde{\widetilde{a}}})$ [resp. $\mathcal{N}_Q(y^{*}_{\widetilde{\widetilde{b}}})$]. Furthermore, $\beta_Q(x^{*}_{\widetilde{\widetilde{a}}})$ [resp. $\beta_Q(y^{*}_{\widetilde{\widetilde{b}}})$] is a countable $*$-MBJNQNB. Also, note that if there is no condition for MBJNN $\widetilde{\widetilde{a}}$, then $\mathcal{N}_Q(x^{*}_{\widetilde{\widetilde{a}}}) = \{\dot{X}\}$ for each $x_{\widetilde{\widetilde{a}}} \in MBJN_P(X)$. Thus (X, τ) is $*$-Q-C_I. Also, note that by placing constraints on MBJNN $\widetilde{\widetilde{a}}$, we can make $\mathcal{N}_Q(x^{*}_{\widetilde{\widetilde{a}}})$ have more members of τ.

(3) Let X be an infinite set and let τ be the MBJ-neutrosophic $\circ$-[resp. $*$-]cofinite topology on X. Assume that $\beta(x^{\circ}_{\widetilde{\widetilde{a}}}) = \{\mathcal{B}_n : n \in \mathbb{N}\}$ is a $\circ$-MBJNNB for $x^{\circ}_{\widetilde{\widetilde{a}}}$. Let $y^{\circ}_{\widetilde{\widetilde{b}}} \in MBJN_P(X)$ such that $x \neq y$. Then clearly, $y^{\circ}_{\widetilde{\widetilde{b}}} \in \mathcal{N}(x^{\circ}_{\widetilde{\widetilde{a}}})$. Thus there is $n \in \mathbb{N}$ such that $y^{\circ}_{\widetilde{\widetilde{b}}} \notin \mathcal{B}_n$. So $\sqcap_{n\in\mathbb{N}}\mathcal{B}_n = \{x^{\circ}_{\widetilde{\widetilde{a}}}\}$. On the other hand, we have

$$x^{\circ,c}_{\widetilde{\widetilde{a}}} = [\sqcap_{n\in\mathbb{N}}\mathcal{B}_n]^c = \sqcup_{n\in\mathbb{N}}\mathcal{B}^c_n.$$

Since $\mathcal{B}^c_n$ is $\circ$-finite for each $n \in \mathbb{N}$, $\sqcup_{n\in\mathbb{N}}\mathcal{B}^c_n$ is countable. So $x^{\circ,c}_{\widetilde{\widetilde{a}}}$ is countable. This is a contradiction. Hence (X, τ) is not $\circ$-C_I. Similarly, we can check that (X, τ) is not $*$-C_I.

Proposition 5.19. *Let (X, τ) be a $\circ$-MBJ-neutrosophc topological space. If (X, τ) is $\circ$-C_I, then it is $\circ$-Q-C_I.*

Proof. Suppose (X,τ) is $\circ$-C_I. It is well-known (Proposition 3.1, [16]) that if (X,τ_M) is fuzzy C_I, then it is fuzzy Q-C_I.

Let $x_{\tilde{a}} \in IVF_P(X)$. Consider a sequence $\{\widetilde{a}_n = [a_n^-, a_n^+]\}_{n\in\mathbb{N}}$ of interval numbers converging to $\widetilde{a}^c = [1-a^+, 1-a^-]$ and interval-valued fuzzy points $x_{\tilde{a}_n}$ in X, where $a_n^- \in (1-a^+, 1]$ and $a_n^+ \in (1-a^-, 1]$. Since (X,τ) is $\circ$-C_I, by Remark 5.15 (3), $(X,\tau_{\widetilde{B}})$ is interval-valued fuzzy C_I. Then for each $n \in \mathbb{N}$, there is a countable interval-valued open neighborhood base $\beta_n(x_{\tilde{a}_n})$. Thus for each $\widetilde{B} \in \beta_n(x_{\tilde{a}_n})$, $\widetilde{B}(x) \geq \widetilde{a}_n > \widetilde{a}^c$. So $\widetilde{B} \in \widetilde{N}_{\tau_{\widetilde{B}},Q}(x_{\tilde{a}})$. Let $\beta(x_{\tilde{a}})$ be the collection of all the members of all $\beta_n(x_{\tilde{a}_n})$. It is clear that $\beta(x_{\tilde{a}})$ is a family of interval-valued fuzzy open Q-neighborhoods of $x_{\tilde{a}}$. Let $\widetilde{A} \in \widetilde{N}_{\tau_{\widetilde{B}},Q}(x_{\tilde{a}})$. Then clearly, $\widetilde{A}(x) > \widetilde{a}^c$. Since $\{\widetilde{a}_n\}_{n\in\mathbb{N}}$ is convergent to $\widetilde{a}^c$, there is $m \in \mathbb{N}$ such that $\widetilde{A}(x) \geq \widetilde{a}_m > \widetilde{a}^c$, i.e., $x_{\tilde{a}_m} \in \widetilde{A}$ and $\widetilde{A}$ is an interval-valued fuzzy open neighborhood of $x_{\tilde{a}_m}$. Thus there is $\widetilde{B} \in \beta_n(x_{\tilde{a}_n}) \subset \beta(x_{\tilde{a}})$ such that $\widetilde{B} \subset \widetilde{A}$ and $\widetilde{B}(x) \geq \widetilde{a}_m > \widetilde{a}^c$. So $\beta(x_{\tilde{a}})$ is a countable interval-valued fuzzy Q-neighborhood of $x_{\tilde{a}}$. Hence $(X,\tau_{\widetilde{B}})$ is interval-valued fuzzy Q-C_I.

Now let $x^\circ_{\bar{a}} \in F_P(X)$. Consider a sequence $\{\overline{a}_n\}_{n\in\mathbb{N}}$ in $[0,\overline{a}^c)$ converging to $\overline{a}^c$ and a $\circ$-fuzzy points $x^\circ_{\bar{a}_n}$ in X. Since (X,τ) is $\circ$-C_I, by Remark 5.15 (3), (X,τ_J) is $\circ$-fuzzy C_I. Then for each $n \in \mathbb{N}$, there is a countable $\circ$-fuzzy closed neighborhood base $\beta_n(x^\circ_{\bar{a}_n})$. Thus for each $B \in \beta_n(x^\circ_{\bar{a}_n})$, $B(x) \leq \overline{a}_n < \overline{a}^c$. So $B \in N^\circ_{\tau_J,Q}(x^\circ_{\bar{a}})$. Let $\beta(x_{\bar{a}})$ be the collection of all the members of all $\beta_n(x_{\bar{a}_n})$. It is clear that $\beta(x^\circ_{\bar{a}})$ is a family of $\circ$-fuzzy closed Q-neighborhoods of $x^\circ_{\bar{a}}$. Let $A \in N_{\tau_J,Q}(x^\circ_{\bar{a}})$. Then clearly, $A(x) < \overline{a}^c$. Since $\{\overline{a}_n\}_{n\in\mathbb{N}}$ is convergent to $\overline{a}^c$, there is $m \in \mathbb{N}$ such that $A(x) \leq \overline{a}_m < \overline{a}^c$, i.e., $x^\circ_{\bar{a}_m} \in A$ and A is a $\circ$-fuzzy closed neighborhood of $x^\circ_{\bar{a}_m}$. Thus there is $B \in \beta_n(x^\circ_{\bar{a}_n}) \subset \beta(x^\circ_{\bar{a}})$ such that $B \supset A$ and $B(x) \leq \overline{a}_m < \overline{a}^c$. So $\beta(x^\circ_{\bar{a}})$ is a countable $\circ$-fuzzy Q-neighborhood of $x^\circ_{\bar{a}}$. Hence (X,τ_J) is $\circ$-fuzzy Q-C_I. Therefore by Remark 5.15, (X,τ) is $\circ$-Q-C_I. □

Proposition 5.20. *Let (X,τ) be a $*$-MBJ-neutrosophc topological space. If (X,τ) is $*$-C_I, then it is $*$-Q-C_I.*

Proof. Suppose (X,τ) is a $*$-C_I. Then from the proof of Proposition 5.19, the conditions (i) and (iii) of Remark 5.15. It is sufficient to show that (ii) of Remark 5.15 holds. Let $x^*_{\tilde{a}} \in IVF_P(X)$. Consider a sequence $\{\widetilde{a}_n\}_{n\in\mathbb{N}}$ of interval numbers converging to $\widetilde{a}^c$ and a $*$-interval-valued fuzzy points $x^*_{\tilde{a}_n}$ in X, where $a_n^- \in [0, 1-a_n^+)$ and $a_n^+ \in [0, 1-a_n^-)$. Since (X,τ) is $*$-C_I, by Remark 5.15 (4), $(X,\tau_{\widetilde{B}})$ is a $*$-interval-valued fuzzy C_I. Then for each $n \in \mathbb{N}$, there is a countable $*$-interval-valued fuzzy closed neighborhood base $\beta_n(x^*_{\tilde{a}_n})$. Thus for each $\widetilde{B} \in \beta_n(x^*_{\tilde{a}_n})$, $\widetilde{B}(x) \leq \widetilde{a}_n < \widetilde{a}^c$. So $\widetilde{B} \in N^{*c}_{\tau_{\widetilde{B}},Q}(x^*_{\tilde{a}})$. Let $\beta(x^*_{\tilde{a}})$ be the collection of all the members of all $\beta_n(x^*_{\tilde{a}_n})$. It is clear that $\beta(x^*_{\tilde{a}})$ is a family of $*$-interval-valued fuzzy closed Q-neighborhoods of $x_{\tilde{a}}^*$. Let $\widetilde{A} \in \widetilde{N}^*_{\tau_{\widetilde{B}},Q}(x^*_{\tilde{a}})$. Then clearly, $\widetilde{A}(x) < \widetilde{a}^c$. Since $\{\widetilde{a}_n\}_{n\in\mathbb{N}}$ is convergent to $\widetilde{a}^c$, there is $m \in \mathbb{N}$ such that $\widetilde{A}(x) \leq \widetilde{a}_m < \widetilde{a}^c$, i.e., $x^*_{\tilde{a}_m} \in \widetilde{A}$ and $\widetilde{A}$ is a

$*$-interval-valued fuzzy closed neighborhood of $x^*_{\widetilde{a}_m}$. Thus there is $\widetilde{B} \in \beta_n(x^*_{\widetilde{a}_n}) \subset \beta(x^*_{\widetilde{a}})$ such that $\widetilde{B} \supset widetildeA$ and $\widetilde{B}(x) \leq \widetilde{a}_m < \widetilde{a}^c$. So $\beta(x^*_{\widetilde{a}})$ is a countable $*$-interval-valued fuzzy Q-neighborhood of $x_{\widetilde{a}}^*$. Hence (X, τ_J) is $*$-interval-valued fuzzy Q-C_I. Therefore by Remark 5.15, (X, τ) is $*$-Q-C_I. □

Definition 5.21. Let (X, τ) be a $\circ$-MBJ-neutrosophc topological space or $*$-MBJ-neutrosophc topological space.

(i) (X, τ) is said to *satisfy the $\circ$-second axiom of countability* or to be $\circ$-C_{II}, if there is a countable $\circ$-base β for τ.

(ii) (X, τ) is said to *satisfy the $*$-second axiom of countability* or to be $*$-C_{II}, if there is a countable $*$-base β for τ.

From Remark 4.4 (5), and Definitions 5.2 and 5.3, we can rewrite Definition 5.1 (i) as followings.

Remark 5.22. Let (X, τ) be a $\circ$-MBJ-neutrosophc topological space or $*$-MBJ-neutrosophc topological space.

(1) (X, τ) is $\circ$-C_{II} $\Longleftrightarrow$

(i) (X, τ_M) is fuzzy C_{II}, i.e., there is a countable fuzzy base β_M for τ_M,

(ii) $(X, \widetilde{B})$ is interval-valued fuzzy C_{II}, i.e., there is a countable interval-valued fuzzy base $\beta_{\widetilde{B}}$ for $\tau_{\widetilde{B}}$,

(iii) (X, τ_J) is $\circ$-fuzzy C_{II}, i.e., there is a countable $\circ$-fuzzy base β_J for τ_J.

(2) (X, τ) is $*$-C_{II} $\Longleftrightarrow$

(i) (X, τ_M) is fuzzy C_{II}, i.e., there is a countable fuzzy base β_M for τ_M,

(ii) $(X, \widetilde{B})$ is $*$-interval-valued fuzzy C_{II}, i.e., there is a countable $*$-interval-valued fuzzy base $\beta_{\widetilde{B}}$ for $\tau_{\widetilde{B}}$,

(iii) (X, τ_J) is $\circ$-fuzzy C_{II}, i.e., there is a countable $\circ$-fuzzy base β_J for τ_J.

Proposition 5.23. *Let (X, τ) be a $\circ$-MBJ-neutrosophc topological space or a $*$-MBJ-neutrosophc topological space.*

(1) *If (X, τ) is $\circ$-C_{II}, then it is $\circ$-C_I.*

(2) *If (X, τ) is $*$-C_{II}, then it is $*$-C_I.*

Proof. (1) Suppose (X, τ) is $\circ$-C_{II}. It is well-known (Theorem 3.3, [14]) that if (X, τ_M) is fuzzy C_{II}, then it is fuzzy C_I. Thus (X, τ_M) satisfies the condition (i) of Remark 5.15 (3).

Let $x_{\widetilde{a}} \in IVF_P(X)$. Then by the hypothesis and Remark 5.22 (1), there is a countable interval-valued fuzzy base $\beta_{\widetilde{B}}$ for $\tau_{\widetilde{B}}$. Let $\beta_{\widetilde{B}, x_{\widetilde{a}}}$ be the subfamily of $\beta_{\widetilde{B}}$ given by $\beta_{\widetilde{B}, x_{\widetilde{a}}} = \{\widetilde{B} : x_{\widetilde{a}} \in \widetilde{B} \in \beta_{\widetilde{B}}\}$. Then clearly, $\beta_{\widetilde{B}, x_{\widetilde{a}}}$ is countable. Let $\widetilde{A} \in \tau_{\widetilde{B}}$ such that $x_{\widetilde{a}} \in \widetilde{A}$. Since $\beta_{\widetilde{B}}$ is an interval-valued fuzzy base for $\tau_{\widetilde{B}}$, by Theorem 5.10, there is $\widetilde{B} \in \beta_{\widetilde{B}}$ such that $x_{\widetilde{a}} \in \widetilde{B} \subset \widetilde{A}$. Then by the definition of $\beta_{\widetilde{B},x_{\widetilde{a}}}$, $\widetilde{B} \in \beta_{\widetilde{B},x_{\widetilde{a}}}$. Thus $(X, \tau_{\widetilde{B}}$ is interval-valued fuzzy C_I. So $(X, \tau_{\widetilde{B}}$ satisfies the condition (ii) of Remark 5.15 (3).

Now let $x^{\circ}_{\overline{a}} \in F_P(X)$. Then by the hypothesis and Remark 5.22 (1), there is a countable $\circ$-fuzzy base β_J for τ_J. Let $\beta_{J,x^{\circ}_{\overline{a}}}$ be the subfamily of β_J defined by $\beta_{J,x^{\circ}_{\overline{a}}} = \{B : x^{\circ}_{\overline{a}} \in B \in \beta_J\}$. Then clearly, $\beta_{J,x^{circ}_{\overline{a}}}$ is countable. Let $A \in \tau_J$ such that $x^{\circ}_{\overline{a}} \in A$. Since β_J is a $\circ$-fuzzy base for τ_J, by Theorem 5.10, there is $B \in \beta_J$ such that $x^{\circ}_{\overline{a}} \in B \supset A$. Then by the definition of $\beta_{J,x^{\circ}_{\overline{a}}}$, $B \in \beta_{J,x^{\circ}_{\overline{a}}}$. Thus $(X, \tau_J$ is $\circ$-fuzzy C_I. So $(X, \tau_J$ satisfies the condition (iii) of Remark 5.15 (3). Hence (X, τ) is $\circ$-C_I.

(2) Suppose (X, τ) is $*$-C_{II}. From (1), it is obvious that the conditions (i) and (iii) of Remark 5.15 (4). It is sufficient to prove that (ii) of Remark 5.15 (4) holds. □

The following is an immediate consequence of Propositions 5.19, 5.20 and 5.23.

Corollary 5.24. *Let (X, τ) be a $\circ$-MBJ-neutrosophc topological space or a $*$-MBJ-neutrosophc topological space.*

(1) *If (X, τ) is $\circ$-C_{II}, then it is $\circ$-Q-C_I.*

(2) *If (X, τ) is $*$-C_{II}, then it is $*$-Q-C_I.*

The converse of Proposition 5.23 does not hold in general (See Example 5.33).

Definition 5.25 ([37])**.** Let X be a classical topological space and let $A : X \to I$ be a mapping.

(i) A is said to be *lower semi-continuous* [resp. *upper semi-continuous*] at $a \in X$, if for each $h < A(a)$ [resp. $k > A(a)$], there is a neighborhood V of a such that $h < A(x)$ [resp. $k > A(x)$] for each $x \in V$.

(ii) A is said to be *lower semi-continuous* [resp. *upper semi-continuous*] on X, if it is lower semi-continuous [resp. upper semi-continuous] at each $a \in X$.

It is well-known (6.2, [37]) that A is continuous on X if and only if A is both upper and lower semi-continuous on X. Moreover, A is lower semi-continuous on X if and only if $1 - A$ is upper semi-continuous on X.

For a fuzzy set A in a set X and any $a \in I$, the *weak* [resp. *strong*] *a-cut* or *a-level set* of A, denoted by $[A]_a$ [resp. $[A]^a$], is a subset of X defined as follows:

$$[A]_a = \{x \in X : A(x) \geq a\} \text{ [resp.} [A]^a = \{x \in X : A(x) > a\} \text{ (See [14]).}$$

Definition 5.26 ([14]). Let (X, T) be a classical topological space. Then the *induced fuzzy topology* on X, denoted by $F(T)$, is a family of fuzzy sets in X given as follows:

$$F(T) = \{A \in I^X : A \text{ is lower semi-continuous}\}.$$

It is well-known (Proposition 3.3, [15]) that for a classical topological space (X, T) and $A \in I^X$, A is fuzzy open [resp. closed] in $(X, F(T))$ if and only if for each $a \in I$, $[A]^a \in T$ [resp. $[A]_a \in T^c$]. Then from Definition 5.25 and the above fact,

$$F(T) = \{A \in I^X : [A]^a \in T,\ a \in I\} \text{ and } CF(T) = \{A \in I^X : [A]_a \in T^c,\ a \in I\},$$

where $CF(T)$ will be called the *induced fuzzy cotopology* on X. Furthermore, we can easily see that the family $\{[A, A] \in IVFS(X) : A \in F(T)\}$ is an interval-valued fuzzy topology on X, and it will be called the *induced interval-valued fuzzy topology* on X and denoted by $IVF(T)$. Also, $CIVF(T) = \{[A, A] \in IVFS(X) : A \in CF(T)\}$ is an interval-valued fuzzy cotopology on X.

Remark 5.27. Let (X, T) be a classical topological space and consider the families τ and η of MBJ-neutrosophic sets in X defined as follows:

$$\tau = \{\mathcal{A} \in MBJNS(X) : M_A \in F(T),\ \widetilde{B}_A \in IVF(T),\ J_A \in CF(T)\},$$

$$\eta = \{\mathcal{A} \in MBJNS(X) : M_A \in F(T),\ \widetilde{B}_A \in CIVF(T),\ J_A \in CF(T)\}.$$

Then clearly, $\tau \in MBJNT^{\circ}(X)$ and $\eta \in MBJNT^{*}(X)$. In this case, we will call τ and η as the *induced* $\circ$*-MBJ-neutrosophic topology* and the *induced* $*$*-MBJ-neutrosophic topology* on X.

Result 5.28 (See Lemma 3.1, [16]). Let (X, T) be a classical complete regular topological space. Then for each $B \in F(T)$, there is a family $\beta_F \subset I^X$ each member of which is continuous with respect to T, such that $B = \bigcup_{A \in \beta_F} A$. In other words, the family

$$\beta_F = \{A \in I^X : A \text{ is continuous on } I\}$$

forms a fuzzy base for $F(T)$.

Remark 5.29. Let (X, T) be a classical complete regular topological space and consider the following families:

$$\beta_{IVF} = \{[A, A] \in IVF(X) : A \in \beta_F\},$$

$$\beta^{\circ}_F = \{A \in I^X : A^c \text{ is continuous on } I\},$$

$$\beta^{*}_{IVF} = \{[A, A] \in IVF(X) : A \in \beta^{\circ}_F\}.$$

Then from Result 5.28, we can easily see that β_{IVF}, β°_F and β^{*}_{IVF} form an interval-valued fuzzy base for $IVF(T)$, a $\circ$-fuzzy base for $CF(T)$ and a $*$-interval-valued fuzzy base for $CIVF(T)$ respectively.

From Remarks 5.5 (1), 5.27 and 5.29, and Result 5.28, we have the following.

Lemma 5.30. *Let (X,T) be a classical complete regular topological space and consider the following families:*

$$\beta = \{\mathcal{A} \in MBJNS(X) : M_A \in \beta_F,\ \widetilde{B}_A \in \beta_{IVF},\ J_A \in \beta_F^{\circ}\},$$

$$\beta^* = \{\mathcal{A} \in MBJNS(X) : M_A \in \beta_F,\ \widetilde{B}_A \in \beta^*_{IVF},\ J_A \in \beta_F^{\circ}\}.$$

(1) *β forms a $\circ$-MBJNB for τ.*

(2) *β^* forms a $*$-MBJNB for η.*

Result 5.31 (See Theorem 3.1, [16])**.** Let (X,T) be the subspace I of the real axis and let $F(T)$ be the induced fuzzy topology for T. Then $(X,F(T)$ is fuzzy C_{II} but not fuzzy C_I.

Lemma 5.32 (See Theorem 3.1, [16])**.** *Let (X,T) be the subspace I of the real axis.*

(1) *$(X,IVF(T))$ is interval-valued fuzzy C_{II} but not interval-valued fuzzy C_I.*

(2) *$(X,CF(T))$ is $\circ$-fuzzy C_{II} but not $\circ$-fuzzy C_I.*

(3) *$(X,CIVF(T))$ is $*$-interval-valued fuzzy C_{II} but not $*$-interval-valued fuzzy C_I.*

Proof. The proofs are similar to Result 5.31. □

From Remark 5.26, Lemmas 5.30 and 5.32, and Result 5.31, we can give as an example which the converse of Proposition 5.23 does not hold.

Example 5.33. Let (X,T) be the subspace I of the real axis.

(1) (X,τ) is $\circ$-C_{II} but not $\circ$-C_I.

(2) (X,η) is $*$-C_{II} but not $*$-C_I.

6. Conclusions

Through the study, we obtained several results as follows:

(1) $(MBJNS(X),\sqcup,\sqcap,^{c,i},\ddot{\emptyset},\ddot{X})$ and $(MBJNS(X),\Cup,\Cap,^{c,i},\dot{\emptyset},\dot{X})$ form Boolean algebras except the condition (13) of Proposition 3.7.

(2) An MBJ-neutrosophic neighborhood system generates an MBJ-neutrosophic topology (See Theorem 4.16 and 4.18).

(3) The characterization of MBJ-neutrosophic base (See Theorems 5.6 and 5.8).

(4) A necessary and sufficient condition for a set of MB J-neutrosophic sets to be an MBJ-neutrosophic topology (See Theorems 5.9 and 5.10).

(5) The relationships among $\circ$-C_I, $*$-C_I, $\circ$-C_{II} and $*$-C_{II}.

Before conducting our research, we came across an interesting paper written by Al-shami [38] during a literature search. By defining soft separation axioms in soft topological spaces, he proposed an algorithm for decision-making problems. In the future, we try to study separation axioms based on MBJ-neutrosophic sets and to apply them to decision-making problems. Moreover, we expect that one can apply MBJ-neutrosophic sets to a category theory, a graph theory, a group theory, etc.

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Some Aggregation Operators of Credibility Interval Trapezoidal Fuzzy Neutrosophic Numbers and Their Decision-Making Application of Landslide Control Design Schemes

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Abstract: As a generalization of trapezoidal fuzzy neutrosophic numbers (TFNNs), credibility trapezoidal fuzzy neutrosophic numbers (C-TFNNs) can independently describe true, false, and indeterminate membership degrees and their credibility levels in uncertain and inconsistent scenarios. Since the true, false, and indeterminate membership degrees are closely related to their credibility levels, C-TFNN can ensure the credibility of TFNN, which shows its clear merit. However, C-TFNNs cannot expresses the interval membership degrees of the truth, falsity and indeterminacy and the uncertain credibility levels, which are produced due to human cognitive vagueness, incompleteness, and uncertainty. Furthermore, existing decision models of C-TFNNs cannot perform such a DM issue with both ITFNNs and uncertain credibility levels, which reveals a gap. To compensates for this gap. this paper extends C-TFNNs to credibility interval TFNNs (C-ITFNNs), which strengthens the expression capability of uncertain information. Then, the operational laws and score function of C-ITFNNs are defined to solve the aggregation and sorting issues of C-ITFNNs in decision-making (DM) problems. Subsequently, the C-ITFNN weighted geometric averaging (C-ITFNNWGA) and C-ITFNN weighted arithmetic averaging (C-ITFNNWAA) operators are proposed in view of operational laws of C-ITFNNs. Furthermore, a multi-attribute DM model is established in terms of the two aggregation operators and the score function in the C-ITFNN circumstance. Finally, a DM case of landslide control design schemes is used to reveal the applicability of the proposed DM model in the C-ITFNN scenario. By comparative analysis, the main superiority of our new DM model is that it not only compensates for the gap of existing DM models, but also is more reliable and versatile than existing DM models.

Keywords: Credibility interval trapezoidal fuzzy neutrosophic number, credibility interval trapezoidal fuzzy neutrosophic number weighted arithmetic averaging operator, credibility interval trapezoidal fuzzy neutrosophic number weighted geometric averaging operator, decision making, landslide control design scheme

1. Introduction

In real life, there are many uncertainties and ambiguities, which it is difficult to measure by crisp concepts. Then, fuzzy sets [1] can represent them by membership degrees belonging to [0, 1]. Due to the uncertainty of the membership degrees, they are difficultly described by exact fuzzy values, so the concept of interval-valued fuzzy sets (IVFSs) was proposed to solve this issue [2]. However, since there is true and false information in real life, Atanassov [3] proposed the concept of intuitionistic fuzzy sets (IFSs). Subsequently, IFSs were generalized to interval-valued IFSs (IVIFSs) [4] to facilitate the representation of incomplete and uncertain information. Although IFSs and IVIFSs can better express true and false membership degrees belonging to [0, 1], they cannot represent true, false, and indeterminate membership degrees independently and indeterminate and inconsistent information. To solve these issues, the neutrosophic sets (NSs) presented by Smarandache [5] are the extension of various fuzzy sets. Since NS can easily describe indeterminate and inconsistent information in terms of true, false, and indeterminate membership degrees, it reveals obvious merits. Due to the diversity of neutrosophic expressions (including fuzzy sets, IVFSs, IFSs, IVIFSs), neutrosophic theory has also become a research hotspot of scholars in recent years and has been widely used in many fields, such as risk assessment [6, 7], image processing (segmentation, denoising, and thresholding) [8-10], decision-making (DM) [11-14], and so on. As subsets of NSs, interval NSs (INSs) and single-valued NSs (SVNSs) were proposed by Wang et al. [15, 16]. Then SVNSs and INSs have been widely used in DM problems [17-21], risk assessment [22, 23], and medical diagnosis [24-26] in neutrosophic environments.

In order to extend discrete fuzzy information to continuous fuzzy information, Wang and Zhong [27] proposed the concept of intuitionistic trapezoidal fuzzy numbers (ITFNs) based on true and false trapezoidal fuzzy numbers (TFNs) and defined their operational laws and the weighted geometric and arithmetic averaging operators for DM. Wan and Dong [28] proposed a multi-attribute group DM approach of ITFNs. Li [29] proposed interval-valued intuitionistic trapezoidal fuzzy numbers (IVITFNs) as a further extension of ITFNs. Subsequently, Ye [30] proposed the concept of trapezoidal fuzzy neutrosophic numbers (TFNNs) in view of true, false and indeterminate TFNs and defined some weighted aggregation operators of TNNs for DM. Then, the concept of interval trapezoidal fuzzy neutrosophic numbers (ITFNNs) [31] were proposed to solve the problems of multi-attribute DM [32-34] in the setting of ITFNNs. As a special case of TFNNs, Deli and Şubaş [35] introduced the weighted geometric operators of triangular fuzzy neutrosophic numbers and applied them to DM problems.

However, decision makers are not completely familiar with various attributes in a DM problem when they evaluate them. The accuracy of the evaluation given by decision makers to unfamiliar attributes is not as high as that of familiar attributes, so it will affect the accuracy of the DM results. For this case, we need to consider the decision maker's credibility level to ensure the credibility of the assessment value to each attribute in a DM problem. Therefore, Ye et al. [36] proposed the concept of fuzzy credibility numbers (FCNs) to enrich the evaluation information of multi-attribute DM problems and to ensure their DM credibility. Then, Ye et al. [37] further proposed the concept of credibility TFNNs (C-TFNNs) and established a multi-attribute DM model using the C-TFNN weighted geometric averaging (C-TFNNWGA) and C-TFNN weighted arithmetic averaging (C-TFNNWAA) operators to solve DM problems with C-TFNNs.

In the C-TFNN situation, it is difficult for decision makers to give exact C-TFNNs, but they easily provide ITFNNs and uncertain credibility levels in indeterminate DM problems to meet the uncertain judgments and expressions of decision makers. However, the existing various DM techniques cannot handle such a DM issue with both ITFNNs and uncertain credibility levels. Therefore, we need to make up for this gap. To do so, this paper aims to: (a) propose the concept of credibility ITFNNs (C-ITFNNs) and the C-ITFNN score function and sorting rules, (b) present two basic aggregation operators of C-ITFNNs, (c) establish a multi-attribute DM model in the scenario of C-ITFNNs, and

(d) apply the established DM model to an actual DM case of landslide control design schemes (LCDSs) in the C-ITFNN scenario.

In this original study, the contributions and advantages of this paper are revealed as follows:

(1) The proposed C-ITFNNs can overcome the defect of single-valued/exact C-TFNNs in the expression of indeterminate information.

(2) The proposed C-ITFNN makes the expression of uncertain information more reasonable and reliable.

(3) The proposed C-ITFNNWAA and C-ITFNNWGA operators and score function provide effective DM tools for handling multi-attribute DM problems with C-ITFNNs.

(4) The DM model established in the C-ITFNN setting has stronger DM credibility and more general DM capabilities.

The rest of the paper is given as follows. Section 2 introduces the related concepts of INS and C-TFNNs, the weighted geometric and arithmetic averaging operators of C-TFNNs, and the scoring function of C-TFNN as preliminaries in this study. Section 3 presents some new concepts of C-ITFNNs, including operational laws, score function, and sorting rules of C-ITFNNs. Section 4 introduces the weighted geometric averaging (C-ITFNNWGA) and weighted arithmetic averaging (C-ITFNNWAA) operators for C-ITFNNs and their characteristics. In Section 5, a multi-attribute DM model is established in light of the C-ITFNNWAA and C-ITFNNWGA operators and score function. Section 6 demonstrates the applicability of the proposed DM model through an actual DM case of LCDSs in the C-ITFNN scenarios. Section 7 summarizes the conclusions of this article and future research.

2. Preliminaries

Definition 1 [15]. Let X be a non-empty set. An INS $\tilde{P}$ in X is given by

$$\tilde{P} = \left\{ x, \left\langle T_{\tilde{P}}(x), I_{\tilde{P}}(x), F_{\tilde{P}}(x), \right\rangle x \in X \right\},$$

where $T_{\tilde{P}}(x) \subseteq [0,1]$, $I_{\tilde{P}}(x) \subseteq [0,1]$, and $F_{\tilde{P}}(x) \subseteq [0,1]$ are the true, indeterminate, and false membership functions and then their membership degrees are subject to $0 \le \sup(T_{\tilde{P}}(x)) + \sup(I_{\tilde{P}}(x)) + \sup(F_{\tilde{P}}(x)) \le 3$.

Definition 2 [37]. Let X be a non-empty set. A C-TFNN s is denoted by $s = \left(\left\langle (g_1, g_2, g_3, g_4); T_N(x), I_N(x), F_N(x) \right\rangle, \left\langle (h_1, h_2, h_3, h_4); T_L(x), I_L(x), F_L(x) \right\rangle \right)$. Then, it's true, indeterminate, and false membership functions are denoted as follows:

$$T_N(x) = \begin{cases} \dfrac{x - g_1}{g_2 - g_1} T_N, g_1 \le x < g_2, \\ T_N, g_2 \le x \le g_3, \\ \dfrac{g_4 - x}{g_4 - g_3} T_N, g_3 < x \le g_4, \\ 0, \text{otherwise} \end{cases}$$

$$I_N(x)=\begin{cases}\dfrac{g_2-x+I_N(x-g_1)}{g_2-g_1}, & g_1\le x<g_2,\\ I_N, g_2\le x\le g_3, & \\ \dfrac{x-g_3+I_N(g_4-x)}{g_4-g_3}, g_3<x\le g_4, & \\ 1, \text{otherwise} & \end{cases}$$

$$F_N(x)=\begin{cases}\dfrac{g_2-x+F_N(x-g_1)}{g_2-g_1}, & g_1\le x<g_2,\\ F_N, g_2\le x\le g_3, & \\ \dfrac{x-g_3+F_N(g_4-x)}{g_4-g_3}, g_3<x\le g_4, & \\ 1, \text{otherwise} & \end{cases}$$

and it's true, indeterminate, and false credibility measure functions are denoted as follows:

$$T_L(x)=\begin{cases}\dfrac{x-h_1}{h_2-h_1}T_L, h_1\le x<h_2,\\ T_L, h_2\le x\le h_3,\\ \dfrac{h_4-x}{h_4-h_3}T_L, h_3<x\le h_4,\\ 0, \text{otherwise}\end{cases}$$

$$I_L(x)=\begin{cases}\dfrac{h_2-x+I_L(x-h_1)}{h_2-h_1}, h_1\le x<h_2,\\ I_L, h_2\le x\le h_3,\\ \dfrac{x-h_3+I_L(h_4-x)}{h_4-h_3}, h_3<x\le h_4,\\ 1, \text{otherwise}\end{cases}$$

$$F_L(x)=\begin{cases}\dfrac{h_2-x+F_L(x-h_1)}{h_2-h_1}, h_1\le x<h_2,\\ F_L, h_2\le x\le h_3,\\ \dfrac{x-h_3+F_L(h_4-x)}{h_4-h_3}, h_3<x\le h_4,\\ 1, \text{otherwise}\end{cases}$$

where $T_N, I_N, F_N \in [0,1]$, $T_L, I_L, F_L \in [0,1]$, $0 \le T_N + I_N + F_N \le 3$, $0 \le T_L + I_L + F_L \le 3$, and g_k, h_k $\in$ $\Re$ (k = 1, 2, 3, 4). Then, a C-TFNN s is simply denoted as $s=\left(\left\langle\left(g_1, g_2, g_3, g_4\right); T_N, I_N, F_N\right\rangle, \left\langle\left(h_1, h_2, h_3, h_4\right); T_L, I_L, F_L\right\rangle\right)$.

Definition 3 [37]. Let $s_1 = \begin{pmatrix} \langle (g_{11}, g_{12}, g_{13}, g_{14}); T_{N1}, I_{N1}, F_{N1} \rangle, \\ \langle (h_{11}, h_{12}, h_{13}, h_{14}); T_{L1}, I_{L1}, F_{L1} \rangle \end{pmatrix}$ and $s_2 = \begin{pmatrix} \langle (g_{21}, g_{22}, g_{23}, g_{24}); T_{N2}, I_{N2}, F_{N2} \rangle, \\ \langle (h_{21}, h_{22}, h_{23}, h_{24}); T_{L2}, I_{L2}, F_{L2} \rangle \end{pmatrix}$ be two C-TFNNs and $\lambda > 0$. The operational laws are defined as follows:

(1) $$s_1 \oplus s_2 = \begin{pmatrix} \left\langle \begin{matrix} (g_{11} + g_{21}, g_{12} + g_{22}, g_{13} + g_{23}, g_{14} + g_{24}); \\ T_{N1} + T_{N2} - T_{N1}T_{N2}, I_{N1}I_{N2}, F_{N1}F_{N2} \end{matrix} \right\rangle, \\ \left\langle \begin{matrix} (h_{11} + h_{21}, h_{12} + h_{22}, h_{13} + h_{23}, h_{14} + h_{24}); \\ T_{L1} + T_{L2} - T_{L1}T_{L2}, I_{L1}I_{L2}, F_{L1}F_{L2} \end{matrix} \right\rangle \end{pmatrix},$$

(2) $$s_1 \otimes s_2 = \begin{pmatrix} \left\langle \begin{matrix} (g_{11}g_{21}, g_{12}g_{22}, g_{13}g_{23}, g_{14}a_{24}); \\ T_{N1}T_{N2}, I_{N1} + I_{N2} - I_{N1}I_{N2}, \\ F_{N1} + F_{N2} - F_{N1}F_{N2} \end{matrix} \right\rangle, \\ \left\langle \begin{matrix} (h_{11}h_{21}, h_{12}h_{22}, h_{13}h_{23}, h_{14}h_{24}); \\ T_{L1}T_{L2}, I_{L1} + I_{L2} - I_{L1}I_{L2}, \\ F_{L1} + F_{L2} - F_{L1}F_{L2} \end{matrix} \right\rangle \end{pmatrix},$$

(3) $$\lambda s_1 = \begin{pmatrix} \left\langle \begin{matrix} (\lambda g_{11}, \lambda g_{12}, \lambda g_{13}, \lambda g_{14}); \\ 1 - (1 - T_{N1})^{\lambda}, I_{N1}^{\lambda}, F_{N1}^{\lambda} \end{matrix} \right\rangle, \\ \left\langle \begin{matrix} (\lambda h_{11}, \lambda h_{12}, \lambda h_{13}, \lambda h_{14}); \\ 1 - (1 - T_{L1})^{\lambda}, I_{L1}^{\lambda}, F_{L1}^{\lambda} \end{matrix} \right\rangle \end{pmatrix},$$

(4) $$(s_1)^{\lambda} = \begin{pmatrix} \left\langle (g_{11}^{\lambda}, g_{12}^{\lambda}, g_{13}^{\lambda}, g_{14}^{\lambda}); T_{N1}^{\lambda}, 1 - (1 - I_{N1})^{\lambda}, 1 - (1 - F_{N1})^{\lambda} \right\rangle, \\ \left\langle (h_{11}^{\lambda}, h_{12}^{\lambda}, h_{13}^{\lambda}, h_{14}^{\lambda}); T_{L1}^{\lambda}, 1 - (1 - I_{L1})^{\lambda}, 1 - (1 - F_{L1})^{\lambda} \right\rangle \end{pmatrix}.$$

Regarding a series of C-TFNNs $s_i = \begin{pmatrix} \langle (g_{i1}, g_{i2}, g_{i3}, g_{i4}); T_{Ni}, I_{Ni}, F_{Ni} \rangle, \\ \langle (h_{i1}, h_{i2}, h_{i3}, h_{i4}); T_{Li}, I_{Li}, F_{Li} \rangle \end{pmatrix}$ (*i = 1, 2, …, J*) subject to the weight λ_i of *si* with $0 \le \lambda_i \le 1$ and $\sum_{i=1}^{J} \lambda_i = 1$, the C-TFNNWAA and C-TFNNWGA operators [37] are introduced as follows:

$$\text{C-TFNNWAA}\left(s_1,s_2,...,s_J\right)=\sum_{i=1}^{J}\lambda_i s_i=\begin{pmatrix}\left\langle\left(\sum_{i=1}^{J}\lambda_i g_{i1},\sum_{i=1}^{J}\lambda_i g_{i2},\sum_{i=1}^{J}\lambda_i g_{i3},\sum_{i=1}^{J}\lambda_i g_{i4}\right);\\ 1-\prod_{i=1}^{J}\left(1-T_{Ni}\right)^{\lambda_i},\prod_{i=1}^{J}I_{Ni}^{\lambda_i},\prod_{i=1}^{J}F_{Ni}^{\lambda_i}\right\rangle,\\ \left\langle\left(\sum_{i=1}^{J}\lambda_i h_{i1},\sum_{i=1}^{J}\lambda_i h_{i2},\sum_{i=1}^{J}\lambda_i h_{i3},\sum_{i=1}^{J}\lambda_i h_{i4}\right);\\ 1-\prod_{i=1}^{J}\left(1-T_{Li}\right)^{\lambda_i},\prod_{i=1}^{J}I_{Li}^{\lambda_i},\prod_{i=1}^{J}F_{Li}^{\lambda_i}\right\rangle\end{pmatrix}, \quad (1)$$

$$\text{C-TFNNWGA}\left(s_1,s_2,...,s_J\right)=\prod_{i=1}^{J}s_i^{\lambda_i}=\begin{pmatrix}\left\langle\left(\prod_{i=1}^{J}g_{i1}^{\lambda_i},\prod_{i=1}^{J}g_{i2}^{\lambda_i},\prod_{i=1}^{J}g_{i3}^{\lambda_i},\prod_{i=1}^{J}g_{i4}^{\lambda_i}\right);\\ \prod_{i=1}^{J}T_{Ni}^{\lambda_i},1-\prod_{i=1}^{J}\left(1-I_{Ni}\right)^{\lambda_i},1-\prod_{i=1}^{J}\left(1-F_{Ni}\right)^{\lambda_i}\right\rangle,\\ \left\langle\left(\prod_{i=1}^{J}h_{i1}^{\lambda_i},\prod_{i=1}^{J}h_{i2}^{\lambda_i},\prod_{i=1}^{J}h_{i3}^{\lambda_i},\prod_{i=1}^{J}h_{i4}^{\lambda_i}\right);\\ \prod_{i=1}^{J}T_{Li}^{\lambda_i},1-\prod_{i=1}^{J}\left(1-I_{Li}\right)^{\lambda_i},1-\prod_{i=1}^{J}\left(1-F_{Li}\right)^{\lambda_i}\right\rangle\end{pmatrix}. \quad (2)$$

Definition 4 [37]. To compare C-TFNNs, the score function of C-TFNN is defined as

$$S(s)=\frac{1}{144}\begin{pmatrix}\left(g_1+g_2+g_3+g_4\right)\times\left(2+T_N-I_N-F_N\right)\times\\ \left(h_1+h_2+h_3+h_4\right)\times\left(2+T_L-I_L-F_L\right)\end{pmatrix},\ 0\le S\left(s\right)\le 1. \quad (3)$$

The following sorting rules are given by the score function:

(1) If $S(s_1)<S(s_2)$, then $s_1<s_2$;

(2) If $S(s_1)=S(s_2)$, then $s_1\cong s_2$.

Particularly, when we ignore credibility degrees in C-TFNNs, C-TFNNs becomes TFNNs. Thus, Eqs. (1)-(3) become the TFNN weighted arithmetic averaging (TFNNWAA) and TFNN weighted geometric averaging (TFNNWGA) operators and the score function [30]:

$$\begin{aligned}&\text{TFNNWAA}\left(s_1,s_2,...,s_J\right)=\sum_{i=1}^{J}\lambda_i s_i\\ &=\left(\left\langle\sum_{i=1}^{J}\lambda_i g_{i1},\sum_{i=1}^{J}\lambda_i g_{i2},\sum_{i=1}^{J}\lambda_i g_{i3},\sum_{i=1}^{J}\lambda_i g_{i4}\right\rangle;1-\prod_{i=1}^{J}\left(1-T_{Ni}\right)^{\lambda_i},\prod_{i=1}^{J}I_{Ni}^{\lambda_i},\prod_{i=1}^{J}F_{Ni}^{\lambda_i}\right),\end{aligned} \quad (4)$$

$$\begin{aligned}&\text{TFNNWGA}\left(s_1,s_2,...,s_J\right)=\prod_{i=1}^{J}s_i^{\lambda_i}\\ &=\left(\left\langle\prod_{i=1}^{J}g_{i1}^{\lambda_i},\prod_{i=1}^{J}g_{i2}^{\lambda_i},\prod_{i=1}^{J}g_{i3}^{\lambda_i},\prod_{i=1}^{J}g_{i4}^{\lambda_i}\right\rangle;\prod_{i=1}^{J}T_{Ni}^{\lambda_i},1-\prod_{i=1}^{J}\left(1-I_{Ni}\right)^{\lambda_i},1-\prod_{i=1}^{J}\left(1-F_{Ni}\right)^{\lambda_i}\right),\end{aligned} \quad (5)$$

$$S''(s)=\frac{1}{12}\left(\left(g_1+g_2+g_3+g_4\right)\times\left(2+T_N-I_N-F_N\right)\right), S''(s)\in[0,1]. \quad (6)$$

If there are no credibility degrees, $g_1 = g_2 = g_3 = g_4 = 1$ and $h_1 = h_2 = h_3 = h_4 = 1$ (without considering TFNs) in C-TFNNs, C-TFNNs become single-valued neutrosophic numbers (SVNNs). Thus, Eqs. (1)-(3) become the SVNN weighted arithmetic averaging (SVNNWAA) and SVNN weighted geometric averaging (SVNNWGA) operators and the score function [19]:

$$\mathrm{SVNNWAA}\left(s_1,s_2,\ldots,s_J\right)=\sum_{i=1}^{J}\lambda_i s_i=\left(1-\prod_{i=1}^{J}\left(1-T_{Ni}\right)^{\lambda_i},\prod_{i=1}^{J}I_{Ni}^{\lambda_i},\prod_{i=1}^{J}F_{Ni}^{\lambda_i}\right), \quad (7)$$

$$\mathrm{SVNNWGA}\left(s_1,s_2,\ldots,s_J\right)=\prod_{i=1}^{J}s_i^{\lambda_i}=\left(\prod_{i=1}^{J}T_{Ni}^{\lambda_i},1-\prod_{i=1}^{J}\left(1-I_{Ni}\right)^{\lambda_i},1-\prod_{i=1}^{J}\left(1-F_{Ni}\right)^{\lambda_i}\right), \quad (8)$$

$$S'(s)=\frac{1}{3}\left(2+T_N-I_N-F_N\right), S'(s)\in[0,1]. \quad (9)$$

3. C-ITFNNs

As an extension of C-TFNNs, this section proposes C-ITFNNs, some operational laws of C-ITFNNs, and a score function for comparing C-TFNNs.

Definition 5. Set X as a non-empty set. A C-ITFNN $\tilde{a}$ can be defined as $\tilde{a}=\left(\left\langle (g_1,g_2,g_3,g_4);\tilde{T}_N(x),\tilde{I}_N(x),\tilde{F}_N(x)\right\rangle,\left\langle (h_1,h_2,h_3,h_4);\tilde{T}_L(x),\tilde{I}_L(x),\tilde{F}_L(x)\right\rangle\right)$ for $x \in X$. Then, it's true, indeterminate, and false membership functions and their corresponding credibility measure functions are indicated, respectively, as follows:

$$\tilde{T}_N(x)=\begin{cases}\left[\frac{x-g_1}{g_2-g_1}\tilde{T}_N^-,\frac{x-g_1}{g_2-g_1}\tilde{T}_N^+\right], g_1\le x<g_2,\\ \left[\tilde{T}_N^-,\tilde{T}_N^+\right], g_2\le x\le g_3,\\ \left[\frac{g_4-x}{g_4-g_3}\tilde{T}_N^-,\frac{g_4-x}{g_4-g_3}\tilde{T}_N^+\right], g_3<x\le g_4,\\ [0,0] \text{ otherwise}\end{cases},$$

$$\tilde{I}_N(x)=\begin{cases}\left[\frac{g_2-x+\tilde{I}_N^-(x-g_1)}{g_2-g_1},\frac{g_2-x+\tilde{I}_N^+(x-g_1)}{g_2-g_1}\right], g_1\le x<g_2,\\ \left[\tilde{I}_N^-,\tilde{I}_N^+\right], g_2\le x\le g_3,\\ \left[\frac{x-g_3+\tilde{I}_N^-(g_4-x)}{g_4-g_3},\frac{x-g_3+\tilde{I}_N^+(g_4-x)}{g_4-g_3}\right], g_3<x\le g_4,\\ [1,1] \text{ otherwise}\end{cases},$$

$$\tilde{F}_N(x)=\begin{cases}\left[\dfrac{g_2-x+\tilde{F}_N^-(x-g_1)}{g_2-g_1},\dfrac{g_2-x+\tilde{F}_N^+(x-g_1)}{g_2-g_1}\right], g_1\le x<g_2,\\ \left[\tilde{F}_N^-,\tilde{F}_N^+\right], g_2\le x\le g_3,\\ \left[\dfrac{x-g_3+\tilde{F}_N^-(g_4-x)}{g_4-g_3},\dfrac{x-g_3+\tilde{F}_N^+(g_4-x)}{g_4-g_3}\right], g_3<x\le g_4,\\ \langle 1,1\rangle, \text{otherwise}\end{cases},$$

and it's true, indeterminate and false credibility measure functions are indicated as follows:

$$\tilde{T}_L(x)=\begin{cases}\left[\dfrac{x-h_1}{h_2-h_1}\tilde{T}_L^-,\dfrac{x-h_1}{h_2-h_1}\tilde{T}_L^+\right], h_1\le x<h_2,\\ \left[\tilde{T}_L^-,\tilde{T}_L^+\right], h_2\le x\le h_3,\\ \left[\dfrac{h_4-x}{h_4-h_3}\tilde{T}_L^-,\dfrac{h_4-x}{h_4-h_3}\tilde{T}_L^+\right], h_3<x\le h_4,\\ \langle 0,0\rangle, \text{otherwise}\end{cases},$$

$$\tilde{I}_L(x)=\begin{cases}\left[\dfrac{h_2-x+\tilde{I}_L^-(x-h_1)}{h_2-h_1},\dfrac{h_2-x+\tilde{I}_L^+(x-h_1)}{h_2-h_1}\right], h_1\le x<h_2,\\ \left[\tilde{I}_L^-,\tilde{I}_L^+\right], h_2\le x\le h_3,\\ \left[\dfrac{x-h_3+\tilde{I}_L^-(h_4-x)}{h_4-h_3},\dfrac{x-h_3+\tilde{I}_L^+(h_4-x)}{h_4-h_3}\right], h_3<x\le h_4,\\ \langle 1,1\rangle, \text{otherwise}\end{cases},$$

$$\tilde{F}_L(x)=\begin{cases}\left[\dfrac{h_1-x+\tilde{F}_L^-(x-h_1)}{h_2-h_1},\dfrac{h_1-x+\tilde{F}_L^+(x-h_1)}{h_2-h_1}\right], \ h_1\le x<h_2,\\ \left[\tilde{F}_L^-,\tilde{F}_L^+\right], h_2\le x\le h_3,\\ \left[\dfrac{x-h_3+\tilde{F}_L^-(h_4-x)}{h_4-h_3},\dfrac{x-h_3+\tilde{F}_L^+(h_4-x)}{h_4-h_3}\right], h_3<x\le h_4,\\ \langle 1,1\rangle, \text{otherwise}\end{cases},$$

where $\left[\tilde{T}_N^-,\tilde{T}_N^+\right]\subseteq[0,1],\left[\tilde{I}_N^-,\tilde{I}_N^+\right]\subseteq[0,1],\left[\tilde{F}_N^-,\tilde{F}_N^+\right]\subseteq[0,1]$ and $\left[\tilde{T}_L^-,\tilde{T}_L^+\right]\subseteq[0,1],\left[\tilde{I}_L^-,\tilde{I}_L^+\right]\subseteq[0,1],\left[\tilde{F}_L^-,\tilde{F}_L^+\right]\subseteq[0,1]$ are the true, indeterminate, false interval membership degrees and credibility levels in the C-ITFNN $\tilde{a}$ subject to $0\le\tilde{T}_N^++\tilde{I}_N^++\tilde{F}_N^+\le 3$ and $0\le\tilde{T}_L^++\tilde{I}_L^++\tilde{F}_L^+\le 3$, then g_k, $h_k \in \Re$ (k = 1, 2, 3, 4) are the parameters of ITFNNs in the C-ITFNN $\tilde{a}$.

For the convenient representation of the C-ITFNN $\tilde{a}$, it is simply expressed as $\tilde{a}=\left(\left\langle(g_1,g_2,g_3,g_4);\tilde{T}_N,\tilde{I}_N,\tilde{F}_N\right\rangle,\left\langle(h_1,h_2,h_3,h_4);\tilde{T}_L,\tilde{I}_L,\tilde{F}_L\right\rangle\right)$.

Then, the two special cases of C-ITFNN are indicated below:

(1) If the upper and lower endpoints of the interval values $\left[\tilde{T}_N^-,\tilde{T}_N^+\right]$, $\left[\tilde{I}_N^-,\tilde{I}_N^+\right]$, $\left[\tilde{F}_N^-,\tilde{F}_N^+\right]$, $\left[\tilde{T}_L^-,\tilde{T}_L^+\right]$, $\left[\tilde{I}_L^-,\tilde{I}_L^+\right]$, and $\left[\tilde{F}_L^-,\tilde{F}_L^+\right]$ in the C-ITFNN $\tilde{a}$ are equal, C-ITFNN becomes C-TFNN.

(2) If $g_1 = g_2 = g_3 = g_4 = 1$ and $h_1 = h_2 = h_3 = h_4 = 1$ in the C-ITFNN $\tilde{a}$, C-ITFNN becomes the credibility interval neutrosophic number.

Definition 6. Let $\tilde{a}_1 = \begin{pmatrix} \left\langle (g_{11}, g_{12}, g_{13}, g_{14}); [\tilde{T}_{N1}^-, \tilde{T}_{N1}^+], [\tilde{I}_{N1}^-, \tilde{I}_{N1}^+], [\tilde{F}_{N1}^-, \tilde{F}_{N1}^+] \right\rangle \\ \left\langle (h_{11}, h_{12}, h_{13}, h_{14}); [\tilde{T}_{L1}^-, \tilde{T}_{L1}^+], [\tilde{I}_{L1}^-, \tilde{I}_{L1}^+], [\tilde{F}_{L1}^-, \tilde{F}_{L1}^+] \right\rangle \end{pmatrix}$ and $\tilde{a}_2 = \begin{pmatrix} \left\langle (g_{21}, g_{22}, g_{23}, g_{24}); [\tilde{T}_{N2}^-, \tilde{T}_{N2}^+], [\tilde{I}_{N2}^-, \tilde{I}_{N2}^+], [\tilde{F}_{N2}^-, \tilde{F}_{N2}^+] \right\rangle \\ \left\langle (h_{21}, h_{22}, h_{23}, h_{24}); [\tilde{T}_{L2}^-, \tilde{T}_{L2}^+], [\tilde{I}_{L2}^-, \tilde{I}_{L2}^+], [\tilde{F}_{L2}^-, \tilde{F}_{L2}^+] \right\rangle \end{pmatrix}$ be two C-ITFNNs and $\lambda > 0$.

Then, their operational laws are satisfied below:

$$(1)\ \tilde{a}_1 \oplus \tilde{a}_2 = \begin{pmatrix} \left\langle \begin{matrix} (g_{11}+g_{21}, g_{12}+g_{22}, g_{13}+g_{23}, g_{14}+g_{24}); \\ [\tilde{T}_{N1}^- + \tilde{T}_{N2}^- - \tilde{T}_{N1}^-\tilde{T}_{N2}^-, \tilde{T}_{N1}^+ + \tilde{T}_{N2}^+ - \tilde{T}_{N1}^+\tilde{T}_{N2}^+], \\ [\tilde{I}_{N1}^-\tilde{I}_{N2}^-, \tilde{I}_{N1}^+\tilde{I}_{N2}^+], [\tilde{F}_{N1}^-\tilde{F}_{N2}^-, \tilde{F}_{N1}^+\tilde{F}_{N2}^+] \end{matrix} \right\rangle, \\ \left\langle \begin{matrix} (h_{11}+h_{21}, h_{12}+h_{22}, h_{13}+h_{23}, h_{14}+h_{24}); \\ [\tilde{T}_{L1}^- + \tilde{T}_{L2}^- - \tilde{T}_{L1}^-\tilde{T}_{L2}^-, \tilde{T}_{L1}^+ + \tilde{T}_{L2}^+ - \tilde{T}_{L1}^+\tilde{T}_{L2}^+], \\ [\tilde{I}_{L1}^-\tilde{I}_{L2}^-, \tilde{I}_{L1}^+\tilde{I}_{L2}^+], [\tilde{F}_{L1}^-\tilde{F}_{L2}^-, \tilde{F}_{L1}^+\tilde{F}_{L2}^+] \end{matrix} \right\rangle \end{pmatrix};$$

$$(2)\ \tilde{a}_1 \otimes \tilde{a}_2 = \begin{pmatrix} \left\langle \begin{matrix} (g_{11}g_{21}, g_{12}g_{22}, g_{13}g_{23}, g_{14}g_{24}); [\tilde{T}_{N1}^-\tilde{T}_{N2}^-, \tilde{T}_{N1}^+\tilde{T}_{N2}^+], \\ [\tilde{I}_{N1}^- + \tilde{I}_{N2}^- - \tilde{I}_{N1}^-\tilde{I}_{N2}^-, \tilde{I}_{N1}^+ + \tilde{I}_{N2}^+ - \tilde{I}_{N1}^+\tilde{I}_{N2}^+], \\ [\tilde{F}_{N1}^- + \tilde{F}_{N2}^- - \tilde{F}_{N1}^-\tilde{F}_{N2}^-, \tilde{F}_{N1}^+ + \tilde{F}_{N2}^+ - \tilde{F}_{N1}^+\tilde{F}_{N2}^+] \end{matrix} \right\rangle, \\ \left\langle \begin{matrix} (h_{11}h_{21}, h_{12}h_{22}, h_{13}h_{23}, h_{14}h_{24}); [\tilde{T}_{L1}^-\tilde{T}_{L2}^-, \tilde{T}_{L1}^+\tilde{T}_{L2}^+], \\ [\tilde{I}_{L1}^- + \tilde{I}_{L2}^- - \tilde{I}_{L1}^-\tilde{I}_{L2}^-, \tilde{I}_{L1}^+ + \tilde{I}_{L2}^+ - \tilde{I}_{L1}^+\tilde{I}_{L2}^+], \\ [\tilde{F}_{L1}^- + \tilde{F}_{L2}^- - \tilde{F}_{L1}^-\tilde{F}_{L2}^-, \tilde{F}_{L1}^+ + \tilde{F}_{L2}^+ - \tilde{F}_{L1}^+\tilde{F}_{L2}^+] \end{matrix} \right\rangle \end{pmatrix};$$

$$(3)\ \lambda\tilde{a}_1 = \begin{pmatrix} \left\langle \begin{matrix} (\lambda g_{11}, \lambda g_{12}, \lambda g_{13}, \lambda g_{14}); \left[1-(1-\tilde{T}_{N1}^-)^\lambda, 1-(1-\tilde{T}_{N1}^+)^\lambda\right], \\ \left[(\tilde{I}_{N1}^-)^\lambda, (\tilde{I}_{N1}^+)^\lambda\right], \left[(\tilde{F}_{N1}^-)^\lambda, (\tilde{F}_{N1}^+)^\lambda\right] \end{matrix} \right\rangle, \\ \left\langle \begin{matrix} (\lambda h_{11}, \lambda h_{12}, \lambda h_{13}, \lambda h_{14}); \left[1-(1-\tilde{T}_{L1}^-)^\lambda, 1-(1-\tilde{T}_{L1}^+)^\lambda\right], \\ \left[(\tilde{I}_{L1}^-)^\lambda, (\tilde{I}_{L1}^+)^\lambda\right], \left[(\tilde{F}_{L1}^-)^\lambda, (\tilde{F}_{L1}^+)^\lambda\right] \end{matrix} \right\rangle \end{pmatrix};$$

$$(4)\ \left(\tilde{a}_1\right)^{\lambda}=\begin{pmatrix}\left\langle\begin{array}{l}\left(g_{11}^{\lambda},g_{12}^{\lambda},g_{13}^{\lambda},g_{14}^{\lambda}\right);\left[\left(\tilde{T}_{N1}^{-}\right)^{\lambda},\left(\tilde{T}_{N1}^{+}\right)^{\lambda}\right],\\ \left[1-\left(1-\tilde{I}_{N1}^{-}\right)^{\lambda},1-\left(1-\tilde{I}_{N1}^{+}\right)^{\lambda}\right],\\ \left[1-\left(1-\tilde{F}_{N1}^{-}\right)^{\lambda},1-\left(1-\tilde{F}_{N1}^{+}\right)^{\lambda}\right]\end{array}\right\rangle,\\ \left\langle\begin{array}{l}\left(h_{11}^{\lambda},h_{12}^{\lambda},h_{13}^{\lambda},h_{14}^{\lambda}\right);\left[\left(\tilde{T}_{L1}^{-}\right)^{\lambda},\left(\tilde{T}_{L1}^{+}\right)^{\lambda}\right],\\ \left[1-\left(1-\tilde{I}_{L1}^{-}\right)^{\lambda},1-\left(1-\tilde{I}_{L1}^{+}\right)^{\lambda}\right],\\ \left[1-\left(1-\tilde{F}_{L1}^{-}\right)^{\lambda},1-\left(1-\tilde{F}_{L1}^{+}\right)^{\lambda}\right]\end{array}\right\rangle\end{pmatrix}.$$

To compare C-ITFNNs, the score function and sorting rules of C-ITFNNs are defined in terms of the score function of C-TFNN [37].

Definition 7. Set $\tilde{a}=\begin{pmatrix}\left\langle\left(g_1,g_2,g_3,g_4\right);\left[\tilde{T}_N^-,\tilde{T}_N^+\right],\left[\tilde{I}_N^-,\tilde{I}_N^+\right],\left[\tilde{F}_N^-,\tilde{F}_N^+\right]\right\rangle\\ \left\langle\left(h_1,h_2,h_3,h_4\right);\left[\tilde{T}_L^-,\tilde{T}_L^+\right],\left[\tilde{I}_L^-,\tilde{I}_L^+\right],\left[\tilde{F}_L^-,\tilde{F}_L^+\right]\right\rangle\end{pmatrix}$ as C-ITFNN. The score function is defined below:

$$\begin{aligned}S\left(\tilde{a}\right)&=\frac{1}{24}\left(\left(g_1+g_2+g_3+g_4\right)\times\left(4+\tilde{T}_N^-+\tilde{T}_N^+-\tilde{I}_N^--\tilde{I}_N^+-\tilde{F}_N^--\tilde{F}_N^+\right)\right)\times\\ &\frac{1}{24}\left(\left(h_1+h_2+h_3+h_4\right)\times\left(4+\tilde{T}_L^-+\tilde{T}_L^+-\tilde{I}_L^--\tilde{I}_L^+-\tilde{F}_L^--\tilde{F}_L^+\right)\right)\\ &=\frac{1}{576}\left(\left(g_1+g_2+g_3+g_4\right)\times\left(4+\tilde{T}_N^-+\tilde{T}_N^+-\tilde{I}_N^--\tilde{I}_N^+-\tilde{F}_N^--\tilde{F}_N^+\right)\right)\times\\ &\left(\left(h_1+h_2+h_3+h_4\right)\times\left(4+\tilde{T}_L^-+\tilde{T}_L^+-\tilde{I}_L^--\tilde{I}_L^+-\tilde{F}_L^--\tilde{F}_L^+\right)\right).\end{aligned}\tag{10}$$

For two C-ITFNNs $\tilde{a}_1$ and $\tilde{a}_2$, if $S\left(\tilde{a}_1\right)>S\left(\tilde{a}_2\right)$, then $\tilde{a}_1>\tilde{a}_2$; if $S\left(\tilde{a}_1\right)=S\left(\tilde{a}_2\right)$, then $\tilde{a}_1\cong\tilde{a}_2$.

Example 1. Set two C-ITFNNs as $\tilde{a}_1=\begin{pmatrix}\left\langle(0.4,0.5,0.6,0.7);[0.7,0.9],[0.1,0.3],[0.2,0.3]\right\rangle,\\ \left\langle(0.3,0.4,0.5,0.6);[0.5,0.6],[0.2,0.3],[0.1,0.2]\right\rangle\end{pmatrix}$ and $\tilde{a}_2=\begin{pmatrix}\left\langle(0.5,0.6,0.7,0.8);[0.6,0.8],[0.3,0.5],[0.1,0.2]\right\rangle;\\ \left\langle(0.6,0.7,0.8,0.9);[0.5,0.7],[0.2,0.3],[0.1,0.3]\right\rangle\end{pmatrix}$. Thus, the two C-ITFNNs are sorted by the score function of Eq. (10):

Since $S(\tilde{a}_1)=\dfrac{1}{576}\begin{pmatrix}(0.4+0.5+0.6+0.7)\times(4+0.7+0.9-0.1-0.3-0.2-0.3)\\ \times(0.3+0.4+0.5+0.6)\times(4+0.5+0.6-0.2-0.3-0.1-0.2)\end{pmatrix}=0.1389$

and $S(\tilde{a}_2)=\dfrac{1}{576}\begin{pmatrix}(0.5+0.6+0.7+0.8)\times(4+0.6+0.8-0.3-0.5-0.1-0.2)\\ \times(0.6+0.7+0.8+0.9)\times(4+0.5+0.7-0.2-0.3-0.1-0.3)\end{pmatrix}=0.2504$, the sorting of both is $\tilde{a}_1<\tilde{a}_2$.

4. Two Basic Aggregation Operators of C-ITFNNs

As an important tool of aggregation operators for DM modeling, this section proposes two basic aggregation operators for C-ITFNNs by extending the weighted arithmetic and geometric averaging operators of C-TFNNs proposed by Ye et al. [37].

Definition 8. Set $\tilde{a}_i = \begin{pmatrix} \left\langle (g_{i1}, g_{i2}, g_{i3}, g_{i4}); [\tilde{T}^-_{Ni}, \tilde{T}^+_{Ni}], [\tilde{I}^-_{Ni}, \tilde{I}^+_{Ni}], [\tilde{F}^-_{Ni}, \tilde{F}^+_{Ni}] \right\rangle \\ \left\langle (h_{i1}, h_{i2}, h_{i3}, h_{i4}); [\tilde{T}^-_{Li}, \tilde{T}^+_{Li}], [\tilde{I}^-_{Li}, \tilde{I}^+_{Li}], [\tilde{F}^-_{Li}, \tilde{F}^+_{Li}] \right\rangle \end{pmatrix}$ (*i* = 1, 2, …, *J*) as a group of C-ITFNNs with the weight λ_i of $\tilde{a}_i$ for $0 \le \lambda_i \le 1$ and $\sum_{i=1}^{J} \lambda_i = 1$. The C-ITFNNWAA operator is defined as follows:

$$C - ITFNNWAA(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_J) = \sum_{i=1}^{J} \lambda_i \tilde{a}_i \ . \tag{11}$$

In view of the operational laws of C-ITFNNs and Eq. (11), we can obtain the C-ITFNNWAA operator.

Theorem 1. Set $\tilde{a}_i = \begin{pmatrix} \left\langle (g_{i1}, g_{i2}, g_{i3}, g_{i4}); [\tilde{T}^-_{Ni}, \tilde{T}^+_{Ni}], [\tilde{I}^-_{Ni}, \tilde{I}^+_{Ni}], [\tilde{F}^-_{Ni}, \tilde{F}^+_{Ni}] \right\rangle \\ \left\langle (h_{i1}, h_{i2}, h_{i3}, h_{i4}); [\tilde{T}^-_{Li}, \tilde{T}^+_{Li}], [\tilde{I}^-_{Li}, \tilde{I}^+_{Li}], [\tilde{F}^-_{Li}, \tilde{F}^+_{Li}] \right\rangle \end{pmatrix}$ (*i* = 1, 2, …, *J*) as a group of C-ITFNNs with the weight λ_i of $\tilde{a}_i$ for $0 \le \lambda_i \le 1$ and $\sum_{i=1}^{J} \lambda_i = 1$. On the basis of Eq. (11) and the operational laws of C-ITFNNs, the C-ITFNNWAA operator can be expressed as follows:

$$C - ITFNNWAA(\tilde{a}_1, \tilde{a}_2, ...\tilde{a}_J) = \sum_{i=1}^{J} \lambda_i \tilde{a}_i = \begin{pmatrix} \left\langle \begin{array}{l} \left(\sum_{i=1}^{J} \lambda_i g_{i1}, \sum_{i=1}^{J} \lambda_i g_{i2}, \sum_{i=1}^{J} \lambda_i g_{i3}, \sum_{i=1}^{J} \lambda_i g_{i4} \right); \\ \left[1 - \prod_{i=1}^{J} \left(1 - \tilde{T}^-_{Ni}\right)^{\lambda_i}, 1 - \prod_{i=1}^{J} \left(1 - \tilde{T}^+_{Ni}\right)^{\lambda_i} \right], \\ \left[\prod_{i=1}^{J} \left(\tilde{I}^-_{Ni}\right)^{\lambda_i}, \prod_{i=1}^{J} \left(\tilde{I}^+_{Ni}\right)^{\lambda_i} \right], \left[\prod_{i=1}^{J} \left(\tilde{F}^-_{Ni}\right)^{\lambda_i}, \prod_{i=1}^{J} \left(\tilde{F}^+_{Ni}\right)^{\lambda_i} \right] \end{array} \right\rangle, \\ \left\langle \begin{array}{l} \left(\sum_{i=1}^{J} \lambda_i h_{i1}, \sum_{i=1}^{J} \lambda_i h_{i2}, \sum_{i=1}^{J} \lambda_i h_{i3}, \sum_{i=1}^{J} \lambda_i h_{i4} \right); \\ \left[1 - \prod_{i=1}^{J} \left(1 - \tilde{T}^-_{Li}\right)^{\lambda_i}, 1 - \prod_{i=1}^{J} \left(1 - \tilde{T}^+_{Li}\right)^{\lambda_i} \right], \\ \left[\prod_{i=1}^{J} \left(\tilde{I}^-_{Li}\right)^{\lambda_i}, \prod_{i=1}^{J} \left(\tilde{I}^+_{Li}\right)^{\lambda_i} \right], \left[\prod_{i=1}^{J} \left(\tilde{F}^-_{Li}\right)^{\lambda_i}, \prod_{i=1}^{J} \left(\tilde{F}^+_{Li}\right)^{\lambda_i} \right] \end{array} \right\rangle \end{pmatrix}. \tag{12}$$

Proof: Here, Theorem 1 is proved in light of mathematical induction.

(1) When *J* = 2, the aggregated result of the two C-ITFNNs is obtained as follows:

$$C-ITFNNWAA(\tilde{a}_1,\tilde{a}_2)=\sum_{i=1}^{2}\lambda_i\tilde{a}_i=\lambda_1\tilde{a}_1\oplus\lambda_2\tilde{a}_2$$

$$=\left(\begin{array}{l}\left\langle\begin{array}{l}\left(\lambda_1 g_{11}+\lambda_2 g_{21},\lambda_1 g_{12}+\lambda_2 g_{22},\lambda_1 g_{13}+\lambda_2 g_{23},\lambda_1 g_{14}+\lambda_2 g_{24}\right);\\ \left[\begin{array}{l}1-\left(1-\tilde{T}_{N1}^{-}\right)^{\lambda_1}+1-\left(1-\tilde{T}_{N2}^{-}\right)^{\lambda_2}-\left(1-\left(1-\tilde{T}_{N1}^{-}\right)^{\lambda_1}\right)\left(1-\left(1-\tilde{T}_{N2}^{-}\right)^{\lambda_2}\right),\\ 1-\left(1-\tilde{T}_{N1}^{+}\right)^{\lambda_1}+1-\left(1-\tilde{T}_{N2}^{+}\right)^{\lambda_2}-\left(1-\left(1-\tilde{T}_{N1}^{+}\right)^{\lambda_1}\right)\left(1-\left(1-\tilde{T}_{N2}^{+}\right)^{\lambda_2}\right)\end{array}\right],\\ \left[\left(\tilde{I}_{N1}^{-}\right)^{\lambda_1}\left(\tilde{I}_{N2}^{-}\right)^{\lambda_2},\left(\tilde{I}_{N1}^{+}\right)^{\lambda_1}\left(\tilde{I}_{N2}^{+}\right)^{\lambda_2}\right],\left[\left(\tilde{F}_{N1}^{-}\right)^{\lambda_1}\left(\tilde{F}_{N2}^{-}\right)^{\lambda_2},\left(\tilde{F}_{N1}^{+}\right)^{\lambda_2}\left(\tilde{F}_{N2}^{+}\right)^{\lambda_2}\right]\end{array}\right\rangle,\\ \left\langle\begin{array}{l}\left(\lambda_1 h_{11}+\lambda_2 h_{21},\lambda_1 h_{12}+\lambda_2 h_{22},\lambda_1 h_{13}+\lambda_2 h_{23},\lambda_1 h_{14}+\lambda_2 h_{24}\right);\\ \left[\begin{array}{l}1-\left(1-\tilde{T}_{L1}^{-}\right)^{\lambda_1}+1-\left(1-\tilde{T}_{L2}^{-}\right)^{\lambda_2}-\left(1-\left(1-\tilde{T}_{L1}^{-}\right)^{\lambda_1}\right)\left(1-\left(1-\tilde{T}_{L2}^{-}\right)^{\lambda_2}\right),\\ 1-\left(1-\tilde{T}_{L1}^{+}\right)^{\lambda_1}+1-\left(1-\tilde{T}_{L2}^{+}\right)^{\lambda_2}-\left(1-\left(1-\tilde{T}_{L1}^{+}\right)^{\lambda_1}\right)\left(1-\left(1-\tilde{T}_{L2}^{+}\right)^{\lambda_2}\right)\end{array}\right],\\ \left[\left(\tilde{I}_{L1}^{-}\right)^{\lambda_1}\left(\tilde{I}_{L2}^{-}\right)^{\lambda_2},\left(\tilde{I}_{L1}^{+}\right)^{\lambda_1}\left(\tilde{I}_{L2}^{+}\right)^{\lambda_2}\right],\left[\left(\tilde{F}_{L1}^{-}\right)^{\lambda_1}\left(\tilde{F}_{L2}^{-}\right)^{\lambda_2},\left(\tilde{F}_{L1}^{-}\right)^{\lambda_1}\left(\tilde{F}_{L2}^{-}\right)^{\lambda_2}\right]\end{array}\right\rangle\end{array}\right)$$

$$=\left(\begin{array}{l}\left\langle\begin{array}{l}\left(\sum_{i=1}^{2}\lambda_i g_{i1},\sum_{i=1}^{2}\lambda_i g_{i2},\sum_{i=1}^{2}\lambda_i g_{i3},\sum_{i=1}^{2}\lambda_i g_{i4}\right);\\ \left[1-\prod_{i=1}^{2}\left(1-\tilde{T}_{Ni}^{-}\right)^{\lambda_i},1-\prod_{i=1}^{2}\left(1-\tilde{T}_{Ni}^{+}\right)^{\lambda_i}\right],\\ \left[\prod_{i=1}^{2}\left(\tilde{I}_{Ni}^{-}\right)^{\lambda_i},\prod_{i=1}^{2}\left(\tilde{I}_{Ni}^{+}\right)^{\lambda_i}\right],\left[\prod_{i=1}^{2}\left(\tilde{F}_{Ni}^{-}\right)^{\lambda_i},\prod_{i=1}^{2}\left(\tilde{F}_{Ni}^{+}\right)^{\lambda_i}\right]\end{array}\right\rangle,\\ \left\langle\begin{array}{l}\left(\sum_{i=1}^{2}\lambda_i h_{i1},\sum_{i=1}^{2}\lambda_i h_{i2},\sum_{i=1}^{2}\lambda_i h_{i3},\sum_{i=1}^{2}\lambda_i h_{i4}\right);\\ \left[1-\prod_{i=1}^{2}\left(1-\tilde{T}_{Li}^{-}\right)^{\lambda_i},1-\prod_{i=1}^{2}\left(1-\tilde{T}_{Li}^{+}\right)^{\lambda_i}\right],\\ \left[\prod_{i=1}^{2}\left(\tilde{I}_{Li}^{-}\right)^{\lambda_i},\prod_{i=1}^{2}\left(\tilde{I}_{Li}^{+}\right)^{\lambda_i}\right],\left[\prod_{i=1}^{2}\left(\tilde{F}_{Li}^{-}\right)^{\lambda_i},\prod_{i=1}^{2}\left(\tilde{F}_{Li}^{+}\right)^{\lambda_i}\right]\end{array}\right\rangle\end{array}\right). \tag{13}$$

(2) If J = n, the aggregated result of n C-ITFNNs is given as follows:

$$C-ITFNNWAA(\tilde{a}_1,\tilde{a}_2,...\tilde{a}_n)=\sum_{i=1}^{n}\lambda_i\tilde{a}_i=\left(\begin{array}{l}\left\langle\begin{array}{l}\left(\sum_{i=1}^{n}\lambda_i g_{i1},\sum_{i=1}^{n}\lambda_i g_{i2},\sum_{i=1}^{n}\lambda_i g_{i3},\sum_{i=1}^{n}\lambda_i g_{i4}\right);\\ \left[1-\prod_{i=1}^{n}\left(1-\tilde{T}_{Ni}^{-}\right)^{\lambda_i},1-\prod_{i=1}^{n}\left(1-\tilde{T}_{Ni}^{+}\right)^{\lambda_i}\right],\\ \left[\prod_{i=1}^{n}\left(\tilde{I}_{Ni}^{-}\right)^{\lambda_i},\prod_{i=1}^{n}\left(\tilde{I}_{Ni}^{+}\right)^{\lambda_i}\right],\left[\prod_{i=1}^{n}\left(\tilde{F}_{Ni}^{-}\right)^{\lambda_i},\prod_{i=1}^{n}\left(\tilde{F}_{Ni}^{+}\right)^{\lambda_i}\right]\end{array}\right\rangle,\\ \left\langle\begin{array}{l}\left(\sum_{i=1}^{n}\lambda_i h_{i1},\sum_{i=1}^{n}\lambda_i h_{i2},\sum_{i=1}^{n}\lambda_i h_{i3},\sum_{i=1}^{n}\lambda_i h_{i4}\right);\\ \left[1-\prod_{i=1}^{n}\left(1-\tilde{T}_{Li}^{-}\right)^{\lambda_i},1-\prod_{i=1}^{n}\left(1-\tilde{T}_{Li}^{+}\right)^{\lambda_i}\right],\\ \left[\prod_{i=1}^{n}\left(\tilde{I}_{Li}^{-}\right)^{\lambda_i},\prod_{i=1}^{n}\left(\tilde{I}_{Li}^{+}\right)^{\lambda_i}\right],\left[\prod_{i=1}^{n}\left(\tilde{F}_{Li}^{-}\right)^{\lambda_i},\prod_{i=1}^{n}\left(\tilde{F}_{Li}^{+}\right)^{\lambda_i}\right]\end{array}\right\rangle\end{array}\right). \quad (14)$$

(3) If $J = n + 1$, according to Eqs. (12) and (13), we can get the following result:

$$C-ITFNNWAA(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_n,\tilde{a}_{n+1})=\sum_{i=1}^{n+1}\lambda_i\tilde{a}_i=\sum_{i=1}^{n}\lambda_i\tilde{a}_i\oplus\lambda_{n+1}\tilde{a}_{n+1}$$

$$=\left(\begin{array}{l}\left\langle\begin{array}{l}\left(\sum_{i=1}^{n+1}\lambda_i g_{i1},\sum_{i=1}^{n+1}\lambda_i g_{i2},\sum_{i=1}^{n+1}\lambda_i g_{i3},\sum_{i=1}^{n+1}\lambda_i g_{i4}\right);\\ \left[\begin{array}{l}1-\prod_{i=1}^{n}\left(1-\tilde{T}_{Ni}^{-}\right)^{\lambda_i}+1-\left(1-\tilde{T}_{Nn+1}^{-}\right)^{\lambda_{n+1}}-\left(1-\prod_{i=1}^{n}\left(1-\tilde{T}_{Ni}^{-}\right)^{\lambda_i}\right)\left(1-\left(1-\tilde{T}_{Nn+1}^{-}\right)^{\lambda_{n+1}}\right),\\ 1-\prod_{i=1}^{n}\left(1-\tilde{T}_{Ni}^{+}\right)^{\lambda_i}+1-\left(1-\tilde{T}_{Nn+1}^{+}\right)^{\lambda_{n+1}}-\left(1-\prod_{i=1}^{n}\left(1-\tilde{T}_{Ni}^{+}\right)^{\lambda_i}\right)\left(1-\left(1-\tilde{T}_{Nn+1}^{+}\right)^{\lambda_{n+1}}\right)\end{array}\right],\\ \left[\prod_{i=1}^{n+1}\left(\tilde{I}_{Ni}^{-}\right)^{\lambda_i},\prod_{i=1}^{n+1}\left(\tilde{I}_{Ni}^{+}\right)^{\lambda_i}\right],\left[\prod_{i=1}^{n+1}\left(\tilde{F}_{Ni}^{-}\right)^{\lambda_i},\prod_{i=1}^{n+1}\left(\tilde{F}_{Ni}^{+}\right)^{\lambda_i}\right]\end{array}\right\rangle,\\ \left\langle\begin{array}{l}\left(\sum_{i=1}^{n+1}\lambda_i h_{i1},\sum_{i=1}^{n+1}\lambda_i h_{i2},\sum_{i=1}^{n+1}\lambda_i h_{i3},\sum_{i=1}^{n+1}\lambda_i h_{i4}\right);\\ \left[\begin{array}{l}1-\prod_{i=1}^{n}\left(1-\tilde{T}_{Li}^{-}\right)^{\lambda_i}+1-\left(1-\tilde{T}_{Ln+1}^{-}\right)^{\lambda_{n+1}}-\left(1-\prod_{i=1}^{n}\left(1-\tilde{T}_{Li}^{-}\right)^{\lambda_i}\right)\left(1-\left(1-\tilde{T}_{Ln+1}^{-}\right)^{\lambda_{n+1}}\right),\\ 1-\prod_{i=1}^{n}\left(1-\tilde{T}_{Li}^{+}\right)^{\lambda_i}+1-\left(1-\tilde{T}_{Ln+1}^{+}\right)^{\lambda_{n+1}}-\left(1-\prod_{i=1}^{n}\left(1-\tilde{T}_{Li}^{+}\right)^{\lambda_i}\right)\left(1-\left(1-\tilde{T}_{Ln+1}^{+}\right)^{\lambda_{n+1}}\right)\end{array}\right],\\ \left[\prod_{i=1}^{n+1}\left(\tilde{I}_{Li}^{-}\right)^{\lambda_i},\prod_{i=1}^{n+1}\left(\tilde{I}_{Li}^{+}\right)^{\lambda_i}\right],\left[\prod_{i=1}^{n+1}\left(\tilde{F}_{Li}^{-}\right)^{\lambda_i},\prod_{i=1}^{n+1}\left(\tilde{F}_{Li}^{+}\right)^{\lambda_i}\right]\end{array}\right\rangle\end{array}\right) \quad (15)$$

$$=\left(\left\langle\begin{array}{l}\left(\sum_{i=1}^{n+1}\lambda_i g_{i1},\sum_{i=1}^{n+1}\lambda_i g_{i2},\sum_{i=1}^{n+1}\lambda_i g_{i3},\sum_{i=1}^{n+1}\lambda_i g_{i4}\right);\\ \left[1-\prod_{i=1}^{n+1}\left(1-\tilde{T}_{Ni}^{-}\right)^{\lambda_i},1-\prod_{i=1}^{n+1}\left(1-\tilde{T}_{Ni}^{+}\right)^{\lambda_i}\right],\\ \left[\prod_{i=1}^{n+1}\left(\tilde{I}_{Ni}^{-}\right)^{\lambda_i},\prod_{i=1}^{n+1}\left(\tilde{I}_{Ni}^{+}\right)^{\lambda_i}\right],\left[\prod_{i=1}^{n+1}\left(\tilde{F}_{Ni}^{-}\right)^{\lambda_i},\prod_{i=1}^{n+1}\left(\tilde{F}_{Ni}^{+}\right)^{\lambda_i}\right]\end{array}\right\rangle,\left\langle\begin{array}{l}\left(\sum_{i=1}^{n+1}\lambda_i h_{i1},\sum_{i=1}^{n+1}\lambda_i h_{i2},\sum_{i=1}^{n+1}\lambda_i h_{i3},\sum_{i=1}^{n+1}\lambda_i h_{i4}\right);\\ \left[1-\prod_{i=1}^{n+1}\left(1-\tilde{T}_{Li}^{-}\right)^{\lambda_i},1-\prod_{i=1}^{n+1}\left(1-\tilde{T}_{Li}^{+}\right)^{\lambda_i}\right],\\ \left[\prod_{i=1}^{n+1}\left(\tilde{I}_{Li}^{-}\right)^{\lambda_i},\prod_{i=1}^{n+1}\left(\tilde{I}_{Li}^{+}\right)^{\lambda_i}\right],\left[\prod_{i=1}^{n+1}\left(\tilde{F}_{Li}^{-}\right)^{\lambda_i},\prod_{i=1}^{n+1}\left(\tilde{F}_{Li}^{+}\right)^{\lambda_i}\right]\end{array}\right\rangle\right).$$

In view of the above results, Eq. (12) is true for any *J*.

Then, the C-ITFNNWAA operator of Eq. (12) has the following characteristics:

(1) Idempotency: Let $\tilde{a}_i$ ($i = 1, 2, \ldots, J$) be a group of C-ITFNNs. If $\tilde{a}_i = \tilde{a}$ for $i = 1, 2, \ldots, J$, there is $C-ITFNNWAA\left(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_J\right)=\tilde{a}$.

(2) Boundedness: Let $\tilde{a}_i$ ($i = 1, 2, \ldots, J$) be a group of C-ITFNNs and let the minimum and maximum C-ITFNNs:

$$\tilde{a}_{\min} = \left(\begin{array}{l} \left\langle \begin{array}{l} \left(\min_i(g_{i1}), \min_i(g_{i2}), \min_i(g_{i3}), \min_i(g_{i4}) \right); \\ \left[\min_i(\tilde{T}_{Ni}^-), \min_i(\tilde{T}_{Ni}^+) \right], \left[\max_i(\tilde{I}_{Ni}^-), \max_i(\tilde{I}_{Ni}^+) \right], \left[\max_i(\tilde{F}_{Ni}^-), \max_i(\tilde{F}_{Ni}^+) \right] \end{array} \right\rangle, \\ \left\langle \begin{array}{l} \left(\min_i(h_{i1}), \min_i(h_{i2}), \min_i(h_{i3}), \min_i(h_{i4}) \right); \\ \left[\min_i(\tilde{T}_{Li}^-), \min_i(\tilde{T}_{Li}^+) \right], \left[\max_i(\tilde{I}_{Li}^-), \max_i(\tilde{I}_{Li}^+) \right], \left[\max_i(\tilde{F}_{Li}^-), \max_i(\tilde{F}_{Li}^+) \right] \end{array} \right\rangle \end{array} \right),$$

$$\tilde{a}_{\max} = \left(\begin{array}{l} \left\langle \begin{array}{l} \left(\max_i(g_{i1}), \max_i(g_{i2}), \max_i(g_{i3}), \max_i(g_{i4}) \right); \\ \left[\max_i(\tilde{T}_{Ni}^-), \max_i(\tilde{T}_{Ni}^+) \right], \left[\min_i(\tilde{I}_{Ni}^-), \min_i(\tilde{I}_{Ni}^+) \right], \left[\min_i(\tilde{F}_{Ni}^-), \min_i(\tilde{F}_{Ni}^+) \right] \end{array} \right\rangle, \\ \left\langle \begin{array}{l} \left(\max_i(h_{i1}), \max_i(h_{i2}), \max_i(h_{i3}), \max_i(h_{i4}) \right); \\ \left[\max_i(\tilde{T}_{Li}^-), \max_i(\tilde{T}_{Li}^+) \right], \left[\min_i(\tilde{I}_{Li}^-), \min_i(\tilde{I}_{Li}^+) \right], \left[\min_i(\tilde{F}_{Li}^-), \min_i(\tilde{F}_{Li}^+) \right] \end{array} \right\rangle \end{array} \right).$$

Then $\tilde{a}_{\min} \le C-ITFNNWAA(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_J) \le \tilde{a}_{\max}$.

(3) Monotonicity: Let $\tilde{a}_i$ (i = 1, 2, …, J) be a group of C-ITFNNs. If $\tilde{a}_i \le \tilde{a}_i'$ for i = 1, 2, …, J, there is $C-ITFNNWAA(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_J) \le C-ITFNNWAA(\tilde{a}_1', \tilde{a}_2', ..., \tilde{a}_J')$.

Proof: (1) Let $\tilde{a}_i = \tilde{a}$ for i = 1, 2, …, J. Then, the aggregated result of Eq. (12) is given below:

$$C-ITFNNWAA(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_J)=\sum_{i=1}^{J}\lambda_i\tilde{a}_i$$

$$=\left(\begin{array}{l}\left\langle\begin{array}{l}\left(\sum_{i=1}^{J}\lambda_i g_{i1},\sum_{i=1}^{J}\lambda_i g_{i2},\sum_{i=1}^{J}\lambda_i g_{i3},\sum_{i=1}^{J}\lambda_i g_{i4}\right);\\ \left[1-\prod_{i=1}^{J}\left(1-\tilde{T}_{Ni}^{-}\right)^{\lambda_i},1-\prod_{i=1}^{J}\left(1-\tilde{T}_{Ni}^{+}\right)^{\lambda_i}\right],\\ \left[\prod_{i=1}^{J}\left(\tilde{I}_{Ni}^{-}\right)^{\lambda_i},\prod_{i=1}^{J}\left(\tilde{I}_{Ni}^{+}\right)^{\lambda_i}\right],\left[\prod_{i=1}^{J}\left(\tilde{F}_{Ni}^{-}\right)^{\lambda_i},\prod_{i=1}^{J}\left(\tilde{F}_{Ni}^{+}\right)^{\lambda_i}\right]\end{array}\right\rangle,\\ \left\langle\begin{array}{l}\left(\sum_{i=1}^{J}\lambda_i h_{i1},\sum_{i=1}^{J}\lambda_i h_{i2},\sum_{i=1}^{J}\lambda_i h_{i3},\sum_{i=1}^{J}\lambda_i h_{i4}\right);\\ \left[1-\prod_{i=1}^{J}\left(1-\tilde{T}_{Li}^{-}\right)^{\lambda_i},1-\prod_{i=1}^{J}\left(1-\tilde{T}_{Li}^{+}\right)^{\lambda_i}\right],\\ \left[\prod_{i=1}^{J}\left(\tilde{I}_{Li}^{-}\right)^{\lambda_i},\prod_{i=1}^{J}\left(\tilde{I}_{Li}^{+}\right)^{\lambda_i}\right],\left[\prod_{i=1}^{J}\left(\tilde{F}_{Li}^{-}\right)^{\lambda_i},\prod_{i=1}^{J}\left(\tilde{F}_{Li}^{+}\right)^{\lambda_i}\right]\end{array}\right\rangle\end{array}\right)$$

$$=\left(\begin{array}{l}\left\langle\begin{array}{l}\left(g_1\sum_{i=1}^{J}\lambda_i,g_2\sum_{i=1}^{J}\lambda_i,g_3\sum_{i=1}^{J}\lambda_i,g_4\sum_{i=1}^{J}\lambda_i\right);\\ \left[1-\left(1-\tilde{T}_{N}^{-}\right)^{\sum_{i=1}^{J}\lambda_i},1-\left(1-\tilde{T}_{N}^{+}\right)^{\sum_{i=1}^{J}\lambda_i}\right],\\ \left[\left(\tilde{I}_{N}^{-}\right)^{\sum_{i=1}^{J}\lambda_i},\left(\tilde{I}_{N}^{+}\right)^{\sum_{i=1}^{J}\lambda_i}\right],\left[\left(\tilde{F}_{N}^{-}\right)^{\sum_{i=1}^{J}\lambda_i},\left(\tilde{F}_{N}^{+}\right)^{\sum_{i=1}^{J}\lambda_i}\right]\end{array}\right\rangle,\\ \left\langle\begin{array}{l}\left(h_1\sum_{i=1}^{J}\lambda_i,h_2\sum_{i=1}^{J}\lambda_i,h_3\sum_{i=1}^{J}\lambda_i,h_4\sum_{i=1}^{J}\lambda_i\right);\\ \left[1-\left(1-\tilde{T}_{L}^{-}\right)^{\sum_{i=1}^{J}\lambda_i},1-\left(1-\tilde{T}_{L}^{+}\right)^{\sum_{i=1}^{J}\lambda_i}\right],\\ \left[\left(\tilde{I}_{L}^{-}\right)^{\sum_{i=1}^{J}\lambda_i},\left(\tilde{I}_{L}^{+}\right)^{\sum_{i=1}^{J}\lambda_i}\right],\left[\left(\tilde{F}_{L}^{-}\right)^{\sum_{i=1}^{J}\lambda_i},\left(\tilde{F}_{L}^{+}\right)^{\sum_{i=1}^{J}\lambda_i}\right]\end{array}\right\rangle\end{array}\right)$$

$$=\left(\begin{array}{l}\left\langle\left(g_1,g_2,g_3,g_4\right);\left[1-\left(1-\tilde{T}_{N}^{-}\right),1-\left(1-\tilde{T}_{N}^{+}\right)\right],\left[\tilde{I}_{N}^{-},\tilde{I}_{N}^{+}\right],\left[\tilde{F}_{N}^{-},\tilde{F}_{N}^{+}\right]\right\rangle,\\ \left\langle\left(h_1,h_2,h_3,h_4\right);\left[1-\left(1-\tilde{T}_{L}^{-}\right),1-\left(1-\tilde{T}_{L}^{+}\right)\right],\left[\tilde{I}_{L}^{-},\tilde{I}_{L}^{+}\right],\left[\tilde{F}_{L}^{-},\tilde{F}_{L}^{+}\right]\right\rangle\end{array}\right)=\tilde{a}.$$

(2) Since $\tilde{a}_{\min}$ and $\tilde{a}_{\max}$ are the minimum and maximum C-ITFNNs, respectively, there is $\tilde{a}_{\min}\le\tilde{a}_i\le\tilde{a}_{\max}$, then $\sum_{i=1}^{J}\lambda_i\tilde{a}_{\min}\le\sum_{i=1}^{J}\lambda_i\tilde{a}_i\le\sum_{i=1}^{J}\lambda_i\tilde{a}_{\max}$ also exists. On the basis of the characteristic (1), there is $\tilde{a}_{\min}\le\sum_{i=1}^{J}\lambda_i\tilde{a}_i\le\tilde{a}_{\max}$, i.e., $\tilde{a}_{\min}\le C-ITFNNWAA\left(\tilde{a}_1,\tilde{a}_2,..,\tilde{a}_J\right)\le\tilde{a}_{\max}$.

(3) Owning to $\tilde{a}_i\le\tilde{a}_i^{'}$ for i = 1, 2, …, J, there is $\sum_{i=1}^{J}\lambda_i\tilde{a}_i\le\sum_{i=1}^{J}\lambda_i\tilde{a}_i^{'}$, namely, $C-ITFNNWAA\left(\tilde{a}_1,\tilde{a}_2,..,\tilde{a}_J\right)\le C-ITFNNWAA\left(\tilde{a}_1',\tilde{a}_2',..,\tilde{a}_J'\right)$.

Example 2. Set two C-ITFNNs as $\tilde{a}_1 = \begin{pmatrix} \langle (0.2, 0.3, 0.4, 0.5); [0.1, 0.2], [0.2, 0.3], [0.3, 0.4] \rangle, \\ \langle (0.3, 0.4, 0.5, 0.6); [0.1, 0.3], [0.2, 0.4], [0.3, 0.5] \rangle \end{pmatrix}$ and $\tilde{a}_2 = \begin{pmatrix} \langle (0.4, 0.5, 0.6, 0.7); [0.2, 0.4], [0.1, 0.3], [0.5, 0.7] \rangle, \\ \langle (0.5, 0.6, 0.7, 0.8); [0.2, 0.3], [0.3, 0.4], [0.4, 0.5] \rangle \end{pmatrix}$ with their weight values 0.3 and 0.7. Then, the calculational result using Eq. (12) is given below:

$$C-ITFNVWAA(\tilde{a}_1, \tilde{a}_2) = \sum_{i=1}^{2} \lambda_i \tilde{a}_i = \lambda_1 \tilde{a}_1 \oplus \lambda_2 \tilde{a}_2$$

$$= \begin{pmatrix} \left\langle \begin{matrix} \begin{pmatrix} 0.2\times0.3+0.4\times0.7, 0.3\times0.3+0.5\times0.7, \\ 0.4\times0.3+0.6\times0.7, 0.5\times0.3+0.7\times0.7 \end{pmatrix}; \\ \left[1-(1-0.1)^{0.3}\times(1-0.2)^{0.7}, 1-(1-0.2)^{0.3}\times(1-0.4)^{0.7}\right], \\ \left[0.2^{0.3}\times0.1^{0.7}, 0.3^{0.3}\times0.3^{0.7}\right], \left[0.3^{0.3}\times0.5^{0.7}, 0.4^{0.3}\times0.7^{0.7}\right] \end{matrix} \right\rangle, \\ \left\langle \begin{matrix} \begin{pmatrix} 0.3\times0.3+0.5\times0.7, 0.4\times0.3+0.6\times0.7, \\ 0.5\times0.3+0.7\times0.7, 0.6\times0.3+0.8\times0.7 \end{pmatrix}; \\ \left[1-(1-0.1)^{0.3}\times(1-0.2)^{0.7}, 1-(1-0.3)^{0.3}\times(1-0.3)^{0.7}\right], \\ \left[0.2^{0.3}\times0.3^{0.7}, 0.4^{0.3}\times0.4^{0.7}\right], \left[0.3^{0.3}\times0.4^{0.7}, 0.5^{0.3}\times0.5^{0.7}\right] \end{matrix} \right\rangle \end{pmatrix}$$

$$= \begin{pmatrix} \langle (0.34, 0.44, 0.54, 0.64); [0.171, 0.346], [0.123, 0.3], [0.429, 0.592] \rangle, \\ \langle (0.44, 0.54, 0.64, 0.74); [0.171, 0.3], [0.266, 0.4], [0.367, 0.5] \rangle \end{pmatrix}.$$

Definition 4.2. Set $\tilde{a}_i = \begin{pmatrix} \langle (g_{i1}, g_{i2}, g_{i3}, g_{i4}); [\tilde{T}_{Ni}^-, \tilde{T}_{Ni}^+], [\tilde{I}_{Ni}^-, \tilde{I}_{Ni}^+], [\tilde{F}_{Ni}^-, \tilde{F}_{Ni}^+] \rangle \\ \langle (h_{i1}, h_{i2}, h_{i3}, h_{i4}); [\tilde{T}_{Li}^-, \tilde{T}_{Li}^+], [\tilde{I}_{Li}^-, \tilde{I}_{Li}^+], [\tilde{F}_{Li}^-, \tilde{F}_{Li}^+] \rangle \end{pmatrix}$ (*i* = 1, 2, …, *J*) as a group of C-ITFNNs with the weight λ_i of $\tilde{a}_i$ for $0 \le \lambda_i \le 1$ and $\sum_{i=1}^{J} \lambda_i = 1$. The C-ITFNNWGA operator is defined as follows:

$$C-ITFNNWGA(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_J) = \prod_{i=1}^{J} \tilde{a}^{\lambda_i}. \quad (16)$$

In terms of the operational laws of C-ITFNNs and Eq. (16), we can obtain the C-ITFNNWGA operator.

Theorem 2. Set $\tilde{a}_i = \begin{pmatrix} \langle (g_{i1}, g_{i2}, g_{i3}, g_{i4}); [\tilde{T}_{Ni}^-, \tilde{T}_{Ni}^+], [\tilde{I}_{Ni}^-, \tilde{I}_{Ni}^+], [\tilde{F}_{Ni}^-, \tilde{F}_{Ni}^+] \rangle \\ \langle (h_{i1}, h_{i2}, h_{i3}, h_{i4}); [\tilde{T}_{Li}^-, \tilde{T}_{Li}^+], [\tilde{I}_{Li}^-, \tilde{I}_{Li}^+], [\tilde{F}_{Li}^-, \tilde{F}_{Li}^+] \rangle \end{pmatrix}$ (*i* = 1, 2, …, *J*) as a group of C-ITFNNs with the weight λ_i of $\tilde{a}_i$ for $0 \le \lambda_i \le 1$ and $\sum_{i=1}^{J} \lambda_i = 1$. On the basis of Eq. (16) and the operational laws of C-ITFNNs, the C-ITFNNWGA operator can be expressed as follows:

$$C-ITFNNWGA\left(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_J\right)=\prod_{i=1}^{J}\tilde{a}_i^{\lambda_i}$$

$$=\left(\begin{array}{l}\left\langle\begin{array}{l}\left(\prod_{i=1}^{J}g_{i1}^{\lambda_i},\prod_{i=1}^{J}g_{i2}^{\lambda_i},\prod_{i=1}^{J}g_{i3}^{\lambda_i},\prod_{i=1}^{J}g_{i4}^{\lambda_i}\right);\\ \left[\prod_{i=1}^{J}\left(\tilde{T}_{Ni}^{-}\right)^{\lambda_i},\prod_{i=1}^{J}\left(\tilde{T}_{Ni}^{+}\right)^{\lambda_i}\right],\left[1-\prod_{i=1}^{J}\left(1-\tilde{I}_{Ni}^{-}\right)^{\lambda_i},1-\prod_{i=1}^{J}\left(1-\tilde{I}_{Ni}^{+}\right)^{\lambda_i}\right],\\ \left[1-\prod_{i=1}^{J}\left(1-\tilde{F}_{Ni}^{-}\right)^{\lambda_i},1-\prod_{i=1}^{J}\left(1-\tilde{F}_{Ni}^{+}\right)^{\lambda_i}\right]\end{array}\right\rangle,\\ \left\langle\begin{array}{l}\left(\prod_{i=1}^{J}h_{i1}^{\lambda_i},\prod_{i=1}^{J}h_{i2}^{\lambda_i},\prod_{i=1}^{J}h_{i3}^{\lambda_i},\prod_{i=1}^{J}h_{i4}^{\lambda_i}\right);\\ \left[\prod_{i=1}^{J}\left(\tilde{T}_{Li}^{-}\right)^{\lambda_i},\prod_{i=1}^{J}\left(\tilde{T}_{Li}^{+}\right)^{\lambda_i}\right],\left[1-\prod_{i=1}^{J}\left(1-\tilde{I}_{Li}^{-}\right)^{\lambda_i},1-\prod_{i=1}^{J}\left(1-\tilde{I}_{Li}^{+}\right)^{\lambda_i}\right],\\ \left[1-\prod_{i=1}^{J}\left(1-\tilde{F}_{Li}^{-}\right)^{\lambda_i},1-\prod_{i=1}^{J}\left(1-\tilde{F}_{Li}^{+}\right)^{\lambda_i}\right]\end{array}\right\rangle\end{array}\right). \quad (17)$$

Since the proof method of Theorem 2 is similar to that of Theorem 1, it can also be proved in light of mathematical induction, which will not be repeated here.

Then, the C-ITFNNWGA operator has the following characteristics:

Idempotency: Let $\tilde{a}_i$ (i = 1, 2, …, J) be a group of C-ITFNNs. If $\tilde{a}_i=\tilde{a}$ for i = 1, 2, …, J, there is $C-ITFNNWGA\left(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_j\right)=\tilde{a}$.

Boundedness: Let $\tilde{a}_i$ (i = 1, 2, …, J) be a group of C-ITFNNs and let the minimum and maximum C-ITFNNs:

$$\tilde{a}_{\min}=\left(\begin{array}{l}\left\langle\begin{array}{l}\left(\min_i\left(g_{i1}\right),\min_i\left(g_{i2}\right),\min_i\left(g_{i3}\right),\min_i\left(g_{i4}\right)\right);\\ \left[\min_i\left(\tilde{T}_{Ni}^{-}\right),\min_i\left(\tilde{T}_{Ni}^{+}\right)\right],\left[\max_i\left(\tilde{I}_{Ni}^{-}\right),\max_i\left(\tilde{I}_{Ni}^{+}\right)\right],\left[\max_i\left(\tilde{F}_{Ni}^{-}\right),\max_i\left(\tilde{F}_{Ni}^{+}\right)\right]\end{array}\right\rangle,\\ \left\langle\begin{array}{l}\left(\min_i\left(h_{i1}\right),\min_i\left(h_{i2}\right),\min_i\left(h_{i3}\right),\min_i\left(h_{i4}\right)\right);\\ \left[\min_i\left(\tilde{T}_{Li}^{-}\right),\min_i\left(\tilde{T}_{Li}^{+}\right)\right],\left[\max_i\left(\tilde{I}_{Li}^{-}\right),\max_i\left(\tilde{I}_{Li}^{+}\right)\right],\left[\max_i\left(\tilde{F}_{Li}^{-}\right),\max_i\left(\tilde{F}_{Li}^{+}\right)\right]\end{array}\right\rangle\end{array}\right),$$

$$\tilde{a}_{\max}=\left(\begin{array}{l}\left\langle\begin{array}{l}\left(\max_i\left(g_{i1}\right),\max_i\left(g_{i2}\right),\max_i\left(g_{i3}\right),\max_i\left(g_{i4}\right)\right);\\ \left[\max_i\left(\tilde{T}_{Ni}^{-}\right),\max_i\left(\tilde{T}_{Ni}^{+}\right)\right],\left[\min_i\left(\tilde{I}_{Ni}^{-}\right),\min_i\left(\tilde{I}_{Ni}^{+}\right)\right],\left[\min_i\left(\tilde{F}_{Ni}^{-}\right),\min_i\left(\tilde{F}_{Ni}^{+}\right)\right]\end{array}\right\rangle,\\ \left\langle\begin{array}{l}\left(\max_i\left(h_{i1}\right),\max_i\left(h_{i2}\right),\max_i\left(h_{i3}\right),\max_i\left(h_{i4}\right)\right);\\ \left[\max_i\left(\tilde{T}_{Li}^{-}\right),\max_i\left(\tilde{T}_{Li}^{+}\right)\right],\left[\min_i\left(\tilde{I}_{Li}^{-}\right),\min_i\left(\tilde{I}_{Li}^{+}\right)\right],\left[\min_i\left(\tilde{F}_{Li}^{-}\right),\min_i\left(\tilde{F}_{Li}^{+}\right)\right]\end{array}\right\rangle\end{array}\right).$$

Then $\tilde{a}_{\min}\le C-ITFNNWGA\left(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_J\right)\le\tilde{a}_{\max}$.

For $\tilde{a}_i \le \tilde{a}_i^{'}$ (i = 1, 2, …, J), there is $\sum_{i=1}^{J}\lambda_i\tilde{a}_i \le \sum_{i=1}^{J}\lambda_i\tilde{a}_i^{'}$, namely, $C-ITFNNWGA\left(\tilde{a}_1, \tilde{a}_2, .., \tilde{a}_J\right) \le C-ITFNNWGA\left(\tilde{a}_1^{'}, \tilde{a}_2^{'}, .., \tilde{a}_J^{'}\right)$.

However, the proof method of the C-ITFNNWGA operator is similar to that of the C-ITFNNWAA operator, which is omitted here.

Example 3. Set two C-ITFNNs as $\tilde{a}_1 = \begin{pmatrix} \langle(0.4, 0.5, 0.6, 0.7);[0.1, 0.4],[0.2, 0.5],[0.3, 0.6]\rangle, \\ \langle(0.5, 0.6, 0.7, 0.8);[0.2, 0.3],[0.3, 0.4],[0.4, 0.5]\rangle \end{pmatrix}$ and $\tilde{a}_2 = \begin{pmatrix} \langle(0.3, 0.4, 0.5, 0.6);[0.1, 0.2],[0.1, 0.3],[0.1, 0.4]\rangle, \\ \langle(0.6, 0.7, 0.8, 0.9);[0.2, 0.5],[0.3, 0.6],[0.4, 0.7]\rangle \end{pmatrix}$ with their weight values 0.4 and 0.6. Then, the calculational result using Eq. (17) is given below:

$$C-ITFNNWGA\left(\tilde{a}_1, \tilde{a}_2\right) = \prod_{i=1}^{2}\tilde{a}_i^{\lambda_i} = \tilde{a}_1^{\lambda_1} \otimes \tilde{a}_2^{\lambda_2}$$

$$= \begin{pmatrix} \left\langle \begin{matrix} \left(0.4^{0.4}\times 0.3^{0.6}, 0.5^{0.4}\times 0.4^{0.6}, 0.6^{0.4}\times 0.5^{0.6}, 0.7^{0.4}\times 0.6^{0.6}\right); \\ \left[0.1^{0.4}\times 0.1^{0.6}, 0.4^{0.4}\times 0.2^{0.6}\right], \\ \left[1-(1-0.2)^{0.4}\times(1-0.1)^{0.6}, 1-(1-0.5)^{0.4}\times(1-0.3)^{0.6}\right], \\ \left[1-(1-0.3)^{0.4}\times(1-0.1)^{0.6}, 1-(1-0.6)^{0.4}\times(1-0.4)^{0.6}\right] \end{matrix} \right\rangle, \\ \left\langle \begin{matrix} \left(0.5^{0.4}\times 0.6^{0.6}, 0.6^{0.4}\times 0.7^{0.6}, 0.7^{0.4}\times 0.8^{0.6}, 0.8^{0.4}\times 0.9^{0.6}\right); \\ \left[0.2^{0.4}\times 0.2^{0.6}, 0.3^{0.4}\times 0.5^{0.6}\right], \\ \left[1-(1-0.3)^{0.4}\times(1-0.3)^{0.6}, 1-(1-0.4)^{0.4}\times(1-0.6)^{0.6}\right], \\ \left[1-(1-0.4)^{0.4}\times(1-0.4)^{0.6}, 1-(1-0.5)^{0.4}\times(1-0.7)^{0.6}\right] \end{matrix} \right\rangle \end{pmatrix}$$

$$= \begin{pmatrix} \langle(0.337, 0.437, 0.538, 0.638);[0.1, 0.264],[0.141, 0.388],[0.186, 0.4898]\rangle, \\ \langle(0.558, 0.658, 0.758, 0.859);[0.2, 0.408],[0.3, 0.53],[0.4, 0.632]\rangle \end{pmatrix}.$$

Particularly, when there are no credibility degrees in C-ITFNNs, C-ITFNNs become ITFNNs. Thus, Eqs. (12), (17), and (10) become the ITFNN weighted arithmetic averaging (ITFNNWAA) and ITFNN weighted geometric averaging (ITFNNWGA) operators and the score function:

$$ITFNNWAA(\tilde{a}_1, \tilde{a}_2, ...\tilde{a}_J) = \sum_{i=1}^{J}\lambda_i\tilde{a}_i$$

$$= \begin{pmatrix} \left\langle \sum_{i=1}^{J}\lambda_i g_{i1}, \sum_{i=1}^{J}\lambda_i g_{i2}, \sum_{i=1}^{J}\lambda_i g_{i3}, \sum_{i=1}^{J}\lambda_i g_{i4} \right\rangle; \left[1-\prod_{i=1}^{J}\left(1-\tilde{T}_{Ni}^{-}\right)^{\lambda_i}, 1-\prod_{i=1}^{J}\left(1-\tilde{T}_{Ni}^{+}\right)^{\lambda_i}\right], \\ \left[\prod_{i=1}^{J}\left(\tilde{I}_{Ni}^{-}\right)^{\lambda_i}, \prod_{i=1}^{J}\left(\tilde{I}_{Ni}^{+}\right)^{\lambda_i}\right], \left[\prod_{i=1}^{J}\left(\tilde{F}_{Ni}^{-}\right)^{\lambda_i}, \prod_{i=1}^{J}\left(\tilde{F}_{Ni}^{+}\right)^{\lambda_i}\right] \end{pmatrix}, \quad (18)$$

$$ITFNNWGA(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_J) = \prod_{i=1}^{J} \tilde{a}_i^{\lambda_i}$$
$$= \left(\begin{array}{l} \left\langle \prod_{i=1}^{J} g_{i1}^{\lambda_i}, \prod_{i=1}^{J} g_{i2}^{\lambda_i}, \prod_{i=1}^{J} g_{i3}^{\lambda_i}, \prod_{i=1}^{J} g_{i4}^{\lambda_i} \right\rangle; \left[\prod_{i=1}^{J} \left(\tilde{T}_{Ni}^-\right)^{\lambda_i}, \prod_{i=1}^{J} \left(\tilde{T}_{Ni}^+\right)^{\lambda_i} \right], \\ \left[1-\prod_{i=1}^{J}\left(1-\tilde{I}_{Ni}^-\right)^{\lambda_i}, 1-\prod_{i=1}^{J}\left(1-\tilde{I}_{Ni}^+\right)^{\lambda_i}\right], \left[1-\prod_{i=1}^{J}\left(1-\tilde{F}_{Ni}^-\right)^{\lambda_i}, 1-\prod_{i=1}^{J}\left(1-\tilde{F}_{Ni}^+\right)^{\lambda_i}\right] \end{array} \right), \quad (19)$$

$$S'(\tilde{a}) = \frac{1}{24}\left((g_1+g_2+g_3+g_4)\times\left(4+\tilde{T}_N^- + \tilde{T}_N^+ - \tilde{I}_N^- - \tilde{I}_N^+ - \tilde{F}_N^- - \tilde{F}_N^+\right)\right), S'(\tilde{a}) \in [0,1]. \quad (20)$$

5. DM Model with C-ITFNNs

In this section, we use the C-ITFNNWAA and C-ITFNNWGA operators and the score function to establish the multi-attribute DM model in the C-ITFNN circumstance.

Let a set of alternatives be $E = \{e_1, e_2, …, e_p\}$ and a set of attributes be $G = \{g_1, g_2, …, g_J\}$. The weight vector of the attributes is $\lambda = (\lambda_1, \lambda_2, …, \lambda_J)$, which indicates the importance of various attributes. Decision makers can give their satisfactory linguistic evaluation values and credibility degrees through the set of linguistic terms L = {Very bad, Bad, Fairly bad, Medium, Fairly good, Good, Very good}. In view of Table 1, we can obtain the linguistic values of each alternative er (r = 1, 2, …, p) over the attributes g_i (i = 1, 2, …, J) and express them as the following C-ITFNN:

$$\tilde{a}_{ri} = \left(\begin{array}{l} \left\langle (g_{ri1}, g_{ri2}, g_{ri3}, g_{ri4}); \left[\tilde{T}_{Nri}^-, \tilde{T}_{Nri}^+\right], \left[\tilde{I}_{Nri}^-, \tilde{I}_{Nri}^+\right], \left[\tilde{F}_{Nri}^-, \tilde{F}_{Nri}^+\right] \right\rangle, \\ \left\langle (h_{ri1}, h_{ri2}, h_{ri3}, h_{ri4}); \left[\tilde{T}_{Lri}^-, \tilde{T}_{Lri}^+\right], \left[\tilde{I}_{Lri}^-, \tilde{I}_{Lri}^+\right], \left[\tilde{F}_{Lri}^-, \tilde{F}_{Lri}^+\right] \right\rangle \end{array} \right).$$

Thus, we can establish the C-ITFNN decision matrix $N = (\tilde{a}_{ri})_{p\times J}$.

Table 1. Linguistic terms and linguistic values of ITFNNs

Linguistic term	Linguistic value of ITFNNs
Very bad (VB)	<(0.1,0.1,0.1,0.1);[0.1,0.2],[0.9,1.0],[0.9,1.0]>
Bad (B)	<(0.2,0.3,0.4,0.5);[0.2,0.3],[0.8,0.9],[0.8,0.9]>
Fairy bad (FB)	<(0.3,0.4,0.5,0.6);[0.3,0.4],[0.7,0.8],[0.7,0.8]>
Medium (M)	<(0.4,0.5,0.6,0.7);[0.5,0.6],[0.4,0.6],[0.4,0.6]>
Fairy good (FG)	<(0.5,0.6,0.7,0.8);[0.7,0.8],[0.2,0.3],[0.2,0.3]>
Good (G)	<(0.6,0.7,0.8,0.9);[0.8,0.9],[0.1,0.2],[0.1,0.2]>
Very good (VG)	<(1.0,1.0,1.0,1.0);[0.9,1],[0,0.1],[0,0.1]>

Then, we use C-ITFNNWAA and C-ITFNNWGA operators and the score function to establish the multi-attribute DM model with C-ITFNN information and to select the best alternative. The DM process is indicated below.

Step 1: Give the aggregated C-ITFNNs $\tilde{a}_r$ for $e_r(r = 1, 2, ..., p)$ by the C-ITNNWAA operator or the C-ITNNWGA operator:

$$\tilde{a}_r = C-ITFNNWAA(\tilde{a}_{r1},\tilde{a}_{r2},...\tilde{a}_{rJ}) = \sum_{i=1}^{J}\lambda_i\tilde{a}_{rJ}$$
$$= \left(\begin{array}{l} \left\langle \begin{array}{l} \left(\sum_{i=1}^{J}\lambda_i g_{ri1},\sum_{i=1}^{J}\lambda_i g_{ri2},\sum_{i=1}^{J}\lambda_i g_{ri3},\sum_{i=1}^{J}\lambda_i g_{ri4}\right); \\ \left[1-\prod_{i=1}^{J}\left(1-\tilde{T}_{Nri}^{-}\right)^{\lambda_i},1-\prod_{i=1}^{J}\left(1-\tilde{T}_{Nri}^{+}\right)^{\lambda_i}\right], \\ \left[\prod_{i=1}^{J}\tilde{I}_{Nri}^{-\ \lambda_i},\prod_{i=1}^{J}\tilde{I}_{Nri}^{+\ \lambda_i}\right],\left[\prod_{i=1}^{J}\tilde{F}_{Nri}^{-\ \lambda_i},\prod_{i=1}^{J}\tilde{F}_{Nri}^{+\ \lambda_i}\right] \end{array} \right\rangle, \\ \left\langle \begin{array}{l} \left(\sum_{i=1}^{J}\lambda_i h_{ri1},\sum_{i=1}^{J}\lambda_i h_{ri2},\sum_{i=1}^{J}\lambda_i h_{ri3},\sum_{i=1}^{J}\lambda_i h_{ri4}\right); \\ \left[1-\prod_{i=1}^{J}\left(1-\tilde{T}_{Lri}^{-}\right)^{\lambda_i},1-\prod_{i=1}^{J}\left(1-\tilde{T}_{Lri}^{+}\right)^{\lambda_i}\right], \\ \left[\prod_{i=1}^{J}\tilde{I}_{Lri}^{-\ \lambda_i},\prod_{i=1}^{J}\tilde{I}_{Lri}^{+\ \lambda_i}\right],\left[\prod_{i=1}^{J}\tilde{F}_{Lri}^{-\ \lambda_i},\prod_{i=1}^{J}\tilde{F}_{Lri}^{+\ \lambda_i}\right] \end{array} \right\rangle \end{array} \right) \tag{21}$$

or

$$\tilde{a}_r = C-ITFNNWGA\left(\tilde{a}_{r1},\tilde{a}_{r2},...\tilde{a}_{rJ}\right) = \prod_{i=1}^{J}\tilde{a}_{ri}^{\lambda_i}$$
$$= \left(\begin{array}{l} \left\langle \begin{array}{l} \left(\prod_{i=1}^{J} g_{ri1}^{\lambda_i},\prod_{i=1}^{J} g_{ri2}^{\lambda_i},\prod_{i=1}^{J} g_{ri3}^{\lambda_i},\prod_{i=1}^{J} g_{ri4}^{\lambda_i}\right); \\ \left[\prod_{i=1}^{J}\tilde{T}_{Nri}^{-\ \lambda_i},\prod_{i=1}^{J}\tilde{T}_{Nri}^{+\ \lambda_i}\right],\left[1-\prod_{i=1}^{J}\left(1-\tilde{I}_{Nri}^{-}\right)^{\lambda_i},1-\prod_{i=1}^{J}\left(1-\tilde{I}_{Nri}^{+}\right)^{\lambda_i}\right], \\ \left[1-\prod_{i=1}^{J}\left(1-\tilde{F}_{Nri}^{-}\right)^{\lambda_i},1-\prod_{i=1}^{J}\left(1-\tilde{F}_{Nri}^{+}\right)^{\lambda_i}\right] \end{array} \right\rangle, \\ \left\langle \begin{array}{l} \left(\prod_{i=1}^{J} h_{ri1}^{\lambda_i},\prod_{i=1}^{J} h_{ri2}^{\lambda_i},\prod_{i=1}^{J} h_{ri3}^{\lambda_i},\prod_{i=1}^{J} h_{ri4}^{\lambda_i}\right); \\ \left[\prod_{i=1}^{J}\tilde{T}_{Lri}^{-\ \lambda_i},\prod_{i=1}^{J}\tilde{T}_{Lri}^{+\ \lambda_i}\right],\left[1-\prod_{i=1}^{J}\left(1-\tilde{I}_{Lri}^{-}\right)^{\lambda_i},1-\prod_{i=1}^{J}\left(1-\tilde{I}_{Lri}^{+}\right)^{\lambda_i}\right], \\ \left[1-\prod_{i=1}^{J}\left(1-\tilde{F}_{Lri}^{-}\right)^{\lambda_i},1-\prod_{i=1}^{J}\left(1-\tilde{F}_{Lri}^{+}\right)^{\lambda_i}\right] \end{array} \right\rangle \end{array} \right). \tag{22}$$

Step 2: Calculate the score values of the aggregated values $\tilde{a}_r$ by Eq. (10).

Step 3: Sort the alternatives and determine the optimal one with the largest score value.

Step 4: End.

6. Actual DM Example

6.1 DM Case of LCDSs

With the rapid development of China economy in recent years, the scale of engineering activities has become larger and larger, and the problem of landslides has become more and more serious. This section gives an actual DM case of LCDSs to illustrate the feasibility and applicability of the proposed DM model in the C-ITFNN scenario. The terrain of the landslide area is high in the west and low in the east, high in the north and low in the south, generally inclined from west to east, and the terrain fluctuates greatly. Referring to the experience of landslide control, four potential LVDSs are provided for some landslide treatment in Shaoxing City, China, which are indicated by a set of the four alternatives $E = \{e_1, e_2, e_3, e_4\}$. In the scheme e_1, graded slope toes and retaining wall and lattices are used in the central and northern areas, while slope unloading, anti-sliding piles and anchor-cable anti-sliding piles in the southern area. In the scheme e_2, graded slope toes, retaining wall, and anchor rod lattice are used in the central and northern areas, and double-row anti-sliding piles and anchor-cable anti-sliding piles are used in the southern area. In the scheme e_3, graded slope toes and supporting anti-sliding piles are used in the central and northern areas, and anchor-cable anti-sliding piles are used in the southern area. In the scheme e_4, graded slope toes and retaining walls and anchor rod lattices are used in the central and northern areas, and anchor cable anti-sliding piles are used in the southern area. Then, a satisfactory evaluation of the four alternatives is subject to the three important conditions (attributes): technical difficulty (g_1), environmental impact (g_2), and governance cost (g_3). The importance of the three conditions is assigned by the weight vector λ = (0.2, 0.3, 0.5).

In the DM issue of LCDSs, experts are invited to give the satisfactory degrees and credibility levels of the four alternatives with respect to the three attributes by the linguistic terms obtained from the set of linguistic terms L = {Very bad, Bad, Fairly bad, Medium, Fairly good, Good, Very good}, then the given linguistic terms are shown in Table 2.

Table 2. Linguistic terms of the satisfactory degrees and credibility levels

	g_1	g_2	g_3
e_1	(B, M)	(FB, G)	(FG, G)
e_2	(G, VG)	(FG, G)	(FB, G)
e_3	(B, FG)	(G, G)	(M, VG)
e_4	(FG, M)	(VB, FG)	(FB, FG)

Thus, the linguistic terms of the satisfactory degrees and credibility levels in Table 2 can be converted into C-ITFNNs in view of the corresponding linguistic values in Table 1, which are constructed as the decision matrix:

$$N = \begin{bmatrix} \begin{pmatrix} \langle (0.2,0.3,0.4,0.5);[0.2,0.3],[0.8,0.9],[0.8,0.9]\rangle, \\ \langle (0.4,0.5,0.6,0.7);[0.5,0.6],[0.4,0.6],[0.4,0.6]\rangle \end{pmatrix} & \begin{pmatrix} \langle (0.3,0.4,0.5,0.6);[0.3,0.4],[0.7,0.8],[0.7,0.8]\rangle, \\ \langle (0.6,0.7,0.8,0.9);[0.8,0.9],[0.1,0.2],[0.1,0.2]\rangle \end{pmatrix} & \begin{pmatrix} \langle (0.5,0.6,0.7,0.8);[0.7,0.8],[0.2,0.3],[0.2,0.3]\rangle, \\ \langle (0.6,0.7,0.8,0.9);[0.8,0.9],[0.1,0.2],[0.1,0.2]\rangle \end{pmatrix} \\ \begin{pmatrix} \langle (0.6,0.7,0.8,0.9);[0.8,0.9],[0.1,0.2],[0.1,0.2]\rangle, \\ \langle (1.0,1.0,1.0,1.0);[0.9,1.0],[0,0.1],[0,0.1]\rangle \end{pmatrix} & \begin{pmatrix} \langle (0.5,0.6,0.7,0.8);[0.7,0.8],[0.2,0.3],[0.2,0.3]\rangle, \\ \langle (0.6,0.7,0.8,0.9);[0.8,0.9],[0.1,0.2],[0.1,0.2]\rangle \end{pmatrix} & \begin{pmatrix} \langle (0.3,0.4,0.5,0.6);[0.3,0.4],[0.7,0.8],[0.7,0.8]\rangle, \\ \langle (0.6,0.7,0.8,0.9);[0.8,0.9],[0.1,0.2],[0.1,0.2]\rangle \end{pmatrix} \\ \begin{pmatrix} \langle (0.2,0.3,0.4,0.5);[0.2,0.3],[0.8,0.9],[0.8,0.9]\rangle, \\ \langle (0.5,0.6,0.7,0.8);[0.7,0.8],[0.2,0.3],[0.2,0.3]\rangle \end{pmatrix} & \begin{pmatrix} \langle (0.6,0.7,0.8,0.9);[0.8,0.9],[0.1,0.2],[0.1,0.2]\rangle, \\ \langle (0.6,0.7,0.8,0.9);[0.8,0.9],[0.1,0.2],[0.1,0.2]\rangle \end{pmatrix} & \begin{pmatrix} \langle (0.4,0.5,0.6,0.7);[0.5,0.6],[0.4,0.6],[0.4,0.6]\rangle, \\ \langle (1.0,1.0,1.0,1.0);[0.9,1.0],[0,0.1],[0,0.1]\rangle \end{pmatrix} \\ \begin{pmatrix} \langle (0.5,0.6,0.7,0.8);[0.7,0.8],[0.2,0.3],[0.2,0.3]\rangle, \\ \langle (0.4,0.5,0.6,0.7);[0.5,0.6],[0.4,0.6],[0.4,0.6]\rangle \end{pmatrix} & \begin{pmatrix} \langle (0.1,0.1,0.1,0.1);[0.1,0.2],[0.9,1],[0.9,1]\rangle, \\ \langle (0.5,0.6,0.7,0.8);[0.7,0.8],[0.2,0.3],[0.2,0.3]\rangle \end{pmatrix} & \begin{pmatrix} \langle (0.3,0.4,0.5,0.6);[0.3,0.4],[0.7,0.8],[0.7,0.8]\rangle, \\ \langle (0.5,0.6,0.7,0.8);[0.7,0.8],[0.2,0.3],[0.2,0.3]\rangle \end{pmatrix} \end{bmatrix}.$$

To deal with the DM problem, we give the following decision process.

Step 1: Utilize the C-ITFNNWAA operator of Eq. (21) and get the aggregated values $\tilde{a}_r$ for e_r (r =1, 2, …, p):

$$\tilde{a}_1 = \begin{pmatrix} \langle (0.38,0.48,0.58,0.68);[0.5293,0.6427],[0.3843,0.5016],[0.3843,0.5016]\rangle, \\ \langle (0.56,0.66,0.76,0.86);[0.7598,0.868],[0.132,0.2491],[0.132,0.2491]\rangle \end{pmatrix},$$

$$\tilde{a}_2 = \begin{pmatrix} \langle (0.42,0.52,0.62,0.72);[0.5774,0.6984],[0.3257,0.4517],[0.3257,0.4517]\rangle, \\ \langle (0.68,0.76,0.84,0.92);[0.8259,1],[0,0.1741],[0,0.1741]\rangle \end{pmatrix},$$

$$\tilde{a}_3 = \begin{pmatrix} \langle (0.42, 0.48, 0.58, 0.68); [0.5827, 0.7048], [0.3031, 0.468], [0.3031, 0.468] \rangle, \\ \langle (0.78, 0.83, 0.88, 0.93); [0.8466, 1], [0, 0.1534], [0, 0.1534] \rangle \end{pmatrix},$$

$$\tilde{a}_4 = \begin{pmatrix} \langle (0.28, 0.48, 0.58, 0.68); [0.3628, 0.4749], [0.5875, 0.703], [0.5875, 0.703] \rangle, \\ \langle (0.48, 0.58, 0.68, 0.78); [0.6677, 0.7703], [0.2297, 0.3446], [0.2297, 0.3446] \rangle \end{pmatrix}.$$

Or use the C-ITFNNWGA operator of Eq. (22) and get the aggregated values $\tilde{a}_r$ for e_r (r =1, 2, …, p):

$$\tilde{a}_1 = \begin{pmatrix} \langle (0.3571, 0.4625, 0.5658, 0.668); [0.4226, 0.5341], [0.5483, 0.6743], [0.5483, 0.6743] \rangle, \\ \langle (0.5533, 0.6544, 0.7553, 0.8559); [0.7282, 0.8299], [0.1701, 0.3036], [0.1701, 0.3036] \rangle \end{pmatrix},$$

$$\tilde{a}_2 = \begin{pmatrix} \langle (0.4017, 0.5052, 0.6076, 0.7093); [0.4707, 0.5792], [0.4984, 0.6157], [0.4984, 0.6157] \rangle, \\ \langle (0.6645, 0.7518, 0.8365, 0.9192); [0.8191, 0.9192], [0.0808, 0.1809], [0.0808, 0.1809] \rangle \end{pmatrix},$$

$$\tilde{a}_3 = \begin{pmatrix} \langle (0.3933, 0.4994, 0.6031, 0.7057); [0.4793, 0.5899], [0.4561, 0.6268], [0.4561, 0.6268] \rangle, \\ \langle (0.7469, 0.8113, 0.8709, 0.9266); [0.8262, 0.9266], [0.0734, 0.1738], [0.0734, 0.1738] \rangle \end{pmatrix},$$

$$\tilde{a}_4 = \begin{pmatrix} \langle (0.239, 0.2862, 0.33, 0.3713); [0.2556, 0.3732], [0.7375, 1], [0.7375, 1] \rangle, \\ \langle (0.4782, 0.5785, 0.6787, 0.7789); [0.6544, 0.7553], [0.2447, 0.3741], [0.2447, 0.3741] \rangle \end{pmatrix}.$$

Step 2: Use Eq. (10) and calculate the score values of the aggregated values $\tilde{a}_r$ (r =1, 2, …, p):

$S(\tilde{a}_1) = 0.1729$, $S(\tilde{a}_2) = 0.2582$, $S(\tilde{a}_3) = 0.2661$, and $S(\tilde{a}_4) = 0.0855$.

Or $S(\tilde{a}_1) = 0.1164$, $S(\tilde{a}_2) = 0.1802$, $S(\tilde{a}_3) = 0.1954$, and $S(\tilde{a}_4) = 0.0258$.

Step 3: Give the sorting of the four LCDSs: $e_3 > e_2 > e_1 > e_4$. Hence, the optimal choice is the scheme e_3.

It can be found that the sorting and optimal choice results obtained by the C-ITFNNWAAA operator and the C-ITFNNWGA operator are consistent.

6.2 Comparison of the Proposed DM Model with Previous DM Models in the Scenarios of C-TFNNs, ITFNNs, TFNNs, and SVNNs

To indicate the efficiency of the proposed DM model in the C-ITFNN scenarios, we compare the DM model proposed in this paper with the previous DM models in the C-TFNN, ITFNN, TFNN and SVNN scenarios. Since the previous DM models cannot perform the DM issue of C-ITFNNs, we only consider the situations of C-TFNNs, ITFNNs, TFNNs, and SVNNs as four special cases of C-ITFNNs for convenient comparison. Therefore, we assume that all interval values in the C-ITFNN decision matrix N are converted into their average values to produce the C-TFNN matrix $N_{C\text{-}TFNN}$:

$$N_{C-TFNN} = \begin{bmatrix} \begin{pmatrix} \langle (0.2, 0.3, 0.4, 0.5); 0.25, 0.85, 0.85 \rangle, \\ \langle (0.4, 0.5, 0.6, 0.7); 0.55, 0.5, 0.5 \rangle \end{pmatrix} & \begin{pmatrix} \langle (0.3, 0.4, 0.5, 0.6); 0.35, 0.75, 0.75 \rangle, \\ \langle (0.6, 0.7, 0.8, 0.9); 0.85, 0.15, 0.15 \rangle \end{pmatrix} & \begin{pmatrix} \langle (0.5, 0.6, 0.7, 0.8); 0.75, 0.25, 0.25 \rangle, \\ \langle (0.6, 0.7, 0.8, 0.9); 0.85, 0.15, 0.15 \rangle \end{pmatrix} \\ \begin{pmatrix} \langle (0.6, 0.7, 0.8, 0.9); 0.85, 0.15, 0.15 \rangle, \\ \langle (1.0, 1.0, 1.0, 1.0); 0.95, 0.05, 0.05 \rangle \end{pmatrix} & \begin{pmatrix} \langle (0.5, 0.6, 0.7, 0.8); 0.75, 0.25, 0.25 \rangle, \\ \langle (0.6, 0.7, 0.8, 0.9); 0.85, 0.15, 0.15 \rangle \end{pmatrix} & \begin{pmatrix} \langle (0.3, 0.4, 0.5, 0.6); 0.35, 0.75, 0.75 \rangle, \\ \langle (0.6, 0.7, 0.8, 0.9); 0.85, 0.15, 0.15 \rangle \end{pmatrix} \\ \begin{pmatrix} \langle (0.2, 0.3, 0.4, 0.5); 0.25, 0.85, 0.85 \rangle, \\ \langle (0.5, 0.6, 0.7, 0.8); 0.75, 0.25, 0.25 \rangle \end{pmatrix} & \begin{pmatrix} \langle (0.6, 0.7, 0.8, 0.9); 0.85, 0.15, 0.15 \rangle, \\ \langle (0.6, 0.7, 0.8, 0.9); 0.85, 0.15, 0.15 \rangle \end{pmatrix} & \begin{pmatrix} \langle (0.4, 0.5, 0.6, 0.7); 0.55, 0.5, 0.5 \rangle, \\ \langle (1.0, 1.0, 1.0, 1.0); 0.95, 0.05, 0.05 \rangle \end{pmatrix} \\ \begin{pmatrix} \langle (0.5, 0.6, 0.7, 0.8); 0.75, 0.25, 0.25 \rangle, \\ \langle (0.4, 0.5, 0.6, 0.7); 0.55, 0.5, 0.5 \rangle \end{pmatrix} & \begin{pmatrix} \langle (0.1, 0.1, 0.1, 0.1); 0.15, 0.95, 0.95 \rangle, \\ \langle (0.5, 0.6, 0.7, 0.8); 0.75, 0.25, 0.25 \rangle \end{pmatrix} & \begin{pmatrix} \langle (0.3, 0.4, 0.5, 0.6); 0.35, 0.75, 0.75 \rangle, \\ \langle (0.5, 0.6, 0.7, 0.8); 0.75, 0.25, 0.25 \rangle \end{pmatrix} \end{bmatrix}.$$

If we do not consider the credibility levels in *N*, *N* becomes the ITFNN matrix N_{ITFNN}:

$$N_{ITFNN}=\begin{bmatrix} \langle(0.2,0.3,0.4,0.5);[0.2,0.3],[0.8,0.9],[0.8,0.9]\rangle & \langle(0.3,0.4,0.5,0.6);[0.3,0.4],[0.7,0.8],[0.7,0.8]\rangle & \langle(0.5,0.6,0.7,0.8);[0.7,0.8],[0.2,0.3],[0.2,0.3]\rangle \\ \langle(0.6,0.7,0.8,0.9);[0.8,0.9],[0.1,0.2],[0.1,0.2]\rangle & \langle(0.5,0.6,0.7,0.8);[0.7,0.8],[0.2,0.3],[0.2,0.3]\rangle & \langle(0.3,0.4,0.5,0.6);[0.3,0.4],[0.7,0.8],[0.7,0.8]\rangle \\ \langle(0.2,0.3,0.4,0.5);[0.2,0.3],[0.8,0.9],[0.8,0.9]\rangle & \langle(0.6,0.7,0.8,0.9);[0.8,0.9],[0.1,0.2],[0.1,0.2]\rangle & \langle(0.4,0.5,0.6,0.7);[0.5,0.6],[0.4,0.6],[0.4,0.6]\rangle \\ \langle(0.5,0.6,0.7,0.8);[0.7,0.8],[0.2,0.3],[0.2,0.3]\rangle & \langle(0.1,0.1,0.1,0.1);[0.1,0.2],[0.9,1],[0.9,1]\rangle & \langle(0.3,0.4,0.5,0.6);[0.3,0.4],[0.7,0.8],[0.7,0.8]\rangle \end{bmatrix}.$$

If we do not consider the credibility levels in $N_{C\text{-}TFNN}$, $N_{C\text{-}TFNN}$ becomes the TFNN matrix N_{TFNN}:

$$N_{TFNN}=\begin{bmatrix} ((0.2,0.3,0.4,0.5);0.25,0.85,0.85) & ((0.3,0.4,0.5,0.6);0.35,0.75,0.75) & ((0.5,0.6,0.7,0.8);0.75,0.25,0.25) \\ ((0.6,0.7,0.8,0.9);0.85,0.15,0.15) & ((0.5,0.6,0.7,0.8);0.75,0.25,0.25) & ((0.3,0.4,0.5,0.6);0.35,0.75,0.75) \\ ((0.2,0.3,0.4,0.5);0.25,0.85,0.85) & ((0.6,0.7,0.8,0.9);0.85,0.15,0.15) & ((0.4,0.5,0.6,0.7);0.55,0.5,0.5) \\ ((0.5,0.6,0.7,0.8);0.75,0.25,0.25) & ((0.1,0.1,0.1,0.1);0.15,0.95,0.95) & ((0.3,0.4,0.5,0.6);0.35,0.75,0.75) \end{bmatrix}.$$

If we do not consider TFNs in N_{TFNN}, N_{TFNN} becomes the SVNN matrix N_{SVNN}:

$$N_{SVNN}=\begin{bmatrix} (0.25,0.85,0.85) & (0.35,0.75,0.75) & (0.75,0.25,0.25) \\ (0.85,0.15,0.15) & (0.75,0.25,0.25) & (0.35,0.75,0.75) \\ (0.25,0.85,0.85) & (0.85,0.15,0.15) & (0.55,0.5,0.5) \\ (0.75,0.25,0.25) & (0.15,0.95,0.95) & (0.35,0.75,0.75) \end{bmatrix}.$$

In the scenarios of C-TFNNs, ITFNNs, TFNNs, and SVNNs, we apply Eqs. (1), (2), (18), (19), (4), (5), (7), and (8) for the decision matrices of C-TFNNs, ITFNNs, TFNNs, and SVNNs to obtain their aggregated values. Then, the score values of their aggregated values are obtained by the score functions of Eqs. (3), (6), (9), and (20) in the corresponding scenarios. For clear comparison, all decision results are given in Table 3.

Table 3. Decision results based on different DM models in the scenarios of SVNNs, TFNNs, ITFNNs, C-TFNNs, and C-ITFNNs

Method	Score value	Sorting	Optimal one
DM model using the SVNNWAA operator [19]	0.5657,0.618,0.6223,0.3752	$e_3 > e_2 > e_1 > e_4$	e_3
DM model using the SVNNWGA operator [19]	0.4203,0.4711,0.4856,0.2335	$e_3 > e_2 > e_1 > e_4$	e_3
DM model using the TFNNWAA operator [30]	0.2998, 0.3523,0.336,0.1895	$e_2 > e_3 > e_1 > e_4$	e_2
DM model using the TFNNWGA operator [30]	0.2158,0.2619,0.2673,0.0716	$e_3 > e_2 > e_1 > e_4$	e_3
DM model using the ITFNNWAA operator [34]	0.3004,0.3535,0.3371,0.1899	$e_2 > e_3 > e_1 > e_4$	e_2
DM model using the ITFNNWAA operator [34]	0.2149,0.2615,0.2663,0.059	$e_3 > e_2 > e_1 > e_4$	e_3
DM model using the C-TFNNWAA operator [37]	0.1725,0.2479,0.2597,0.0853	$e_3 > e_2 > e_1 > e_4$	e_3
DM model using the C-TFNNWGA operator [37]	0.1170,0.1805,0.1965,0.0313	$e_3 > e_2 > e_1 > e_4$	e_3
DM model using the C-ITFNNWAA operator	0.1729,0.2582,0.2661,0.0855	$e_3 > e_2 > e_1 > e_4$	e_3
DM model using the C-ITFNNWGA operator	0.1164,0.1802,0.1954,0.0258	$e_3 > e_2 > e_1 > e_4$	e_3

From the sorting results in Table 3, it can be seen that there are differences in the DM results obtained based on different aggregation operators of SVNNs, TFNNs, C-TFNNs, and C-ITFNNs. In the cases of TFNNs and ITFNNs, the optimal schemes obtained by different aggregation operators are inconsistent, where the optimal scheme obtained by using the weighted arithmetic averaging operators is e_2 and the optimal scheme obtained by using the weighted geometric averaging operators is e_3. Moreover, the optimal DM result obtained in the scenarios of SVNNs, C-TFNNs and C-ITFNNs is e_3. However, the level of credibility plays a key role in the sorting of the alternatives because it can ensure the credibility of the assessment information of TFNNs and ITFNNs. The previous DM models with SVNNs, TFNNs, ITFNNs [19, 30, 34] may result in unreasonable/distorted DM results because they are difficult to ensure the credibility of SVNNs, TFNNs, and ITFNNs. In addition, in the scenarios of C-TFNNs and C-ITFNNs, the proposed DM model of C-ITFNNs obtains the same DM results as the DM model of C-TFNNs [37], which also proves the rationality and efficiency of the proposed DM model in the scenario of C-ITFNNs. The reason is that the C-TFNN matrix obtained by taking the average value of the interval values in C-ITFNNs is only a special case of the C-ITFNN matrix. Therefore, it can be seen that the proposed DM model of C-ITFNNs generalizes the previous DM model of C-TFNNs [37], while the previous DM model of C-TFNNs [37] is only a special case of the proposed DM model of C-ITFNNs. In general, the proposed DM model of C-ITFNNs makes DM applications more general and practical, demonstrating the clear advantages in the setting of C-ITFNNs.

7. Conclusion

As an extension of C-TFNNs, this paper first proposed C-ITFNNs in view of ITFNNs and credibility levels, which are expressed by an ordered pair of ITFNNs. Then, we defined some operational laws of C-ITFNNs and the score function of C-ITFNN and presented the C-ITFNNWAA and C-ITFNNWGA operators and their properties. Furthermore, the C-ITFNNWAA and C-ITFNNWGA operators and the score function were used for a multi-attribute DM model of C-ITFNNs. Lastly, the proposed DM model was applied to the DM case of LCDSs in the scenario of C-ITFNNs and verified its feasibility. By comparative analysis of the different DM models in the scenarios of C-ITFNNs, C-TFNNs, TFNNs, and SVNNs, the proposed DM model revealed the superiority of DM generalization in the scenario of C-ITFNNs since the previous DM models are only the special cases of the proposed DM model of C-ITFNNs.

Generally, the information representation, operation and DM techniques of C-ITFNNs reveal their original contributions in this study. Then, the main superiority of our new DM model is that it not only compensates for the gap of existing DM models, but also is more reliable and versatile than existing DM models. As future research, the techniques proposed in this paper can be extended to slope stability/risk assessment, mine risk/safety assessment, and image processing in a C-ITFNN circumstance.

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Method for Comprehensive Evaluation of Urban Smart Traffic Management System Based on the 2-tuple Linguistic Neutrosophic Numbers

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Abstract: In the process of the rapid development of big data and the Internet of Things in recent years, in order to create a "strong transportation" construction goals, intelligent transportation projects have become the key carrier of the development of China's transportation industry, the national and local transportation level, economic development and the improvement of people's living standards have an important role. However, although the construction of intelligent transportation projects around the world is in full swing, but the actual operation effect is not ideal. The comprehensive evaluation of urban smart traffic management systems (USTMS) is a classical MADM issues. In this paper, the MADM issues are studied with defined 2-tuple linguistic neutrosophic sets (2TLNSs). Then, connected traditional GRA with 2TLNSs, the 2TLNN-GRA method is elaborated for MADM. Finally, an example for comprehensive evaluation of USTMS was given and some comparisons was elaborated the 2TLNN-GRA method

Keywords: Multiple attribute decision making (MADM); 2TLNS; GRA method; Urban smart traffic management system(USTMS)

1. Introduction

Generally speaking, decision making refers to making a decision based on the realization of conditions, whether it is a major decision made by the state or corporate policy, or a decision made by people in ordinary daily life[1-3]. Therefore, decision making is widely elaborated in various fields of life and production, and has gained more and more attention, such as a company needs to improve a new product, a government department bidding activity, or an individual's choice of occupation or the purchase of goods, all of which are of decision making significance[4-7]. In fact, human beings inevitably face a variety of complex decision-making problems, involving artificial intelligence and other fields, network data, granularity of computing[8-11]. Nowadays, decision making is one of the quite common activities in people's daily life, which aims at ranking a limited number of alternatives by the decision maker according to the value of the evaluation index of each alternative[12-16]. Multi-Attribute Decision Making (MADM) is a branch of decision making that is considered as a cognitive-based human activity. The first step in decision-making is to build mathematical models to describe the uncertain information from different levels, and MADM is one of the processes to find the best solution among all feasible alternatives[17-19]. According to certain attributes, decision information of all alternatives, their corresponding values are represented by some precise value, however, it is believed that most real-life decisions are made in environments with inaccurate or imprecise goals and constraints, which are inherently ambiguous and thus cannot represent preferences with precise values, and most decision makers, due to time decision pressure and lack of full data, may have limited information processing capabilities [20-27]. To cope with this situation, fuzzy set theory has been widely used to deal with uncertainty and vague information [28-32]. After the successful application of fuzzy set decision theory, researchers have worked on the extensions and applications of fuzzy set theory, among which intuitionistic fuzzy sets(IFSs) [33-43]and neutrosophic sets (NSs) [44-55] theory is one of the most important extensions and has been fully applied to MADM. Furthermore, Wang, Wei and Wei [56] devised the 2TLNSs which fuzzy decision information are elaborated with 2TLs[57-63].

With the acceleration of China's urbanization process, urban population, housing and industrial agglomeration on a large scale, urban traffic problems are becoming more and more prominent: traffic congestion is serious, resulting in increased travel time and huge energy consumption; traffic safety problems are serious, accidents are frequent; vehicle emissions and environmental pollution and traffic noise pollution more serious; the increase in vehicle ownership brings parking facilities gradually intensify the contradiction between supply and demand, etc.. Intelligent transportation is proposed in this context, but whether the development of intelligent transportation in a city meets the target requirements requires a complete evaluation system to judge. At present, the concept of intelligent transportation, the technology required for intelligent transportation, intelligent transportation evaluation system and standards are not in-depth research, how to develop intelligent transportation, how to establish an appropriate intelligent transportation evaluation system is an urgent issue to be solved. The problems of comprehensive evaluation of USTMS are MADM problems. In this elaborated paper, the 2TLNN-GRA is constructed based on GRA [64-70] and 2TLNSs. Finally, an example for comprehensive evaluation of USTMS was given and some comparisons were elaborated the 2TLNN-GRA. In order to conduct so, the reminder of such paper elaborates. The definition of 2TLNNSs is elaborated in Sec. 2. The 2TLNN-GRA is elaborated for MADM are elaborated in Sec. 3. An example for comprehensive evaluation of USTMS is elaborated the 2TLNN-GRA in Sec. 4. Sec. 5 lists the conclusions.

2. Preliminaries

Wang et al. [56] elaborated the 2TLNSs.

Definition 1 [56]. Let $f\delta = \{ fs_i \,|\, i = 0,1,2,\cdots,H \}$ be the LTSs. The fs_i elaborates a possible linguistic value, and $f\delta = \{ fs_0 = \textit{exceedingly terrible}, fs_1 = \textit{very terrible}, fs_2 = \textit{terrible}, fs_3 = \textit{medium}, fs_4 = \textit{well}, fs_5 = \textit{very well}, fs_6 = \textit{exceedingly well}.\}$, then the 2TLNSs is described as:

$$f\delta = \left\langle \left(fs_t, f\alpha \right), \left(fs_i, f\beta \right), \left(fs_f, f\chi \right) \right\rangle \tag{1}$$

where $\Delta^{-1}\left(fs_t, f\alpha, \right), \Delta^{-1}\left(fs_i, f\beta \right), \Delta^{-1}\left(fs_f, f\chi \right) \in [0, H]$ elaborate truth membership, indeterminacy membership and falsity membership with 2-tuple linguistic decision information, $0 \le \Delta^{-1}\left(fs_\alpha, f\phi \right) + \Delta^{-1}\left(fs_\beta, f\varphi \right) + \Delta^{-1}\left(fs_\chi, f\gamma \right) \le 3H$.

Definition 2[56]. Let $f\delta_1 = \left\langle \left(fs_{t_1}, f\alpha_1, \right), \left(fs_{i_1}, f\beta_1 \right), \left(fs_{f_1}, f\chi_1 \right) \right\rangle$, $f\delta_2 = \left\langle \left(fs_{t_2}, f\alpha_2, \right), \left(fs_{i_2}, f\beta_2 \right), \left(fs_{f_2}, f\chi_2 \right) \right\rangle$, the given operation is elaborated:

(1) $$f\delta_1 \oplus f\delta_2 = \left\{ \begin{array}{l} \Delta\left(H\left(\dfrac{\Delta^{-1}\left(fs_{t_1}, f\alpha_1, \right)}{H} + \dfrac{\Delta^{-1}\left(fs_{t_2}, f\alpha_2, \right)}{H} - \dfrac{\Delta^{-1}\left(fs_{t_1}, f\alpha_1, \right)}{H} \cdot \dfrac{\Delta^{-1}\left(fs_{t_2}, f\alpha_2, \right)}{H} \right) \right), \\ \Delta\left(H\left(\dfrac{\Delta^{-1}\left(fs_{i_1}, f\beta_1 \right)}{H} \cdot \dfrac{\Delta^{-1}\left(fs_{i_2}, f\beta_2 \right)}{H} \right) \right), \Delta\left(k\left(\dfrac{\Delta^{-1}\left(fs_{f_1}, f\chi_1 \right)}{H} \cdot \dfrac{\Delta^{-1}\left(fs_{f_1}, f\chi_1 \right)}{H} \right) \right) \end{array} \right\};$$

(2) $$f\delta_1 \otimes f\delta_2 = \left\{ \begin{array}{l} \Delta\left(H\left(\frac{\Delta^{-1}\left(fs_{t_1}, f\alpha_1, \right)}{H} \cdot \frac{\Delta^{-1}\left(fs_{t_2}, f\alpha_2, \right)}{H} \right) \right), \\ \Delta\left(H\left(\frac{\Delta^{-1}\left(fs_{i_1}, f\beta_1 \right)}{H} + \frac{\Delta^{-1}\left(fs_{i_2}, f\beta_2 \right)}{H} - \frac{\Delta^{-1}\left(fs_{i_1}, f\beta_1 \right)}{H} \cdot \frac{\Delta^{-1}\left(fs_{i_2}, f\beta_2 \right)}{H} \right) \right), \\ \Delta\left(H\left(\frac{\Delta^{-1}\left(fs_{f_1}, f\chi_1 \right)}{H} + \frac{\Delta^{-1}\left(fs_{f_2}, f\chi_2 \right)}{H} - \frac{\Delta^{-1}\left(fs_{f_1}, f\chi_1 \right)}{H} \cdot \frac{\Delta^{-1}\left(fs_{f_2}, f\chi_2 \right)}{H} \right) \right) \end{array} \right\};$$

(3) $$\lambda f\delta_1 = \left\{ \begin{array}{l} \Delta\left(H\left(1 - \left(1 - \frac{\Delta^{-1}\left(fs_{t_1}, f\alpha_1, \right)}{H} \right)^{\lambda} \right) \right), \Delta\left(H\left(\frac{\Delta^{-1}\left(fs_{i_1}, f\beta_1 \right)}{H} \right)^{\lambda} \right), \\ \Delta\left(H\left(\frac{\Delta^{-1}\left(fs_{f_1}, f\chi_1 \right)}{H} \right)^{\lambda} \right) \end{array} \right\}, \lambda > 0;$$

(4) $$f\delta_1^{\lambda} = \left\{ \begin{array}{l} \Delta\left(H\left(\frac{\Delta^{-1}\left(fs_{t_1}, f\alpha_1, \right)}{M} \right)^{\lambda} \right), \Delta\left(H\left(1 - \left(1 - \frac{\Delta^{-1}\left(fs_{i_1}, f\beta_1 \right)}{H} \right)^{\lambda} \right) \right), \\ \Delta\left(H\left(1 - \left(1 - \frac{\Delta^{-1}\left(fs_{f_1}, f\chi_1 \right)}{H} \right)^{\lambda} \right) \right) \end{array} \right\}, \lambda > 0.$$

Definition 3[71]. Let $f\delta_1 = \left\langle \left(fs_{t_1}, f\alpha_1, \right), \left(fs_{i_1}, f\beta_1 \right), \left(fs_{f_1}, f\chi_1 \right) \right\rangle$, $f\delta_2 = \left\langle \left(fs_{t_2}, f\alpha_2, \right), \left(fs_{i_2}, f\beta_2 \right), \left(fs_{f_2}, f\chi_2 \right) \right\rangle$, then the Euclidean distance is:

$$ED\left(f\delta_1, f\delta_2 \right) = \sqrt{\frac{1}{3}\left(\begin{array}{l} \left| \frac{\Delta^{-1}\left(fs_{t_1}, f\alpha_1 \right) - \Delta^{-1}\left(fs_{t_2}, f\alpha_2 \right)}{H} \right|^2 + \left| \frac{\Delta^{-1}\left(fs_{i_1}, f\beta_1 \right) - \Delta^{-1}\left(fs_{i_2}, f\beta_2 \right)}{H} \right|^2 \\ + \left| \frac{\Delta^{-1}\left(fs_{f_1}, f\chi_1 \right) - \Delta^{-1}\left(fs_{f_2}, f\chi_2 \right)}{H} \right|^2 \end{array} \right)} \quad (2)$$

Definition 4[56]. Let $f\delta = \left\langle \left(fs_t, f\alpha, \right), \left(fs_i, f\beta \right), \left(fs_f, f\chi \right) \right\rangle$, the score and accuracy functions of $f\delta$ is elaborated:

$$SF\left(l\delta \right) = \frac{\left(2H + \Delta^{-1}\left(fs_t, f\alpha, \right) - \Delta^{-1}\left(fs_i, f\beta \right) - \Delta^{-1}\left(fs_f, f\chi \right) \right)}{3H}, SF\left(f\delta \right) \in \left[0,1 \right] \quad (3)$$

$$HF\left(l\delta \right) = \frac{1}{H}\left(\Delta^{-1}\left(fs_t, f\alpha, \right) - \Delta^{-1}\left(fs_f, f\chi \right) \right), HF\left(f\delta \right) \in \left[-1,1 \right] \quad (4)$$

For $f\delta_1$ and $f\delta_2$, then

(1) $if\ SF(f\delta_1) \prec SF(f\delta_2),\ f\delta_1 \prec f\delta_2;$

(2) $if\ SF(f\delta_1) = SF(f\delta_2), HF(f\delta_1) \prec HF(f\delta_2),\ f\delta_1 \prec f\delta_2;$

(3) $if\ SF(f\delta_1) = SF(f\delta_2), HF(f\delta_1) = HF(f\delta_2),\ f\delta_1 = f\delta_2.$

3. 2TLNN-GRA method for MADM

The 2TLNN-GRA is elaborated for MADM. Suppose m defined decision alternatives $\{DA_1, DA_2, \ldots, DA_m\}$, n given attributes $\{GO_1, GO_2, \ldots, GO_n\}$, $fw = (fw_1, fw_2, \cdots, fw_n)$ is weight GO_j, where $fw_j \in [0,1], \sum_{j=1}^{n} fw_j = 1$. The 2TLNN-GRA for MADM are elaborated.

Step 1. Elaborate the 2TLNN-matrix $F = [f\phi_{ij}]_{m\times n}$.

$$F = [f\phi_{ij}]_{m\times n} = \begin{matrix} & GO_1 & GO_2 & \cdots & GO_n \\ DA_1 & f\phi_{11} & f\phi_{12} & \cdots & f\phi_{1n} \\ DA_2 & f\phi_{21} & f\phi_{22} & \cdots & f\phi_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ DA_m & f\phi_{m1} & f\phi_{m2} & \cdots & f\phi_{mn} \end{matrix} \tag{5}$$

$$f\phi_{ij} = \left\{ \left(fs_{t_{ij}}, f\alpha_{ij} \right), \left(fs_{i_{ij}}, f\beta_{ij} \right), \left(fs_{f_{ij}}, f\chi_{ij} \right) \right\} \tag{6}$$

Step 2. Elaborate normalized $F = [f\phi_{ij}]_{m\times n}$ to $NF = [nf\phi_{ij}]_{m\times n}$.

Aimed at benefit decision attributes:

$$\begin{aligned} nf\phi_{ij} &= \left\{ \left(nfs_{t_{ij}}, nf\alpha_{ij} \right), \left(nfs_{i_{ij}}, nf\beta_{ij} \right), \left(nfs_{f_{ij}}, nf\chi_{ij} \right) \right\} \\ &= \left\{ \left(fs_{t_{ij}}, f\alpha_{ij} \right), \left(fs_{i_{ij}}, f\beta_{ij} \right), \left(fs_{f_{ij}}, f\chi_{ij} \right) \right\}, \end{aligned} \tag{7}$$

Aimed at cost decision attributes:

$$\begin{aligned} nf\phi_{ij} &= \left\{ \left(nfs_{t_{ij}}, nf\alpha_{ij} \right), \left(nfs_{i_{ij}}, nf\beta_{ij} \right), \left(nfs_{f_{ij}}, nf\chi_{ij} \right) \right\} \\ &= \left\{ \begin{array}{l} \Delta\left(H - \Delta^{-1}\left(fs_{t_{ij}}, f\alpha_{ij} \right) \right), \Delta\left(H - \Delta^{-1}\left(fs_{i_{ij}}, f\beta_{ij} \right) \right), \\ \Delta\left(H - \Delta^{-1}\left(fs_{f_{ij}}, f\chi_{ij} \right) \right) \end{array} \right\} \end{aligned} \tag{8}$$

Step 3. Elaborate the 2TLNN positive ideal alternative (2TLNNPIA) and 2TLNN negative ideal alternative (2TLNNNIA) with Eq. (9-14):

$$2TLNNPIA = \{2TLNNPIA_j\} \tag{9}$$

$$2TLNNNIA = \left\{ 2TLNNNIA_j \right\} \tag{10}$$

$$2TLNNPIA_j = \left\{ \left(nfs_{t_j}^{+}, nf\,\alpha_j^{+} \right), \left(nfs_{i_j}^{+}, nf\,\beta_j^{+} \right), \left(nfs_{f_j}^{+}, nf\,\chi_j^{+} \right) \right\}, \tag{11}$$

$$2TLNNNIA_j = \left\{ \left(nfs_{t_j}^{-}, nf\,\alpha_j^{-} \right), \left(nfs_{i_j}^{-}, nf\,\beta_j^{-} \right), \left(nfs_{f_j}^{-}, nf\,\chi_j^{-} \right) \right\}, \tag{12}$$

$$\begin{aligned} & SV\left\{ \left(nfs_{t_j}^{+}, nf\,\alpha_j^{+} \right), \left(nfs_{i_j}^{+}, nf\,\beta_j^{+} \right), \left(nfs_{f_j}^{+}, nf\,\chi_j^{+} \right) \right\} \\ & = \max_i SV\left(\left\{ \left(nfs_{t_{ij}}, nf\,\alpha_{ij} \right), \left(nfs_{i_{ij}}, nf\,\beta_{ij} \right), \left(nfs_{f_{ij}}, nf\,\chi_{ij} \right) \right\} \right) \end{aligned} \tag{13}$$

$$\begin{aligned} & SV\left\{ \left(nfs_{t_j}^{-}, nf\,\alpha_j^{-} \right), \left(nfs_{i_j}^{-}, nf\,\beta_j^{-} \right), \left(nfs_{f_j}^{-}, nf\,\chi_j^{-} \right) \right\} \\ & = \min_i SV\left(\left\{ \left(nfs_{t_{ij}}, nf\,\alpha_{ij} \right), \left(nfs_{i_{ij}}, nf\,\beta_{ij} \right), \left(nfs_{f_{ij}}, nf\,\chi_{ij} \right) \right\} \right) \end{aligned} \tag{14}$$

Step 4. Elaborate the grey rational coefficients (GRC) from the 2TLNNPIA and 2TLNNNIA as:

$$\begin{aligned} & 2TLNNPIAGRC\left(\xi_{ij} \right) \\ & = \frac{\min\limits_{1 \le i \le m} ED\left(nf\,\phi_{ij}, 2TLNNPIA_j \right) + \rho \max\limits_{1 \le i \le m} ED\left(nf\,\phi_{ij}, 2TLNNPIA_j \right)}{ED\left(nf\,\phi_{ij}, 2TLNNPIS_j \right) + \rho \max\limits_{1 \le i \le m} ED\left(nf\,\phi_{ij}, 2TLNNPIA_j \right)} \end{aligned} \tag{15}$$

$$\begin{aligned} & 2TLNNNIAGRC\left(\xi_{ij} \right) \\ & = \frac{\min\limits_{1 \le i \le m} ED\left(nf\,\phi_{ij}, 2TLNNNIA_j \right) + \rho \max\limits_{1 \le i \le m} ED\left(nf\,\phi_{ij}, 2TLNNNIA_j \right)}{ED\left(nf\,\phi_{ij}, 2TLNNNIA_j \right) + \rho \max\limits_{1 \le i \le m} ED\left(nf\,\phi_{ij}, 2TLNNNIA_j \right)} \end{aligned} \tag{16}$$

Step 5. Elaborate the grey relation degree (GRD) for 2TLNNPIA and 2TLNNNIA:

$$2TLNNPIAGRD\left(\xi_i \right) = \sum_{j=1}^{n} fw_j\, 2TLNNPIAGRC\left(\xi_{ij} \right) \tag{17}$$

$$2TLNNNIAGRD\left(\xi_i \right) = \sum_{j=1}^{n} fw_j\, 2TLNNNIAGRC\left(\xi_{ij} \right) \tag{18}$$

Step 6. Obtain the defined 2TLNN relative relational degree (2TLNNRRD) for 2TLNNPIA:

$$2TLNNRRD\left(\xi_i \right) = \frac{2TLNNNIAGRD\left(\xi_i \right)}{2TLNNPIAGRD\left(\xi_i \right) + 2TLNNNIAGRD\left(\xi_i \right)} \tag{19}$$

Step 7. The optimal alternative is obtained with higher $2TLNNRRD\left(\xi_i \right)$ value.

4. An example and comparisons

4.1. An example for comprehensive evaluation of USTMS

Since the invention of the automobile, human beings have continuously conducted research on urban transportation and its evaluation. In traditional urban transportation planning, the evaluation has focused on the ability and level of the transportation system to solve traffic problems. In the 1930s, since Greenshields proposed the traffic flow theory, people began to use speed, flow, density and other traffic indicators to analyze and study the traffic operation. The idea of traffic demand prediction theory is to make traffic trip OD generation prediction and traffic demand prediction by establishing the basic relationship between traffic and land use, combining with land use information, and then applying network analysis techniques to allocate traffic (shortest path traffic allocation and multi-path traffic allocation) and formulate road traffic planning schemes. With the continuous development of urban transportation, traffic problems also emerge, and in order to solve various traffic problems that appear at different development stages, different traffic development concepts are proposed one after another, and even multiple traffic development concepts appear at the same time in one period. The new transportation development concept is proposed and involved in the construction of transportation, there must be a corresponding evaluation system to judge the development status to better guide the practice. For example, green transportation and low-carbon transportation are proposed on the basis of serious traffic pollution and high carbon emissions from motor vehicles, both of which have in common the concept of focusing on the development of public transportation, reducing energy consumption and achieving environmentally friendly development. Their evaluation systems, in addition to traffic function evaluation, mainly focus on traffic demand, improvement of environmental quality, and rational use of resources. Intelligent transportation is a transportation development concept proposed in the context of serious traffic congestion and road resource scarcity, focusing on the use of information technology and sensors to achieve a highly efficient and intelligent transportation system. Its evaluation focuses on the level of road infrastructure development, the level of intelligence, etc. Nowadays, the evaluation of urban transportation system is very rich and contains many aspects: traffic impact evaluation, traffic function evaluation, road traffic infrastructure level evaluation, traffic economic benefit evaluation, environmental impact evaluation, traffic management evaluation, road traffic safety evaluation, etc., involving all aspects of transportation. Although the evaluation contents are various and diverse, the evaluation objects and evaluation purposes are the same. The evaluation objects are all urban traffic systems, and the evaluation purposes are to diagnose the current situation of urban traffic development and provide reference opinions for further development, so as to promote the benign development of urban traffic in the target direction. The problems of comprehensive evaluation of USTMS are classical MADM problems. In this elaborated section, we provide an example about comprehensive evaluation of USTMS with 2TLNN-GRA. Aimed at five possible USTMSs $DA_i\left(i=1,2,3,4,5\right)$ to be elaborated with four attributes:

①PQ is the information service level: The realization of intelligent transportation requires various types of IoT infrastructure, as well as various sensing equipment to collect traffic information. Therefore, the state of basic infrastructure directly affects the rapid development speed of intelligent transportation, which is the most basic content of intelligent transportation evaluation, and is also the key content of evaluation.

②DC is the transport infrastructure: Intelligent transportation is supported by a new generation of information technology, which gives transportation "wisdom" and provides people with "humanized" transportation information services. To provide people with "humanized" traffic information services, so the level of information services of intelligent transportation is the most important basis for judging the level of rapid development of intelligent transportation. The level of information service includes the strength of people's attention to traffic information, the diversity of government related departments to provide traffic information channels of diversity, real-time and accuracy, people's satisfaction with public transport services The level of information service includes people's attention to transportation information, the diversity, real-time and accuracy of transportation information channels provided by government departments, and people's satisfaction with public transportation services.

③SL is the green environmental protection level: Intelligent transportation inherits the advantages of green transportation, low-carbon transportation and sustainable transportation environment Environmentally friendly, reduce carbon emissions and other advantages to achieve green transportation and sustainable development of transportation. Therefore, the evaluation of green environment protection. The evaluation of green level is also an aspect of the evaluation of intelligent transportation.

④EP is the security condition evaluation: Nowadays, with the surge in the number of motor vehicles, traffic accidents occur and have a great threat to people's lives. The problem of traffic safety cannot be ignored at any time. Smart transportation provides real-time and accurate traffic information to travelers by improving road. The smart traffic can achieve traffic safety by improving road infrastructure, providing real-time and accurate traffic information to travelers, and improving vehicle design. Only by focusing on safety and reducing traffic accidents can intelligent transportation develop in a positive way.

The five possible USTMSs $DA_i\left(i=1,2,3,4,5\right)$ are to be elaborated with defined 2TLNNs under elaborated four attributes with $fw=\left(0.19,0.26,0.32,13\right)$. The 2TLNN-GRA is elaborated to cope with the comprehensive evaluation of USTMS.

Step 1. Elaborate the built 2TLNN-matrix $F=\left[f\phi_{ij}\right]_{m\times n}$ (See Table 1).

Table 1. 2TLNN matrix $F=\left[f\phi_{ij}\right]_{m\times n}$

	PQ	DC
DA_1	{(fs_3, 0.21), (fs_4, 0.03), (fs_2, 0.34)}	{(fs_3, 0.26), (fs_5, 0.12), (fs_2, 0.15)}
DA_2	{(fs_4, 0.07), (fs_1, 0.15), (fs_2, 0.27)}	{(fs_5, 0.03), (fs_3, 0.23), (fs_2, 0.15)}
DA_3	{(fs_3, 0.31), (fs_2, 0.04), (fs_5, 0.29)}	{(fs_1, 0.05), (fs_3, 0.16), (fs_4, 0.07)}
DA_4	{(fs_2, 0.31), (fs_4, 0.06), (fs_1, 0.19)}	{(fs_3, 0.04), (fs_5, 0.23), (fs_2, 0.37)}
DA_5	{(fs_2, 0.32), (fs_3, 0.14), (fs_4, 0.08)}	{(fs_4, 0.03), (fs_2, 0.06), (fs_1, 0.19)}
	SL	EP
DA_1	{(fs_5, 0.42), (fs_2, 0.07), (fs_1, 0.16)}	{(fs_2,0.13), (fs_4, 0.05), (fs_3, 0.03)}
DA_2	{(fs_4, 0.14), (fs_3, 0.08), (fs_1, 0.11)}	{(fs_4, 0.16), (fs_2, 0.24), (fs_1, 0.18)}

DA_3	{(fs_1, 0.31), (fs_2, 0.04), (fs_4, 0.03)}	{(fs_2, 0.06), (fs_4, 0.15), (fs_5, 0.17)}
DA_4	{(fs_2, 0.16), (fs_2, 0.12), (fs_3, 0.03)}	{(fs_2, 0.18), (fs_4, 0.09), (fs_3, 0.12)}
DA_5	{(fs_2, 0.32), (fs_1, 0.01), (fs_5, 0.05)}	{(fs_2, 0.07), (fs_3, 0.06), (fs_4, 0.09)}

Step 2. Normalize $F=\left[f\phi_{ij}\right]_{m\times n}$ to $NF=\left[nf\phi_{ij}\right]_{m\times n}$, for all the defined attributes are benefit, the defined decision normalization is omitted.

Table 2. 2TLNN matrix $NF=\left[nf\phi_{ij}\right]_{m\times n}$

	PQ	DC
DA_1	{(fs_3, 0.21), (fs_4, 0.03), (fs_2, 0.34)}	{(fs_3, 0.26), (fs_5, 0.12), (fs_2, 0.15)}
DA_2	{(fs_4, 0.07), (fs_1, 0.15), (fs_2, 0.27)}	{(fs_5, 0.03), (fs_3, 0.23), (fs_2, 0.15)}
DA_3	{(fs_3, 0.31), (fs_2, 0.04), (fs_5, 0.29)}	{(fs_1, 0.05), (fs_3, 0.16), (fs_4, 0.07)}
DA_4	{(fs_2, 0.31), (fs_4, 0.06), (fs_1, 0.19)}	{(fs_3, 0.04), (fs_5, 0.23), (fs_2, 0.37)}
DA_5	{(fs_2, 0.32), (fs_3, 0.14), (fs_4, 0.08)}	{(fs_4, 0.03), (fs_2, 0.06), (fs_1, 0.19)}
	SL	EP
DA_1	{(fs_5, 0.42), (fs_2, 0.07), (fs_1, 0.16)}	{(fs_2,0.13), (fs_4, 0.05), (fs_3, 0.03)}
DA_2	{(fs_4, 0.14), (fs_3, 0.08), (fs_1, 0.11)}	{(fs_4, 0.16), (fs_2, 0.24), (fs_1, 0.18)}
DA_3	{(fs_1, 0.31), (fs_2, 0.04), (fs_4, 0.03)}	{(fs_2, 0.06), (fs_4, 0.15), (fs_5, 0.17)}
DA_4	{(fs_2, 0.16), (fs_2, 0.12), (fs_3, 0.03)}	{(fs_2, 0.18), (fs_4, 0.09), (fs_3, 0.12)}
DA_5	{(fs_2, 0.32), (fs_1, 0.01), (fs_5, 0.05)}	{(fs_2, 0.07), (fs_3, 0.06), (fs_4, 0.09)}

Step 3. Elaborate the 2TLNNPIA and 2TLNNNIA (See Table 3).

Table 3. The 2TLNNPIS and 2TLNNNIS

	PQ	DC
2TLNNPIA	{(fs_4, 0.07), (fs_1, 0.15), (fs_2, 0.27)}	{(fs_5, 0.03), (fs_3, 0.23), (fs_2, 0.15)}
2TLNNNIA	{(fs_2, 0.32), (fs_3, 0.14), (fs_4, 0.08)}	{(fs_1, 0.05), (fs_3, 0.16), (fs_4, 0.07)}
	SL	EP
2TLNNPIS	{(fs_5, 0.42), (fs_2, 0.07), (fs_1, 0.16)}	{(fs_4, 0.16), (fs_2, 0.24), (fs_1, 0.18)}
2TLNNNIS	{(fs_1, 0.31), (fs_2, 0.04), (fs_4, 0.03)}	{(fs_2, 0.07), (fs_3, 0.06), (fs_4, 0.09)}

Step 4. Compute the $2TLNNPIAGRC\left(\xi_{ij}\right)$ and $2TLNNNIAGRC\left(\xi_{ij}\right)$ (See Table 4-5).

Table 4. The $2TLNNPIAGRC\left(\xi_{ij}\right)$

Alternatives	PQ	DC	SL	EP
DA_1	0.5902	1.0000	0.2410	0.3172
DA_2	0.4520	0.3420	0.3252	0.4020
DA_3	0.6126	0.5902	1.0000	1.0000
DA_4	0.4258	0.4020	0.3420	0.3908
DA_5	1.0000	0.3420	0.3252	0.4258

Table 5. The $2TLNNNIAGRC\left(\xi_{ij}\right)$

Alternatives	PQ	DC	SL	EP
DA_1	0.6493	0.4809	0.5488	0.5866
DA_2	0.9216	1.0000	1.0000	0.9654
DA_3	0.5733	0.4980	0.4654	0.5268
DA_4	1.0000	0.6493	0.8464	1.0000
DA_5	0.5607	0.5866	0.6160	0.6090

Step 5. Compute the $2TLNNPIAGRD\left(\xi_{i}\right)$ and $2TLNNNIAGRD\left(\xi_{i}\right)$ (See Table 6):

Table 6. The $2TLNNPIAGRD\left(\xi_{i}\right)$ and $2TLNNNIAGRD\left(\xi_{i}\right)$

	$2TLNNPIAGRD(\xi_i)$	$2TLNNNIAGRD(\xi_i)$
DA_1	0.3236	0.3082
DA_2	0.4420	0.5744
DA_3	0.6570	0.2912
DA_4	0.3880	0.1905
DA_5	0.4077	0.2726

Step 6. Compute the $2TLNNRRD(\xi_i)$ (See Table 7).

Table 7. The $2TLNNRRD(\xi_i)$

	$2TLNNRRD(\xi_i)$	Order
DA_1	0.5069	2
DA_2	0.5877	1
DA_3	0.3186	5
DA_4	0.3366	4
DA_5	0.4144	3

Step 7. Form $2TLNNRRD(\xi_i)$, the decision order is: $DA_2 > DA_1 > DA_5 > DA_4 > DA_3$ and DA_2 is the best USTMSs.

4.2. Comparing 2TLNN-GRA with defined 2TLNNs decision operators

The 2TLNN-GRA is fully compared with 2TLNWHM and 2TLNWDHM operator[72]. The fused information values are elaborated within Table 8.

Table 8. The comparisons with 2TLNNs operators

	2TLNWHM	2TLNWDHM
DA_1	$\{(fs_2, 0.23), (fs_2, 0.18), (fs_3, 0.12)\}$	$\{(fs_3, 0.17), (fs_5, 0.27), (fs_2, 0.42)\}$
DA_2	$\{(fs_5, 0.49), (fs_2, 0.12), (fs_1, 0.25)\}$	$\{(fs_5, 0.15), (fs_3, 0.29), (fs_2, 0.21)\}$

DA_3	{(fs_5, 0.46), (fs_2, 0.16), (fs_1, 0.21)}	{(fs_4, 0.16), (fs_2, 0.09), (fs_1, 0.23)}
DA_4	{(fs_1, 0.36), (fs_2, 0.22), (fs_4, 0.08)}	{(fs_1, 0.11), (fs_3, 0.26), (fs_4, 0.29)}
DA_5	{(fs_4, 0.18), (fs_3, 0.17), (fs_1, 0.19)}	{(fs_3, 0.28), (fs_5, 0.09), (fs_2, 0.03)}

According to score of 2TLNNs, the score is elaborated in Table 9.

Table 9. Scores of given USTMSs

	2TLNWHM	2TLNWDHM
$SF(DA_1)$	0.7494	0.4464
$SF(DA_2)$	0.8678	0.6126
$SF(DA_3)$	0.7828	0.5294
$SF(DA_4)$	0.6738	0.4259
$SF(DA_5)$	0.7635	0.5108

The order is elaborated in Table 10.

Table 10. Order by 2TLNNs operators

	order
2TLNWHM operator [72]	$DA_2 > DA_3 > DA_5 > DA_1 > DA_4$
2TLNWDHM operator [72]	$DA_2 > DA_3 > DA_5 > DA_1 > DA_4$
2TLNN-GRA method	$DA_2 > DA_1 > DA_5 > DA_4 > DA_3$

Comparing the results of the 2TLNN-GRA method with 2TLNWHM & 2TLNWDHM fused operators, the obtained results are slightly different and the chosen best USTMS is same.

5. Conclusion

With the continuous development of China's economy, people's income level is increasing, and more and more families have the ability to buy private cars, which leads to a sharp increase in the number and frequency of urban motor vehicle ownership and use, and the contradiction between the effective supply and demand of people, vehicles and roads is becoming more and more prominent, resulting in urban traffic congestion and other urban traffic problems are becoming more and more obvious. The traditional methods of alleviating traffic problems are no longer applicable to the contradiction between people's traffic demand and traffic infrastructure supply in modern times. Recently, in the context of smart cities, scholars at home and abroad have started to study smart transportation, and with rapid development of new generation technologies such as cloud computing, Internet of Things, big data and 5G, more and more scholars have started to study smart transportation, which is an important part of smart cities. The comprehensive evaluation of USTMS is the MADM. In this elaborated paper, the 2TLNN-GRA is elaborated for MADM. Finally, an example for comprehensive evaluation of USTMS was given to elaborate the 2TLNN-GRA and the elaborated comparisons are also executed to elaborate the 2TLNN-GRA. In the future works, the 2TLNN-GRA shall be applied to existed risk decision [73-76], existed selection decision[77-83] and other existed MADM under different uncertain environments[84-88].

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TOPSIS Method-Based Decision-Making Model of Simplified Neutrosophic Indeterminate Sets for Teaching Quality Evaluation

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Abstract: Recently, colleges/universities have paid a lot of attention to the teaching quality evaluation (TQE) of teachers in China. TQE is an essential way to improve teachers' teaching ability and quality in the teaching process. Then, the TQE of teaching supervisors is a multi-attribute decision-making (MADM) problem with vague, inconsistent, and indeterminate information. The simplified neutrosophic indeterminate set/element (SNIS/SNIE) is an appropriate form to express the indeterminate decision-making information in the TQE process. Therefore, this article presents an improved ranking method based on maximizing deviations principle and technique for order of preference by similarity (TOPSIS) for SNIS and applies it to evaluate teachers' teaching quality. First, the Hamming distance between two SNIEs is defined. Then, attribute weights are obtained by maximizing deviation method and the TOPSIS method-based decision-making model is developed for the MADM applications with unknown attribute weights. Finally, we perform the developed MADM model for a TQE case and compare it with existing related models to indicate the feasibility and rationality of the proposed model with unknown attribute weights in the SNIE circumstance.

Keywords: simplified neutrosophic indeterminate set; maximizing deviation; TOPSIS method; teaching quality evaluation

1. Introduction

Cultivating qualified talents is the central task of colleges/universities. During the talent training process, the teaching quality evaluation (TQE) is a key task. Then, TQE is one of the main tasks of teaching administration in various colleges/universities. A teaching evaluation system includes two aspects: evaluation framework and evaluation method. However, TQE is a multi-attribute decision-making (MADM) problem, which implies vagueness and uncertainty in the evaluation process. In recent years, various fuzzy evaluation methods have been applied to TQE [1–5].

Recently, neutrosophic set (NS) [6] has become the most popular topic for describing indeterminate and inconsistent information. The true, indeterminate, and false membership functions in NS are independent components. Compared with a fuzzy set (FS) [7] and an intuitionistic FS (IFS) [8, 9], NS can be used to express the corresponding inconsistent, indeterminate, and incomplete information in real decision-making (DM) problems. In practical applications, NS has been simplified into many forms, for example, single-valued NS (SvNS) [10], interval NS (INS) [11], and simplified NS (SNS) [12]. They are widely used in engineering and science fields. However, the true, indeterminate, and false membership degrees are specified by single values or interval values in SNS. In complicated MADM problems, the true, indeterminate, and false membership degrees may be partly certain and partly uncertain. In this case, a neutrosophic number (NN) [13] can describe them by $p = \rho + \mu\xi$ for $\rho, \mu \in \Re$ and $\xi \in [\xi^-, \xi^+]$, where ρ is the certain term and $\mu\xi$ is the uncertain term. Du et al. [14] put forward a simplified neutrosophic indeterminate set/element (SNIS/SNIE) combining SNS with NN. Each SNIE consists of the true NN, the false NN, and the indeterminate NN. SNIS can be transformed to SvNS or INS according to $\xi^- = \xi^+$ or $\xi^- \neq \xi^+$ with the value/range of indeterminacy $\xi \in [\xi^-, \xi^+]$.

The TOPSIS method proposed by Hwang and Yoon [15] is a kind of distance-based rank method. Better choices are closer to positive ideals and farther away from negative ideals. Then, it has been applied to various fuzzy DM environments. For example, the TOPSIS method was used to solve the supplier selection problem in intuitionistic fuzzy environment [16]. Later, many researchers [17–20] developed the DM methods using TOPSIS methods in hesitant FS, IFS, and interval-valued IFS environments. In indeterminate and inconsistent circumstances, Sahin and Yiider [21] introduced a modified TOPSIS method with SvNSs for the group DM. Chi and Liu [22] extended the TOPSIS method to the INS environment.

Although some researchers have developed several operational rules for SNIEs and SNIS. Existing SNIS decision-making (DM) methods [14, 23, 24] used a specified weight vector of attributes to solve the MADM problems of SNISs. So far, no researchers consider the influence of indeterminate degrees on attribute weights in MADM problems of SNISs. In a real DM situation, the weights of attributes may be indeterminate or unknown. In this article, we extend a MADM model which combines the determining method of unknown attribute weights with the TOPSIS method of SNISs and use it for TQE. The rest of the article is as follows.

In Section 2, the Hamming distance of SNIEs is introduced. Section 3 presents a method for determining unknown attribute weights and an extended TOPSIS method for SNISEs. In Section 4, we apply the proposed model to a TQE case and analyze the influence of indeterminate ranges in SNISEs on decision results. Then, the extend TOPSIS method is compared with the related models in Section 5. The article is summarized in Section 6.

2. Distance of SNIEs

This section presents the Hamming distance between SNIEs and its properties.

First, we introduce the notions of SNIS and SNIE [14].

Definition 1 [14]. Let $S = \{s_1, s_2, \ldots, s_n\}$ be a universe set. A SNIS B in S is described as $B = \{<s_k, P(s_k, \xi), N(s_k, \xi), Q(s_k, \xi)> | s_k \in S\}$, where $P(s_k, \xi) = \rho_k+\mu_k\xi \subseteq [0, 1]$, $N(s_k, \xi) = \delta_k+\Phi_k\xi \subseteq [0, 1]$, and $Q(s_k, \xi) = \lambda_k+\nu_k\xi \subseteq [0, 1]$ for $s_k \in S$ ($k = 1, 2, \ldots, n$) and $\xi \in [\xi^-, \xi^+]$. Then, $P(s_k, \xi)$, $N(s_k, \xi)$, and $Q(s_k, \xi)$ are the true NN, the indeterminate NN, and the false NN. Each component $<s_k, P(s_k, \xi), N(s_k, \xi), Q(s_k, \xi)>$ in B is called SNIE, which can be represented as the simple form $b_k = <P_k(\xi), N_k(\xi), Q_k(\xi)> = <\rho_k+\mu_k\xi, \delta_k+\Phi_k\xi, \lambda_k+\nu_k\xi>$.

Then, we present the Hamming distance of SNIEs below.

Definition 2. Suppose that $b_1 = <P_1(\xi), N_1(\xi), Q_1(\xi)> = <\rho_1+\mu_1\xi, \delta_1+\Phi_1\xi, \lambda_1+\nu_1\xi>$ and $b_2 = <P_2(\xi), N_2(\xi), Q_2(\xi)> = <\rho_2+\mu_2\xi, \delta_2+\Phi_2\xi, \lambda_2+\nu_2\xi>$ are two SNIEs for $\xi \in [\xi^-, \xi^+]$. Thus, the Hamming distance between b_1 and b_2 are defined as follows:

$$l_H(b_1,b_2)=\frac{1}{6}\begin{pmatrix}\left|P_1(\xi^-)-P_2(\xi^-)\right|+\left|P_1(\xi^{\circledcirc})-P_2(\xi^{\circledcirc})\right|+\left|N_1(\xi^-)-N_2(\xi^-)\right|+\\ \left|N_1(\xi^{\circledcirc})-N_2(\xi^{\circledcirc})\right|+\left|Q_1(\xi^-)-Q_2(\xi^-)\right|+\left|Q_1(\xi^{\circledcirc})-Q_2(\xi^{\circledcirc})\right|\end{pmatrix}$$
$$\circledcirc\frac{1}{6}\begin{pmatrix}\left|\rho_1-\rho_2\circledcirc(\mu_1-\mu_2)\xi^-\right|+\left|\rho_1-\rho_2\circledcirc(\mu_1-\mu_2)\xi^{\circledcirc}\right|+\left|\delta_1-\delta_2\circledcirc(\phi_1-\phi_2)\xi^-\right|+\\ \left|\delta_1-\delta_2\circledcirc(\phi_1-\phi_2)\xi^{\circledcirc}\right|+\left|\lambda_1-\lambda_2\circledcirc(\nu_1-\nu_2)\xi^-\right|+\left|\lambda_1-\lambda_2\circledcirc(\nu_1-\nu_2)\xi^{\circledcirc}\right|\end{pmatrix} \quad (1)$$

Proposition 1. Set b_1 = <$P_1(\xi)$, $N_1(\xi)$, $Q_1(\xi)$> = <$\rho_1+\mu_1\xi$, $\delta_1+\Phi_1\xi$, $\lambda_1+\nu_1\xi$>, b_2 = <$P_2(\xi)$, $N_2(\xi)$, $Q_2(\xi)$> = <$\rho_2+\mu_2\xi$, $\delta_2+\Phi_2\xi$, $\lambda_2+\nu_2\xi$ >, and b_3 = <$P_3(\xi)$, $N_3(\xi)$, $Q_3(\xi)$> = <$\rho_3+\mu_3\xi$, $\delta_3+\Phi_3\xi$, $\lambda_3+\nu_3\xi$ > as three SNIEs for $\xi\in[\xi^-,\xi^+]$. The Hamming distance between them meets the following properties:

(1) $0\le l_H(b_1,b_2)\le 1$;

(2) $l_H(b_1,b_2)=l_H(b_2,b_1)$;

(3) $l_H(b_1,b_3)\le l_H(b_1,b_2)+l_H(b_2,b_3)$.

Proof:

(1) Since $P(\xi)$, $N(\xi)$, $Q(\xi)\subseteq[0,1]$, $|P_1(\xi^+)-P_2(\xi^+)|$, $|P_1(\xi^-)-P_2(\xi^-)|$, $|N_1(\xi^+)-N_2(\xi^+)|$, $|N_1(\xi^-)-N_2(\xi^-)|$, $|Q_1(\xi^+)-Q_2(\xi^+)|$, $|Q_1(\xi^-)-Q_2(\xi^-)|\subseteq[0,1]$, then there is $0\le l_H(b_1,b_2)\le 1$.

(2) The proof is obvious.

(3) Since there is the following inequality:

$$l_H(b_1,b_3)=\frac{1}{6}\begin{pmatrix}\left|P_1(\xi^-)-P_3(\xi^-)\right|+\left|P_1(\xi^{\circledcirc})-P_3(\xi^{\circledcirc})\right|+\left|N_1(\xi^-)-N_3(\xi^-)\right|+\\ \left|N_1(\xi^{\circledcirc})-N_3(\xi^{\circledcirc})\right|+\left|Q_1(\xi^-)-Q_3(\xi^-)\right|+\left|Q_1(\xi^{\circledcirc})-Q_3(\xi^{\circledcirc})\right|\end{pmatrix}$$
$$\circledcirc\frac{1}{6}\begin{pmatrix}\left|P_1(\xi^-)-P_2(\xi^-)+P_2(\xi^-)-P_3(\xi^-)\right|+\left|P_1(\xi^{\circledcirc})-P_2(\xi^{\circledcirc})+P_2(\xi^{\circledcirc})-P_3(\xi^{\circledcirc})\right|+\\ \left|N_1(\xi^-)-N_2(\xi^-)+N_2(\xi^-)-N_3(\xi^-)\right|+\left|N_1(\xi^{\circledcirc})-N_2(\xi^{\circledcirc})+N_2(\xi^{\circledcirc})-N_3(\xi^{\circledcirc})\right|+\\ \left|Q_1(\xi^-)-Q_2(\xi^-)+Q_2(\xi^-)-Q_3(\xi^-)\right|+\left|Q_1(\xi^{\circledcirc})-Q_2(\xi^{\circledcirc})+Q_2(\xi^{\circledcirc})-Q_3(\xi^{\circledcirc})\right|\end{pmatrix}$$
$$\le\frac{1}{6}\begin{pmatrix}\left|P_1(\xi^-)-P_2(\xi^-)\right|+\left|P_2(\xi^-)-P_3(\xi^-)\right|+\left|P_1(\xi^{\circledcirc})-P_2(\xi^{\circledcirc})\right|+\left|P_2(\xi^{\circledcirc})-P_3(\xi^{\circledcirc})\right|+\\ \left|N_1(\xi^-)-N_2(\xi^-)\right|+\left|N_2(\xi^-)-N_3(\xi^-)\right|+\left|N_1(\xi^{\circledcirc})-N_2(\xi^{\circledcirc})\right|+\left|N_2(\xi^{\circledcirc})-N_3(\xi^{\circledcirc})\right|+\\ \left|Q_1(\xi^-)-Q_2(\xi^-)\right|+\left|Q_2(\xi^-)-Q_3(\xi^-)\right|+\left|Q_1(\xi^{\circledcirc})-Q_2(\xi^{\circledcirc})\right|+\left|Q_2(\xi^{\circledcirc})-Q_3(\xi^{\circledcirc})\right|\end{pmatrix}$$
$$=\frac{1}{6}\begin{pmatrix}\left|P_1(\xi^-)-P_2(\xi^-)\right|+\left|P_1(\xi^{\circledcirc})-P_2(\xi^{\circledcirc})\right|+\left|N_1(\xi^-)-N_2(\xi^-)\right|+\left|N_1(\xi^{\circledcirc})-N_2(\xi^{\circledcirc})\right|\\ +\left|Q_1(\xi^-)-Q_2(\xi^-)\right|+\left|Q_1(\xi^{\circledcirc})-Q_2(\xi^{\circledcirc})\right|\end{pmatrix}$$
$$+\frac{1}{6}\begin{pmatrix}\left|P_2(\xi^-)-P_3(\xi^-)\right|+\left|P_2(\xi^{\circledcirc})-P_3(\xi^{\circledcirc})\right|+\left|N_2(\xi^-)-N_3(\xi^-)\right|+\left|N_2(\xi^{\circledcirc})-N_3(\xi^{\circledcirc})\right|+\\ \left|Q_2(\xi^-)-Q_3(\xi^-)\right|+\left|Q_2(\xi^{\circledcirc})-Q_3(\xi^{\circledcirc})\right|\end{pmatrix}$$
$$\circledcirc l_H(b_1,b_2)+l_H(b_2,b_3)$$

Thus $l_H(b_1,b_3)\le l_H(b_1,b_2)+l_H(b_2,b_3)$ holds.

3. An Extended TOPSIS Method for SNIEs

For a MADM problem, suppose that $E=\{E_1, E_2, \ldots, E_m\}$ and $C=\{C_1, C_2, \ldots, C_n\}$ are a set of alternatives and a set of attributes, respectively. But the attribute weight vector $\beta=\{\beta_1, \beta_2, \ldots, \beta_n\}$ is unknown for $\sum_{j=1}^{n}\beta_j=1$ and $\beta_j\in[0,1]$. The decision matrix is $B=[b_{ij}]_{m\times n}$, where b_{ij} = <$P_{ij}(\xi)$, $N_{ij}(\xi)$, $Q_{ij}(\xi)$> = <$\rho_{ij}+\mu_{ij}\xi$, $\delta_{ij}+\Phi_{ij}\xi$, $\lambda_{ij}+\nu_{ij}\xi$> is SNIE as the assessment value of the alternative E_i on the attribute C_j. The MADM method is described by the following steps.

Step 1. Normalize SNIEs for different attribute types.

In the DM problems, attributes were classified as benefit and cost. For the benefit type, it is better with a higher attribute value; For the cost type, it is worse with a higher attribute value. We generally adopt the beneficial type in most DM situations. If the attribute C_j is the cost type, the SNIE b_{ij} needs to be converted to its complement $\overline{b}_{ij}$ by $\overline{b}_{ij}$ = <$[\lambda_{ij}+\nu_{ij}\xi^-, \lambda_{ij}+\nu_{ij}\xi^+]$, $[1-(\delta_{ij}+\Phi_{ij}\xi^+), 1-(\delta_{ij}+\Phi_{ij}\xi^-)]$, $[\rho_{ij}+\mu_{ij}\xi^-, \rho_{ij}+\mu_{ij}\xi^+]$>.

Step 2. Determine attribute weights by the maximizing deviation model.

When the attribute weight information is incomplete or completely unknown in MADM problems. The maximizing deviation model [25] is a commonly used method to determine attribute weights. The model is based on the following principle. In MADM, the evaluation values of all alternatives for each attribute are generally different. For an attribute, the difference between the assessment values of all alternatives demonstrates the importance of the attribute. The greater the deviation between attribute values, the greater influence this attribute will have on the ranking of alternatives. Thus, this attribute is set as a higher weight. On the contrary, the smaller the deviation between attribute values, the lower the weight.

We determine the weight vector in view of the following steps:

(i) Define the deviation of E_i (i = 1, 2, …, m) to all alternatives for the attribute C_j (j = 1, 2, …, n) by

$$L_{ij}\left(\beta_j\right)=\sum_{k=1}^{m} l_H(b_{ij},b_{kj})\beta_j \tag{2}$$

(ii) Define the deviation of all the alternatives for the attribute C_j for (j = 1, 2, …, n) by

$$L_j\left(\beta_j\right)=\sum_{i=1}^{m} L_{ij}\left(\beta_j\right)=\sum_{i=1}^{m}\sum_{k=1}^{m} l_H(b_{ij},b_{kj})\beta_j \tag{3}$$

(iii) Define the deviation for all attributes by

$$L\left(\beta_j\right)=\sum_{j=1}^{n} L_j\left(\beta_j\right)=\sum_{j=1}^{n}\sum_{i=1}^{m} L_{ij}\left(\beta_j\right)=\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m} l_H(b_{ij},b_{kj})\beta_j \tag{4}$$

(iv) Construct a model that maximizes all deviations to determine the weights by

$$\max L\left(\beta_j\right)=\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m} l_H(b_{ij},b_{kj})\beta_j \tag{5}$$

$$s.t\ \begin{cases}\sum_{j=1}^{n}\beta_j^2=1\\ 0\le\beta_j\le 1,\ j=1,2,\ldots n\end{cases}$$

(v) Construct a Lagrange function by

$$S(\beta_j,\varphi)=\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m} l_H(b_{ij},b_{kj})\beta_j \circledcirc \frac{\varphi}{2}\left(\sum_{j=1}^{n}\beta_j^2-1\right) \tag{6}$$

(vi) Take the partial derivative with respect to β_j and φ by

$$\frac{\partial S(\beta_j,\varphi)}{\partial\beta_j}=\sum_{i=1}^{m}\sum_{k=1}^{m} l_H(b_{ij},b_{kj})\circledcirc\varphi\beta_j=0$$

$$\frac{\partial S(\beta_j,\varphi)}{\partial\varphi}=\frac{1}{2}\left(\sum_{j=1}^{n}\beta_j^2-1\right)=0$$

$$\beta_j = \frac{\sum_{i=1}^{m}\sum_{k=1}^{m} l_H(b_{ij}, b_{kj})}{\sqrt{\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\sum_{k=1}^{m} l_H(b_{ij}, b_{kj})\right)^2}} \quad (7)$$

(vii) Normalize the attribute weights by

$$\beta_j = \frac{\sum_{i=1}^{m}\sum_{k=1}^{m} l_H(b_{ij}, b_{kj})}{\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m} l_H(b_{ij}, b_{kj})} \quad (8)$$

Step 3. Rank all alternatives with the TOPSIS method.

(i) Determine two sets of positive and negative ideal solutions $b^+ = \{b_1^+, b_2^+, \ldots, b_n^+\}$ and $b^- = \{b_1^-, b_2^-, \ldots, b_n^-\}$ by the following equations:

$$b_j^+ = \left\langle \left[\max_i P_{ij}(\xi^-), \max_i P_{ij}(\xi^+)\right], \left[\min_i N_{ij}(\xi^-), \min_i N_{ij}(\xi^+)\right], \left[\min_i Q_{ij}(\xi^-), \min_i Q_{ij}(\xi^+)\right] \right\rangle, \quad (9)$$

$$b_j^- = \left\langle \left[\min_i P_{ij}(\xi^-), \min_i P_{ij}(\xi^+)\right], \left[\max_i N_{ij}(\xi^-), \max_i N_{ij}(\xi^+)\right], \left[\max_i Q_{ij}(\xi^-), \max_i Q_{ij}(\xi^+)\right] \right\rangle \quad (10)$$

(ii) Calculate the weighted distances of all alternatives to the positive ideal set $h^+ = \{h_1^+, h_2^+, \ldots, h_m^+\}$ and the weighted distances of all alternatives to the negative ideal set $h^- = \{h_1^-, h_2^-, \ldots, h_m^-\}$ by

$$\begin{cases} h_i^+ = \sum_{j=1}^{n} \beta_j l_H(b_{ij}, b_j^+) \\ h_i^- = \sum_{j=1}^{n} \beta_j l_H(b_{ij}, b_j^-) \end{cases} \quad (11)$$

(iii) Calculate the correlation coefficient h_i for each alternative by

$$h_i = \frac{h_i^-}{h_i^+ + h_i^-} \quad (12)$$

(iv) Rank alternatives

Alternatives are ranked according to their correlation coefficient values.

4. An Indeterminate DM Case about TQE

4.1 Problem Description of a TQE Case

This study is to apply the proposed TOPSIS method to TQE in the teaching assessment of teachers. Teaching skill competitions are often held in universities to promote teaching quality and improve teaching level. Teachers participating in the competition hold open classes based on their courses. In China, each university usually establishes a teaching evaluation system and specifies a group of teaching experts as an assessment committee. In the assessment process, teaching evaluation system commonly includes the attributes/criteria of teaching content, method, and attitude. Each attribute is briefly described as follows.

The teaching content means the rationality of teaching design. An excellent teaching design can strengthen the key points and important knowledge of the teaching content, which should be related to the knowledge levels and the learning ability of students.

The teaching method reflects that heuristic teaching is adopted. The teachers focus on teaching feedback and ability cultivation. Various teaching means are used in the classroom to maintain a good classroom atmosphere.

The teaching attitude is reflected in good appearance, good manners and fluent teaching language. Teachers must strictly adhere to the teaching norms. Teachers must be strict with students and manage classroom order well.

In the TQE case, the School of Mechanical and Electrical Engineering of Shaoxing University in China will offer teaching excellence awards to outstanding teachers among the four finalists E_1, E_2, E_3 and E_4 (the four alternatives) in the final round. Teaching supervisors observe their teaching one time in any class. Experts give their assessments according to the three attributes: C_1 (the teaching content design), C_2 (the teaching method), and C_3 (the teaching attitude). The teaching experts were invited to assess each teacher participating in the competition process. However, since the weights of the three attributes are not specified, they are unknown. Then, they give the assessment values of the SNIEs b_{ij} = $<P_{ij}(\xi), N_{ij}(\xi), Q_{ij}(\xi)>$ = $<\rho_{ij}+\mu_{ij}\xi, \delta_{ij}+\Phi_{ij}\xi, \lambda_{ij}+\nu_{ij}\xi>$ (i = 1, 2, 3, 4; j = 1, 2, 3) for $\xi \in [0, 1.5]$. The decision matrix is indicated as below.

$$B=\begin{bmatrix} <0.7+0.2\xi,0.2+0.1\xi,0.2+0.2\xi> & <0.7+0.2\xi,0.1+0.3\xi,0.1+0.1\xi> & <0.6+0.2\xi,0.2+0.2\xi,0.2+0.2\xi> \\ <0.7+0.2\xi,0.2+0.1\xi,0.3+0.1\xi> & <0.8+0.1\xi,0.1+0.2\xi,0.1+0.3\xi> & <0.7+0.1\xi,0.2+0.2\xi,0.1+0.1\xi> \\ <0.8+0.1\xi,0.2+0.1\xi,0.1+0.2\xi> & <0.7+0.1\xi,0.2+0.1\xi,0.1+0.2\xi> & <0.7+0.2\xi,0.3+0.1\xi,0.2+0.1\xi> \\ <0.7+0.1\xi,0.1+0.2\xi,0.2+0.1\xi> & <0.8+0.1\xi,0.1+0.2\xi,0.2+0.1\xi> & <0.7+0.1\xi,0.2+0.1\xi,0.2+0.2\xi> \end{bmatrix}$$

4.2 Ranking the Alternatives

The proposed TOPSIS method is applied to the TQE case and gives the following steps.

Step 1. Since the attributes are all benefit types in this case**,** their assessed values do not need to be converted.

Firstly, we assume that ξ is an indeterminate range of $\xi \in [0, 0.5]$, then the decision matrix is produced as follows:

$$B^{*}=\begin{bmatrix} <[0.7,0.8],[0.2,0.25],[0.2,0.3]> & <[0.7,0.8],[0.1,0.25],[0.1,0.15]> & <[0.6,0.7],[0.2,0.3],[0.2,0.3]> \\ <[0.7,0.8],[0.2,0.25],[0.3,0.35]> & <[0.7,0.85],[0.1,0.2],[0.1,0.25]> & <[0.7,0.75],[0.2,0.3],[0.1,0.15]> \\ <[0.8,0.85],[0.2,0.25],[0.1,0.2]> & <[0.7,0.75],[0.2,0.25],[0.1,0.2]> & <[0.7,0.8],[0.3,0.35],[0.2,0.25]> \\ <[0.7,0.75],[0.1,0.2],[0.2,0.25]> & <[0.8,0.85],[0.1,0.2],[0.2,0.25]> & <[0.7,0.75],[0.2,0.25],[0.2,0.3]> \end{bmatrix}$$

Step 2. Calculate and normalize the attribute weights β_j.

(i) According to Eq. (2), we calculate the Hamming distances between b_{ij} and b_{kj}, which are listed in Table 1.

(ii) According to Eq. (9), we can get the normalized attribute weights

β_1= 0.3554, β_2= 0.3140, and β_3= 0.3306.

Step 3. Rank the alternatives with the TOPSIS method.

(i) According to Eqs. (10)-(11) and the decision matrix B^*, we can determine two sets of the positive and negative ideal solutions b^+ and b^-:

b^+ = {<[0.8,0.85],[0.1,0.2],[0.1,0.2]>,<[0.8,0.85],[0.1,0.2],[0.1,0.15]>,<[0.7,0.8].[0.20,0.25],[0.10,0.15]>},

b^- = {<[0.7,0.75],[0.2,0.25],[0.3,0.35]>,<[0.7,0.75],[0.2,0.25],[0.2,0.25]>,<[0.6,0.7].[0.3,0.35],[0.2,0.3]>}

(ii) According to Eq. (12), we calculate the weighted distances of all alternatives to the positive and negative ideal solutions:

h_1^+ = 0.0676, h_2^+ = 0.0492, h_3^+ = 0.0519, and h_4^+ = 0.0477;

h_1^- = 0.0384, h_2^- = 0.0568, h_3^- = 0.0542, and h_4^- = 0.0583.

(iii) By Eq. (13), we get the correlation coefficient h_i as follows:

h_1 = 0.3623, h_2 = 0.5357, h_3 = 0.5110, and h_4 = 0.5500.

(iv) In terms of the ranking rules, the alternative is better if the coefficient value is greater. Therefore, the ranking order from the best to worst is $E_4 > E_2 > E_3 > E_1$. The best teacher is E_4 when the indeterminacy ξ is in the range of [0, 0.5].

Table 1. The Hamming distances between b_{ij} and b_{kj}

$l_H(b_{ij}, b_{kj})$				
		j = 1	j = 2	j = 3
i ◎ 1	k ◎ 1	0.0000	0.0000	0.0000
	k ◎ 2	0.0250	0.0500	0.0667
	k ◎ 3	0.0583	0.0333	0.0667
	k ◎ 4	0.0417	0.0667	0.0333
i ◎ 2	k ◎1	0.0250	0.0500	0.0667
	k ◎ 2	0.0000	0.0000	0.0000
	k ◎ 3	0.0833	0.0667	0.0667
	k ◎ 4	0.0667	0.0167	0.0500
i ◎ 3	k ◎ 1	0.0583	0.0333	0.0667
	k ◎ 2	0.0833	0.0667	0.0667
	k ◎ 3	0.0000	0.0000	0.0000
	k ◎ 4	0.0833	0.0833	0.0500
i ◎ 4	k ◎ 1	0.0417	0.0667	0.0333
	k ◎ 2	0.0667	0.0167	0.0500
	k ◎ 3	0.0833	0.0833	0.0500
	k = 4	0.0000	0.0000	0.0000

4.3 Sensitivity Analysis

In the MADM method proposed above, by sensitivity analysis, we reveal that different indeterminate ranges of ξ can change weight values and decision results.

The relationship between the indeterminate range of ξ and the weight value is shown in Fig. 1. The corresponding relationship between the indeterminate range of ξ and the decision result is exhibited in Fig. 2. Fig. 1 reflects that the weight values will change slightly as the indeterminate range of ξ changes. In Fig.2, when the range of ξ is less than [0, 0.6], the ranking order of the alternatives is $E_4 > E_2 > E_3 > E_1$. Then, the ranking order of the alternatives becomes $E_2 > E_4 > E_3 > E_1$ when the range of ξ is between [0, 0.7] and [0, 1.2]. In other ranges of ξ, the ranking order is $E_3 > E_2 > E_4 > E_1$.

5. Comparison with Existing Related MADM Models

We compare the developed TOPSIS method with existing MADM models of SNIEs [14, 23]. In the existing DM models [14, 23], the weight vector $\beta = (\beta_1, \beta_2, \beta_3)$ is specified as β = (0.30,0.36,0.34), which is not related to the indeterminate range of ξ. It is seen from Table 2 that the ranking results of the developed TOPSIS method are mostly different from those of existing DM methods [14, 23]. Comparing the results of Fig. 1 and Fig. 2, we find that the ranking order is $E_2 > E_4 > E_3 > E_1$ when the range of ξ is between [0, 0.7] and [0, 1.2], while the corresponding weight vector in Fig. 1 is gradually close to the specified weight vector β = (0.30,0.36,0.34). The decision result of this extended TOPSIS method is almost the same as that of other aggregation approaches regarding the same weight vector. It demonstrates that the developed method is effective.

In DM problems of SINEs, the decision results change with the change in the indeterminate range of ξ, but the existing DM models all use a specified weight vector, which is not related to the indeterminate range. Then, the developed DM model fully reflects the influence of the range of ξ on the weight vector and the ranking order. This new approach demonstrates the importance of decision makers' indeterminate levels. Obviously, there are three kinds of indeterminate levels in the TQE case, such as the low indeterminate level for $\xi \in \{[0, 0], [0, 0.6]\}$, the moderate indeterminate level for $\xi \in \{[0, 0.7], [0, 1.2]\}$, and the high indeterminate level for $\xi \in \{[0, 1.3], [0, 1.5]\}$. Since different indeterminate levels reflect different ranking orders, decision makers can choose the decision result based on some indeterminate level.

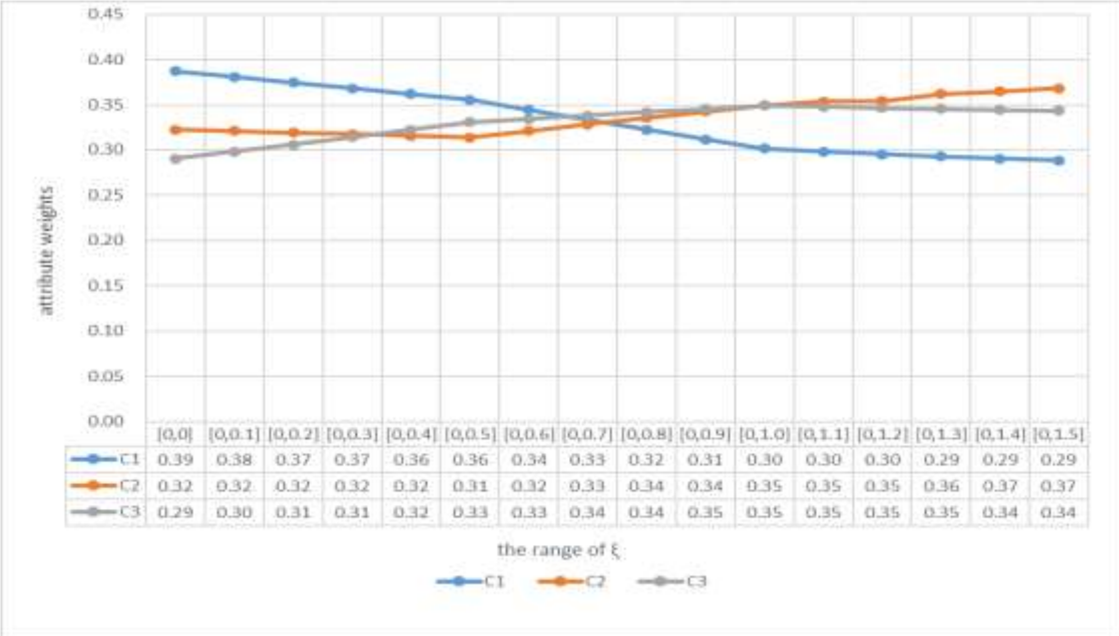

Fig. 1. Relationship between the range of ξ and the weight values

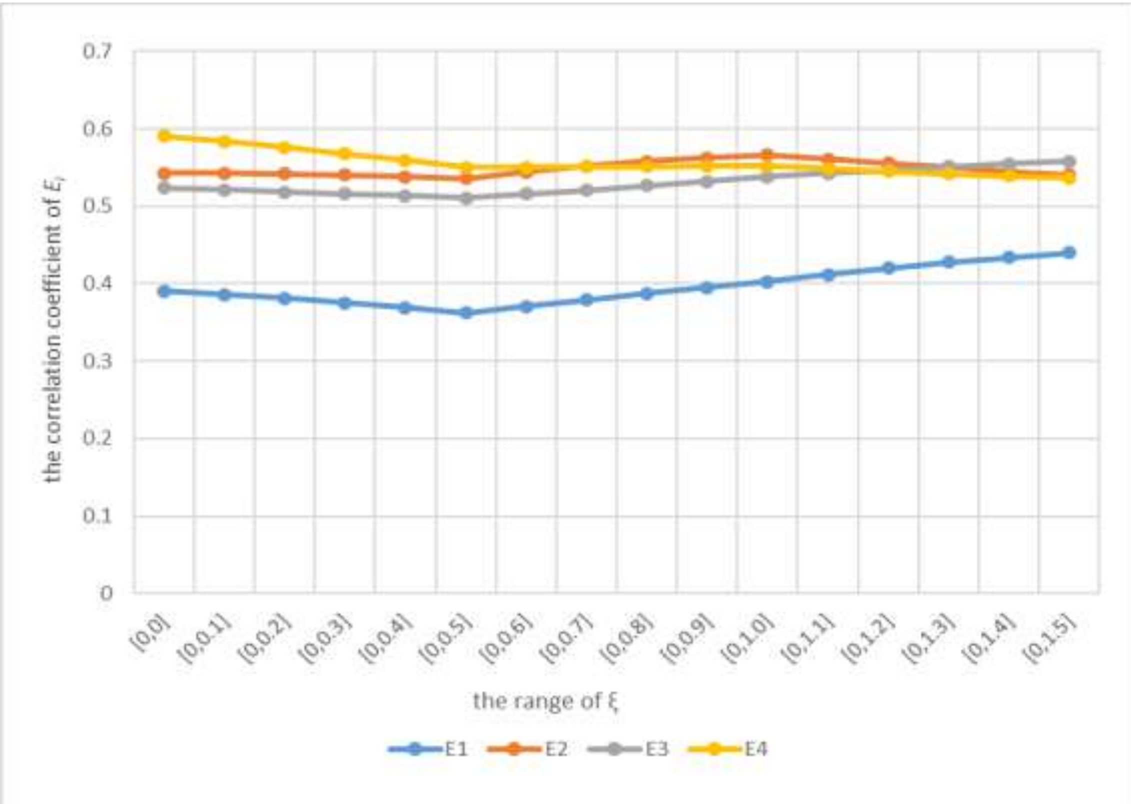

Fig. 2. Relationship between the range of ξ and the decision results

Table 2. Ranking results of different methods

Approaches	Ranking order with different indeterminacy ξ			
	ξ= [0,0]	ξ= [0,0.5]	ξ= [0,1]	ξ= [0,1.5]
SNIEWAA [14]	$E_2 > E_4 > E_3 > E_1$.	$E_2 > E_4 > E_3 > E_1$.	$E_2 > E_4 > E_3 > E_1$	$E_2 > E_3 > E_4 > E_1$
SNIEWGA [14]	$E_2 > E_4 > E_3 > E_1$.	$E_2 > E_4 > E_3 > E_1$.	$E_4 > E_2 > E_3 > E_1$	$E_3 > E_4 > E_2 > E_1$
SNIEEWA [23]	$E_2 > E_4 > E_3 > E_1$.	$E_2 > E_4 > E_3 > E_1$.	$E_2 > E_4 > E_3 > E_1$	$E_2 > E_3 > E_4 > E_1$
SNIEEWG [23]	$E_2 > E_4 > E_3 > E_1$.	$E_2 > E_4 > E_3 > E_1$.	$E_2 > E_4 > E_3 > E_1$	$E_3 > E_4 > E_2 > E_1$
TOPSIS	$E_4 > E_2 > E_3 > E_1$.	$E_4 > E_2 > E_3 > E_1$.	$E_2 > E_4 > E_3 > E_1$	$E_3 > E_2 > E_4 > E_1$

6. Conclusions

This paper first defined the Hamming distance between two SNIEs. According to the distance of SNIEs, we proposed the maximizing deviation method for determining the weight vector and the TOPSIS method for ranking alternatives. Then, the extended TOPSIS method-based MADM model was developed in the SNIE circumstance. Next, the developed MADM model was applied to a TQE case. By the case analysis, we not only obtained the decision results corresponding to the specified indeterminate ranges of ξ, but also analyzed the influence of the indeterminate range of ξ on the weight vector and the ranking order. Through comparing the developed model and the existing models, the results demonstrated that the developed model with unknown weights is valid by considering the weight vector obtained in the indeterminate range of ξ. The extended TOPSIS method-based MADM model can be widely used for these areas such as quality evaluation, service evaluation, project optimization in the environment of SNIEs.

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Fundamental group and complete parts of Neutrosophic Quadruple H_v-groups

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Abstract. It is well known that H_v-groups and groups are connected through regular relations. The purpose of this paper is to find a similar connection between neutrosophic H_v-groups and neutrosophic groups by using the concept of fundamental relations on Hv-groups. First, we characterize the complete parts of neutrosophic Hv-groups. Then we study their (strongly) regular relations. Finally, we find their fundamental group.

Keywords: neutrosophic quadruple number; neutrosophic quadruple H_v-group; fundamental group; complete part

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1. **Introduction**

Fuzzy set theory [1] has many real-world applications, but it is not suitable for simulate an indeterminate issue in an abstract situation. By giving indeterminates representation, neutrosophic theory has advanced an important concept. One of the essential aspects in practically all problems in the real world is uncertainty or indeterminacy. Fuzzy theory is used to model uncertainty, whereas neutrosophic theory is employed to represent indeterminacy. In 1995, F. Smarandache created the concept of neutrosophy to represent problems involving indeterminates. For further background on neutrosophy and neutrosophic algebraic structures, see [2–6]. By assuming that the result of "interaction" between two elements of a non-empty set is a non-void set of elements, one is obviously generalizing the definition of a group. (and not only one element, as for groups), A hypergroup was first proposed in 1934 at the eighth congress of Scandinavian mathematicians by the French mathematician Frederic Marty [7]. The rule that describes such a structure is known as a "hyperoperation", and the theory of algebraic structures with at least one multi-valued operation is recognized as the Hyperstructure Theory. Marty's motivation to introduce hypergroups is that the quotient of a group modulo any subgroup (not necessarily normal) is a hypergroup. This theory has been studied in the following decades and nowadays by many mathematicians. As a generalization of algebraic hyperstructures, Vougiouklis, in 1990, at the fourth A.H.A. congress [8,9] introduced the notion of H_v-structures.

Our paper presents a connection between hypergroups, fundamental groups and neutrosophy and it is constructed as follows: After an Introduction, in Section 2, we present some definitions related to (weak) hyperstructures that are used throughout the paper. In Section 3, we use the concept of neutrosophic H_v-group, defined by the authors in [10] and classify its complete parts and its (strongly) regular relations. Finally, in Section 4, we find the fundamental neutrosophic group of neutrosophic H_v-groups.

2. Complete parts and regular relations in neurtosphic H_v-groups

We use the notion of neutrosophic H_v-group, defined by the authors in [10] and classify its complete parts as well as its (strongly) regular relations. For basic definitions about algebraic hyperstructures we refer to [11–14]. In neutrosophy, $< X >$, $< antiX >$, and $< neutX >$ are paired two by two, as well as all three at once, to create the NeutroSynthesis. Neutrosophy lays out the universal relationships between $< X >$, $< antiX >$, and $< neutX >$. $< X >$ is the thesis, $< antiX >$ the antithesis, and $< neutX >$ the neutrothesis (neither $< X >$ nor $< antiX >$, but the neutrality in between them).

Definition 2.1. [4] Let B be a nonempty set. A neutrosophic quadruple B-number is an ordered quadruple (b_1, b_2T, b_3I, b_4F) where $b_1, b_2, b_3, b_4 \in B$ and T, I, F have their usual neutrosophic logic meanings.
The set of all neutrosophic quadruple B-numbers is denoted by NQ(B), that is,

$$NQ(B) = \{(b_1, b_42T, b_3I, b_4F) : b_1, b_2, b_3, b_4 \in B\}.$$

Let $(H\ \ , +)$ be an H_v-group with identity " 0" and define "$\oplus$" on $NQ(H\ \ )$ as follows:

$$(x_1, x_2T, x_3I, x_4F) \oplus (y_1, y_2T, y_3I, y_4F)$$
$$= \{(a, bT, cI, dF) :\ a \in x_1 + y_1, b \in x_2 + y_2, c \in x_3 + y_3, d \in x_4 + y_4\}.$$

Throughout this section, $(H, +)$ is an H_v-group with identity "0" and $0 + 0 = 0$.

Theorem 2.2. *[10] Let H be any set with a hyperoperation $+$. Then $(NQ(H), \oplus)$ is a neutrosophic quadruple H_v-group if and only if $(H, +)$ is an H_v-group with identity "$0 \in H$" and $0 + 0 = 0$.*

Theorem 2.3. *[10] Let H be any set with a hyperoperation $+$. Then $(NQ(H), \oplus)$ is a neutrosophic quadruple hypergroup if and only if $(H, +)$ is a hypergroup with identity "$0 \in H$" and $0 + 0 = 0$.*

Theorem 2.4. *Let $(H, +)$ be an H_v-group with identity "0", $0 + 0 = 0$ and X be a non empty subset of $NQ(H)$. Then X is a complete part in $NQ(H)$ if and only if there exist $A_1, A_2, A_3, A_4 \subseteq H$ such that $X = \{(a, bT, cI, dF) : a \in A_1, b \in A_2, c \in A_3, d \in A_4\}$ and that A_i is a complete part in H for $i = 1, 2, 3, 4$.*

Proof. Let X be a complete part in $NQ(H)$. Then there exist $A_1, A_2, A_3, A_4 \subseteq H$ such that $X = \{(a, bT, cI, dF) : a \in A_1, b \in A_2, c \in A_3, d \in A_4\}$. We prove that A_1 is a complete part in H and the others are done in a similar manner. Let $x \in A_1 \cap P \neq \emptyset$. Then there exist $x_i \in H$ with $i = 1, 2, \ldots, k$ such that $x \in x_1 + \ldots + x_k$. Let $y \in A_2$, $z \in A_3$ and $w \in A_4$. It is clear that

$$(x, yT, zI, wF) \in X \cap ((x_1, yT, zI, wF) \oplus (x_2, 0T, 0I, 0F) \oplus \ldots \oplus (x_k, 0T, 0I, 0F)).$$

The latter and having X a complete part in $NQ(H)$ imply that $((x_1, yT, zI, wF) \oplus (x_2, 0T, 0I, 0F) \ldots \oplus (x_k, 0T, 0I, 0F) \subseteq X$. Consequently, we get $x_1 + \ldots + x_k \subseteq A_1$.

Conversely, let $A_1, A_2, A_3, A_4 \subseteq H$ be complete parts in H , $X = \{(a, bT, cI, dF) : a \in A_1, b \in A_2, c \in A_3, d \in A_4\}$ and $(a, bT, cI, dF) \in X \cap ((x_1, y_1T, z_1I, w_1F) \oplus \ldots \oplus (x_k, y_kT, z_kI, w_kF))$. Then $a \in A_1 \cap (x_1 + \ldots + x_k)$, $b \in A_2 \cap (y_1 + \ldots + y_k)$, $c \in A_1 \cap (z_1 + \ldots + z_k)$ and $d \in A_1 \cap (w_1 + \ldots + w_k)$. Having A_i a complete part in H for $i = 1, 2, 3, 4$ implies that $x_1 + \ldots + x_k \subseteq A_1$, $y_1 + \ldots + y_k \subseteq A_2$, $z_1 + \ldots + z_k \subseteq A_3$ and $w_1 + \ldots + w_k \subseteq A_4$. The latter implies that $(x_1, y_1T, z_1I, w_1F) \oplus \ldots \oplus (x_k, y_kT, z_kI, w_kF) \subseteq X$. □

Corollary 2.5. *Let $(H, +)$ be an H_v-group with identity "0" and $0+0 = 0$ and A be a complete part in H . Then $NQ(A)$ is a complete part in $NQ(H)$.*

Proof. By setting $A_i = A$ for $i = 1, 2, 3, 4$ and $X = \{(a, bT, cI, dF) : a \in A_1, b \in A_2, c \in A_3, d \in A_4\}$, Theorem 2.4 asserts that $NQ(A) = X$ is a complete part in $NQ(H)$. □

The authors in [15, 16] considered the set of arithmetic functions H and defined a hyperoperation $+$ on it as follows:

$$\alpha + \beta(n) = \{\alpha(d) + \beta(\frac{n}{d}) : d|n\}.$$

Let $0_\star(n) = 0$ for all $n \in \mathbb{N}$. It is clear that $0_\star + 0_\star = 0_\star$. The authors proved that $(H, +)$ is a hypergroup with identity $0_\star$ and characterized all complete parts in H as: $S = \bigcup_{r \in M} A_r$ where M is a non empty subset of the set of complex numbers $\mathbb{C}$ and $A_r = \{\alpha \in H : \alpha(1) = r\}$.

Proposition 2.6. *Let* $(H, +)$ *be the hypergroup of arithmetic functions under the above hyperoperation. Then* $(NQ(H), \oplus)$ *is a neutrosophic hypergroup.*

Proof. The proof follows from Theorem 2.3. □

Proposition 2.7. *Let* $(H, +)$ *be the hypergroup of arithmetic functions defined in [16] and* X *be a complete part in* $NQ(H)$*. Then there exist non empty subsets* M_i *of* $\mathbb{C}$ *with* $i = 1, 2, 3, 4$ *such that* $X = \{(a, bT, cI, dF) : a \in S_1, b \in S_2, c \in S_3, d \in S_4\}$ *and* $S_i = \cup_{r \in M_i} A_r$ *for* $i = 1, 2, 3, 4$*.*

Proof. The proof follows from Theorem 2.4. □

Corollary 2.8. *Let* $(H, +)$ *be the hypergroup of arithmetic functions defined in [16] and* r *be any complex number. Then* $NQ(A_r)$ *is a complete part in* $NQ(H)$*.*

Let $(H, +)$ be an H_v-group with identity "0", $0 + 0 = 0$ and R_i be a relation on H for $i = 1, 2, 3, 4$. We define ρ on $NQ(H)$ as follows:

$$(a, bT, cI, dF)\rho(a', b'T, c'I, d'F) \Leftrightarrow aR_1a', bR_2b', cR_3c', dR_4d'.$$

Proposition 2.9. *Let* $(H, +)$ *be an* H_v*-group with identity "0" and* $0 + 0 = 0$*. Then* ρ *is an equivalence relation on* $NQ(H)$ *if and only if* R_i *is an equivalence relation on* H^* *for* $i = 1, 2, 3, 4$*.*

Proof. The proof is straightforward. □

Theorem 2.10. *Let* $(H, +)$ *be an* H_v*-group with identity "0" and* $0 + 0 = 0$*. Then* ρ *is a regular relation on* $NQ(H)$ *if and only if* R_i *is a regular relation on* H *for* $i = 1, 2, 3, 4$*.*

Proof. Let ρ be a regular relation on $NQ(H)$ and $a, a' \in H$ with aR_1a'. Then $(a, 0T, 0I, 0F)\rho(a', 0T, 0I, 0F)$ (as $0R_i0$ for $i = 2, 3, 4$). We need to show that $a + x\overline{R_1}a' + x$ and $x + a\overline{R_1}x + a'$ for all $x \in H$. We prove that $a + x\overline{R_1}a' + x$. Let $b \in a + x$. Then $(b, 0T, 0I, 0F) \in (a, 0T, 0I, 0F) \oplus (x, 0T, 0I, 0F)$. Having $(a, 0T, 0I, 0F)\rho(a', 0T, 0I, 0F)$ and ρ a regular relation on $NQ(H)$ imply that

$$(a, 0T, 0I, 0F) \oplus (x, 0T, 0I, 0F)\overline{\rho}(a', 0T, 0I, 0F) \oplus (x, 0T, 0I, 0F).$$

The latter implies that there exist $(y, 0T, 0I, 0F) \in (a', 0T, 0I, 0F) \oplus (x, 0T, 0I, 0F)$ such that $(z, 0T, 0I, 0F)\rho(y, 0T, 0I, 0F)$ for every $(z, 0T, 0I, 0F) \in (a, 0T, 0I, 0F) \oplus (x, 0T, 0I, 0F)$. We get now that for every $z \in a + x$ there exists $y \in a' + x$ such that zR_1y. Thus, R_1 is a regular relation on H . In a similar manner, we can prove that R_i is a regular relation on H for $i = 2, 3, 4$.

Conversely, let R_i be a regular relation on H for $i = 1, 2, 3, 4$,

$$(a, bT, cI, dF)\rho(a', b'T, c'I, d'F) \text{ and } (x, yT, zI, wF) \in NQ(H).$$

We need to show that

$$(a, bT, cI, dF) \oplus (x, yT, zI, wF)\overline{\rho}(a', b'T, c'I, d'F) \oplus (x, yT, zI, wF).$$

Let $(e, fT, gI, hF) \in (a, bT, cI, dF) \oplus (x, yT, zI, wF)$. Then $e \in a + x$, $f \in b + y$, $g \in c + z$ and $h \in d + w$. Having aR_1a', bR_2b', cR_3c', dR_4d' and R_i a regular relation on H for $i = 1, 2, 3, 4$ imply that $a + x\overline{R_1}a' + x$, $b + y\overline{R_2}b' + y$, $c + z\overline{R_3}c' + z$, $d + w\overline{R_4}d' + w$. The latter implies that there exist $e' \in a' + x$, $f' \in b' + y$, $g' \in c' + z$, $h' \in d' + w$ such that eR_1e', fR_2f', gR_3g' and hR_4h'. We get now that $(e', f'T, g'I, h'F) \in (a', b'T, c'I, d'F) \oplus (x, yT, zI, wF)$ with $(e, fT, gI, hF)\rho(e', f'T, g'I, h'F)$. □

Theorem 2.11. *Let $(H, +)$ be an H_v-group with identity "0" and $0 + 0 = 0$. Then ρ is a strongly regular relation on $NQ(H)$ if and only if R_i is a strongly regular relation on H for $i = 1, 2, 3, 4$.*

Proof. Let ρ be a strongly regular relation on $NQ(H)$ and $a, a' \in H$ with aR_1a'. Then $(a, 0T, 0I, 0F)\rho(a', 0T, 0I, 0F)$ (as $0R_i0$ for $i = 2, 3, 4$). We need to show that $a + x\overline{\overline{R_1}}a' + x$ and $x + a\overline{\overline{R_1}}x + a'$ for all $x \in H$. We prove that $a + x\overline{\overline{R_1}}a' + x$ and the other is done in a similar manner. Let $b \in a + x$. Then $(b, 0T, 0I, 0F) \in (a, 0T, 0I, 0F) \oplus (x, 0T, 0I, 0F)$. Having $(a, 0T, 0I, 0F)\rho(a', 0T, 0I, 0F)$ and ρ a strongly regular relation on $NQ(H)$ imply that $(a, 0T, 0I, 0F) \oplus (x, 0T, 0I, 0F)\overline{\overline{\rho}}(a', 0T, 0I, 0F) \oplus (x, 0T, 0I, 0F)$. The latter implies that for all $(y, 0T, 0I, 0F) \in (a', 0T, 0I, 0F) \oplus (x, 0T, 0I, 0F)$ and for all $(z, 0T, 0I, 0F) \in (a, 0T, 0I, 0F) \oplus (x, 0T, 0I, 0F)$, we have $(z, 0T, 0I, 0F)\rho(y, 0T, 0I, 0F)$. We get now that for every $z \in a + x$ and for all $y \in a' + x$ such that zR_1y. Thus, R_1 is a strongly regular relation on H^*. In a similar manner, we can prove that R_i is a strongly regular relation on H for $i = 2, 3, 4$.

Conversely, let R_i be a strongly regular relation on H for $i = 1, 2, 3, 4$, $(a, bT, cI, dF)\rho(a', b'T, c'I, d'F)$ and $(x, yT, zI, wF) \in NQ(H)$. We need to show that $(a, bT, cI, dF) \oplus (x, yT, zI, wF)\overline{\overline{\rho}}(a', b'T, c'I, d'F) \oplus (x, yT, zI, wF)$. Let $(e, fT, gI, hF) \in (a, bT, cI, dF) \oplus (x, yT, zI, wF)$. Then $e \in a + x$, $f \in b + y$, $g \in c + z$ and $h \in d + w$. Having aR_1a', bR_2b', cR_3c', dR_4d' and R_i a strongly regular relation on H for $i = 1, 2, 3, 4$ imply that $a + x\overline{\overline{R_1}}a' + x$, $b + y\overline{\overline{R_2}}b' + y$, $c + z\overline{\overline{R_3}}c' + z$, $d + w\overline{\overline{R_4}}d' + w$. The latter implies that for all $e' \in a' + x$, $f' \in b' + y$, $g' \in c' + z$, $h' \in d' + w$, we have eR_1e', fR_2f', gR_3g' and hR_4h'. We get now that $(e', f'T, g'I, h'F) \in (a', b'T, c'I, d'F) \oplus (x, yT, zI, wF)$ with $(e, fT, gI, hF)\rho(e', f'T, g'I, h'F)$. □

Example 2.12. Let $\mathbb{Q}$ be the set of rational numbers, $(H,+)$ be the hypergroup of arithmetic functions and define the strongly regular equivalence relation R_i for $i = 1,2,3,4$ on H as follows:

$$\alpha R_1 \gamma \Leftrightarrow \alpha(1) = \gamma(1) + q; q \in \mathbb{Q},$$

and for $i = 2,3,4$

$$\alpha R_i \gamma \Leftrightarrow \alpha(1) = \gamma(1).$$

Applying Theorem 2.11, we get ρ a strongly regular equivalence relation on $NQ(H)$, where ρ is defined as follows:

$$(\alpha, \beta T, \gamma I, \lambda F)\rho(\alpha', \beta' T, \gamma' I, \lambda' F)$$
$$\Leftrightarrow \alpha(1) = \alpha'(1) + q; q \in \mathbb{Q}, \beta(1) = \beta'(1), \gamma(1) = \gamma'(1), \lambda(1) = \lambda'(1).$$

3. Fundamental group of neutrosophic quadruple H_v-groups

In this part, we investigate the fundamental relation on neutrosophic H v-groups and determine its fundamental neutrosophic group..

In [3], Akinleye et. al. conducted their investigation of neutrosophic quadruple algebraic structures on quadruple numbers based on the set of real numbers. And they proved that if G is a group of real numbers then $NQ(G) = \{(g_1, g_2T, g_3I, g_4F) : g_1, g_2, g_3, g_4 \in G\}$ is a neutrosophic group under the operation "$\oplus$" given by:

$$g_1, g_2T, g_3I, g_4F) \oplus (g_1{}', g_2{}'T, g_3{}'I, g_4{}'F) = (g_1 + g_1', (g_2 + g_2{}')T, (g_3 + g_3{}')I, (g_4 + g_4{}')F).$$

The following theorem generalizes their result to any group.

Theorem 3.1. *Let G be a set with operation $+$ and $0 \in G$. Then $(NQ(G), \oplus)$ is a group if and only if $(G,+)$ is a group.*

Proof. The proof is similar to that of Theorem 2.1 in [3]. □

Proposition 3.2. *Let $(G,+)$ and $(G',+')$ be isomorphic groups. Then $(NQ(G), \oplus)$ and $(NQ(G'), \oplus')$ are isomorphic neutrosophic groups.*

Proof. Let $(G,+)$ and $(G',+')$ be isomorphic groups. Then there exists a group isomorphism $f : G \to G'$. Let $\phi : NQ(G) \to NQ(G')$ be defined as follows:

$$\phi((a, bT, cI, dF)) = (f(a), f(b)T, f(c)I, f(d)F).$$

One can easily see that ϕ is a group isomorphism. □

Definition 3.3. [13] For all $n > 1$, we define the relation β_n on a semihypergroup $(H, \circ)$ as follows:

$$x\beta_n y \text{ if there exist } a_1, \ldots, a_n \text{ in } H \text{ such that } \{x, y\} \subseteq \prod_{i=1}^{n} a_i$$

and we set $\beta = \bigcup_{n \geq 1} \beta_n$, where $\beta_1 = \{(x, x) \mid x \in H\}$ is the diagonal relation on H^*.

This relation was established by Koskas [17] and studied by many authors. Clearly, the relation β is reflexive and symmetric. Denote by β^* the transitive closure of β.

Throughout this section, $\beta, \beta^\star$ are the relations on H and $\beta_N, \beta_N^\star$ are the relations on $NQ(H)$.

Theorem 3.4. *Let $(H, +)$ be an H_v-group with identity "0" and $0 + 0 = 0$ and let $a, a', b, b', c, c', d, d' \in H$. Then $(a, bT, cI, dF)\beta_N(a', b'T, c'I, d'F)$ if and only if $a\beta a'$, $b\beta b'$, $c\beta c'$ and $d\beta d'$.*

Proof. Let $(a, bT, cI, dF)\beta_N(a', b'T, c'I, d'F)$. Then there exist (x_i, y_iT, z_iI, w_iF) with $i = 1, \ldots, k$ such that $\{(a, bT, cI, dF), (a', b'T, c'I, d'F)\} \subseteq (x_1, y_1T, z_1I, w_1F) \oplus \ldots \oplus (x_k, y_kT, z_kI, w_kF)$. The latter implies that $a, a' \in x_1 + \ldots + x_k$, $b, b' \in y_1 + \ldots + y_k$, $c, c' \in z_1 + \ldots + z_k$ and $d \in w_1 + \ldots + w_k$. Thus, $a\beta a'$, $b\beta b'$, $c\beta c'$ and $d, d'\beta d'$.

Conversely, let $a\beta a'$, $b\beta b'$, $c\beta c'$ and $d\beta d'$. Then there exist $k_1, k_2, k_3, k_4 \in \mathbb{N}$ and $x_1, \ldots, x_{k_1}$, $y_1, \ldots, y_{k_2}$, $z_1, \ldots, z_{k_3}$, $w_1, \ldots, w_{k_4} \in H$ such that $a, a' \in x_1 + \ldots + x_{k_1}$, $b, b' \in y_1 + \ldots + y_{k_2}$, $c, c' \in z_1 + \ldots + z_{k_3}$ and $d, d' \in w_1 + \ldots + w_{k_4}$. By setting $k = \max\{k_1, k_2, k_3, k_4\}$ and $x_i = 0$ for $k_1 < i \leq k$, $y_i = 0$ for $k_2 < i \leq k$, $z_i = 0$ for $k_3 < i \leq k$ and $w_i = 0$ for $k_4 < i \leq k$ and using the fact that $x \in 0 + x \cap x + 0$ for all $x \in H$, we get $a, a' \in x_1 + \ldots + x_k$, $b, b' \in y_1 + \ldots + y_k$, $c, c' \in z_1 + \ldots + z_k$ and $d, d' \in w_1 + \ldots + w_k$. The latter implies that $\{(a, bT, cI, dF), (a', b'T, c'I, d'F)\} \subseteq (x_1, y_1T, z_1I, w_1F) \oplus \ldots \oplus (x_k, y_kT, z_kI, w_kF)$. Thus, $(a, bT, cI, dF)\beta_N(a', b'T, c'I, d'F)$. □

Theorem 3.5. *Let $(H\ , +)$ be an H_v^*-group with identity "0" and $0 + 0 = 0$ and let $a, a', b, b', c, c', d, d' \in H$. Then $(a, bT, cI, dF)\beta_N^\star(a', b'T, c'I, d'F)$ if and only if $a\beta^\star a'$, $b\beta^\star b'$, $c\beta^\star c'$ and $d\beta^\star d'$.*

Proof. Let $(a, bT, cI, dF)\beta_N^\star(a', b'T, c'I, d'F)$. Then there exist $(x_i, y_iT, z_iI, w_iF) \in NQ(H)$ with $i = 1, \ldots, k$ such that
$(a, bT, cI, dF)\beta_N(x_1, y_1T, z_1I, w_1F)$, $(x_i, y_iT, z_iI, w_iF)\beta_N(x_{i+1}, y_{i+1}T, z_{i+1}I, w_{i+1}F)$ for $i = 1, \ldots, k-1$ and $(x_k, y_kT, z_kI, w_kF)\beta_N(a', b'T, c'I, d'F)$. Theorem 3.4 implies that $a\beta x_1$, $x_i\beta x_{i+1}$ for $i = 1, \ldots, k-1$, $x_k\beta a'$, $b\beta y_1$, $y_i\beta y_{i+1}$ for $i = 1, \ldots, k-1$, $y_k\beta b'$, $c\beta z_1$, $z_i\beta z_{i+1}$ for $i = 1, \ldots, k-1$, $z_k\beta c'$, $d\beta w_1$, $w_i\beta w_{i+1}$ for $i = 1, \ldots, k-1$, $w_k\beta d'$. Thus, $a\beta^\star a'$, $b\beta^\star b'$, $c\beta^\star c'$ and $d\beta^\star d'$.

Conversely, let $a\beta^\star a'$, $b\beta^\star b'$, $c\beta^\star c'$ and $d\beta^\star d'$. Then there exist $x_i, y_i, z_i, w_i \in H$ such that $a\beta x_1$, $x_i\beta x_{i+1}$ for $i = 1,\ldots,k-1$, $x_k\beta a'$, $b\beta y_1$, $y_i\beta y_{i+1}$ for $i = 1,\ldots,l-1$, $y_l\beta b'$, $c\beta z_1$, $z_i\beta z_{i+1}$ for $i = 1,\ldots,m-1$, $z_m\beta c'$, $d\beta w_1$, $w_i\beta w_{i+1}$ for $i = 1,\ldots,s-1$, $w_s\beta d'$. By setting $t = \max\{k,l,m,s\}$ and $x_i = a'$ for $k < i \leq t$, $y_i = b'$ for $t < i \leq t$, $z_i = c'$ for $m < i \leq t$ and $w_i = d'$ for $s < i \leq t$. The latter implies that

$$(a, bT, cI, dF)\beta_N(x_1, y_1T, z_1I, w_1F), \quad (x_i, y_iT, z_iI, w_iF)\beta_N(x_{i+1}, y_{i+1}T, z_{i+1}I, w_{i+1}F),$$

for $i = 1,\ldots,t-1$ and

$$(x_t, y_tT, z_tI, w_tF)\beta_N(a', b'T, c'I, d'F).$$

Thus, $(a, bT, cI, dF)\beta_N^\star(a', b'T, c'I, d'F)$. □

Theorem 3.6. *Let $(H\ ,+)$ be an H_v^*-group with identity "0" and $0 + 0 = 0$. Then $NQ(H)/\beta_N^\star \cong NQ(H/\beta^\star)$.*

Proof. Let $\phi : NQ(H)/\beta_N^\star \to NQ(H/\beta^\star)$ be defined as

$$\phi(\beta_N^\star((a, bT, cI, dF))) = (\beta^\star(a), \beta^\star(b)T, \beta^\star(c)I, \beta^\star(d)F).$$

Theorem 3.5 implies that ϕ is well-defined and one-to-one. Also, it is clear that ϕ is onto. We need to show that ϕ is a group homomorphism. Since

$$\beta_N^\star((a, bT, cI, dF)) \boxplus' \beta_N^\star((a', b'T, c'I, d'F)) = \beta_N^\star((x, yT, zI, wF))$$

where $(x, yT, zI, wF) \in (a, bT, cI, dF) \oplus (a', b'T, c'I, d'F) = (a + a', (b + b')T, (c + c')I, (d + d')F)$, it follows that $\phi(\beta_N^\star((a, bT, cI, dF)) \boxplus \beta_N^\star((a', b'T, c'I, d'F))) = \phi((x, yT, zI, wF)) = (\beta^\star(x), \beta^\star(y)T, \beta^\star(z)I, \beta^\star(w)F)$. Having $\beta^\star(x) = \beta^\star(a) \boxplus \beta^\star(a')$, $\beta^\star(y) = \beta^\star(b) \boxplus \beta^\star(b')$, $\beta^\star(z) = \beta^\star(c) \boxplus \beta^\star(c')$ and $\beta^\star(w) = \beta^\star(d) \boxplus \beta^\star(d')$ imply that $\phi(\beta_N^\star((a, bT, cI, dF)) \boxplus' \beta_N^\star((a', b'T, c'I, d'F))) = \phi(\beta_N^\star((a, bT, cI, dF)) \oplus' \phi(\beta_N^\star((a', b'T, c'I, d'F)))$. □

Theorem 3.7. *Let $(H, +)$ be an H_v^*-group with identity "0" and $0 + 0 = 0$. If G is the fundamental group of H^* (up to isomorphism) then $NQ(G)$ is the fundamental group of $NQ(H)$ (up to isomorphism).*

Proof. Since G is the fundamental group of H (up to isomorphism), it follows that $H/\beta^\star \cong G$. The latter and Proposition 3.2 imply that $NQ(H/\beta^\star) \cong NQ(G)$. Theorem 3.6 completes the proof. □

Example 3.8. Let $(H, +)$ be the hypergroup of arithmetic functions defined in [15] with the group of complex numbers $(\mathbb{C}, +)$ under standard addition as a fundamental group (up to isomorphism). Then $(NQ(\mathbb{C}), +)$ is the fundamental group of $NQ(H)$ up to isomorphism.

Corollary 3.9. *Let $(H,+)$ be an H_v-group with identity "0" and $0+0=0$. If H has a trivial fundamental group then $NQ(H)$ has a trivial fundamental group.*

Proof. The proof is straightforward by applying Corollary 3.7. □

Example 3.10. Let $H_1 = \{0,1\}$ and define $(H_1,+_1)$ as follows:

$+_1$	0	1
0	0	1
1	1	H_1

Then $(NQ(H_1),\oplus)$ is a quadruple H_v-group. Since $0,1 \in 1+_1 1$, it follows that $x\beta y$ for all $x,y \in H_1$. Thus, H_1 has trivial fundamental group. Corollary 3.9 asserts that $(NQ(H_1),\oplus)$ has trivial fundamental group.

Theorem 3.11. *Let $(H,+)$ be an H_v-group with identity "0" and $0+0=0$ and w_H be the heart of H . Then $NQ(w_H)$ is the heart of $NQ(H)$.*

Proof. Let $w_H = \{x \in H : \beta^\star(x) = \beta^\star(0)\}$ be the heart of H . Having $w_{NQ(H)} = \{(a,bT,cI,dF) \in NQ(H) : \beta^\star_N((a,bT,cI,dF)) = \beta^\star_N(0,0T,0I,0F)\}$ and applying Theorem 3.5, we get that $\beta^\star(a)=\beta^\star(b)=\beta^\star(c)=\beta^\star(d)=\beta^\star(0)$. Thus, $w_{NQ(H)} = \{(a,bT,cI,dF) \in NQ(H) : a,b,c,d \in w_H\} = NQ(w_H)$. □

Example 3.12. Let $(H,+)$ be the hypergroup of arithmetic functions defined in [15]. Then $NQ(A_0)$ is the heart of $NQ(H)$.

4. **Conclusion**

This paper connected neutrosophic H_v-groups and neutrosophic groups by means of complete parts and regular relations. More precisely, it used the complete parts and the fundamental relation of H_v-groupls to reach its main results that are summarized in Theorems 4.6 and 4.7. The results were supported by non-trivial illustrative examples. For future research, it is interesting to find a connection between other types of neutrosophic H_v-structures and neutrosophic algebraic structures. One can investigate the connection between neutrosophic H_v-rings and neutrosophic rings or the connection between neutrosophic H_v-modules and neutrosophic modules.

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Possibility Neutrosophic Hypersoft Set (PNHSS)

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Abstract: Soft set developed by Smarandache in 2018 to Hypersoft set (HSS) to deal with multi-argument approximate functions. The soft set cannot deal with cases when attributes are required to be further divided into disjoint attribute-valued sets. Neutrosophic Hypersoft Set (NHSS) is the most effective and useful method to handle the environment which involved more than one attribute. Neutrosophic Hypersoft Set introduced by combining Hypersoft Set and Neutrosophic Soft Set. In this paper, we first define the concept of Possibility Neutrosophic Hypersoft Set (PNHSS in short) which is combination of PNSS and HSS. Certain essential basic characteristics as subset, equal and complement are studied with illustrative examples. Basic operations such as: union, intersection and some properties such as commutative, associative, distributive low and De Morgan's low are discussing. Also, we introduce AND and OR operation of PNHSS with suitable examples and some propositions.

Keywords: Soft Set, Neutrosophic Soft Set, Hypersoft Set, Neutrosophic Hypersoft Set, Possibility Neutrosophic Set and Possibility Neutrosophic Hybersoft Set.

1. Introduction

Fuzzy sets were developed by Zadeh [1] to solve problems which contain uncertain information. Some cases cannot deal with fuzzy set, so Turksen [2] introduced interval-valued fuzzy set. Atanassove [3] extended fuzzy set to Intuitionistic fuzzy set. Which more general than fuzzy set.

Neutrosophy introduced by Smarandache [4] which is a new tool for dealing with problems containing imprecise, indeterminacy and inconsistent data.

Neutrosophic sets which introduced by Smarandach in 2005 [5] is a generalization of the Intuitionistic fuzzy set.

Soft Set defined by Molodtsov [6] as another commonly used method in handling uncertainties in the data. Soft Set extended and introduced some of its operations and properties by Maji [8]. Sezgin et al. [11] were proved De Morgan s Law on Soft Set.

The concept of fuzzy soft set introduced by Maji [7]. Fuzzy soft set extended to Generalized fuzzy soft sets by Majumdar and Samanta in 2010 [9]. They joined the degree with the parameterization of fuzzy soft sets while defining a fuzzy soft set. Here for each parameter e_i and $\forall i = 1, \dots, n$, $F_\mu(e_i) = (F(e_i), \mu(e_i))$ indicates not only the degree of belongingness of the elements of U in $F(e_i)$ but also the degree of possibility of such belongingness which is represented by $\mu(e_i)$. The concept of possibility fuzzy soft set introduced by Alkahazaleh et al. [10] by assigning a possibility degree to each number of fuzzy sets.

Neutrosophic Soft Set NSS with basic basic operation and properties proposed by Maji [12]. The new concept Generalised neutrosophic soft set GNSS which introduced by Sahin [13], was extension of the concept NSS defined by Maji [8]. NSS was also extended by Karaaslan [14] and defined Possibility Neutrosophic Soft Set. NSS developed by Broumi [15] to Generalised Neutrosophic Soft Set with basic definitions and operations. He used this concept for solving decision making problems. Recently the researchers [16–20] extended the theory of neutrosophic soft set and developed it by discussion and applications in decision making.

In 2018, Soft Set developed to hypersoft Set by converting a single attribute- valued function to multi-attribute valued function by Smarandache [21]. In 2019, Saqlain et al. [22] extended this concept to deals with the Generalization of TOPSIS for NHSS, by using accuracy function.

In 2020, the concept of HSS was generalized and the fundamentals of HSS with some relations and operations on HSS by Saeed et al. [23, 24]. The concept of fuzzy plithogenic hypersoft set in matrix introduced with some basic operations and properties in [25]. The combination of two concepts: Plithogenic set and hypersoft set gave a new concept, which was Plithogenic hypersoft set introduced in [26].

The concept of hypersoft point defined in different environments such as; fuzzy hypersoft set, Intuitionistic fuzzy hypersoft set, neutrosophic hypersoft set and gave some basic operation of hypersoft points in these environments by Majahid et al. [27].

Aggregate operators of NHSS were discussed in some cases by Saqlain et al. [28] with examples.

Zulqarnain et al. [29] developed the Aggregate operators of NHSS with examples and properties.

The concept of Complex hypersoft set defined by Rahman et al. [30]. They generalized the hybrids of hypersoft set with complex fuzzy and its generalized structure. Rahman et al. [31] introduced the concept of Convex and Concave hypersoft Sets with some properties and sutiable examples. In 2021, Rahman et al. [32] introduced an application in decision making based on fuzzy parametrized hypersoft set theory. They made the existing literature regarding fuzzy parametrized soft set in line with the need of multi-attribute function. Another application to solve problems in decision making based on neutrosophic parametrized hypersoft set theory introduced in [33]. Numerous researchers discussed the concept of Rough soft set which was combination between rough set and soft set. Rahman et al. [34] introduced development of rough hypersoft set with application in decision making for the best choice of chemical material. They proposed a new algorithm to solve decision making problems with illustrative examples. Saeed et al. [35] defined the concept of mapping hypersoft classes. They developed some properties of mapping on hypersoft set classes such as hypersoft images and hypersoft images.

In 2022, Debanath [36] presented the notion of Interval-valued intuitionistic fuzzy hypersoft sets (IVIFHSSs), which was combining interval-valued intuitionistic fuzzy sets (IVIFSS) and hypersoft sets (HSSs). He also, discussed some different operators of this concept such as complement, union, intersection, AND and OR. He introduced a new algorithm based on (IVIFHSSs). Finally, he introduced a numerical example to check the reliability and validity of the algorithm.

Florentin Smarandache [37] introduced for the first time the concept of IndetermSoft as extention of soft set, that deals with indeterminate data, where 'Indeterm' stands for 'Indeterminate'. Similarly, he extended hypersoft set to IndetermHypersoft set. At the end, he presented an application of the IndetermHyperSoft Set. Ihsan et. [38] defined expert set on Neutrosophic hypersoft set. This model solved the problem of dealing with one expert and solved the problem of different parametric-valued sets parallel to different characteristics. They discussed basic characteristics, aggregation operation, and results with examples. Finally, they presented an application to NHSES in decision making problem. Neutrosophic hypersoft set are developed and an application is discussed in decision making, which appear from [39]-[43].

The organization of this paper as follows: Section 2 present the basic definitions of neutrosophic set, soft set, Neutrosophic soft set, Hypersoft set, Possibility neutrosophic soft set, Neutrosophic Hypersoft set and some relative definitions used in this work. Section 3 define the new concept of possibility neutrosophic hypersoft set with related definitions and suitable examples. Section 4 describes the basic operations of PNHSS. Section 5 discusses AND and OR operation. Section 6 presents conclude of this paper with suggested future work.

2. Preliminary

In this section, we present some definitions required in this paper.

Definition 1 [5] Neutrosophic Set.

A neutrosophic set A on the universe of discourse X is defined as $A = \{\langle x: T_A(x), I_A(x), F_A(x)\rangle; \mathrm{x} \in \mathrm{X}\}$ where $T; I; F : X \to [0,1]$ and $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$.

Definition 2 [8] Soft Set.

Let U be a universe and E be a set of parameters. Let $P(U)$ denote the power set of U and $A \subseteq E$. A pair (F, A) is called a soft set over U, where F is a mapping $F: A \to P(U)$.

Definition 3 [7] Fuzzy soft set.

Let U be an initial universal set and let E be a set of parameters. Let I^U denote the power set of all fuzzy subsets of U. Let $A \subseteq E$. A pair (F, E) is called a fuzzy soft set over U where F is a mapping given by $F : A \to I^U$.

Definition 4 [21] Neutrosophic Soft Set.

Let U be an initial universe set and E be a set of parameters. Consider $A \subseteq E$. Let $P(U)$ denotes the set of all neutrosophic sets of U. The collection (F, A) is termed to be the soft neutrosophic set over U, where F is a mapping given by $F: A \to P(U)$.

Definition 5 [21] Hypersoft Set.

Let U be a universal set and $P(U)$ be the all neutrosophic subset of U and for $n \ge 1$, there are n distinct attributes such as $\ell_1, \ell_2, \dots, \ell_n$ and $L_1, L_2, \dots, L_n$ are sets for corresponding values attributes respectively with following conditions such as $L_i \cap L_j = \emptyset$ for $i \ne j$ and $i, j \in \{1, 2, \dots, n\}$.

Then the pair $(\psi, L_1 \times L_1 \times \dots \times L_n)$ is said to be Hypersoft set over U where ψ is a mapping from $L_1 \times L_1 \times \dots \times L_n$ to $P(U)$.

Definition 6 [14] Possibility Neutrosophic Soft Set (GNSS).

Let U be an initial universe and E be a set of parameters. Let $N\ (U)$ be the set of all neutrosophic sets Of U and I^U is collection of all fuzzy subset of U. A possibility neutrosophic soft set f_μ over U is defined by the set of ordered pairs

$f_\mu(e) = \left\{\left(e_k, \left\{\left(\frac{u_j}{f(e_k)(u_j)}, \mu(e_k)(u_j)\right) : u\_j \in U\right\}\right) : e\_k \in E\right\}$, or a mapping defined by

$f_\mu: E \to N(U) \times I^U$ where μ is a fuzzy set such that $\mu: E \to I = [0,1]$ and f_μ is a mapping defined by $f_\mu: E \to N(U)$.

Definition 7 [27] Neutrosophic Hypersoft Set (NHSS).

Let U be a universal set and $P(U)$ be a power set of U and for $n \ge 1$, there are n distinct attributes such as $\ell_1, \ell_2, \dots, \ell_n$ and $L_1, L_2, \dots, L_n$ are sets for corresponding values attributes respectively with following conditions such as $L_i \cap L_j = \emptyset$ for $i \ne j$ and $i, j \in \{1, 2, \dots, n\}$. Then the pair $(\psi\ , \Lambda)$ is said to be NHSS over U if there exists a relation $L_1 \times L_1 \times \dots \times L_n = \Lambda$. ψ is a mapping from $L_1 \times L_1 \times \dots \times L_n$ to $P(U)$ and $\psi_\Lambda(L_1 \times L_1 \times \dots \times L_n) = \{\langle u,\ T_\Lambda(u), I_\Lambda(u), F_\Lambda(u)\rangle : u \in U\}$ where T, I, F are membership values for truthness, indeterminacy, and falsity respectively such that $T, I, F: U \to [0,1]$ and $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$.

Definition 8 [28] Neutrosophic Hypersoft subset (NHSS).

For two Neutrosophic Hypersoft subsets (NHSs) $\psi_{\Lambda_1}$and ψ_{Λ_2}over U, ψ_{Λ_1} is called a neutrosophic hypersoft subset of ψ_{Λ_2} if $T(\psi_{\Lambda_1}) \leq T(\psi_{\Lambda_2}), I(\psi_{\Lambda_1}) \leq I(\psi_{\Lambda_2}), F(\psi_{\Lambda_1}) \geq F(\psi_{\Lambda_2})$.

Definition 9 [28] Neutrosophic Hypersoft set equal.

Two Neutrosophic Hypersoft subsets (NHSs) $\psi_{\Lambda_1}$and ψ_{Λ_2} over U, are said to be equal if $\psi_{\Lambda_1}$is a NHSs of ψ_{Λ_2} and ψ_{Λ_2} is a NHSs of ψ_{Λ_1}.

Definition 10 [28] Neutrosophic Hypersoft set complement.

The complement of a Neutrosophic Hypersoft Set ψ_Λ is denoted by $(\psi_\Lambda)^c$ is defined by $(\psi_\Lambda)^c$ such that $(\psi_\Lambda)^c = \{\langle u, T(\psi_\Lambda{}^c) = F(\psi_\Lambda), I(\psi_\Lambda{}^c) = 1 - I(\psi_\Lambda), F(\psi_\Lambda{}^c) = T(\psi_\Lambda)\rangle \ , u \in U\}$.

Definition 11 [29] Neutrosophic Hypersoft set union.

The union of two NHSs ψ_{Λ_1} and ψ_{Λ_2}over the common universe U. denoted by $\psi_{\Lambda_1} \cup \psi_{\Lambda_2}$ is the NHS and is given as follows:

$$T(\psi_{\Lambda_1} \cup \psi_{\Lambda_2}) = \begin{cases} T(\psi_{\Lambda_1}) & if\ u \in \Lambda_1 - \Lambda_2, \\ T(\psi_{\Lambda_2}) & if\ u \in \Lambda_2 - \Lambda_1, \\ max\left(T(\psi_{\Lambda_1}), T(\psi_{\Lambda_2})\right) & if\ u \in \Lambda_1 \cap \Lambda_2, \end{cases}$$

$$I(\psi_{\Lambda_1} \cup \psi_{\Lambda_2}) = \begin{cases} I(\psi_{\Lambda_1}) & if\ u \in \Lambda_1 - \Lambda_2, \\ I(\psi_{\Lambda_2}) & if\ u \in \Lambda_2 - \Lambda_1, \\ min\left(I(\psi_{\Lambda_1}), I(\psi_{\Lambda_2})\right) & if\ u \in \Lambda_1 \cap \Lambda_2, \end{cases}$$

$$F(\psi_{\Lambda_1} \cup \psi_{\Lambda_2}) = \begin{cases} F(\psi_{\Lambda_1}) & if\ u \in \Lambda_1 - \Lambda_2, \\ F(\psi_{\Lambda_2}) & if\ u \in \Lambda_2 - \Lambda_1, \\ min\left(F(\psi_{\Lambda_1}), F(\psi_{\Lambda_2})\right) & if\ u \in \Lambda_1 \cap \Lambda_2. \end{cases}$$

Definition 12 [29] Neutrosophic Hypersoft intersection.

The intersection of two NHSs ψ_{Λ_1} and ψ_{Λ_2}over the common universe U. denoted by $\psi_{\Lambda_1} \cap \psi_{\Lambda_2}$ is the NHS and is given as follows:

$$T(\psi_{\Lambda_1} \cap \psi_{\Lambda_2}) = \begin{cases} T(\psi_{\Lambda_1}) & if\ u \in \Lambda_1 - \Lambda_2, \\ T(\psi_{\Lambda_2}) & if\ u \in \Lambda_2 - \Lambda_1, \\ min\left(T(\psi_{\Lambda_1}), T(\psi_{\Lambda_2})\right) & if\ u \in \Lambda_1 \cap \Lambda_2, \end{cases}$$

$$I(\psi_{\Lambda_1} \cap \psi_{\Lambda_2}) = \begin{cases} I(\psi_{\Lambda_1}) & if\ u \in \Lambda_1 - \Lambda_2, \\ I(\psi_{\Lambda_2}) & if\ u \in \Lambda_2 - \Lambda_1, \\ max\left(I(\psi_{\Lambda_1}), I(\psi_{\Lambda_2})\right) & if\ u \in \Lambda_1 \cap \Lambda_2, \end{cases}$$

$$F(\psi_{\Lambda_1} \cap \psi_{\Lambda_2}) = \begin{cases} F(\psi_{\Lambda_1}) & if\ u \in \Lambda_1 - \Lambda_2, \\ F(\psi_{\Lambda_2}) & if\ u \in \Lambda_2 - \Lambda_1, \\ max\left(F(\psi_{\Lambda_1}), F(\psi_{\Lambda_2})\right) & if\ u \in \Lambda_1 \cap \Lambda_2. \end{cases}$$

Definition 13 [29] AND-Operation of Two Neutrosophic Hypersoft Set.

Let ψ_{Λ_1} and ψ_{Λ_2} be two NHSs over the common universe U, then $\psi_{\Lambda_1} \wedge \psi_{\Lambda_2} = \psi_{\Lambda_1 \times \Lambda_2}$ is given as follows:

$T(\psi_{\Lambda_1 \times \Lambda_2}) = min\left(T(\psi_{\Lambda_1}), T(\psi_{\Lambda_2})\right)$,

$I(\psi_{\Lambda_1\times\Lambda_2}) = max\left(I(\psi_{\Lambda_1}), I(\psi_{\Lambda_2})\right),$

$F(\psi_{\Lambda_1\times\Lambda_2}) = max\left(I(\psi_{\Lambda_1}), I(\psi_{\Lambda_2})\right).$

Definition 14 [29] OR-Operation of Two Neutrosophic Hypersoft Set.

Let ψ_{Λ_1} and ψ_{Λ_2} be two NHSs over the common universe U, then $\psi_{\Lambda_1} \vee \psi_{\Lambda_2} = \psi_{\Lambda_1\times\Lambda_2}$ is given as follows:

$T(\psi_{\Lambda_1\times\Lambda_2}) = max\left(T(\psi_{\Lambda_1}), T(\psi_{\Lambda_2})\right),$

$I(\psi_{\Lambda_1\times\Lambda_2}) = min\left(I(\psi_{\Lambda_1}), I(\psi_{\Lambda_2})\right),$

$F(\psi_{\Lambda_1\times\Lambda_2}) = min\left(I(\psi_{\Lambda_1}), I(\psi_{\Lambda_2})\right).$

3.Fundamental of Possibility Neutrosophic Hypersoft Set

Definition 15 Possibility Neutrosophic Hypersoft Set (PNHSS)

Let $\mathfrak{I}$ be the universal set and $N(\mathfrak{I})$ be set of all neutrosophic subset of $\mathfrak{I}$. For $n \geq 1$, let $\ell_1, \ell_2, \dots, \ell_n$ be n well-defined attributes, whose corresponding attributive are respectively the set $L_1, L_2, \dots, L_n$ with $L_i \cap L_j = \emptyset$ for $i \neq j$ and $i, j \in \{1, 2, \dots, n\}$ and their relation $L_1 \times L_1 \times \dots \times L_n = \Lambda$. The pair (ψ^{μ}, Λ) is said to be possibility neutrosophic hypersoft set over $\mathfrak{I}$ where

$\psi^{\mu}_{\Lambda}(e) = \{\langle x, (\psi_{\Lambda}(e)(\mathrm{x}), \mu(e))\rangle : x \in \mathfrak{I}, \psi_{\Lambda}(e)(\mathrm{x}) \in \in N(\mathfrak{I}) \text{ and } \mu(e) \in I = [0,1]\}$. Where ψ_{Λ} is a mapping given by $\psi_{\Lambda}: L_1 \times L_1 \times \dots \times L_n \longrightarrow N(\mathfrak{I})$ and μ is a fuzzy set such that $\mu: \Lambda \longrightarrow I$. Here ψ^{μ}_{Λ} is a mapping defined

$\psi^{\mu}_{\Lambda}: L_1 \times L_1 \times \dots \times L_n \longrightarrow N(\mathfrak{I}) \times I.$

Example 1

Let $\mathfrak{I}$ be the set of decision makers to decide best car given as $\mathfrak{I} = \{d_1, d_2, d_3, d_4\}$ and a set $M = \{d_1, d_2\} \subset \mathfrak{I}$. Also consider the set of attributes as

$L_1 = Car\ type, L_2 = Engine\ capacity, L_3 = Saftey, L_4 = Performace$ and their respective attributes are given as follows:

$L_1 = Car\ type = \{Mercedes - Benz, BMW, Volvo, Ford\}$

$L_2 = Engine\ capacity = \{1500cc, 1800cc, 2000cc, 2500cc\}$

$L_3 = Saftey = \{APS, Air\ bag\}$

$L_4 = Performace = \{car\ torque, speeds\}$

Let $\psi^{\mu}_{\Lambda}: L_1 \times L_2 \times L_3 \times L_4 \to N(\mathfrak{I}) \times I$

And $\mu: \Lambda \longrightarrow I$. Assume that the customer concentrate on type of car is BMW with engine capacity which provide air bag and speed. Then PNHSS is defined as follows:

$\psi^{\mu}_{\Lambda}(L_1 \times L_2 \times L_3 \times L_4) = \psi^{\mu}_{\Lambda}(BMW, 2000cc, Air\ bag, Speed) = \{d_1, d_2\}$

Then the relation of above PNHSS is given as

$$\psi^{\mu}_{\Lambda}(BMW, 2000cc, Air\ Bag, Speed) = \{\langle d_1, (\langle 0.6,0.4,0.5\rangle, \langle 0.8,0.3,0.4\rangle, \langle 0.8,0.1,0.2\rangle, \langle 0.4,0.7,0.5\rangle, (0.2))\rangle, \\ , \langle d_2, (\langle 0.4,0.3,0.2\rangle, \langle 0.9,0.4,0.1\rangle, \langle 0.3,0.7,0.6\rangle, \langle 0.7,0.4,0.2\rangle), (0.5)\rangle\}.$$

Definition 16

Let $\psi^{\mu}_{\Lambda_1}$ & $\psi^{\eta}_{\Lambda_2}$ be two PNHSS over $\mathfrak{I}$. Then $\psi^{\mu}_{\Lambda_1}$ is the GNHS subset of $\psi^{\eta}_{\Lambda_2}$ if:

1) μ is fuzzy subset of η
2) Λ_1 is a subset of Λ_2.
3) $\forall e \in \Lambda_1 \cap \Lambda_2$, $\psi_{\Lambda_1}(e)$ is a NHSS $\psi_{\Lambda_2}(e)$.

Example 2

Consider the two PNHSS $\psi^{\mu}_{\Lambda_1}$ & $\psi^{\eta}_{\Lambda_2}$ over the same universe $\mathfrak{I} = \{d_1, d_2, d_3, d_4, d_5\}$.

Then $(\psi^{\eta}, \Lambda_2) \subset (\psi^{\mu}, \Lambda_1)$.

Where, $(\psi^{\eta}, \Lambda_2) = \{\langle d_1, (\langle 0.4,0.3,0.7\rangle, \langle 0.7,0.2,0.5\rangle, \langle 0.6,0,0.4\rangle, \langle 0.2,0.3,0.7\rangle, (0.1))\rangle\}$

is a GNHS subset of $(\psi^{\mu}, \Lambda_1) = \{\langle d_1, (\langle 0.6,0.4,0.5\rangle, \langle 0.8,0.3,0.4\rangle, \langle 0.8,0.1,0.2\rangle, \langle 0.4,0.7,0.5\rangle, (0.2))\rangle,$

$, \langle d_2, (\langle 0.4,0.3,0.2\rangle, \langle 0.9,0.4,0.1\rangle, \langle 0.3,0.7,0.6\rangle, \langle 0.7,0.4,0.2\rangle), (0.5)\rangle\}.$

Definition 17

Let $\psi^{\mu}_{\Lambda_1}$ & $\psi^{\eta}_{\Lambda_2}$ be two PNHSS over $\mathfrak{I}$. Then $\psi^{\mu}_{\Lambda_1}$ & $\psi^{\eta}_{\Lambda_2}$ *are called* GNHS equal, denoted by $\psi^{\mu}_{\Lambda_1} = \psi^{\eta}_{\Lambda_2}$ if ψ_{Λ_1} is a GNHS subset of ψ_{Λ_2} & ψ_{Λ_2} is a GNHS subset of ψ_{Λ_1}.

Definition 18

The complement of a PNHSS ψ^{μ}_{Λ} is denoted by $(\psi^{\mu}_{\Lambda})^c$ and define

$(\psi^{\mu}_{\Lambda})^c = \{\langle x, \psi^c_{\Lambda}(e)(x), \mu^{(c)}(e)\rangle : x \in \mathfrak{I}, \psi_{\Lambda}(e)(x) \in N(\mathfrak{I}) \text{ and } \mu(e) \in I = [0,1]\}$

where, $\mu^{(c)}(e) = 1 - \mu(e)$ *and* ψ^c_{Λ} = neutrosophic soft complement with

$$T^{(c)}_{\Lambda}(e) = F_{\Lambda}(e), I^{(c)}_{\Lambda}(e) = 1 - I_{\Lambda}(e), F^{(c)}_{\Lambda}(e) = T_{\Lambda}(e)$$

Example 3

Let $\psi^{\mu}_{\Lambda} = \{\langle d_1, (\langle 0.6,0.4,0.5\rangle, \langle 0.8,0.3,0.4\rangle, \langle 0.8,0.1,0.2\rangle, \langle 0.4,0.7,0.5\rangle, (0.2))\rangle,$

$, \langle d_2, (\langle 0.4,0.3,0.2\rangle, \langle 0.9,0.4,0.1\rangle, \langle 0.3,0.7,0.6\rangle, \langle 0.7,0.4,0.2\rangle), (0.5)\rangle\}$

By using the PNHSS complement, we obtain the complement given by

$(\psi^{\mu}_{\Lambda})^c = \{\langle d_1, (\langle 0.5,0.6,0.6\rangle, \langle 0.4,0.7,0.8\rangle, \langle 0.2,0.9,0.8\rangle, \langle 0.5,0.3,0.4\rangle, (0.8))\rangle,$

$, \langle d_2, (\langle 0.2,0.7,0.4\rangle, \langle 0.1,0.6,0.9\rangle, \langle 0.6,0.3,0.3\rangle, \langle 0.2,0.6,0.7\rangle), (0.5)\rangle\}.$

Proposition 1

Let ψ^{μ}_{Λ} be PNHSS, then $((\psi^{\mu}_{\Lambda})^c)^c = \psi^{\mu}_{\Lambda}$.

Proof. Let $(\psi^{\mu}_{\Lambda})^c = \{\langle x, \psi^c_{\Lambda}(e)(x), \mu^{(c)}(e)\rangle : x \in \mathfrak{I}\}$

$$= \{\langle x, (F_{\Lambda}(e), I^{(c)}_{\Lambda}(e), T_{\Lambda}(e)), 1 - \mu(e)\rangle : x \in \mathfrak{I}\}$$

$$= \{\langle x, (F_{\Lambda}(e), 1 - I_{\Lambda}(e), T_{\Lambda}(e)), 1 - \mu(e)\rangle : x \in \mathfrak{I}\}$$

Then, $((\psi^{\mu}_{\Lambda})^c)^c = [\{\langle x, (F_{\Lambda}(e), 1 - I_{\Lambda}(e), T_{\Lambda}(e)), 1 - \mu(e)\rangle : x \in \mathfrak{I}\}]^c$

$$= \{\langle x, (T_{\Lambda}(e), 1 - (1 - I_{\Lambda}(e)), F_{\Lambda}(e)), 1 - (1 - \mu(e))\rangle : x \in \mathfrak{I}\}$$

$= \{\langle x, (T_\Lambda(e),\ I_\Lambda(e), F_\Lambda(e)), 1-\ \mu(e)\rangle : x \in \mathfrak{I}\} = \psi_\Lambda^\mu \quad , \forall e \in \Lambda, \mu(e) \in [0,1].$

OR

$((\psi_\Lambda^\mu)^c)^c = \left[\langle u, \left(T_\Lambda^{(c)}(e) = F_\Lambda(e), I_\Lambda^{(c)}(e) = 1-\ I_\Lambda(e), F_\Lambda^{(c)}(e) = T_\Lambda(e), \mu^{(c)}(e) = 1-\mu(e)\right)\rangle : u \in \mathfrak{I}\right]^c$

$= [\langle u, (T_\Lambda(e) = F_\Lambda^{(c)}(e), I_\Lambda(e) = 1-\ I_\Lambda^{(c)}(e), T_\Lambda(e) = F_\Lambda^{(c)}(e), \mu(e) = 1-\mu^c(e)\ )\rangle : u \in \mathfrak{I}]$

$= \left[\langle u, \left(T_\Lambda(e) = F_\Lambda^{(c)}(e), I_\Lambda(e) = 1-[1-I_\Lambda(e)], T_\Lambda(e) = F_\Lambda^{(c)}(e), \mu(e) = 1-(1-\mu(e))\right)\rangle : u \in \mathfrak{I}\right]$

$= \left[\langle u, \left(T_\Lambda(e) = F_\Lambda^{(c)}(e), I_\Lambda(e) = I_\Lambda(e), T_\Lambda(e) = F_\Lambda^{(c)}(e), \mu(e) = \mu(e)\right)\rangle : u \in \mathfrak{I}\ \right]$

$= \psi_\Lambda^\mu \quad , \forall e \in \Lambda, \mu(e) \in [0,1].$

4. Basic Operations

In this section, we present some basic operation with illustrative examples and propositions.

Definition 19

The union of two PNHSS $\psi_{\Lambda_1}^\mu$ & $\psi_{\Lambda_2}^\eta$ over $\mathfrak{I}$ is a PNHSS ψ_Λ^λ defined as $\psi(\Lambda, \lambda)$ where $\Lambda = \Lambda_1 \cup \Lambda_2$ and $\lambda(e) = max(\mu(e), \eta(e))$ and $\forall e \in \Lambda$ we have the follow:

$$\psi_\Lambda^\lambda = \psi_{\Lambda_1}^\mu\ \hat{\cup}\ \psi_{\Lambda_2}^\eta \quad \text{where} \quad e\, \epsilon \Lambda_1 \cap \Lambda_2$$

Where $\hat{\cup}$ is a NHSS union.

Example 4

Assume that two PNHSS $\psi_{\Lambda_1}^\mu$ & $\psi_{\Lambda_2}^\eta$ over the same universe $\mathfrak{I} = \{d_1, d_2, d_3, d_4\}$ are defined as follows:

$\psi_{\Lambda_1}^\mu = \{\langle d_1, (\langle 0.6,0.4,0.5\rangle, \langle 0.8,0.3,0.4\rangle, \langle 0.8,0.1,0.2\rangle, \langle 0.4,0.7,0.5\rangle, (0.2))\rangle,$
$\quad , \langle d_2, (\langle 0.4,0.3,0.2\rangle, \langle 0.9,0.4,0.1\rangle, \langle 0.3,0.7,0.6\rangle, \langle 0.7,0.4,0.2\rangle), (0.5)\rangle\}$

$\psi_{\Lambda_2}^\eta = \{\langle d_1, (\langle 0.8,0.6,0.3\rangle, \langle 0.9,0.5,0.2\rangle, \langle 0.3,0.2,0.4\rangle, \langle 0.3,0.2,0.7\rangle, (0.3))\rangle,$
$\quad , \langle d_3, (\langle 0.5,0.3,0.4\rangle, \langle 0.2,0.5,0.7\rangle, \langle 0.7,0.1,0.5\rangle, \langle 0.4,0.5,0.2\rangle), (0.4)\rangle\}.$

Then,

$\psi_\Lambda^\lambda = \psi_{\Lambda_1}^\mu\ \hat{\cup}\ \psi_{\Lambda_2}^\eta = \{\langle d_1, (\langle 0.8,0.4,0.3\rangle, \langle 0.9,0.3,0.2\rangle, \langle 0.8,0.1,0.2\rangle, \langle 0.4,0.2,0.5\rangle, (0.3))\rangle$
$\quad \langle d_2, (\langle 0.4,0.3,0.2\rangle, \langle 0.9,0.4,0.1\rangle, \langle 0.3,0.7,0.6\rangle, \langle 0.7,0.4,0.2\rangle), (0.5)\rangle$
$\quad , \langle d_3, (\langle 0.5,0.3,0.4\rangle, \langle 0.2,0.5,0.7\rangle, \langle 0.7,0.1,0.5\rangle, \langle 0.4,0.5,0.2\rangle), (0.4)\rangle\}.$

Proposition 2

Let $\psi_{\Lambda_1}^\mu, \psi_{\Lambda_2}^\eta$ & $\psi_{\Lambda_3}^\delta$ are PNHSS over $\mathfrak{I}$. Then

1) $\psi_{\Lambda_1}^\mu\ \hat{\cup}\ \psi_{\Lambda_2}^\eta = \psi_{\Lambda_2}^\eta\ \hat{\cup}\ \psi_{\Lambda_1}^\mu$ (Commutative law)
2) $(\psi_{\Lambda_1}^\mu\ \hat{\cup}\ \psi_{\Lambda_2}^\eta)\ \hat{\cup}\ \psi_{\Lambda_3}^\delta = \psi_{\Lambda_1}^\mu\ \hat{\cup}\ (\psi_{\Lambda_2}^\eta\ \hat{\cup}\ \psi_{\Lambda_3}^\delta)$ (Associative law)

Proof. In the following proof first two cases are trivial, we consider only the third case.

1) $\psi_{\Lambda_1}^\mu\ \hat{\cup}\ \psi_{\Lambda_2}^\eta$

$= \left\{\langle x, \left((max\{T(\psi_{\Lambda_1}^\mu\ ), T(\psi_{\Lambda_2}^\eta\ )\}, min\{I(\psi_{\Lambda_1}^\mu\ ), I(\psi_{\Lambda_2}^\eta\ )\}, min\{F(\psi_{\Lambda_1}^\mu\ ), F(\psi_{\Lambda_2}^\eta\ )\}\right), max\{\mu(e\ ), \eta(e)\}\rangle\right\}$

$= \left\{ \langle x, \left((max\{T(\psi^{\eta}_{\Lambda_2}), T(\psi^{\mu}_{\Lambda_1})\}, min\{I(\psi^{\eta}_{\Lambda_2}), I(\psi^{\mu}_{\Lambda_1})\}, min\{F(\psi^{\eta}_{\Lambda_2}), F(\psi^{\mu}_{\Lambda_1})\}\right), max\{\eta(e), \mu(e)\}\rangle \right\}$

$= \psi^{\eta}_{\Lambda_2} \ \widehat{\cup}\ \psi^{\mu}_{\Lambda_1}$.

2) $\psi^{\mu}_{\Lambda_1} \ \widehat{\cup}\ \psi^{\eta}_{\Lambda_2}$

$= \left\{ \langle x, \left((max\{T(\psi^{\mu}_{\Lambda_1}), T(\psi^{\eta}_{\Lambda_2})\}, min\{I(\psi^{\mu}_{\Lambda_1}), I(\psi^{\eta}_{\Lambda_2})\}, min\{F(\psi^{\mu}_{\Lambda_1}), F(\psi^{\eta}_{\Lambda_2})\}\right), max\{\mu(e), \eta(e)\}\rangle \right\}$

Then $\left(\psi^{\mu}_{\Lambda_1} \ \widehat{\cup}\ \psi^{\eta}_{\Lambda_2}\right) \widehat{\cup}\ \psi^{\delta}_{\Lambda_3}$

$= \left\{ \langle x, \left((max\{max\{T(\psi^{\mu}_{\Lambda_1}), T(\psi^{\eta}_{\Lambda_2})\}, T(\psi^{\delta}_{\Lambda_3})\}, min\{min\{I(\psi^{\mu}_{\Lambda_1}), I(\psi^{\eta}_{\Lambda_2})\}, I(\psi^{\delta}_{\Lambda_3})\}, \right.\right.$

$\left.\left., min\{F(\psi^{\mu}_{\Lambda_1}), F(\psi^{\eta}_{\Lambda_2})\}, F(\psi^{\delta}_{\Lambda_3})\}\right), max\{max\{\mu(e), \eta(e)\}, \delta(e)\}\rangle \right\}$.

$= \left\{ \langle x, \left((max\{T(\psi^{\mu}_{\Lambda_1}), T(\psi^{\eta}_{\Lambda_2}), T(\psi^{\delta}_{\Lambda_3})\}, min\{min\{I(\psi^{\mu}_{\Lambda_1}), I(\psi^{\eta}_{\Lambda_2}), I(\psi^{\delta}_{\Lambda_3})\}, \right.\right.$

$\left.\left. min\{F(\psi^{\mu}_{\Lambda_1}), F(\psi^{\eta}_{\Lambda_2}), (\psi^{\delta}_{\Lambda_3})\}\right), max\{\mu(e), \eta(e), \delta(e)\}\rangle \right\}$.

$= \left\{ \langle x, \left((max\{T(\psi^{\mu}_{\Lambda_1}), max\{T(\psi^{\eta}_{\Lambda_2}), T(\psi^{\delta}_{\Lambda_3})\}\right\}, \left((min\{I(\psi^{\mu}_{\Lambda_1}), min\{I(\psi^{\eta}_{\Lambda_2}), I(\psi^{\delta}_{\Lambda_3})\}\right\}, \right.$

$\left. \left((min\{F(\psi^{\mu}_{\Lambda_1}), min\{F(\psi^{\eta}_{\Lambda_2}), F(\psi^{\delta}_{\Lambda_3})\}\right\}, ((max\{\mu(e), max\{F\eta(e), \delta(e)\}\}\rangle \right\}$

$= \psi^{\mu}_{\Lambda_1} \ \widehat{\cup}\ \left(\psi^{\eta}_{\Lambda_2} \ \widehat{\cup}\ \psi^{\delta}_{\Lambda_3}\right)$.

Definition 20

The intersection of two PNHSS $\psi^{\mu}_{\Lambda_1}$ & $\psi^{\eta}_{\Lambda_2}$ over $\mathfrak{I}$ is a PNHSS ψ^{λ}_{Λ} defined as $\psi(\varepsilon, \lambda)$ where $\Lambda = \Lambda_1 \cap \Lambda_2$ and $\varepsilon(e) = min(\mu(e), \eta(e))$ and $\forall e \in \Lambda$ we have the follow:

$$\psi^{\varepsilon}_{\Lambda} = \psi^{\mu}_{\Lambda_1} \ \widehat{\cap}\ \psi^{\eta}_{\Lambda_2} \quad \text{where} \quad e \in \Lambda_1 \cap \Lambda_2$$

Where $\widehat{\cap}$ is a NHSS intersection.

Example 5

Consider example 4. By using basic neutrosophic intersection we can easily verify that $\psi^{\varepsilon}_{\Lambda} = \psi^{\mu}_{\Lambda_1} \ \widehat{\cap}\ \psi^{\eta}_{\Lambda_2}$, where

$\psi^{\varepsilon}_{\Lambda} = \psi^{\mu}_{\Lambda_1} \ \widehat{\cap}\ \psi^{\eta}_{\Lambda_2} = \left\{ \langle d_1, \left(\langle 0.6,0.6,0.5\rangle, \langle 0.8,0.5,0.4\rangle, \langle 0.3,0.2,0.2\rangle, \langle 0.3,0.7,0.7\rangle, (0.2)\right)\rangle \right\}$.

Proposition 3

Let $\psi^{\mu}_{\Lambda_1}, \psi^{\eta}_{\Lambda_2}$ & $\psi^{\delta}_{\Lambda_3}$ are PNHSS over $\mathfrak{I}$. Then

1) $\psi^{\mu}_{\Lambda_1} \ \widehat{\cap}\ \psi^{\eta}_{\Lambda_2} = \psi^{\eta}_{\Lambda_2} \ \widehat{\cap}\ \psi^{\mu}_{\Lambda_1}$ (Commutative law)
2) $\left(\psi^{\mu}_{\Lambda_1} \ \widehat{\cap}\ \psi^{\eta}_{\Lambda_2}\right) \widehat{\cap}\ \psi^{\delta}_{\Lambda_3} = \psi^{\mu}_{\Lambda_1} \ \widehat{\cap}\ \left(\psi^{\eta}_{\Lambda_2} \ \widehat{\cap}\ \psi^{\delta}_{\Lambda_3}\right)$ (Associative law)

Proof. Similar to proposition 2.

Proposition 4

Let $\psi^{\mu}_{\Lambda_1}$ & $\psi^{\eta}_{\Lambda_2}$ are PNHSS over $\mathfrak{I}$. Then

1) $\left(\psi^{\mu}_{\Lambda_1} \ \widehat{\cup}\ \psi^{\eta}_{\Lambda_2}\right)^c = \left(\psi^{\eta}_{\Lambda_2}\right)^c \ \widehat{\cap}\ \left(\psi^{\mu}_{\Lambda_1}\right)^c$.
2) $\left(\psi^{\mu}_{\Lambda_1} \ \widehat{\cap}\ \psi^{\eta}_{\Lambda_2}\right)^c = \left(\psi^{\eta}_{\Lambda_2}\right)^c \ \widehat{\cup}\ \left(\psi^{\mu}_{\Lambda_1}\right)^c$.

Proof. The proof is straightforward from Definitions 18 and 19.

Proposition 5

Let $\psi^{\mu}_{\Lambda_1}$ & $\psi^{\eta}_{\Lambda_2}$ are PNHSS over $\mathfrak{I}$. Then

1) $\psi^{\mu}_{\Lambda_1}$ GNHS subset of $\psi^{\mu}_{\Lambda_1} \hat{\cup} \psi^{\eta}_{\Lambda_2}$
2) $\psi^{\mu}_{\Lambda_1} \hat{\cap} \psi^{\eta}_{\Lambda_2}$ GNHS subset of $\psi^{\mu}_{\Lambda_1}$

Proof. It's clear from definition.

Proposition 6

Let $\psi^{\mu}_{\Lambda_1}, \psi^{\eta}_{\Lambda_2}$ & $\psi^{\delta}_{\Lambda_3}$ are PNHSS over $\mathfrak{I}$. Then

1) $\left(\psi^{\mu}_{\Lambda_1} \hat{\cup} \psi^{\eta}_{\Lambda_2}\right) \hat{\cap} \psi^{\delta}_{\Lambda_3} = \left(\psi^{\mu}_{\Lambda_1} \hat{\cap} \psi^{\delta}_{\Lambda_3}\right) \hat{\cup} \left(\psi^{\eta}_{\Lambda_2} \hat{\cap} \psi^{\delta}_{\Lambda_3}\right)$.
2) $\left(\psi^{\mu}_{\Lambda_1} \hat{\cap} \psi^{\eta}_{\Lambda_2}\right) \hat{\cup} \psi^{\delta}_{\Lambda_3} = \left(\psi^{\mu}_{\Lambda_1} \hat{\cup} \psi^{\delta}_{\Lambda_3}\right) \hat{\cap} \left(\psi^{\eta}_{\Lambda_2} \hat{\cup} \psi^{\delta}_{\Lambda_3}\right)$.

Proof. The proof can be easily obtained from relative definitions.

5. AND and OR Operation.

In this section, we introduce the definitions of AND and OR operations for Possibility neutrosophic hypersoft set, derive their properties, and give some examples.

Definition 21

Let $\psi^{\mu}_{\Lambda_1}$ & $\psi^{\eta}_{\Lambda_2}$ are PNHSS over $\mathfrak{I}$. Then $\psi^{\mu}_{\Lambda_1}$ AND $\psi^{\eta}_{\Lambda_2}$ denoted by $\psi^{\mu}_{\Lambda_1} \hat{\wedge} \psi^{\eta}_{\Lambda_2}$ is given as

$$\psi^{\mu}_{\Lambda_1} \hat{\wedge} \psi^{\eta}_{\Lambda_2} = \psi^{\varepsilon}_{\Lambda_1 \times \Lambda_2}$$

such that $\psi^{\varepsilon}_{\Lambda_1 \times \Lambda_2}(\alpha, \beta) = \psi^{\mu}_{\Lambda_1}(\alpha) \hat{\cap} \psi^{\eta}_{\Lambda_2}(\beta), \forall (\alpha, \beta) \in \Lambda_1 \times \Lambda_2$

Where $\hat{\cap}$ is a NHSS intersection and $\varepsilon(e) = min\left(\mu(e), \eta(e)\right)$.

Example 6

Consider example 4. Then we can easily verify $\psi^{\mu}_{\Lambda_1} \hat{\wedge} \psi^{\eta}_{\Lambda_2} = \psi^{\varepsilon}_{\Lambda_1 \times \Lambda_2}$ where

$\psi^{\varepsilon}_{\Lambda_1 \times \Lambda_2} = \{\langle (d_1, d_1)\left(\langle 0.6,0.6,0.5\rangle, \langle 0.8,0.5,0.4\rangle, \langle 0.3,0.2,0.4\rangle, \langle 0.3,0.7,0.7\rangle, (0.2)\right)\rangle$,
$\langle (d_1, d_3), \left(\langle 0.5,0.4,0.5\rangle, \langle 0.2,0.5,0.7\rangle, \langle 0.7,0.1,0.5\rangle, \langle 0.4,0.7,0.5\rangle, (0.2)\right)\rangle$,
$\langle (d_2, d_1), \left(\langle 0.4,0.6,0.3\rangle, \langle 0.9,0.5,0.2\rangle, \langle 0.3,0.7,0.6\rangle, \langle 0.3,0.4,0.7\rangle, (0.3)\right)\rangle$,
$\langle (d_2, d_3), \left(\langle 0.4,0.3,0.4\rangle, \langle 0.2,0.5,0.7\rangle, \langle 0.3,0.7,0.6\rangle, \langle 0.4,0.5,0.2\rangle, (0.4)\right)\rangle\}$.

Definition 22

Let $\psi^{\mu}_{\Lambda_1}$ & $\psi^{\eta}_{\Lambda_2}$ are PNHSS over $\mathfrak{I}$. Then $\psi^{\mu}_{\Lambda_1}$ OR $\psi^{\eta}_{\Lambda_2}$ denoted by $\psi^{\mu}_{\Lambda_1} \hat{\vee} \psi^{\eta}_{\Lambda_2}$ is given as

$$\psi^{\mu}_{\Lambda_1} \hat{\vee} \psi^{\eta}_{\Lambda_2} = \psi^{\lambda}_{\Lambda_1 \times \Lambda_2}$$

such that $\psi^{\lambda}_{\Lambda_1 \times \Lambda_2}(\alpha, \beta) = \psi^{\mu}_{\Lambda_1}(\alpha) \hat{\cup} \psi^{\eta}_{\Lambda_2}(\beta), \forall (\alpha, \beta) \in \Lambda_1 \times \Lambda_2$

Where $\hat{\cup}$ is a GNSS union and $\lambda(e) = max\left(\mu(e), \eta(e)\right)$.

Example 7

Consider example 4. Then we can easily verify $\psi^{\mu}_{\Lambda_1} \hat{\vee} \psi^{\eta}_{\Lambda_2} = \psi^{\lambda}_{\Lambda_1 \times \Lambda_2}$ where

$\psi^{\lambda}_{\Lambda_1 \times \Lambda_2} = \{\langle (d_1, d_1)\left(\langle 0.8,0.4,0.3\rangle, \langle 0.9,0.3,0.2\rangle, \langle 0.8,0.1,0.4\rangle, \langle 0.4,0.2,0.5\rangle, (0.3)\right)\rangle$,

$\langle(d_1, d_3), (\langle 0.6,0.3,0.4\rangle, \langle 0.8,0.3,0.4\rangle, \langle 0.8,0.1,0.2\rangle, \langle 0.4,0.5,0.2\rangle, (0.4))\rangle$,
$\langle(d_2, d_1), (\langle 0.8,0.3,0.2\rangle, \langle 0.9,0.4,0.1\rangle, \langle 0.3,0.2,0.4\rangle, \langle 0.7,0.2,0.2\rangle, (0.5))\rangle$,
$\langle(d_2, d_3), (\langle 0.5,0.3,0.2\rangle, \langle 0.9,0.4,0.1\rangle, \langle 0.7,0.1,0.5\rangle, \langle 0.7,0.4,0.2\rangle, (0.5))\rangle\}$.

Proposition 7

Let $\psi^{\mu}_{\Lambda_1}$ & $\psi^{\eta}_{\Lambda_2}$ are PNHSS, then

1) $\left(\psi^{\mu}_{\Lambda_1} \hat{\vee}\ \psi^{\eta}_{\Lambda_2}\right)^c = \left(\psi^{\mu}_{\Lambda_1}\right)^c \hat{\wedge} \left(\psi^{\eta}_{\Lambda_2}\right)^c$.
2) $\left(\psi^{\mu}_{\Lambda_1} \hat{\wedge}\ \psi^{\eta}_{\Lambda_2}\right)^c = \left(\psi^{\mu}_{\Lambda_1}\right)^c \hat{\vee} \left(\psi^{\eta}_{\Lambda_2}\right)^c$.

Proof. The proof is straightforward from Definitions 18, 21 and 22.

4. Conclusions

In this paper we have introduced the concept of of Possibility Neutrosophic Hypersoft Set and studied some of its properties like: subset, equal, complement with detailed examples. Basic operation of PNHSS are established like: union, intersection with illustrative examples. Some basic laws such as commutative, associative, distributive and De Morgens low are discussed. AND and OR operation of PNHSS are defined with suitable examples and some propositions.

In the future we use the new concept of PNHSS in decision making problem and in medical diagnosis. Also, the authors may extend this Possibility Neutrosophic Hypersoft Set to algebraic structure such as group, ring and field and their generalizations may be studied.

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Group Decision-Making Model Using the Exponential Similarity Measure of Confidence Neutrosophic Number Cubic Sets in a Fuzzy Multi-Valued Circumstance

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Abstract: Fuzzy decision-making is a critical research topic in uncertain decision-making issues. Under uncertain scenarios, a group of decision makers/experts presents the fuzzy evaluation data of the criteria to an alternative. In this case, we can use a fuzzy multi-valued set (FMVS) to express them. To solve the operation problem between different fuzzy sequence lengths in FMVSs and ensure some confidence level of fuzzy assessment values from the perspective of probability, this paper first proposes a transformation technique from FMVS to a confidence neutrosophic number cubic set (CNNCS) based on confidence levels and normal distribution of fuzzy values in FMVS. Then, we present an exponential similarity measure between CNNCSs and its group DM model with some confidence levels and normal distribution in a FMVS circumstance. Finally, the developed group DM model is applied to the selection of intelligent manufacturing equipment, and then the decision results corresponding to the 90%, 95%, and 99% confidence levels reveal the decision flexibility and rationality/reliability.

Keywords: fuzzy multi-valued set; confidence neutrosophic number cubic set; exponential similarity measure; group decision-making

1. Introduction

In uncertain decision-making (DM) issues, fuzzy DM is a critical one of DM research topics. Fuzzy sets (FSs) [1] have been applied in various DM areas, such as social science, economics and engineering management [2–6]. As an extension of FS that contains almost one occurrence of each element, Yager [7] presented a fuzzy multi-set (FMS) or bag, where permit multiple occurrences of the elements with identical or different membership degrees. Since then, the fuzzy multisets have been applied to group DM [8, 9] and clustering analysis [10–12] and so on. To avoid aggregation operations between different fuzzy sequence lengths in FMSs, Fu et al. [13] introduced a transformation technique from FMS to an entropy fuzzy set in terms of the mean and Shannon/probability entropy of fuzzy sequences, and then developed a group DM model using the Aczel-Alsina aggregation operators of entropy fuzzy elements and used it for renal cancer surgery options with FMS information.

In view of the hybrid form of interval fuzzy values (uncertain fuzzy values) and fuzzy values (exact fuzzy values), Jun et al. [14, 15] proposed (fuzzy) cubic sets (CSs). Then, CSs have been applied in many DM problems [16–18]. Moreover, there are some extension forms of CSs, such as cubic hesitant fuzzy sets [19–21], fuzzy credibility cubic numbers [22], and cubic fuzzy-consistency sets transformed from cubic fuzzy multi-valued sets [23], and their DM applications in existing literature. Since CS shows its obvious merit in the hybrid information expression of interval fuzzy values and fuzzy values, it is more useful than FS in multi-criteria group DM problems.

In uncertain problems, a neutrosophic number (NN) $N = h + uI = [h + uI^-, h + uI^+]$ for an indeterminacy $I = [I^-, I^+]$ and $h, u \in \Re$ was proposed by Smarandache [24–26]. NN implies its main merit in the indeterminate information representation of changeable interval values or fuzzy values corresponding to different indeterminate ranges of I. Hence, it shows better flexibility and generalization in the representation and processing capability of uncertain information in multi-criteria DM problems [27, 28]. Recently, Lv et al. [29] presented the concepts of NN probability and confidence neutrosophic numbers (CNNs) (confidence intervals) in light of confidence levels and normal and log-normal probability distributions of multi-valued datasets from the perspective of probability, and then developed CNN linear programming methods based on normal and log-normal probability distributions to carry out production planning problems in uncertain scenarios.

In the setting of FMSs, Fu et al. proposed a transformation technique from FMS to entropy fuzzy elements based on the mean and Shannon/probability entropy of fuzzy sequences in FMS. Then, from the perspective of probability estimation, the transformation technique does not consider a confidence level and certain probability distribution of fuzzy sequences/data, which shows its defect. To avoid this defect, this paper proposes a new transformation technique from a fuzzy multi-valued set (FMVS) to a confidence neutrosophic number cubic set (CNNCS) and group DM model using an exponential similarity measure (ESM) of CNNCSs to solve group DM problems in view of the conditions of some confidence levels and normal distribution in a FMVS circumstance.

This paper contains remaining structures. The second section introduces the definitions of FMVS and CNNCS and some basic relationships of CNNCEs. The third section proposes an ESM between CNNCSs and a weighted ESM of CNNCSs. The fourth section develops a group DM model based on the weighted ESM of CNNCSs in a FMVS circumstance. The fifth section utilizes the developed group DM model to perform the selection of intelligent manufacturing equipment. The sixth section provides decision results and discussions corresponding to the 90%, 95%, and 99% confidence levels to reveal the decision flexibility and rationality/reliability. The last section summarizes the conclusions and future research directions.

2. FMVS and CNNCS

This section gives the definitions of FMVS and CNNCS and then defines some basic relationships of confidence neutrosophic number cubic elements (CNNCEs).

Definition 1. A FMVS H on a finite set $Z = \{z_1, z_2, \ldots, z_q\}$ is defined as

$$H = \left\{ \left\langle z_k, M_H(z_k) \right\rangle | z_k \in Z \right\}, \tag{1}$$

where $M_H(z_k)$ contains multiple membership degrees of each element z_k to the set H, denoted as a fuzzy sequence $M_H(z_k) = (h_{k1}, h_{k2}, \ldots, h_{kr_k})$ with identical and/or different fuzzy values for $z_k \in Z$ and $h_{ki} \in [0, 1]$ ($k = 1, 2, \ldots, q$; $i = 1, 2, \ldots, r_k$).

For convenience, each element $\left\langle z_k, M_H(z_k) \right\rangle$ in H is denoted as a fuzzy multi-valued element (FMVE) $h_k = \left\langle z_k, (h_{k1}, h_{k2}, \ldots, h_{kr_k}) \right\rangle$ with increasing fuzzy sequence. Especially when $r_k = 1$, the FMVS H becomes FS.

According to the confidence interval with a $(1-\varphi)100\%$ confidence level [29], we present a transformation technique from FMVS to CNNCS, which is defined below.

Definition 2. Set FMVS as H_1 = {<z_1, (h_{11}, h_{12}, …, h_{1r_1})>, <z_2, (h_{21}, h_{22}, …, h_{2r_2})>,…, <z_q, (h_{q1}, h_{q2}, …, h_{qr_q})>} in a finite set Z = {z_1, z_2, …, z_q}. Thus, CNNCS can be defined as

$$G_{1\varphi}=\left\{\begin{array}{l}\left\langle z_1,\left[h_{11}^-(I_\varphi),h_{11}^+(I_\varphi)\right],h_{m11}\right\rangle,\left\langle z_2,\left[h_{12}^-(I_\varphi),h_{12}^+(I_\varphi)\right],h_{m12}\right\rangle,\ldots,\\ \left\langle z_q,\left[h_{1q}^-(I_\varphi),h_{1q}^+(I_\varphi)\right],h_{m1q}\right\rangle \mid I_\varphi=[-t_{\varphi/2},t_{\varphi/2}]\end{array}\right\}, \tag{2}$$

where $\left[h_{1k}^-(I_\varphi),h_{1k}^+(I_\varphi)\right]$ (k = 1, 2, …, q) is CNN, which is obtained by

$$\left[h_{1k}^-(I_\varphi),h_{1k}^+(I_\varphi)\right]=\left[h_{m1k}+u_{1k}I_\varphi^-,h_{m1k}+u_{1k}I_\varphi^+\right]=\left[h_{m1k}-\frac{\sigma_{1k}}{\sqrt{r_k}}t_{\varphi/2},h_{m1k}+\frac{\sigma_{1k}}{\sqrt{r_k}}t_{\varphi/2}\right]; \tag{3}$$

$I_\varphi=[I_\varphi^-,I_\varphi^+]=[-t_{\varphi/2},t_{\varphi/2}]$ is an indeterminate interval depending on a specified value of $t_{\varphi/2}$; u_{1k} is an indeterminate parameter; then h_{m1k} and σ_{1k} are the average value and standard deviation of a fuzzy sequence in H_1, which are yielded by the formulae:

$$h_{m1k}=\frac{1}{r_k}\sum_{i=1}^{r_k}h_{1i}, \tag{4}$$

$$\sigma_{1k}=\sqrt{\frac{1}{r_k-1}\sum_{i=1}^{r_k}(h_{1i}-h_{m1k})^2}. \tag{5}$$

Remark 1. The specified values of $t_{\varphi/2}$ are related to (1−φ)100% confidence levels [29], which are usually specified as $t_{\varphi/2}$ = 1.645, 1.960, 2.576 for the levels of φ = 0.1, 0.05, 0.01 in actual applications [29].

From a probabilistic viewpoint and the estimation of small example data in some distribution situation, the CNN of Eq. (3) with a (1−φ)100% confidence level reveals the probability of fuzzy values falling within CNN (confidence interval). For example, considering the 90% confidence level, the 90% probability of all fuzzy values will occur within CNN, while the 10% probability of all fuzzy values will occur outside CNN.

Example 1. Assume that there is the FMVS H_1 = {<z_1, (0.5, 0.6, 0.7, 0.9)>, <z_2, (0.6, 0.7, 0.7, 0.8, 0.9)>} in a finite set Z = {z_1, z_2}, where fuzzy data are in the normal distribution situation. Considering the 90% confidence level with the specified value of $t_{\varphi/2}$ = 1.645, the FMVS H_1 can be transformed into the CNNCS $G_{\varphi 1}$ by Eqs. (3)–(5), which is described by the calculational process below.

Using Eqs. (4) and (5), the average values and standard deviations of two fuzzy sequences in H_1 are given as follows:

h_{m11} = 0.675, h_{m12} = 0.74, σ_{11} = 0.1708, and σ_{12} = 0.114.

Using Eq. (3), two CNNs are produced as follows:

$$\left[h_{11}^-(I_\varphi),h_{11}^+(I_\varphi)\right]=\left[0.675-\frac{0.1708}{\sqrt{4}}\times1.645,0.675+\frac{0.1708}{\sqrt{4}}\times1.645\right]=[0.5345,0.8155],$$

$$\left[h_{11}^-(I_\varphi),h_{11}^+(I_\varphi)\right]=\left[0.74-\frac{0.114}{\sqrt{5}}\times1.645,0.74+\frac{0.114}{\sqrt{5}}\times1.645\right]=[0.6561,0.8239].$$

Thus, the CNNCS $G_{1\varphi}$ for φ = 0.1 is obtained below:

$G_{1\varphi=0.1}$ = {<z_1, [0.5345, 0.8155], 0.675>, <z_2, [0.6561, 0.8239], 0.74>|I_φ = [−1.645, 1.645]}.

Then, each element $\left\langle z_1,\left[h_{1k}^-(I_\varphi),h_{1k}^+(I_\varphi)\right],h_{m1k}\right\rangle$ in the CNNCS $G_{1\varphi}$ is simply represented as the CNNCE $g_{1k}(I_\varphi)=\left\langle\left[h_{\varphi 1k}^-,h_{\varphi 1k}^+\right],h_{m1k}\right\rangle$ (k = 1, 2, …, q).

Definition 3. Set two CNNCEs as $g_{1k}(I_\varphi)=\left\langle\left[h_{\varphi 1k}^-,h_{\varphi 1k}^+\right],h_{m1k}\right\rangle$ and $g_{2k}(I_\varphi)=\left\langle\left[h_{\varphi 2k}^-,h_{\varphi 2k}^+\right],h_{m2k}\right\rangle$ (k = 1, 2, …, q). Then, their basic relationships are defined below:

(1) $g_{1k}(I_\varphi) \subseteq g_{2k}(I_\varphi) \Leftrightarrow [h^-_{\varphi 1k}, h^+_{\varphi 1k}] \subseteq [h^-_{\varphi 2k}, h^+_{\varphi 2k}]$ and $h_{m1k} \le h_{m2k}$;

(2) $g_{1k}(I_\varphi) = g_{2k}(I_\varphi) \Leftrightarrow g_{1k}(I_\varphi) \subseteq g_{2k}(I_\varphi)$ and $g_{1k}(I_\varphi) \supseteq g_{2k}(I_\varphi)$, i.e., $h^-_{\varphi 1k} = h^-_{\varphi 2k}$, $h^+_{\varphi 1k} = h^+_{\varphi 2k}$, and $h_{m1k} = h_{m2k}$;

(3) $g_{1k}(I_\varphi) \cup g_{2k}(I_\varphi) = \left\langle [h^-_{\varphi 1k} \vee h^-_{\varphi 2k}, h^+_{\varphi 1k} \vee h^+_{\varphi 2k}], h_{m1k} \vee h_{m2k} \right\rangle$;

(4) $g_{1k}(I_\varphi) \cap g_{2k}(I_\varphi) = \left\langle [h^-_{\varphi 1k} \wedge h^-_{\varphi 2k}, h^+_{\varphi 1k} \wedge h^+_{\varphi 2k}], h_{m1k} \wedge h_{m2k} \right\rangle$;

(5) $g^c_{1k}(I_\varphi) = \left\langle [1 - h^+_{\varphi 1k}, 1 - h^-_{\varphi 1k}], 1 - h_{m1k} \right\rangle$ (Complement of $g_{1k}(I_\varphi)$).

3. ESM of CNNCSs

In this section, we present the ESM of CNNCSs, the weighted ESM of CNNCSs, and their characteristics.

Definition 4. Set $G_{1\varphi} = \{g_{11}(I_\varphi), g_{12}(I_\varphi), \ldots, g_{1q}(I_\varphi)\}$ and $G_{2\varphi} = \{g_{21}(I_\varphi), g_{22}(I_\varphi), \ldots, g_{2q}(I_\varphi)\}$ as two CNNCSs, where $g_{1k}(I_\varphi) = \left\langle \left[h^-_{\varphi 1k}, h^+_{\varphi 1k}\right], h_{m1k} \right\rangle$ and $g_{2k}(I_\varphi) = \left\langle \left[h^-_{\varphi 2k}, h^+_{\varphi 2k}\right], h_{m2k} \right\rangle$ (k = 1, 2, …, q) are two collections of CNNCEs. Thus, the ESM of two CNNCSs $G_{\varphi 1}$ and $G_{\varphi 2}$ is defined as

$$E_\varphi(G_{1\varphi}, G_{2\varphi}) = \frac{1}{q}\sum_{k=1}^{q} \frac{\exp\left(-\left((h^-_{\varphi 1k} - h^-_{\varphi 2k})^2 + (h^+_{\varphi 1k} - h^+_{\varphi 2k})^2 + (h_{m1k} - h_{m2k})^2\right)\right) - \exp(-3)}{1 - \exp(-3)}. \quad (6)$$

Proposition 1. The ESM $E_\varphi(G_{1\varphi}, G_{2\varphi})$ contains the following characteristics:

(a) $E_\varphi(G_{1\varphi}, G_{2\varphi}) = E_\varphi(G_{2\varphi}, G_{1\varphi})$;

(b) $0 \le E_\varphi(G_{1\varphi}, G_{2\varphi}) \le 1$;

(c) $E_\varphi(G_{1\varphi}, G_{2\varphi}) = 1$ if and only if $G_{1\varphi} = G_{2\varphi}$;

(d) If $G_{1\varphi} \subseteq G_{2\varphi} \subseteq G_{3\varphi}$ for any three CNNCSs $G_{1\varphi}$, $G_{2\varphi}$, and $G_{3\varphi}$, then $E_\varphi(G_{1\varphi}, G_{2\varphi}) \ge E_\varphi(G_{1\varphi}, G_{3\varphi})$ and $E_\varphi(G_{2\varphi}, G_{3\varphi}) \ge E_\varphi(G_{1\varphi}, G_{3\varphi})$ exist.

Proof:

(a) This characteristic is obvious.

(b) Since there is the inequality $0 \le (h^-_{\varphi 1k} - h^-_{\varphi 2k})^2 + (h^+_{\varphi 1k} - h^+_{\varphi 2k})^2 + (h_{m1k} - h_{m2k})^2 \le 3$, the inequality $\exp(0) = 1 \le \exp\left(-\left((h^-_{\varphi 1k} - h^-_{\varphi 2k})^2 + (h^+_{\varphi 1k} - h^+_{\varphi 2k})^2 + (h_{m1k} - h_{m2k})^2\right)\right) \le \exp(-3)$ also exists. Therefore, the value of Eq. (6) belongs to [0, 1], i.e., $0 \le E_\varphi(G_{1\varphi}, G_{2\varphi}) \le 1$.

(c) When $G_{1\varphi}, = G_{2\varphi}$, $g_{1k}(I_\varphi) = g_{2k}(I_\varphi)$ (k = 1, 2, …, q) exists. Thus, there are $h^-_{\varphi 1k} = h^-_{\varphi 2k}$, $h^+_{\varphi 1k} = h^+_{\varphi 2k}$, and $h_{m1k} = h_{m2k}$ (k = 1, 2, …, q). In this case, there is exp(0) = 1 in Eq. (6), and then $E_\varphi(G_{1\varphi}, G_{2\varphi}) = 1$ exists.

When $E_\varphi(G_{1\varphi}, G_{2\varphi}) = 1$, there is exp(0) = 1 in Eq. (6). Hence, $h^-_{\varphi 1k} = h^-_{\varphi 2k}$, $h^+_{\varphi 1k} = h^+_{\varphi 2k}$, and $h_{m1k} = h_{m2k}$ exist. In this case, there is $g_{1k}(I_\varphi) = g_{2k}(I_\varphi)$ (k = 1, 2, …, q), and then $G_{1\varphi} = G_{2\varphi}$ can hold.

(d) For $G_{1\varphi} \subseteq G_{2\varphi} \subseteq G_{3\varphi}$, there is $g_{1k}(I_\varphi) \subseteq g_{2k}(I_\varphi) \subseteq g_{3k}(I_\varphi)$, and then $[h^-_{\varphi 1k}, h^+_{\varphi 1k}] \subseteq [h^-_{\varphi 2k}, h^+_{\varphi 2k}] \subseteq [h^-_{\varphi 3k}, h^+_{\varphi 3k}]$ and $h_{m1k} \le h_{m2k} \le h_{m3k}$ (k = 1, 2, …, q) exist. Thus, there are the following inequalities:

$$(h^-_{\varphi 1k} - h^-_{\varphi 2k})^2 \le (h^-_{\varphi 1k} - h^-_{\varphi 3k})^2, (h^+_{\varphi 1k} - h^+_{\varphi 2k})^2 \le (h^+_{\varphi 1k} - h^+_{\varphi 3k})^2,$$

$$(h^-_{\varphi 2k} - h^-_{\varphi 3k})^2 \le (h^-_{\varphi 1k} - h^-_{\varphi 3k})^2, (h^+_{\varphi 2k} - h^+_{\varphi 3k})^2 \le (h^+_{\varphi 1k} - h^+_{\varphi 3k})^2,$$

$$(h_{m1k} - h_{m2k})^2 \le (h_{m1k} - h_{m3k})^2, \ (h_{m2k} - h_{m3k})^2 \le (h_{m1k} - h_{m3k})^2.$$

Since exp(−y) for $y \ge 0$ is a decreasing function, $E_\varphi(G_{1\varphi}, G_{2\varphi}) \ge E_\varphi(G_{1\varphi}, G_{3\varphi})$ and $E_\varphi(G_{2\varphi}, G_{3\varphi}) \ge E_\varphi(G_{1\varphi}, G_{3\varphi})$ can hold.

Considering the weight of $g_{jk}(I_\varphi)$ (k = 1, 2, …, q; j = 1, 2), it is given by $\lambda_k \in [0, 1]$ for $\sum_{k=1}^{q}\lambda_k = 1$. Thus, the weighted ESM of the CNNCSs $G_{1\varphi}$ and $G_{2\varphi}$ is established below:

$$E_{W\varphi}(G_{1\varphi}, G_{2\varphi}) = \sum_{k=1}^{q}\lambda_k \frac{\exp\left(-\left((h_{\varphi 1k}^{-} - h_{\varphi 2k}^{-})^2 + (h_{\varphi 1k}^{+} - h_{\varphi 2k}^{+})^2 + (h_{m1k} - h_{m2k})^2\right)\right) - \exp(-3)}{1-\exp(-3)}. \quad (7)$$

Proposition 2. The weighted ESM $E_{W\varphi}(G_{1\varphi}, G_{2\varphi})$ also contains these characteristics:

(a) $E_{W\varphi}(G_{1\varphi}, G_{2\varphi}) = E_{W\varphi}(G_{2\varphi}, G_{1\varphi})$;

(b) $0 \le E_{W\varphi}(G_{1\varphi}, G_{2\varphi}) \le 1$;

(c) $E_{W\varphi}(G_{1\varphi}, G_{2\varphi}) = 1$ if and only if $G_{1\varphi} = G_{2\varphi}$;

(d) If $G_{1\varphi} \subseteq G_{2\varphi} \subseteq G_{3\varphi}$ for any three CNNCSs $G_{1\varphi}$, $G_{2\varphi}$, and $G_{3\varphi}$, then there are $E_{W\varphi}(G_{1\varphi}, G_{2\varphi}) \ge E_{W\varphi}(G_{1\varphi}, G_{3\varphi})$ and $E_{W\varphi}(G_{2\varphi}, G_{3\varphi}) \ge E_{W\varphi}(G_{1\varphi}, G_{3\varphi})$.

Based on the similar proof process of Proposition 1, Proposition 2 can be easily verified (omitted).

4. Group DM Model Based on the ESM of CNNCSs

A multi-criteria group DM problem usually contains a group of possible alternatives $Me = \{Me_1, Me_2, …, Me_p\}$ and a group of main assessment criteria $Z = \{z_1, z_2, …, z_q\}$. Taking into account the weights of different criteria, their weight vector is expressed as $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, …, \lambda_q)$. In the group DM problem, the group DM model can be developed and reflected by the decision procedure below.

Step 1: In the suitability assessment of the alternatives, the fuzzy evaluation values of each alternative satisfying the criteria are assigned by a group of experts/decision makers and constructed as the FMVS $H_j = \{h_{jk} | k = 1, 2, ..., q\}$ containing the q FMVEs $h_{jk} = \langle z_k, (h_{jk1}, h_{jk2}, ..., h_{jkr_k})\rangle$ (k = 1, 2, …, q; j = 1,2, …, p) for $z_k \in Z$. Then, all FMVSs can be formed as their decision matrix $D_H = (h_{jk})_{p\times q}$.

Step 2: Using Eqs. (3)–(5) for the 90%, 95% and 99% confidence levels with the specified values of $t_{\varphi/2}$ = 1.645, 1.96, 2.576, the FMVSs H_j (j = 1, 2, …, p) can be transformed into the CNNCSs $G_{j\varphi} = \{g_{j1}(I_\varphi), g_{j2}(I_\varphi), …, g_{jq}(I_\varphi)\}$ containing the q CNNCEs $g_{jk}(I_\varphi) = \langle [h_{\varphi jk}^{-}, h_{\varphi jk}^{+}], h_{mjk}\rangle$ (j = 1, 2, …, p; k = 1, 2, …, q) for φ = 0.1, 0.05, 0.01. Thus, their decision matrix is denoted as $D_\varphi = (g_{jk}(I_\varphi))_{p\times q}$.

Step 3: Set the ideal solution/CNNCS as $G^* = \{<z_1, [1, 1], 1>, <z_2, [1, 1], 1>, …, <z_q, [1, 1], 1>\}$. Then, the weighted ESM values of $E_{W\varphi}(G_{j\varphi}, G^*)$ (j = 1, 2, …, p) are given by

$$E_{W\varphi}(G_{j\varphi}, G^*) = \sum_{k=1}^{q}\lambda_k \frac{\exp\left(-\left((h_{\varphi jk}^{-} - 1)^2 + (h_{\varphi jk}^{+} - 1)^2 + (h_{mjk} - 1)^2\right)\right) - \exp(-3)}{1-\exp(-3)}. \quad (8)$$

Step 4: The alternatives are sorted, and the optimal choice is determined by the largest weighted ESM value.

Step 5: End.

5. DM Example

5.1 Selection of intelligent manufacturing equipment

This section provides a DM example on the selection of intelligent manufacturing equipment in a manufacturing company to reflect the practicability and efficiency of the developed group DM model in the scenario of FMVSs.

To improve intelligent manufacturing capability in a manufacturing company, the manufacturing company wants to purchase a type of intelligent manufacturing equipment from possible equipment providers. In this case, the technology department preliminarily selects possible six types of intelligent manufacturing equipment (six alternatives) from possible equipment providers, which are denoted as a set of six alternatives $Me = \{Me_1, Me_2, Me_3, Me_4, Me_5, Me_6\}$. To assess their suitability, the technology department chooses four assessment criteria: cost (z_1), intelligent degree (z_2), technical advancement level (z_3), and manufacturing performance and capability (z_4). Then, the decision department invites five experts to select the optimal type of intelligent manufacturing equipment (the optimal alternative) by the suitability assessment of each alternative with respect to the four criteria. The weight vector of the four criteria **λ** = (0.2, 0.3, 0.2, 0.3) is presented by experts/decision makers.

For the DM example, the developed group DM model can be applied to the selection problem of intelligent manufacturing equipment and depicted by the decision procedure below.

Step 1: Five experts present their fuzzy evaluation values of each alternative Me_j (j = 1, 2, 3, 4, 5, 6) satisfying the criteria z_k (k = 1, 2, 3, 4). Then, their assessed fuzzy values are constructed as the FMVS decision matrix:

$$D_H = \begin{bmatrix} \langle z_1,(0.7,0.7,0.8,0.8,0.9)\rangle & \langle z_2,(0.6,0.7,0.7,0.7,0.7)\rangle & \langle z_3,(0.7,0.8,0.8,0.9,0.9)\rangle & \langle z_4,(0.7,0.8,0.8,0.8,0.8)\rangle \\ \langle z_1,(0.7,0.7,0.7,0.8,0.8)\rangle & \langle z_2,(0.6,0.6,0.7,0.7,0.8)\rangle & \langle z_3,(0.7,0.7,0.8,0.8,0.8)\rangle & \langle z_4,(0.6,0.7,0.7,0.8,0.8)\rangle \\ \langle z_1,(0.6,0.6,0.6,0.7,0.7)\rangle & \langle z_2,(0.6,0.7,0.8,0.8,0.9)\rangle & \langle z_3,(0.7,0.8,0.8,0.8,0.9)\rangle & \langle z_4,(0.6,0.6,0.6,0.7,0.8)\rangle \\ \langle z_1,(0.6,0.7,0.7,0.7,0.8)\rangle & \langle z_2,(0.6,0.6,0.7,0.8,0.8)\rangle & \langle z_3,(0.6,0.7,0.7,0.7,0.8)\rangle & \langle z_4,(0.6,0.6,0.7,0.7,0.8)\rangle \\ \langle z_1,(0.7,0.7,0.8,0.8,0.8)\rangle & \langle z_2,(0.7,0.7,0.7,0.7,0.7)\rangle & \langle z_3,(0.6,0.7,0.7,0.7,0.7)\rangle & \langle z_4,(0.5,0.6,0.7,0.7,0.7)\rangle \\ \langle z_1,(0.6,0.7,0.7,0.7,0.8)\rangle & \langle z_2,(0.6,0.7,0.7,0.8,0.8)\rangle & \langle z_3,(0.6,0.6,0.6,0.7,0.7)\rangle & \langle z_4,(0.5,0.6,0.8,0.8,0.9)\rangle \end{bmatrix}.$$

Step 2: The specified values for φ = 0.1, 0.05, 0.01 are $t_{\varphi/2}$ = 1.645, 1.96, 2.576 [29]. Using Eqs. (3)–(5) with the 90%, 95% and 99% confidence levels, the FMVS decision matrix D_H can be transformed into the following three CNNCS matrices:

$$D_{\varphi=0.1} = \begin{bmatrix} \langle [0.7184, 0.8416], 0.78\rangle & \langle [0.6471, 0.7129], 0.68\rangle & \langle [0.7584, 0.8816], 0.82\rangle & \langle [0.7471, 0.8129], 0.78\rangle \\ \langle [0.6997, 0.7803], 0.74\rangle & \langle [0.6184, 0.7416], 0.68\rangle & \langle [0.7197, 0.8003], 0.76\rangle & \langle [0.6584, 0.7816], 0.72\rangle \\ \langle [0.5997, 0.6803], 0.64\rangle & \langle [0.6761, 0.8439], 0.76\rangle & \langle [0.7480, 0.8520], 0.80\rangle & \langle [0.5942, 0.7258], 0.66\rangle \\ \langle [0.6480, 0.7520], 0.70\rangle & \langle [0.6264, 0.7736], 0.70\rangle & \langle [0.6480, 0.7520], 0.70\rangle & \langle [0.6184, 0.7416], 0.68\rangle \\ \langle [0.7197, 0.8003], 0.76\rangle & \langle [0.7000, 0.7000], 0.70\rangle & \langle [0.6471, 0.7129], 0.68\rangle & \langle [0.5742, 0.7058], 0.64\rangle \\ \langle [0.6480, 0.7520], 0.70\rangle & \langle [0.6584, 0.7816], 0.72\rangle & \langle [0.5997, 0.6803], 0.64\rangle & \langle [0.5991, 0.8409], 0.72\rangle \end{bmatrix}.$$

$$D_{\varphi=0.05} = \begin{bmatrix} \langle [0.7067, 0.8533], 0.78\rangle & \langle [0.6408, 0.7192], 0.68\rangle & \langle [0.7467, 0.8933], 0.82\rangle & \langle [0.7408, 0.8192], 0.78\rangle \\ \langle [0.6920, 0.7880], 0.74\rangle & \langle [0.6067, 0.7533], 0.68\rangle & \langle [0.7120, 0.8080], 0.76\rangle & \langle [0.6467, 0.7933], 0.72\rangle \\ \langle [0.5920, 0.6880], 0.64\rangle & \langle [0.6601, 0.8599], 0.76\rangle & \langle [0.7380, 0.8620], 0.80\rangle & \langle [0.5816, 0.7384], 0.66\rangle \\ \langle [0.6380, 0.7620], 0.70\rangle & \langle [0.6123, 0.7877], 0.70\rangle & \langle [0.6380, 0.7620], 0.70\rangle & \langle [0.6067, 0.7533], 0.68\rangle \\ \langle [0.7120, 0.8080], 0.76\rangle & \langle [0.7000, 0.7000], 0.70\rangle & \langle [0.6408, 0.7192], 0.68\rangle & \langle [0.5616, 0.7184], 0.64\rangle \\ \langle [0.6380, 0.7620], 0.70\rangle & \langle [0.6467, 0.7933], 0.72\rangle & \langle [0.5920, 0.6880], 0.64\rangle & \langle [0.5760, 0.8640], 0.72\rangle \end{bmatrix},$$

$$D_{\varphi=0.01} = \begin{bmatrix} \langle [0.6836, 0.8764], 0.78\rangle & \langle [0.6285, 0.7315], 0.68\rangle & \langle [0.7236, 0.9164], 0.82\rangle & \langle [0.7285, 0.8315], 0.78\rangle \\ \langle [0.6769, 0.8031], 0.74\rangle & \langle [0.5836, 0.7764], 0.68\rangle & \langle [0.6969, 0.8231], 0.76\rangle & \langle [0.6236, 0.8164], 0.72\rangle \\ \langle [0.5769, 0.7031], 0.64\rangle & \langle [0.6286, 0.8914], 0.76\rangle & \langle [0.7185, 0.8815], 0.80\rangle & \langle [0.5570, 0.7630], 0.66\rangle \\ \langle [0.6185, 0.7815], 0.70\rangle & \langle [0.5848, 0.8152], 0.70\rangle & \langle [0.6185, 0.7815], 0.70\rangle & \langle [0.5836, 0.7764], 0.68\rangle \\ \langle [0.6969, 0.8231], 0.76\rangle & \langle [0.7000, 0.7000], 0.70\rangle & \langle [0.6285, 0.7315], 0.68\rangle & \langle [0.5370, 0.7430], 0.64\rangle \\ \langle [0.6185, 0.7815], 0.70\rangle & \langle [0.6236, 0.8164], 0.72\rangle & \langle [0.5769, 0.7031], 0.64\rangle & \langle [0.5307, 0.9093], 0.72\rangle \end{bmatrix}.$$

Step 3: Using Eq. (8), the weighted ESM values of $Ew_\varphi(G_{j\varphi}, G^*)$ are shown in Table 1.

Table 1. Decision results corresponding to the 90%, 95% and 99% confidence levels

φ	$t_{\varphi/2}$	$Ew_\varphi(G_{j\varphi}, G^*)$	Sorting order	Optimal choice
0.1	1.645	0.8220, 0.7735, 0.7587, 0.7361, 0.7318, 0.7397	$Me_1 > Me_2 > Me_3 > Me_6 > Me_4 > Me_5$	Me_1
0.05	1.96	0.8203, 0.7715, 0.7557, 0.7336, 0.7306, 0.7354	$Me_1 > Me_2 > Me_3 > Me_6 > Me_4 > Me_5$	Me_1
0.01	2.576	0.8163, 0.7666, 0.7485, 0.7274, 0.7278, 0.7250	$Me_1 > Me_2 > Me_3 > Me_5 > Me_4 > Me_6$	Me_1

Step 4: The six alternatives are sorted and the optimal choice is determined by the largest weighted ESM value, then all decision results corresponding to the 90%, 95%, and 99% confidence levels are shown in Table 1.

5.2 Results and discussions

In view of the decision results in Table 1, different confidence levels can impact on the sorting orders of the six alternatives, then the optimal alternative always is Me_1. By comparing existing DM models in the scenarios of FMSs and CSs [13, 16, 17, 18], our new DM model reveals the following main merits:

(i) The proposed information transformation technique from FMVSs to CNNCSs can make the information expression more reasonable and confident and avoid operation problems between different fuzzy sequence lengths in FMVSs since CNNCS contains CNNs (confidence intervals) and average values. Then, CNN can reflect the probabilistic estimation of fuzzy values related to some confidence level to ensure the probabilistic reliability of fuzzy values falling within CNN.

(ii) Our new group DM model based on the weighted ESM of CNNCSs can reflect its decision flexibility depending on specified confidence levels. Then, decision makers can choose their optimal alternative according to their preference for confidence levels so as to satisfy some actual applications or requirements.

(iii) To some extent, existing CS is only a special case of CNNCS. In terms of a probabilistic viewpoint, existing CSs lack a confidence level in group DM problems, which shows its defect in the probabilistic estimation of the group evaluation values; while CNNCS contains both CNNs and average values, which can reflect the confidence level and magnitude of the group evaluation values. Therefore, our new group DM model indicates its obvious superiority over the existing DM models in the scenarios of FMSs and CSs.

6. Conclusions

Based on a confidence level of small sample data (the collection of several fuzzy values), this paper proposed a transformation technique from FMVSs to CNNCSs to reasonably express the mixed information of CNN and mean of fuzzy sequences. In the group DM process, the advantage of CNNCSs is that CNNCSs can effectively ensure the group evaluation data and mean falling within CNNs (confidence intervals) in light of a confidence level and a distribution status of the group evaluation data and solve the operational issue between different fuzzy sequence lengths in the scenario of FMVSs. Then, the proposed ESM of CNNCSs can make the similarity measure more reasonable and confident since it is closely related to confidence levels and normal distribution. Moreover, it also implies the measure flexibility corresponding to different confidence levels. The developed group DM model based on the proposed ESM of CNNCSs can not only make decision results more flexible and confident depending on certain confidence level, but also ensure the credibility and effectiveness of the DM results from the perspective of probability estimation in the scenario of FMVSs. It is obvious that the developed group DM model of CNNCSs reveals its obvious superiority over the existing DM models of FMSs/CSs in the information conversion/expression and DM methods.

Since this original study proposed the transformation technique from FMVSs to CNNCSs and the group DM model of CNNCSs for the first time, they are only suitable for group DM problems under the normal distribution condition of the group evaluation data (FMVSs), which shows their limitation in group DM applications. Therefore, we shall further develop other transformation techniques and group DM models and their applications, such as medical diagnosis, image processing, and production programming problems, as future research directions.

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Neutrosophic model for vehicular malfunction detection

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Abstract: The internet of vehicular things (IOVT) is an important modern technology that offers many advantages and facilities; however, if vehicular malfunctions are not detected in a timely manner, it may cause many dangers and serious accidents. To achieve safe self-driving vehicles, safety and security measures must be taken. In this work, a safety and security model are proposed to evaluate the level of vehicular malfunctions and determine the corresponding danger in terms of road safety. The proposed model presents the optimal actions and alternatives for self-driving vehicles to avoid crises. The objective of this study to develop a hybrid model for multicriteria decision-making problems using neutrosophic theory to handle vehicular malfunctions that occur in the IOVT environment under uncertain conditions and conflicting information. In addition, the technique for order of preference by similarity to the ideal solution is used to prioritize the corresponding alternatives in the case of vehicular malfunction. A case study considering four likely vehicular defects is presented to ensure the applicability and availability of the proposed model.

Keywords: internet of vehicular things (IOVT), vehicular malfunction detection, multi-criteria decision making (MCDM), neutrosophic theory, analytical hierarchy process (AHP), TOPSIS.

1. Introduction

Self-driving vehicles have become one of the most important technological advances in the world [1]. To reduce risks that result from vehicles, self-driving vehicles are expected to be relied upon in more countries and cities [2]. By 2040, 40% of vehicles are expected to be self-driving [3]. According to the World Health Organization, many people at risk of serious injury or death each year from accidents due to undiscovered vehicular defects [4]. The problem of defect detection in vehicles, especially while driving, is the target of this study, with the aim of preventing accidents on the road based on statistics regarding the causes of accidents. This paper proposes the most important problems that may cause accidents, which are classified into four categories depicted in Figure 1.

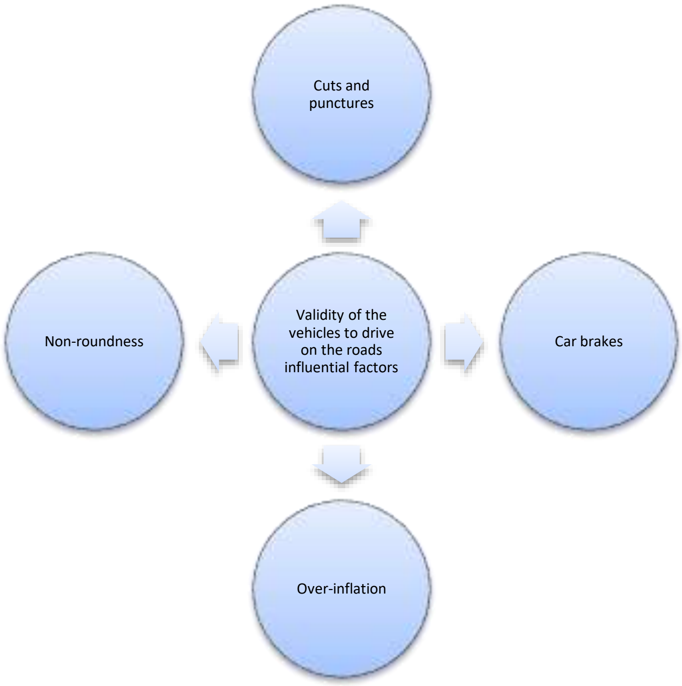

Figure 1. Factors and obstacles that influence the roadworthiness of vehicles.

Tires are important sources of vehicular risks and accidents. Hence, three of the problems considered in this study are related to tires: 1) overinflation, when the pressure exceeds the normal range [5],[6]; 2) non-roundness, deformations or extensions of the tire [7]; and 3) cuts and punctures, which can occur if the vehicle drives over hazards on the road [8]. The fourth problem is related to brakes [9]. Malfunctions and defects of the brakes can cause vague and inconsistent information for self-driving vehicles [1], which may lead to difficulty in making consistent and accurate driving decisions.

This paper discusses some of the defects that affect vehicles depending on internet of vehicular things (IOVT) technology by integrating multicriteria decision-making (MCDM) method, analytical hierarchy process (AHP), neutrosophic sets, and the technique for order of preference by similarity to ideal solution (TOPSIS) based on intelligent techniques [10]. AHP is among the most popular methods to deal with complicated MCDM problems [11] and can be summarized as a process of decomposition, calculating the weights for the decision criteria, and finally calculating the priority for alternatives [11]. This classical AHP method can determine the priorities for criteria and is also able compare and grade alternatives, but it is unable to handle ambiguous information [10] and the Saaty comparison matrix cannot determine whether it is in a consistent or inconsistent state because it has no systemic methodology [10]. To overcome this problem, fuzzy AHP (FAHP) combines fuzzy set theory and AHP [12]. This method can handle conflict, but decision makers cannot determine the membership function permanently. This paper proposes a neutrosophic technique combined with AHP to help decision makers handle uncertainty and determine influential factors to better handle vehicular defects. The neutrosophic set is expressed as truth, falsity, and indeterminacy (T, I, F) membership [13]. Based on this, uncertainty, conflict, and vague and incomplete information can be handled. TOPSIS methods, including AHP TOPSIS, depend on classifying alternatives into two parts: positive and negative solutions, where the optimal solution is the solution near the set of positive solutions that is farthest from the set of negative solutions [10,14]. This proposed model examined four risks that could affect self-driving vehicles that rely on IOVT technology to determine the optimal action that must be taken at the right time. IOVT is a network of vehicles that contain software, sensors, and other important techniques and among the most influential factors in autonomous vehicles [15,16]. This paper aims to achieve the following objectives:

1. Determine whether IOVT can overcome vehicular problems and accidents.
2. Discuss some defects and malfunctions that may affect vehicles and lead problems and accidents.
3. Assess the influence of criteria in attempt to help experts and decision makers reach optimal solutions.
4. Propose solutions to deal with MCDM problems.
5. Propose a hybrid model that integrates AHP, neutrosophic models, and TOPSIS to recommend the best option out of three proposed alternatives.
6. Apply the proposed framework in a case study of self-driving vehicles that depend on IOVT.
7. Conduct a sensitivity analysis to ensure the robustness and reliability of decision-making by IOVT.

The next sections of this paper are as follows. Section 2 presents a literature review. Section 3 introduces the framework of this study. Section 4 presents the methodology of the proposed model. Section 5 concludes the paper with insights obtained from this work and future considerations.

2. Literature Review

Many researchers have suggested the importance of the internet of things (IOT) to connect devices via the internet, and IOVT has become one of the most important modern technical developments in the era in both academic and industrial fields [17]. The rapid development of the intelligent transportation system (ITS) helps to provide utilities to consumers, including safe traffic management [18].

The daily use of roads causes some dangers for drivers [19]. There are many causes of vehicle accidents arising from a lack of experience in dealing with emergency situations. These situations include tire problems, such as overinflation, cuts and punctures, and non-roundness, as well as problems with the vehicle's brakes [20]. IOVT helps to predict these malfunctions in a timely manner and make an appropriate decision to avoid accidents. Many recent studies on vehicular defect detection in intelligent transportation systems introduce risk assessment of vehicles to propose a theoretical basis to prevent accidents that results from vehicular malfunctions [21]. In [22], a unified diagnostics service protocol (UDS) proposes a semiautomatic approach to brake pedal testing and diagnostics. In [23], radiography is used to detect defects in vehicle tires, and [24] describes three vehicular defects, including changes in tire pressure. Effective methods have been developed to detect aquaplaning detection using a small group of sensors, stability-based electronic control, and drive torques [5]. In [25], IOT and deep learning are combined to produce an integrated self-diagnostic system for self-driving vehicles. In [17] discuss federated learning issue and aims to develop IOVT applications which is characterized by confidentiality and security. IOT and ITS have been merged to improve the efficiency and effectiveness of ITS. Information about malfunctions that vehicles may be exposed may be incomplete and uncertainty [26]. The authors of [27] propose techniques to handle uncertainty when predicting crashes in self-driving vehicles.

MCDM methods have become an important issue for decision makers, as they are used to prioritize criteria and alternatives to help solve the problems of uncertainty and incomplete information [11]. In addition, several studies have been presented based on fuzzy sets. For example, [28] presents a theory of sets to manage uncertainty, and [4] presents an FAHP method to evaluate the roadworthiness of vehicles. When AHP methods are integrated with fuzzy techniques, they can better handle uncertainty information, but they still cannot handle indeterminate values. FAHP is very convenient for evaluating alternatives. FAHP can evaluate the current state of the vehicle, but it has some limitations. For example, when input data are expressed in linguistic terms depending on the experience and opinions of decision makers, it cannot obtain actual relations between the criteria and alternatives [11]. MCDM methods use neutrosophic sets to offer solutions under ambiguous and conflicting information by proposing truth, indeterminate, and falsity (T, I, F). In [29], MCDM with single value neutrosophic sets is proposed to calculate values between options and available choices. The neutrosophic set proposes three membership functions to calculate the weights of criteria and alternatives and choose the optimal alternative, and its integration with TOPSIS is a new development to enable the selection of an ideal choice [30]. In [31] researchers present a realistic empirical example of Starbucks company to develop strategies for its development and uses a model that combines AHP and Neutrosophic theory. In [32] researchers present a model that combines AHP and Neutrosophic theory. In [33] researchers discuss the problem of choosing the best learning management system (LMS) because there are many (LMS) available in the marketplace therefore, decision -making to choose the best system is a multi-criteria problem. so, this research applies neutrosophic AHP method. The main contribution of this study is the application of the MCDM method using a neutrosophic set, AHP, and TOPSIS to produce an effective model that can handle the problems of IOVT.

3. Framework

The integration of AHP, MCDM, neutrosophic, and TOPSIS techniques is an effective way to help decision makers face the problems of uncertainty and confusion of information to make appropriate decisions. Neutrosophic and TOPSIS methods have been used in recent studies to help determine ideal solutions. AHP is a method to solve confusion and complex problems [34] and is characterized by its simplicity, as it decomposes problems into subproblems [35]. This study proposes the integration of AHP and neutrosophic techniques to analyze the factors that influence the safety of vehicles [36]. The resulting system outputs a warning if defects or malfunctions are detected based on multiple data sets obtained from sensors in the tires, which are connected to each other and the warning system using IOT [37].

This section describes four main criteria that cause vehicular malfunctions, which may lead to injuries and accidents. Tires are a major source of problems that result in accidents; therefore, tires must be replaced or repaired as soon as a problem is detected to avoid accidents. Three of the four criteria considered herein are tire defects: overinflation, non-roundness, and cuts or punctures. The fourth defect is malfunction of the brakes [38]. The correct action must be selected in the event of any of these defects from the following three options: stop the vehicle immediately, stop the vehicle at the nearest repair station, or continue (there is no danger).

The main criteria are measured as follows:

1. Overinflation:

A sensor is used to measure the air temperature and pressure changes inside the tire [39]. If the pressure reaches the critical pressure, the sensor sends warning. The sensor utilizes multiple previously constructed datasets to determine the critical value.

2. Non-roundness:

A sensor is used to detect stretching and changes in the tire radius.

3. Cuts and punctures:

A moving sensor detects any cuts or punctures in the tire.

4. Brakes:

Braking condition is a well-established influential factor that must be constantly examined in IOT environments [40]. A warning is sent to the vehicle if the sensor detects any abnormal conditions.

The steps of the proposed method, as depicted in Figure 2, can be divided into three stages. The first stage is to specify the criteria and actions. In the second stage, the criteria are evaluated using neutrosophic scales to help decision makers determine the optimal action. The third stage applies TOPSIS through the following steps:

5. Normalize the criteria and actions.
6. Find the positive and negative regions.
7. Find the positive and negative Euclidian regions and determine the relative proximity.

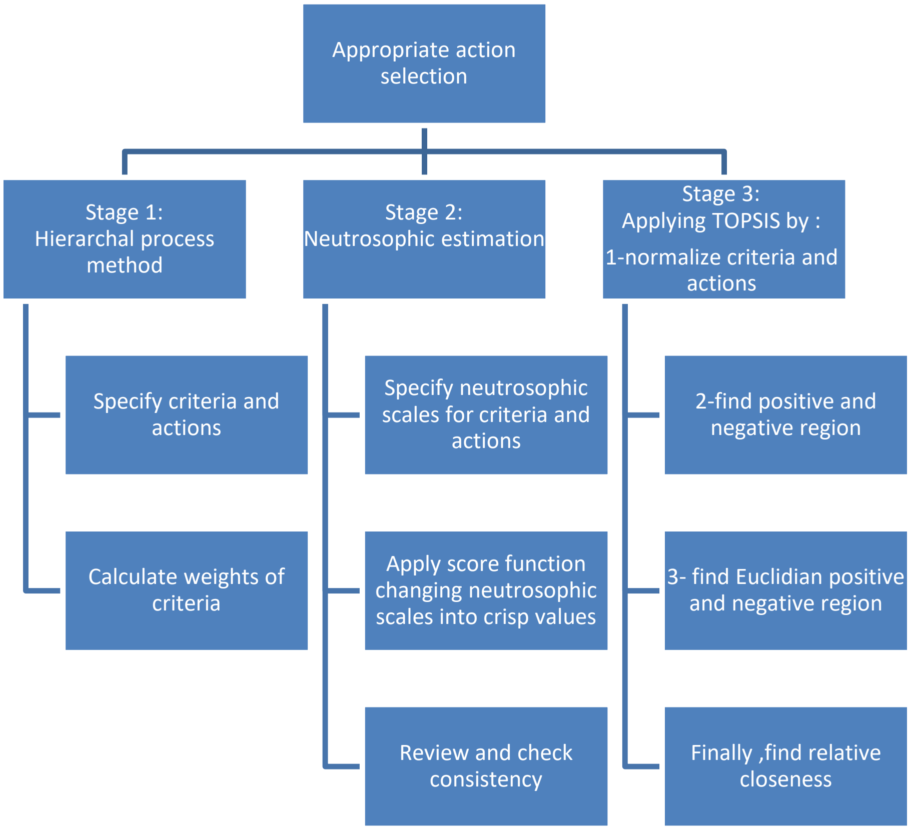

Figure 2. Conceptual steps of the proposed method.

Then, the optimal action is determined by the following steps:
Step 1: Specify the criteria using the AHP model according to Table 1.
Step 2: Compare the criteria and actions based on the neutrosophic scale in Table 1.
If criterion 1 is strongly significant than criterion 2, the value of the neutrosophic scale is written as $\langle(4,5,6)0.80,0.15,0.20\rangle$; conversely, if criterion 2 is strongly significant than criterion 1, the neutrosophic scale is the inverse of <4,5,6>, which is1/ $\langle(4,5,6)0.80,0.15,0.20\rangle$.
The pairwise comparison matrix between the different criteria is

$$A^k = \begin{bmatrix} x_{11}^k & x_{12}^k & x_{1n}^k \\ - & - & - \\ x_{n1}^k & - & x_{mn}^k \end{bmatrix}, \quad (1)$$

where x_{mn}^k. k represents the decision maker's number depending on the preference of the n^{th} criterion over the m^{th}. For example, in the form of the neutrosophic triangular, the decision maker's sight is presented as <<4, 5, 6>; <0.80, 0.15, 0.20>>, where the neutrosophic triangular scale values are referenced as the lower, median, and upper values.
The decision maker's degree of certainty is represented as <0.80, 0.15, 0,20> truth, indeterminacy, and falsity. Thus, the triangular neutrosophic scale structure is < (L, m, u); T, I, F x_{mn}^k, where l, m, and u refer to the lower, median, and upper neutrosophic triangular scale values. I_{mn}^k , F_{mn}^k T_{mn}^k are the truth, indeterminacy, and falsity, which represent the certainty of the decision maker's perspective.
For example, x_{24}^3 refers to the comparison of criteria 2 and 4 from the perspective of the third decision maker.

Step 3: Aggregate the decision makers' preference relations between the criteria.

To achieve certainty, multiple decision makers evaluate the preference relations between the criteria. The aggregated s_{ij} is

$$s_{ij} = \frac{\sum_{k=1}^{k} < (l_{ij}^{k}, m_{ij}^{k}, u_{ij}^{k}); T_{ij}^{k}, I_{ij}^{k}, F_{ij}^{k} >}{k}, \tag{2}$$

and the aggregated pairwise comparison matrix is

$$G = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{mn} \end{bmatrix}. \tag{3}$$

Then, the neutrosophic scales are transformed into crisp values using the score function of (s_{ij}).

$$S\ (s_{ij}) = \left| \left(l_{ij} \times m_{ij} \times u_{ij}\right)^{\frac{T_{ij} + I_{ij} + F_{ij}}{9}} \right|. \tag{4}$$

The scale of the neutrosophic numbers represented by l, m, u and T, I, F symbolize lower, median, upper and truth, indeterminacy, and falsity membership functions of the triangular neutrosophic number.

Step 4: According to the previous matrix, the weights and priorities are calculated.
First, calculate the sum of the average row.

$$w_i = \frac{\sum_{i=1}^{n} (x_{ij})}{n}, \tag{5}$$

where i = 1, 2, 3, 4, …, m and j = 1, 2, 3, 4, …, n.
Second, normalize the crisp value using

$$w_i^{m} = \frac{w_i}{\sum_{i=1}^{m} w_i} \ ; i = 1.2.3.4 \dots\dots m. \tag{6}$$

Step 5: Verify the decision maker's decision.

$$CR = \frac{CI}{RI}, \tag{7}$$

where CR, CI, and RI denote the consistency rate, consistency index, and random consistency index, respectively. The result achieved an accepted consistency of 1%.

Table 1. Triangular neutrosophic scales corresponding to linguistic phrases.

Score	Linguistic Phrase	Neutrosophic Triangular Scale
1	Equally significant	$1 = \langle(1,1,1); 0.50, 0.50, 0.50\rangle$
3	Slightly significant	$3 = \langle(2,3,4); 0.30, 0.75, 0.70\rangle$
5	Strongly significant	$5 = \langle(4,5,6); (0.80, 0.15, 0.20\rangle$
7	very strongly significant	$7 = \langle(6,7,8), 0.90, 0.10, 0.10\rangle$
9	Absolutely significant	$9 = \langle(9,9,0); 1.00, 0.00, 0.00\rangle$
2	Sporadic values between two close scales	$2 = \langle(1,2,3); 0.40, 0.60, 0.65\rangle$
4		$4 = \langle(3,4,5); 0.35, 0.60, 0.40\rangle$
6		$6 = \langle(5,6,7); 0.70, 0.25, 0.30\rangle$
8		$8 = \langle(7,8,9); 0.85, 0.10, 0.15\rangle$

Step 6: Upgrade the consistency in the neutrosophic AHP by collecting the inconsistent elements in the pairwise comparison matrix using the induced matrix, as mentioned in [38]. Then calculate the normalized decision matrix as follows:

$$r_{ij} = \frac{z_{ij}}{\sqrt{\sum_{i=1} z_{ij}^2}}. \quad (8)$$

Step 7: Multiply each alternative by its corresponding weight considering its corresponding criterion to obtain an action score using:

$$z_{ij} = w_j \times r_{ij} \quad (9)$$

Step 8: Select the best decision according to the rankings of the alternatives. This process is implemented in several steps:

Step 8.1: Calculate the positive and negative regions using Eqs. (10) and (11), respectively:

$$Q^+ = \left\{ < min\ (y_{ij} | i=1,2,\dots\dots,m0) j \in j^- < max(y_{ij} | i=1,2,\dots\dots,m) | j \in j^+ >, \right. \quad \textbf{(10)}$$

$$Q^- = \left\{ < max\ (y_{ij} | i = 1,2,3,\dots.m) | j \in j^- > \quad < min(y_{ij} | i = 1,2,3,,\dots m) | j \in j^+ >, \right. \quad \textbf{(11)}$$

Step 8.2: Compute the Euclidian distance between the positive (d_i^+) and negative (d_i^-) optimal solutions using Eqs. (12) and (13), respectively:

$$d_i^+ = \sqrt{\sum_{i=1}^{n} (y_{ij} - y_j^+)^2}, i = 1,2,3,4,\dots.m, \quad (12)$$

$$d_i^- = \sqrt{\sum_{i=1}^{n} (y_{ij} - y\ )^2}, i = 1,2,3,4,\dots.m. \quad (13)$$

Step 8.3: Compose a final ranking of actions and select the ideal action. For this purpose, calculate the relative closeness as

$$R_j = \frac{d_i^-}{d_i^+ + d_i^-}; \quad 1,2,3,\dots.,m \quad (14)$$

Step 8.4: Choose the optimal action.

4. Empirical Application

As an empirical application of the neutrosophic model, we consider a vehicle manufacturing company in Egypt. This company hopes to introduce IOT technologies to self-driving vehicles, which will detect vehicle malfunctions during an early stage of driving on the road and make an appropriate decision for accident and disaster avoidance. The company employs an expert panel of four decision makers (Table 2). During a meeting, the expert panel proposed the following four criteria for identifying vehicle malfunctions:

C1: overinflation.
C2: non-roundness.
C3: cuts and punctures.
C4: brake malfunctions.

Decision makers select one of three actions:
1- stop the car immediately.
2- stop the car at the nearest repair station.
3- continue driving (no problem or cause for concern).

The proposed model proceeds through the following steps:

Step1: Select an expert panel of four decision makers. The credentials and demographic information of the experts are listed in Table 2 and the four main criteria and actions related to vehicle malfunctions are proposed in Figure 3.

Table 2. Demographic information of the expert committee.

Demographic information	**Job title**	Qualifications	Age	Gender
First expert	Financial consultant	Master	45	Female
Second expert	Mechatronics engineer	PhD	50	Male
Third expert	Quality and safety manger	Master	35	Female
Fourth expert	Mechanical engineer	Bachelor	40	Male

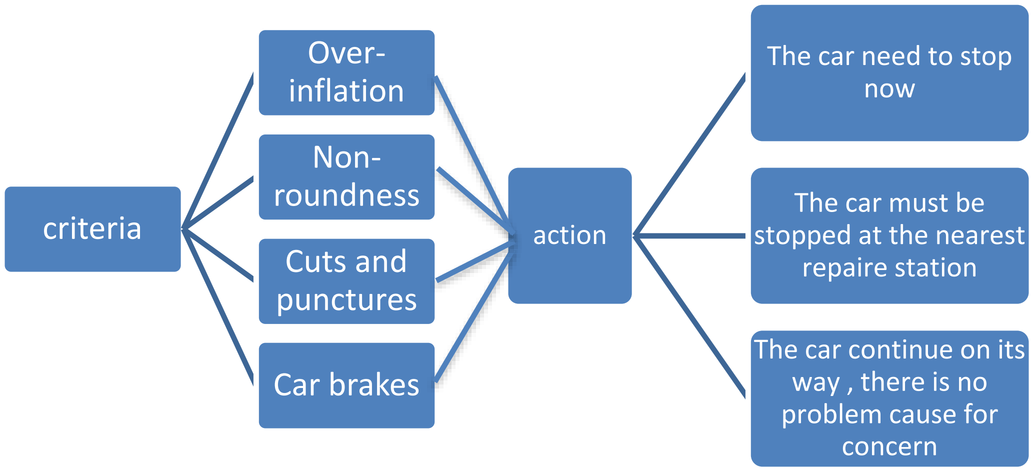

Figure 3. AHP structure of the presented criteria and actions.

Step 2: Map the decision maker's perspectives onto the neutrosophic scale using Eq. (1). The experts' decisions are aggregated using Eq. (2) and are expressed in the format of Eq. (3) in Table 3.

Table 3. Proposed collected perspectives of the decision makers of criteria.

Criterion	C1	C2	C3	C4
C1	<<1,1,1>;0.50,0.50,0.50>	<<4,5,6>;<0.80,0.15,0.20>	1/<<2,3,4>;0.30,0.75,0.70>	<<1,2,3>;0.40,0,60,0.65>
C2	1/<<4,5,6>;<0.80,0.5,0.20 >	<<1,1,1>;<0.50,0.50,0.50>	<<3,4,5>;0.35,0.60,0.40>	<<6,7,8>;0.90,0.10,0.10>
C3	1/<<2,3,4>;0.30,0.75,0.70>	1/<<3,4,5>;0.35,0.60,0.40>	<<1,1,1>;0.50,0.50,0.50>	<<7,8,9>;0.85,0.10,0.15>
C4	1/<<1,2,3>;0.40,0,60,0.65>	1/<<6,7,8>;0.90,0.10,0.10>	1/<<7,8,9>;0.85,0.10,0.15>	<<1,1,1>;0.50,0.50,0.50>

Step 3: For simplicity, convert the neutrosophic aggregated perspectives into crisp values using Eq. (4). The results are shown in Table 4.

Step 4: Compute the weights of the criteria using Eqs. (5) and (6). The results are listed in Table 5 and visualized as a pie chart in Figure 4.

Step 5: Compute the consistency rate using Eq. (7). The consistency was determined as 1%.

Table 4. Crisp values of criteria according to the perspectives of the decision makers.

Criterion	C1	C2	C3	C4
C1	1	1.843	1.855	1.388
C2	0.542	1	1.848	1.450
C3	0.539	0.541	1	2.139
C4	0.720	0.689	0.467	1

Table 5. Criteria weights.

Criteria	Weights
C1	0.377
C2	0.268
C3	0.234
C4	0.159

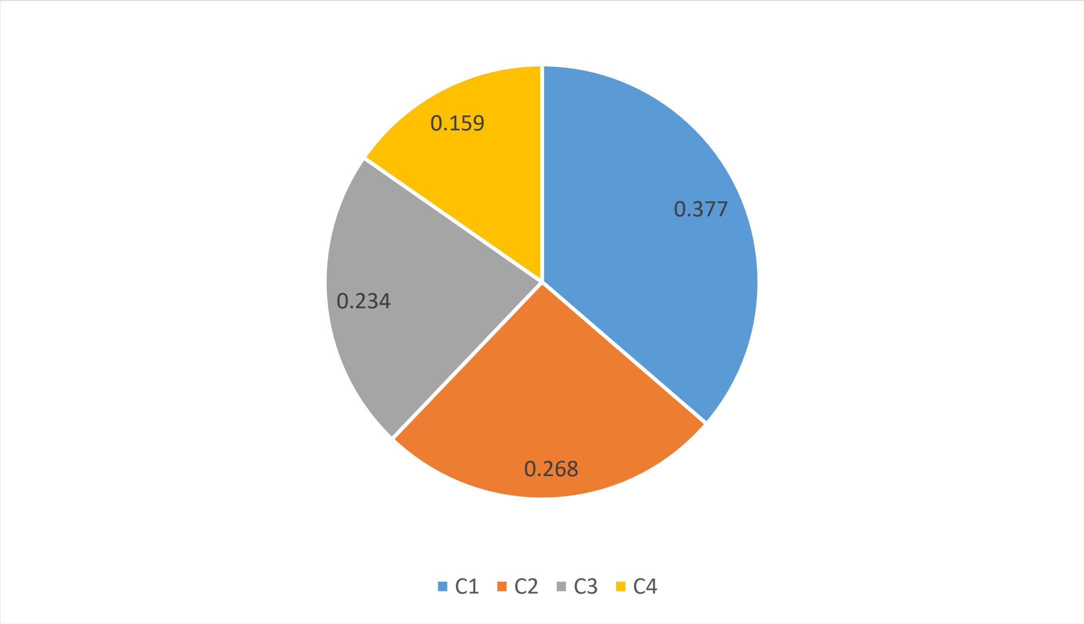

Figure 4. Pie chart of IOVT malfunctions criteria weights.

Step 6: Gain the perspectives of the decision makers on the presented actions and criteria (Table 6), then calculate the crisp neutrosophic values of the decision makers using Eq. (4) (Table 7). Finally, normalize the decision matrix as $r_{ij} = \frac{x_{ij}}{\sqrt{\sum_{i=1} x_{ij}^2}}$. The normalized results are listed in Table 8.

Table 6. Proposed decision matrix of criteria and actions for decision makers.

Criteria	C1	C2	C3	C4
A1	<<6,7,8>;0.90,0.10,0.10>	<<2,3,4>;0.30,0.75,0.70>	<<4,5,6>;0.80,0.15,0.20>	<<1,1,1>;0.50,0.50,0.50>
A2	<<1,1,1>;0.50,0.50,0.50>	<<1,2,3>;0.40,0.60,0.65>	<<2,3,4>;0.30,0.75,0.70>	<<9,9,9>;1.00,0.00,0.00>
A3	<<2,3,4>;0.30,0.75,0.70>	<<4,5,6>;0.80,0.15,0.20>	<<6,7,8>;0.90,0.10,0.10>	<<1,1,1>;0.50,0.50,0.50>

Table 7. Crisp neutrosophic values for decision makers.

Criteria	C1	C2	C3	C4
A1	2.03	1.85	1.84	1
A2	1	1.38	1.85	2.08
A3	1.85	1.84	2.03	1

Table 8. Normalization of decision matrix by applying $r_{ij} = \frac{z_{ij}}{\sqrt{\sum_{i=1} z_{ij}^2}}$.

Criteria	C1	C2	C3	C4
A1	0.415	0.364	0.321	0.245
A2	0.204	0.272	0.323	0.509
A3	0.379	0.362	0.354	0.245

Step 7: To calculate the weighted matrix, multiply the criteria weights obtained from the neutrosophic AHP by the normalized decision matrix [Eq. (9)]. The results are tabulated in Table 9 and presented in Figure 5.

Table 9. Weighted matrix obtained by applying $z_{ij} = w_j \times r_{ij}$ to multiply the criteria weights obtained from the neutrosophic AHP by the normalized decision matrix.

Criteria	C1	C2	C3	C4
A1	0.156	0.097	0.075	0.038
A2	0.076	0.072	0.075	0.080
A3	0.142	0.097	0.082	0.038

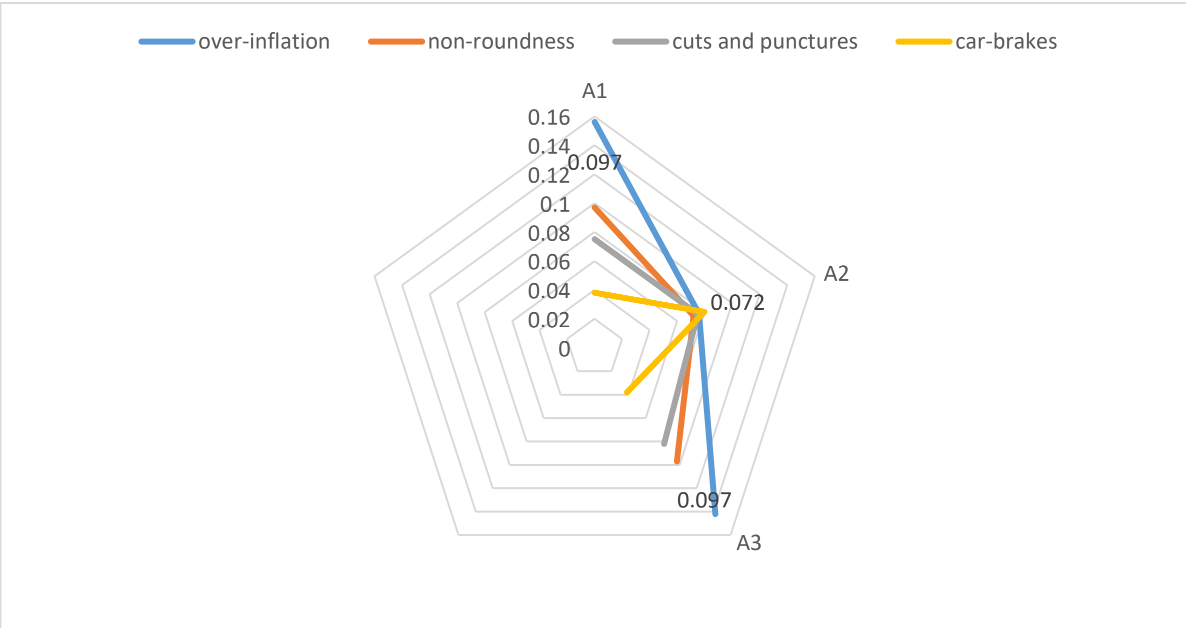

Figure 5. Comparison of the three alternatives based on different criteria Table 9.

Step 8: Calculate the positive and negative regions using Eqs. (10) and (11), respectively, then calculate the Euclidian distances between the positive (d_i^+) and negative (d_i^-) optimal solutions to present actions using Eqs. (12) and (13), respectively. Finally, rank the actions using Eq. (14). The ranked results are listed in Table 10.

$A^+ = \{0.156, 0.097, 0.082, 0.080\}$

$A^- = \{0.076, 0.072, 0.075, 0.038\}$

Table 10. Final ranks of actions.

	d_i^+	d_i^-	c_i	rank
A1	0.042	0.083	0.664	1
A2	0.084	0.042	0.33	3
A3	0.044	0.070	0.614	2

Actions A1 and A2 are considered to be the best and worst choices, respectively, in the opinion of the decision makers. That is, the best action for the driver to take is action A1—stop the car immediately—and the worst action, which the driver must not take, is to stop the car at the nearest repair station, as this action may cause danger or accidents.

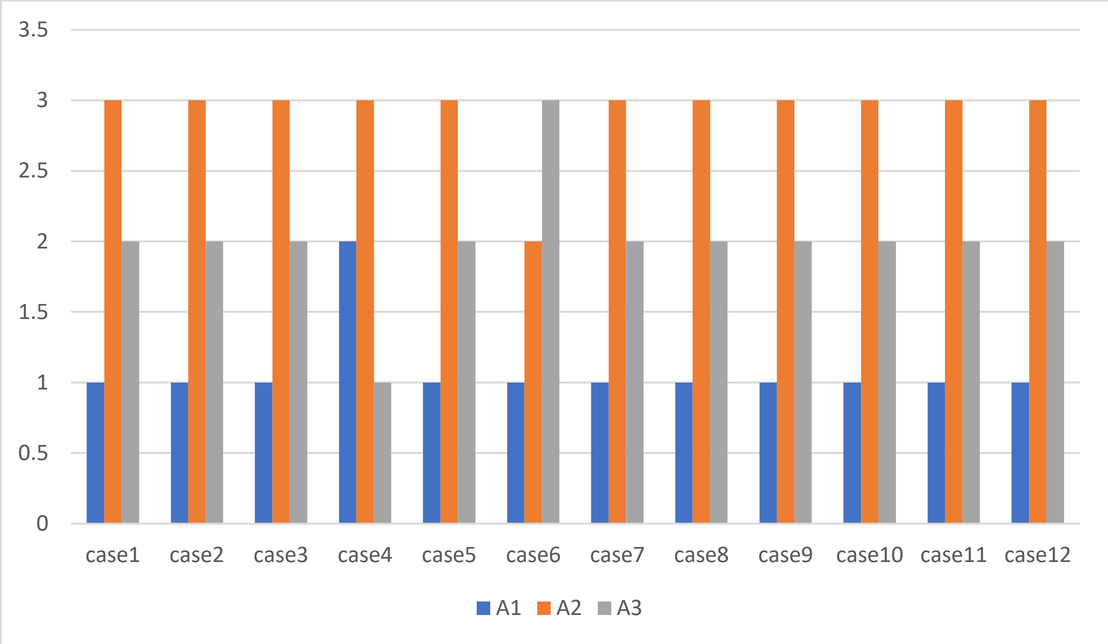

Figure 6. Sensitivity analysis of weights of alternatives depending on various priorities of criteria.

Case#	A1	A2	A3
1	1	3	2
2	1	3	2
3	1	3	2
4	2	3	1
5	1	3	2
6	1	2	3
7	1	3	2
8	1	3	2
9	1	3	2
10	1	3	2
11	1	3	2
12	1	3	2

Table 11. Final ranks of the alternatives for different priorities of criteria A1, A2, and A3

5. Sensitivity analysis

A sensitivity analysis studies the effect of the variance of each input measure on the model output. It is useful for prioritizing the selection of the best alternatives. During a sensitivity analysis, the model is assumed sufficiently precise to reproduce the behavior of the system. The present study conducts a sensitivity analysis on the criteria (attribute) ranking. Specifically, it demonstrates how the prioritization of the criteria affects the final rank of the alternatives. To obtain efficient and accurate results, we selected 12 random cases for the sensitivity analysis (three alternatives and four criteria; see Table (11). Figure 6 illustrates how the final rank of alternatives changes after changing the priority order of the criteria.

The sensitivity analysis clarified that in all cases except Case 6, A1 is the best alternative and A2 is the worst alternative. A3 ranked medium in most cases.

6. Comparative analysis

This part of the study compares the results of our suggested approach that integrates AHP, neutrosophic theory, and topsis with those of another approach that assumes a fuzzy environment [41],[42],[43]. Applying the fuzzy approach to select the best action of an autonomous vehicle in our case study, the alternatives are ranked as follows: A3> A1>A2 (Figure 7).

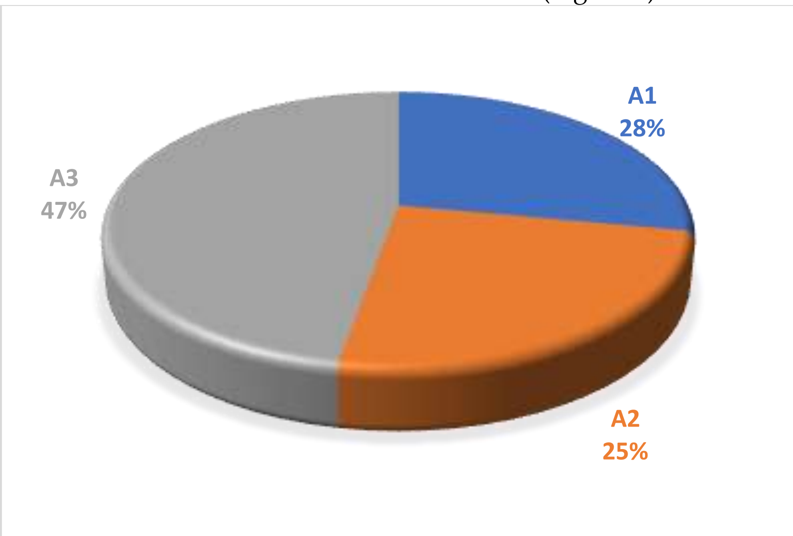

Figure 7. Alternative ranking based on the fuzzy approach.

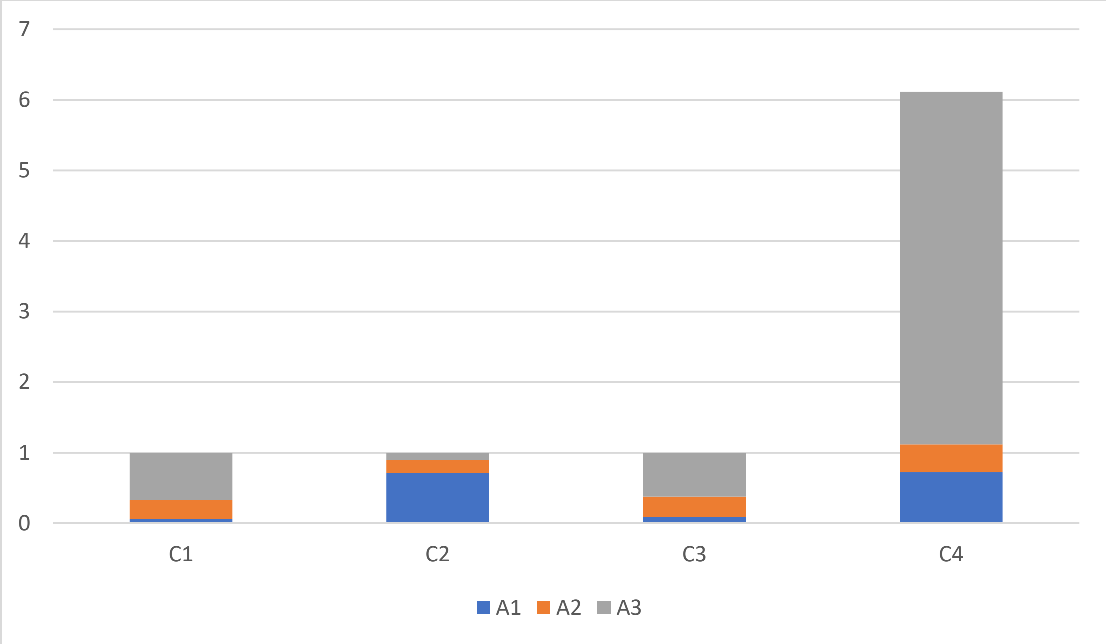

Figure 8. Aggregated results for each alternative according to each criteria.

Table 12 lists the final ranks of the alternatives using the fuzzy approach presented in [41] and [42] and our suggested neutrosophic topsis approach.

Table 12. Final rankings of alternatives based on our proposed approach and a fuzzy approach.

Alternatives	AHP neutrosophic topsis approach	Fuzzy approach
A1	1	2
A2	3	3
A3	2	1

To compare the ranks of the neutrosophic topsis approach and fuzzy approach, we applied Spearman's correlation method [44], which estimates whether two continuous variables are correlated or uncorrelated:

$$S_m = 1- \left[\frac{6.\sum_{r=1}^{t}(d_r)^2}{t.(t^2-1)}\right] \quad (15)$$

In this formula, t denotes the number of alternatives and d_r is the difference between two ranks of alternatives. If Sm is +1 or -1, the correlation is strong and if S_m is 0, the variables are uncorrelated. The Pearson's correlation indicates the degree of linear correlation between two variables. It ranges from -1 (completely negatively correlated) through 0 (completely uncorrelated) to +1 (completely positively correlated). The Pearson correlation is calculated as

$$P_{(a,b)} = \frac{cov(a,b)}{\sigma_a \sigma_b}, \quad (16)$$

where cov (a, b) denotes the covariance of a and b, and σ_a and σ_b denote the standard deviations of a and b, respectively. The Spearman's correlation coefficient was computed as 0.5, indicating a strong correlation between our proposed approach and the fuzzy approach. The Pearson's correlation coefficient between the two approaches was also 0.5. By ranking the weights of the criteria and alternatives and comparing the results of our proposed and fuzzy approaches, we find that our proposed approach simplifies the application as follows:

1. As the fuzzy approach requires more equations than our approach, it is necessarily more complex, time-consuming, and storage-demanding than the AHP neutrosophic–topsis approach.
2. Our proposed approach depends on the truth degree, falsity degree, and indeterminacy degree whereas the fuzzy approach depends only on the truth and falsity degrees. Therefore, our proposed approach can handle ambiguous and conflicting information which cannot be efficiently handled by the fuzzy approach. Moreover, the neutrosophic–topsis approach can simulate natural human thinking.
3. The fuzzy approach depends on linguistic variables, so is restricted in scale and cannot provide a logical confirmation degree. In contrast, our proposed approach allows decision makers to use suitable linguistic variables and confirmation degree.

7. Applications

The study proposes an intelligent hybrid model that merges AHP, neutrosophic theory, and topsis. The model handles MCDM problems and optimizes decision making to overcome the problems introduced by uncertainties and incomplete information. Although IOVT is being rapidly developed, its many advantages are partly offset by the increased risk of accidents caused by vehicle malfunctions that are undetected and not corrected by an appropriate action in a timely manner. The objective of this study was to ensure safety and security on the roads by discovering malfunctions in self-driving vehicles and quickly implementing the optimal action. Factories, companies, manufacturers and developers of self-driving vehicles will benefit from this model because it identifies and prioritizes the proper attributes and actions in the event of any problem or danger.

8. Conclusions and Future Work

The study proposes an intelligent hybrid model that merges AHP, neutrosophic, and TOPSIS techniques to solve MCDM problems and help decision makers overcome the problems of uncertainty and incomplete information. Many countries are witnessing significant developments in IOVT, which has some disadvantages and risks that arise from undetected vehicular malfunctions. These risks can be mitigated by taking the appropriate action in a timely manner. The objective of this study was to achieve safety and security on the roads by discovering malfunctions in self-driving vehicles in a timely manner and implementing the optimal action.

Comparing our proposed approach with the fuzzy approach, we concluded that our neutrosophic topsis approach is more effective and simpler to implement than the fuzzy approach; moreover, it simulates natural human thinking.

In the future, we will update this technique to predict more vehicular defects using diverse multicriteria decision-analysis methods. Further, we will improve this method by applying evolutionary algorithms to determine the most effective criteria. we will apply many methods such as VIKOR, ENTROPY, and DEMATEL method in the future to this problem.

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Neutrosophic Statistical Analysis of Temperature of Different Cities of Pakistan

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Abstract: In this paper, neutrosophic statistical analysis of temperature data of five different cities of Pakistan is given. The neutrosophic mean and neutrosophic coefficient of variation are computed using the temperature data. From the analysis, it is concluded that on average the temperature of Lahore city is higher than the temperature of other cities. Also, the temperature of Karachi city is more consistent compared to other cities. In addition, the neutrosophic results are compared with results under classical statistics. The neutrosophic statistical analysis is found to be more informative than classical statistics results.

Keywords: temperature; indeterminacy; uncertainty; statistical; analysis

1. Introduction

The weather temperature has a serious effect on the human body. An increase in weather temperature can be harmful to humans and other living things. It also affects the reduction of productivity of agriculture (Janjua et al., 2020). Recently, Pakistan has been experiencing an unexpected extreme change in climate which cause a lot of damage to health and livelihoods in the country (Eckstein, 2018). In recent years many heat strokes have been recorded which has caused many problems in the environment. The animals are dying because of water due to environmental change. The statistical methods have been widely applied for the prediction and estimation of temperature. Several researchers also studied different aspects of temperature. Iqbal et al. (2016) presented a study on the recent changes in maximum and minimum temperatures in Pakistan. This analysis deals with the trends in both variables at a monthly, seasonal, and annual resolution. Dawood (2017) presented a Spatio-statistical analysis of temperature fluctuations, this analysis deals with the variation in temperature such that positive tends with mean maximum temperature and negative tends with mean minimum temperature and slope magnitude. Amin et al. (2018) dealt with the analysis of historical temperature (1996–2015) and projected (2030–2060) climate in Pakistan, presented the possible variations for both minimum and maximum temperature. Khan et al. (2019) presented the analysis of both minimum and maximum temperature trends and the significant increase in a heat wave. This analysis shows that the intense heat wave occurred in southwest Pakistan. Abid et al. (2019) presented the Farmers perception of climate change, observed trends, and adaptation of agriculture in Pakistan. This analysis deals with the perception of increasing mean temperature with locally recorded data. Tariq at al. (2020) presented the analysis of seasonal land surface temperature and land use land cover change using optical multi-temporal satellite data of Faisalabad, Pakistan. Saleem et al. (2021) presented the annual and seasonal trends of extreme temperature and pacific variability during 1980-2019. Rafiq et al. (2022) presented the analysis of the variability of mean monthly, seasonal and annual temperature of Baluchistan province, Pakistan.

These analyses are done by using classical statistics. More information on analysis can be seen in Iqbal and Quamar (2011).

Classical statistics deals with determinate and exact data, crisp arguments charts, diagrams, probability distributions, algorithms, functions, parametric and non-parametric whereas; neutrosophic statistics is an advanced form of classical statistics that deal with indeterminacy, uncertainty, unclear and incomplete form of data, and also a generalization of interval statistics, see Smarandache (2014). According to Smarandache (2022) "Neutrosophic Statistics is an extension of the Interval Statistics, since it may deal with all types of indeterminacies (with respect to the data, inferential procedures, probability distributions, graphical representations, etc.), it allows the reduction of indeterminacy, and it uses the neutrosophic probability that is more general than imprecise and classical probabilities, and has more detailed corresponding probability density functions. While Interval Statistics only deals with indeterminacy that can be represented by intervals. Not all indeterminacies (uncertainties) may be represented by intervals. Also, in some applications, we should better use hesitant sets (that have less indeterminacy) instead of intervals. Neutrosophic statistics is a generalization of interval statistics, because of, among others, while interval statistics is based on interval analysis, neutrosophic statistics is based on set Analysis (meaning all kinds of sets, not only intervals)"To deal with neutrosophic data or data in intervals the various applications can be viewed in Broumi and Smarandache (2014) presented the neutrosophic set of new cosine similarities between two intervals. Aslam and Khan (2021) worked on the normality test of temperature in Jeddah city using Cochran's test under indeterminacy. Afzal et al. (2021) presented the analysis of resistance depending on the temperature variance of conducting material under the neutrosophic statistical analysis. Further, Janjua et al. (2022) worked on the climate variability and wheat crop under a neutrosophic environment. Afzal et al. (2022) presented the work on the fabrication of temperature flexibility on robot skin.

In this paper, we will apply neutrosophic statistics to the temperature data collected from different cities in Pakistan. We will present the neutrosophic statistical analysis of the temperature data of Gujranwala, Lahore, Karachi, Islamabad and Sialkot enumerated by Pakistan Meteorological Department. We will compare the result of classical statistics with the result of neutrosophic statistics using the temperature data of different cities in Pakistan.

2. Methodology

Suppose that $X_N = X_L + X_U I_N; I_N \epsilon [I_L, I_U]$ be a neutrosophic random variable which represents the temperature of different cities of Pakistan, where X_L is the lower temperature and $X_U I_N$ is the upper temperature and $I_N \epsilon [I_L, I_U]$ be the interval of indeterminacy. By following, Chen et al. (2017), Chen et al. (2017) and Aslam (2019), the neutrosophic average of temperature data $X_N \epsilon [X_L, X_U]$ can be calculated as $\bar{X}_N = \bar{X}_L + \bar{X}_U I_N; I_N \epsilon [I_L, I_U]$, where $\bar{X}_L = \frac{1}{n_N}\sum_{i=1}^{n_N} X_{iL}$, $\bar{X}_U = \frac{1}{n_N}\sum_{i=1}^{n_N} X_{iU}$ and $n_N \epsilon [n_L, n_U]$ be a neutrosophic sample. The neutrosophic standard deviation can be computed as follows

$$\sum_{i=1}^{n_N}(X_i - \bar{X}_{iN})^2 = \sum_{i=1}^{n_N}\begin{bmatrix} \min\begin{pmatrix} (a_i + b_i I_L)(\bar{a} + \bar{b} I_L), (a_i + b_i I_L)(\bar{a} + \bar{b} I_U) \\ (a_i + b_i I_U)(\bar{a} + \bar{b} I_L), (a_i + b_i I_L)(\bar{a} + \bar{b} I_L) \end{pmatrix} \\ \max\begin{pmatrix} (a_i + b_i I_L)(\bar{a} + \bar{b} I_L), (a_i + b_i I_L)(\bar{a} + \bar{b} I_U) \\ (a_i + b_i I_U)(\bar{a} + \bar{b} I_L), (a_i + b_i I_U)(\bar{a} + \bar{b} I_U) \end{pmatrix} \end{bmatrix}, I \epsilon [l_L, l_U] \quad (1)$$

Note that $a_i = X_L$ and $b_i = X_U$. We will use the symbols a_i and b_i to present the lower and upper values, respectively throughout the paper. The neutrosophic sample variance can be computed by;

$$S_N^2 = \frac{\sum_{i=1}^{n_N}(X_i - \bar{X}_{iN})^2}{n_N}; \; S_N^2 \epsilon [S^2{}_L, S^2{}_U] \quad (2)$$

The neutrosophic form of $S_N^2 \epsilon [S^2{}_L, S^2{}_U]$ can be written as

$$a_S + b_S I_{NS}; I_{NS} \epsilon [I_{LS}, I_{US}] \quad (3)$$

The neutrosophic coefficient of variation (CV_N) can be applied to see the consistency of the temperature in the different cities of Pakistan. A city having a smaller value of CV_N means more consistent than the other city in temperature. The CV_N can be computed by;

$$CV_N = \frac{\sqrt{S_N^2}}{\bar{X}_N} \times 100; CV_N \epsilon [CV_L, CV_U] \quad (4)$$

The neutrosophic form of CV_N is

$$a_V + b_v I_{Nv}; I_{Nv} \epsilon [I_{Lv}, I_{Uv}] \quad (5)$$

3. Data collection

We used temperature data of different big cities of Pakistan like Gujranwala, Lahore, Karachi, Islamabad and Sialkot. Our aim is to investigate which city on average has the higher temperature and which city temperature is more consistent. We used daily data of temperature for the month of July 2022 from https://www.gismeteo.com/. The data is reported in Table 1. Table 1 presents low and high values of the temperature data. The temperature data given in the interval cannot be analysed using classical statistics. The interval data can be analyzed using neutrosophic statistics. The neutrosophic statistical analysis for the temperature data is shown in Section 4.

Table 1: The temperature data (in C°) of different cities in Pakistan

Day	Date	Gujranwala		Lahore		Karachi		Islamabad		Sialkot	
		Low	High	Low	High	Low	High	Low	High	Low	High
Monday	4	29	40	29	40	28	36	28	35	28	37
Tuesday	5	34	43	36	43	29	34	31	39	31	41
Wednesday	6	29	39	31	38	28	31	26	36	30	35
Thursday	7	28	36	29	39	28	33	27	35	28	36
Friday	8	31	41	31	41	29	32	29	37	31	41
Saturday	9	30	35	29	34	29	33	28	34	29	35
Sunday	10	28	37	29	39	30	35	26	33	29	37
Monday	11	28	37	29	37	29	33	27	35	28	37
Tuesday	12	30	36	31	37	29	31	28	34	27	34
Wednesday	13	27	34	28	36	29	32	28	34	27	34
Thursday	14	28	37	27	38	28	31	27	33	26	36
Friday	15	27	36	26	35	29	32	27	35	27	36
Saturday	16	29	38	29	37	29	33	27	35	29	37
Sunday	17	29	38	30	40	29	36	27	32	28	38
Monday	18	31	40	32	40	28	32	26	37	31	40
Tuesday	19	33	41	33	41	28	31	30	37	31	40
Wednesday	20	33	40	34	41	28	30	29	37	29	36
Thursday	21	30	38	32	40	28	31	25	35	26	33
Friday	22	29	33	31	35	28	32	27	34	26	29
Saturday	23	28	34	30	36	28	32	26	34	25	30
Sunday	24	28	36	30	38	28	32	26	35	25	32
Monday	25	29	39	31	42	29	32	27	37	26	34
Tuesday	26	32	41	35	43	29	31	28	39	28	36
Wednesday	27	32	40	35	43	29	31	29	39	28	35
Thursday	28	32	39	35	42	29	30	29	38	28	35
Friday	29	29	36	31	39	28	30	28	34	26	31
Saturday	30	29	37	32	40	28	30	26	35	25	32
Sunday	31	29	37	32	40	28	30	26	34	25	32

4. Result and interpretation

We performed the neutrosophic statistical analysis using the temperature data. The neutrosophic mean of temperature is shown in Table 2. The neutrosophic standard deviation is shown in Table 3. The neutrosophic coefficient variation is shown in Table 4. From Table 2, the neutrosophic form of the average temperature of Gujranwala city is

$29.6+37.79I_N;I_N\epsilon[0,0.20]$. It means that the average temperature of Gujranwala city is between 29.68 to 37.79 and the measure of indeterminacy is 0.20, a neutrosophic form of average temperature of Lahore city is $30.97+39.08I_N;I_N\epsilon[0,0.20]$. It means that the average temperature of Lahore city is between 30.97 to 39.08 and the measure of indeterminacy is 0.20, a neutrosophic form of the average temperature of Karachi city is $28.52+32.00I_N;I_N\epsilon[0,0.10]$.

It means that the average temperature of Karachi city is between 28.52 to 32.00 and the measure of indeterminacy is 0.10, a neutrosophic form of the average temperature of Islamabad city is $27.41+35.41I_N;I_N\epsilon[0,0.26]$. It means that the average temperature of Islamabad city is between 27.41 to 35.41 and the measure of indeterminacy is 0.26 and, neutrosophic form of average temperature of Sialkot city is $27.75+35.31I_N;I_N\epsilon[0,0.20]$. It means that the average temperature of between 27.75 to 35.31 and the measure of indeterminacy is 0.20. On average, the temperature of Lahore city is higher than other cities as it has the maximum monthly average temperature as compared to Gujranwala city.

Table 2. The neutrosophic mean of the temperature of different cities of Pakistan.

Cities	$\bar{X}_N$	$a_{\bar{X}}+b_{\bar{X}}I_{N\bar{X}};I_{N\bar{X}}\epsilon[I_{L\bar{X}},I_{U\bar{X}}]$
Gujranwala	[29.68,37.79]	$29.6+37.79I_N;I_N\epsilon[0,0.20]$
Lahore	[30.97,39.08]	$30.97+39.08I_N;I_N\epsilon[0,0.20]$
Karachi	[28.52,32.00]	$28.52+32.00I_N;I_N\epsilon[0,0.10]$
Islamabad	[27.41,35.41]	$27.41+35.41I_N;I_N\epsilon[0,0.26]$
Sialkot	[27.75,35.31]	$27.75+35.31I_N;I_N\epsilon[0,0.20]$

Table 3. The neutrosophic standard deviation of the temperature of different cities of Pakistan.

Cities	S_N	$a_S+b_SI_{NS};I_{NS}\epsilon[I_{LS},I_{US}]$
Gujranwala	[1.887,2.439]	$1.887+2.439I_N;I_N\epsilon[0,0.23]$
Lahore	[2.487,2.538]	$2.487+2.538I_N;I_N\epsilon[0,0.02]$
Karachi	[0.575,1.678]	$0.575+1.678I_N;I_N\epsilon[0,0.66]$

Islamabad	[1.398,1.893]	$1.398+1.893I_N; I_N\varepsilon[0,0.26]$
Sialkot	[1.935,3.104]	$27.75+35.31I_N; I_N\varepsilon[0,0.38]$

Table 4. Neutrosophic coefficient of variation of temperature of different cities in Pakistan.

Cities	CV_N	$a_V+b_vI_{Nv}; I_{Nv}\epsilon[I_{Lv}, I_{Uv}]$
Gujranwala	[6.36,6.45]	6.36+6.45I_{Nv}; $I_{Nv}\epsilon[0,0.0149]$
Lahore	[8.03,6.49]	8.03-6.49I_{Nv}; $I_{Nv}\epsilon[0,0.2365]$
Karachi	[2.02,5.24]	2.02+5.24I_{Nv}; $I_{Nv}\epsilon[0,0.6155]$
Islamabad	[5.10,5.35]	5.10+5.35I_{Nv}; $I_{Nv}\epsilon[0,0.0459]$
Sialkot	[6.97,8.79]	6.97+8.79I_{Nv}; $I_{Nv}\epsilon[0,0.2068]$

From Table 4, the coefficient of variation of temperature of Gujranwala city is between 6.36 to 6.45, coefficient of variation of temperature of Lahore city is between 8.03 to 6.49, coefficient of variation of temperature of Karachi city is between 2.02 to 5.24, coefficient of variation of temperature of Islamabad city is between 5.10 to 5.35, coefficient of variation of temperature of Sialkot city is between 6.97 to 8.79. The measures of indeterminacy associated with the coefficient of variation are also shown in Table 4. Based on the analysis, it can be concluded that the values of the coefficient of variation of temperature in Karachi is minimum. Therefore, the temperature of Karachi city is more consistent than the other cities in Pakistan.

5. Comparative study

The neutrosophic statistical analysis is the generalization of the classical statistical analysis. The neutrosophic statistical analysis reduces to classical statistical analysis when no indeterminacy is found in the data or data is not recorded in the intervals. Note here that temperature data is always recorded in intervals and therefore adequately analysed by the neutrosophic statistics. We now compare the results obtained using neutrosophic statistics with the results of classical statistics. The neutrosophic form of the temperature of Karachi city is CV_N=2.02+5.24I_{Nv}; $I_{Nv}\epsilon[0,0.6155]$. The first value (determinate) 2.02 of this neutrosophic shows the analysis from the classical statistics while the second part 5.24I_{Nv} of the neutrosophic form shows the indeterminate part. From the analysis, it can be seen that the values CV_N ranges from 2.02% to 5.24% with the measure of indeterminacy or uncertainty at 0.6155. Note that when $I_{Nv}=0$, the neutrosophic statistical results reduce to the results under classical statistics. Based on the comparative study, it can be concluded that neutrosophic statistical results are more adequate, flexible and more informative than the classical statistics

5. Concluding remarks

In this paper, we applied neutrosophic statistical analysis to temperature data of different cities of Pakistan. We observed that neutrosophic analysis of temperature data provided the estimated results of temperature in intervals rather the result of temperature in exact values. Therefore, the neutrosophic result is more flexible than classical statistics result. The government of Pakistan should take serious steps to reduce global warming by planting more trees, especially in Lahore city. The neutrosophic statistical analysis can be applied to analyse the interval data more adequately than classical statistics.

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Integrity and Domination Integrity in Neutrosophic Soft Graphs

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Abstract: Vagueness and uncertainty are two distinct models are represented by Fuzzy sets and Soft sets. The combination of Soft sets and simple graphs produces soft graphs which is also an interesting concept to deal with uncertainty problems. Any communication network can be modeled as a graph whose nodes are the processors (stations) and a communication link as an edge between corresponding nodes. The stability of a communication network is a very important factor for the network designers to reconstruct the it after the failure of certain stations or communication links. Two essential quantities in an analysis of the vulnerability of a communication network are (1) the number of nodes that are not functioning and (2) the size of a maximum order of a remaining sub network within which mutual communications can still occur. C. A. Barefoot, et. al. [13] introduced the concept of integrity. The extension of such a vulnerability parameter is studied in fuzzy graphs. Since neutrosophic soft graphs are the most generalized network structure where we can define and study the importance of the vulnerability parameters is made in this manuscript. Also, we introduce the domination integrity of neutrosophic soft graphs and explain with suitable examples. Few bounds are obtained.

Keywords: Soft graph, Neutrosophic soft graphs, Integrity, Domination integrity.

1. Introduction

The problems deal with vagueness and uncertainty can be modelled by using two different soft tools namely f**uzzy set** defined by Zadeh [48] in 1965 and **soft set** defined by Molodtsov [31] in 1999. The **intuitionistic fuzzy set** is the generalization of fuzzy set was introduced by Atanassov [2-4]. It depends on a membership function and a non membership function. Any real time problems which consist of involving imprecise, indeterminacy and inconsistent data can be represented as the **neutrosophic se**t, introduced by Smarandache [38]. This is the generalization of classical sets and fuzzy sets. The degree of acceptance deals in fuzzy sets, membership (truth) function and a non-membership (falsity) function deals in intuitionistic fuzzy set, neutrosophic set deals truth-membership, indeterminacy-membership, and falsity-membership. The **rough soft sets, soft rough sets, and soft-rough fuzzy sets** are obtained from soft sets with rough sets and fuzzy sets. Feng et al. [18 -20] and Ali [7] introduced these soft tools in the consecutive years 2010 and 2011. In 2014, Rajesh Thumbakara et. al.[33] introduced **soft graphs.** They defined soft graph homomorphism, soft tree and soft complete graph and discussed their properties also. Ali et al. [7] discussed the fuzzy sets and fuzzy soft sets induced by soft sets.

In 1736, **graph theory** was defined by Euler. **Fuzzy graph** was introduced by Azriel Rosenfied in 1975[29 & 35]. Muhammad Akram et.al. [6] defined **fuzzy soft graphs i**n 2015**.** Also, they have investigated the properties **of strong, complete** and **regular fuzzy soft graphs.** Guven et. al. [25] introduced an idea about neutrosophic soft graphs and its application. Shannon and Atanassov [37] defined the **intuitionistic fuzzy graph** (IFG). A.M.Shyla [46] introduced the concept of **Intuitionistic Fuzzy Soft graph** in 2016. Ghorai. G. et. al.[21] modelled **the neutrosophic graphs** in 2017. Akram [6] established the certain notions including **neutrosophic soft graphs**, strong neutrosophic soft graphs, and complete neutrosophic soft graphs.

Graphs are the most important and essential tool in the modern communication world which has communication nodes and links. The stability of such communication networks can be measured by vulnerability parameters like connectivity, toughness [11], tenacity [16], rupture degree, scattering number, integrity [13-15], domination integrity [39-42], etc. Two essential quantities in an analysis of the vulnerability of a communication network are (1) the number of nodes that are not functioning and (2) the size of a maximum order of a remaining sub network within which mutual communications can still occur. C. A. Barefoot, et. al. [13-14] introduced the concept of integrity. It is a useful measure of vulnerability and it is defined as follows. $I(G) = min\{|S| + m(G - S) : S \subset V(G)\}$, where $m(G - S)$ denotes the order of the largest component in G – S.

Integrity measures not only the difficulty to break down the network but also the damage caused. A small group of people have effective communication links with other members of the organization and they take important decisions in an administrative set up. Domination in graphs provides a model for such a concept. A minimum dominating set of nodes provides a link with the rest of the nodes in a network, If the removal of such a set, results huge impact in the network. That is, the decision-making process is paralyzed but also the communication between the remaining members is minimized. The damage will be more when the dominating sets of nodes are under attack.

This motivated to study the concept of domination integrity when the sets of nodes disturbed are dominating sets. Sundareswaran et. al. introduced the concept of Domination Integrity of a graph and studied in [39] as another measure of vulnerability of a graph which is defined as follows $DI(G) = min\{|S| + m(G - S)\}$, where S is a dominating set of G and $m(G - S)$ denotes the order of the largest component in $G - S$ and is denoted by $DI(G)$. M. Saravanan et. al. extended the idea of vulnerability parameters in fuzzy graphs [42 - 44]. They explained a real time application for the domination integrity [45]. There are different versions of domination integrity were introduced in the literature such as Domination Weak Integrity in graphs [47], Geodomination integrity [12], Connected domination integrity in graphs [27] and Total Edge Domination Integrity in graphs [8].

This motivated us to introduce the concept of integrity and domination integrity in neutrosophic fuzzy soft graphs. Also, we prove certain properties of these new parameter concepts are described with suitable examples.

In the second section, we provide all the basic definitions and results related to our article. The definitions of the Integrity and Domination integrity in Fuzzy graphs were stated in the third section and in the fourth section, we introduce the concept of Integrity and Domination integrity in Neutrosophic graphs. At the end of the article, we give the conclusion of our work and discuss the future work.

2. Preliminaries

In this section, we provide all the basic definitions and results in the literature.

Definition 2.1 [21]

A **neutrosophic graph** is of the form $G^* = (V, \sigma, \mu)$ where $\sigma = (T_1, I_1, F_1)$ & $\mu = (T_2, I_2, F_2)$

(i) $V = \{v_1, v_2, v_3, \dots v_n\}$ such that $T_1 : V \longrightarrow [0,1]$, $I_1 : V \longrightarrow [0,1]$ and $F_1 : V \rightarrow [0,1]$ denote the degree of truth-membership function , indeterminacy –membership function and falsity-membership function of the vertex $v_1 \in V$ respectively and $0 \leq T_i(v) + I_i(v) + F_1(v) \leq 3, \forall\, v_i \in V$ ($i = 1,2\,,3, \dots n$).

(ii) $T_3 : V \times V \longrightarrow [0,1]$, $I_2 : V \times V \longrightarrow [0,1]$ and $F_2 : V \times V \rightarrow [0,1]$ where $T_2(v_i, v_j), I_2(v_i, v_j)$ and $F_2(v_i, v_j)$ denote the degree of truth-membership function , indeterminacy –membership function and falsity-membership function of the edge (v_i, v_j) respectively such that for every edge (v_i, v_j),

$$T_2(v_i, v_j) \leq \min\{T_1(v_i), T_1(v_j)\},$$
$$I_2(v_i, v_j) \leq \min\{I_1(v_i), I_1(v_j)\},$$
$$F_2(v_i, v_j) \leq \max\{F_1(v_i), F_1(v_j)\},$$

and $T_2(v_i, v_j) + I_2(v_i, v_j) + F_2(v_i, v_j) \leq 3$

Definition 2.2 [33]

Let $G = (V, E)$ be a simple graph, A any nonempty set. Let R an arbitrary relation between elements of A and elements of V. That is $\subseteq A \times V$. A set valued function $F : A \dashrightarrow P(V)$ can be defined as $F(x) = y \in V \mid xRy\}$. The pair (F, A) is a soft set over V. Let (F, A) be a soft set over V. Then (F, A) is said to be a **soft graph** of G if the subgraph induced by F(x) in $G, F(x)$ is a connected subgraph of G for all $x \in A$. The set of all soft graph of G is denoted by $SG(G)$.

Definition 2.3 [6]

A **neutrosophic soft graph** $G = (G^*, F, K, A)$ is an ordered four tuple if it satisfies the following conditions:

i. $G^* = (V, E)$ is a simple graph,
ii. A is a nonempty set of parameters,
iii. (F, A) is a neutrosophic soft set over V,
iv. (K, A) is a neutrosophic soft set over E,
v. $(F(e), K(e))$ is a neutrosophic graph of $G^*$for all $e \in A$. That is

$$T_{K(e)}(xy) \leq \min\{T_{F(e)}(x), T_{F(e)}(y)\} ;$$
$$I_{K(e)}(xy) \leq \min\{I_{F(e)}(x), I_{F(e)}(y)\} ;$$
$$F_{K(e)}(xy) \leq \max\{F_{F(e)}(x), F_{F(e)}(y)\} ;$$

such that $0 \leq T_{K(e)}(xy) + I_{K(e)}(xy) + F_{K(e)}(xy) \leq 3, \forall e \in A, x, y \in V$.

S. Satham Hussain et. al. defined in [36] degree and total degree of a vertex v in a neutrosophic soft graph G, order and size of a neutrosophic soft graph G. Also, they introduced vertex, edge and cardinality of a neutrosophic graph G.

Definition 2.4 [36]

Let $G=(G^{*},J,K,A)$ be a neutrosophic soft graph. Then the **degree of a vertex** $u\in G$ is a sum of degree truth membership, sum of indeterminacy membership and sum of falsity membership of all those edges which are incident on vertex u denoted by $d(u)=(d_{TJ(e)}(u),d_{IJ(e)}(u),d_{FJ(e)}(u))$ where

$d_{TJ(e)}(u)=\Sigma_{e\in A}(\Sigma_{u\neq v\in V}T_{K(e)}(u,v))$ called the degree of truth membership vertex

$d_{IJ(e)}(u)=\Sigma_{e\in A}(\Sigma_{u\neq v\in V}I_{K(e)}(u,v))$ called the degree of indeterminacy membership vertex

$d_{FJ(e)}(u)=\Sigma_{e\in A}(\Sigma_{u\neq v\in V}F_{K(e)}(u,v))$ called the degree of falsity membership vertex for all $e\in A,u,v\in V$.

Definition 2.5 [36]

Let $G=(G^{*},J,K,A)$ be a neutrosophic soft graph. Then the **total degree of a vertex** $u\in G$ is a sum of degree truth membership, sum of indeterminacy membership and sum of falsity membership of all those edges which are incident on vertex u denoted by $td(u)=(td_{TJ(e)}(u),td_{IJ(e)}(u),td_{FJ(e)}(u))$ where

$td_{TJ(e)}(u)=\Sigma_{e\in A}(\Sigma_{u\neq v\in V}T_{K(e)}(u,v)+T_{J(e)}(u,v))$ called the degree of truth membership vertex

$td_{IJ(e)}(u)=\Sigma_{e\in A}(\Sigma_{u\neq v\in V}I_{K(e)}(u,v)+\ I_{J(e)}(u,v)))$ called the degree of indeterminacy membership vertex

$td_{FJ(e)}(u)=\Sigma_{e\in A}(\Sigma_{u\neq v\in V}F_{K(e)}(u,v)+\ F_{J(e)}(u,v))$ called the degree of falsity membership vertex for all $e\in A,u,v\in V$.

Definition 2.6 [36]

The **order** of a neutrosophic soft graph G is

$$Ord(G)=\Sigma_{e_i\in A}(\Sigma_{x\in V}T_{J(e_i)}(e_i)(x),\Sigma_{x\in V}I_{J(e_i)}(ei)(x),\Sigma_{x\in V}I_{J(e_i)}(ei)(x).$$

Definition 2.7 [36]

The **size** of a neutrosophic soft graph G is

$$S(G)=\Sigma_{e_i\in A}(\Sigma_{xy\in V}T_{K_{e_i}}(e_i)(xy),xy\in V,\Sigma_{xy\in V}I_{K_{e_i}}(e_i)(xy),\Sigma_{xy\in V}F_{e_i}(e_i)(xy)$$

Definition 2.8 [36]

Let $G=(G^{*},J,K,A)$ be an neutrosophic soft graph. Then **cardinality** of G is defined to be

$$|G|=\Sigma_{e\in A}\,|\Sigma_{v_i\in V}\frac{1+T_{J(e)}(x)+I_{J(e)}(x)-F_{J(e)}(x)}{2}|+|\,\Sigma_{v_i,v_j\in V}\ \frac{1+T_{J(e)}(xy)+I_{J(e)}(xy)-F_{J(e)}(xy)}{2}|$$

Definition 2.9 [36]

Let $G=(G,J,K,A)$ be an neutrosophic soft graph, then **vertex cardinality of *G*** is defined to be

$$|V|=\Sigma_{e\in A}|\Sigma_{v_i\in V}\frac{1+T_{J(e)}(x)+I_{J(e)}(x)-F_{J(e)}(x)}{2}\ |$$

Definition 2.10 [36]

Let $G=(G*,J,K,A)$ be an neutrosophic soft graph, then **edge cardinality of G** is defined to be

$$|E|=\Sigma_{e\in A}|\Sigma_{xy\in E}\frac{1+T_{K(e)}(xy)+I_{K(e)}(xy)-F_{K(e)}(xy)}{2}|$$

Definition 2.11 [36]

An arc (u,v) is said to be **strong arc**, if $T_{K(e)}(u,v)\geq T^{\infty}_{K(e)}(u,v)$ and $I_{K(e)}(u,v)\geq I^{\infty}_{K(e)}(u,v)$ and $F_{K(e)}(u,v)\geq F^{\infty}_{K(e)}(u,v)$.

Clearly, if u,v are connected by means of path of length k then $T^{k}_{K(e)}(v_i,v_j)$ is defined as

$$sup\{T_{K(e)}(u,v_1)\wedge T_{K(e)}(v_1,v_2)\wedge T_{K(e)}(v_1,v_3)\wedge\ldots\wedge T_{K(e)}(v_{k-1},v_k)/u,v,v_1,\ldots v_{k-1},v\in V\},$$

$I^k_{K(e)}(v_i, v_j)$ is defined as

$$inf\{I_{K(e)}(u,v_1) \vee I_{K(e)}(v_1,v_2) \vee I_{K(e)}(v_1,v_3) \vee \ldots \vee I_{K(e)}(v_{k-1},v_k)/u,v,v_1,\ldots v_{k-1},v \in V\}$$ and

$F^k_{K(e)}(v_i, v_j)$ is defined as

$$inf\{F_{K(e)}(u,v_1) \vee F_{K(e)}(v_1,v_2) \vee F_{K(e)}(v_1,v_3) \vee \ldots \vee F_{K(e)}(v_{k-1},v_k)/u,v,v_1,\ldots v_{k-1},v \in V\}, e \in A.$$

Definition 2.12 [36]

Let $G = (G*, J, K, A)$be a neutrosophic soft graph on V. Let $u, v \in V$, we say that u dominates v in G if there exists a strong arc between them.

Definition 2.13 [36]

Given $S \subset V$ is called a **dominating set** in G if for every vertex $v \in V - S$ there exists a vertex $u \in S$ such that u dominates v. for all $e \in A, u, v \in V$.

Definition 2.14 [36]

A **dominating set** S of a neutrosophic soft graph $G = (G*, J, K, A)$ is said to be **minimal dominating set** if no proper subset of S is a dominating set, for all $e \in A, u, v \in V$.

Definition 2.15 [R.Dhavaseelan et. al.17]

A neutrosophic graph $\mathrm{G} = (\mathrm{G}*, \mathrm{J}, \mathrm{K}, \mathrm{A})$ is called **Strong Neutrosophic graph** if

$$T_{K(e)}(xy) = \min\{T_{F(e)}(x), T_{F(e)}(y)\} ;$$
$$I_{K(e)}(xy) = \min\{I_{F(e)}(x), I_{F(e)}(y)\} ;$$
$$F_{K(e)}(xy) = \max\{F_{F(e)}(x), F_{F(e)}(y)\} \ \forall e \in A, x, y \in V$$

Definition 2.16 [36]

A neutrosophic soft graph G is a **strong neutrosophic soft graph** if H(e) is a strong neutrosophic graph for all e $\in$ A.

Definition 2.17 [36]

Let $G = (G*, J, K, A)$ be a **strong neutrosophic soft graph** and $v \in V$. Then the strong degree and the **strong neighborhood degree** of v are defined, respectively

$$d_S(v) = \Sigma_{e \in A}(\Sigma_{u \in N_s(v)} T_{K(e)}(uv), \Sigma_{u \in N_s(v)} I_{K(e)}(uv), \Sigma_{u \in N_s(v)} F_{K(e)}(uv))$$
$$d_s N(v) = \Sigma_{e \in A}(\Sigma_{u \in N_s(v)} T_{J(e)}(uv), \Sigma_{u \in N_s(v)} I_{J(e)}(uv), \Sigma_{u \in N_s(v)} F_{J(e)}(uv))$$

The strong degree cardinality of v are defined by

$$|d_s(v)| = \sum_{e \in A}\Big(\sum_{u \in N_s(v)} \frac{1 + T_{K(e)}(u,v) + I_{K(e)}(u,v) - F_{K(e)}(u,v)}{2}\Big)$$

The minimum and maximum strong degree of G are defined, respectively as

$\delta_s(G) = \wedge |d_s(v)|, \forall v \in V$ and $\Delta_s(v) = \vee |d_s(v)|, \forall v \in V, e \in A$

Definition 2.18 [36]

The **strong degree** cardinality and the strong neighborhood degree cardinality of v are defined by

$$|d_s(v)| = \Sigma_{e \in A}(\Sigma_{u \in N_s(v)} \frac{1+T_{K(e)}(u,v)+I_{K(e)}(u,v)-F_{K(e)}(u,v)}{2})$$

$$|d_S N(v)| = \Sigma_{e\in A}(\Sigma_{u\in N_s(v)} \frac{1+T_{J(e)}(u,v)+I_{J(e)}(u,v)-F_{J(e)}(u,v)}{2})$$

Definition 2.19 [36]

Two vertices in a neutrosophic soft graph $G = (G *, J, K, A)$are said to be an **independent** if there is no strong arc between them.

Definition 2.20 [36]

Given $S \subset V$ is said to be **independent set** of G if $T_{K(e)}(u,v) < T^{\infty}_{K(e)}(u,v)$ and $I_{K(e)}(u,v) < I^{\infty}_{K(e)}(u,v)$ and $F_{K(e)}(u,v) < F^{\infty}_{K(e)}(u,v)$ $\forall e \in A, u, v \in S$.

Definition 2.21 [36]

An independent set S of G in a neutrosophic soft graph is said to be **maximal independent**, if for every vertex $v \in V - S$, the set $S \cup \{v\}$ is not independent.

Definition 2.22 [36]

The minimum cardinality among all maximal independent set is called lower independence number of G, and it is denoted by $\Sigma_{e\in A}(iNS(G))$. The maximum cardinality among all maximal independent set is called lower independence number of G, and it is denoted by $\Sigma_{e\in A}(INS(G))$.

Muhammad Akram and Sundas Shahzadi gave the following definitions [6]

Definition 2.23 [6]

A neutrosophic soft graph $G' = = (G *, J', K', A')$ is called a **neutrosophic soft subgraph** of G = $(G *, J, K, A)$ if i. $A' \subseteq A$

$ii. K'_e \subseteq K_e$, that is $T_{K'_e}(x) \leq T_{K_e}(x), , I_{K'_e}(x) \leq I_{K_e}(x), F_{K'_e}(x) \geq F_{K_e}(x)$

$iii. J'_e \subseteq J_e$, that is $T_{J'_e}(x) \leq T_{J_e}(x), , I_{J'_e}(x) \leq I_{J_e}(x), F_{J'_e}(x) \geq F_{J_e}(x)$ for all $e \in A$.

Definition 2.24 [6]

The neutrosophic soft graph $G_1 = (G *, J_1, K_1, B)$ is called **spanning neutrosophic soft subgraph** of $G = (G *, J, K, A)$ if

(i) $B \subseteq A$,

(ii) $T_{F_1(e)}(v) = T_{J(e)}(v), I_{J_1(e)}(v) = I_{J(e)}(v), F_{J_1(e)}(v) = F_{J(e)}(v)$ for all $e \in A, v \in V$

Definition 2.25 [6]

The **complement of a neutrosophic soft graph** $G = (J, K, A)$ denoted by $G^c = (J^c, K^c, A^c)$ is defined as follows:

(i) $A^c = A$,

$(ii) J^c(e) = J(e)$,

$(iii) T_{K^c}(e)(u,v) = T_{J(e)}(u) \wedge T_{J(e)}(v) - T_{K(e)}(u,v)$,

$(iv) I_{K^c}(e)(u,v) = I_{J(e)}(u) \wedge I_{J(e)}(v) - I_{K(e)}(u,v)$,

$(v) F_{K^c}(e)(u,v) = F_{J(e)}(u) \vee F_{J(e)}(v) - F_{K(e)}(u,v)$, for all $u, v \in V, e \in A$.

Definition 2.26 [6]
A neutrosophic soft graph G is **self-complementary** if $G \approx G^c$.

Definition 2.27 [6]
A neutrosophic soft graph G is a complete neutrosophic soft graph if $H(e)$ is a **complete neutrosophic graph** of G for all $e \in A$,

$$T_{K(e)}(uv) = min\{T_{F(e)}(u), T_{F(e)}(v)\}$$
$$I_{K(e)}(uv) = \min\{I_{F(e)}(u), I_{F(e)}(v)\}\ and$$
$$F_{K(e)}(uv) = max\{F_{F(e)}(u), F_{F(e)}(v)\}$$
$$\forall u, v \in V, e \in A.$$

3. Integrity and Domination integrity in Fuzzy graphs

Saravanan et. al.[33 - 36] introduced the idea of the vulnerability parameter namely integrity and domination integrity in fuzzy graphs.

Definition 3. 1 [41]
Let $G = (\sigma, \mu)$ be a fuzzy graph. The **integrity** of G, denoted by $\check{I}(G)$, is defined as $\check{I}(G) = \min\{|S| + m(G-S)\}$ where $|S| = \sum_{u\in S}\sigma(u)$ denotes the cardinality of S, and $m(G-S) = \sum_{u\in V(G-S)}\sigma(v)$ is order of the biggest component of $G - S$ [41 - 43].

Definition 3.2 [35]
Let $G = (\sigma, \mu)$ be a fuzzy graph. The **domination integrity** of G, denoted by $\widetilde{DI}(G)$, is defined as $\widetilde{DI}(G), = \min\{|S| + m(G-S)\}$, S is the dominating set of G and $|S| = \sum_{u\in S}\sigma(u)$ denotes the cardinality of S, and $m(G-S) = \sum_{u\in V(G-S)}\sigma(v)$ is order of the biggest component of $G - S$ [33 - 36].

4. Integrity and Domination integrity in Neutrosophic soft graphs

In the crisp graph, membership values of vertex and edge are the same. In fuzzy, intuitionistic fuzzy graphs and neutrosophic graph, the membership values of vertices and edges have their own importance depending on the situation like uncertainty, indeterminacy, and falsity. This motivates to define these vulnerability parameters in neutrosophic fuzzy graphs. Also, it gives more accurate values in the real time problems especially in decision making process.

Definition 4.1
Let $G = (G^*, J, K, A)$ be a neutrosophic soft graph. The **integrity** of G, denoted by $\ddot{I}(G)$ is defined as $\ddot{I}(G) = \min\{|S| + m(G-S)\}$ where $|S| = \Sigma_{e\in A}\left|\Sigma_{v_i\in S}\frac{1+T_{J(e)}(x)+I_{J(e)}(x)-F_{J(e)}(x)}{2}\right|$ denotes the cardinality of S, and $m(G-S) = \Sigma_{e\in A}|\Sigma_{v_i\in V(G-S)}\frac{1+T_{J(e)}(x)+I_{J(e)}(x)-F_{J(e)}(x)}{2}|$ is order of the biggest component of $G - S$.

Definition 4.2
Let $G = (G^*, J, K, A)$ be a neutrosophic soft graph. The **domination integrity** of G, denoted by $\ddot{DI}(G)$, is defined as $\ddot{DI}(G) = \min\{|S| + m(G-S)\}$ and S is a dominating set of G, where $|S| =$

$\Sigma_{e\in A}|\Sigma_{v_i\in S}\frac{1+T_{J(e)}(x)+I_{J(e)}(x)-F_{J(e)}(x)}{2}|$ denotes the cardinality of S, and $m(G-S) = \Sigma_{e\in A}|\Sigma_{v_i\in V(G-S)}\frac{1+T_{J(e)}(x)+I_{J(e)}(x)-F_{J(e)}(x)}{2}|$ is order of the biggest component of $G - S$.

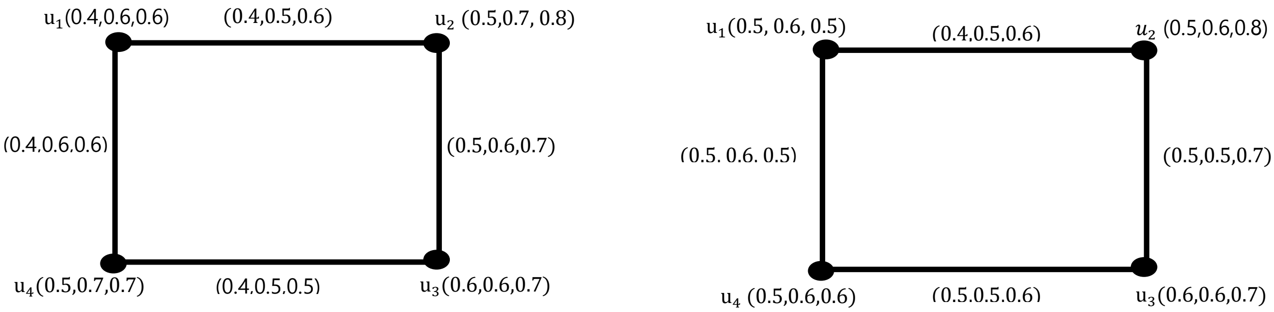
Definition 4.3

An Ï **-set** of $G = (G^*, J, K, A)$ is any (strict) subset S of V(G) for which Ï(G) = min{ |S| + m(G − S)}.

Definition 4.4

An D̈Ï(G)**-set** of $G = (G^*, J, K, A)$ is any (strict) subset S of V(G) for which D̈Ï(G) = min{ |S| + m(G − S)}.

Example : 4.5



$H(e_1)$ Figure 1 $H(e_2)$

S	$\|S\|$	$m(G-S)$	Ï(G)
$S_1 = \{u_1, u_3\}$	1.4	.7 for $\{u_2\}$ .75 for $\{u_4\}$	2.1 2.15
$S_2 = \{u_2, u_4\}$	1.45	.7 for $\{u_1\}$ .75 for $\{u_3\}$	2.1 2.15
$S_3 = \{u_1, u_2\}$	1.4	1.5 for $\{u_3, u_4\}$	2.9
$S_4 = \{u_1, u_4\}$	1.45	1.45 for $\{u_2, u_3\}$	2.9

Among all these subsets, S_1 is a Ï **-set** of G and Ï(G) = 2.1 corresponding to the parameter e_1

For e_2

S	$\|S\|$	$m(G-S)$	Ï(G)
$S_1 = \{u_1, u_3\}$	1.55	.65 for $\{u_2\}$ .75 for $\{u_4\}$	2.2 2.3
$S_2 = \{u_2, u_4\}$	1.4	.8 for $\{u_1\}$ .75 for $\{u_3\}$	2.2 2.15
$S_3 = \{u_1, u_2\}$	1.45	1.5 for $\{u_3, u_4\}$	2.95
$S_4 = \{u_1, u_4\}$	1.55	1.4 for $\{u_2, u_3\}$	2.95

Among all these subsets, S_1 and S_2 are the Ï **-sets** of G and Ï(G) = 2.2 corresponding to the parameter e_2

Example: 4.6

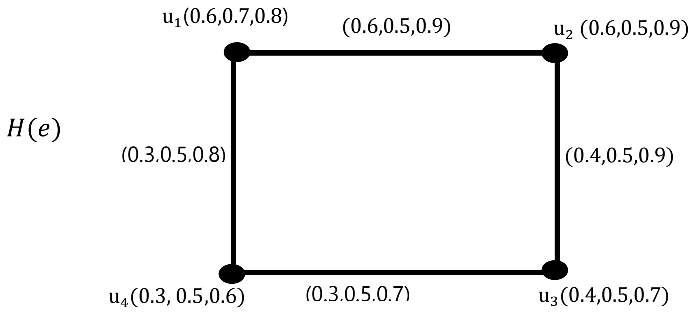


Figure 2

In Figure 2, corresponding to the parameter $H(e)$, $\{(u_1,u_2),(u_2,u_3),(u_3,u_4),(u_1,u_4)\}$ are the dominating sets.

S	$\|S\|$	$m(G-S)$	D̈Ï(G)
$S_1=\{u_1,u_2\}$	1.35	1.2 for $\{u_3,u_4\}$	2.55
$S_2=\{u_2,u_3\}$	1.2	1.35 for $\{u_1,u_4\}$	2.35
$S_3=\{u_3,u_4\}$	1.2	1.5 for $\{u_1,u_2\}$	2.9
$S_4=\{u_1,u_4\}$	1.35	1.45 for $\{u_2,u_3\}$	2.9
$S_5=\{u_1,u_3\}$	1.35	0.6 for $\{u_2\}$ 0.6 for $\{u_4\}$	1.95
$S_5=\{u_2,u_4\}$	1.2	0.6 for $\{u_3\}$ 0.75 for $\{u_1\}$	$Min\{1.8,1.95\}$ $=$ 1.8

Among all these subsets, S_5 is a D̈Ï **-set** of G and D̈Ï(G) $=1.8$ corresponding to the parameter e. In this neutrosophic graph G $\ddot{I}(G)=\ddot{D}\ddot{I}(G)$.

Example: 4.7

$H(e)$

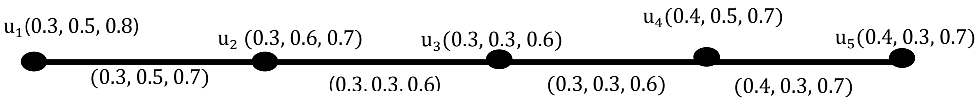


Figure 3

In Figure 2, corresponding to the parameter $H(e)$, $\{(u_2,u_4)\}$ are the dominating sets

S	$\|S\|$	$m(G-S)$	D̈Ï(G)
$S_1=\{u_2,u_4\}$	1.2	.5 for $\{u_1\}$ .5 for $\{u_3\}$ .7 for $\{u_5\}$	1.7

S	$\|S\|$	$m(G-S)$	Ï(G)

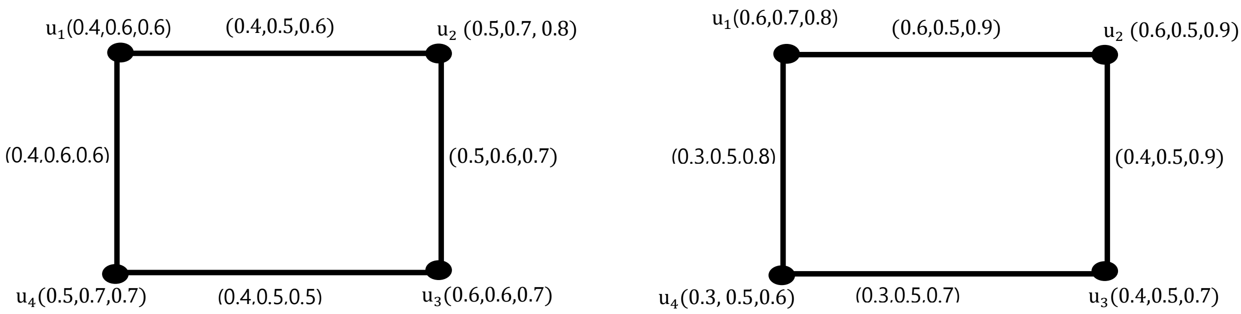
$S_1 = \{u_3\}$	.5	1.1 for $\{u_1, u_2\}$ 1.1 for $\{u_4, u_5\}$	1.6 1.6

In crisp graph $, I(G) \leq DI(G)$. But there is no relationship between these parameters in Fuzzy as well Neutrosophic soft graphs.

Definition 4.8

Let $G = (G^*, J, K, A)$ be a neutrosophic soft graph. A subset $S \subset V(G)$ is said to be a **vertex covering** of G if S contains at least one end of every strong arcs of G. A vertex covering S of G is called a minimal vertex covering if no subset of S is a vertex covering. The minimum cardinality of among all minimal vertex covering of G is called its vertex covering number and is denoted by $\Sigma_{e\in A}(cNS(G))$. .

Note: In Neutrosophic soft graphs, independent set may contain arcs which are not a strong arcs.



$H(e_1)$ $H(e_2)$

In $H(e_1)$, u_3u_4 is not a strong. So, independent set $S = \{u_3, u_4\}$ and vertex covering set $W = \{u_1, u_2\}$

In $H(e_2)$, all are strong arcs. Therefore, independent set $S = \{u_1, u_3\}$ and vertex covering set $W = \{u_3, u_4\}$

Theorem 4.9

Let G be neutrosophic soft graph. Then $\Sigma_{e\in A}\big(INS(G)\big) + \Sigma_{e\in A}\big(cNS(G)\big) = |V(G)|$.

Proof.

Let S be a maximum independent set of a neutrosophic soft graph G and W be a minimum vertex covering of G. Hence $\Sigma_{e\in A}\big(INS(G)\big) + \Sigma_{e\in A}\big(cNS(G)\big) = |V(G)|$.

Definition 4.10

A neutrosophic soft graph G is said to be **strong arc neutrosophic soft graph** if every arc in G is a strong arc.

Theorem 4.11

Let G be strong arc neutrosophic soft graph. Then $\ddot{I}(G) \leq \ddot{D}\ddot{I}(G) \leq |V(G)|$. Also $\ddot{I}(G) \leq \ddot{D}\ddot{I}(G) \leq |V(G)| - \Sigma_{e\in A}\big(cNS(G)\big) + 1$.

Proof.

In strong neutrosophic graph, every arc is a strong arc. Therefore, $\ddot{I}(G) \leq \ddot{D}\ddot{I}(G)$. Let S be vertex covering in G. Then, clearly the induced graph of $G - S$ is an independent set, say T. Hence the removal of S results totally independent vertices (isolates). Therefore, $m(G - S) = 1$. Hence $|V(G)| - \Sigma_{e\in A}(cNS(G)) + 1$.

Theorem 4.12

For any neutrosophic soft graph, $\Sigma_{e\in A}\left(d_{NS}(G)\right) \leq \ddot{D}\ddot{I}(G)$.

Proof.

The domination integrity number of a neutrosophic soft graph G depends upon the dominating set S and the corresponding maximum order of the component of $G - S$. This implies that $\Sigma_{e\in A}\left(d_{NS}(G)\right) < \ddot{D}\ddot{I}(G)$. The equality holds only when all the vertices of a neutrosophic soft graph. Hence $\Sigma_{e\in A}\left(d_{NS}(G)\right) \leq \ddot{D}\ddot{I}(G)$.

Theorem 4.13

For any strong arc neutrosophic soft graph, $\delta_s(G) + 1 \leq \ddot{I}(G) \leq \ddot{D}\ddot{I}(G)$.

Proof.

Let G be a strong neutrosophic soft graph. Let S be a subset of V(G) . Let $u \in V(G)$ be a minimum strong degree vertex of G. Let $|d_s(v)| = \delta_s(G)$. Then, after the removal of the vertices in S from G, we get $m(G - S) \geq 1$ which gives the result $\delta_s(G) + 1 \leq \ddot{I}(G)$.

Theorem 4.14

Let $G' == (G*, J', K', A')$ is called a neutrosophic soft subgraph of $G = (G*, J, K, A)$. Then $\ddot{I}(H) \leq \ddot{I}(G)$.

Proof.

Let $G' == (G*, J', K', A')$ is called a neutrosophic soft subgraph of $G = (G*, J, K, A)$. Clearly, $|V(H)| \leq |V(G)|$ (by subgraph definition, at least one vertex, $v \in H$ which has less membership value comparing with membership value of G, otherwise $|V(G)| \leq |V(H)|$). Moreover, for any neutrosophic soft graph H, $\ddot{I}(H) \leq |H| < |G|$.

Suppose $\ddot{I}(G) > \ddot{I}(H)$ for an integrity set S of H. Then $m(H - S) < \ddot{I}(G) - |S|$. If S is also an integrity set of G, then $m(H - S) < m(G - S)$, which is impossible, since H is sub set of G. If S is not an integrity set of G then $\ddot{I}(G) - |S| < m(G - S)$, this is a contradiction. Hence any integrity set S of G is such that $\ddot{I}(H) \leq \ddot{I}(G)$.

Theorem 4.15

Let $G = (G*, J, K, A)$ be a complete neutrosophic soft graph. Then $\ddot{I}(G) = |V(G)| = \ddot{D}\ddot{I}(G)$.

Proof.

Clearly, in complete neutrosophic soft graph, all the vertices are adjacent with the remaining set of vertices. Therefore, after the removal of any subset S of vertices from G, $m(G - S) = |V(G)| - |S|$.

Theorem 4.16

If $G = (J, K, A)$ is a strong neutrosophic soft graph and its complement $G^c = (J^c, K^c, A^c)$, then $I(G \cup G^c) = |V(G)|$.

Proof.

Let G be a strong neutrosophic graph and G^c be the complement of G. By proposition 3.34[6], $G \cup G^c$ is a complete neutrosophic soft graph. Hence $I(G \cup G^c) = |V(G)|$.

Theorem 4.17

Let G_1 and G_2 be two connected neutrosophic soft graphs and $G = G_1 \cup G_2$ with $|G_1| \geq |G_2|$, then vertex integrity of G is given by

$\ddot{I}(G) = min\{|G_1|, \ddot{I}(G_1), |S| + |V(G_2)|, |S| + max\{m(G_1 - S), m(G_2 - S)\}\}$ where S is $\ddot{I}$- set of G.

Proof.

Let G_1 and G_2 be two connected neutrosophic soft graphs and $G = G_1 \cup G_2$ with $|G_1| \geq |G_2|$. Assume that $|G_1| > |G_2|$. In this case integrity set S of G is either vertices from G1 or G2 or both or empty. Since $|G_1| \geq |G_2|, S$ cannot contain vertices from G_2 alone.

Based on each case which is mentioned above, we get the result.

Theorem 4.18

Let G_1 and G_2 be two connected neutrosophic soft graphs and $G = G_1 + G_2$ with $V_1 \cap V_2 \neq \emptyset$. Then $\ddot{I}(G) = min\{\ddot{I}(G_1) + |V(G_2)|, \ddot{I}(G_2) + |V(G_1)|\}$.

Proof.

Let G_1 and G_2 be two complete neutrosophic soft graphs. Clearly, G is a complete neutrosophic soft graph. Therefore, $\ddot{I}(G) = \ddot{I}(G_1) + \ddot{I}(G_2) = \ddot{I}(G_1) + |V(G_2)| = |V(G_1)| + \ddot{I}(G_2)$. If we take all the vertices of G_1 in the $\ddot{I}$-set of G, then induced graph G_2 is a single connected component, since every vertex from G_1 is linked with G_2 with an edge. In the similar manner, we consider G_2. Moreover, other subsets of $V(G)$, $m(G - S)$contains all the remaining vertices of G. Hence the theorem

5. Conclusion

In this present work, we introduced the concept of integrity and domination integrity in neutrosophic soft graphs and calculated the certain bounds of these new parameters. In our future work, we will study the applications of these new parameters in neutrosophic real time networks for decision making problems.

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An efficient framework for evaluating the usability of academic websites: Calibration, validation, analysis, and methods

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Abstract
The success of an organization nowadays is heavily dependent on the usability of its website. Offering educational content and services online is becoming more commonplace in the higher education sector. University websites serve a wide range of users, like students, faculty, parents, staff, etc. Hence, the website must address the different needs of these users while maintaining good usability. Good usability makes it easier for users to find what they are looking for, understand how to use the website, and navigate through the content. This helps improve user satisfaction and engagement with the website, which can lead to increased productivity and better outcomes. Therefore, usability testing and analysis is the unspoken metric for success. Understanding the many factors contributing to the usability of academic websites is a multi-criteria decision-making (MCDM) topic. In this paper, we propose a framework for evaluating the usability of academic websites using the Entropy and Weighted Aggregated Sum Product Assessment (WASPAS) MCDM methods under type-2 neutrosophic sets. The entropy method is used to compute the objective weights of four main criteria contributing to the usability of academic websites, namely (Content, Organization, Presentation and Interaction, and Trustworthiness), with 31 sub-criteria. The WASPAS method is then used to rank five Egyptian university websites and select the best one in terms of usability. This framework will help designers understand the important criteria to consider while designing for university websites in addition to providing them with a usability evaluation method tailored to university websites.

Keywords: University websites; Usability; Neutrosophic; WASPAS; Website Evaluation; MCDM

1. Introduction

Currently, the internet is the most used medium of communication and service delivery by people or entities, especially post-Covid-19. Users search the internet for information they need daily, whether for business, health, education, or governance purposes. With websites acting as a powerful platform for information distribution, many institutions have resorted to the web channel for access. One important type of website that attracts a lot of users is the university website. University websites are crucial for current and prospective students, faculty, and parents. It provides important information and can act as a marketing or public relations tool to attract potential students. Academic websites have changed the way information is stored and accessed. They have made accessing information related to admission, courses, and exams easier. This has removed many of the boundaries that once limited these processes, such as geography and time [1]. While university websites have always been a source of valuable information, they have become even more essential in recent years. With their ability to provide quick and easy access to up-to-date academic information, university websites are now among the most comprehensive information platforms available. University websites are intended to provide services to a diverse audience; therefore, it’s critical to maintain the accessibility and usability of these sites for all groups of users to satisfy their intended use and to provide the users with the most intuitive and quality experience. Yet, many university websites suffer from poor designs, difficult interactions, and problems with the findability of core information buried in a sea of pages. There are several factors contributing to this problem. One such factor is that universities usually have large websites with hundreds of sub-sites and thousands of pages, and the need to serve multiple distinct audiences with different needs and questions to answer. These factors lead to academic websites failing to meet the expectations of users or not providing the users with quality information and quality in look and feel. Hence, the effort and cost put into maintaining and hosting these websites become useless and wasteful.

Organizations place a great deal of importance on their website design, working to create sites that not only look good but are also easy and usable. Usability, according to ISO 9241-11, is defined as the effectiveness, efficiency, and satisfaction with which a set of users can achieve a set of tasks in a defined environment [2]. Nevertheless, the users' satisfaction and the interfaces' usability are questionable and vague. Hence, there is a growing need for tools to support and help designers make better decisions and go in the right direction to achieve maximum user satisfaction. We need a way to make user interfaces quantifiable, thus allowing for automatic calculation of how good an interface is and easily comparing different versions of designs without involving end users. To this end, it is highly important to understand the impact of the different criteria contributing toward user satisfaction with academic websites which is a MCDM problem. The neutrosophic approach is a promising method to deal with uncertainty. That makes it highly suitable for addressing the usability of academic websites. This research focuses on implementing a framework for evaluating the usability of academic websites using MCDM methods under type-2 neutrosophic sets.

The remainder of the paper is organized as follows. Technical background and literature review in Section 2. Section 3 presents the research methodology. Section 4 presents the case study and analysis. Section 5 presents the sensitivity analysis. Finally, we conclude this paper in Section 6.

2. Technical background and literature review

In this section, we give a quick overview of usability and usability evaluation methods, then a literature review of previous work.

2.1 Concepts and terminologies

Usability

According to ISO 25000, Usability is “the degree to which specified users can use a product or system to achieve specified goals with effectiveness, efficiency, and satisfaction in a specified context of use”. And they summarize usability in 6 characteristics: Recognizability, Learnability, Operability, Error protection, Aesthetics, and Accessibility [3]. Nielsen defined usability as how easy an interface is to use, and he defined usability through 5 characteristics: Learnability, Efficiency, memorability, Errors, and Satisfaction. [4]

So, a website’s usability is the website’s ability to enable users to find the information they’re looking for most efficiently and delightfully to deliver user satisfaction. Therefore, website usability is achieved through multiple criteria, such as efficiency, learnability, memorability, delightfulness, and error tolerance, etc., and the criteria would differ based on the target users, their needs, and the situation. Hence, Usability is one of the major factors determining a website's success. It is important, therefore, to have some guidelines to ensure websites’ usability, and to have a way to assess the usability of websites.

Usability evaluation techniques

An essential part of the design process of user interfaces is their evaluation by users to enhance their usability, as users are becoming less willing to interact with difficult or uncomfortable interfaces. A usability evaluation method is a procedure used to collect user interaction data with software to assess the degree of usability achieved by the properties of that software [5].

There are several methods to assess the usability of user interfaces, we could classify them into two general types based on end-users involvement: empirical methods and inspection methods. Empirical methods require end-user presence to complete some tasks using the software or prototype and to capture his interactions and usage data to detect usability issues, this end-user presence makes empirical methods costly and restricts their conduction till the software is developed. Inspection methods on the other side don't require end users but are performed by experts who review the user interface with respect to some predefined set of principles and guidelines to assess their usability and detect any usability violations, which makes it more cost-efficient, this method has the advantage that problems can be ironed out before considerable effort and resources have been expended on the design process. However, inspection methods could be affected by evaluators' expertise, biases and opinions, and the quality of the evaluation guidelines, which could leave out the real user needs. One of the most famous inspection evaluation methods is the heuristic evaluation method by Nielsen [6], [7] used for finding usability problems in a user interface by following a set of usability heuristics "principles" and checking if the interface violates any of them [8]. It is not justifiable to standardize usability guidelines across different design situations, as different organizations have distinct business goals and end-users. Organizations should design websites focusing on who their end-users are, what information they need, and how they can easily retrieve this information.

Usability evaluation studies

Many studies have addressed the problem of website usability assessment, either using traditional assessment methods like questionnaires, Likert scales, and heuristic evaluations or using multi-criteria decision-making methods such as AHP and FAHP, etc. These studies identified various factors that affect the usability of academic websites.

Astani [9] evaluated the effectiveness of the top 50 universities' websites in the U.S. and analyzed the weaknesses and strengths of these websites, using traditional assessment methods such as a questionnaire and a list of 6 predefined usability characteristics from a literature review rated using a five-point Likert scale (Information, Content, Navigation, Usability, Customization, Download Speed, Security). Similarly, Manzoor & Hussain [10] evaluated the usability of higher education websites in Asia using a survey and performed some analysis on the results to propose a "WUEM" Web usability evaluation model consisting of 4 main usability criteria (Web design, page design, navigation, accessibility) and a total of 17 sub-criteria. Another study evaluated the usability of the Namik Kamel university's website using 5 usability criteria (attractiveness, controllability, helpfulness, efficiency, and learnability) defined by WAMMI (Website Analysis and Measurement Inventory)[11]. A similar study developed a set of 7 principles and heuristics to evaluate 12 Saudi Arabia university websites, including: (visual design and consistency, links and navigation, data entry forms, information truth and precision, privacy and security, search functionality, help, feedback, and error tolerance). These principles were based on Nielsen's heuristics and ISO standards [12]. Hasan [13] is another researcher that employed the heuristic evaluation method to evaluate the usability of 3 Jordan university websites using a set of 5 usability criteria related to educational websites (Navigation, architecture/organization, ease of use and communication, design, content). Based on that heuristic evaluation, a list of 34 specific types of usability problems was identified. Roy et al. [14] used questionnaire-based evaluation and performance-based evaluation to evaluate 3 academic websites based on 4 criteria (Task success, Task completion time, Number of clicks, and satisfaction metrics). Five high-level quality factors (functionality, usability, reliability, presentation, content) and 20 sub-quality factors based on ISO 9126-1 for evaluating academic websites were identified by Devi & Sharma [15]'s framework. EduGate, an online academic portal of King Saud University, was evaluated by 3 experts using a heuristic checklist based on Nielsen's heuristics [16]. Vakkalanka et al. [17] proposed a tool for evaluating academic websites, based on a set of 6 main criteria developed from previous models and ISO 9126 (content, usability, reliability, maintenance, functionality) with a total of 24 sub-criteria. According to the systematic literature review of university website usability evaluation conducted in 2022, most usability problems found were related to interface design, navigation, content, and performance and accessibility issues [18]. A comparison of the criteria used in these studies is presented in table 1.

As we have seen, many researchers paid attention to the problem of academic website evaluation. However, all the above studies used questionnaires, automated accessibility tools, and heuristic rules to evaluate usability. But usability is a more complex problem that is influenced by many criteria, these criteria in most real-life scenarios can be conflicting, and there will be a tradeoff between them, like aesthetics and simplicity, paying more attention to the website's design using more colors, images and visual content could sometimes make the design more complex to use. That's why we need to empower designers with a framework of the most suitable criteria for academic website usability with their relative importance, this becomes a multi-criteria decision-making problem (MCDM), and within the last few years, some studies have assessed academic websites usability using some MCDM methods like Analytical hierarchy process (AHP), fuzzy analytical hierarchy process (FAHP), fuzzy TOPSIS, PROMETHEE.

Nagpal et al. [19] proposed a rule-based system using the ANFIS method to assess website usability from the perspective of end users. A survey was used to identify the factors affecting usability, and 4 factors were chosen (Ease of use, information, response time, and ease of navigation). In Nagpal, Bhatia, et al. [20], the same authors employed a FAHP approach to evaluate the weights of usability criteria of an educational institute website and used the proposed approach to rank 4 websites based on their evaluated usability score. They used the same 4 criteria used in [19]. In Nagpal, Bhatia, et al. [20] Also, FAHP was used to evaluate the criteria affecting the usability of a website, and a fuzzy TOPSIS method was used to rank 4 websites based on the usability criteria (Ease of use, informative, response time, ease of navigation). An AHP-based usability evaluation technique is proposed by Roy et al. [21] to measure the usability score of a website. A questionnaire was used to measure users' satisfaction degree on 5 usability criteria (attractiveness, controllability, efficiency, helpfulness, and learnability), and the results were analyzed using AHP. In Nagpal et al. [22] a metric is proposed integrating objective and subjective usability evaluation approaches, using fuzzy AHP and entropy methods respectively, on 5 usability criteria (Ease of use, information, response time, ease of navigation, and contrast errors). Response time (RT) was suggested by the entropy as the main contributor to usability, and Ease of use (EOU) was suggested as the main contributor by FAHP. RT was the main contributor to the evaluation of the usability of academic websites according to the combined approach. In Shayganmehr et al. [23] hybrid MCDM approaches (AHP and PROMETHEE) are used to determine the importance of criteria and sub-criteria contributing to the usability of E-services of Iranian universities' websites. Nine indexes (criteria) were used (website design, responsiveness quality, security, trust, content and information quality, participation, support and maintenance, services, and usability). A framework for evaluating university websites was proposed by Gharibe Niazi et al. [1]. It used the Delphi technique, systematic review, and meta-analysis approaches. The proposed framework included 10 criteria (credibility, reliability, usability, website design, functionality, content, page design, efficiency, and webometrics) that are suggested for university website evaluation. This study suggested that credibility is the most important factor in the evaluation of university websites. In Muhammad et al. [24], a FAHP approach is used to evaluate the usability of academic websites with 3 usability criteria (usability, navigation, content) and 9 sub-criteria (ease of use, interactivity, learnability, ease of navigation, accessibility, efficiency, informative, accuracy, user satisfaction), The fuzzy extent analysis technique was used to rank 5 university websites. A comparison of the different criteria used in these studies is presented in table 2.

Table 1. Universities' website usability criteria.

Usability Criteria	Studies									
	[9]	[13]	[25]	[10]	[16]	[14]	[26]	[18]	[15]	[17]
Content	✓	✓	✓	✓	✓				✓	✓
Navigation	✓	✓		✓	✓			✓		✓
Usability	✓				✓				✓	✓
Customization	✓									
Security	✓		✓							
Download speed	✓									
Architecture/ Organization		✓		✓				✓		
Ease of use and communication		✓		✓						
Design		✓	✓	✓				✓	✓	✓
Consistency			✓	✓	✓					
Links			✓	✓	✓			✓		
Data entry forms			✓							
Search functionality			✓						✓	✓
Help, feedback, and error tolerance, recoverability			✓		✓				✓	✓
Sitemap				✓						
Concise News and Events				✓					✓	
Multiple Language Support				✓	✓				✓	✓
Accurate Page title				✓	✓					✓
Page headings				✓	✓					
Avoid Page scrolling				✓						
Link logo to homepage				✓	✓					
Home page navigation in the main menu				✓	✓					
Adequate text-to-background contrast				✓						
Font size/spacing is easy to read				✓						
Attractiveness							✓		✓	✓
Controllability							✓			
Efficiency							✓			
Helpfulness					✓		✓			✓
Learnability					✓		✓		✓	
Reliability									✓	✓
Functionality									✓	✓
Understandability									✓	✓
Interactivity									✓	✓
Availability					✓				✓	✓
Memorability					✓					
User satisfaction					✓	✓				
Few errors					✓					

Table 2. Usability criteria used in MCDM studies.

Usability Criteria	Studies								
	[19]	[20]	[29]	[21]	[30]	[22]	[27]	[1]	[24]
Ease of use	✓	✓	✓	✓		✓			✓
Informative	✓	✓	✓		✓	✓			
Response time	✓	✓	✓	✓		✓	✓		
Ease of navigation	✓	✓	✓		✓	✓			✓
Attractiveness				✓					
Controllability				✓					
Efficiency				✓	✓			✓	✓
Helpfulness				✓					
Learnability				✓	✓			✓	✓
Contrast errors						✓			
Website Design							✓	✓	
Responsiveness							✓		
Security							✓		
Trust							✓		
Content and information quality				✓	✓		✓	✓	✓
Participation							✓		
Support and maintenance							✓		
Services							✓	✓	
Usability							✓	✓	✓
Reliability								✓	
Web credibility								✓	
Functionality								✓	
Systematic cues								✓	
Page design								✓	
Webometric								✓	
Interactivity					✓			✓	✓
Accessibility	✓				✓	✓		✓	✓
Accuracy									✓
User satisfaction	✓				✓				✓
User-friendliness	✓								
Personalization	✓								

3. Research methodology

The methodology used in this research consists of 3 phases: Criteria identification through literature review and surveying the stakeholders and UX experts and academics to identify the key relevant usability criteria for academic websites to consider in our framework, Computing the criteria objective weights using Shannon's entropy method, and finally, ranking five university websites based on the usability criteria weights using the WASPAS method. The usability evaluation of a university website is a multi-criteria scenario that considers different criteria, with varying importance, and it's a challenging task to try to consider the different needs and criteria, so tradeoffs will often be made when designing, hence, it is necessary to understand how each criterion contributes to the overall usability of the website, so that educated design decisions can be made. And when presented with different designs, we can evaluate and select the most usable design. The proposed methodology used the neutrosophic environment to overcome vague and incomplete information. The type-2 neutrosophic sets (TNSs) are used in this study. The graphical representation of the research methodology is depicted in Fig. 1. The steps of the proposed framework are organized as follows:

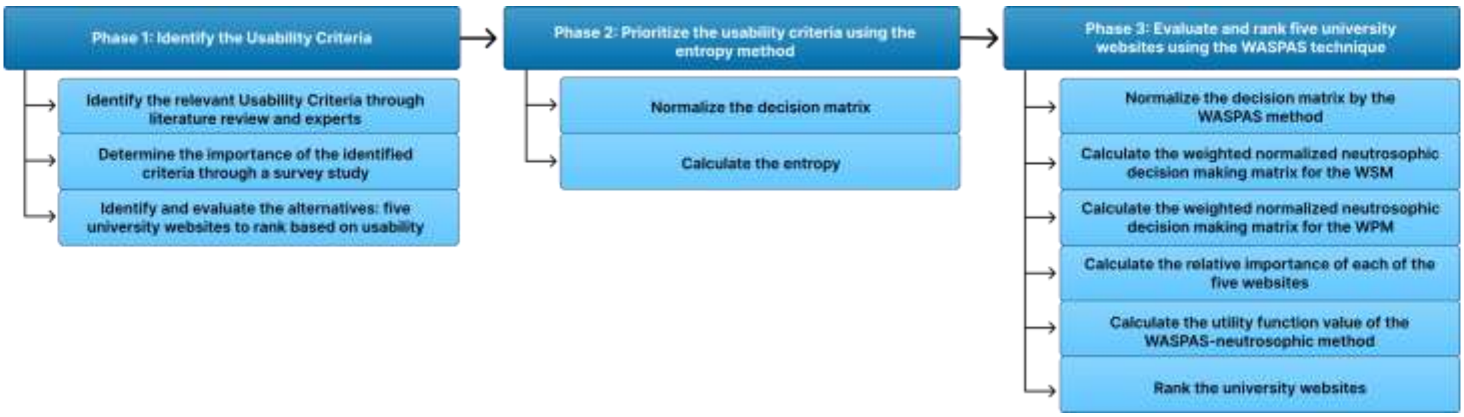


Figure 1. Research Methodology

Phase 1: Identify the usability criteria

Step 1: In order to identify the relevant usability criteria, a comprehensive literature review was conducted and UX experts and academics were consulted to identify the key usability criteria relevant to academic websites.

Step 2: A survey study was conducted to determine the importance of the identified criteria to help focus on the most relevant ones. The participants were UX field experts with experience in usability evaluation, academics with Ph.D. and experience in the field of Human-Computer Interaction, stakeholders, current and prospective students using academic websites, faculty members, etc.

Step 3: Five university websites were selected for usability evaluation, and three UX and usability evaluation experts evaluated the five websites based on the selected criteria from step 2, and a decision matrix for each of the three experts was constructed using the type-2 neutrosophic sets. Then the opinions of experts were converted from linguistic terms into crisp values. Finally, we aggregated the opinions of the three experts into one matrix.

Phase 2: Prioritize the usability criteria using the entropy method

3.1 Shannon's entropy method

Assuming we have m alternatives $(A_1, A_2, \dots a)$ and b criteria $(C_1, C_2, \dots b)$ for a decision problem

Step 4: Normalize the aggregated decision matrix

$$nr_{cd} = \frac{x_{cd}}{\sum_{c=1}^{a} x_{cd}} \quad (1)$$

Where $c = 1,2,3, \dots . a;\ and\ d = 1,2,3, \dots . . b$

Step 5: Calculate the entropy

$$r_d = -L \sum_{c=1}^{a} nr_{cd} \ln nr_{cd} \quad (2)$$

Where $L = 1/\ln a$

Step 6: Compute the objective weights of the criteria

$$w_d = \frac{1 - r_d}{\sum_{c=1}^{b}(1 - r_d)} \quad (3)$$

Phase 3: Evaluate and rank the five university websites using the WASPAS technique

3.2 WASPAS method

The weighted aggregated sum product WASPAS is a decision-making method that combines the weighted sum model (WSM) and the weighted product model (WPM) to help identify the ranking of the different alternatives to solve the decision-making problem. The WSM approach calculates the total score of the alternative as a weighted sum of the criteria. The WPM approach was created to prevent alternatives that have poor attributes or criterion values. The WASPAS method can check the consistency in the overall ranking of the alternatives using the λ coefficient. Apply steps 1 to 3.

Step 7: Normalize the decision matrix by the WASPAS method as:

$$X^*_{cd} = \frac{x_{cd}}{\max_c x_{cd}} \ for\ beneficial\ criteria, \quad (4)$$

$$X^-_{cd} = \frac{\min_c x_{cd}}{x_{cd}} \ for\ non\ beneficial\ criteria. \quad (5)$$

Step 8: Calculate the weighted normalized neutrosophic decision-making matrix for the WSM:

$$WX^*_{cd} = X^*_{cd} * W_d \quad (6)$$

Step 9: Calculate the weighted normalized neutrosophic decision-making matrix for the WPM:

$$WX^-_{cd} = X^-_{cd} * W_d \quad (7)$$

Step 10: Calculate the total relative importance based on:

The WSM for c alternative:

$$S_c^1 = \sum_{d=1}^{b} WX^*_{cd} \quad (8)$$

The WPM for c alternative:

$$S_c^2 = \prod_{d=1}^{b}(X^-_{cd})^{W_d} \quad (9)$$

Step 11: To improve the ranking accuracy, the utility function value of the WASPAS-neutrosophic method is calculated as follows:

$$S_c = \alpha S_c^1 + (1 - \alpha) S_c^2 \quad (10)$$

Where $\alpha\ is\ between\ 0\ and\ 1$

The alternatives are ranked based on the values of S_c , the alternatives having the highest values being the most significant.

4. Case study and analysis

The proposed methodology results of the entropy and WASPAS methods under a neutrosophic environment are demonstrated through this case study.

4.1 Description of criteria

Considering section 2, After conducting a comprehensive literature review of the previous studies on the usability evaluation of academic websites and talking to usability experts working in the field of UX design, we considered the most important and effective criteria for evaluating the usability of academic websites to be 31 criteria (shown in figure 2) organized under four main Categories: Content, Organization, Presentation and Interaction, Trustworthiness. All criteria are positive except Broken links, Load time, and response time. Appendix table 1 shows the 4 main criteria and 31 sub-criteria.

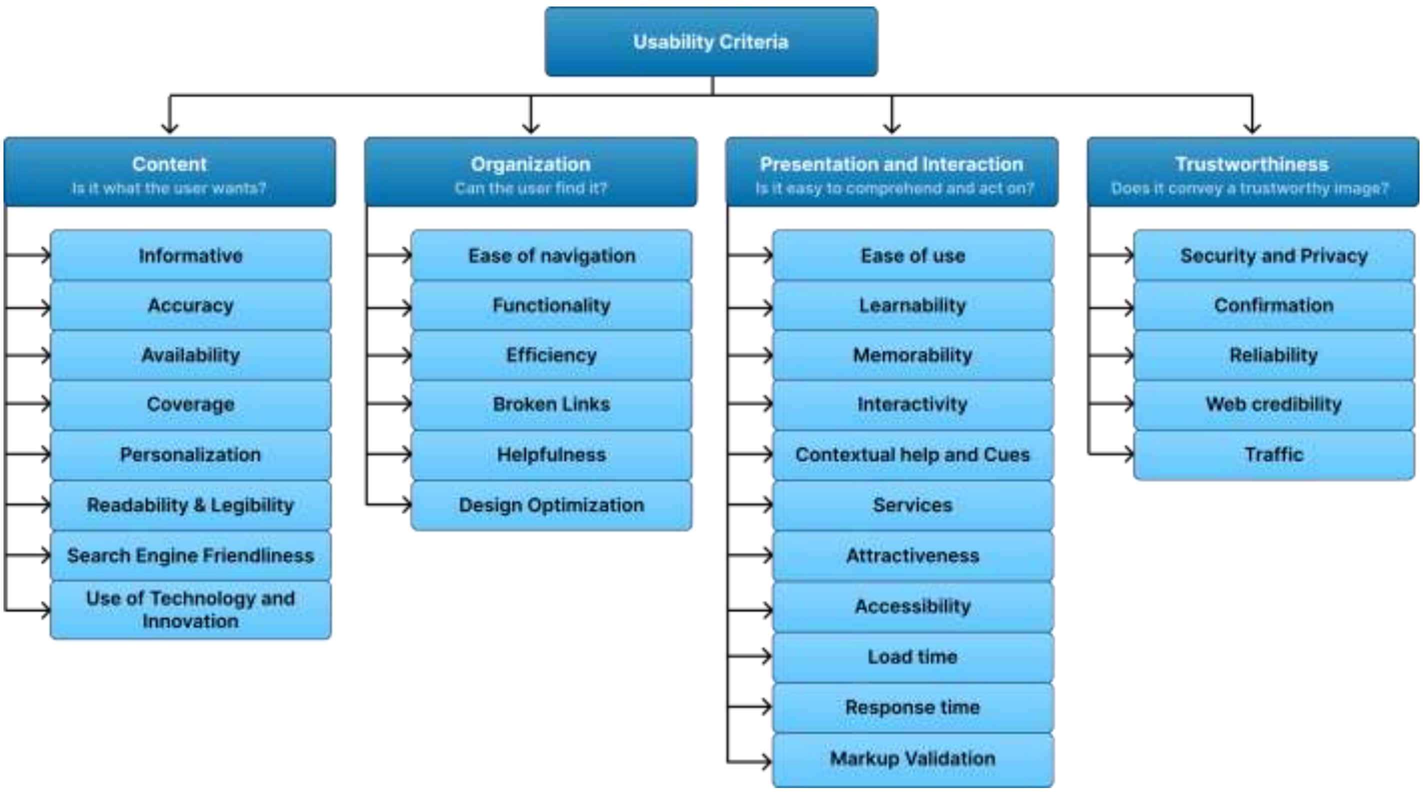


Figure 2. The Identified usability criteria, 4 main criteria, 31 sub-criteria

The criteria are introduced and explained below.

Category 1: Content. In this category, all criteria are related to a website's content, including text, images, videos, audio data, etc., which answers the question: Is it what the user wants?

1. **Informative:** Users come to a website looking for specific kinds of information. Informative refers to how the content on the website provides current, relevant, complete, valuable, and quality information. The content should be comprehensive, appropriate, and within the expected level of detail [28], which is a significant usability factor for a university website, as providing clear and understandable content will encourage users to keep returning to the website.
2. **Accuracy:** In a university website, it's important that the content is always accurate, reliable, correct, and authentic so that it builds trust. This can be done by checking for spelling or grammatical errors that could alter the meaning of information, providing images and multimedia of appropriate quality, using accurate page titles, and providing precise and trustworthy content [20].

3. **Availability:** Availability is a measure of the readiness of content. Content should always be ready and available for users to access. This also includes the ability to reach past and archived content easily.
4. **Coverage:** Coverage refers to the degree that topics of interest are successfully addressed, with clearly presented arguments and adequate support to substantiate them. This can refer to the diversity of services and academic activities covered on the website.
5. **Personalization:** Personalization and customization are other significant factors of a university website due to the diversity of its audience. This is the website's ability to offer customized content to the user based on different criteria like location etc., using the user's data to suggest and serve up related content and allowing the users to customize their experiences.
6. **Readability and Legibility:** Readability and legibility refer to how the users read and view the content of a page or screen; good readability helps users read the content more efficiently and understand the message more clearly. On the other hand, good legibility makes the presentation edible and allows users to quickly understand what is on the page or screen. The use of appropriate Typography, whitespace, hierarchy, etc., can help achieve these two [31].
7. **Search Engine Friendliness:** A university website needs to have a strong presence in search engines. Multiple factors contribute to that, such as clear web page structure, using a good mix of visual media, conserving the website's storage to improve site speed, and having a responsive design [32].
8. **Use of Technology and Innovation:** This criterion refers to how the website adapts to accommodate the latest technological advancements, such as Augmented reality (AR) and Virtual reality (VR), using chatbots and voice search.
9. **Updates:** A university website needs to have up-to-date content. An outdated website will cause confusion and loss of credibility, so the content and style presented in a website should be frequently updated, displaying the latest update date.

Category 2: Organization. In this category, all criteria are related to the organization and structure of the information on the academic website. This category answers: Can the user find it?

10. **Ease of Navigation:** A university website is a collection of large and diverse amounts of information. It's essential that the users can navigate through it and find information quickly and easily. As users will discontinue using the site if it is complex to navigate or if too many clicks are required to retrieve the required information. Ease of Navigation depends on how the information is organized and arranged, the presence of navigational aids, and providing alternative navigational ways. This helps overcome the navigational complexity, especially in the case of big websites like university websites [22].
11. **Functionality:** Functionality refers to the degree that the website provides functions that meet and cover all the stated or implied needs, tasks, and objectives of the users [33]. Examples of functionality are: Searching and retrieving mechanisms, navigational prediction, and online services.
12. **Efficiency:** Efficiency measures how quickly and easily the users can locate and achieve their goals without putting in much cognitive effort. Users can experience a measurable decline in efficiency when they lose their sense of location on a website or feel disoriented [34]. According to Jakob's Usability Heuristic, efficiency refers to how quickly users can perform tasks after learning to use the design by allowing for flexibility, shortcuts, and reducing the number of clicks [35].

13. **Broken Links:** Broken links affect navigation significantly, and university websites should have no broken links and no orphan pages. Broken links frustrate and drive users away and negatively affect the website's SEO, search ranking, and quality scores.
14. **Helpfulness:** Helpfulness refers to how helpful the website is to the users, reducing their cognitive effort. Hence, it is essential to help users during each visit step (before, during, and after). A high level of helpfulness corresponds with the users' expectations about the content and structure [21].
15. **Design optimization:** A university website must be compatible with and perform well in different browsers and platforms. The website design and organization should also be consistent and accessible through all browsers and platforms (responsiveness).

Category 3: Presentation and Interaction. In this category, all criteria are related to how the website supports the user in terms of presentation and interaction. It answers the question: Is it easy to comprehend, and can the user act on it?

16. **Ease of Use:** Ease of use is an essential factor in assessing the usability of a university website [19]. It measures how intuitively and easily the user can use the website. Consistent design, clear instructions, help, using simple terms and conventions are examples of factors contributing to the ease of use of an interface.
17. **Learnability:** Refers to how easy the system is to learn, which is an important factor for university websites, as these websites have diverse audiences, and not all are frequent users. So, their design should be self-descriptive encouraging users to quickly become familiar with and learn how to perform different academic tasks through the website [36].
18. **Memorability:** Memorability refers to how easily users can remember how to use the website and re-establish proficiency after a period of absence, which is crucial for university websites that are used infrequently. Users need to be reminded how to do tasks and find the information they are looking for. There are many ways of designing a website to support memorability. For example, using meaningful icons, obvious names, and menu options and structuring the content in a relevant way.
19. **Interactivity:** Interactivity refers to how engaging and interactive the website is, which is related to support; Hence, the website should provide means of interaction with the website's functions, error prevention mechanism, visible controls, hints, and a feedback mechanism to assist and encourage the users during their visit [30].
20. **Contextual help and cues:** This factor refers to providing support for the users relative to the area they are currently interacting with through tooltips, visual prompts, walkthroughs, inline instructions, partial content, sound, and vibration to help guide users to the most significant elements and equally, move away from the least significant ones.
21. **Services:** The criteria refer to the number of academic services delivered through the website [37].
22. **Attractiveness:** The content on the website should be attractive to retain and interest the users. Attractiveness is a usability attribute that measures the visual aspect of the website. According to [10], the appearance of a website is a crucial factor in improving the perception of information in terms of better cognitive mapping and easy assessment of decisions. Thus, the website should be aesthetic, pleasant, fun, well-organized, and clean.
23. **Accessibility:** Accessibility measures how easily and intuitively accessible is the website's information for any user. Examples of accessibility include multiple language support, Adequate text-to-background contrast, proper font size/spacing, images having appropriate ALT tags, and compliance with WCAG accessibility guidelines. A university website needs to be accessed efficiently anytime and anywhere for the users to benefit from it [23].

24. **Load Time:** Load time refers to the time it takes to download and display an entire webpage, including all page elements, such as HTML, scripts, CSS, images, and third-party resources. Compressing images used on the website, compressing your images, removing unnecessary custom fonts and plugins, etc. can help speed up load time.
25. **Response Time:** Response Time measures how quickly a website responds to a request. Response times play a critical role in university websites, as delays in accessing the information cause users to be highly unsatisfied, particularly at times of enrollment, result declaration, etc. thus, response time affects inversely to the website's usability. Various parameters affect response time like network bandwidth, download, query processing time, etc. [28].
26. **Markup Validation:** Markup validation ensures that the HTML of the website is clean, well-structured, and used in a way that is compliant with the HTML specifications, as it supports assistive technologies, browser compatibility, and website usability.

Category 4: Trustworthiness. In this category, all criteria are related to how trustworthy the website is perceived to be.

27. **Security and Privacy:** Security and privacy are very important, especially on university websites, as they deal with sensitive information, confidential information should be well protected, and privacy and security policies should be presented to users. Factors like using secure protocols and data encryption methods help protect privacy and security from the infrastructure dimension while using security code images, a virtual keyboard for entering a password, and sending alarm messages when an unknown user logs into other users' accounts assure privacy and security from the interface dimension.
28. **Confirmation:** Confirmation messages are an important key to enforcing a trustworthy image. These are messages which require users to confirm an action they are trying to perform [38]. Confirmation messages are important to use to communicate information the user must confirm before an action is completed. A balance between transparency and excess information is needed.
29. **Reliability:** The performance of the website starts with how reliable it is and its ability to recover quickly from problems; reliability, according to [33], is the "degree to which a system, product or component performs specified functions under specified conditions for a specified period of time". Reliability can be measured by: Fault tolerance, recoverability, and availability.
30. **Web Credibility:** According to a Stanford study on web credibility [39], credibility is "perceived trustworthiness + perceived expertise". This can be measured by factors such as having a professional website appearance, providing information about the university, showing total transparency, listing communication information visible on the site, testimonials, highlighting professional accomplishments, showing social proof, ratings, and reviews, etc.
31. **Traffic:** The success of a website is measured by the number of its visitors and its conversion rate, which is affected by factors such as engaging content, impressive design, optimization for mobile, SEO, smooth navigation, etc.

4.2 Prioritizing the usability criteria using the entropy method

Step 1: Five websites were selected for usability evaluation in this study, to preserve confidentiality the websites are referred to as (WebA1, WebA2, WebA3, WebA4, WebA5).

Step 2: Three UX and usability experts with PhD and experience not less than 15 years in this field are selected to evaluate the five selected websites in terms of the identified usability criteria to compute the objective weights of the criteria using the entropy method as mentioned in step 4,5,6.

Step 3: The three experts used the type-2 neutrosophic numbers [40] to evaluate the websites based on the identified 31 criteria, using linguistic terms. Then we converted the opinions of experts (linguistic terms) to neutrosophic numbers as shown in appendix tables 2-4. After that, the type-2 neutrosophic numbers were converted to crisp values [40]. Then we aggregated the different opinions of the experts into one matrix by the average method.

Applying the Entropy method

Step 4: The aggregated decision matrix was normalized by applying Eq. (1), as shown in Table 3.

Table 3. The normalization matrix by the entropy method

	WebA1	WebA2	WebA3	WebA4	WebA5
WebC1	0.225727	0.084278	0.203056	0.203056	0.283884
WebC2	0.129779	0.289738	0.207243	0.207243	0.165996
WebC3	0.224561	0.236647	0.224561	0.08655	0.22768
WebC4	0.161665	0.230492	0.208884	0.255302	0.143657
WebC5	0.134974	0.228662	0.265582	0.19611	0.174672
WebC6	0.211221	0.181151	0.211221	0.245325	0.151082
WebC7	0.226939	0.235102	0.201633	0.134694	0.201633
WebC8	0.225599	0.099942	0.240795	0.19287	0.240795
WebC9	0.205703	0.293279	0.163951	0.113035	0.224033
WebC10	0.178349	0.132399	0.192368	0.260514	0.236371
WebC11	0.176494	0.231047	0.136382	0.234256	0.221821
WebC12	0.098404	0.231383	0.231383	0.231383	0.207447
WebC13	0.169065	0.258993	0.205935	0.25	0.116007
WebC14	0.160131	0.120915	0.155773	0.313725	0.249455
WebC15	0.188555	0.275422	0.132565	0.237135	0.166324
WebC16	0.14726	0.328767	0.167808	0.188356	0.167808
WebC17	0.127075	0.382942	0.127075	0.127075	0.235833
WebC18	0.191388	0.138388	0.211999	0.246227	0.211999
WebC19	0.30139	0.168818	0.202085	0.151936	0.17577
WebC20	0.148084	0.264373	0.148084	0.291376	0.148084
WebC21	0.11677	0.357764	0.160248	0.204969	0.160248
WebC22	0.129665	0.108159	0.163188	0.364326	0.234662
WebC23	0.157229	0.251787	0.321056	0.122045	0.147883
WebC24	0.249679	0.094912	0.185977	0.246259	0.223172
WebC25	0.146366	0.267144	0.113613	0.298874	0.174002
WebC26	0.213915	0.185877	0.213915	0.271028	0.115265
WebC27	0.128755	0.224737	0.192743	0.261022	0.192743
WebC28	0.253074	0.169057	0.132172	0.234631	0.211066
WebC29	0.207191	0.136502	0.201097	0.207191	0.24802
WebC30	0.24184	0.284866	0.127596	0.16815	0.177547
WebC31	0.224037	0.079736	0.224037	0.248152	0.224037

Step 5: Then, the entropy is computed using Eq. (2).

Step 6: The objective weights of the criteria are computed by Eq. (3). From the results, the presentation and interaction category scored the highest weight compared to the other three main criteria, while the organization category scored the lowest weight. Fig. 2 shows the weights of the criteria. From Fig.3, we see that "ease of use" is of the highest importance in all 31 criteria, and "updates" are of the lowest importance.

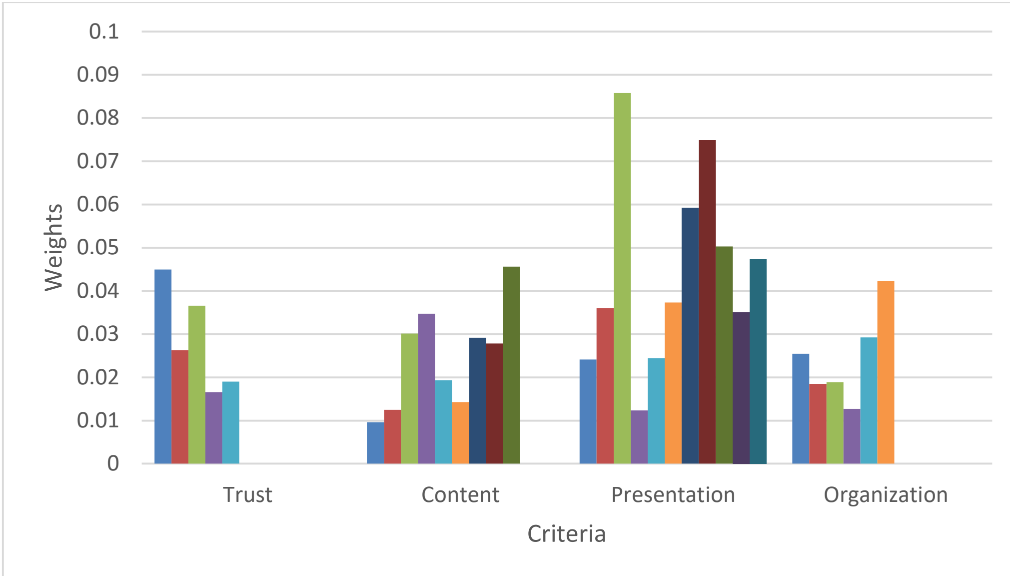


Figure 3. The weights of criteria.

4.3 Ranking the five university websites

The proposed framework was used to rank five Egyptian university websites using the WASPAS method.

Applying the WASPAS method

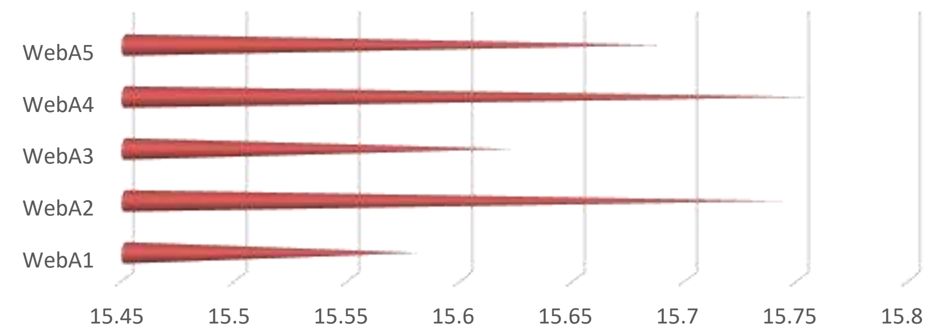
Step 7: The decision matrix was normalized by Eqs. (4,5) as shown in table 4.

Step 8: The WSM matrix is computed by Eq. (6), as shown in Table 5.

Step 10: The WPM matrix is computed by Eq. (7), as shown in Table 6.

Step 11: The total relative importance of the alternatives is calculated by Eqs. (8,9)

Step 12: Finally, the utility function was computed by Eq. (10). We use $\alpha = 0.5$. Then the alternatives are ranked based on the highest value of the utility function, as shown in Fig. 3. From Fig. 4. website 4 has the highest rank and website 1 has the lowest rank.



	WebA1	WebA2	WebA3	WebA4	WebA5
Rank	15.58025851	15.74179416	15.62137732	15.75467735	15.68864893

Figure 4. The rank of alternatives.

Table 4. The normalization matrix by the WASPAS method.

	WebA1	WebA2	WebA3	WebA4	WebA5
WebC1	0.795139	0.296875	0.715278	0.715278	1
WebC2	0.447917	1	0.715278	0.715278	0.572917
WebC3	0.948929	1	0.948929	0.365733	0.962109
WebC4	0.633229	0.902821	0.818182	1	0.562696
WebC5	0.508221	0.860987	1	0.738416	0.657698
WebC6	0.860987	0.738416	0.860987	1	0.615845
WebC7	0.965278	1	0.857639	0.572917	0.857639
WebC8	0.936893	0.415049	1	0.800971	1
WebC9	0.701389	1	0.559028	0.385417	0.763889
WebC10	0.684604	0.508221	0.738416	1	0.907324
WebC11	0.753425	0.986301	0.582192	1	0.946918
WebC12	0.425287	1	1	1	0.896552
WebC13	0.652778	1	0.795139	0.965278	0.447917
WebC14	0.510417	0.385417	0.496528	1	0.795139
WebC15	1.42236	2.07764	1	1.78882	1.254658
WebC16	1	2.232558	1.139535	1.27907	1.139535
WebC17	0.331839	1	0.331839	0.331839	0.615845
WebC18	0.77728	0.562033	0.860987	1	0.860987
WebC19	1	0.560132	0.670511	0.504119	0.583196
WebC20	0.508221	0.907324	0.508221	1	0.508221
WebC21	0.326389	1	0.447917	0.572917	0.447917
WebC22	0.355903	0.296875	0.447917	1	0.644097
WebC23	0.489726	0.784247	1	0.380137	0.460616
WebC24	1	0.380137	0.744863	0.986301	0.893836
WebC25	0.489726	0.893836	0.380137	1	0.582192
WebC26	0.789272	0.685824	0.789272	1	0.425287
WebC27	0.493274	0.860987	0.738416	1	0.738416
WebC28	1	0.668016	0.522267	0.927126	0.834008
WebC29	0.835381	0.550369	0.810811	0.835381	1
WebC30	0.848958	1	0.447917	0.590278	0.623264
WebC31	0.902821	0.321317	0.902821	1	0.902821

Table 5. The WSM matrix by the WASPAS method.

	WebA1	WebA2	WebA3	WebA4	WebA5
WebC1	0.035726	0.013339	0.032137	0.032137	0.04493
WebC2	0.011753	0.026238	0.018768	0.018768	0.015032
WebC3	0.034678	0.036545	0.034678	0.013366	0.03516
WebC4	0.010504	0.014975	0.013571	0.016587	0.009334
WebC5	0.009642	0.016334	0.018972	0.014009	0.012478
WebC6	0.008283	0.007104	0.008283	0.009621	0.005925
WebC7	0.012076	0.01251	0.010729	0.007167	0.010729
WebC8	0.028194	0.01249	0.030094	0.024104	0.030094
WebC9	0.024336	0.034698	0.019397	0.013373	0.026505
WebC10	0.013224	0.009817	0.014263	0.019316	0.017526
WebC11	0.010744	0.014064	0.008302	0.01426	0.013503
WebC12	0.012396	0.029148	0.029148	0.029148	0.026132
WebC13	0.018143	0.027794	0.0221	0.026829	0.012449
WebC14	0.023281	0.017579	0.022647	0.045611	0.036267
WebC15	0.034281	0.050075	0.024102	0.043114	0.03024
WebC16	0.036001	0.080375	0.041025	0.046048	0.041025
WebC17	0.028472	0.085801	0.028472	0.028472	0.05284
WebC18	0.009602	0.006943	0.010636	0.012354	0.010636
WebC19	0.024398	0.013666	0.016359	0.0123	0.014229
WebC20	0.018969	0.033865	0.018969	0.037324	0.018969
WebC21	0.019344	0.059265	0.026546	0.033954	0.026546
WebC22	0.026655	0.022234	0.033546	0.074894	0.048239
WebC23	0.024606	0.039404	0.050244	0.0191	0.023143
WebC24	0.03503	0.013316	0.026093	0.034551	0.031311
WebC25	0.023155	0.042262	0.017974	0.047282	0.027527
WebC26	0.020078	0.017446	0.020078	0.025439	0.010819
WebC27	0.009136	0.015947	0.013676	0.018521	0.013676
WebC28	0.018877	0.01261	0.009859	0.017502	0.015744
WebC29	0.010602	0.006985	0.01029	0.010602	0.012691
WebC30	0.02479	0.029201	0.01308	0.017237	0.0182
WebC31	0.038146	0.013576	0.038146	0.042252	0.038146

Table 6. The WPM matrix by the WASPAS method.

	WebA1	WebA2	WebA3	WebA4	WebA5
WebC1	0.989753	0.946897	0.985057	0.985057	1
WebC2	0.979147	1	0.991247	0.991247	0.985491
WebC3	0.998086	1	0.998086	0.963909	0.998589
WebC4	0.99245	0.998306	0.996677	1	0.990507
WebC5	0.987241	0.997164	1	0.994263	0.992082
WebC6	0.998561	0.997087	0.998561	1	0.995347
WebC7	0.999558	1	0.998081	0.993056	0.998081
WebC8	0.99804	0.973884	1	0.993344	1
WebC9	0.987768	1	0.980024	0.96746	0.990698
WebC10	0.992708	0.987012	0.99416	1	0.998123
WebC11	0.995971	0.999803	0.992316	1	0.999223

WebC12	0.975387	1	1	1	0.996822
WebC13	0.988215	1	0.993649	0.999018	0.977924
WebC14	0.969791	0.957445	0.968571	1	0.989599
WebC15	1.008528	1.01778	1	1.014115	1.005483
WebC16	1	1.029336	1.004714	1.0089	1.004714
WebC17	0.909693	1	0.909693	0.909693	0.95926
WebC18	0.996892	0.992907	0.998153	1	0.998153
WebC19	1	0.985959	0.990295	0.983428	0.98693
WebC20	0.975054	0.996377	0.975054	1	0.975054
WebC21	0.935796	1	0.953516	0.967527	0.953516
WebC22	0.925545	0.913059	0.941622	1	0.967591
WebC23	0.964766	0.987863	1	0.952564	0.9618
WebC24	1	0.966685	0.989735	0.999517	0.996076
WebC25	0.966808	0.994707	0.955298	1	0.974747
WebC26	0.993998	0.990452	0.993998	1	0.978485
WebC27	0.986996	0.997232	0.994399	1	0.994399
WebC28	1	0.992413	0.987813	0.998573	0.996579
WebC29	0.99772	0.99245	0.997342	0.99772	1
WebC30	0.99523	1	0.97682	0.984724	0.986289
WebC31	0.99569	0.953162	0.99569	1	0.99569

In this paper Shannon's entropy method is used to rank the usability criteria of academic websites by calculating their objective weights, The main criteria contributing to the usability of academic websites were earlier identified as Content, Organization, Presentation and Interaction, and Trustworthiness, with 31 sub-criteria that were rated by usability experts using the linguistic term to incorporate the vagueness in the experts' opinions. Based on the evaluation of the three experts, the "Presentation and Interaction" criteria are of the highest importance, followed by "Content" and then "Trustworthiness". The "Organization" came with the lowest importance compared to the other three main criteria. In the sub-criteria, "Ease-of-use" scored the highest importance, followed by "Interactivity" and "Attractiveness", While "Updates" scored the lowest importance in 31 criteria. The entropy method was found easier for decision-makers and more meaningful than performing pair-wise comparisons between the sub-criteria which is a tedious task and can be subject to personal opinions. Also, it wouldn't make sense to ask someone for example whether they think content or organization is more important, unlike the process followed in the entropy method where the experts evaluate existing websites according to the usability criteria and based on that the objective weights of the criteria are computed.

The WASPAS was later used to evaluate five Egyptian university websites based on the weights of the 31 usability criteria identified by the entropy method. Based on this evaluation, the fourth website scored the highest rank, followed by the second website, while the first website had the lowest rank. This method makes it easier to choose the best design from multiple alternatives based on their usability.

5. Sensitivity Analysis

In this section, the sensitivity analysis of the α value is performed to show the robustness and reliability of the proposed entropy and WASPAS model. The goal of the sensitivity analysis is to show how the rank of alternatives change when changing the α value. The α value changes between 0.1 and 0.9. Table 8 shows the rank of alternatives according to the different α values. From Figure 4, The rank of the alternatives doesn't change in the α value between 0.1 and 0.7. But when the α value was equal to 0.8 and 0.9, the rank of the alternatives changed, the second website became the best, followed by the fourth website, with the first website being the worst. All α values resulted in the first alternative being the worst, and all α values between 0.1 and 0.7 agreed the fourth alternative is the best. However, the α value of 0.8 and 0.9 agreed the second alternative is the best. From this analysis, the alternatives' rank is not sensitive to the α change.

Table 8. The rank of alternatives after changing the α value

	$\alpha = 0.1$	$\alpha = 0.2$	$\alpha = 0.3$	$\alpha = 0.4$	$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$
WebA1	27.52037	24.53534	21.55031	18.56529	15.58026	12.59523	9.610204	6.625177	3.64015
WebA2	27.68274	24.69751	21.71227	18.72703	15.74179	12.75656	9.77132	6.786082	3.800845
WebA3	27.57273	24.58489	21.59705	18.60922	15.62138	12.63354	9.6457	6.657862	3.670023
WebA4	27.71423	24.72434	21.73445	18.74456	15.75468	12.76479	9.774902	6.785015	3.795127
WebA5	27.65553	24.66381	21.67209	18.68037	15.68865	12.69693	9.705207	6.713487	3.721766

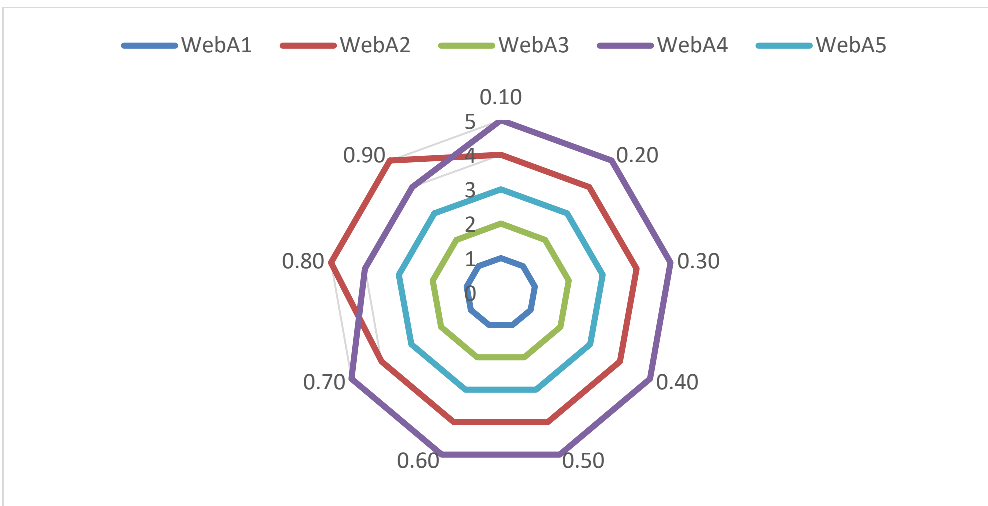


Figure 5. The rank of alternatives by sensitivity analysis.

6. Conclusion and Future work

This study was conducted methodically to propose a multi-stage MCDM framework using Shannon's entropy method and the WASPAS method under the type-2 neutrosophic environment for evaluating the usability of academic websites. First, the usability criteria relevant to academic websites were identified through secondary research and literature review and were further validated by usability experts and websites' users through a survey study which narrowed them down to four main usability criteria namely, Content, Organization, Presentation and Interaction, and Trustworthiness, with 31 sub-criteria. The usability criteria weights were then computed using the entropy method to understand their relative importance. This study found that the most important criteria among the four main criteria were Presentation and Interaction; The least critical criteria of the main criteria was Organization. Meaning that the website's organization is important but only after the content provided meets the users' needs and is easy to comprehend and act on and can be trusted. Only then the organization would make sense. In terms of the sub-criteria, Ease of use scored the highest importance while updates were the least important sub-criteria. Five Egyptian university websites were ranked using the WASPAS method based on the criteria weights identified.

The major strength of this work in relation to previous studies to the best of our knowledge is that it is the first framework to address this large number of criteria; 31, covering almost all aspects of a university website in detail and precisely, instead of addressing few major usability criteria that can be interpreted differently by different designers.

To further verify the soundness of this framework regarding the ranking of the usability criteria, it will be tested with a larger number of university website users. As for future work, the DEMATEL method can be used to explain the dependency between the identified usability criteria which will further help designers understand their contribution to the overall usability of academic websites.

The contribution of this research study can be summarized as follows:
1. The proposed framework provides an evaluation tool to diagnose weak usability areas of academic websites, so designers, developers, and universities can use it to improve the experiences provided through their websites. 2. This framework will help designers understand the key usability criteria to consider when designing new or evaluating existing academic websites, which is more suitable compared to the general usability heuristic rules used currently. 3. Providing designers with the relative importance of the different usability criteria contributing to academic websites, will help them prioritize and make educated design decisions and tradeoffs between the criteria, which in real-life scenarios it can be challenging to address all these criteria as they can be conflicting.

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Appendix

Table 1. The 4 main criteria and 31 sub criteria

Trustworthiness WebC1	Content WebC2	Presentation & Interaction WebC3	Organization WebC4
Does the website convey a good and trustworthy image?	Is it what the user wants?	Is it easy to comprehend and can the user act on it?	Can the user find it?
Security & privacy	Updates	Load time	Design optimization
Confirmation	Personalization	Response time	Helpfulness
Reliability	Accuracy	Ease of use	Functionality
Web credibility	Use of technology and innovation	Systematic cues	Broken links
Traffic	Coverage	Memorability	Efficiency
	Readability & legibility	Services	Ease of navigation
	Availability	Attractiveness	
	Search engine friendliness	Interactivity	
	Informative	Accessibility	
		Markup validation	
		Learnability	

Table 2. The decision matrix by the first expert

	WebA1	WebA2	WebA3	WebA4	WebA5
WebC1	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.2,0.2,0.1,0.65,0.8,0.85,0.45,0.8,0.7	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2
WebC2	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6
WebC3	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05
WebC4	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05	0.2,0.2,0.1,0.65,0.8,0.85,0.45,0.8,0.7
WebC5	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.5,0.45,0.5,0.4,0.35,0.5,0.35,0.3,0.45
WebC6	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6
WebC7	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6
WebC8	0.5,0.45,0.5,0.4,0.35,0.5,0.35,0.3,0.45	0.2,0.2,0.1,0.65,0.8,0.85,0.45,0.8,0.7	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2
WebC9	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6
WebC10	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05
WebC11	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05
WebC12	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2
WebC13	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6
WebC14	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2
WebC15	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2
WebC16	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6
WebC17	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6
WebC18	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2
WebC19	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.2,0.2,0.1,0.65,0.8,0.85,0.45,0.8,0.7	0.2,0.2,0.1,0.65,0.8,0.85,0.45,0.8,0.7
WebC20	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2
WebC21	0.2,0.2,0.1,0.65,0.8,0.85,0.45,0.8,0.7	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6
WebC22	0.2,0.2,0.1,0.65,0.8,0.85,0.45,0.8,0.7	0.2,0.2,0.1,0.65,0.8,0.85,0.45,0.8,0.7	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05
WebC23	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.2,0.2,0.1,0.65,0.8,0.85,0.45,0.8,0.7
WebC24	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2
WebC25	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65
WebC26	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65
WebC27	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6
WebC28	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2
WebC29	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.2,0.2,0.1,0.65,0.8,0.85,0.45,0.8,0.7	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65
WebC30	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.2,0.2,0.1,0.65,0.8,0.85,0.45,0.8,0.7
WebC31	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2

Table 3. The decision matrix by the second expert

	WebA1	WebA2	WebA3	WebA4	WebA5
WebC1	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.2,0.2,0.1,0.65,0.8,0.85,0.45,0.8,0.7	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2
WebC2	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6
WebC3	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05
WebC4	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2
WebC5	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2
WebC6	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6
WebC7	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2
WebC8	0.5,0.45,0.5,0.4,0.35,0.5,0.35,0.3,0.45	0.2,0.2,0.1,0.65,0.8,0.85,0.45,0.8,0.7	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6
WebC9	0.5,0.45,0.5,0.4,0.35,0.5,0.35,0.3,0.45	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.5,0.45,0.5,0.4,0.35,0.5,0.35,0.3,0.45	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.5,0.45,0.5,0.4,0.35,0.5,0.35,0.3,0.45
WebC10	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2
WebC11	0.5,0.45,0.5,0.4,0.35,0.5,0.35,0.3,0.45	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.5,0.45,0.5,0.4,0.35,0.5,0.35,0.3,0.45	0.5,0.45,0.5,0.4,0.35,0.5,0.35,0.3,0.45
WebC12	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.5,0.45,0.5,0.4,0.35,0.5,0.35,0.3,0.45	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.5,0.45,0.5,0.4,0.35,0.5,0.35,0.3,0.45
WebC13	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65
WebC14	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2
WebC15	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65
WebC16	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6
WebC17	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2
WebC18	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2
WebC19	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05
WebC20	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65
WebC21	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65
WebC22	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.2,0.2,0.1,0.65,0.8,0.85,0.45,0.8,0.7	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65
WebC23	0.5,0.45,0.5,0.4,0.35,0.5,0.35,0.3,0.45	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.5,0.45,0.5,0.4,0.35,0.5,0.35,0.3,0.45	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.5,0.45,0.5,0.4,0.35,0.5,0.35,0.3,0.45
WebC24	0.5,0.45,0.5,0.4,0.35,0.5,0.35,0.3,0.45	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.5,0.45,0.5,0.4,0.35,0.5,0.35,0.3,0.45	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.5,0.45,0.5,0.4,0.35,0.5,0.35,0.3,0.45
WebC25	0.5,0.45,0.5,0.4,0.35,0.5,0.35,0.3,0.45	0.5,0.45,0.5,0.4,0.35,0.5,0.35,0.3,0.45	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.5,0.45,0.5,0.4,0.35,0.5,0.35,0.3,0.45	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65
WebC26	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.5,0.45,0.5,0.4,0.35,0.5,0.35,0.3,0.45	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.5,0.45,0.5,0.4,0.35,0.5,0.35,0.3,0.45	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65
WebC27	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2
WebC28	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6
WebC29	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6
WebC30	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2
WebC31	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2

Table4. The decision matrix by the third expert

	WebA1	WebA2	WebA3	WebA4	WebA5
WebC1	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.2,0.2,0.1,0.65,0.8,0.85,0.45,0.8,0.7	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2
WebC2	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6
WebC3	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.5,0.45,0.5,0.4,0.35,0.5,0.35,0.3,0.45
WebC4	0.5,0.45,0.5,0.4,0.35,0.5,0.35,0.3,0.45	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.5,0.45,0.5,0.4,0.35,0.5,0.35,0.3,0.45	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6
WebC5	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6
WebC6	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2
WebC7	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2
WebC8	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.2,0.2,0.1,0.65,0.8,0.85,0.45,0.8,0.7	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6
WebC9	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2
WebC10	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2
WebC11	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2
WebC12	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.5,0.45,0.5,0.4,0.35,0.5,0.35,0.3,0.45	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.5,0.45,0.5,0.4,0.35,0.5,0.35,0.3,0.45	0.5,0.45,0.5,0.4,0.35,0.5,0.35,0.3,0.45
WebC13	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65
WebC14	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.5,0.45,0.5,0.4,0.35,0.5,0.35,0.3,0.45	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65
WebC15	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05	0.5,0.45,0.5,0.4,0.35,0.5,0.35,0.3,0.45	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.5,0.45,0.5,0.4,0.35,0.5,0.35,0.3,0.45
WebC16	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65
WebC17	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6
WebC18	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2
WebC19	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.2,0.2,0.1,0.65,0.8,0.85,0.45,0.8,0.7	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65
WebC20	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65
WebC21	0.2,0.2,0.1,0.65,0.8,0.85,0.45,0.8,0.7	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65
WebC22	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.2,0.2,0.1,0.65,0.8,0.85,0.45,0.8,0.7	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65
WebC23	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65
WebC24	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2
WebC25	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2
WebC26	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65
WebC27	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2
WebC28	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6
WebC29	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.2,0.2,0.1,0.65,0.8,0.85,0.45,0.8,0.7	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05
WebC30	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6	0.35,0.35,0.1,0.5,0.75,0.8,0.5,0.75,0.65	0.4,0.3,0.35,0.5,0.45,0.6,0.45,0.4,0.6
WebC31	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.2,0.2,0.1,0.65,0.8,0.85,0.45,0.8,0.7	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2	0.95,0.9,0.95,0.1,0.1,0.05,0.05,0.05,0.05	0.7,0.75,0.8,0.15,0.2,0.25,0.1,0.15,0.2

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Failure analysis of pump piping system using DEMATEL SVN methodology

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Abstract: Piping systems tend to malfunction when a set parameter goes off the mark. The parameters which directly affect the structure and working of a pump piping system are known as critical factors. The processes influenced by these factors and consequently the underlying cause of failure have been thoroughly investigated through research. The interrelationship between them, however, remains a mystery. The ability of the plant supervisor or worker to combine their vast theoretical knowledge with actual data will be enhanced by exposing these cause-effect correlations among the problems commonly encountered in pump piping systems. The SVN DEMATEL (Decision-Making Trial-and-Evaluation Laboratory) approach is used in this study to determine the predominant causes of pump piping system failures. By considering a group of expert perspectives to create a cause-and-effect relationship diagram, the DEMATEL approach allows one to determine and assess the most significant element. SVN sets in DEMATEL, likewise, eliminate uncertainty when making conclusions concerning failure relationships from the judgements provided by experts [1]. The focus of the failure analysis was divided into six groups: selection of pump, design of pump, construction, operation/maintenance, piping errors and commissioning of the system. A total of 26 factors were identified and were assigned to a relevant group. Each factor was further categorized into four levels based on the degree of influence using the methodology presented in this study. It was found that "Temperature" has the highest degree of influence over the other criteria whereas criteria like "Pressure" and "Power Supply" tend to be influenced by other factors. The proposed SVN-DEMATEL method would be suitable for qualitative analysis of different industrial systems.

Keywords: DEMATEL model; Pump piping system; failure analysis; single-valued neutrosophic sets; linguistic variable.

1. Introduction

The decision-making trial and evaluation laboratory (DEMATEL) is a comprehensive analytical method for constructing a structural model indicating the causal relationships existent between complex factors [3]. The initial goal was to identify integrated solutions employing matrices and graphs to the fragmented and conflicting phenomena of world civilizations [4], because it is practical to see the structure of complicated causal interactions. The DEMATEL approach has become quite popular among diverse fields. It is based on digraphs, which may split the related components into two groups: cause and effect. Digraphs, or directed graphs, are more useful than directionless graphs which can show the directed links between subsystems. A digraph can be used to show a communication network or a dominance relationship between people [5]. The digraph depicts a basic notion of contextual relation among the system's elements, with the numeral indicating the degree of influence [6]. Hence, this method can convert the relationship between the causes and effects of factors into an intelligible structural model of the system.

The DEMATEL approach uses linguistic variables to perform a weighted analysis of the decision maker's opinion. It was conclusively shown that DEMATEL has a higher Spearman ranking coefficient in comparison to other multi criteria decision making (MCDM) analysis [7]. This implies that the relationship between two variables can be described using a monotonic function. This makes it more relevant to the industry as most of the parameters under the scope of analysis have a nonlinear connection and regressive in nature.

However, the linguistic terms pose a major setback. To begin with, linguistic terms are not ideally suited to provide an in-depth analysis and a judgement further on since the information provided is often vague and incomplete [8]. Because of this incomplete information, the judgements of the decision makers might be misconstrued. To deal with the ambiguities that come with such estimation, it's a good idea to transform these linguistic terms into fuzzy numbers. A linguistic variable contains unique values (linguistic values) that depict the form of phrases or sentences one can find in a natural language [9]. This is commonly known as the fuzzy set. The generalization of fuzzy set lead to the development of another important analysis tool called "neutrosophic set". To curtail its application to real life scientific problems, it was further developed to single values neutrosophic set (SVNS). Owing to the ease in application of SVNSs, they sets have been adapted in other scientific areas such as information technology, information system and decision support system for example, relational database systems, semantic web services, financial data set detection, new economy's growth, decline analysis and etc [10-15].

1.1. Literature Review

The following section summarizes the results and gaps identified by various other research work related to neutrosophic sets, DEMATEL and pipeline failure analysis. Previous work mainly focuses on the integration of DEMATEL with other MCDM analysis methods. However, SVNS were seldom included. There has been certain research work aimed at different applications which used SVNS, but it was still lacking in proficiency as quadrant analysis was not used. The table given below summarizes the recent research and the corresponding research gaps. Owing to these limitations, this paper aims to provide an integrated SVN-DEMATEL method to investigate pipeline failure.

The eight-step procedure of DEMATEL is followed wherein the linguistic variables are designated by truth, indeterminacy, and falsity membership [16].

Table 1. Literature Review Summary.

Authors	Method	Application	Research gap
Betty Chang [17]	Fuzzy DEMATEL method	Supply chain management (SCM)	Does not employ SVNs to state assumptions in the dataset– No quadrant analysis
GülçinBüyüközkan [18]	Integrated DEMATEL-ANP approach	Renewable energy resources	Integration of DEMATEL with fuzzy logic – No quadrant analysis
EmreAkyuz [19]	fuzzy DEMATEL method	Shipboard operations (Operational hazards)	Shows only cause-effect diagram – does not portray the basic concept of contextual relationship and strengths of influence among the elements or criteria.
Yuan-WeiDu [20]	Hierarchical DEMATEL method	Complex Systems	Integration of DEMTEL with other methods (fuzzy logic, SVNs) This paper establishes a method to approach complex problems with several factors.

1.2. Future Scope

The presented study can be used as a basis for ranking data based on their significance, which consecutively can be used to generate machine learning models for the system. The output of this method could be used to further train an AI model to make an informed decision about a certain process. This could be incorporated in adaptive control systems. On a much rudimentary level, the machine can be trained to display the most significant control parameters that can be controlled to reduce errors using which, a trained professional can make the decisions.

2. SVN-DEMATEL

The algorithm used in DEMATEL is the framework of the proposed SVN-DEMATEL. Instead of real numbers representing the linguistic variables as seen in the traditional methodology, SVNS are used to deal with uncertainty. Apart from the inclusion of SVNS, the proposed method also incorporates the relative importance of the decision-maker weight. Boran et al. [21] suggested a proportion equation to approximate the relative weights of each decision maker. This bears accurate and crisper computational results rather than using equal weights for all decision makers. The three memberships of the linguistic variables are re-defined into real numbers to aid mathematical calculations. Radwan and Fouda [22] proposed the concept of average as an equation which is employed in this proposed method. Unlike the original DEMATEL, this method rules out the need to find multiplicative inverse of a matrix. The establishment of four types of criteria, also known as quadrant analysis, is extended by this method using causal-effect diagram.

2.1 Proposed SVN-DEMATEL algorithm:

Step-1. Construction of direct-relation matrix

The critical factors relevant to the study are identified and group into a matrix X of size M x M where M is the number of criteria. This matrix depicts the interrelation between pairs of elements using a linguistic scale. Hence, a total of N^2 relations are obtained [23].

Step-2. Finding the relative weights of decision-makers.

The aggregated crisp matrix is formulated in consideration of the weight of each decision-maker's judgment. Based on the importance of each decision maker, an SVNN is assigned which is used to calculate the overall distinctive weights [24]. Because decision-makers' work experience and expertise fluctuate regularly, this is critical to the success of research analysis.

The linguistic variables used for relative importance weights of decision-makers and their respective SVNNs are shown in Table 2 [25]. If the SVNN for the kth expert's relative importance is $\lambda_k = (T_k, I_k, F_k)$, then the value of the relative weight for the kth expert can be calculated using the equation:

$$\lambda_k = \frac{T_k(x) + I_k(x)\big((T_k(x)/T_k(x) + F_k(x))\big)}{\sum_{k=1}^{l} T_k(x) + I_k(x)\big((T_k(x)/T_k(x) + F_k(x))\big)} \quad \text{------------(1)}$$

($\lambda_k \geq 0, \sum_{k=1}^{l} \lambda_k = 1$, l is the number of decision makers)

Sample Calculation:

Value of denominator $(A) = [0.9 + 0.1(0.9/1)] + [0.5 + 0.4(0.5/0.95)] + [0.35 + 0.6(0.35/1.05)] = 2.250526316$

For example, substituting the values of Linguistic Variable "Very important" and that of A

$$\lambda_{k1} = [0.9 + 0.1(0.9/1)]/A = \mathbf{0.4398971001}$$

Step-3. Construction of aggregated direct-relation matrix (AGDRM)

The opinions of each decision makers are compiled into initial direct relation matrices. By including the relative weights, they can be merged into a collective matrix that represent the overall correlations between the factors. Let the SVN given by the kth expert on the assessment of criterion i on j be $z_k\ (i,j)\ =\ (T_k\ (i,j), I_k\ (i,j), F_k\ (i,j))$ [20]. The procedure of Single Valued Neutrosophic Set Weighted Aggregation (SVNSWA) is used [21]. Note that x(i,j) represents the influence level of criterion i on j.

$$a_{ij}\ =\ SVNSWA\ (z_1\ [i,j], z_2\ [i,j], \dots, z_k\ [i,j]\ )$$

$$= \sum \lambda_k z_{ij}^k\ =\ [1 - \prod_{k=1}^{l}(1 - T_j)^{w_j}, \prod_{k=1}^{l}(I_j)^{w_j}, \prod_{k=1}^{l}(F_j)^{w_j}\ ] \qquad \text{------------(2)}$$

$$i\ =\ 1,2,3,4,\dots,m;\ j\ =\ 1,2,3,4,\dots,n,$$

Step-4.. Convert the SVNNs to real numbers

Using the following equation, convert the aggregated single neutrosophic relation matrix to a real number matrix:

$$P(z)\ =\ [3\ +\ T\ -\ 2I - F]\ /\ 4 \qquad \text{-------------(3)}$$

Step-5. Normalization of AGDRM

Normalized DRM (matrix N) is computed using the equation:

$$N\ =\ k\ \times\ R \qquad \text{-------------(4)}$$

where, $k = \min((1/max \sum_{j=1}^{n}|a_{ij}|\ ),\ (1/max \sum_{i=1}^{n}|a_{ij}|\ ))$, i, $j \in$ {1,2,3, . . ., n}
and R is the Aggregated DRM with real numbers. -------------(5)

Step-6.. The Total Relation Matrix, T, is computed using the equation below:

$$T\ =\ N\ (I\ -\ N)^{-1} \qquad \text{-------------(6)}$$

('I' is an identity matrix of rank M)

Step-7. Construct a causal diagram. Using the following equation, calculate H and V from TRM.

Given T,

$$T\ =\ \left[\ t_y\right]_{n\ X\ n}\quad i,j\ =\ 1,2,\dots.n, \textit{and using the following equation:}$$

$$\text{H}= \left[\sum_{i=1}^{n} t_y\right]_{l\ X\ n}\ =\ [t(j)]_{l\ X\ n}, \qquad \text{------------(7)}$$

$$\text{V}=\left[\sum_{i=1}^{n} t_y\right]_{n\ X\ l} = [t(i)]_{n\ X\ l} \qquad \text{------------(8)}$$

where H signifies the total number of rows in the matrix and V denotes the total number of columns. As a result, the values of $(H + V)$ and $(H - V)$ are computed separately in different columns. If (H-V) is positive and $(H + V)$ is large, a criterion is classified as a cause group. It's classified as an effect group if $(H - V)$ is negative and $(H + V)$ is small [28].

8. Segregation of criteria

The values of $(H+V)$ and $(H-V)$ are employed as Cartesian plane coordinates $(H+V, H-V)$ to divide the criteria into four classes [29].

The case of a pump piping system failure is presented in detail.

2.2 Criteria and Linguistic scale

Criteria that influence the pump piping system are identified and it is represented in the form of fish bone diagram shown in Figure 1. There are 26 such criteria based on:

Selection of pump: Working fluid (F1), Design pressure (F2), Temperature (F3), Geographical location (F4),

Construction: Misalignment (F5), Support placement (F6), Valve placement (F7), Pipe strain (F8), Coupling (F9),

Operation and Maintenance: Power overloading (F10), Instrument malfunction (F11), Pressure head (F12), Lubrication (F13), Temperature (F14),

Piping errors: Valve placement (F15), Temporary strainer (F16), Layout (F17), Pipe run (F18), Water hammering (F19),

Design of Pump: Working fluid (F20), Vibration damping (F21), Material of casing (F22), Pressure head (F23),

Commissioning: Nature of priming (F24), Power supply (F25) and Strainer clogging (F26).

Table 2 represents the five-point linguistic scale using which these criteria will be assessed. The pairwise comparison method is made use in judgments.

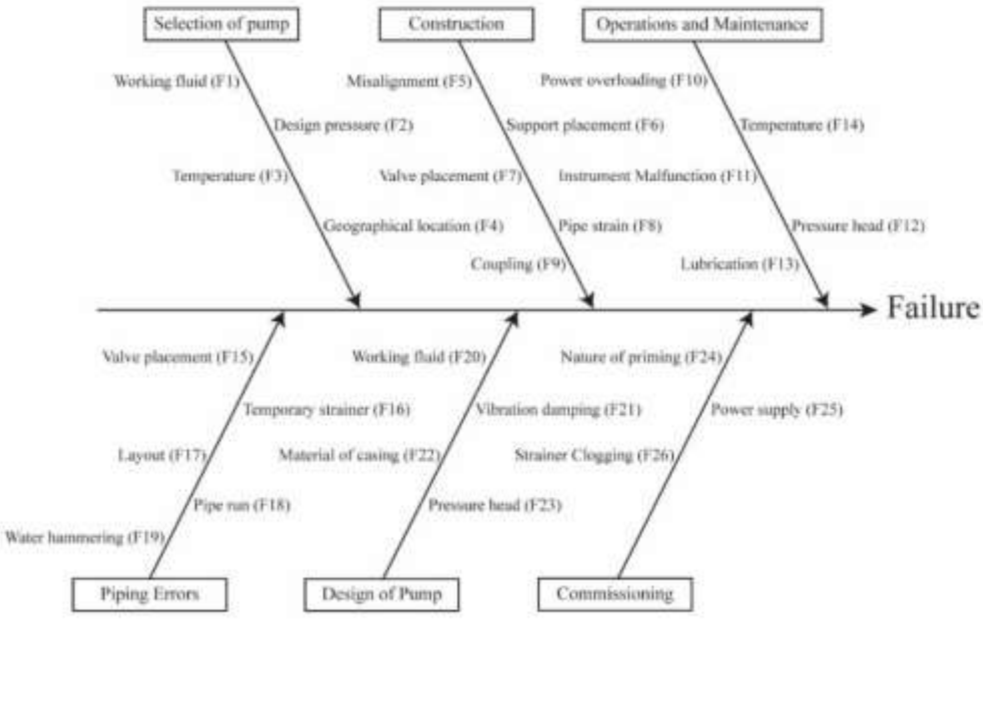


Fig. 1. Ishikawa Diagram of the Pump piping system failures.

Linguistic variable for relative importance	**SVNN {T, I, F }**
Very important (VI)	{0.90, 0.10, 0.10}
Important (I)	{0.80, 0.20, 0.15}
Medium (M)	{0.50, 0.40, 0.45}
Unimportant (UI)	{0.35, 0.60, 0.70}
Very unimportant (VUI)	{0.10, 0.80, 0.90}

Table 2 Linguistic Variable for relative importance of decision makers [19]

Linguistic terms for level of influence	**SVNS {T, I, F }**
None (1)	{0.00, 1.00, 1.00}
Low (2)	{0.20, 0.85, 0.80}
Medium (3)	{0.40, 0.65, 0.60}
High (4)	{0.60, 0.35, 0.40}
Very high (5)	{0.80, 0.15, 0.20}

Table 3 Linguistic Scale used in study [24]

2.3 Illustrations

Piping Engineer:

Piping engineers are engineering professionals who are responsible for the design of piping systems that transport fluids such as oil, gas, water, and waste from one location to another. Their work involves design, material selection, stress analysis and commissioning of piping systems.

Project Manager:

Project managers oversee planning, procurement, and execution of any activity with a defined scope, start, and end date.

Quality Engineer:

A quality engineer is a professional who manages and implements the quality assurance and control systems of a company. Piping engineers don't work independently but rather work as a team comprising of members from piping, mechanical, process instrumentation divisions. To ensure the smooth coordination within the team as well as suggest corrective measures, a quality engineer is crucial to the team.

Maintenance Engineer:

In industries, maintenance engineers oversee keeping equipment and machinery working smoothly. They are required to constantly upkeep the support equipment such as valves and FRL while keeping an eye on the pipe layout. Since long maintenance times could prove to be a costly affair to the company, maintenance engineers need to have a solid understanding about the system to perform quick actions when needed.

All decision-makers specialize in assessing pump piping system. The decision-makers were formally approached by letter to rank criteria based on the degree of influence over other factors utilizing the linguistic scale to evaluate pipeline failures. The data thus obtained from them are applied to the proposed SVNS-DEMATEL method.

2.4 Implementation

The following computations are carried out using the suggested algorithm:

Step - 1. Construction of Direct Relation Matrix for each individual decision-maker i.e., DM1, DM2 and DM3 give their judgment regarding the influence of the criteria on failure in pump piping systems. The following table shows the initial direct relation matrix i.e., judgments of one of the decision-maker. Linguistic terms are involved in the matrix to represent the correlations between the criteria as mentioned below.

Step -2. Finding the relative weights of decision-makers. λ, which represents the relative weights of decision-makers are computed using equation 1.

Table 4. Judgments of criteria (DM1)

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**25**	**26**
1	x	5	5	5	2	2	2	2	2	2	2	2	2	2	3	3	3	3	3	5	5	5	5	2	2	2
2	5	X	5	5	2	2	2	2	2	2	2	2	2	2	3	3	3	3	3	5	5	5	5	2	2	2
3	5	5	x	5	2	2	2	2	2	2	2	2	2	2	3	3	3	3	3	5	5	5	5	2	2	2
4	3	3	3	x	3	3	3	3	3	4	4	4	4	4	4	4	4	4	4	3	3	3	3	3	3	3
5	1	1	1	1	x	5	5	5	5	5	5	5	5	5	2	2	2	2	2	2	2	2	2	4	4	4
6	1	1	1	1	5	x	5	5	5	3	3	3	3	3	3	3	3	3	3	2	2	2	2	3	3	3
7	1	1	1	1	3	3	x	3	3	3	3	3	3	3	4	4	4	4	4	1	1	1	1	3	3	3
8	3	3	3	3	3	3	3	x	3	3	3	3	3	3	3	3	3	3	3	1	1	1	1	3	3	3
9	3	3	3	3	3	3	3	3	x	3	3	3	3	3	2	2	2	2	2	3	3	3	3	4	4	4
10	3	3	3	3	3	3	3	3	4	x	5	5	5	5	1	1	1	1	1	2	2	2	2	5	5	5
11	1	1	1	1	4	4	4	4	4	5	x	5	5	5	3	3	3	3	3	1	1	1	1	5	5	5
12	5	5	5	5	5	5	5	5	5	5	5	x	5	5	3	3	3	3	3	5	5	5	5	5	5	5
13	3	3	3	3	3	3	3	3	3	4	4	4	x	4	2	2	2	2	2	1	1	1	1	2	2	2
14	5	5	5	5	2	2	2	2	2	4	4	4	4	X	2	2	2	2	2	5	5	5	5	2	2	2
15	2	2	2	2	3	3	3	3	3	3	3	3	3	3	x	4	4	4	4	1	1	1	1	3	3	3
16	4	4	4	4	4	4	4	4	4	4	4	4	4	4	2	X	2	2	2	2	2	2	2	4	4	4
17	3	3	3	3	4	4	4	4	4	4	4	4	4	4	4	4	x	4	4	2	2	2	2	3	3	3
18	4	4	4	4	4	4	4	4	4	4	4	4	4	4	3	3	3	x	3	2	2	2	2	3	3	3
19	4	4	4	4	4	4	4	4	4	5	5	5	5	5	5	5	5	5	X	3	3	3	3	4	4	4
20	5	5	5	5	2	2	2	2	2	3	3	3	3	3	3	3	3	3	3	x	5	5	5	2	2	2
21	2	2	2	2	3	3	3	3	3	4	4	4	4	4	2	2	2	2	2	2	x	2	2	3	3	3
22	4	4	4	4	3	3	3	3	3	2	2	2	2	2	1	1	1	1	1	5	5	x	5	2	2	2
23	5	5	5	5	5	5	5	5	5	5	5	5	5	5	3	3	3	3	3	5	5	5	x	5	5	5
24	3	3	3	3	5	5	5	5	5	4	4	4	4	4	3	3	3	3	3	1	1	1	1	x	5	5
25	4	4	4	4	4	4	4	4	4	5	5	5	5	5	1	1	1	1	1	4	4	4	4	5	X	5
26	1	1	1	1	4	4	4	4	4	5	5	5	5	5	2	2	2	2	2	1	1	1	1	5	5	x

From Table 4, note that the diagonal elements are taken as x, purely for programming purposes only.

Step -**3. Construct the AGDRM using Equation (2)**

Table 5. Aggregated DRM

Criteria	F1	F2 -----------	F25	F26
F1	0.0000 1.0000 1.0000	0.6648 0.2932 0.3352	0.1998 0.8380 0.8002	0.5640 0.3958 0.4360
F2	0.6061 0.3359 0.3939	0.000 1.000 1.000	0.1998 0.8380 0.8002	0.1998 0.8380 0.8002
F3	0.7051 0.2412 0.2949	0.6648 0.2932 0.3352	0.1998 0.8380 0.8002	0.1998 0.8380 0.8002
F4	0.2949 0.7448 0.7051	0.2949 0.7448 0.7051	0.2949 0.7448 0.7051	0.2949 0.7448 0.7051
F5 . .	0.2006 0.7738 0.7994	0.2006 0.7738 0.7994	0.5583 0.4072 0.4417	0.5583 0.4072 0.4417
F22	0.5999 0.3501 0.4001	0.5999 0.3501 0.4001	0.2543 0.7961 0.7457	0.2543 0.7961 0.7457
F23	0.7630 0.1846 0.2370	0.7630 0.1846 0.2370	0.7383 0.2147 0.2617	0.7383 0.2147 0.2617
F24	0.3999 0.6501 0.6001	0.3999 0.6501 0.6001	0.7383 0.2147 0.2617	0.7383 0.2147 0.2617
F25	0.5583 0.4072 0.4417	0.5583 0.4072 0.4417	0.0000 1.0000 1.0000	0.7383 0.2147 0.2617
F26	0.1173 0.9001 0.8827	0.1173 0.9001 0.8827	0.7383 0.2147 0.2617	0.0000 1.0000 1.0000

Step -4. Construction of DRM with real numbers.

Using equation (3), the aggregated neutrosophic matrix is translated into a DRM with real numbers. The matrices are tabulated as below [25].

Table 6. DRM with real numbers

	F1	F2	F3........................	F25	F26
F1	0.00	0.6	0.59	0.33	0.55

F2	0.58	0.00	0.58	0.33	0.33
F3 . .	0.63	0.6	0.00	0.33	0.33
F25	0.55	0.55	0.55	0.00	0.64
F26	0.3	0.3	0.3	0.64	0.00

Step -5. Construct normalized DRM.

To begin with, the summation of rows and columns of DRM is made to construct the normalized.

Table 7. Summation of Rows and Columns

Criteria	Row Sum	Column Sum
F1	11.00	23.68
F2	11.12	23.76
F3 . .	11.25	23.94
F24	12.21	23.50
F25	13.13	23.58
F26	11.38	24.06

The maximum number derived from the summation of rows and the summation of columns is determined. Then, equation (5) is used to compute k using these maximum numbers.

By multiplying the aggregated DRM by the value of k, the DRM is normalised (equation (4)). The normalised DRM with real values is represented as a 26 x 26 matrix as shown in the table below.

Table 8. Normalized DRM

	F1	F2	F3 ………	F25	F26
F1	0.00	0.04	0.04	0.02	0.04
F2	0.04	0.00	0.04	0.02	0.02
F3 . .	0.04	0.04	0.00	0.03	0.03
F25	0.04	0.04	0.04	0.00	0.04
F26	0.02	0.02	0.02	0.04	0.00

Step -6. Obtaining total-relation matrix (TRM)

Equation (6) is used to compute TRM. TRM is calculated by multiplying the inverse of DRM with the difference of identity matrix and DRM The results are tabulated as follows:

Table 9. Total Relation Matrix

	F1	F2	F3	F21	F22
F1	0.137452	0.17820384	0.17853866	0.1590871	0.17605889

F2	0.17801681	0.13949323	0.17950661	0.1603709	0.16320226
F3 . .	0.18218722	0.18071384	0.14142555	0.16118724	0.16408375
F21	0.19875082	0.19925168	0.2004106	0.1620496	0.20802736
F22	0.16159569	0.16198612	0.16307218	0.18533314	0.14491719

Step -7. Plot casual diagram.

The sum of rows (H) and the sum of columns (V) are calculated to obtain the Cause-and-Effect diagram. $H+V$ and $H-V$ are calculated using these two summations. The results can be acquired by translating the $(H+V, H-V)$ data set onto the cartesian plane. Table 5 gives the calculated performance of criteria.

Table 5 Performance of the Criteria

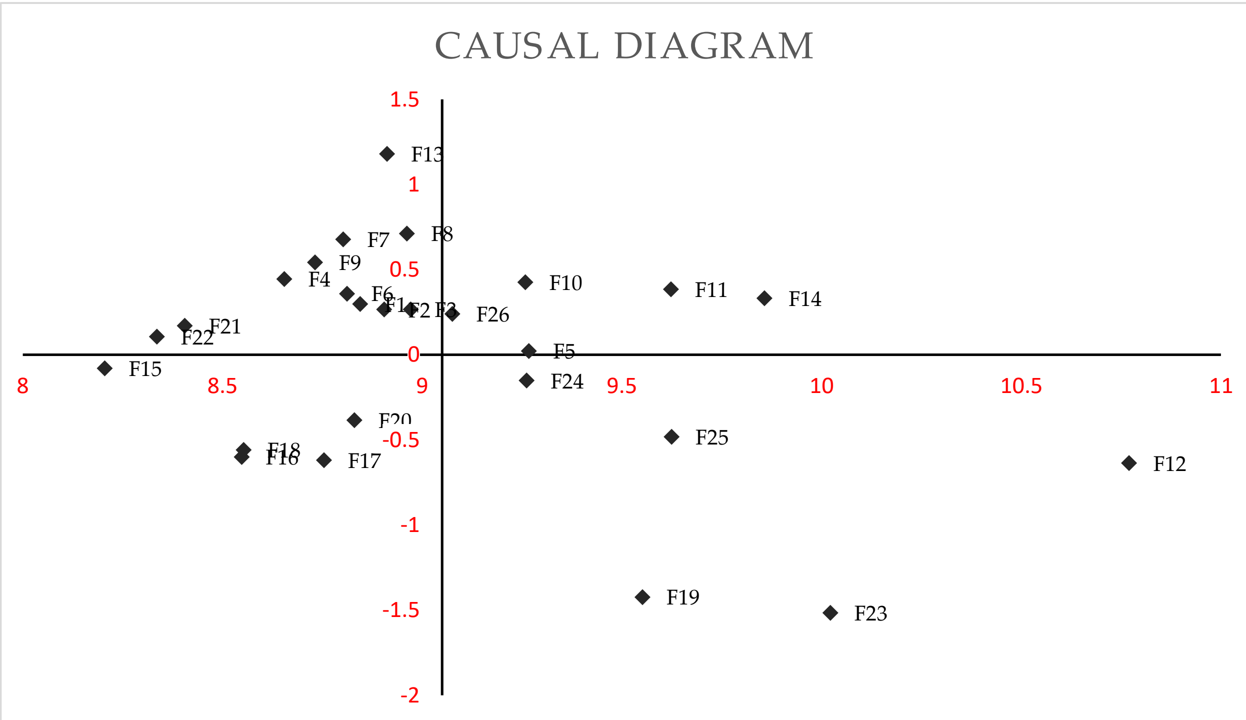
Summation of Rows of TRM [H]	Summation of Columns of TRM [V]	[H+V]	[H-V]
4.273448111482863	4.571708319732535	8.845156431215397	0.2982602082496717
4.319454924785043	4.585681466062835	8.905136390847877	0.266226541277792
4.35176372479797	4.619417399098856	8.971181123896827	0.2676536743008855
4.104556749171913	4.550125169053278	8.65468191822519	0.445568419881365
4.622340474030425	4.645100601369393	9.267441075399818	0.022760127338967706
4.226040326244855	4.586126600106721	8.812166926351576	0.36008627386186554
4.061933426960683	4.740230474655422	8.802163901616105	0.6782970476947394
4.124326337331055	4.837887343076521	8.962213680407576	0.7135610057454667
4.094015191818948	4.63753832340008	8.731553515219028	0.5435231315811322
4.416001893528742	4.842298184588803	9.258300078117546	0.42629629106006117
4.619182296562277	5.003907315135014	9.623089611697292	0.38472501857273667
5.7028723886376484	5.0671875895313025	10.770059978168952	-0.6356847991063459
3.865264956267218	5.046966406364441	8.912231362631658	1.181701450097223
4.761954782723879	5.09534063462179	9.857295417345668	0.33338585189791115
4.141719045979212	4.063762855590824	8.205481901570035	-0.07795619038838808
4.573492093775418	3.9744198005428304	8.547911894318249	-0.599072293232588

4.68620928200716	4.067925222137253	8.754134504144414	-0.6182840598699064
4.555388693643281	3.9973845247800885	8.55277321842337	-0.5580041688631927
5.486932197999035	4.064814466974208	9.551746664973244	-1.4221177310248274
4.606972932704049	4.223800692369891	8.83077362507394	-0.383172240334158
4.116863397367269	4.288953433924168	8.405816831291437	0.1720900365568987
4.114405090712399	4.22123009801596	8.335635188728359	0.10682500730356104
5.768493768672305	4.253444220034323	10.021937988706629	-1.5150495486379816
4.70633151280972	4.555342307719526	9.261673820529246	-0.15098920509019376
5.053124279140549	4.571215860265098	9.624340139405646	-0.48190841887545144
4.417188040006047	4.658466610008801	9.075654650014847	0.24127857000275377

3. Result and Discussion

The $H + V$ and $H - V$ values are plotted into a casual diagram. The Fig. 2 shows the casual diagram between the cause- and-effect group of criteria, being separated by the $H + V$ axis

Fig. 2 Casual diagram for Criteria



The causal diagram helps us in visualizing the cause criteria and the effect criteria. The cause criteria are Working fluid**,** Design pressure, Temperature, Geographical location, Misalignment, Support placement, Pipe strain, Coupling, Power overloading, Instrument malfunction, Lubrication, Temperature, Vibration damping, Material of casing, Strainer clogging **as their H-V values is positive. The effect criteria** Pressure head, Valve placement, Temporary strainer, Layout, Pipe run, Water hammering, working fluid, Pressure head, Nature of priming, Power supply **as their H-V values are negative.**

3.1 Segregation of criteria.

The coordinates of $(H+V, H-V)$ can be used to better analyze the figure. There are four different sorts of criterion. Based on the coordinates of $(H+V, H-V)$, all criteria in this study may be divided into four quadrants. There are four basic types of instances. Fig. 3 shows the criteria details in quadrant analysis. Note that the value of $(H+V)$ is taken as large or small in comparison to the mean of all the values (9.0523)

Case(i): When $(H + V)$ is large and $(H + V)$ is positive, the first kind occurs. This suggests that the factors are cause criteria, as well as a driving factor to resolve critical issues. Hence, the criterion "Temperature" is the most governing element on other factors.

Case(ii): When $H-V$ is positive and $H+V$ is small. It demonstrates that factors are self-contained and can only impact a small number of others. In the selection of the factor for studying the pump piping system failure, the criterion "Material of casing" is a stand-alone criterion that has no bearing on other factors.

Case(iii): When $H-V$ is negative and $H+V$ is large. It shows that the factors turn out to be an effect criterion, which can be enhanced. The factor "Pressure head" is an effect criterion which is highly contingent to other factors.

Case(iv): When $H-V$ is negative and $H+V$ is small. It demonstrates that the factors are self-contained and are scarcely affected by other factors. In our case, "Valve placement" is considered as an independent criterion.

From the above cases, we can conclude that "Temperature" and "Pressure head" are the most important factors which have to be taken into consideration while making pump piping system failure analysis.

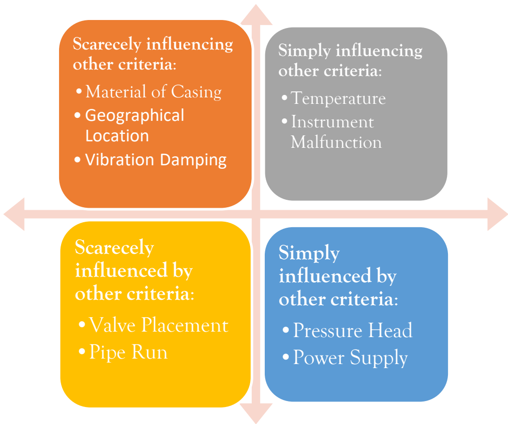


Fig.3 Quadrant Analysis

4. Conclusion

The DEMATEL method has proved itself to be a powerful tool in the industry ever since it was first proposed. It is applicable to almost all engineering systems and provides comprehensible results which can be used to improve the system. A study indicated that DEMATEL is most used in conjunction with ANP. This is followed by the integrated Fuzzy-DEMATEL method which has a major advantage in situations dealing with uncertainty. The present study proposes an incorporation of neutrosophic sets into the classical DEMATEL to understand the fundamentals of the application in a much more sophisticated manner. More specifically, Single Valued Neutrosophic (SVN) sets are used which has the advantage of its three membership functions to tackle indeterminacy. However, the method is not without limitations. Despite the incorporation of neutrosophic sets to reduce 'vagueness', the opinions of the decision makers may vary based on their mood, judgement, and accuracy of perception. The reliability of the input data needs to be verified before proceeding with the analysis. In this paper, the proposed method is utilized in assessing the piping failure where a total of 26 critical factors were identified. The extensive review of criteria by the Truth, Indeterminacy, and Falsity memberships of SVNS successfully divides these criteria into two groups: cause and effect. The results indicate that the criterion "Temperature" is the most important cause in influencing other factors that must be studied during the pump piping system failure. This is in line with the available theoretical knowledge. Temperature change creates expansion or contraction which can lead to thermal stresses thus altering the entire working conditions of the piping system. Higher temperatures can also imply increased corrosion rate. The criterion "Pressure" on the other hand, is highly influenced by changes in other factors. This is a particularly useful detail since by placing tighter control mechanisms on the pressure head, the efficiency of the entire system can be directly improved. The results also show that even though factors like material, pipe layout and valve placement can directly contribute to the failure of the system, their inter-relationship with other factors are negligibly low and need not be prioritised over others. The procedure implemented in this paper can be successfully applied to various other systems and obtain intelligible results despite any level of uncertainty involved.

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Generalized Neutrosophic Sampling Strategy for Elevated estimation of Population Mean

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Abstract: One of the disadvantages of the point estimate in survey sampling is that it fluctuates from sample to sample due to sampling error, as the estimator only provides a point value for the parameter under discussion. The neutrosophic approach, pioneered by Florentin Smarandache, is an excellent tool for estimating the parameters under consideration in sampling theory since it yields interval estimates in which the parameter lies with a very high probability. As a result, the neutrosophic technique, which is a generalization of classical approach, is used to deal with ambiguous, indeterminate, and uncertain data. In this investigation, we suggest a new general family of ratio and exponential ratio type estimators for the elevated estimation of neutrosophic population mean of the primary variable utilizing known neutrosophic auxiliary parameters. For the first degree approximation, the bias and Mean Squared Error (MSE) of the suggested estimators are computed. The neutrosophic optimum values of the characterizing constants are determined, as well as the minimum value of the neutrosophic MSE of the suggested estimator is obtained for these optimum values of the characterizing scalars. Because the minimum MSE of the classical estimators of population mean lies inside the estimated interval of the neutrosophic estimators, the neutrosophic estimators are better than the equivalent classical estimators. The empirical investigation, which used both real and simulated data sets, backs up the theoretical findings. For practical utility in various areas of applications, the estimator with the lowest MSE or highest Percentage Relative Efficiency (PRE) is recommended.

Keywords: Classical Ratio Estimators, Neutrosophic Estimators, Bias, MSE, PRE, Simulation.

1. Introduction

Due to time and financial constraints, sampling becomes unavoidable when the population is big. The most apt estimator for the parameter under consideration is the corresponding statistic and so is the sample mean ($\bar{y}$) for the population mean ($\bar{Y}$) of main variable Y . Although $\bar{y}$ is an unbiased estimator of $\bar{Y}$, its sampling variance is rather high, hence the sampling distribution of $\bar{y}$ will not be very close to the genuine $\bar{Y}$. Therefore, we look for a population mean estimator that is even biased yet has a sampling distribution closer to the true $\bar{Y}$. The employment of an auxiliary variable (X) having a high degree of positive or negative association with $\bar{Y}$ achieves the goal of finding efficient estimators. The use of supplementary information to elevate the effectiveness of the estimators of the parameters under consideration is well established in sampling theory. For elevated estimation of $\bar{Y}$ using positively and negatively correlated auxiliary information with main variable, respectively, ratio and product technique of estimation processes are utilized with the condition that the line of regression pass through origin. If the line does not cross through the origin, the regression method of estimation is favored above the ratio and product approaches. The ratio method is preferred in real-world applications due to its broad applicability; for example, area and production in crop yield applications, income and investment in business and economics, hospital infrastructure and health are some examples of applications, where ratio estimators are used to estimate $\overline{Y}$. As a result, the current research focuses on estimating $\overline{Y}$ using known positively associated auxiliary variable.

1.2. Estimation under Classical Sampling Theory

In estimation methods of classical sampling theory, the data utilized for elevated estimation of $\overline{Y}$ using ratio, product, or regression type estimators are known and produced by crisp numbers. In classical statistics, various authors worked on numerous estimators of $\overline{Y}$ in the presence of known X and suggested various ratio type estimators. In classical sampling theory, [1] introduced the conventional ratio estimator of $\overline{Y}$ using the positively correlated X . As an auxiliary parameter, he utilized the known population mean ($\overline{X}$) of X . Various authors later used well-known auxiliary parameters like coefficient of variation (CV), coefficient of skewness, coefficient of kurtosis, standard deviation, quartiles, and so on to improve the estimation of $\overline{Y}$. [2] worked on a modified ratio estimator of $\overline{Y}$ utilizing the known CV of X . For the elevated estimation of $\overline{Y}$, [3] proposed the exponential ratio estimator employing a known X . [4] proposed two ratio estimators for more efficient estimation of $\overline{Y}$, utilizing known coefficient of kurtosis and the CV of X . [5] focused on improving $\overline{Y}$ estimate utilizing known population correlation coefficient between Y and X , and their results outperformed rival estimators. For increased estimate of $\overline{Y}$, [6] suggested the modifications on ratio estimator of $\overline{Y}$, that makes the use of known coefficient of kurtosis of X . [7] proposed several modified ratio estimators of $\overline{Y}$ based on known information on some well-known auxiliary parameters. [8] suggested two ratio type estimators of $\overline{Y}$ utilizing known skewness and kurtosis of X , which outperformed rival estimators. [9] presented an increased estimation approach for population mean using auxiliary parameters on characteristic. [10] worked in the direction of improving a family of ratio and product estimators of $\overline{Y}$ with known parameters of X and [11] worked on a generic family of estimators of $\overline{Y}$ using transformed X . [12] proposed a generalized family of dual to ratio-cum-product $\overline{Y}$ estimators with known auxiliary parameters. [13] developed a new ratio estimator for $\overline{Y}$ utilizing linear transformation of X as minimum and maximum values. Using auxiliary parameters, [14] provided several efficient estimators for $\overline{Y}$. [15] introduced a new family of $\overline{Y}$ estimators based on the main variable's known population median and shown improvement over the estimators in competition. [16] proposed a new modified ratio type estimator based on an auxiliary variable's exponential parameter. [17] proposed an improved family of $\overline{Y}$ estimators utilizing known parameters of Y and X for improving the efficiency of the estimators, [18] used some well-known traditional and non-traditional auxiliary parameters. Many more authors have attempted to improve $\overline{Y}$ estimation using known data on traditional and non-traditional, robust and non-robust auxiliary parameters in classical sampling theory.

1.3. Estimation under Neutrosophic Sampling Theory

The data in classical sampling theory is mostly deterministic with no uncertainty in the measurements of the observations for the characteristics under investigation, however, we frequently encounter difficulties in everyday life where the data for the attributes under examination are not determined, for instance the measurement of temperature at any place along with other applications including information technology, information systems, decision support systems, financial data set detection, new economy growth, decline analysis, and more. In such cases, we seek alternate ways for dealing with undetermined data, and the fuzzy logic pioneered by Prof. Lofti A. Zadeh in 1965 gives a solution for dealing with such data when exact measurements of the variable under examination are unavailable. Although fuzzy statistics deals with ambiguous, unclear, or imprecise data, it does not take into account the indeterminacy measurements. Neutosophic logic, further, is a generalized fuzzy logic that measures indeterminacy together with the determinate component of the observations and is utilized to analyze when the observations are imprecise or ambiguous, [19, 20]. [21] utilized the fuzzy logic in decision making for more precise decisions. Later different procedures using fuzzy logic have been developed and utilized extensively for making decisions in different areas of applications, [22-26]. [27] mentioned that the complex fuzzy sets are the advanced fuzzy sets and its generalization is the complex neutrosophic set. [28] suggested a diagram of fuzzy sets along with the generalizations of the sets and utilized the interval-valued neutrosophic sets for making decisions.

According to [29], Neutosophic statistics are used when data has some indeterminacy. Neutrosophic statistics is the extended form of classical statistics and are applied when the observations in the population or sample are imprecise, indeterminate, or vague. Further he mentioned that the methods of Neutrosophic statistics are utilized to analyze Neutrosophic data, which is indeterminate to some degree and the sample size may not be an exact number. In their works, [30] and [19] argue that neutrosophic statistics are particularly useful and acceptable for use in the system with the uncertainty. [31] used neutrosophic statistics to analyze the effect on scale and anisotropy for neutrosophic numbers of rock joint roughness coefficient. [20] focused on a Neutrosophic analysis of variance for university student data. [32] used a neutrosophic soft matrix (NSM) and relative weights of experts to develop an algorithmic strategy for group decision making (GDM) challenges. [33] used neutrosophic statistics to examine data from diabetes patients who had undergone a new diagnosis test. [34] worked on the estimation of the ratio of a crisp variable and a neutrosophic variable and shown improvement over the classical ratio method of estimation. [35] employed NEWMA chart and recurrent sampling to monitor road traffic crashes using neutrosophic statistics and in his research, [36] used neutrosophic statistics to develop a new goodness of fit test utilizing unclear parameters. In a study of skewness and kurtosis estimators of wind speed distributions under indeterminacy, [37] employed neutrosophic statistics. [38] devised a decision-making approach for determining the best fit of those damages in a neutrosophic environment, with the badly damaged machine receiving preference. [39] developed several new single-valued neutrosophic graph (SVNG) concepts, stating that the fuzzy set and the neutrosophic set are two effective instruments for dealing with the uncertainties and ambiguity of any real-world scenario.

When dealing with the uncertainties of a real-life scenario, the neutrosophic set outperforms the fuzzy set. [40] used neutrosophic parameterized hypersoft set theory to develop a decision-making application. They first conceptualized the neutrosophic parameterized hypersoft set, as well as some of its basic features and operations, and then used this theory to construct a decision-making-based method. For both one and two sample hypothesis testing situations, [41] suggested a modified Sign test that takes into account the indeterminate condition and true data form. They evaluated the suggested improved Sign test using two real data sets: covid-19 reproduction rate and covid-positive daily cases in ICU in Pakistan, and found that the suggested methodologies are appropriate for the problems of nonparametric in decision-making involving interval-valued data. To handle medical diagnostics and decision-making difficulties, [42] worked on algorithms for a generalization of multipolar neutrosophic soft set with measures of information. They proposed a general multipolar neutrosophic soft set, complete with operations and fundamental features. Later, they extended it to tackle decision-making problems by introducing various information measures for the generalization of multipolar neutrosophic soft set, such as distance, similarity, and correlation coefficient. [43] mentioned that in traditional survey sample studies where data is definite, certain, and unambiguous, the estimates are a single valued crisp results that may be incorrect, overestimated, or understated, which might be a disadvantage. There are a variety of scenarios where data is neutrosophic in nature, and this is when Neutrosophic statistics is used instead of traditional approaches.

Uncertain and ambiguous values of the variables, non-clear contentions, and imprecise interval values are examples of neutrosophic data. As a result, data from trials or populations may be interval-valued neutrosophic numbers. The factual observation, that was ambiguous at the time of collecting, was thought to be a value within that range. There are more indeterminate data than definite data available in real life. As a result, more statistical techniques that are neutrosophic are needed. In real life, there are so many research variables that gathering information is quite costly, especially when the information is confusing. Thus, using traditional methods for indeterminate data to determine the unknown real value of the parameter will be dangerous and costly. After a thorough review of the literature, no study in sample surveys for ratio method of estimation for $\bar{Y}$ utilizing known X under neutrosophic data has been found. There are not enough promising articles in this subject of statistics yet. There was no available solution to tackle the issue using ratio estimation when Y and X were neutrosophic in nature. As a result, [43] presented a neutrosophic ratio-type estimation approach as the initial step in this direction. Further [43] mentioned that Neutrosophic Statistical analysis aids in the study of data with a degree of indeterminacy or insufficient knowledge, as well as conflicting beliefs. For the problem of indeterminacy, traditional statistics unsucceeded to analyze the data since certain observations were presented in a range of unknown values with the possibility of including a factual measurement within that range. As a result, in an uncertain environment, neutrosophic statistics is used, which is a more flexible alternative to and generalization of classical statistics. There have been numerous studies in the field of sample surveys under the Neutrosophy, where the method of ratio estimation is still new and necessitates a great deal of attention to the uncertain data system. For instance, the measurements of a machine product such as nuts or bolts may have slight measurement or manufacturing errors, and we may accept such product if it falls within the specified measurement range. Marks in grade system and health parameters through different testing procedures may be the areas of applications where neutrosophic statistics may be a better choice than the traditional one. Thus it is clear that in many situations, discussed above, the Neutrosophic estimators are used for improved estimation of population mean over the classical estimators where the observations of the study variable are not deterministic rather these are nondeterministic.

In this investigative work, we suggest a novel generalized neutrosophic ratio estimator for enhanced estimation of $\bar{Y}$ utilizing the known parameters of X . The sampling properties of the suggested estimator are studied for the first degree of approximation. The complete manuscript is being presented in different sections from introduction to the references.

1.4. Observations in Neutrosophic Environment and Notations

Quantitative neutrosophic data, where a number may lie in an uncertain interval [a, b], is one sort of observation in the neutrosophic environment, [30]. Neutosophic numbers' interval value can be represented in a variety of ways. [43] have defined neutrosophic interval values as $Z_N = Z_L + Z_U I_N$, where, $I_N \in [I_L, I_U]$. We also use the same notations of [43] for the considered neutrosophic data, which are in the interval form as $Z_N \in [Z_L, Z_U]$, where Z_L and Z_L are the lower and upper values of the neutrosophic variable Z_N. Let the neutrosophic population consists of N distinct units $(P_1, P_2, ..., P_N)$ and a neutrosophic random sample of size $n_N \in [n_L, n_U]$ is taken from the above population using simple random sampling without replacement (srswor) technique. Let $y_N(i)$ be the observation on the ith unit of the sample for the neutrosophic data under consideration for the main variable y_N, of the form $y_N(i) \in [y_L, y_U]$ and by the same way for the auxiliary variable $x_N(i) \in [x_L, x_U]$. Let $\bar{y}_N(i) \in [\bar{y}_L, \bar{y}_U]$ be the sample mean for the neutrosophic study variable y_N and $\bar{x}_N(i) \in [\bar{x}_L, \bar{x}_U]$ be sample mean for the neutrosophic x_N which is correlated with y_N. Further let $\bar{Y}_N \in [\bar{Y}_L, \bar{Y}_U]$ and $\bar{X}_N \in [\bar{X}_L, \bar{X}_U]$ be the population means for the neutrosophic variables y_N and x_N respectively, which are the overall averages of the neutrosophic data set. The neutrosophic coefficients of variation of y_N and x_N are given as $C_{yN} \in [C_{yNL}, C_{yNU}]$ and $C_{xN} \in [C_{xNL}, C_{xNU}]$ respectively. The correlation coefficient between the neutrosophic variables y_N and x_N is represented as $\rho_{yxN} \in [\rho_{yxNL}, \rho_{yxNU}]$. The neutrosophic coefficients of skewness and kurtosis for x_N are given by $\beta_{1(x)N} \in [\beta_{1(x)NL}, \beta_{1(x)NU}]$ and $\beta_{2(x)N} \in [\beta_{2(x)NL}, \beta_{2(x)NU}]$ respectively. The neutrosophic quartiles of x_N are given by $Q_{iN} \in [Q_{iNL}, Q_{iNU}]$, $i = 1, 3$ and the neutrosophic median of auxiliary variable as $M_{dN} \in [M_{dNL}, M_{dNU}]$.

1.5. Flow Chart of the Study

The graph given below represents the flow chart of the suggested study using neutrosophic numbers. The following chart is a recreated flow chart suggested by [43].

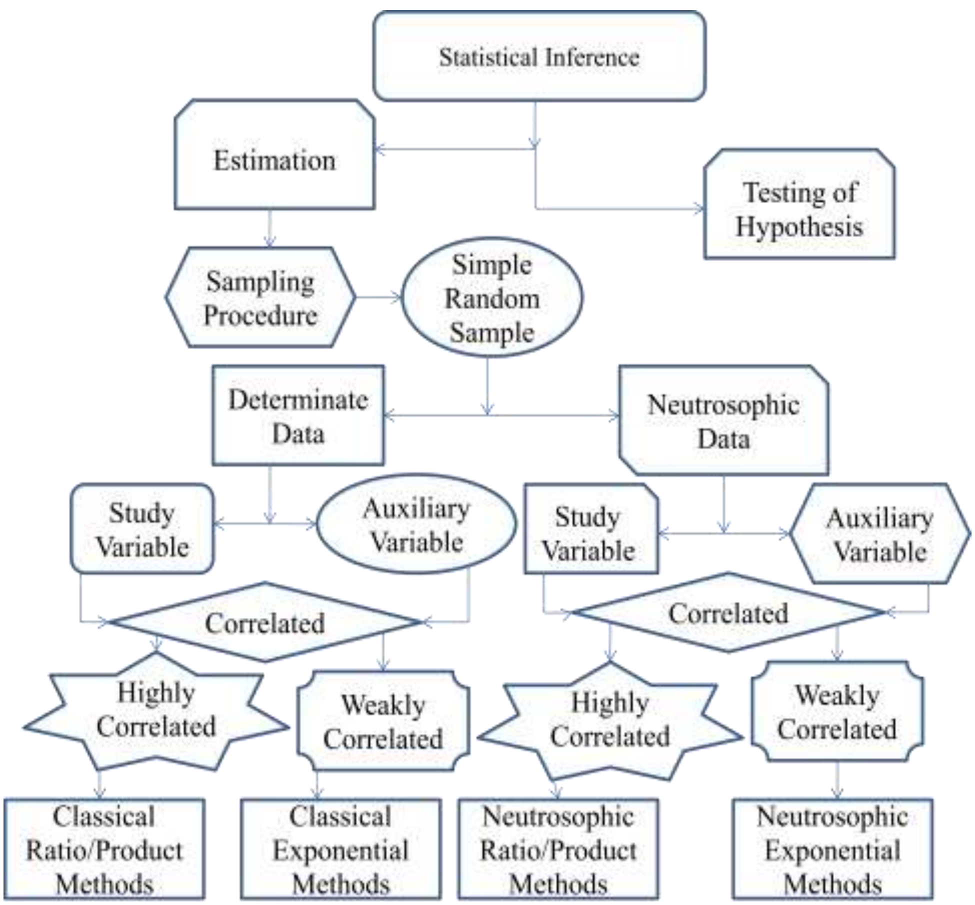

Figure-1: Flow chart of the study

1.6. Standard Approximations

Following are some standard approximations used for the sampling properties of the neutrosophic estimators, suggested by [43] as,

Let $\bar{e}_{yN} \in [\bar{e}_{yL}, \bar{e}_{yU}]$ and $\bar{e}_{xN} \in [\bar{e}_{xL}, \bar{e}_{xU}]$ be the mean errors for the study and the auxiliary neutrosophic variables along with $\bar{e}_{yN}(i) = \bar{y}_N(i) - \bar{Y}_N$ and $\bar{e}_{xN}(i) = \bar{x}_N(i) - \bar{X}_N$ respectively. The expectations of these errors for different orders are defined as;

$E(\bar{e}_{yN}) = E(\bar{e}_{xN}) = 0$ and,

$E(\bar{e}_{yN}^2) = \theta_N \bar{Y}_N^2 C_{yN}^2$, $E(\bar{e}_{xN}^2) = \theta_N \bar{X}_N^2 C_{xN}^2$, $E(\bar{e}_{yN}\bar{e}_{xN}) = \theta_N \bar{X}_N \bar{Y}_N \rho_{yxN} C_{yN} C_{xN}$

Where,

$\bar{e}_{yN} \in [\bar{e}_{yL}, \bar{e}_{yU}]$, $\bar{e}_{xN} \in [\bar{e}_{xL}, \bar{e}_{xU}]$, $\bar{e}_{yN}\bar{e}_{xN} \in [\bar{e}_{yL}\bar{e}_{xL}, \bar{e}_{yU}\bar{e}_{xU}]$, $\bar{e}_{yN}^2 \in [\bar{e}_{yL}^2, \bar{e}_{yU}^2]$, $\bar{e}_{xN}^2 \in [\bar{e}_{xL}^2, \bar{e}_{xU}^2]$,

$C_{xN}^2 = \dfrac{\sigma_{xN}^2}{\bar{X}_N^2}$, $C_{yN}^2 = \dfrac{\sigma_{yN}^2}{\bar{Y}_N^2}$, $C_{xN}^2 \in [C_{xL}^2, C_{xU}^2]$, $C_{yN}^2 \in [C_{yL}^2, C_{yU}^2]$, $\rho_{yxN} = \dfrac{\sigma_{yxN}}{\sigma_{yN}\sigma_{xN}}$, $\rho_{yxN} \in [\rho_{yxL}, \rho_{yxU}]$,

$\theta_N = \dfrac{1 - f_N}{n_N}$, $\theta_N \in [\theta_L, \theta_U]$, $n_N \in [n_L, n_U]$, $\sigma_{xN}^2 \in [\sigma_{xL}^2, \sigma_{xU}^2]$, $\sigma_{yN}^2 \in [\sigma_{yL}^2, \sigma_{yU}^2]$, $\sigma_{yxN} \in [\sigma_{yxL}, \sigma_{yxU}]$

On the basis of the errors of the neutrosophic variables, bias and the Mean Squared Error (MSE) of the introduced and the competing estimators are obtained for an approximation of order one. The Bias and the MSE in neutrosophic environment are defined as, $Bias(\bar{y}_N) \in [Bias_L, Bias_U]$ and $MSE(\bar{y}_N) \in [MSE_L, MSE_U]$. Further the correlated auxiliary variables are used for the elevated estimation of $\bar{Y}_N$ and neutrosophic ratio type estimators are applied when there is indeterminacy in the data.

1.7. Review of Neutrosophic Estimators

The most appropriate neutrosophic estimator for the neutrosophic $\bar{Y}_N$ of Y is the corresponding neutrosophic sample mean and is given by,

$t_0 = \bar{y}_N$

The variance of the neutrosophic sample mean for the first degree of approximation is,

$$V(t_0) = \theta_N \bar{Y}_N^2 C_{yN}^2 \tag{1}$$

Where, $t_{0N} \in [t_{0L}, t_{0U}]$

Using [1], [43] suggested the usual neutrosophic ratio estimator of $\bar{Y}_N$ using the known neutrosophic population mean of X as,

$$t_{RN} = \bar{y}_N \left(\frac{\bar{X}_N}{\bar{x}_N} \right)$$

The bias and MSE of the neutrosophic ratio estimator t_{RN}, for an approximation of degree one respectively are,

$\mathrm{Bias}(t_{RN}) = \theta_N \bar{Y}_N [C_{xN}^2 - C_{yxN}]$, where, $C_{yxN} = \rho_{yxN} C_{yN} C_{xN}$

$$\mathrm{MSE}(t_{RN}) = \theta_N \bar{Y}_N^2 [C_{yN}^2 + C_{xN}^2 - 2C_{yxN}] \tag{2}$$

Where, $t_{RN} \in [t_{RL}, t_{RU}]$

Motivated by [2], [43] suggested the following neutrosophic ratio estimator using CV of neutrosophic variable X as,

$$t_{1N} = \bar{y}_N \left(\frac{\bar{X}_N + C_{xN}}{\bar{x}_N + C_{xN}} \right)$$

The bias and MSE of the neutrosophic ratio estimator t_{1N}, for an approximation of order one respectively are,

$$\text{Bias}(t_{1N}) = \theta_N \bar{Y}_N \left[\lambda_{1N}^2 C_{xN}^2 - \lambda_{1N} C_{yxN} \right]$$

$$\text{MSE}(t_{1N}) = \theta_N \bar{Y}_N^2 \left[C_{yN}^2 + \lambda_{1N}^2 C_{xN}^2 - 2\lambda_{1N} C_{yxN} \right] \tag{3}$$

Where, $\lambda_{1N} = \dfrac{\bar{X}_N}{\bar{X}_N + C_{xN}}$ and $t_{1N} \in [t_{1L}, t_{1U}]$

Based on [3], [43] proposed the following neutrosophic exponential ratio estimator as,

$$t_{2N} = \bar{y}_N \exp\left(\frac{\bar{X}_N - \bar{x}_N}{\bar{X}_N + \bar{x}_N} \right)$$

The bias and MSE of the neutrosophic exponential ratio estimator t_{2N}, for an approximation of order one respectively are,

$$\text{Bias}(t_{2N}) = \theta_N \bar{Y}_N \left[\frac{3}{8} C_{xN}^2 - \frac{1}{2} C_{yxN} \right]$$

$$\text{MSE}(t_{2N}) = \theta_N \bar{Y}_N^2 \left[C_{yN}^2 + \frac{C_{xN}^2}{4} - C_{yxN} \right] \tag{4}$$

Where, $t_{2N} \in [t_{2L}, t_{2U}]$

Motivated by [4], the two neutrosophic ratio estimators using CV and coefficient of kurtosis of X may be given as,

$$t_{3N} = \bar{y}_N \left(\frac{C_{xN} \bar{X}_N + \beta_{2(x)N}}{C_{xN} \bar{x}_N + \beta_{2(x)N}} \right)$$

$$t_{4N} = \bar{y}_N \left(\frac{\beta_{2(x)N} \bar{X}_N + C_{xN}}{\beta_{2(x)N} \bar{x}_N + C_{xN}} \right)$$

The biases and MSEs of the neutrosophic ratio estimators t_{3N} and t_{4N}, for an approximation of order one respectively are,

$$\text{Bias}(t_{3N}) = \theta_N \bar{Y}_N \left[\lambda_{3N}^2 C_{xN}^2 - \lambda_{3N} C_{yxN} \right]$$

$$\text{Bias}(t_{4N}) = \theta_N \bar{Y}_N \left[\lambda_{4N}^2 C_{xN}^2 - \lambda_{4N} C_{yxN} \right]$$

$$\text{MSE}(t_{3N}) = \theta_N \bar{Y}_N^2 \left[C_{yN}^2 + \lambda_{3N}^2 C_{xN}^2 - 2\lambda_{3N} C_{yxN} \right] \tag{5}$$

$$\text{MSE}(t_{4N}) = \theta_N \bar{Y}_N^2 \left[C_{yN}^2 + \lambda_{4N}^2 C_{xN}^2 - 2\lambda_{4N} C_{yxN} \right] \tag{6}$$

Where, $\lambda_{3N} = \dfrac{C_{xN} \bar{X}_N}{C_{xN} \bar{X}_N + \beta_{2(x)N}}$, $\lambda_{4N} = \dfrac{\beta_{2(x)N} \bar{X}_N}{\beta_{2(x)N} \bar{X}_N + C_{xN}}$ and $t_{3N} \in [t_{3L}, t_{3U}]$, $t_{4N} \in [t_{4L}, t_{4U}]$

Motivated by [5], the neutrosophic ratio estimator t_{5N}, using known population coefficient of correlation may be given as,

$$t_{5N} = \bar{y}_N \left(\frac{\bar{X}_N + \rho_{yxN}}{\bar{x}_N + \rho_{yxN}} \right)$$

The bias and MSE of the neutrosophic ratio estimator t_{5N}, for an approximation of order one respectively are,

$$\text{Bias}(t_{5N}) = \theta_N \bar{Y}_N \left[\lambda_{5N}^2 C_{xN}^2 - \lambda_{5N} C_{yxN} \right]$$

$$\text{MSE}(t_{5N}) = \theta_N \bar{Y}_N^2 \left[C_{yN}^2 + \lambda_{5N}^2 C_{xN}^2 - 2\lambda_{5N} C_{yxN} \right] \tag{7}$$

Where, $\lambda_{5N} = \dfrac{\bar{X}_N}{\bar{X}_N + \rho_{yxN}}$ and $t_{5N} \in \left[t_{5L}, t_{5U} \right]$

[43] suggested the following neutrosophic ratio estimator by adapting the estimator by [6], using coefficient of kurtosis of X as,

$$t_{6N} = \bar{y}_N \left(\frac{\bar{X}_N + \beta_{2(x)N}}{\bar{x}_N + \beta_{2(x)N}} \right)$$

The bias and MSE of the neutrosophic ratio estimator t_{6N}, for an approximation of order one respectively are,

$$\text{Bias}(t_{6N}) = \theta_N \bar{Y}_N \left[\lambda_{6N}^2 C_{xN}^2 - \lambda_{6N} C_{yxN} \right]$$

$$\text{MSE}(t_{6N}) = \theta_N \bar{Y}_N^2 \left[C_{yN}^2 + \lambda_{6N}^2 C_{xN}^2 - 2\lambda_{6N} C_{yxN} \right] \tag{8}$$

Where, $\lambda_{6N} = \dfrac{\bar{X}_N}{\bar{X}_N + \beta_{2(x)N}}$ and $t_{6N} \in \left[t_{6L}, t_{6U} \right]$

Motivated by [44], the two neutrosophic ratio estimators using first and third quartiles of X may be given as,

$$t_{7N} = \bar{y}_N \left(\frac{\bar{X}_N + Q_{1N}}{\bar{x}_N + Q_{1N}} \right)$$

$$t_{8N} = \bar{y}_N \left(\frac{\bar{X}_N + Q_{3N}}{\bar{x}_N + Q_{3N}} \right)$$

The biases and MSEs of the neutrosophic ratio estimators t_{7N} and t_{8N}, for an approximation of order one respectively are,

$$\text{Bias}(t_{7N}) = \theta_N \bar{Y}_N \left[\lambda_{7N}^2 C_{xN}^2 - \lambda_{7N} C_{yxN} \right]$$

$$\text{Bias}(t_{8N}) = \theta_N \bar{Y}_N \left[\lambda_{8N}^2 C_{xN}^2 - \lambda_{8N} C_{yxN} \right]$$

$$\text{MSE}(t_{7N}) = \theta_N \bar{Y}_N^2 \left[C_{yN}^2 + \lambda_{7N}^2 C_{xN}^2 - 2\lambda_{7N} C_{yxN} \right] \tag{9}$$

$$\text{MSE}(t_{8N}) = \theta_N \bar{Y}_N^2 \left[C_{yN}^2 + \lambda_{8N}^2 C_{xN}^2 - 2\lambda_{8N} C_{yxN} \right] \tag{10}$$

Where, $\lambda_{7N} = \dfrac{\bar{X}_N}{\bar{X}_N + Q_{1N}}$, $\lambda_{8N} = \dfrac{\bar{X}_N}{\bar{X}_N + Q_{3N}}$ and $t_{7N} \in \left[t_{7L}, t_{7U} \right]$, $t_{8N} \in \left[t_{8L}, t_{8U} \right]$

Motivated by [8], the two neutrosophic ratio estimators using coefficients of skewness and kurtosis of X may be represented as,

$$t_{9N}=\bar{y}_N\left(\frac{\bar{X}_N+\beta_{1(x)N}}{\bar{x}_N+\beta_{1(x)N}}\right)$$

$$t_{10N}=\bar{y}_N\left(\frac{\beta_{1(x)N}\bar{X}_N+\beta_{2(x)N}}{\beta_{1(x)N}\bar{x}_N+\beta_{2(x)N}}\right)$$

The biases and MSEs of the neutrosophic ratio estimators t_{9N} and t_{10N}, for an approximation of order one respectively are,

$$\text{Bias}(t_{9N})=\theta_N\bar{Y}_N\left[\lambda_{9N}^2C_{xN}^2-\lambda_{9N}C_{yxN}\right]$$

$$\text{Bias}(t_{10N})=\theta_N\bar{Y}_N\left[\lambda_{10N}^2C_{xN}^2-\lambda_{10N}C_{yxN}\right]$$

$$MSE(t_{9N})=\theta_N\bar{Y}_N^2\left[C_{yN}^2+\lambda_{9N}^2C_{xN}^2-2\lambda_{9N}C_{yxN}\right] \quad (11)$$

$$MSE(t_{10N})=\theta_N\bar{Y}_N^2\left[C_{yN}^2+\lambda_{10N}^2C_{xN}^2-2\lambda_{10N}C_{yxN}\right] \quad (12)$$

Where, $\lambda_{9N}=\dfrac{\bar{X}_N}{\bar{X}_N+\beta_{1(x)N}}$, $\lambda_{10N}=\dfrac{\beta_{1(x)N}\bar{X}_N}{\beta_{1(x)N}\bar{X}_N+\beta_{2(x)N}}$ and $t_{9N}\in[t_{9L},t_{9U}]$, $t_{10N}\in[t_{10L},t_{10U}]$

Motivated by [45], the two neutrosophic ratio estimators using median and coefficients of variation of X, we may define as,

$$t_{11N}=\bar{y}_N\left(\frac{\bar{X}_N+M_{d(x)N}}{\bar{x}_N+M_{d(x)N}}\right)$$

$$t_{12N}=\bar{y}_N\left(\frac{C_{xN}\bar{X}_N+M_{d(x)N}}{C_{xN}\bar{x}_N+M_{d(x)N}}\right)$$

The biases and MSEs of the neutrosophic ratio estimators t_{11N} and t_{12N}, for an approximation of order one respectively are,

$$\text{Bias}(t_{11N})=\theta_N\bar{Y}_N\left[\lambda_{11N}^2C_{xN}^2-\lambda_{11N}C_{yxN}\right]$$

$$\text{Bias}(t_{12N})=\theta_N\bar{Y}_N\left[\lambda_{12N}^2C_{xN}^2-\lambda_{12N}C_{yxN}\right]$$

$$MSE(t_{11N})=\theta_N\bar{Y}_N^2\left[C_{yN}^2+\lambda_{11N}^2C_{xN}^2-2\lambda_{11N}C_{yxN}\right] \quad (13)$$

$$MSE(t_{12N})=\theta_N\bar{Y}_N^2\left[C_{yN}^2+\lambda_{12N}^2C_{xN}^2-2\lambda_{12N}C_{yxN}\right] \quad (14)$$

Where, $\lambda_{11N}=\dfrac{\bar{X}_N}{\bar{X}_N+M_{d(x)N}}$, $\lambda_{12N}=\dfrac{C_{xN}\bar{X}_N}{C_{xN}\bar{X}_N+M_{d(x)N}}$ and $t_{1N}\in[t_{11L},t_{11U}]$, $t_{12N}\in[t_{12L},t_{12U}]$

Motivated by [46], the neutrosophic ratio estimator t_{13N}, using known population coefficient of correlation may be given as,

$$t_{13N}=\bar{y}_N\left(\frac{\bar{X}_N+n_N}{\bar{x}_N+n_N}\right)$$

The bias and MSE of the neutrosophic ratio estimator t_{13N}, for an approximation of order one respectively are,

$$\text{Bias}(t_{13N}) = \theta_N \bar{Y}_N \left[\lambda_{13N}^2 C_{xN}^2 - \lambda_{13N} C_{yxN}\right]$$

$$MSE(t_{13N}) = \theta_N \bar{Y}_N^2 \left[C_{yN}^2 + \lambda_{13N}^2 C_{xN}^2 - 2\lambda_{13N} C_{yxN}\right] \tag{15}$$

Where, $\lambda_{13N} = \dfrac{\bar{X}_N}{\bar{X}_N + n_N}$ and $t_{13N} \in [t_{13L}, t_{13U}]$

Motivated by [47], [43] suggested the following neutrosophic modified exponential ratio estimator as,

$$t_{14N} = \bar{y}_N \exp\left(\frac{(a\bar{X}_N + b) - (a\bar{x}_N + b)}{(a\bar{X}_N + b) + (a\bar{x}_N + b)}\right)$$

where, a and b are the neutrosophic auxiliary parameters.

The bias and MSE of the neutrosophic exponential ratio estimator t_{14N}, for an approximation of order one respectively are,

$$\text{Bias}(t_{14N}) = \theta_N \bar{Y}_N \left[\frac{3}{8}\lambda_{14N}^2 C_{xN}^2 - \frac{1}{2}\lambda_{13N} C_{yxN}\right]$$

$$MSE(t_{14N}) = \theta_N \bar{Y}_N^2 \left[C_{yN}^2 + \lambda_{14N}^2 C_{xN}^2 - 2\lambda_{14N} C_{yxN}\right] \tag{16}$$

Where, $\lambda_{14N} = \dfrac{a\bar{X}_N}{2(a\bar{X}_N + b)}$ and $t_{14N} \in [t_{14L}, t_{14U}]$

Motivated by [48], [43] proposed the following generalized neutrosophic exponential ratio estimator as,

$$t_{15N} = \bar{y}_N \exp\left[\alpha\left(\frac{\bar{X}_N^{\frac{1}{h}} - \bar{x}_N^{\frac{1}{h}}}{\bar{X}_N^{\frac{1}{h}} + (a-1)\bar{x}_N^{\frac{1}{h}}}\right)\right]$$

where, α and h are the real known constants with $-\infty < \alpha < \infty$ and $h > 0$. The characterizing scalar a ($a \neq 0$) is determined so that the MSE of t_{15N} is minimum.

The bias and MSE of the neutrosophic generalized exponential ratio estimator t_{15N}, for an approximation of order one respectively are,

$$\text{Bias}(t_{15N}) = \theta_N \bar{Y}_N \left[\frac{\alpha C_{xN}^2}{ah^2} - \frac{\alpha C_{xN}^2}{a^2h^2} + \frac{\alpha^2 C_{xN}^2}{2a^2h^2} - \frac{\alpha C_{yxN}}{ah}\right]$$

$$MSE(t_{15N}) = \theta_N \bar{Y}_N^2 \left[C_{yN}^2 + \frac{\alpha^2 C_{xN}^2}{a^2h^2} - \frac{2\alpha C_{yxN}}{ah}\right] \tag{17}$$

The optimum value of the characterizing constant a is obtained by minimizing $MSE(t_{15N})$ and the optimum value is,

$$a_{opt} = \frac{\alpha C_{xN}^2}{h C_{yxN}} \tag{18}$$

The minimum value of the $MSE(t_{15N})$ for the optimum value of a_{opt} is,

$$MSE_{\min}(t_{15N}) = \theta_N \bar{Y}_N^2 C_{yN}^2 (1 - \rho_{yxN}^2) \tag{19}$$

2. Material and Methods

Motivated by [49], we suggest a ratio cum exponential ratio class of neutrosophic main variable using the neutrosophic auxiliary parameters as,

$$t_{pN}=\kappa_1\bar{y}_N\left(\frac{a\bar{X}_N+b}{a\bar{x}_N+b}\right)+\kappa_2\bar{y}_N\exp\left(\frac{(a\bar{X}_N+b)-(a\bar{x}_N+b)}{(a\bar{X}_N+b)+(a\bar{x}_N+b)}\right)$$

Where, κ_1 and κ_2 are the characterizing scalars to be determine such that the MSE of t_{pN} is minimum. It is worth notable that,

(i) If $\kappa_2=0$, then the introduced estimator t_{pN} reduces to [49] ratio type estimators having different estimators by different authors as its special cases.

(ii) If $\kappa_2=0$ and $\kappa_1=1$, the introduced estimator t_{pN} reduces to ratio type estimators having different estimators by different authors as its special cases.

(iii) If $\kappa_1=0$, then the suggested class of estimators t_{pN} reduces to [49] exponential ratio type estimators having different estimators by different authors as its special cases.

(iv) If $\kappa_1=0$ and $\kappa_2=1$, the suggested family of estimators t_{pN} reduces to exponential ratio type estimators having different estimators by different authors as its special cases.

Expressing the introduced estimator in terms of $\bar{e}_{yN}$ and $\bar{e}_{xN}$, we have

$$t_{pN}=\kappa_1\bar{Y}_N(1+\bar{e}_{yN})\left(\frac{a\bar{X}_N+b}{a\bar{X}_N(1+\bar{e}_{xN})+b}\right)+\kappa_2\bar{Y}_N(1+\bar{e}_{yN})\exp\left(\frac{(a\bar{X}_N+b)-(a\bar{X}_N(1+\bar{e}_{xN})+b)}{(a\bar{X}_N+b)+(a\bar{X}_N(1+\bar{e}_{xN})+b)}\right)$$

$$=\bar{Y}_N(1+\bar{e}_{yN})\left[\kappa_1\left(\frac{a\bar{X}_N+b}{a\bar{X}_N(1+\bar{e}_{xN})+b}\right)+\kappa_2\exp\left(\frac{(a\bar{X}_N+b)-(a\bar{X}_N(1+\bar{e}_{xN})+b)}{(a\bar{X}_N+b)+(a\bar{X}_N(1+\bar{e}_{xN})+b)}\right)\right]$$

$$=\bar{Y}_N(1+\bar{e}_{yN})\left[\kappa_1(1+\lambda\bar{e}_{xN})^{-1}+\kappa_2\exp\left(-\frac{\lambda\bar{e}_{xN}}{2}(1+\frac{\lambda\bar{e}_{xN}}{2})^{-1}\right)\right]$$

Expanding the terms on the right hand side and simplifying and retaining the terms for the first degree of approximation, we get

$$t_{pN}=\bar{Y}_N\left\{\kappa_1(1+\bar{e}_{yN}-\lambda\bar{e}_{xN}-\lambda\bar{e}_{yN}\bar{e}_{xN}+\lambda^2\bar{e}_{xN}^2)+\kappa_2(1+\bar{e}_{yN}-\frac{\lambda\bar{e}_{xN}}{2}-\frac{\lambda\bar{e}_{yN}\bar{e}_{xN}}{2}+\frac{3}{8}\lambda^2\bar{e}_{xN}^2)\right\}$$

Subtracting $\bar{Y}_N$ on both sides of the above equation, we have

$$t_{pN}-\bar{Y}_N=\bar{Y}_N\left\{\kappa_1(1+\bar{e}_{yN}-\lambda\bar{e}_{xN}-\lambda\bar{e}_{yN}\bar{e}_{xN}+\lambda^2\bar{e}_{xN}^2)+\kappa_2(1+\bar{e}_{yN}-\frac{\lambda\bar{e}_{xN}}{2}-\frac{\lambda\bar{e}_{yN}\bar{e}_{xN}}{2}+\frac{3}{8}\lambda^2\bar{e}_{xN}^2)-1\right\} \quad (20)$$

Taking expectations on both sides of (20) and putting values of different expectations, we get the bias of t_{pN} as,

$$Bias(t_{pN})=\bar{Y}_N\left\{\kappa_1(1-\lambda\theta_N\bar{Y}_N\bar{X}_NC_{yxN}+\lambda^2\theta_N\bar{X}_N^2C_{xN}^2)+\kappa_2(1-\frac{\lambda}{2}\theta_N\bar{Y}_N\bar{X}_NC_{yxN}+\frac{3}{8}\lambda^2\theta_N\bar{X}_N^2C_{xN}^2)-1\right\} \quad (21)$$

Squaring on both sides of (20), simplifying for the first degree of approximation, we get

$$(t_{pN}-\bar{Y}_N)^2=\bar{Y}_N^2\left\{\begin{array}{l}1+\kappa_1^2(1+\bar{e}_{yN}^2+3\lambda^2\bar{e}_{xN}^2-4\lambda\bar{e}_{yN}\bar{e}_{xN})+\kappa_2^2(1+\bar{e}_{yN}^2+\lambda^2\bar{e}_{xN}^2-2\lambda\bar{e}_{yN}\bar{e}_{xN})\\-2\kappa_1(1+\lambda^2\bar{e}_{xN}^2-\lambda\bar{e}_{yN}\bar{e}_{xN})-2\kappa_2(1+\frac{3}{8}\lambda^2\bar{e}_{xN}^2-\frac{\lambda}{2}\bar{e}_{yN}\bar{e}_{xN})\\+2\kappa_1\kappa_2(1+\bar{e}_{yN}^2+\frac{15}{8}\lambda^2\bar{e}_{xN}^2-3\lambda\bar{e}_{yN}\bar{e}_{xN})\end{array}\right\}$$

Putting values of different expectations after taking expectation on both sides, we get the MSE of t_{pN} as,

$$MSE(t_{pN}) = \bar{Y}_N^2 \left\{ \begin{array}{l} 1+\kappa_1^2(1+\theta_N \bar{Y}_N^2 C_{yN}^2 + 3\lambda^2\theta_N \bar{X}_N^2 C_{xN}^2 - 4\lambda\theta_N \bar{Y}_N \bar{X}_N C_{yxN}) \\ +\kappa_2^2(1+\theta_N \bar{Y}_N^2 C_{yN}^2 + \lambda^2\theta_N \bar{X}_N^2 C_{xN}^2 - 2\lambda\theta_N \bar{Y}_N \bar{X}_N C_{yxN}) \\ -2\kappa_1(1+\lambda^2\theta_N \bar{X}_N^2 C_{xN}^2 - \lambda\theta_N \bar{Y}_N \bar{X}_N C_{yxN}) \\ -2\kappa_2(1+\frac{3}{8}\lambda^2\theta_N \bar{X}_N^2 C_{xN}^2 - \frac{\lambda}{2}\theta_N \bar{Y}_N \bar{X}_N C_{yxN}) \\ +2\kappa_1\kappa_2(1+\theta_N \bar{Y}_N^2 C_{yN}^2 + \frac{15}{8}\lambda^2\theta_N \bar{X}_N^2 C_{xN}^2 - 3\lambda\theta_N \bar{Y}_N \bar{X}_N C_{yxN}) \end{array} \right\} \tag{22}$$

$$MSE(t_{pN}) = \bar{Y}_N^2 \left[1 + A\kappa_1^2 + B\kappa_2^2 - 2C\kappa_1 - 2D\kappa_2 + 2F\kappa_1\kappa_2 \right] \tag{23}$$

Where,

$$A = (1+\theta_N \bar{Y}_N^2 C_{yN}^2 + 3\lambda^2\theta_N \bar{X}_N^2 C_{xN}^2 - 4\lambda\theta_N \bar{Y}_N \bar{X}_N C_{yxN})$$

$$B = (1+\theta_N \bar{Y}_N^2 C_{yN}^2 + \lambda^2\theta_N \bar{X}_N^2 C_{xN}^2 - 2\lambda\theta_N \bar{Y}_N \bar{X}_N C_{yxN})$$

$$C = (1+\lambda^2\theta_N \bar{X}_N^2 C_{xN}^2 - \lambda\theta_N \bar{Y}_N \bar{X}_N C_{yxN})$$

$$D = (1+\frac{3}{8}\lambda^2\theta_N \bar{X}_N^2 C_{xN}^2 - \frac{\lambda}{2}\theta_N \bar{Y}_N \bar{X}_N C_{yxN})$$

$$F = (1+\theta_N \bar{Y}_N^2 C_{yN}^2 + \frac{15}{8}\lambda^2\theta_N \bar{X}_N^2 C_{xN}^2 - 3\lambda\theta_N \bar{Y}_N \bar{X}_N C_{yxN})$$

The optimum values of the characterizing constants κ_1 and κ_2 which minimizes the MSE of the suggested estimator t_{pN} respectively are,

$\kappa_{1(opt)} = \dfrac{(DF - BC)}{(F^2 - AB)}$ and $\kappa_{2(opt)} = \dfrac{(CF - AD)}{(F^2 - AB)}$

The minimum value of the MSE of t_{pN} for these optimum values of κ_1 and κ_2 is,

$$MSE_{\min}(t_{pN}) = \bar{Y}_N^2 \left\{ 1 - \frac{\begin{bmatrix} 2C(DF-BC)(F^2-AB) + 2D(CF-AD)(F^2-AB) \\ -2F(DF-BC)(CF-AD) - (DF-BC) - (CF-AD) \end{bmatrix}}{(F^2-AB)^2} \right\} \tag{24}$$

$$MSE_{\min}(t_{pN}) = \bar{Y}_N^2 \left\{ 1 - \frac{P}{Q} \right\} \tag{25}$$

Where,

$$P = \begin{bmatrix} 2C(DF-BC)(F^2-AB) + 2D(CF-AD)(F^2-AB) \\ -2F(DF-BC)(CF-AD) - (DF-BC) - (CF-AD) \end{bmatrix}$$

$$Q = (F^2 - AB)^2$$

3. Theoretical Efficiency Comparison

Under this section, we have compared the introduced neutrosophic estimator with the competing neutrosophic estimators of $\overline{Y}$ using the neutrosophic auxiliary parameters. The efficiency of the introduced estimator has been compared in terms of MSEs and the efficiency condition of the introduced estimator to be more efficient than the competing one is obtained.

The suggested estimator t_{pN} is more efficient than t_0 for the condition if,

$V(t_0) - MSE_{\min}(t_{pN}) > 0$ or,

$$\theta_N C_{yN}^2 - \left\{1 - \frac{P}{Q}\right\} > 0$$

The introduced estimator t_{pN} has lesser MSE than estimator t_{RN} for the following condition.

$MSE(t_{RN}) - MSE_{\min}(t_{pN}) > 0$ or,

$$\theta_N \left[C_{yN}^2 + C_{xN}^2 - 2C_{yxN}\right] - \left\{1 - \frac{P}{Q}\right\} > 0$$

The suggested estimator t_{pN} is better than the estimator t_{1N} by [43] under the restriction if,

$MSE(t_{1N}) - MSE_{\min}(t_{pN}) > 0$ or,

$$\theta_N \left[C_{yN}^2 + \lambda_{1N}^2 C_{xN}^2 - 2\lambda_{1N} C_{yxN}\right] - \left\{1 - \frac{P}{Q}\right\} > 0$$

The suggested estimator t_{pN} is better than the exponential ratio type estimator t_{2N} by [43] for the condition if,

$MSE(t_{2N}) - MSE_{\min}(t_{pN}) > 0$ or,

$$\theta_N \left[C_{yN}^2 + \frac{C_{xN}^2}{4} - C_{yxN}\right] - \left\{1 - \frac{P}{Q}\right\} > 0$$

The introduced estimator t_{pN} performs better than the estimator t_{3N} if,

$MSE(t_{3N}) - MSE_{\min}(t_{pN}) > 0$ or,

$$\theta_N \left[C_{yN}^2 + \lambda_{3N}^2 C_{xN}^2 - 2\lambda_{3N} C_{yxN}\right] - \left\{1 - \frac{P}{Q}\right\} > 0$$

The introduced estimator t_{pN} has lesser MSE than the estimator t_{4N} if it satisfies,

$MSE(t_{4N}) - MSE_{\min}(t_{pN}) > 0$ or,

$$\theta_N \left[C_{yN}^2 + \lambda_{4N}^2 C_{xN}^2 - 2\lambda_{4N} C_{yxN}\right] - \left\{1 - \frac{P}{Q}\right\} > 0$$

The proposed estimator t_{pN} is better than the estimator t_{5N} if,

$MSE(t_{5N}) - MSE_{\min}(t_{pN}) > 0$ or,

$$\theta_N \left[C_{yN}^2 + \lambda_{5N}^2 C_{xN}^2 - 2\lambda_{5N} C_{yxN}\right] - \left\{1 - \frac{P}{Q}\right\} > 0$$

The suggested estimator t_{pN} performs better than the ratio estimator t_{6N} by [43] if,

$$\mathrm{M}SE(t_{6N}) - MSE_{\min}(t_{pN}) > 0 \text{ or,}$$

$$\theta_N \circledcirc C_{yN}^2 + \lambda_{6N}^2 C_{xN}^2 - 2\lambda_{6N} C_{yxN} \circledcirc - \left\{1 - \frac{P}{Q}\right\} > 0$$

The introduced estimator t_{pN} is better than the ratio estimator t_{7N} under the condition if,

$$\mathrm{M}SE(t_{7N}) - MSE_{\min}(t_{pN}) > 0 \text{ or,}$$

$$\theta_N \circledcirc C_{yN}^2 + \lambda_{7N}^2 C_{xN}^2 - 2\lambda_{7N} C_{yxN} \circledcirc - \left\{1 - \frac{P}{Q}\right\} > 0$$

The proposed estimator t_{pN} has lesser MSE than the ratio estimator t_{8N} for the condition if,

$$\mathrm{M}SE(t_{8N}) - MSE_{\min}(t_{pN}) > 0 \text{ or,}$$

$$\theta_N \circledcirc C_{yN}^2 + \lambda_{8N}^2 C_{xN}^2 - 2\lambda_{8N} C_{yxN} \circledcirc - \left\{1 - \frac{P}{Q}\right\} > 0$$

The introduced estimator t_{pN} performs better than the estimator t_{9N} if,

$$\mathrm{M}SE(t_{9N}) - MSE_{\min}(t_{pN}) > 0 \text{ or,}$$

$$\theta_N \circledcirc C_{yN}^2 + \lambda_{9N}^2 C_{xN}^2 - 2\lambda_{9N} C_{yxN} \circledcirc - \left\{1 - \frac{P}{Q}\right\} > 0$$

The introduced estimator t_{pN} has lesser MSE than the estimator t_{10N} if,

$$\mathrm{M}SE(t_{10N}) - MSE_{\min}(t_{pN}) > 0 \text{ or,}$$

$$\theta_N \circledcirc C_{yN}^2 + \lambda_{10N}^2 C_{xN}^2 - 2\lambda_{10N} C_{yxN} \circledcirc - \left\{1 - \frac{P}{Q}\right\} > 0$$

The proposed estimator t_{pN} has lesser MSE in comparison to the ratio estimator t_{11N} under the condition if,

$$\mathrm{M}SE(t_{11N}) - MSE_{\min}(t_{pN}) > 0 \text{ or,}$$

$$\theta_N \circledcirc C_{yN}^2 + \lambda_{11N}^2 C_{xN}^2 - 2\lambda_{11N} C_{yxN} \circledcirc - \left\{1 - \frac{P}{Q}\right\} > 0$$

The introduced estimator t_{pN} is better than the ratio estimator t_{12N} for the condition if,

$$\mathrm{M}SE(t_{12N}) - MSE_{\min}(t_{pN}) > 0 \text{ or,}$$

$$\theta_N \circledcirc C_{yN}^2 + \lambda_{12N}^2 C_{xN}^2 - 2\lambda_{12N} C_{yxN} \circledcirc - \left\{1 - \frac{P}{Q}\right\} > 0$$

The proposed estimator t_{pN} perform better the estimator t_{13N} if,

$$\mathrm{M}SE(t_{13N}) - MSE_{\min}(t_{pN}) > 0 \text{ or,}$$

$$\theta_N \circledcirc C_{yN}^2 + \lambda_{13N}^2 C_{xN}^2 - 2\lambda_{13N} C_{yxN} \circledcirc - \left\{1 - \frac{P}{Q}\right\} > 0$$

The suggested estimator t_{pN} has lesser MSE with that of the ratio estimator t_{14N} under the condition if,

$$\mathrm{M}SE(t_{14N}) - MSE_{\min}(t_{pN}) > 0 \text{ or,}$$

$$\theta_N \left(C_{yN}^2 + \lambda_{14N}^2 C_{xN}^2 - 2\lambda_{14N} C_{yxN}\right) - \left\{1 - \frac{P}{Q}\right\} > 0$$

The introduced estimator t_{pN} is better than that of the estimator t_{15N} if,

$MSE_{\min}(t_{15N}) - MSE_{\min}(t_{pN}) > 0$ or,

$$\theta_N C_{yN}^2 (1 - \rho_{yxN}^2) - \left\{1 - \frac{P}{Q}\right\} > 0$$

4. Simulation Study

To verify the theoretical efficiency conditions and evaluate the efficiencies of the suggested and competing neutrosophic estimator of $\bar{Y}$ utilizing known auxiliary parameters, we have simulated a neutrosophic data set using the same parameters of [43]. To generate the neutrosophic data, we have considered that the neutrosophic main and auxiliary random variables Y_N and X_N follow the neutrosophic normal distributions. Thus $Y_N \sim NN(\mu_{yN}, \sigma_{yN}^2)$; $Y_N \in (Y_L, Y_U)$, $\mu_{yN} \in (\mu_{yL}, \mu_{yU})$, $\sigma_{yN}^2 \in (\sigma_{yL}^2, \sigma_{yU}^2)$ and $X_N \sim NN(\mu_{xN}, \sigma_{xN}^2)$; $X_N \in (X_L, X_U)$, $\mu_{xN} \in (\mu_{xL}, \mu_{xU})$, $\sigma_{xN}^2 \in (\sigma_{xL}^2, \sigma_{xU}^2)$. For the numerical illustration, we have taken $Y_N \sim NN([76.0, 84.9], [(12.9)^2, (17.2)^2])$, where, $\mu_{yN} \in (76.0, 84.9)$, $\sigma_{yN} \in (12.9, 17.2)$ and $X_N \sim NN([171.2, 180.4], [(5.8)^2, (6.7)^2])$, where, $\mu_{xN} \in (171.2, 180.4)$, $\sigma_{xN} \in (5.8, 6.7)$ and generated 1000 normal random observation for both the variables. The descriptive statistics for the simulated data is presented in Table 1.

Table 1. Descriptive statistics of the simulated data for the neutrosophic data

Parameter	Neutrosophic Value	Parameter	Neutrosophic Value
N_N	[1000, 1000]	C_{xN}	[0.0332, 0.0369]
n_N	[20, 20]	$\beta_{1(x)N}$	[0.0020, 0.0051]
μ_{yN}	[76.20, 85.63]	$\beta_{2(x)N}$	[3.0227, 2.9539]
μ_{xN}	[171.08, 180.34]	$Q_{1(x)N}$	[167.3941, 176.1144]
σ_{yN}	[12.79, 17.37]	$M_{d(x)N}$	[170.9067, 180.3451]
σ_{xN}	[5.67, 6.65]	$Q_{3(x)N}$	[174.9269, 184.7586]
C_{yN}	[0.1679, 0.2028]	ρ_{yxN}	[0.01933, 0.00703]

The Table 2 is representing the neutrosophic MSEs of different competing along with the suggested estimator of population mean.

Table 2. Neutrosophic MSEs of different competing and suggested estimator

SR. No.	Estimators	MSE
1.	t_0	[8.019213, 14.77799]
2.	t_{RN}	[17.39673, 27.98680]
3.	t_{1N}	[17.39674, 27.98681]
4.	t_{2N}	[8.066852, 14.8812]
5.	t_{3N}	[17.39709, 27.98701]

6.	t_{4N}	[17.39674, 27.98681]
7.	t_{5N}	[17.39674, 27.98681]
8.	t_{6N}	[17.3978, 27.98741]
9.	t_{7N}	[17.42703, 28.00546]
10.	t_{8N}	[17.4277, 28.00592]
11.	t_{9N}	[17.39673, 27.98681]
12.	t_{10N}	[17.4517, 28.01563]
13.	t_{11N}	[17.42734, 28.00569]
14.	t_{12N}	[17.45602, 28.02323]
15.	t_{13N}	[17.40314, 27.99058]
16.	t_{14N}(a=1,b=0)	[17.42736, 28.00569]
17.	t_{14N}(a=1,b=1)	[17.42754, 28.00579]
18.	t_{15N}	[8.016216, 14.77726]
19.	t_{pN}	**[7.864525, 13.821846]**

5. Results and Discussion

From Table 2, it may clearly be observed that the estimator t_0 of $\overline{Y}_N$ has its neutrosophic sampling variance as [8.019213, 14.77799] and the neutrosophic MSE of the exponential ratio estimator t_{2N} is [8.066852, 14.8812] while the neutrosophic MSEs of all the mentioned ratio type estimators lie in the interval [17.45602, 28.02323]. The neutrosophic ratio type estimators have high MSEs than the neutrosophic estimator t_0 because of the low neutrosophic correlation between neutrosophic y and x. The neutrosophic MSE of the suggested class of estimators is [7.864525, 13.821846], which is the minimum among the group of all neutrosophic estimators of $\overline{Y}_N$ in competition.

6. Conclusion

In this scripture, we have suggested a novel family of neutrosophic estimators of $\overline{Y}_N$ for the elevated estimation of neutrosophic $\overline{Y}_N$ using the known neutrosophic auxiliary parameters. We studied the neutrosophic sampling properties mainly bias and MSE of the proposed family of estimators for the approximation of degree one. The neutrosophic optimum values of the characterizing scalars of the introduced estimator are obtained and the neutrosophic minimum MSE of the suggested estimator has also been obtained for these neutrosophic optimum values of the characterizing scalars. The introduced estimator has been compared with the neutrosophic competing estimators theoretically and the efficiency condition over the competing estimators have been obtained. These efficiency conditions are verified using a neutrosophic simulated data set. The results in Table-2 are showing that the suggested estimator is most efficient among the class of all neutrosophic competing estimators of $\overline{Y}_N$. Thus the introduced class of estimators may be recommended for elevated estimation of neutrosophic $\overline{Y}_N$ in different areas of applications. It is to be mentioned here that the neutrosophic estimators are most suitable for improved estimation of population mean for the situations where the observations of the study variable are nondeterministic but for the situation where its observations are deterministic, it may be inferior to the classical estimators.

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On r-Edge Regular Neutrosophic Graphs

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Abstract: We approach learning characteristic on a neutrosophic graph such as r-edge regular neutrosophic graph, strongly edge regular neutrosophic graph and absolute degree of vertex since a neutrosophic set $NS = \{\langle x, NS_{\mathfrak{T}}(x), NS_{\mathrm{I}}(x), NS_{\mathfrak{F}}(x)\rangle; x \in X\}$ of a universe set A. We discuss different aspects of these graphics in this article. We've also included several examples to help you understand these concepts.

Keywords: Neutrosophic set, r-edge regular neutrosophic graph, strongly edge regular neutrosophic graph, absolute degree of vertex.

1. Introduction

By changing the definition of the fuzzy set, Smarandache [2] presented the neutrosophic set. Any vague real-life problem can be solved using the neutrosophic set, which can function with uncertain, indeterminate, unclear, and inconsistent details. It's essentially a hybrid of the crisp set, Type 1 fuzzy set, and the IFS. The truth, indeterminate, and false membership degrees of any object are used to define it. These three membership degrees are independent of one another and always fall within the range of [0, 1+], i.e. a nonstandard unit interval. Numerous scholars have long become more interested in neutrosophic graph theory, such as Ye [3] and Yang et al. [5]. Borzooei [1], Azadi et al. [9], Arkam [6] and Poulik, S., Ghorai, G [9-14] . The vertex degree is a useful way to define a vertex's total number of relationships in a graph, and it can also be utilised evaluate the graph. In a fuzzy graph, Gani and Lathi raised the concepts of irregularity, total irregularity, and total degree. Maheswari and Sekar suggested the d_2-vertex term and defined several assets of the d_2-vertex degree of a fuzzy graph. Darabian et al. introduced the d_m-regular vague graph, the td_m-regular vague graph, the m-highly irregular vague graph, and the m-highly complete irregular vague graph, as well as some of their attributes. In this article, we look at neutrosophic graphs using certain r-edge regularity and absolute degree of vertex properties. The purpose of this work is to generalise an idea from neutrosophic graph.

2. Preliminaries

2.1. Definition [7]

A graph $G = (V, E)$ is really an ordered pair made up of a non-empty vertex set V, another edge set E, and a link that connects each edge across two end points.

2.2. Definition [7]
Consider the graph $G = (V, E)$. Since $\mathfrak{A} \subseteq V$and $\mathfrak{B} \subseteq E$ then $\vartheta_G = (\mathfrak{A}, \mathfrak{B})$ could very well be a sub graph of G.

2.3. Definition [11]
A function $\mu : \mathfrak{A} \rightarrow [0, 1]$. Defines a fuzzy set on a set $\mathfrak{A}$.

2.4. Definition
A fuzzy graph $G = (\sigma, \mu)$ is called complete fuzzy graph if $\mu(a, b) = \min\{\sigma(a), \sigma(b)\}, \forall a, b \epsilon \sigma$.

2.5. Definition
A fuzzy graph $G = (\sigma, \mu)$ is called strong fuzzy graph if $\mu(a, b) = \min\{\sigma(a), \sigma(b)\}\ \forall a, b \epsilon \mu$.

2.6. Definition
The complement of a fuzzy graph $G = (\sigma, \mu)$ is a fuzzy graph and it is represented as $G^c = (\sigma^c, \mu^c)$, where $\sigma^c = \sigma$ and $\mu^c(a, b) = min\{\sigma(a), \sigma(b)\} - \mu(a, b)$.

2.7. Definition [7]
A fuzzy graph $\mathfrak{A}_G = (V, \lambda, \mu)$ is a non-empty set V together with pair of functions $\lambda: V \rightarrow [0,1]$ and $\mu: V \times V \rightarrow [0,1]$ such that for all $a, b \in V, \mu(a, b) \leq \min\{\lambda(a), \lambda(b)\}$ where $\lambda(a)$ and $\mu(ab)$ represent the membership value of the vertex a and the edge a, b in $\mathfrak{A}_G$, respectively. The underlying crisp graph of the fuzzy graph $\mathfrak{A}_G = (V, \lambda, \mu)$ is denoted by $\mathfrak{A}_{G^*} = (V, \lambda^*, \mu^*)$ where $\lambda^* = \{a \in V; \lambda(a) > 0\}$ and $\mu^* = \{ab \in V \times V; \mu(ab) > 0\}$. Thus for underlying fuzzy graph $\lambda^* = V$.

2.8. Definition [14]
An intuitionistic fuzzy graph is a pair Let $G = (V, E)$ of a graph $G^* = (V, E)$ where $\mathfrak{A} = (\mathfrak{A}_\mu, \mathfrak{A}_\lambda)$an intuitionistic fuzzy set on V is and $\mathfrak{B} = (\mathfrak{B}_\mu, \mathfrak{B}_\lambda)$ is an intuitionistic fuzzy relation on E such that $\mathfrak{B}_\mu(ab) \leq \min\{\mathfrak{A}_\mu(a), \mathfrak{A}_\mu(b)\}, \mathfrak{B}_\lambda(ab) \geq \max\{\mathfrak{A}_\lambda(a), \mathfrak{A}_\lambda(b)\}$ for all $a, b\ in\ V$. The underlying crip graph of $G = (\mathfrak{A}, \mathfrak{B})$ is the crisp graph $G^* = (V, E)$,where $V = \{a; \mathfrak{A}_\lambda(a) > 0\ or\ \mathfrak{A}_\lambda(a) = 0\}$ and $E = \{ab; \mathfrak{B}_\mu(ab) > 0\ or\ \mathfrak{B}_\mu(ab) = 0\}$

2.9. Definition [3]
A neutrosophic graph is of the form $G = (V, E)$ where

1. V such that $\mathfrak{T}_1: \mathfrak{A} \rightarrow [0,1]$, $\mathfrak{l}_1: \mathfrak{A} \rightarrow [0,1]$ and $\mathfrak{F}_1: \mathfrak{A} \rightarrow [0,1]$ denote the degree of membership, degree of indeterminacy and non-membership of the element $v_i \in V$, respectively, and $0 \leq \mathfrak{T}_i(v_i) + \mathfrak{l}_i(v_i) + \mathfrak{F}_i(v_i) \leq 3$, for every $v_i \in V, (i = 1,2, \dots \dots, n)$.

2. $E \subseteq V \times V$, Where $\mathfrak{T}_2: \mathfrak{A} \rightarrow [0,1]$, $\mathfrak{l}_2: \mathfrak{A} \rightarrow [0,1]$ and $\mathfrak{F}_2: \mathfrak{A} \rightarrow [0,1]$
such that $\mathfrak{T}_2(v_i, v_j) \leq \min\{\mathfrak{T}_1(v_i), \mathfrak{T}_1(v_j)\}, \mathfrak{l}_2(v_i, v_j) \geq \max\{\mathfrak{l}_1(v_i), \mathfrak{l}_1(v_j)\}$, and
$\mathfrak{F}_2(v_i, v_j) \geq \max\ \{\mathfrak{F}_1(v_i), \mathfrak{F}_1(v_j)\}$ and $0 \leq \mathfrak{T}_i(v_i, v_j) + \mathfrak{l}_i(v_i, v_j) + \delta_i(v_i, v_j) \leq 3$ for every $v_i, v_j \in E, (i, j = 1,2, \dots \dots, n)$.

2.10. Example
Consider a neutrosophic graph G, such that $\mathfrak{A} = \{a, b, c, d, e\}$ and $\mathfrak{B} = \{ab, ac, cb, ce, ed, bd, cd, eb\}$ as in Figure 1.

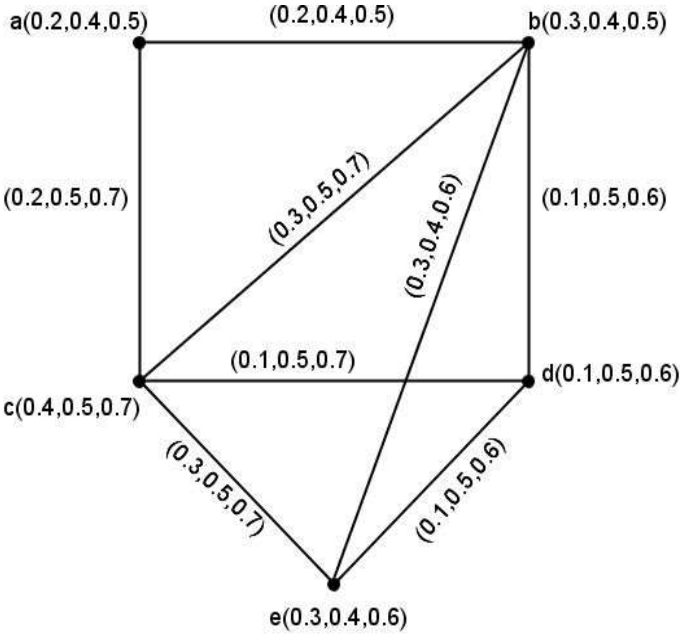


Figure 1: An example of neutrosophic graph

3. Neutrosophic graph in r-edge regular

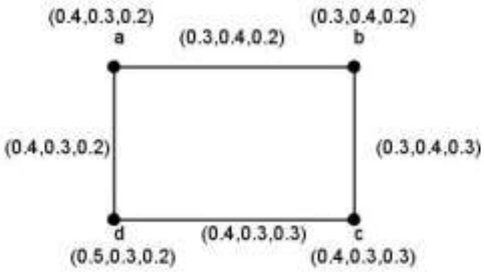
3.1. Definition

A graph $G^* = (\mathfrak{A}, \mathfrak{B})$ with a neutrosophic graph $G = (\mathfrak{A}, \mathfrak{B})$ is said to be strong if, for all $ab \in \mathfrak{A}$ and

1. $\mathfrak{B}_{\mathfrak{T}}(ab) = \min\{\mathfrak{A}_{\mathfrak{T}}(a), \mathfrak{A}_{\mathfrak{T}}(b)\}$
2. $\mathfrak{B}_{I}(ab) = \max\{\mathfrak{A}_{I}(a), \mathfrak{A}_{I}(b)\}$
3. $\mathfrak{B}_{\mathfrak{F}}(ab) = \max\{\mathfrak{A}_{\mathfrak{F}}(a), \mathfrak{A}_{\mathfrak{F}}(b)\}$

3.2. Example

Consider a neutrosophic graph G, such that $\mathfrak{A} = \{a, b, c, d\}$ and $\mathfrak{B} = \{ab, bc, cd, da\}$ as in figure 2.



Strong neutrosophic graph

Figure 2

3.3. Definition

A graph $G^* = (\mathfrak{A}, \mathfrak{B})$ with a neutrosophic graph $G = (\mathfrak{A}, \mathfrak{B})$ is said to be complete if, for all $ab \in \mathfrak{B}$ and

1. $\mathfrak{B}_{\mathfrak{T}}(ab) = \min\{\mathfrak{A}_{\mathfrak{T}}(a), \mathfrak{A}_{\mathfrak{T}}(b)\}$
2. $\mathfrak{B}_{I}(ab) = \max\{\mathfrak{A}_{I}(a), \mathfrak{A}_{I}(b)\}$
3. $\mathfrak{B}_{\mathfrak{F}}(ab) = \max\{\mathfrak{A}_{\mathfrak{F}}(a), \mathfrak{A}_{\mathfrak{F}}(b)\}$.

3.4. Example

Consider a neutrosophic graph G, such that $\mathfrak{A} = \{a, b, c, d\}$ and $\mathfrak{B} = \{ab, bc, cd, da, ac, db\}$ as in figure 3.

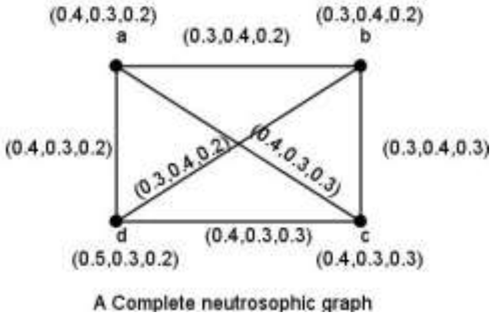


Figure 3

3.5. Definition

A complement of a neutrosophic graph $G = (\mathfrak{A}, \mathfrak{B})$ is a neutrosophic graph $\bar{G} = (\bar{\mathfrak{A}}, \bar{B})$, where $\bar{\mathfrak{A}} = (\bar{\mathfrak{A}}_{\mathfrak{T}}(a), \bar{\mathfrak{A}}_I(a), \bar{\mathfrak{A}}_{\mathfrak{F}}(a))$ and $\bar{\mathfrak{B}} = (\bar{\mathfrak{B}}_{\mathfrak{T}}(a), \bar{\mathfrak{B}}_I(a), \bar{\mathfrak{B}}_{\mathfrak{F}}(a))$

Here,

1. $\bar{\mathfrak{B}}_{\mathfrak{T}}(ab) = \min\{\mathfrak{A}_{\mathfrak{T}}(a), \mathfrak{A}_{\mathfrak{T}}(b)\} - \mathfrak{B}_{\mathfrak{T}}(ab)$
2. $\bar{\mathfrak{B}}_{I}(ab) = \max\{\mathfrak{A}_{I}(a), \mathfrak{A}_I(b)\} - \mathfrak{B}_{I}(ab)$
3. $\bar{\mathfrak{B}}_{\mathfrak{F}}(ab) = \max\{\mathfrak{A}_{\mathfrak{F}}(a), \mathfrak{A}_{\mathfrak{F}}(b)\} - \mathfrak{B}_{\mathfrak{F}}(ab)$ for all $a, b \in \mathfrak{B}$.

3.6. Example

Consider a neutrosophic graph G, such that $\mathfrak{A} = \{a, b, c, d\}$ and $\mathfrak{B} = \{ac, ad, db, bc\}$ as in figure 4.

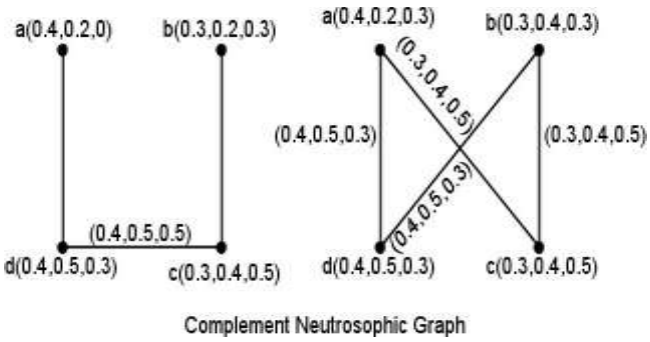


Figure 4

3.7. Definition

The absolute degree of any vertex an is determined by if $G = (\mathfrak{A}, \mathfrak{B})$ is a neutrosophic graph.

$\mathfrak{D}_a = (\mathfrak{T}_D(a), I_D(a), F_D(a))$, where

1. $\mathfrak{T}_D(a) = \sum \mathfrak{B}_{\mathfrak{T}}(ab); a \neq b, ab \in E$
2. $I_D(a) = \sum \mathfrak{B}_I(ab); a \neq b, ab \in E$
3. $\mathfrak{F}_D(a) = \sum \mathfrak{B}_{\mathfrak{F}}(ab); a \neq b, ab \in E$

And hence $\mathfrak{D}_a = \left| \sum_{\substack{a \neq b \\ a \in V}} \mathfrak{B}_{\mathfrak{T}}(ab) - \sum_{\substack{a \neq b \\ a \in V}} \mathfrak{B}_I(ab) - \sum_{\substack{a \neq b \\ a \in V}} \mathfrak{B}_{\mathfrak{F}}(ab) \right|$

3.8. Example

Let $G^* = (\mathfrak{A}, \mathfrak{B})$, where $\mathfrak{A} = \{\mathrm{a, b, c, d, e}\}$ and $\mathfrak{B} = \{\mathrm{ab, ae, bd, ad, de}\}$, then

$$D_{a} = |(0.3 + 0.2) - (0.5 + 0.5) - (0.6 + 0.6)| = 1.8$$
$$D_{b} = |(0.4 + 0.3) - (0.6 + 0.5) - (0.7 + 0.6)| = 1.7$$
$$D_{c} = |(0.3 + 0.4 + 0.2) - (0.6 + 0.6 + 0.6) - (0.7 + 0.7 + 0.7)| = 3$$
$$D_{d} = |(0.2 + 0.2 + 0.2) - (0.5 + 0.5 + 0.6) - (0.6 + 0.6 + 0.7)| = 2.9$$
$$D_{e} = |(0.3 + 0.2) - (0.6 + 0.5) - (0.7 + 0.6)| = 1.9$$

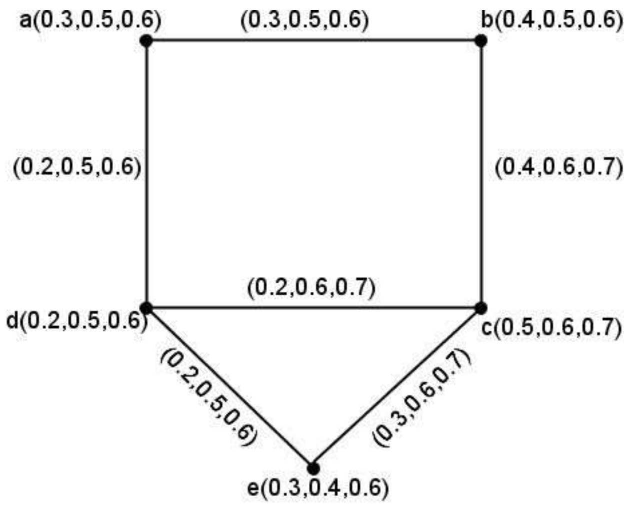


Figure 2: Graph G which id defined in Example 3.8

Figure 5

3.9 Definition

If $G^* = (V, E)$ is a crisp graph and i = a, b is an edge in G^*, then $\mathrm{D_e} = \mathrm{D_a} + \mathrm{D_b} - 2$ is the degree of the e ∈ E.

3.10. Definition

Let $G = (\mathfrak{A}, \mathfrak{B})$ be a neutrosophic graph. $\mathrm{D}_N(\mathrm{a}) = (N_{\mathfrak{A}}(\mathrm{a}) + N_{\mathfrak{B}}(\mathrm{a}))$ is the degree neighbourhood of a vertex. Where $N_{\mathfrak{A}}(\mathrm{a}) = \sum_{\mathrm{b} \in N(\mathrm{a})} \mathfrak{A}_{\mathfrak{T}}(\mathrm{b}), N_{\mathfrak{A}}(\mathrm{a}) = \sum_{\mathrm{b} \in N(\mathrm{a})} \mathfrak{A}_{\mathrm{I}}(\mathrm{b})$. $\mathrm{a}nd\ N_B(\mathrm{a}) = \sum_{\mathrm{b} \in N(\mathrm{a})} \mathfrak{B}_{\mathfrak{F}}(\mathrm{b})$.

3.11. Definition

In a neutrosophic graph G = (𝔄, 𝔅) an edge's total open neighbourhood degree ab ∈ E is known as $\mathfrak{T}_{\mathrm{D}}(\mathrm{ab}) = \left(T_{\mathrm{D}}\mathfrak{A}_{\mathfrak{T}}(\mathrm{ab}), T_{\mathrm{D}}\mathfrak{A}_{\mathrm{I}}(\mathrm{ab}), T_{\mathrm{D}}\mathfrak{A}_{\mathfrak{F}}(\mathrm{ab})\right)$

Where,

$$T_{\mathrm{D}}\mathfrak{A}_{\mathfrak{T}}(\mathrm{ab}) = T_{\mathrm{D}}(\mathrm{a}) + T_D(\mathrm{b}) - \mathfrak{B}_{\mathfrak{T}}(\mathrm{ab})$$
$$T_{\mathrm{D}}\mathfrak{A}_{\mathrm{I}}(\mathrm{ab}) = T_{\mathrm{D}}(\mathrm{a}) + T_{\mathrm{D}}(\mathrm{b}) - \mathfrak{B}_{\mathrm{I}}(\mathrm{ab})$$
$$T_{\mathrm{D}}\mathfrak{A}_{\mathfrak{F}}(\mathrm{ab}) = T_{\mathrm{D}}(\mathrm{a}) + T_{\mathrm{D}}(\mathrm{b}) - \mathfrak{B}_{\mathfrak{F}}(\mathrm{ab})$$

An edge's minimum total open neighbourhood degree is equal to $\Delta_{TE} = \min\{T_{\mathrm{D}}(\mathrm{ab}); \mathrm{ab} \in E\}$
An edge's minimum total open neighbourhood degree is equal to $\Delta_{IE} = \min\{I_{\mathrm{D}}(\mathrm{ab}); \mathrm{ab} \in E\}$ and
An edge's maximum open neighbourhood degree is known as $F_{TE} = \max\{F_{\mathrm{D}}(\mathrm{ab}); \mathrm{ab} \in E\}$.

3.12. Definition

Let G = (𝔄, 𝔅) be a neutrosophic graph. The degree neighbourhood of a vertex a is defined as $D_N = (N_T(a), N_I(a), N_F(a))$, Where $N_T(a) = \sum_{b \in N_T(a)} T(b)$, $N_I(a) = \sum_{b \in N_I(a)} I(b)\ and\ N_F(a) = \sum_{b \in N_F(a)} F(b)$.

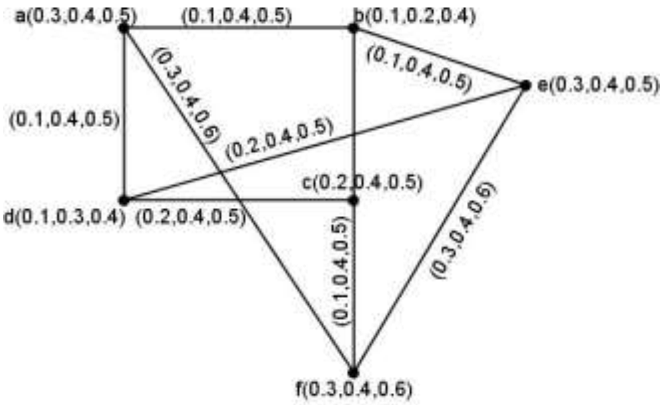
3.13. Definition

Assume $G = (\mathfrak{A}, \mathfrak{B})$ is a neutrosophic graph on G^*.

1. G also is an r-edge regular neutrosophic graph if all of its edges get the same neighbourhood degree r.
   In a neutrosophic graph $G = (\mathfrak{A}, \mathfrak{B})$, the open neighbourhood degree of an edge $\mathrm{ab} \in E$ is classified as $\mathcal{D}_{\mathrm{ab}} = (\mathcal{D}_{\mathfrak{A}\mathfrak{T}}(\mathrm{ab}), \mathcal{D}_{\mathfrak{A}\mathrm{I}}(\mathrm{ab}), \mathcal{D}_{\mathfrak{A}\mathfrak{F}}(\mathrm{ab})$ such that;
   $\mathcal{D}_{\mathfrak{A}\mathfrak{T}}(\mathrm{ab}) = \mathfrak{T}_{\mathrm{D}}(\mathrm{a}) + \mathfrak{T}_{\mathrm{D}}(\mathrm{b}) - 2\mathfrak{B}_{\mathfrak{T}}(\mathrm{ab}), \mathcal{D}_{\mathfrak{A}\mathrm{I}}(\mathrm{ab}) = \mathrm{I}_{\mathrm{D}}(\mathrm{a}) + \mathrm{I}_{\mathrm{D}}(\mathrm{b}) - 2\mathfrak{B}_{\mathrm{I}}(\mathrm{ab})$,
   and $\mathcal{D}_{\mathfrak{A}\mathfrak{F}}(\mathrm{ab}) = \mathfrak{F}_{\mathrm{D}}(\mathrm{a}) + \mathfrak{F}_{\mathrm{D}}(\mathrm{b}) - 2\mathfrak{B}_{\mathfrak{F}}(\mathrm{ab})$.

3.14. Example

Consider a neutrosophic graph $G = (\mathfrak{A}, \mathfrak{B})$ such that



An example of a neutrosophic graph

Figure 6

$\mathcal{D}_{\mathrm{a}} = (0.5, 1.2, 1.6)$ $\mathcal{D}_{\mathrm{b}} = (0.3, 1.2, 1.5)$ $\mathcal{D}_{\mathrm{c}} = (0.5, 1.2, 1.6)$

$\mathcal{D}_{d} = (0.5, 1.2, 1.5)$ $\mathcal{D}_{e} = (0.6, 1.2, 1.6)$ $\mathcal{D}_{f} = (0.8, 1.2, 1.8)$

$$\mathcal{D}_{\mathfrak{A}T}(\mathrm{ab}) = \mathfrak{T}_{\mathrm{D}}(\mathrm{a}) + \mathfrak{T}_{\mathrm{D}}(\mathrm{b}) - 2\mathfrak{B}_{\mathfrak{T}}(\mathrm{ab}) = 0.5 + 0.3 - 2(0.1) = 0.6$$
$$\mathcal{D}_{\mathfrak{A}\mathrm{I}}(\mathrm{ab}) = \mathrm{I}_{\mathrm{D}}(\mathrm{a}) + \mathrm{I}_{\mathrm{D}}(\mathrm{b}) - 2\mathfrak{B}_{\mathrm{I}}(\mathrm{ab}) = 1.2 + 1.2 - 2(0.4) = 1.6$$
$$\mathcal{D}_{\mathfrak{A}\mathfrak{F}}(\mathrm{ab}) = \mathfrak{F}_{\mathrm{D}}(\mathrm{a}) + \mathfrak{F}_{\mathrm{D}}(\mathrm{b}) - 2\mathfrak{B}_{\mathfrak{F}}(\mathrm{ab}) = 1.6 + 1.5 - 2(0.5) = 2.1$$
$$\mathcal{D}_{\mathrm{ab}} = (\mathcal{D}_{\mathfrak{A}\mathfrak{T}}(\mathrm{ab}), \mathcal{D}_{A\mathrm{I}}(\mathrm{ab})\ \mathcal{D}_{AF}(\mathrm{ab})) = (0.6, 1.6, 2.1)$$

$$\mathcal{D}_{\mathfrak{A}\mathfrak{T}}(\mathrm{bc}) = \mathfrak{T}_{\mathrm{D}}(\mathrm{b}) + \mathfrak{T}_{D}(\mathrm{c}) - 2\mathfrak{B}_{\mathfrak{T}}(\mathrm{bc}) = 0.3 + 0.5 - 2(0.1) = 0.6$$
$$\mathcal{D}_{\mathfrak{A}\mathrm{I}}(\mathrm{bc}) = \mathrm{I}_{\mathrm{D}}(\mathrm{b}) + \mathrm{I}_{\mathrm{D}}(\mathrm{c}) - 2\mathfrak{B}_{\mathrm{I}}(\mathrm{bc}) = 1.2 + 1.2 - 2(0.4) = 1.6$$
$$\mathcal{D}_{\mathfrak{A}\mathfrak{F}}(\mathrm{bc}) = \mathfrak{F}_{\mathrm{D}}(\mathrm{b}) + \mathfrak{F}_{\mathrm{D}}(\mathrm{c}) - 2\mathfrak{B}_{\mathfrak{F}}(\mathrm{bc}) = 1.5 + 1.6 - 2(0.5) = 2.1$$
$$\mathcal{D}_{\mathrm{bc}} = (\mathcal{D}_{\mathfrak{A}\mathfrak{T}}(\mathrm{bc}), \mathcal{D}_{\mathfrak{A}\mathrm{I}}(\mathrm{bc})\ \mathcal{D}_{\mathfrak{A}\mathfrak{F}}(\mathrm{bc})) = (0.6, 1.6, 2.1)$$

$$\mathcal{D}_{\mathfrak{A}\mathfrak{T}}(\mathrm{c}d) = \mathfrak{T}_{\mathrm{D}}(\mathrm{c}) + \mathfrak{T}_{\mathrm{D}}(d) - 2\mathfrak{B}_{\mathfrak{T}}(\mathrm{c}d) = 0.5 + 0.5 - 2(0.2) = 0.6$$
$$\mathcal{D}_{\mathfrak{A}\mathrm{I}}(\mathrm{c}d) = \mathrm{I}_{\mathrm{D}}(\mathrm{c}) + \mathrm{I}_{\mathrm{D}}(d) - 2\mathfrak{B}_{\mathrm{I}}(\mathrm{c}d) = 1.2 + 1.2 - 2(0.4) = 1.6$$
$$\mathcal{D}_{\mathfrak{A}\mathfrak{F}}(\mathrm{c}d) = \mathfrak{F}_{\mathrm{D}}(\mathrm{c}) + \mathfrak{F}_{\mathrm{D}}(d) - 2\mathfrak{B}_{\mathfrak{F}}(\mathrm{c}d) = 1.6 + 1.5 - 2(0.5) = 2.1$$
$$\mathcal{D}_{\mathrm{c}d} = (\mathcal{D}_{\mathfrak{A}T}(\mathrm{c}d), \mathcal{D}_{\mathfrak{A}\mathrm{I}}(\mathrm{c}d)\ \mathcal{D}_{\mathfrak{A}\mathfrak{F}}(\mathrm{c}d)) = (0.6, 1.6, 2.1)$$

$$\mathcal{D}_{\mathfrak{A}\mathfrak{T}}(\mathrm{da}) = \mathfrak{T}_{\mathrm{D}}(\mathrm{d}) + \mathfrak{T}_{\mathrm{D}}(\mathrm{a}) - 2\mathfrak{B}_{\mathfrak{T}}(\mathrm{da}) = 0.4 + 0.4 - 2(0.1) = 0.6$$
$$\mathcal{D}_{\mathfrak{A}\mathrm{I}}(\mathrm{da}) = \mathrm{I}_{\mathrm{D}}(\mathrm{d}) + \mathrm{I}_{\mathrm{D}}(\mathrm{a}) - 2\mathfrak{B}_{\mathrm{I}}(\mathrm{da}) = 0.9 + 1.3 - 2(0.4) = 1.4$$
$$\mathcal{D}_{\mathfrak{A}\mathfrak{F}}(\mathrm{da}) = \mathfrak{F}_{\mathrm{D}}(\mathrm{d}) + \mathfrak{F}_{\mathrm{D}}(\mathrm{a}) - 2\mathfrak{B}_{\mathfrak{F}}(\mathrm{da}) = 1.9 + 2.1 - 2(0.6) = 2.8$$
$$\mathcal{D}_{d\mathrm{a}} = (\mathcal{D}_{\mathfrak{A}\mathfrak{T}}(\mathrm{da}), \mathcal{D}_{\mathfrak{A}\mathrm{I}}(\mathrm{da}), \mathcal{D}_{\mathfrak{A}\mathfrak{F}}(\mathrm{da})) = (0.6, 1.4, 2.8)$$

$$\mathcal{D}_{\mathfrak{A}\mathfrak{T}}(\mathrm{a}f) = \mathfrak{T}_{\mathrm{D}}(\mathrm{a}) + \mathfrak{T}_{\mathrm{D}}(f) - 2\mathfrak{B}_{\mathfrak{T}}(\mathrm{a}f) = 0.5 + 0.8 - 2(0.3) = 1.1$$

$$\mathcal{D}_{\mathfrak{A}\mathrm{I}}(af) = \mathrm{I}_{\mathcal{D}}(a) + \mathrm{I}_{\mathcal{D}}(f) - 2\mathfrak{B}_{\mathrm{I}}(af) = 1.2 + 1.2 - 2(0.4) = 1.6$$
$$\mathcal{D}_{\mathfrak{A}\mathfrak{F}}(af) = \mathfrak{F}_{\mathcal{D}}(a) + \mathfrak{F}_{\mathcal{D}}(f) - 2\mathfrak{B}_{\mathfrak{F}}(af) = 1.6 + 1.8 - 2(0.6) = 2.2$$

$$\mathcal{D}_{af} = (\mathcal{D}_{\mathfrak{A}\mathfrak{T}}(af), \mathcal{D}_{\mathfrak{A}\mathrm{I}}(af)\ \mathcal{D}_{\mathfrak{A}F}(af)) = (1.1, 1.6, 2.2)$$

$$\mathcal{D}_{\mathfrak{A}\mathfrak{T}}(ef) = \mathfrak{T}_{\mathcal{D}}(e) + \mathfrak{T}_{\mathcal{D}}(f) - 2\mathfrak{B}_{\mathfrak{T}}(ef) = 0.6 + 0.8 - 2(0.3) = 0.8$$
$$\mathcal{D}_{\mathfrak{A}\mathrm{I}}(ef) = \mathrm{I}_{\mathcal{D}}(e) + \mathrm{I}_{\mathcal{D}}(f) - 2\mathfrak{B}_{\mathrm{I}}(ef) = 1.2 + 1.2 - 2(0.4) = 1.6$$
$$\mathcal{D}_{\mathfrak{A}\mathfrak{F}}(ef) = \mathfrak{F}_{\mathcal{D}}(e) + \mathfrak{F}_{\mathcal{D}}(f) - 2\mathfrak{B}_{\mathfrak{F}}(ef) = 1.6 + 1.8 - 2(0.6) = 2.2$$
$$\mathcal{D}_{ef} = (\mathcal{D}_{\mathfrak{A}\mathfrak{T}}(ef), \mathcal{D}_{\mathfrak{A}\mathrm{I}}(ef)\ \mathcal{D}_{\mathfrak{A}\mathfrak{F}}(ef)) = (0.8, 1.6, 2.2)$$

$$\mathcal{D}_{\mathfrak{A}\mathfrak{T}}(\text{ç}f) = \mathfrak{T}_{\mathcal{D}}(\text{ç}) + \mathfrak{T}_{\mathcal{D}}(f) - 2\mathfrak{B}_{\mathfrak{T}}(\text{ç}f) = 0.5 + 0.8 - 2(0.2) = 0.9$$
$$\mathcal{D}_{\mathfrak{A}\mathrm{I}}(\text{ç}f) = \mathrm{I}_{\mathcal{D}}(\text{ç}) + \mathrm{I}_{\mathcal{D}}(f) - 2\mathfrak{B}_{\mathrm{I}}(\text{ç}f) = 1.2 + 1.2 - 2(0.4) = 1.6$$
$$\mathcal{D}_{\mathfrak{A}\mathfrak{F}}(\text{ç}f) = \mathfrak{F}_{\mathcal{D}}(\text{ç}) + \mathfrak{F}_{\mathcal{D}}(f) - 2\mathfrak{B}_{\mathfrak{F}}(\text{ç}f) = 1.6 + 1.8 - 2(0.6) = 2.2$$
$$\mathcal{D}_{\text{ç}e} = (\mathcal{D}_{\mathfrak{A}\mathfrak{T}}(\text{ç}f), \mathcal{D}_{\mathfrak{A}\mathrm{I}}(\text{ç}f)\ \mathcal{D}_{\mathfrak{A}\mathfrak{F}}(\text{ç}f)) = (0.9, 1.6, 2.2)$$

$$\mathcal{D}_{\mathfrak{A}\mathfrak{T}}(be) = \mathfrak{T}_{\mathcal{D}}(b) + \mathfrak{T}_{\mathcal{D}}(e) - 2\mathfrak{B}_{\mathfrak{T}}(be) = 0.3 + 0.6 - 2(0.1) = 0.7$$
$$\mathcal{D}_{\mathfrak{A}\mathrm{I}}(be) = \mathrm{I}_{\mathcal{D}}(b) + \mathrm{I}_{\mathcal{D}}(e) - 2\mathfrak{B}_{\mathrm{I}}(be) = 1.2 + 1.2 - 2(0.4) = 1.6$$
$$\mathcal{D}_{\mathfrak{A}\mathfrak{F}}(be) = \mathfrak{F}_{\mathcal{D}}(b) + \mathfrak{F}_{\mathcal{D}}(e) - 2\mathfrak{B}_{\mathfrak{F}}(be) = 1.5 + 1.6 - 2(0.5) = 2.1$$
$$\mathcal{D}_{be} = (\mathcal{D}_{\mathfrak{A}\mathfrak{T}}(be), \mathcal{D}_{\mathfrak{A}\mathrm{I}}(be)\ \mathcal{D}_{\mathfrak{A}\mathfrak{F}}(be)) = (0.7, 1.6, 2.1)$$

$$\mathcal{D}_{\mathfrak{A}\mathfrak{T}}(de) = \mathfrak{T}_{\mathcal{D}}(d) + \mathfrak{T}_{\mathcal{D}}(e) - 2\mathfrak{B}_{\mathfrak{T}}(de) = 0.5 + 0.6 - 2(0.2) = 0.7$$
$$\mathcal{D}_{\mathfrak{A}\mathrm{I}}(de) = \mathrm{I}_{\mathcal{D}}(d) + \mathrm{I}_{\mathcal{D}}(e) - 2\mathfrak{B}_{\mathrm{I}}(de) = 1.2 + 1.2 - 2(0.4) = 1.6$$
$$\mathcal{D}_{\mathfrak{A}\mathfrak{F}}(de) = \mathfrak{F}_{\mathcal{D}}(d) + \mathfrak{F}_{\mathcal{D}}(e) - 2\mathfrak{B}_{\mathfrak{F}}(de) = 1.5 + 1.6 - 2(0.5) = 2.1$$
$$\mathcal{D}_{be} = (\mathcal{D}_{\mathfrak{A}\mathfrak{T}}(de), \mathcal{D}_{\mathfrak{A}\mathrm{I}}(de)\ \mathcal{D}_{\mathfrak{A}\mathfrak{F}}(de)) = (0.7, 1.6, 2.1)$$

$$\mathcal{D}_{\mathfrak{A}T}(ab) = \mathfrak{T}_{\mathcal{D}}(a) + \mathfrak{T}_{\mathcal{D}}(b) - \mathfrak{B}_{\mathfrak{T}}(ab) = 0.5 + 0.3 - (0.1) = 0.7$$
$$\mathcal{D}_{\mathfrak{A}\mathrm{I}}(ab) = \mathrm{I}_{\mathcal{D}}(a) + \mathrm{I}_{\mathcal{D}}(b) - \mathfrak{B}_{\mathrm{I}}(ab) = 1.2 + 1.2 - (0.4) = 2.0$$
$$\mathcal{D}_{\mathfrak{A}\mathfrak{F}}(ab) = \mathfrak{F}_{\mathcal{D}}(a) + \mathfrak{F}_{\mathcal{D}}(b) - \mathfrak{B}_{\mathfrak{F}}(ab) = 1.6 + 1.5 - (0.5) = 2.6$$
$$\mathcal{D}_{ab} = (\mathcal{D}_{\mathfrak{A}\mathfrak{T}}(ab), \mathcal{D}_{AI}(ab)\ \mathcal{D}_{AF}(ab)) = (0.7, 2.0, 2.6)$$

$$\mathcal{D}_{\mathfrak{A}\mathfrak{T}}(b\text{ç}) = \mathfrak{T}_{\mathcal{D}}(b) + \mathfrak{T}_{D}(\text{ç}) - \mathfrak{B}_{\mathfrak{T}}(b\text{ç}) = 0.3 + 0.5 - (0.1) = 0.7$$
$$\mathcal{D}_{\mathfrak{A}\mathrm{I}}(b\text{ç}) = \mathrm{I}_{\mathcal{D}}(b) + \mathrm{I}_{\mathcal{D}}(\text{ç}) - \mathfrak{B}_{\mathrm{I}}(b\text{ç}) = 1.2 + 1.2 - (0.4) = 2.0$$
$$\mathcal{D}_{\mathfrak{A}\mathfrak{F}}(b\text{ç}) = \mathfrak{F}_{\mathcal{D}}(b) + \mathfrak{F}_{\mathcal{D}}(\text{ç}) - \mathfrak{B}_{\mathfrak{F}}(b\text{ç}) = 1.5 + 1.6 - (0.5) = 1.6$$
$$\mathcal{D}_{b\text{ç}} = (\mathcal{D}_{\mathfrak{A}\mathfrak{T}}(b\text{ç}), \mathcal{D}_{\mathfrak{A}\mathrm{I}}(b\text{ç})\ \mathcal{D}_{\mathfrak{A}\mathfrak{F}}(b\text{ç})) = (0.7, 2.0, 1.6)$$

$$\mathcal{D}_{\mathfrak{A}\mathfrak{T}}(\text{ç}d) = \mathfrak{T}_{\mathcal{D}}(\text{ç}) + \mathfrak{T}_{\mathcal{D}}(d) - \mathfrak{B}_{\mathfrak{T}}(\text{ç}d) = 0.5 + 0.5 - (0.2) = 0.8$$
$$\mathcal{D}_{\mathfrak{A}\mathrm{I}}(\text{ç}d) = \mathrm{I}_{\mathcal{D}}(\text{ç}) + \mathrm{I}_{\mathcal{D}}(d) - \mathfrak{B}_{\mathrm{I}}(\text{ç}d) = 1.2 + 1.2 - (0.4) = 2.0$$
$$\mathcal{D}_{\mathfrak{A}\mathfrak{F}}(\text{ç}d) = \mathfrak{F}_{\mathcal{D}}(\text{ç}) + \mathfrak{F}_{\mathcal{D}}(d) - \mathfrak{B}_{\mathfrak{F}}(\text{ç}d) = 1.6 + 1.5 - (0.5) = 2.7$$
$$\mathcal{D}_{\text{ç}d} = (\mathcal{D}_{\mathfrak{A}T}(\text{ç}d), \mathcal{D}_{\mathfrak{A}\mathrm{I}}(\text{ç}d)\ \mathcal{D}_{\mathfrak{A}\mathfrak{F}}(\text{ç}d)) = (0.8, 2.0, 2.7)$$

$$\mathcal{D}_{\mathfrak{A}\mathfrak{T}}(da) = \mathfrak{T}_{\mathcal{D}}(d) + \mathfrak{T}_{\mathcal{D}}(a) - \mathfrak{B}_{\mathfrak{T}}(da) = 0.4 + 0.4 - (0.1) = 0.9$$
$$\mathcal{D}_{\mathfrak{A}\mathrm{I}}(da) = \mathrm{I}_{\mathcal{D}}(d) + \mathrm{I}_{\mathcal{D}}(a) - \mathfrak{B}_{\mathrm{I}}(da) = 0.9 + 1.3 - (0.4) = 2.0$$
$$\mathcal{D}_{\mathfrak{A}\mathfrak{F}}(da) = \mathfrak{F}_{\mathcal{D}}(d) + \mathfrak{F}_{\mathcal{D}}(a) - \mathfrak{B}_{\mathfrak{F}}(da) = 1.9 + 2.1 - (0.6) = 2.6$$
$$\mathcal{D}_{da} = (\mathcal{D}_{\mathfrak{A}\mathfrak{T}}(da), \mathcal{D}_{\mathfrak{A}\mathrm{I}}(da), \mathcal{D}_{\mathfrak{A}\mathfrak{F}}(da)) = (0.6, 1.4, 2.8)$$

$$\mathcal{D}_{\mathfrak{A}\mathfrak{T}}(af) = \mathfrak{T}_{\mathcal{D}}(a) + \mathfrak{T}_{\mathcal{D}}(f) - \mathfrak{B}_{\mathfrak{T}}(af) = 0.5 + 0.8 - (0.3) = 1.0$$
$$\mathcal{D}_{\mathfrak{A}\mathrm{I}}(af) = \mathrm{I}_{\mathcal{D}}(a) + \mathrm{I}_{\mathcal{D}}(f) - \mathfrak{B}_{\mathrm{I}}(af) = 1.2 + 1.2 - (0.4) = 2.0$$
$$\mathcal{D}_{\mathfrak{A}\mathfrak{F}}(af) = \mathfrak{F}_{\mathcal{D}}(a) + \mathfrak{F}_{\mathcal{D}}(f) - \mathfrak{B}_{\mathfrak{F}}(af) = 1.6 + 1.8 - (0.6) = 2.8$$
$$\mathcal{D}_{af} = (\mathcal{D}_{\mathfrak{A}\mathfrak{T}}(af), \mathcal{D}_{\mathfrak{A}\mathrm{I}}(af)\ \mathcal{D}_{\mathfrak{A}F}(af)) = (1.0, 2.0, 2.8)$$

$$\mathcal{D}_{\mathfrak{A}\mathfrak{T}}(ef) = \mathfrak{T}_{\mathcal{D}}(e) + \mathfrak{T}_{\mathcal{D}}(f) - \mathfrak{B}_{\mathfrak{T}}(ef) = 0.6 + 0.8 - (0.3) = 1.1$$
$$\mathcal{D}_{\mathfrak{A}\mathrm{I}}(ef) = \mathrm{I}_{\mathcal{D}}(e) + \mathrm{I}_{\mathcal{D}}(f) - \mathfrak{B}_{\mathrm{I}}(ef) = 1.2 + 1.2 - (0.4) = 2.0$$

$$\mathfrak{D}_{\mathfrak{A}\mathfrak{F}}(ef) = \mathfrak{F}_{\mathrm{D}}(e) + \mathfrak{F}_{\mathrm{D}}(f) - \mathfrak{B}_{\mathfrak{F}}(ef) = 1.6 + 1.8 - (0.6) = 2.8$$
$$\mathfrak{D}_{ef} = (\mathfrak{D}_{\mathfrak{A}\mathfrak{T}}(ef), \mathfrak{D}_{\mathfrak{A}\mathrm{I}}(ef)\ \mathfrak{D}_{\mathfrak{A}\mathfrak{F}}(ef)) = (1.1, 2.0, 2.8)$$

$$\mathfrak{D}_{\mathfrak{A}\mathfrak{T}}(çf) = \mathfrak{T}_{\mathrm{D}}(ç) + \mathfrak{T}_{\mathrm{D}}(f) - \mathfrak{B}_{\mathfrak{T}}(çf) = 0.5 + 0.8 - (0.2) = 1.1$$
$$\mathfrak{D}_{\mathfrak{A}\mathrm{I}}(çf) = \mathrm{I}_{\mathrm{D}}(ç) + \mathrm{I}_{\mathrm{D}}(f) - \mathfrak{B}_{\mathrm{I}}(çf) = 1.2 + 1.2 - (0.4) = 2.0$$
$$\mathfrak{D}_{\mathfrak{A}\mathfrak{F}}(çf) = \mathfrak{F}_{\mathrm{D}}(ç) + \mathfrak{F}_{\mathrm{D}}(f) - \mathfrak{B}_{\mathfrak{F}}(çf) = 1.6 + 1.8 - (0.6) = 2.8$$
$$\mathfrak{D}_{çe} = (\mathfrak{D}_{\mathfrak{A}\mathfrak{T}}(çf), \mathfrak{D}_{\mathfrak{A}\mathrm{I}}(çf)\ \mathfrak{D}_{\mathfrak{A}\mathfrak{F}}(çf)) = (1.1, 2.0, 2.8)$$

$$\mathfrak{D}_{\mathfrak{A}\mathfrak{T}}(be) = \mathfrak{T}_{\mathrm{D}}(b) + \mathfrak{T}_{\mathrm{D}}(e) - \mathfrak{B}_{\mathfrak{T}}(be) = 0.3 + 0.6 - (0.1) = 0.8$$
$$\mathfrak{D}_{\mathfrak{A}\mathrm{I}}(be) = \mathrm{I}_{\mathrm{D}}(b) + \mathrm{I}_{\mathrm{D}}(e) - \mathfrak{B}_{\mathrm{I}}(be) = 1.2 + 1.2 - (0.4) = 2.0$$
$$\mathfrak{D}_{\mathfrak{A}\mathfrak{F}}(be) = \mathfrak{F}_{\mathrm{D}}(b) + \mathfrak{F}_{\mathrm{D}}(e) - \mathfrak{B}_{\mathfrak{F}}(be) = 1.5 + 1.6 - (0.5) = 2.6$$
$$\mathfrak{D}_{be} = (\mathfrak{D}_{\mathfrak{A}\mathfrak{T}}(b), \mathfrak{D}_{\mathfrak{A}\mathrm{I}}(e)\ \mathfrak{D}_{\mathfrak{A}\mathfrak{F}}(be)) = (0.8, 2.0, 2.6)$$

$$\mathfrak{D}_{\mathfrak{A}\mathfrak{T}}(de) = \mathfrak{T}_{\mathrm{D}}(d) + \mathfrak{T}_{\mathrm{D}}(e) - \mathfrak{B}_{\mathfrak{T}}(de) = 0.3 + 0.6 - (0.1) = 0.9$$
$$\mathfrak{D}_{\mathfrak{A}\mathrm{I}}(de) = \mathrm{I}_{\mathrm{D}}(d) + \mathrm{I}_{\mathrm{D}}(e) - \mathfrak{B}_{\mathrm{I}}(de) = 1.2 + 1.2 - (0.4) = 2.0$$
$$\mathfrak{D}_{\mathfrak{A}\mathfrak{F}}(de) = \mathfrak{F}_{\mathrm{D}}(d) + \mathfrak{F}_{\mathrm{D}}(e) - \mathfrak{B}_{\mathfrak{F}}(de) = 1.5 + 1.6 - (0.5) = 2.6$$
$$\mathfrak{D}_{be} = (\mathfrak{D}_{\mathfrak{A}\mathfrak{T}}(de), \mathfrak{D}_{\mathfrak{A}\mathrm{I}}(de)\ \mathfrak{D}_{\mathfrak{A}\mathfrak{F}}(de)) = (0.9, 2.0, 2.6)$$

3.15. Definition

A neutrosophic graph G is a totally (r_1, r_2, r_3) -edge regular neutrosophic graph if every edge has the same total degree (r_1, r_2, r_3).

3.16. Definition

A neutrosophic graph is $G = (\mathfrak{A}, \mathfrak{B})$. G is shown to be a regular neutrosophic graph of degree (r_1, r_2, r_3) if every vertex does have the same degree(r_1, r_2, r_3).

3.17. Definition

If any vertex in a neutrosophic graph $G = (\mathfrak{A}, \mathfrak{B})$has the same degree$(r_1, r_2, r_3)$, then G is called a (r_1, r_2, r_3) edge regular neutrosophic graph.

3.18. Theorem

$G = \sum_{a_i a_j \in E} D_{a_i a_j} = \sum_{a_i \in V} \mathfrak{D}_{a_i}$ if G is an edge regular neutrosophic graph on a cycle G^*.

Proof

Since G is an edge regular neutrosophic graph, thus

$$\sum_{i=1}^{n} \mathfrak{D}_{a_i a_{j+1}} = \left(\sum_{i=1}^{n} \mathfrak{D}_{\mathfrak{A}\mathfrak{T}}(a_i a_j), \sum_{i=1}^{n} \mathfrak{D}_{\mathfrak{A}\mathrm{I}}(a_i a_j), \sum_{i=1}^{n} \mathfrak{D}_{\mathfrak{A}\mathfrak{F}}(a_i a_j) \right)$$
$$\sum_{i=1}^{n} \mathfrak{D}_{\mathfrak{A}\mathfrak{T}}(a_i a_j) = \mathfrak{D}_{\mathfrak{A}\mathfrak{T}}(a_1 a_2) + \mathfrak{D}_{\mathfrak{A}\mathfrak{T}}(a_2 a_3) + \cdots + \mathfrak{D}_{\mathfrak{A}\mathfrak{T}}(a_n a_1)$$

Since,

$$\begin{aligned} a_{n+1} &= \mathfrak{D}_{\mathfrak{A}\mathfrak{T}}(a_1) + \mathfrak{D}_{\mathfrak{A}\mathfrak{T}}(a_2) - 2\mathfrak{B}(a_1 a_2) + \mathfrak{D}_{\mathfrak{A}\mathfrak{T}}(a_2) + \mathfrak{D}_{\mathfrak{A}\mathfrak{T}}(a_3) - 2\mathfrak{B}(a_2 a_3) + \cdots \\ &\quad + \mathfrak{D}_{\mathfrak{A}\mathfrak{T}}(a_n) + \mathfrak{D}_{\mathfrak{A}\mathfrak{T}}(a_1) - 2\mathfrak{B}(a_n a_1) \\ &= 2\mathfrak{D}_{\mathfrak{A}\mathfrak{T}}(a_1) + 2\mathfrak{D}_{\mathfrak{A}\mathfrak{T}}(a_2) + 2\mathfrak{D}_{\mathfrak{A}\mathfrak{T}}(a_n) - 2(\mathfrak{B}(a_1 a_2) + \mathfrak{B}(a_2 a_3) + \mathfrak{B}(a_n a_1)) \\ &= 2\sum_{a_i \in V} \mathfrak{D}_{\mathfrak{A}\mathfrak{T}}(a_1) + 2\sum_{i=1}^{n}(a_i a_{i+1}) - 2\sum_{i=1}^{n} B_{\mathfrak{T}}(a_i a_{i+1}) \\ &= \sum_{a_i \in V} \mathfrak{D}_{\mathfrak{A}\mathfrak{T}}(a_i) \end{aligned}$$

Then

$$\sum_{i=1}^{n} \mathfrak{D}_{\mathfrak{A}\mathfrak{T}}(a_i a_{j+1}) = \sum_{a_i \in V} \mathfrak{D}_{\mathfrak{A}\mathfrak{T}}(a_i)$$

Similarly

$$\sum_{i=1}^{n} \mathfrak{D}_{\mathfrak{A}\mathrm{I}}(a_i a_{i+1}) = \sum_{a_i \in V} \mathfrak{D}_{\mathfrak{A}\mathrm{I}}(a_i)$$
$$\sum_{i=1}^{n} \mathfrak{D}_{\mathfrak{A}\mathfrak{F}}(a_i a_{i+1}) = \sum_{a_i \in V} \mathfrak{D}_{\mathfrak{A}\mathfrak{F}}(a_i) \qquad \sum_{i=1}^{n} \mathfrak{D}(a_i a_{i+1}) =$$
$$\left(\sum_{i=1}^{n} \mathfrak{D}_{\mathfrak{A}\mathfrak{T}}(a_i a_{i+1}), \sum_{i=1}^{n} \mathfrak{D}_{\mathfrak{A}\mathrm{I}}(a_i a_{i+1}), \sum_{i=1}^{n} \mathfrak{D}_{\mathfrak{A}\mathfrak{F}}(a_i a_{i+1}) \right)$$

$\sum_{i=1}^{n} \mathcal{D}(a_i a_{i+1}) = \sum_{a_i \in V} \mathcal{D}_{a_i}$.

3.19. Lemma

If G is an edge regular neutrosophic graph on G^*, then $\sum_{a_i a_j \in E}^{n} \mathcal{D}(a_i a_j) = \left(\sum_{a_i a_j \in E}^{n} \mathcal{D}_{G^*}(a_i a_j)\, \mathfrak{B}_{\mathfrak{T}}(a_i a_j), \sum_{a_i a_j \in E}^{n} \mathcal{D}_{G^*}(a_i a_j)\, \mathfrak{B}_{\mathrm{I}}(a_i a_j), \sum_{a_i a_j \in E}^{n} \mathcal{D}_{G^*}(a_i a_j)\, \mathfrak{B}_{\mathfrak{F}}(a_i a_j)\right)$

Since, $\mathcal{D}_{G^*}(a_i a_j) = \mathcal{D}_{G^*}(a_i) + \mathcal{D}_{G^*}(a_j) - 2, for\ all\ a_i a_j \in E$.

3.20. Proposition

If G is an r-regular G^* edge regular neutrosophic graph, then $\sum_{a_i a_j \in E}^{n} \mathcal{D}(a_i a_{i+1}) = \left((r-1)\sum_{a_i} \mathcal{D}_{\mathfrak{A}_{\mathfrak{T}}}(a_i), (r-1)\sum_{a_i} \mathcal{D}_{\mathfrak{A}_I}(a_i), (r-1)\sum_{a_i} \mathcal{D}_{\mathfrak{A}_{\mathfrak{F}}}(a_i)\right)$.

Proof

By lemma we obtain $\sum \mathcal{D}(a_i a_j) =$

$$\left(\sum_{a_i a_j \in E}^{n} \mathcal{D}_{G^*}(a_i a_j)\, \mathfrak{B}_{\mathfrak{T}}(a_i a_j), \sum_{a_i a_j \in E}^{n} \mathcal{D}_{G^*}(a_i a_j)\, \mathfrak{B}_{\mathrm{I}}(a_i a_j), \sum_{a_i a_j \in E}^{n} \mathcal{D}_{G^*}(a_i a_j)\, \mathfrak{B}_{\mathfrak{F}}(a_i a_j)\right)$$

$$= \begin{pmatrix} \sum(\mathcal{D}_{G^*}(a_i) + \mathcal{D}_{G^*}(a_j) - 2)\, \mathfrak{B}_{\mathfrak{T}}(a_i a_j), \\ \sum(\mathcal{D}_{G^*}(a_i) + \mathcal{D}_{G^*}(a_j) - 2)\, \mathfrak{B}_{\mathrm{I}}(a_i a_j), \\ \sum(\mathcal{D}_{G^*}(a_i) + \mathcal{D}_{G^*}(a_j) - 2)\, \mathfrak{B}_{\mathfrak{F}}(a_i a_j) \end{pmatrix}$$

We know G^* is regular then the degree of every vertex in G^* is r , this means that $\mathcal{D}_{G^*}(a_i) = r$ and hence,

$$\sum \mathcal{D}(a_i a_{i+1}) = \left((r + r - 2)\sum \mathfrak{B}_{\mathfrak{T}}(a_i a_j), (r + r - 2)\sum \mathfrak{B}_I(a_i a_j), (r + r - 2)\sum \mathfrak{B}_{\mathfrak{F}}(a_i a_j)\right)$$

$$= \left(2(r-1)\sum \mathfrak{B}_{\mathfrak{T}}(a_i a_j), 2(r-1)\sum \mathfrak{B}_I(a_i a_j), 2(r-1)\sum \mathfrak{B}_F(a_i a_j)\right)$$

$$\sum_{a_i a_j \in E}^{n} \mathcal{D}(a_i a_{i+1}) = \left((r-1)\sum_{a_i} \mathcal{D}_{\mathfrak{A}_{\mathfrak{T}}}(a_i), (r-1)\sum_{a_i} \mathcal{D}_{\mathfrak{A}_I}(a_i), (r-1)\sum_{a_i} \mathcal{D}_{\mathfrak{A}_F}(a_i)\right)$$

3.21. Corollary

$\sum T_{\mathcal{D}}(a_i a_i) = \begin{pmatrix} \sum \mathcal{D}_{G^*}(a_i a_j)\, \mathfrak{B}_{\mathfrak{T}}(a_i a_j) + \sum \mathfrak{B}_{\mathfrak{T}}(a_i a_j), \\ \sum \mathcal{D}_{G^*}(a_i a_j)\, \mathfrak{B}_{\mathrm{I}}(a_i a_j) + \sum \mathfrak{B}_{\mathrm{I}}(a_i a_j), \\ \sum \mathcal{D}_{G^*}(a_i a_j)\, \mathfrak{B}_{\mathfrak{F}}(a_i a_j) + \sum \mathfrak{B}_{\mathfrak{F}}(a_i a_j) \end{pmatrix}$, where G is a regular neutrosophic graph with edges on G^*.

Proof

$$\sum T_{\mathcal{D}}(a_i a_i) = \left(\sum T_{\mathcal{D}}\, \mathfrak{A}_{\mathfrak{T}}(a_i a_j) + \sum T_{\mathcal{D}}\, \mathfrak{A}_I(a_i a_j) + \sum T_{\mathcal{D}}\, \mathfrak{A}_{\mathfrak{F}}(a_i a_j)\right)$$

$$= \left(\sum T_{\mathcal{D}}\, \mathfrak{A}_{\mathfrak{T}}(a_i a_j) + \mathfrak{B}_{\mathfrak{T}}(a_i a_j), \sum T_{\mathcal{D}}\, \mathfrak{A}_{\mathrm{I}}(a_i a_j) + \mathfrak{B}_{\mathrm{I}}(a_i a_j), \sum T_{\mathcal{D}}\, \mathfrak{A}_F(a_i a_j) + \mathfrak{B}_{\mathfrak{F}}(a_i a_j)\right) =$$

$$\left(\sum \mathcal{D}_{\mathfrak{A}_{\mathfrak{T}}}(a_i a_j) + \sum \mathfrak{B}_{\mathfrak{T}}(a_i a_j), \sum \mathcal{D}_{\mathfrak{A}_I}(a_i a_j) + \sum \mathfrak{B}_I(a_i a_j), \sum \mathcal{D}_{\mathfrak{A}_F}(a_i a_j) + \sum \mathfrak{B}_{\mathfrak{F}}(a_i a_j)\right)$$

By lemma, we get

$$\sum T_{\mathcal{D}}(a_i a_i) = \begin{pmatrix} \sum \mathcal{D}_{G^*}(a_i a_j)\, \mathfrak{B}_{\mathfrak{T}}(a_i a_j) + \sum \mathfrak{B}_T(a_i a_j) \\ \sum \mathcal{D}_{G^*}(a_i a_j)\, \mathfrak{B}_{\mathrm{I}}(a_i a_j) + \sum \mathfrak{B}_{\mathrm{I}}(a_i a_j) \\ \sum \mathcal{D}_{G^*}(a_i a_j)\, \mathfrak{B}_F(a_i a_j) + \sum \mathfrak{B}_{\mathfrak{F}}(a_i a_j) \end{pmatrix}$$

3.22. Definition

When a neutrosophic graph G is strongly regular, it means:

1. G is a regular neutrosophic graph (r_1, r_2, r_3)
2. a_i, a_j of is the number of the member values of the general neighbourhood vertices of any pair of adjacent and non-adjacent vertices. G has the same weight and is denoted by the

symbols $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2)$. $SN_G = (r, \alpha, \beta)$ represents a strongly neutrosophic graph G.

3.23. Theorem

If G is a complete neutrosophic graph with constant functions $(\mathfrak{A}_{\mathfrak{T}}, \mathfrak{A}_{\mathrm{I}}, \mathfrak{A}_{\mathfrak{F}})$ and $(\mathfrak{B}_{\mathfrak{T}}, \mathfrak{B}_{\mathrm{I}}, \mathfrak{B}_{\mathfrak{F}})$ then G is a highly normal neutrosophic graph.

Proof

Since that $(\mathfrak{A}_{\mathfrak{T}}, \mathfrak{A}_{\mathrm{I}}, \mathfrak{A}_{\mathfrak{F}})$ and $(\mathfrak{B}_{\mathfrak{T}}, \mathfrak{B}_{\mathrm{I}}, \mathfrak{B}_{\mathfrak{F}})$ are constant function, then
$\mathfrak{A}_{\mathfrak{T}}(a_i) = k$, $A_I(a_i) = s$, $\mathfrak{A}_F(a_i) = t$ and $\mathfrak{A}_{\mathfrak{T}}(a_i a_j) = d_1$, $\mathfrak{A}_{\mathrm{I}}(a_i a_j) = d_2$, $\mathfrak{A}_{\mathfrak{F}}(a_i a_j) = d_3$.
Such that k, s, t, d_1, d_2, d_3 are constant, and G is complete, then

$$\mathcal{D}_{a_i a_j} = \big((n-1)d_1, (n-1)d_2, (n-1)d_3\big),$$

Therefore G is

$$\textstyle\sum \mathfrak{A}_{\mathfrak{T}}(a_i a_j), \sum \mathfrak{A}_{\mathrm{I}}(a_i a_j), \sum \mathfrak{A}_{\mathfrak{F}}(a_i a_j)) = \mathcal{D}_{\mathfrak{A}_{\mathfrak{T}}}(a_i a_j), \mathcal{D}_{\mathfrak{A}_{\mathrm{I}}}(a_i a_j), \mathcal{D}_{\mathfrak{A}_{\mathfrak{F}}}(a_i a_j)$$
$$= \big((n-1)d_1, (n-1)d_2, (n-1)d_3\big),$$

On the other hand, in a regular neutrosophic graph with n vertices, the sum of $\mathfrak{T}$, I,F of common neighbourhood vertices of any pair of adjacent vertices $\alpha = (n-2)k, (n-2)s, (n-2)t$ is equal, and the sum of $\mathfrak{T}$, I,F values common neighbourhood vertices of any pair of non-adjacent vertices $\beta = 0$ is equal.

3.24. Theorem

G^C is a (r_1, r_2, r_3) regular if G is a strongly regular neutrosophic graph that is strong.

Proof

We know G is strong, then

$$\mathfrak{B}^C{}_{\mathfrak{T}}(a_i a_j) = \begin{cases} 0 & a_i a_j \in \mathfrak{B} \\ \min\{\mathfrak{A}_{\mathfrak{T}}(a_i), \mathfrak{A}_{\mathfrak{T}}(a_j)\} & a_i a_j \notin \mathfrak{B} \end{cases}$$
$$\mathfrak{B}^C{}_I(a_i a_j) = \begin{cases} 0 & a_i a_j \in \mathfrak{B} \\ \min\{\mathfrak{A}_{\mathrm{I}}(a_i), \mathfrak{A}_{\mathrm{I}}(a_j)\} & a_i a_j \notin \mathfrak{B} \end{cases}$$
$$\mathfrak{B}^C{}_F(a_i a_j) = \begin{cases} 0 & a_i a_j \in \mathfrak{B} \\ \min\{\mathfrak{A}_{\mathfrak{F}}(a_i), \mathfrak{A}_{\mathfrak{F}}(a_j)\} & a_i a_j \notin \mathfrak{B} \end{cases}$$

Also,

$$\mathcal{D}^C{}_G(a_i) = \big(\mathfrak{T}\mathcal{D}^C{}_G(a_i), \mathrm{I}\mathcal{D}^C{}_G(a_i), \mathfrak{F}\mathcal{D}^C{}_G(a_i)\big),$$

Such that,

$$\mathfrak{T}\mathcal{D}^C{}_G(a_i) = \sum_{a_i \neq a_j} \mathfrak{B}^C{}_{\mathfrak{T}}(a_i a_j)$$
$$= \textstyle\sum_{a_i \neq a_j} min\left(\mathfrak{A}^C{}_{\mathfrak{T}}(a_i), \mathfrak{A}^C{}_{\mathfrak{T}}(a_j)\right) = r_1$$
$$\mathrm{I}\mathcal{D}^C{}_G(a_i) = \sum_{a_i \neq a_j} \mathfrak{B}^C{}_{\mathrm{I}}(a_i a_j)$$
$$= \textstyle\sum_{a_i \neq a_j} min\left(\mathfrak{A}^C{}_{\mathrm{I}}(a_i), \mathfrak{A}^C{}_{\mathrm{I}}(a_j)\right) = r_2$$
$$\mathfrak{F}\mathcal{D}^C{}_G(a_i) = \sum_{a_i \neq a_j} \mathfrak{B}^C{}_{\mathfrak{F}}(a_i a_j)$$
$$= \textstyle\sum_{a_i \neq a_j} min\left(\mathfrak{A}^C{}_{\mathfrak{F}}(a_i), \mathfrak{A}^C{}_{\mathfrak{F}}(a_j)\right) = r_3$$

For all $a_i \in V$. Thus $\mathcal{D}(a_i a_j) = (r_1, r_2, r_3)$. Hence G^C is (r_1, r_2, r_3) regular neutrosophic graph.

Application

The models of graph are used in wide application in much area of computer science, mathematical models of social sciences. These graph models need to incorporate more structure than simply the adjacency between vertices. In the discussion of set behaviour, it is observed that certain people can influence thinking of others. Each element of a set is represented by a node. There is a directed path from node a to node b when the member represented by node a influence the node represented by node b.

In any social set all the nodes can never be members of the group always. Any node can be removed from the set at any time if his/her activity is against the set. Each node of the set is represented by a vertex and every vertex has two values; the first value represents the power of the node in the set which means how much it possess to control the set, the second value represents the power of the node in the set when it became removed itself from the set.

Each path has also two values such that the first component represents the influence by the first node over the second node when the first node is element of the set. The second component represents the influence by the first node over the second node when the first node is non-member of the set. Any different neutrosophic graph needs large data for training to be able to help in decision making technology and science. The new style which is generalized in this research is based on the pattern of unique cases that can help us to make a better choice in the contrast to the established solutions of neutrosophic graph.

Conclusion

The main contribution of this manuscript is to introduce the idea of regularity in neutrosophic graph theory. In it paper, we have described the idea over the on regular neutrosophic graph. Some unique kinds on neutrosophic graphs certain as the regular, regular strong, r-edge regular neutrosophic graph, strongly edge regular, neutrosophic graph and absolute degree of vertex, have been introduced here. We bear additionally provided some sufficient standards for r-edge regular neutrosophic graph and strongly edge regular.

In the future, we pleasure focal point about the education on neutrosophic intersection graphs, neutrosophic interval graphs, neutrosophic hyper graphs, or therefore on. The notion over the neutrosophic graph execute stay ancient of countless areas regarding expert systems, image processing, computer networks, and communal systems.

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Some Single Valued Neutrosophic Queueing Systems with Maple Code

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Abstract:
In this paper we introduce for the first time the concept of single valued neutrosophic queueing systems (SVNQSs) as an extension of crisp and fuzzy queueing systems which are very applicable and important in controlling systems. SVNQSs have been defined assuming that arrival rates and departure rates are single valued neutrosophic trapezoidal numbers, and depending on this assumption probabilities and performance measures were also single valued neutrosophic trapezoidal numbers. Numerical examples were presented and solved successfully, and because of hard computations, a maple code is presented to make calculations easier.

Keywords: Single Valued Neutrosophic Set; Single Valued Trapezoidal Neutrosophic Number; Queueing Systems; Markovian Queues; Performance Measures

1. Introduction

Fuzzy sets presented by L.A. Zadeh assume that each element x belongs to a set $A \subseteq \Omega$ with membership degree $0 \le \mu_A(x) \le 1$ and doesn't belong to the set with non-membership degree $\mu_{A^c}(x)$ where $\mu_{A^c}(x) = 1 - \mu_A(x)$ [1]. This definition was extended by K. Atanassov by assuming that $0 \le \mu_A(x) + \mu_{A^c}(x) \le 1$ or $0 \le \mu_A(x) + \nu_A(x) \le 1$ [2]. Recently F. Smarandache presented what is known by neutrosophic set as a generalization to fuzzy sets and intuitionistic fuzzy sets assuming that $^-0 \le \mu_A(x) + \delta_A(x) + \nu_A(x) \le 3^+$ where $\delta_A(x)$ is degree of indeterminacy, also F. Smarandache supposed that all these three components are subsets of nonstandard real intervals $]^-0, 1^+[$. When these components are taken as subsets of standard real intervals, we have what is named by single valued neutrosophic set, i.e., when $\mu_A(x), \delta_A(x), \nu_A(x) \in [0,1]$ and $0 \le \mu_A(x) + \delta_A(x) + \nu_A(x) \le 3$ [3] [4].

Due to this generalization, many extensions to all branches of science have been made including probability theory, statistics, reliability theory, queueing theory, artificial intelligence, data mining, algebra, linear algebra, mathematical analysis, complex analysis, differential equations, physics, philosophy etc [5] [6] [7] [8] [9] [10] [11] [12] [13] [14].

In probability theory, neutrosophic probability measure was defined by F.Smarandache as a mapping $NP: X \to [0,1]^3$ where X is a neutrosophic sample space and the neutrosophic probability function of an event $A \subseteq \Omega$ is defined by $NP(A) = \big(ch(A), ch(neutA), ch(antiA)\big) = (\alpha, \beta, \gamma)$ where $0 \le \alpha, \beta, \gamma \le 1$ and $0 \le \alpha + \beta + \gamma \le 3$, i.e., $NP(A)$ can be called a single valued probability measure.

researchers introduced many neutrosophic probability distributions like Poisson, exponential, binomial, normal, uniform, Weibull, …etc.

Later, M.B. Zeina and A. Hatip in [15] defined the neutrosophic random variable in the form $X_N = X + I$ where I is indeterminacy which satisfies $I^2 = I$ and then defined many probabilistic properties based on the last definition, which opens new philosophy to the study probability measure and many new definitions and extensions were made depending on it like in [16] [17].

One of the most important applications of probability theory is queueing theory presented by A.K. Erlang when he created mathematical models describing systems of telephone exchanges and then this theory has been applied in many telecommunication systems [18].

Queueing theory was generalized to fuzzy queueing theory to fit uncertainty in telecommunication systems in many papers like [19] [20] [21]. Many neutrosophic queueing systems were defined and studied by M.B. Zeina in many papers like [22] [23] [24] [25].

In the previous studies about neutrosophic queueing systems only indeterminacy was taken in hand where systems have been studied assuming that parameters are of the form $a + bI$ where I is indeterminacy. In this paper we will study neutrosophic queueing systems assuming that parameters (Arrivals and Departures) are single valued trapezoidal neutrosophic numbers as a more general case of fuzzy and intuitionistic fuzzy queueing systems, then we present a maple code that makes calculations easier.

2. Preliminaries

2.1 Neutrosophic Set [4]

Let E be a universe. Then, a neutrosophic set A over E is defined by:

$$A = \{ < x, (T_A(x), I_A(x), F_A(x)) > ;\ x \in E \}$$

Where:

$T_A : E \to]^-0, 1^+[$ is called truth-membership function.

$I_A : E \to]^-0, 1^+[$ indeterminacy-membership function.

$F_A : E \to]^-0, 1^+[$ falsity-membership function.

$$^-0 \le T_A(x) + I_A(x) + F_A(x) \le 3^+$$

2.2 Single Valued Neutrosophic Set [4]

Let E be a universe. Then, a single valued neutrosophic set A over E is defined by:

$$A = \{ < x, (T_A(x), I_A(x), F_A(x)) > ;\ x \in E \}$$

Where:

$T_A : E \to [0,1]$ is called truth-membership function.

$I_A : E \to [0,1]$ indeterminacy-membership function.

$F_A : E \to [0,1]$ falsity-membership function.

$$0 \le T_A(x) + I_A(x) + F_A(x) \le 3$$

This set is used in engineering, biology, decision-making and other applications because its concept is clearer and more applied than neutrosophic sets.

2.3 Single Valued Trapezoidal Neutrosophic Number (SVTNN) [26]

A single valued trapezoidal neutrosophic number is denoted by:

$$\tilde{a} = < (a_1, b_1, c_1, d_1) ; w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}} >$$

Where:

$w_{\tilde{a}}$ is the nucleus of truth membership, $u_{\tilde{a}}$ is the nucleus of indeterminacy membership and $y_{\tilde{a}}$ is the nucleus of falsity membership where:

Truth-membership function is:

$$T_{\tilde{a}}(x) = \begin{cases} \dfrac{(x - a_1)w_{\tilde{a}}}{b_1 - a_1} & ; a_1 \le x < b_1 \\ w_{\tilde{a}} & ; b_1 < x \le c_1 \\ \dfrac{(d_1 - x)w_{\tilde{a}}}{d_1 - c_1} & ; c_1 < x \le d_1 \\ 0 & ; \text{otherwise} \end{cases}$$

Indeterminacy-membership function is:

$$I_{\tilde{a}}(x) = \begin{cases} \dfrac{(b_1 - x + u_{\tilde{a}}(x - a_1))}{b_1 - a_1} & ; a_1 \le x < b_1 \\ u_{\tilde{a}} & ; b_1 < x \le c_1 \\ \dfrac{(x - c_1 + u_{\tilde{a}}(d_1 - x)}{d_1 - c_1} & ; c_1 < x \le d_1 \\ 1 & ; \text{otherwise} \end{cases}$$

Falsity-membership function is:

$$F_{\tilde{a}}(x) = \begin{cases} \dfrac{(b_1 - x + y_{\tilde{a}}(x - a_1))}{b_1 - a_1} & ; a_1 \le x < b_1 \\ y_{\tilde{a}} & ; b_1 < x \le c_1 \\ \dfrac{(x - c_1 + y_{\tilde{a}}(d_1 - x)}{d_1 - c_1} & ; c_1 < x \le d_1 \\ 1 & ; \text{otherwise} \end{cases}$$

2.3 Operations on SVTNNs [26]

Let $\tilde{a} = < (a_1, b_1, c_1, d_1) ; w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}} >, \tilde{b} = < (a_2, b_2, c_2, d_2) ; w_{\tilde{b}}, u_{\tilde{b}}, y_{\tilde{b}} >$ be two SVTNNs and $\gamma \ge 0$ any real number, then:

$\tilde{a} \oplus \tilde{b} = < (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2) ; w_{\tilde{a}} + w_{\tilde{b}} - w_{\tilde{a}} w_{\tilde{b}}, u_{\tilde{a}} u_{\tilde{b}}, y_{\tilde{a}} y_{\tilde{b}} >$

$\tilde{a} \otimes \tilde{b} = < (a_1 a_2, b_1 b_2, c_1 c_2, d_1 d_2) ; w_{\tilde{a}} w_{\tilde{b}}, u_{\tilde{a}} + u_{\tilde{b}} - u_{\tilde{a}} u_{\tilde{b}}, y_{\tilde{a}} + y_{\tilde{b}} - y_{\tilde{a}} y_{\tilde{b}} >$

$\gamma \tilde{a} = < (\gamma a_1, \gamma b_1, \gamma c_1, \gamma d_1) ; 1 - (1 - w_{\tilde{a}})^{\gamma}, u_{\tilde{a}}^{\gamma}, y_{\tilde{a}}^{\gamma} >$

$\tilde{a}^{\gamma} = < (a_1^{\gamma}, b_1^{\gamma}, c_1^{\gamma}, d_1^{\gamma}) ; w_{\tilde{a}}^{\gamma}, 1 - (1 - u_{\tilde{a}})^{\gamma}, 1 - (1 - y_{\tilde{a}})^{\gamma} >$

3. Recall of Some Crisp Queueing Systems [18]

Here we recall definitions and properties of some crisp queueing systems including M/M/1, M/M/c, M/M/c/b.

3.1 M/M/1 Queueing System

In this system we have one server with arrival rate λ customers per time unit and serving rate η customers per time unit.

The probability that we will not find customers in the system is:

$$P_0 = (1 - \rho) \quad ; \quad \rho = \frac{\lambda}{\eta}$$

The probability that we will find n customers in the system is:

$$P_n = (1 - \rho) * \rho^n \quad ; \ n = 0,1,2, \ldots$$

The average number of customers in queue is:

$$L_q = \frac{\rho^2}{1 - \rho}$$

The average number of customers in system is:

$$L_s = \frac{\rho}{1 - \rho}$$

The mean waiting time in queue is:

$$W_q = \frac{Lq}{\lambda}$$

The mean waiting time in system is:

$$W_s = \frac{Ls}{\lambda}$$

3.2 M/M/c Queueing System

In this system we have c servers with arrival rate λ and serving rate η.

The probability that we will not find customers in the system is:

$$P0 = \frac{1}{\sum_{n=0}^{c-1} \frac{\rho^n}{n!} + \frac{\rho^c}{c! * \left(1 - \frac{\rho}{c}\right)}} \quad ; \quad \rho = \frac{\lambda}{\eta} \quad , \quad \frac{\rho}{c} < 1$$

The probability that we will find n customers in the system is:

$$Pn=\begin{cases}\frac{\rho^n}{n!}*P0 & ;\ n=0,1,\dots,c-1\\ \frac{\rho^n}{c!}*\frac{P0}{c^{n-c}} & ;\ n=c,c+1,\dots\end{cases}$$

The average number of customers in queue is:

$$L_q=\frac{\rho^{c+1}}{c!}*\frac{P0}{c*(1-\frac{\rho}{c})^2}$$

The average number of customers in system is:

$$L_s=Lq+\rho$$

The mean waiting time in queue is:

$$W_q=\frac{Lq}{\lambda}$$

The mean waiting time in system is:

$$W_s=\frac{Ls}{\lambda}$$

3.3 M/M/c/b Queueing System

Here we have c servers and the system size is limited by b where $c<b$ with arrival rate λ and serving rate η.

The probability that we will not find customers in the system is:

$$P0=\frac{1}{\sum_{n=0}^{c-1}\frac{\rho^n}{c!}+\frac{\rho^c}{c!}*\left(\frac{1-\left(\frac{\rho}{c}\right)^{b-c+1}}{1-\frac{\rho}{c}}\right)}\quad ;\quad \rho=\frac{\lambda}{\eta}$$

The probability that we will find n customers in the system is:

$$Pn=\begin{cases}\frac{\rho^n}{n!}*P0 & ;\ n=0,1,\dots,c-1\\ \frac{\rho^n}{c!}*\frac{P0}{c^{n-c}} & ;\ n=c,c+1,\dots,b\end{cases}$$

The average number of customers in queue is:

$$L_q=\frac{\rho^c*P0}{c!}*\sum_{n=c+1}^{b}(n-c)*\left(\frac{\rho}{c}\right)^{n-c}$$

The effective arrival rate is:

$$\lambda_e=\lambda*(1-P_b)$$

The average number of customers in system is:

$$L_s=Lq+\frac{\lambda_e}{\eta}$$

The mean waiting time in queue is:

$$W_q=\frac{Lq}{\lambda_e}$$

The mean waiting time in system is:

$$W_s=\frac{Ls}{\lambda_e}$$

4. Single Valued Neutrosophic Queueing Systems

Here we suppose that both of arrival rate and serving rate are SVTNNs given by

$N\lambda=<(\lambda_1,\lambda_2,\lambda_3,\lambda_4);\ w_\lambda,u_\lambda,v_\lambda>,\ N\eta=<(\eta_1,\eta_2,\eta_3,\eta_4);\ w_\eta,u_\eta,v_\eta>$

Where:

$$0\le w_\lambda,u_\lambda,v_\lambda\le 1\quad \&\quad 0\le w_\eta,u_\eta,v_\eta\le 1$$
$$0\le w_\lambda+u_\lambda+v_\lambda\le 3\quad \&\quad 0\le w_\eta+u_\eta+v_\eta\le 3$$

The neutrosophic utilization coefficient is:

$$N\rho = \frac{N\lambda}{N\eta} =< \left(\frac{\lambda_1}{\eta_4},\frac{\lambda_2}{\eta_3},\frac{\lambda_3}{\eta_2},\frac{\lambda_4}{\eta_1}\right);\ w_\lambda * w_\eta, u_\lambda + u_\eta - u_\lambda * u_\eta, v_\lambda + v_\eta - v_\lambda * v_\eta >$$

We will depend on all of our results on neutrosophic operations defined in 2.3

4.1 Single Valued Neutrosophic M/M/1 Queueing System

In this system we have one server and infinite queue size and based on (2.3,3.1) we can find the following:

Neutrosophic probability that we will not find customers in the system

$$NP_0 =< (1-\rho_4, 1-\rho_3, 1-\rho_2, 1-\rho_1);\ w_\rho, u_\rho, v_\rho > \qquad (1)$$

Where w_ρ, u_ρ, v_ρ are truth, indeterminacy and falsity of neutrosophic utilization coefficient respectively.

The neutrosophic probability that we will find n customers in the system will be:

$$NP_n =< \left((1-\rho_4){\rho_1}^n, (1-\rho_3){\rho_2}^n, (1-\rho_2){\rho_3}^n, (1-\rho_1){\rho_4}^n\right); w_\rho * {w_\rho}^n, 1-\left(1-u_\rho\right)^n - \left(1-\left(1-u_\rho\right)^n\right)u_\rho + u_\rho, 1-\left(1-v_\rho\right)^n - \left(1-\left(1-v_\rho\right)^n\right)v_\rho + v_\rho > ;$$
$$n = 0,1,2,\ldots \qquad (2)$$

The neutrosophic average number of customers in queue is:

$$NL_q =< \left(\frac{{\rho^2}_1}{1-\rho_4},\frac{{\rho^2}_2}{1-\rho_3},\frac{{\rho^2}_3}{1-\rho_2},\frac{{\rho^2}_4}{1-\rho_1}\right); {w_\rho}^3, 1-\left(1-u_\rho\right)^2 - \left(1-\left(1-u_\rho\right)^2\right)u_\rho + u_\rho,$$
$$1-\left(1-v_\rho\right)^2 - \left(1-\left(1-v_\rho\right)^2\right)v_\rho + v_\rho > \qquad (3)$$

The neutrosophic average number of customers in system is:

$$NL_s =< \left(\frac{\rho_1}{1-\rho_4},\frac{\rho_2}{1-\rho_3},\frac{\rho_3}{1-\rho_2},\frac{\rho_4}{1-\rho_1}\right);\ {w_\rho}^2, 2u_\rho - {u_\rho}^2, 2v_\rho - {v_\rho}^2 > \qquad (4)$$

The neutrosophic mean waiting time in queue is:

$$NW_q = \frac{NLq}{N\lambda} =< \left(\frac{{\rho^2}_1}{\lambda_4(1-\rho_4)},\frac{{\rho^2}_2}{\lambda_3(1-\rho_3)},\frac{{\rho^2}_3}{\lambda_2(1-\rho_2)},\frac{{\rho^2}_4}{\lambda_1(1-\rho_1)}\right);\quad w_\lambda * {w_\rho}^3,$$
$$1-\left(1-u_\rho\right)^2 - \left(1-\left(1-u_\rho\right)^2\right)u_\rho + u_\rho + u_\lambda - \left(1-\left(1-u_\rho\right)^2 - \left(1-\left(1-u_\rho\right)^2\right)u_\rho + u_\rho\right) * u_\lambda,$$
$$1-\left(1-v_\rho\right)^2 - \left(1-\left(1-v_\rho\right)^2\right)v_\rho + v_\rho + v_\lambda - \left(1-\left(1-v_\rho\right)^2 - \left(1-\left(1-v_\rho\right)^2\right)v_\rho + v_\rho\right) * v_\lambda > \qquad (5)$$

The neutrosophic mean waiting time in system is:

$$NW_s = \frac{NLs}{N\lambda} =< \left(\frac{\rho_1}{\lambda_4(1-\rho_4)},\frac{\rho_2}{\lambda_3(1-\rho_3)},\frac{\rho_3}{\lambda_2(1-\rho_2)},\frac{\rho_4}{\lambda_1(1-\rho_1)}\right);\quad {w_\rho}^3,$$
$$2u_\rho - {u_\rho}^2 + u_\lambda - \left(2u_\rho - {u_\rho}^2\right) * u_\lambda, 2v_\rho - {v_\rho}^2 + v_\lambda - \left(2v_\rho - {v_\rho}^2\right) * v_\lambda > \qquad (6)$$

Example 1 (Single Valued M/M/1 Queue)

Suppose that both arrival rate and serving rate are SVTNNs given by

$N\lambda =< (1,2,3,4);\ 0.8,0.2,0.01 >,\ N\eta =< (5,6,7,8);\ 0.9,0.1,0.01 >$

The neutrosophic probability that we will find no customers in systems using equation (1) will be:

$NP_0 =< (0.2000000000, 0.5000000000, 0.7142857143, 0.8750000000), 0.72, 0.28, 0.0199 >$

Which means that with possibility 72% the probability of finding 0 customers in the system will range between 50% and 71.43% and will never be less than 20% or more than 87.5%.

We are also unsure with percentage of indeterminacy 28% of these results and we may be false by 1.9% falsity degree.

Fig1. Below shows all the possibilities of different truth, indeterminacy and falsity degrees by drawing a horizontal line on the graph, e.g. if we draw a horizontal line at y=0.4 we say that: with truth degree 40% P0 will lay between about 0.36 and about 0.78, with indeterminacy degree 60% (which is 1-y) P0 will lay between 0.45 and 0.74 and with falsity degree 60% (which is also 1-y) P0 will lay between 0.44 and 0.75.

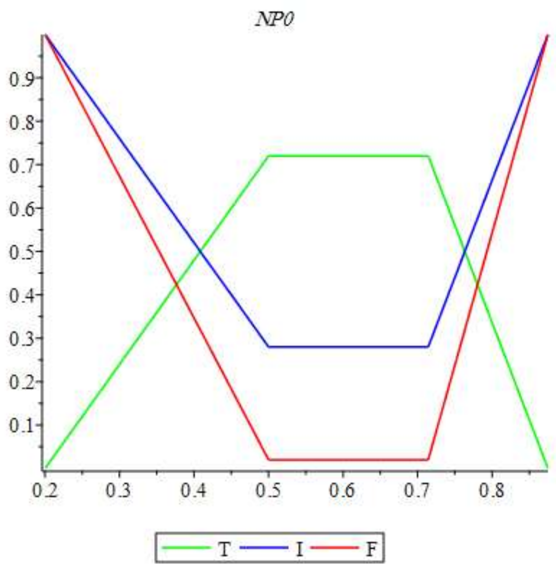

Fig1. NP0 in single valued neutrosophic M/M/1 queue

The neutrosophic probability that we will find 3 customers in system using equation (2) will be:

$$NP_3 = < (0.0003906250000, 0.01166180758, 0.08928571429, 0.4480000000), 0.26873856, 0.73126144, 0.07725530557 >$$

Which means that with possibility 26.9% the probability of finding 3 customers in the system will range between 1.2% and 8.9% and will never be less than 0.04% or more than 44.8%.

We are also unsure with percentage of indeterminacy 73% of these results and we may be false by 7.7% falsity degree.

Fig2. Below shows all the possibilities of different truth, indeterminacy and falsity degrees of NP3:

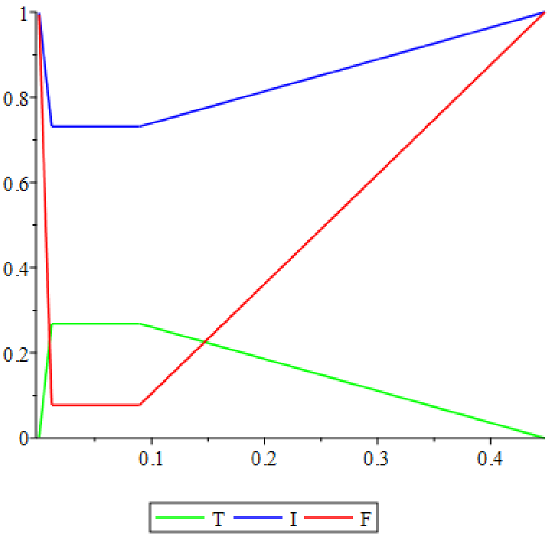

Fig2. NP3 in single valued neutrosophic M/M/1 queue

Neutrosophic performance measures using equations (3) to (6) will be:

$$NL_q = < (0.01785714286, 0.1142857143, 0.5000000000, 3.200000000), 0.373248, 0.626752, 0.05851985060 >$$

Which means that surely 37% the average number of customers in queue will range between 11% and 50% and will never be less than 1.8% or more than 3.20.

We are also unsure with percentage of indeterminacy 62.7% of these results and we may be false by 5.8% falsity degree.

Fig3. Below shows all the possibilities of different truth, indeterminacy and falsity degrees of NLq:

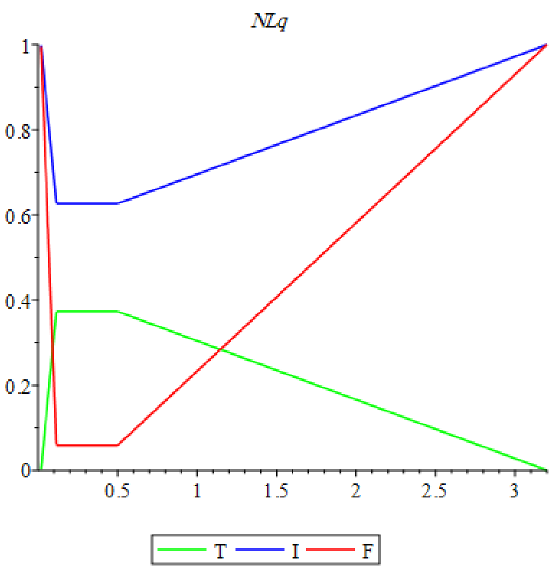

Fig3. NLq in single valued neutrosophic M/M/1 queue

$$NL_s = < (0.1428571429, 0.4000000000, 1., 4.), 0.5184, 0.4816, 0.03940399 >$$

Which means that surely 51.8% the average number of customers in system will range between 40% and 100% and will never be less than 14.3% or more than 4.

We are also unsure with percentage of indeterminacy 48% of these results and we may be false by 3.9% falsity degree.

Fig4. Below shows all the possibilities of different truth, indeterminacy and falsity degrees of NLs:

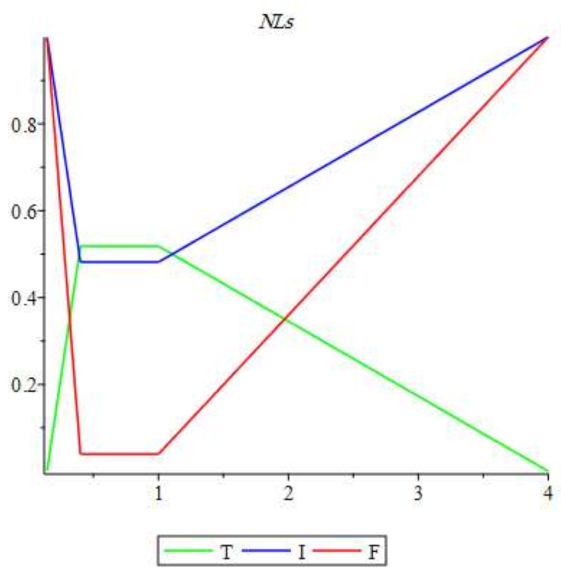

Fig4. NLs in single valued neutrosophic M/M/1 queue

$$NW_q = < (0.004464285715, 0.03809523810, 0.2500000000, 3.200000000), 0.2985984, 0.7014016, 0.06793465209 >$$

Which means that surely 29.8% the mean waiting time in queue will range between 3.8% and 25% and will never be less than 0.44% or more than 3.20.

We are also unsure with percentage of indeterminacy 70% of these results and we may be false by 6.8% falsity degree.

Fig5. Below shows all the possibilities of different truth, indeterminacy and falsity degrees of NWq:

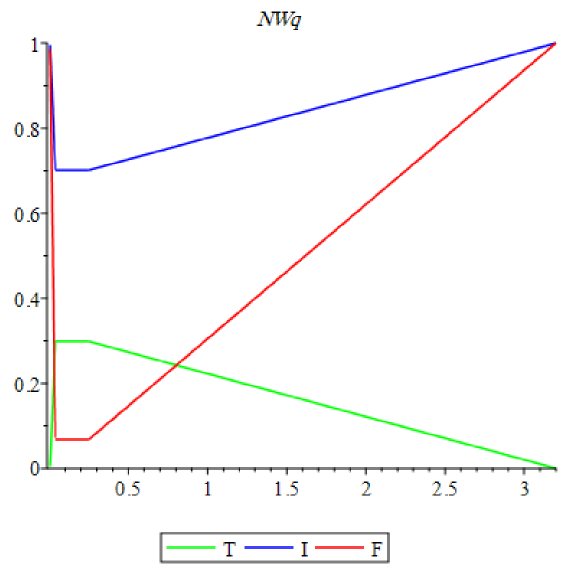

Fig5. NWq in single valued neutrosophic M/M/1 queue

$$NW_s =< (0.03571428572, 0.1333333333, 0.5000000000, 4.), 0.41472, 0.58528, 0.0490099501 >$$

Which means that surely 41% the mean waiting time in system will range between 13.3% and 50% and will never be less than 3.5% or more than 4.

We are also unsure with percentage of indeterminacy 58.5% of these results and we may be false by 4.9% falsity degree.

Fig6. Below shows all the possibilities of different truth, indeterminacy and falsity degrees of NWs:

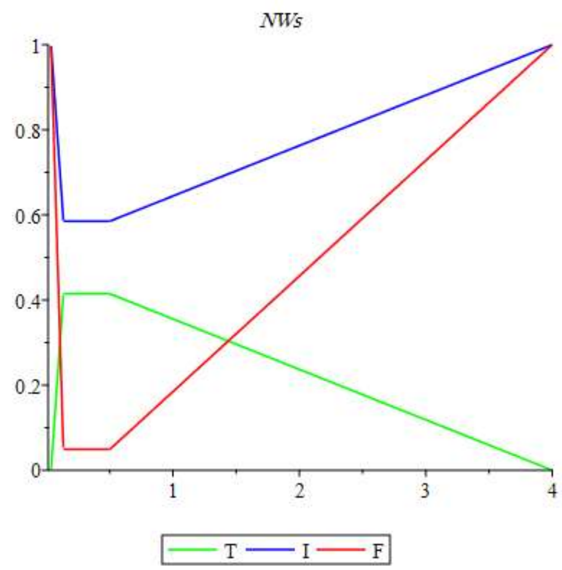

Fig6. NWs in single valued neutrosophic M/M/1 queue

4.2 Single Valued Neutrosophic M/M/c and M/M/c/b Queueing System

In M/M/c system we have c parallel servers working homogeneously with neutrosophic arrivals and neutrosophic departures and unlimited queue size. M/M/c/b is as same as M/M/c except that the first one has limited queue size. Finding probabilities of these queueing systems cannot be done by hand because of complex calculations so we will present a Maple package of code which makes calculations easier.

4.3 Single Valued Neutrosophic Queueing Systems Package

Notice that equations (1-6) are very hard to be applied by hand calculating, so we wrote a maple code depending on our previous neutrosophic package in [14] to handle neutrosophic queues easily.

A general procedure with overloading in parameters named QueueingSystem(NLambda,NMu,n,c,b) calls a procedure to calculate neutrosophic probability of finding 0 customers in the system, neutrosophic probability of finding n customers in the system, neutrosophic average number of customers in the system, neutrosophic average number of customers in the queue, neutrosophic mean time in the system and neutrosophic mean time in the queue then plots these numbers, where:

NLambda is $N\lambda$ (neutrosophic arrival rate)

NMu is $N\mu$ (neutrosophic departure rate)

n is number of customers that we want to calculate the neutrosophic probability of finding them in the system.

c is number of servers working in parallel.

b is size of system (including servers).

We can use this procedure to describe different types of queues as shown in table 1:

Procedure	Neutrosophic Queueing System
QueueingSystem(NLambda,NMu,n,c,b)	NM/NM/c/b
QueueingSystem(NLambda,NMu,n,c)	NM/NM/c/∞ or NM/NM/c
QueueingSystem(NLambda,NMu,n)	NM/NM/1/∞ or NM/NM/1

Table 1. Single valued neutrosophic queueing systems overloaded procedures.

Also, a plotting procedure is programmed to plot the results as follows:

SVTNQPlot(QueueingSystem(NLambda,NMu,OVERLOADED PARAMETERS),measure);

Where 'measure' can be replaced by p0,pn,lq,ls,wq or ws.

Example 2 (Single Valued M/M/c Queue)

suppose that we have a queueing system with arrival rate given as follows:

$N\lambda =< (2,2.5,3,3.5);\ \ 0.8,0.2,0.01 >,\ N\eta =< (5,5.5,6,6.5);\ \ 0.9,0.1,0.01 >$

We can define these rates using the code:

NLambda :=SVTN(2,2.5,3,3.5,0.8,0.2,0.01);NMu:=SVTN(5,5.5,6,6.5,0.9,0.1,0.01);

Suppose that we have 2 servers, and we would like to calculate the following:

1. Probability of finding no customers in the system.
2. Probability of finding 3 customers in the system.
3. Performance measures.

These can be done by typing:

QueueingSystem(NLambda,NMu,3,2);

Which results:

The neutrosophic probability that we will find no customers in systems using equation (1) will be:

$$NP_0 = \begin{bmatrix} [0.4814814815, 0.5714285711, 0.6551724136, 0.7333333331], \\ 0.96224256, 0.03775744, 0.00001560437408 \end{bmatrix}$$

Which means that with possibility 96.2% the probability of finding 0 customers in the system will range between 57% and 65.5% and will never be less than 48% or more than 73.3%.

We are also unsure with percentage of indeterminacy 3.8% of these results and we may be false by 0.002% falsity degree.

SVTNQPlot(QueueingSystem(NLambda,NMu,3,2),p0);

Fig7. Below shows all the possibilities of different truth, indeterminacy and falsity degrees of NP0:

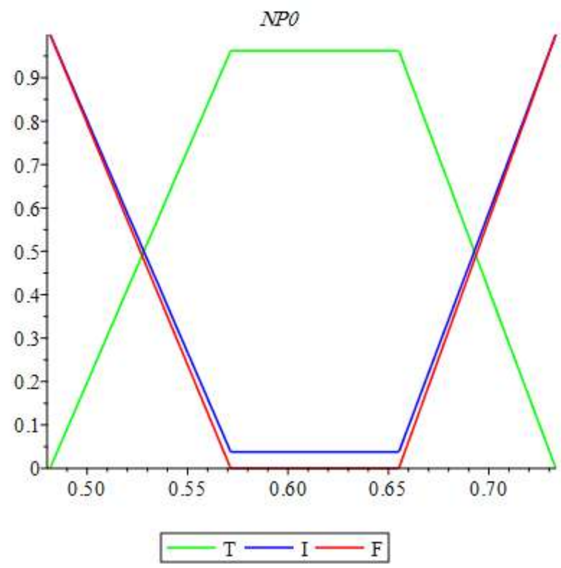

Fig7. NP0 in single valued neutrosophic M/M/c queue

The neutrosophic probability that we will find 3 customers in system using equation (2) will be:

$$NP_3 = \begin{bmatrix} [0.003506465046, 0.01033399471, 0.02658099950, 0.06288333332], \\ 0.1678434279, 0.8321565721, 0.2449034052 \end{bmatrix}$$

Which means that with possibility 16.8% the probability of finding 3 customers in the system will range between 1.04% and 2.7% and will never be less than 0.35% or more than 6.3%.

We are also unsure with percentage of indeterminacy 83% of these results and we may be false by 24.5% falsity degree.

SVTNQPlot(QueueingSystem(NLambda,NMu,3,2),pn);

Fig8. Below shows all the possibilities of different truth, indeterminacy and falsity degrees of NP3:

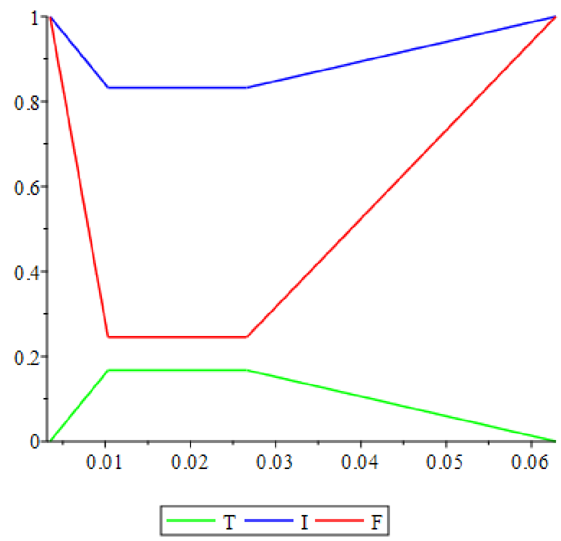

Fig8. NP3 in single valued neutrosophic M/M/c queue

Neutrosophic performance measures using equations (3) to (6) will be:

$$NL_q = \begin{bmatrix} [0.004897459442, 0.01648858989, 0.05025470221, 0.1488362919], \\ 0.2004575439, 0.7995424561, 0.2419205946 \end{bmatrix}$$

Which means that surely 20% the average number of customers in queue will range between 1.65% and 5.03% and will never be less than 0.48% or more than 14.9%.

We are also unsure with percentage of indeterminacy 79.9% of these results and we may be false by 24.2% falsity degree.

SVTNQPlot(QueueingSystem(NLambda,NMu,3,2),lq);

Fig9. Below shows all the possibilities of different truth, indeterminacy and falsity degrees of NLq:

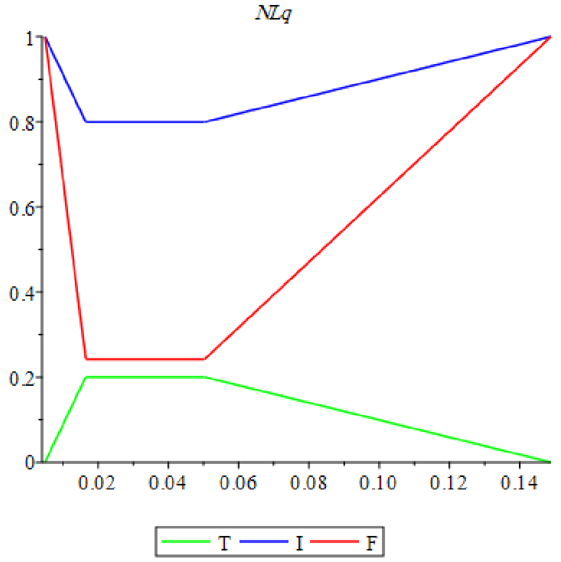

Fig9. NLq in single valued neutrosophic M/M/c queue

$$NL_s = \begin{bmatrix} [0.3125897671, 0.4331552566, 0.5957092477, 0.8488362919], \\ 0.7761281123, 0.2238718877, 0.004814219833 \end{bmatrix}$$

Which means that surely 77.6% the average number of customers in system will range between 43.3% and 59.6% and will never be less than 31.3% or more than 84.9%.

We are also unsure with percentage of indeterminacy 22.4% of these results and we may be false by 0.48% falsity degree.

SVTNQPlot(QueueingSystem(NLambda,NMu,3,2),ls);

Fig10. Below shows all the possibilities of different truth, indeterminacy and falsity degrees of NLs:

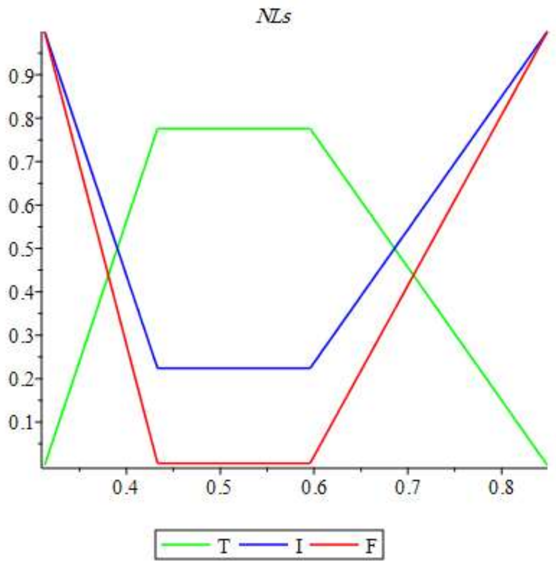

Fig10. NLs in single valued neutrosophic M/M/c queue

$$NW_q = \begin{bmatrix} [0.001399274126, 0.005496196630, 0.02010188088, 0.07441814595e], \\ 0.1603660351, 0.8396339649, 0.2495013887 \end{bmatrix}$$

Which means that surely 16% the mean waiting time in queue will range between 0.55% and 2.01% and will never be less than 0.14% or more than 7.44%.

We are also unsure with percentage of indeterminacy 83.9% of these results and we may be false by 24.9% falsity degree.

SVTNQPlot(QueueingSystem(NLambda,NMu,3,2),wq);

Fig11. Below shows all the possibilities of different truth, indeterminacy and falsity degrees of NWq:

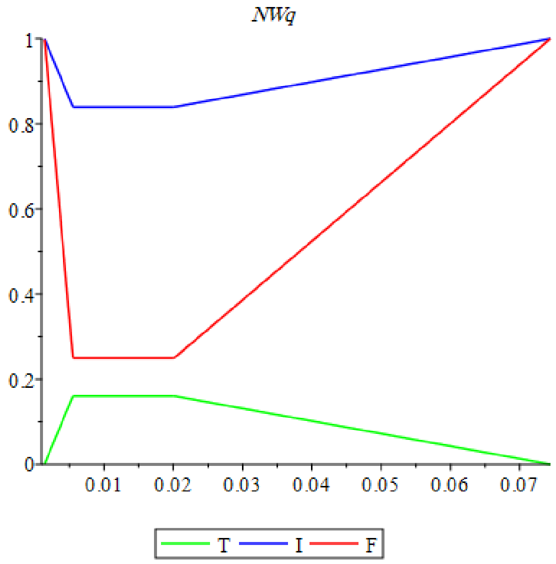

Fig11. NWq in single valued neutrosophic M/M/c queue

$$NW_s = \begin{bmatrix} [0.08931136203, 0.1443850855, 0.2382836991, 0.4244181460], \\ 0.6209024898, 0.3790975102, 0.01476607764 \end{bmatrix}$$

Which means that surely 62% the mean waiting time in system will range between 14.4% and 24% and will never be less than 8.9% or more than 42%.

We are also unsure with percentage of indeterminacy 37.9% of these results and we may be false by 1.4% falsity degree.

SVTNQPlot(QueueingSystem(NLambda,NMu,3,2),ws);

Fig12. Below shows all the possibilities of different truth, indeterminacy and falsity degrees of NWs:

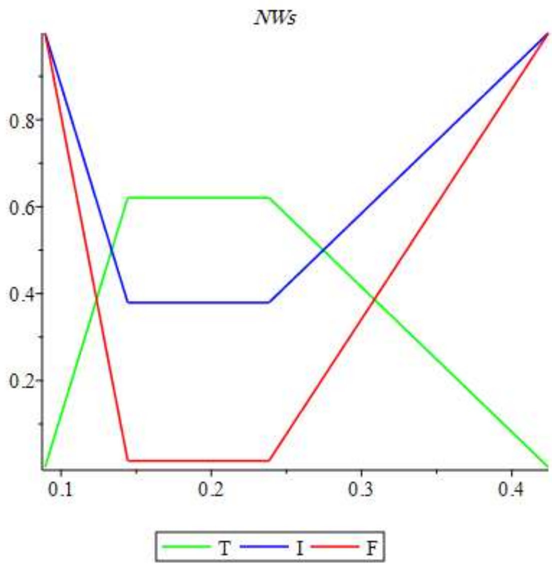

Fig12. NWs in single valued neutrosophic M/M/c queue

Example 3 (Single Valued M/M/c/b Queue)

Suppose that we have a queueing system with:

$N\lambda =< (2,2.5,3,3.5);\ 1,0.01,0.01 >$, $N\eta =< (4,4.5,5,5.5);\ 0.9,0.1,0.01 >$

Suppose that we have 2 servers and system size limited by 4, and we would like to calculate the following:

1.Probability of finding no customers in the system.

2.Probability of finding 4 customers in the system.

3.Performance measures.

These can be done by typing:

NLambda :=SVTN(2,2.5,3,3.5,1,0.01,0.01);NMu:=SVTN(4,4.5,5,5.5,0.9,0.1,0.01);

QueueingSystem(NLambda,NMu,4,2,4);

Which results:

The neutrosophic probability that we will find no customers in systems using equation (1) will be:

$$NP_0 = \begin{bmatrix} [0.3919316843, 0.5013054829, 0.6022304833, 0.6955662590], \\ 0.9912331324, 0.01057498123, 0.0002232497869 \end{bmatrix}$$

Which means that with possibility 99% the probability of finding 0 customers in the system will range between 50% and 60% and will never be less than 39% or more than 69.5%.

We are also unsure with percentage of indeterminacy 1.05% of these results and we may be false by 0.02% falsity degree.

SVTNQPlot(QueueingSystem(NLambda,NMu,4,2,4),p0);

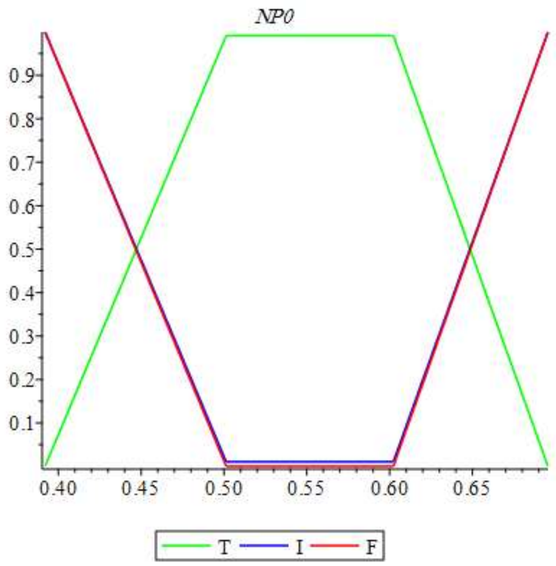

Fig13. NP0 in single valued neutrosophic M/M/c/b queue

The neutrosophic probability that we will find 4 customers in system using equation (2) will be:

$$NP_n = \begin{bmatrix} [0.0008566227644, 0.003916449084, 0.01486988847, 0.05096602137], \\ 0.2870206713, 0.7337556823, 0.3662088099 \end{bmatrix}$$

Which means that with possibility 28.7% the probability of finding 4 customers in the system will range between 0.39% and 1.5% and will never be less than 0.08% or more than 5.09%.

We are also unsure with percentage of indeterminacy 73.4% of these results and we may be false by 36.6% falsity degree.

SVTNQPlot(QueueingSystem(NLambda,NMu,4,2,4),pn);

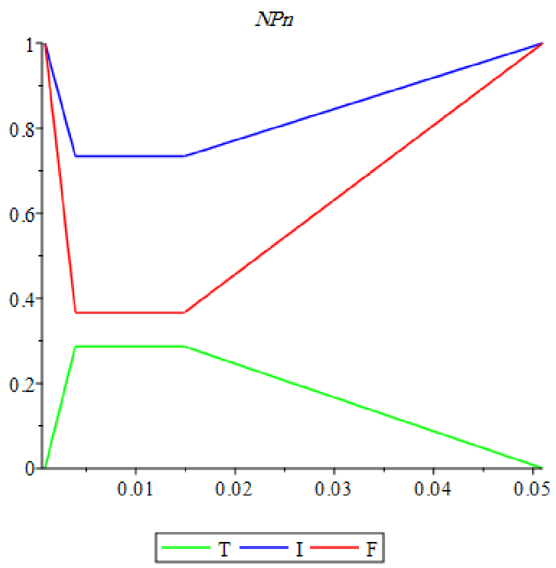

Fig14. NPn in single valued neutrosophic M/M/c/b queue

Neutrosophic performance measures using equations (3) to (6) will be:

$$NL_q = \begin{bmatrix} [0.006424670732, 0.02349869451, 0.07434944242, 0.2184258058], \\ 0.5090336259, 0.5139856353, 0.2064567181 \end{bmatrix}$$

Which means that surely 50% the average number of customers in queue will range between 2.4% and 7.4% and will never be less than 0.64% or more than 21.8%.
We are also unsure with percentage of indeterminacy 51.4% of these results and we may be false by 20.6% falsity degree.
SVTNQPlot(QueueingSystem(NLambda,NMu,4,2,4),lq);

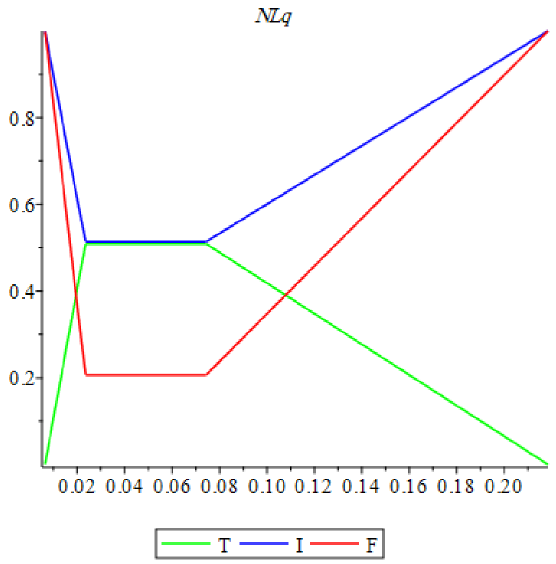

Fig15. NLq in single valued neutrosophic M/M/c/b queue

$$NL_s = \begin{bmatrix} [0.3515279356, 0.5160637503, 0.7384051431, 1.092676261], \\ 0.6358593744, 0.3920560678, 0.07821019297 \end{bmatrix}$$

Which means that surely 63.6% the average number of customers in system will range between 52% and 74% and will never be less than 35% or more than 1.1.
We are also unsure with percentage of indeterminacy 39% of these results and we may be false by 7.8% falsity degree.
SVTNQPlot(QueueingSystem(NLambda,NMu,4,2,4),ls);

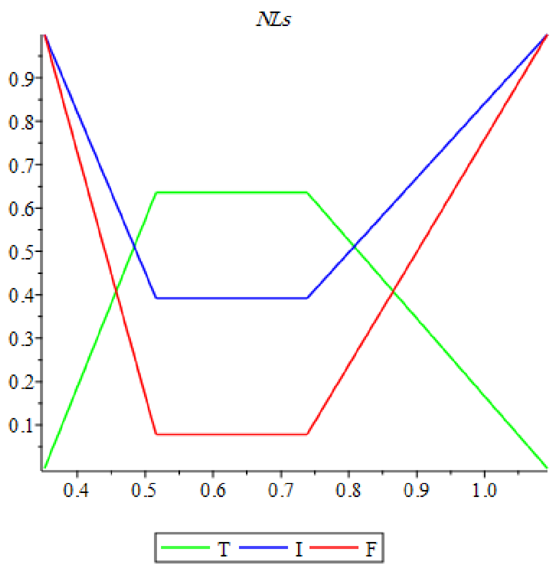

Fig16. NLs in single valued neutrosophic M/M/c/b queue

$$NW_q = \begin{bmatrix} [0.001837193992, 0.007863695934, 0.03018867926, 0.1150779691], \\ 0.1461031730, 0.8718954227, 0.5020886664 \end{bmatrix}$$

Which means that surely 14.6% the mean waiting time in queue will range between 0.8% and 3% and will never be less than 0.2% or more than 11.5%.

We are also unsure with percentage of indeterminacy 87% of these results and we may be false by 50% falsity degree.

SVTNQPlot(QueueingSystem(NLambda,NMu,4,2,4),wq);

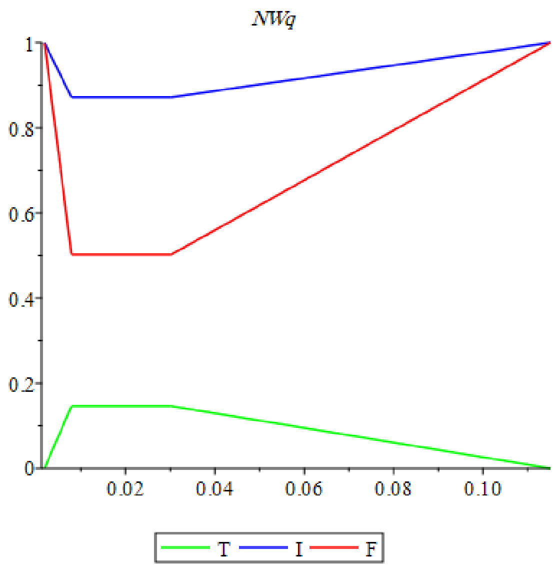

Fig17. NWq in single valued neutrosophic M/M/c/b queue

$$NW_s = \begin{bmatrix} [0.1005226630, 0.1726976115, 0.2998203524, 0.5756781558], \\ 0.1825047845, 0.8397569988, 0.4216199638 \end{bmatrix}$$

Which means that surely 18% the mean waiting time in system will range between 29.9% and 83.9% and will never be less than 10% or more than 57.5%.

We are also unsure with percentage of indeterminacy 83.9% of these results and we may be false by 42% falsity degree.

SVTNQPlot(QueueingSystem(NLambda,NMu,4,2,4),ws);

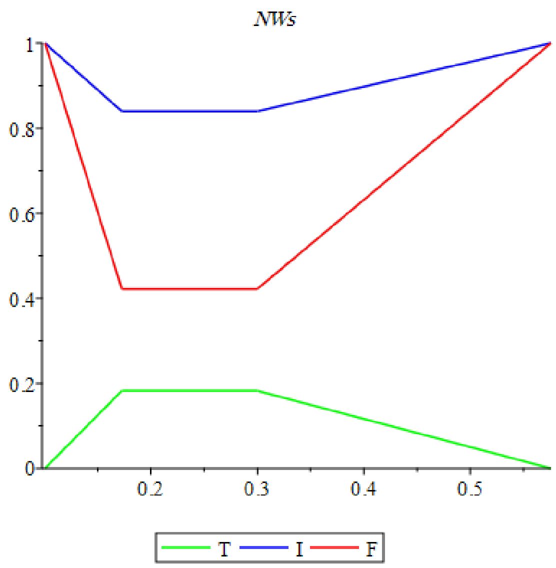

Fig18. NWs in single valued neutrosophic M/M/c/b queue

6. Conclusions

Ignoring indeterminacy may lead decision makers to make wrong decisions especially in controlling systems which is one of the most important applications of queueing theory. We found that neutrosophic queues are more reliable and applicable than crisp queues because of dealing with indeterminacy and uncertainty. We are looking forward to study and define more queuing systems in neutrosophic logic including tandem networks, open Jackson networks, balk queues, etc.

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Appendix (Maple Code)

```
interface(warnlevel = 0):

with(plots):with(plottools):

SVTN := proc (a1, a2, a3, a4, wa, ua, ya)

return [[a1, a2, a3, a4], wa, ua, ya];

end proc:

SVTNNPlotT:=proc(n::list)

n1:=n[1][1];

n2:=n[1][2];

n3:=n[1][3];

n4:=n[1][4];
```

```
t:=n[2];
i:=n[3];
f:=n[4];
lt:=piecewise(n1<w and w<n2,t*(w-n1)/(n2-n1),undefined);
mt:=piecewise(n2<w and w<n3,t,undefined);
rt:=piecewise(n3<w and w<n4,t*(w-n4)/(n3-n4),undefined);
plot([lt(w),mt(w),rt(w)],w=n1..n4,color="green",legend=["T","",""],labels=["",""]);
end proc:
SVTNNPlotI:=proc(n::list)
n1:=n[1][1];
n2:=n[1][2];
n3:=n[1][3];
n4:=n[1][4];
t:=n[2];
i:=n[3];
f:=n[4];
li:=piecewise(n1<u and u<n2,(n2-u+i*(u-n1))/(n2-n1),undefined);
mi:=piecewise(n2<u and u<n3,i,undefined);
ri:=piecewise(n3<u and u<n4,(u-n3+i*(n4-u))/(n4-n3),undefined);
plot([li(u),mi(u),ri(u)],u=n1..n4,color="blue",legend=["I","",""],labels=["",""]);
end proc:
SVTNNPlotF:=proc(n::list)
n1:=n[1][1];
n2:=n[1][2];
n3:=n[1][3];
n4:=n[1][4];
t:=n[2];
i:=n[3];
```

```
f:=n[4];

lf:=piecewise(n1<y and y<n2,(n2-y+f*(y-n1))/(n2-n1),undefined);

mf:=piecewise(n2<y and y<n3,f,undefined);

rf:=piecewise(n3<y and y<n4,(y-n3+f*(n4-y))/(n4-n3),undefined);

plot([lf(y),mf(y),rf(y)],y=n1..n4,color="red",legend=["F","",""],labels=["",""]);

end proc:

SVTNPlot:=proc(n::list,myTitle)

t:=SVTNNPlotT(n):

i:=SVTNNPlotI(n):

f:=SVTNNPlotF(n):

display([t, i, f],title=myTitle);

end proc:

SVTNQPlot:=proc(q::Matrix,property)

properties:=[p0,pn,lq,ls,wq,ws];

i:=ListTools[SearchAll](property,properties);

n:=rhs(op(convert(q[i],list)));

myTitle:=lhs(op(convert(q[i],list)));

SVTNPlot(n,myTitle);

end proc:

CrispNumberSS := proc (n) return SVTN(n, n, n, n, 0, 1, 1); end proc:

CrispNumberMD := proc (n) return SVTN(n, n, n, n, 1, 0, 0); end proc:

CrispNumber:=proc (n,NRho) return SVTN(n, n, n, n, NRho[2], NRho[3], NRho[4]); end proc:

SVTNSum := proc (t1, t2) x := t1; y := t2;

L1 := x[1][1]+y[1][1], x[1][2]+y[1][2], x[1][3]+y[1][3], x[1][4]+y[1][4];

U1 := x[2]+y[2]-x[2]*y[2], x[3]*y[3], x[4]*y[4]; [[L1], U1]; end proc:

SVTNSub := proc (t1, t2) x := t1; y := t2;

L1 := x[1][1]-y[1][4], x[1][2]-y[1][3], x[1][3]-y[1][2], x[1][4]-y[1][1];

U1 := x[2]+y[2]-x[2]*y[2], x[3]*y[3], x[4]*y[4]; [[L1], U1]; end proc:
```

```
SVTNMult := proc (t1, t2) x := t1; y := t2;

L1 := x[1][1]*y[1][1], x[1][2]*y[1][2], x[1][3]*y[1][3], x[1][4]*y[1][4];

U1 := x[2]*y[2], x[3]+y[3]-x[3]*y[3], x[4]+y[4]-x[4]*y[4]; [[L1], U1]; end proc:

SVTNScalarMult := proc (t1, PN) x := t1;

L1 := PN*x[1][1], PN*x[1][2], PN*x[1][3], PN*x[1][4];

U1 := 1-(1-x[2])^PN, x[3]^PN, x[4]^PN; [[L1], U1]; end proc:

SVTNDiv := proc (t1, t2) x := t1; y := t2;

L1 := x[1][1]/y[1][4], x[1][2]/y[1][3], x[1][3]/y[1][2], x[1][4]/y[1][1];

U1 := x[2]*y[2], x[3]+y[3]-x[3]*y[3], x[4]+y[4]-x[4]*y[4]; [[L1], U1]; end proc:

SVTNPower := proc (t1, PN) x := t1;

L1 := x[1][1]^PN, x[1][2]^PN, x[1][3]^PN, x[1][4]^PN;

U1 := x[2]^PN, 1-(1-x[3])^PN, 1-(1-x[4])^PN; [[L1], U1]; end proc:

SVTNSeries:=proc(x,n,NRho)

S:=CrispNumber(1,NRho);

for i from 1 by 1 to n-1 do

S:=SVTNSum(S,SVTNScalarMult(SVTNPower(x,i),1/(i!)));

end do;

S;

end proc:

SVTNSeriesmmcb:=proc(x,c,b)

Lq:=SVTNScalarMult(x,1/c);

for i from c+2 by 1 to b do

Lq:=SVTNSum(Lq,SVTNScalarMult(SVTNPower(SVTNScalarMult(x,1/c),i-c),i-c));

end do;

Lq;

end proc:

QueueingSystem:=overload(

[
```

```
proc(NLambda::list,NMu::list,n::integer,c::integer,b::integer) option overload;

NRho:=SVTNDiv(NLambda, NMu);

NP0:=evalf(SVTNDiv(CrispNumberMD(1),SVTNSum(SVTNSeries(NRho,c,NRho),SVTNMult(SVT
NScalarMult(SVTNPower(NRho,c),1/(c!)),SVTNDiv(SVTNSub(CrispNumberSS(1),SVTNPower(SV
TNScalarMult(NRho,1/c),b-c+1)),SVTNSub(CrispNumberSS(1),SVTNScalarMult(NRho,1/c)))))));

if 0 <= n and n < c then
NPn:=evalf(SVTNMult(SVTNScalarMult(SVTNPower(NRho,n),1/(n!)),NP0));

NPb:=evalf(SVTNMult(SVTNScalarMult(SVTNPower(NRho,b),1/(b!)),NP0));

elif c <= n and n <= b then
NPn:=evalf(SVTNMult(SVTNScalarMult(SVTNPower(NRho,n),1/(c!)),SVTNScalarMult(NP0,1/(c^(n
-c)))));

NPb:=evalf(SVTNMult(SVTNScalarMult(SVTNPower(NRho,b),1/(c!)),SVTNScalarMult(NP0,1/(c^(b
-c)))));

end if;

NLq:=SVTNMult(SVTNMult(SVTNScalarMult(SVTNPower(NRho,c),1/c!),NP0),SVTNSeriesmmcb(
NRho,c,b));

NLambdae:= evalf(SVTNMult(NLambda,SVTNSub(CrispNumberSS(1),NPb)));

NLs:=evalf(SVTNSum(NLq,SVTNDiv(NLambdae,NMu)));

NWs:=evalf(SVTNDiv(NLs,NLambdae));

NWq:=evalf(SVTNDiv(NLq,NLambdae));

<'NP0'=NP0,

'NPn'=NPn,

'NLq'=NLq,

'NLs'=NLs,

'NWq'=NWq,

'NWs'=NWs>;

end proc,

#MMc

proc(NLambda,NMu,n,c) option overload;

NRho:=SVTNDiv(NLambda, NMu);
```

```
NP0:=evalf(SVTNDiv(CrispNumberMD(1),SVTNSum(SVTNSeries(NRho,c,NRho),SVTNDiv(SVTN
Power(NRho,c),SVTNScalarMult(SVTNSub(CrispNumberMD(1),SVTNScalarMult(NRho,1/c)),c!))))
);

NLq:=evalf(SVTNMult(SVTNScalarMult(SVTNPower(NRho,c+1),1/(c!)),SVTNDiv(NP0,SVTNScalar
Mult(SVTNPower(SVTNSub(CrispNumberMD(1),SVTNScalarMult(NRho,1/c)),2),c))));

NLs:=evalf(SVTNSum(NLq,NRho));

NWs:=evalf(SVTNDiv(NLs,NLambda));

NWq:=evalf(SVTNDiv(NLq,NLambda));

if n>=0 and n < c then NPn:= evalf(SVTNMult(SVTNScalarMult(SVTNPower(NRho,n),1/(n!)),NP0));

elif n>=c then NPn:=
evalf(SVTNMult(SVTNScalarMult(SVTNPower(NRho,n),1/(c!)),SVTNScalarMult(NP0,1/(c^(n-c)))));
end if;

<'NP0'=NP0,

'NPn'=NPn,

'NLq'=NLq,

'NLs'=NLs,

'NWq'=NWq,

'NWs'=NWs>;

end proc,

#MM1

proc(NLambda,NMu,n) option overload;

NRho:=SVTNDiv(NLambda, NMu);

NLs:=evalf(SVTNDiv(NRho, SVTNSub(CrispNumberSS(1),NRho)));

NLq:=evalf(SVTNDiv(SVTNPower(NRho,2), SVTNSub(CrispNumberSS(1),NRho)));

NWs:=evalf(SVTNDiv(NLs, NLambda));

NWq:=evalf(SVTNDiv(NLq, NLambda));

NP0:=evalf(SVTNSub(CrispNumberSS(1),NRho));

NPn:=evalf(SVTNMult(NP0, SVTNPower(NRho, n)));

<'NP0'=NP0,

'NPn'=NPn,
```

```
'NLq'=NLq,

'NLs'=NLs,

'NWq'=NWq,

'NWs'=NWs>;

end proc

]

):
```

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An abstract approach to convex and concave sets under refined neutrosophic set environment

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Abstract. A refined neutrosophic set (RNS) is an extension of a neutrosophic set in which all the uncertain belonging-based entities like belonging-grade, non-belonging-grade, and indeterminate-grade are further categorized into their respective sub-belonging grades, sub-non-belonging-grades, and sub-indeterminate-grades, respectively. In other words, the RNS provides multi sub-grades for each uncertain component of theneutrosophic set. This study is aimed to integrate the classical concepts of convexity and concavity with RNS to make the RNS applicable to various optimization problems.Thus, convex RNS and concave RNS are developed. Some of their important aggregation operations and results are investigated and then modified.

Keywords: Sub-belonging grade; Sub non-belonging grade; Sub-indeterminacy grade; Infimum projection; Supremum projection; Ortho-convexity; Ortho-concavity.

---

1. **Introduction**

To deal with uncertainty, Zadeh [1] proposed a fuzzy set (FS) in 1965. Each component of the universe under investigation is given a belonging grade from the range $[0, 1]$ in an FS. Zadeh [2] used his own idea of FSs as the foundation for a theory of possibility. The link between FSs and probability theories was studied by Dubois et al. [4, 5]. For algebraic operations carried out between random set-valued variables, they derived the monotonicity property. Dubois et al. [3] performed research on ranking fuzzy numbers in the context of possibility theory. Beg et al. computed similarities between FSs under specific implications [6–8]. The solution of nonlinear partial differential equations in a fuzzy environment was determined by Osman et al. [9]. Khan et al. [10] envisaged some semi-groups in the context of fuzzy interior intuitionistic ideals. With applications in both the first and second senses, Rahman et al. [11] and Ihsan et al. [26] proposed the conceptual framework of (m,n)-convexity-cum-concavity on fuzzy soft set and fuzzy soft expert set, respectively.

Only being a member is insufficient in some real-world situations. Atanassov conceptualized an intuitionistic fuzzy set (IFS) to make the FSs suitable for the non-belonging grade in 1986 [13, 14]. Each component of the universe of discourse receives an allocation of both belonging value and non-belonging value from a [0,1]. The generalization of the FS, the IFS, has shown to be a very useful tool for academics. With their study of operations, algebra, model operators, and normalization on IFSs, Ejegwa et al. [15] broadened the concept.

Since both Zadeh's FS and Atanassov's IFS are insufficient for the grade of indeterminacy, Smarandache [16] devised the neutrosophic set (NS) to overcome these drawbacks. Additionally, because the NS does not impose the dependency requirement on uncertain components, truthfulness, falseness, and indeterminacy grades are independent and can take on any value inside a closed unit interval.

The concept of a concave FS was presented by Chaudhuri [17, 18]. He also examined some of the sets' valuable qualities and defined some of their related concepts and computing methods. The development of fuzzy geometry and fuzzy structures can benefit from this idea. This idea was improved by Yu-Ru Syau [19] to include convex and concave fuzzy mappings. Concavo-convex FSs were introduced by Sarkar [20], who also established some of its intriguing characteristics. The discussion on convex IFSs given by Ban [21, 22] led to the development of convex temporal IFSs. The collection of convex IFSs was described and its generalized qualities were covered in depth by Dı́az et al. [23]. Sarkar [26] discusses convexity on the NS.

Smarandache [24] introduced refinements in FS-like structures including NS by developing their relevant models with refined settings which categorizes the uncertain grades of these models into their respective sub-grades. Rahman et al. [25] studied the fundamental properties, operations, and results of refined IFSs with examples. The researches [21, 22, 24, 26, 27] have many concepts which lead to the motivation of this study and thus convex and concave sets are generalized under refined NS (RNS). Additionally, few significant properties and results are investigated in this context.

The remaining portion of the paper has been divided into three sections: section 2, section 3, and section 4. Section 2 is about the recalling of some important definitions, section 3 is aimed to investigate the notions of classical convexity and concavity under the RNS environment along with modifications of various results, and the last section summarizes the paper accompanied by future scope.

2. Preliminaries

This portion is aimed to recall few definitions which assist the readers to understand the main concepts. The acronyms $\hat{\Delta}$, $\mathscr{G}$, I, $\hat{\zeta}$, $\hat{\vartheta}$ and $\hat{\xi}$ are meant for initial set of objects, $\mathscr{R}^n$, $[0,1]$, true-belonging, false-belonging and indeterminate-belonging functions respectively.

Definition 2.1. [1,2] A FS $\hat{\Lambda}$ is stated as $\hat{\Lambda} = \left\{ \left(\hat{\wp}, \hat{\zeta}_{\hat{\Lambda}}(\hat{\wp}) \right) : \hat{\wp} \in \hat{\Delta} \right\}$ such that $\hat{\zeta}_{\hat{\Lambda}} : \hat{\Delta} \to [0,1]$ with $\hat{\zeta}_{\hat{\Lambda}}(\hat{\wp}) \in [0,1]$ as belonging-grade of $\hat{\wp}$ in $\hat{\Delta}$. If $\hat{\Lambda}_1$ and $\hat{\Lambda}_2$ are FSs then

(1) $\hat{\Lambda}^c = \left\{ \left(\hat{\wp}, 1 - \hat{\zeta}_{\hat{\Lambda}}(\hat{\wp}) \right) : \hat{\wp} \in \hat{\Delta} \right\}$.
(2) $\hat{\Lambda}_3 = \hat{\Lambda}_1 \cup \hat{\Lambda}_2 = \left\{ \left(\hat{\wp}, max\{\hat{\zeta}_{\hat{\Lambda}_1}(\hat{\wp}), \hat{\zeta}_{\hat{\Lambda}_2}(\hat{\wp})\} \right) : \hat{\wp} \in \hat{\Delta} \right\}$.
(3) $\hat{\Lambda}_4 = \hat{\Lambda}_1 \cap \hat{\Lambda}_2 = \left\{ \left(\hat{\wp}, min\{\hat{\zeta}_{\hat{\Lambda}_1}(\hat{\wp}), \hat{\zeta}_{\hat{\Lambda}_2}(\hat{\wp})\} \right) : \hat{\wp} \in \hat{\Delta} \right\}$.

Definition 2.2. [1] A FS $\hat{\Lambda}$ is stated to be convex FS when its belonging function $\hat{\zeta}_{\hat{\Lambda}}$ satisfies the following inequality $\hat{\zeta}_{\hat{\Lambda}}\left(\hat{\varsigma}\hat{\wp}_1 + (1-\hat{\varsigma})\,\hat{\wp}_2\right) \geq min\left(\hat{\zeta}_{\hat{\Lambda}}(\hat{\wp}_1), \hat{\zeta}_{\hat{\Lambda}}(\hat{\wp}_2)\right)$ with $\hat{\varsigma} \in [0,1]$ and $\hat{\wp}_1, \hat{\wp}_2 \in \hat{\Delta}$.

Definition 2.3. [17] A FS $\hat{\Lambda}$ is stated to be concave FS when its belonging function $\hat{\zeta}_{\hat{\Lambda}}$ satisfies the following inequality $\hat{\zeta}_{\hat{\Lambda}}\left(\hat{\varsigma}\hat{\wp}_1 + (1-\hat{\varsigma})\,\hat{\wp}_2\right) \leq max\left(\hat{\zeta}_{\hat{\Lambda}}(\hat{\wp}_1), \hat{\zeta}_{\hat{\Lambda}}(\hat{\wp}_2)\right)$ with $\hat{\varsigma} \in [0,1]$ and $\hat{\wp}_1, \hat{\wp}_2 \in \hat{\Delta}$.

Definition 2.4. [13] A IFS $\hat{\Gamma}$ is stated as $\hat{\Gamma} = \left\{ \left(\hat{\wp}, \langle \hat{\zeta}_{\hat{\Gamma}}(\hat{\wp}), \hat{\vartheta}_{\hat{\Gamma}}(\hat{\wp}) \rangle \right) : \hat{\wp} \in \hat{\Delta} \right\}$ such that $\hat{\zeta}_{\hat{\Gamma}}, \hat{\vartheta}_{\hat{\Gamma}} : \hat{\Delta} \to [0,1]$ with $\hat{\zeta}_{\hat{\Gamma}}(\hat{\wp}), \hat{\vartheta}_{\hat{\Gamma}}(\hat{\wp}) \in [0,1]$ as belonging-grade and non belonging-grade of $\hat{\wp}$ in $\hat{\Delta}$ such that $0 \leq \hat{\zeta}_{\hat{\Gamma}}(\hat{\wp}) + \hat{\vartheta}_{\hat{\Gamma}}(\hat{\wp}) \leq 1$. If $\hat{\Gamma}_1$ and $\hat{\Gamma}_2$ are IFSs then

(1) $\hat{\Gamma}^c = \left\{ \left(\hat{\wp}, \langle \hat{\vartheta}_{\hat{\Gamma}}(\hat{\wp}), \hat{\zeta}_{\hat{\Gamma}}(\hat{\wp}) \rangle \right) : \hat{\wp} \in \hat{\Delta} \right\}$.
(2) $\hat{\Gamma}_3 = \hat{\Gamma}_1 \cup \hat{\Gamma}_2 = \left\{ \left(\hat{\wp}, \langle max\{\hat{\zeta}_{\hat{\Gamma}_1}(\hat{\wp}), \hat{\zeta}_{\hat{\Gamma}_2}(\hat{\wp})\}, min\{\hat{\vartheta}_{\hat{\Gamma}_1}(\hat{\wp}), \hat{\vartheta}_{\hat{\Gamma}_2}(\hat{\wp})\} \rangle \right) : \hat{\wp} \in \hat{\Delta} \right\}$.
(3) $\hat{\Gamma}_4 = \hat{\Gamma}_1 \cap \hat{\Gamma}_2 = \left\{ \left(\hat{\wp}, \langle min\{\hat{\zeta}_{\hat{\Gamma}_1}(\hat{\wp}), \hat{\zeta}_{\hat{\Gamma}_2}(\hat{\wp})\}, max\{\hat{\vartheta}_{\hat{\Gamma}_1}(\hat{\wp}), \hat{\vartheta}_{\hat{\Gamma}_2}(\hat{\wp})\} \rangle \right) : \hat{\wp} \in \hat{\Delta} \right\}$.

Definition 2.5. [21] A IFS $\hat{\Gamma}$ is stated to be concave IFS when its belonging function $\hat{\zeta}_{\hat{\Gamma}}$ and non belonging function $\hat{\vartheta}_{\hat{\Gamma}}$ satisfy the following inequalities

(1) $\hat{\zeta}_{\hat{\Gamma}}\left(\hat{\varsigma}\hat{\wp}_1 + (1-\hat{\varsigma})\,\hat{\wp}_2\right) \geq min\left(\hat{\zeta}_{\hat{\Gamma}}(\hat{\wp}_1), \hat{\zeta}_{\hat{\Gamma}}(\hat{\wp}_2)\right)$
(2) $\hat{\vartheta}_{\hat{\Gamma}}\left(\hat{\varsigma}\hat{\wp}_1 + (1-\hat{\varsigma})\,\hat{\wp}_2\right) \leq max\left(\hat{\vartheta}_{\hat{\Gamma}}(\hat{\wp}_1), \hat{\vartheta}_{\hat{\Gamma}}(\hat{\wp}_2)\right)$

with $\hat{\varsigma} \in [0,1]$ and $\hat{\wp}_1, \hat{\wp}_2 \in \hat{\Delta}$.

Definition 2.6. [16] A NS $\hat{\aleph}$ is stated as

$$\hat{\aleph} = \left\{ \left(\hat{\wp}, \langle \hat{\zeta}_{\hat{\aleph}}(\hat{\wp}), \hat{\vartheta}_{\hat{\aleph}}(\hat{\wp}), \hat{\xi}_{\hat{\aleph}}(\hat{\wp}) \rangle \right) : \hat{\wp} \in \hat{\Delta}, \hat{\zeta}_{\hat{\aleph}}, \hat{\vartheta}_{\hat{\aleph}}, \hat{\xi}_{\hat{\aleph}} \in \,]^{-}0, 1^{+}[\right\}$$

with $\hat{\zeta}_{\hat{\aleph}}$, $\hat{\vartheta}_{\hat{\aleph}}$ and $\hat{\xi}_{\hat{\aleph}}$ as belonging, non-belonging and indeterminate functions such that $^{-}0 \leq \hat{\zeta}_{\hat{\aleph}}(\hat{\wp}) + \hat{\vartheta}_{\hat{\aleph}}(\hat{\wp}) + \hat{\xi}_{\hat{\aleph}}(\hat{\wp}) \leq 3^{+}$.

Definition 2.7. [26] A NS $\hat{\aleph}$ is stated to be convex NS when its belonging function $\hat{\zeta}_{\hat{\aleph}}$, non belonging function $\hat{\vartheta}_{\hat{\aleph}}$ and indeterminate function $\hat{\xi}_{\hat{\aleph}}$ satisfy the following inequalities

(1) $\hat{\zeta}_{\hat{\aleph}}\left(\varsigma\hat{\wp}_1+(1-\varsigma)\hat{\wp}_2\right)\geq min\left(\hat{\zeta}_{\hat{\aleph}}(\hat{\wp}_1),\hat{\zeta}_{\hat{\aleph}}(\hat{\wp}_2)\right)$
(2) $\hat{\vartheta}_{\hat{\aleph}}\left(\varsigma\hat{\wp}_1+(1-\varsigma)\hat{\wp}_2\right)\leq max\left(\hat{\vartheta}_{\hat{\aleph}}(\hat{\wp}_1),\hat{\vartheta}_{\hat{\aleph}}(\hat{\wp}_2)\right)$
(3) $\hat{\xi}_{\hat{\aleph}}\left(\varsigma\hat{\wp}_1+(1-\varsigma)\hat{\wp}_2\right)\leq max\left(\hat{\xi}_{\hat{\aleph}}(\hat{\wp}_1),\hat{\xi}_{\hat{\aleph}}(\hat{\wp}_2)\right)$

with $\varsigma\in[0,1]$ and $\hat{\wp}_1,\hat{\wp}_2\in\hat{\Delta}$.

Definition 2.8. [24] A *refined FS* $\hat{\Omega}_{RFS}$ is stated as

$$\hat{\Omega}_{RFS}=\left\{\left(\hat{\wp},\left\langle\hat{\zeta}^1_{\hat{\Omega}_{RFS}}(\hat{\wp}),\hat{\zeta}^2_{\hat{\Omega}_{RFS}}(\hat{\wp}),...,\hat{\zeta}^p_{\hat{\Omega}_{RFS}}(\hat{\wp})\right\rangle\right):p\geq 2,\hat{\wp}\in\hat{\Omega}_{RFS}\right\}$$

with $\hat{\zeta}^k_{\hat{\Omega}_{RFS}}$ as sub-belonging grades of k^{th}-type entities of $\hat{\Delta}$ with respect to $\hat{\Omega}_{RFS}$, and for $k\in[1,p]$ and $\sum_{k=1}^{p}\sup\hat{\zeta}^k_{\hat{\wp}}\leq 1,\forall\hat{\wp}\in\hat{\Omega}_{RFS}$.

Definition 2.9. [24] A refined IFS $\hat{\Omega}_{RIFS}$ is stated as

$$\hat{\Omega}_{RIFS}=\left\{\left(\hat{\wp},\left\langle\begin{array}{c}\left(\hat{\zeta}^1_{\hat{\Omega}_{RIFS}}(\hat{\wp}),\hat{\zeta}^2_{\hat{\Omega}_{RIFS}}(\hat{\wp}),...,\hat{\zeta}^p_{\hat{\Omega}_{RIFS}}(\hat{\wp})\right);\\\left(\hat{\vartheta}^1_{\hat{\Omega}_{RIFS}}(\hat{\wp}),\hat{\vartheta}^2_{\hat{\Omega}_{RIFS}}(\hat{\wp}),...,\hat{\vartheta}^s_{\hat{\Omega}_{RIFS}}(\hat{\wp})\right)\end{array}\right\rangle\right),p+s\geqslant 3,\hat{\wp}\in\hat{\Omega}_{RIFS}\right\}$$

with $\hat{\zeta}^k_{\hat{\Omega}_{RIFS}}$ as sub-belonging grades of k^{th}-type entities with respect to $\hat{\Omega}_{RIFS}$, and $\hat{\vartheta}^l_{\hat{\Omega}_{RIFS}}$ as sub non-belonging grades of l^{th}-type entities with respect to $\hat{\Omega}_{RIFS}$ and $\sum_{k=1}^{p}\sup\hat{\zeta}^k+\sum_{l=1}^{s}\sup\hat{\vartheta}^l\leq 1$, and $\hat{\zeta}^k_{\hat{\Omega}_{RIFS}},\hat{\vartheta}^l_{\hat{\Omega}_{RIFS}}\subseteq[0,1]$ for $k\in[1,p]$ and $l\in[1,s]$.

Definition 2.10. [24] A RNS $\hat{\Omega}_{RNS}$ is stated as

$$\hat{\Omega}_{RNS}=\left\{\left(\hat{\wp},\left\langle\begin{array}{c}\left(\hat{\zeta}^1_{\hat{\Omega}_{RNS}}(\hat{\wp}),\hat{\zeta}^2_{\hat{\Omega}_{RNS}}(\hat{\wp}),...,\hat{\zeta}^p_{\hat{\Omega}_{RNS}}(\hat{\wp})\right);\\\left(\hat{\vartheta}^1_{\hat{\Omega}_{RNS}}(\hat{\wp}),\hat{\vartheta}^2_{\hat{\Omega}_{RNS}}(\hat{\wp}),...,\hat{\vartheta}^s_{\hat{\Omega}_{RNS}}(\hat{\wp})\right);\\\left(\hat{\xi}^1_{\hat{\Omega}_{RNS}}(\hat{\wp}),\hat{\xi}^2_{\hat{\Omega}_{RNS}}(\hat{\wp}),...,\hat{\xi}^t_{\hat{\Omega}_{RNS}}(\hat{\wp})\right)\end{array}\right\rangle\right):p+s+t\geqslant 3,\hat{\wp}\in\hat{\Omega}_{RNS}\right\}$$

with $\hat{\zeta}^k_{\hat{\Omega}_{RNS}}$ as sub-belonging grades of k^{th}-type entities, $\hat{\vartheta}^l_{\hat{\Omega}_{RNS}}$ as sub non-belonging grades of l^{th}-type entities and $\hat{\xi}^m_{\hat{\Omega}_{RNS}}$ as sub indeterminate grades of m^{th}-type entities with respect to $\hat{\Omega}_{RNS}$ and ${}^-0\leq\sum_{k=1}^{p}\sup\hat{\zeta}^k_{\hat{\Omega}_{RNS}}+\sum_{l=1}^{s}\sup\hat{\vartheta}^l_{\hat{\Omega}_{RNS}}+\sum_{m=1}^{t}\sup\hat{\xi}^m_{\hat{\Omega}_{RNS}}\leq 3^+$, and $\hat{\zeta}^k_{\hat{\Omega}_{RNS}},\hat{\vartheta}^l_{\hat{\Omega}_{RNS}},\hat{\xi}^m_{\hat{\Omega}_{RNS}}\subseteq\,]^-0,1^+[$ for $k\in[1,p],l\in[1,s],m\in[1,t]$.

3. **Convexity and Concavity on RNSs**

This portion describes the notions of convexity and concavity for RNSs. Throughout the paper, the symbols "RNS" and "$\overline{\hat{z}_1\hat{z}_2}$" are meant for RNS and line-segment correspondingly.

Definition 3.1. In $\mathscr{G}$, a RNS $\hat{\Omega}_{RNS}$ is stated to be convex if the points $\hat{z}_1,\hat{z}_2,\hat{z}_3\in\mathscr{G}$ on $\overline{\hat{z}_1\hat{z}_2}$

$$\hat{\zeta}^k_{\hat{\Omega}_{RNS}}(\hat{z}_3)\geq\min\left(\hat{\zeta}^k_{\hat{\Omega}_{RNS}}(\hat{z}_1),\hat{\zeta}^k_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right),k\in[1,p]$$

$$\hat{\vartheta}^l_{\hat{\Omega}_{RNS}}(\hat{z}_3)\leq\max\left(\hat{\vartheta}^l_{\hat{\Omega}_{RNS}}(\hat{z}_1),\hat{\vartheta}^l_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right),l\in[1,s]$$

$$\hat{\xi}^m_{\hat{\Omega}_{RNS}}(\hat{z}_3) \leq \max\left(\hat{\xi}^m_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\xi}^m_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right), m \in [1,t]$$

where $\hat{\zeta}^k_{\hat{\Omega}_{RNS}}$ is k^{th}-type grade of sub-belonging of the entities with respect to $\hat{\Omega}_{RNS}$, and for $k \in [1,p]$, $\sum_{k=1}^{p} \sup \hat{\zeta}^k \leq 1$, $\hat{\vartheta}^l_{\hat{\Omega}_{RNS}}$ is l^{th}-type grade of sub non-belonging of the entities with respect to $\hat{\Omega}_{RNS}$, and for $l \in [1,s]$ and $\sum_{l=1}^{s} \sup \hat{\vartheta}^l \leq 1$ and $\hat{\xi}^m_{\hat{\Omega}_{RNS}}$ is m^{th}-type grade of sub-indeterminacy of the entities with respect to $\hat{\Omega}_{RNS}$, and for $m \in [1,t]$, $\sum_{m=1}^{t} \sup \hat{\xi}^m \leq 1$ with condition $\sum_{k=1}^{p} \sup \hat{\zeta}^k + \sum_{l=1}^{s} \sup \hat{\vartheta}^l + \sum_{m=1}^{t} \sup \hat{\xi}^m \leq 3$. The symbol $\hat{\Xi}_{CxRNS}$ is meant for family of convex RNSs.

Definition 3.2. In $\mathscr{G}$, a RNS $\hat{\Omega}_{RNS}$ is stated to be ortho-convex if the points $\hat{z}_1, \hat{z}_2, \hat{z}_3 \in \mathscr{G}$ on $\overline{\hat{z}_1\hat{z}_2}$ which is lying on that line which is $\parallel$ axis

$$\hat{\zeta}^k_{\hat{\Omega}_{RNS}}(\hat{z}'_3) \geq \min\left(\hat{\zeta}^k_{\hat{\Omega}_{RNS}}(\hat{z}'_1), \hat{\zeta}^k_{\hat{\Omega}_{RNS}}(\hat{z}'_2)\right), k \in [1,p].$$
$$\hat{\vartheta}^l_{\hat{\Omega}_{RNS}}(\hat{z}'_3) \leq \max\left(\hat{\vartheta}^l_{\hat{\Omega}_{RNS}}(\hat{z}'_1), \hat{\vartheta}^l_{\hat{\Omega}_{RNS}}(\hat{z}'_2)\right), l \in [1,s].$$
$$\hat{\xi}^m_{\hat{\Omega}_{RNS}}(\hat{z}'_3) \leq \max\left(\hat{\xi}^m_{\hat{\Omega}_{RNS}}(\hat{z}'_1), \hat{\xi}^m_{\hat{\Omega}_{RNS}}(\hat{z}'_2)\right), m \in [1,t].$$

with same conditions as provided in Definition 3.1. The symbol $\hat{\Xi}^O_{CxRNS}$ is meant for family of ortho-convex RNSs.

Remark 3.3. If $\hat{\Omega}_{RNS} \in \hat{\Xi}^O_{CxRNS}$ then $\hat{\Omega}_{RNS} \in \hat{\Xi}_{CxRNS}$ but the converse is not true.

Definition 3.4. In $\mathscr{G}$, a RNS $\hat{\Omega}_{RNS}$ is stated to be concave if the points $\hat{z}_1, \hat{z}_2, \hat{z}_3 \in \mathscr{G}$ on $\overline{\hat{z}_1\hat{z}_2}$

$$\hat{\zeta}^k_{\hat{\Omega}_{RNS}}(\hat{z}_3) \leq \max\left(\hat{\zeta}^k_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\zeta}^k_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right), k \in [1,p].$$
$$\hat{\vartheta}^l_{\hat{\Omega}_{RNS}}(\hat{z}_3) \geq \min\left(\hat{\vartheta}^l_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\vartheta}^l_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right), l \in [1,s].$$
$$\hat{\xi}^m_{\hat{\Omega}_{RNS}}(\hat{z}_3) \geq \min\left(\hat{\xi}^m_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\xi}^m_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right), m \in [1,t].$$

where

$\hat{\zeta}^k_{\hat{\Omega}_{RNS}}$ is k^{th}-type grade of sub-belonging of the entities with respect to $\hat{\Omega}_{RNS}$, and is subset of I for $k \in [1,p]$ and $\sum_{k=1}^{p} \sup \hat{\zeta}^k \leq 1$, $\hat{\vartheta}^l_{\hat{\Omega}_{RNS}}$ is l^{th}-type grade of sub non-belonging of the entities with respect to $\hat{\Omega}_{RNS}$, and is subset of I for $l \in [1,s]$ and $\sum_{l=1}^{s} \sup \hat{\vartheta}^l \leq 1$ with condition $\sum_{k=1}^{p} \sup \hat{\zeta}^k + \sum_{l=1}^{s} \sup \hat{\vartheta}^l \leq 1$ and $\hat{\xi}^m_{\hat{\Omega}_{RNS}}$ is m^{th}-type grade of sub indeterminacy of the entities with respect to $\hat{\Omega}_{RNS}$, and is subset of I for $m \in [1,t]$ and $\sum_{m=1}^{t} \sup \hat{\xi}^m \leq 1$ with condition $\sum_{k=1}^{p} \sup \hat{\zeta}^k + \sum_{l=1}^{s} \sup \hat{\vartheta}^l + \sum_{m=1}^{t} \sup \hat{\xi}^m \leq 3$.

The symbol $\hat{\Xi}_{CvRNS}$ is meant for family of concave RNSs.

Definition 3.5. In $\mathscr{G}$, a RNS $\hat{\Omega}_{RNS}$ is stated to be ortho-concave if the points $\hat{z}_1, \hat{z}_2, \hat{z}_3 \in \mathscr{G}$ on $\overline{\hat{z}_1\hat{z}_2}$ that is lying on line which is $\|$ axis

$$\hat{\zeta}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right) \leq \max\left(\hat{\zeta}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right), \hat{\zeta}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right)\right), k \in [1, p].$$

$$\hat{\vartheta}^l_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right) \geq \min\left(\hat{\vartheta}^l_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right), \hat{\vartheta}^l_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right)\right), l \in [1, s].$$

$$\hat{\xi}^m_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right) \geq \min\left(\hat{\xi}^m_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right), \hat{\vartheta}^m_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right)\right), m \in [1, t].$$

with same conditions as provided in Definition 3.4.

The symbol $\hat{\Xi}^O_{CvRNS}$ is meant for family of ortho-concave RNSs.

Remark 3.6. If $\hat{\Omega}_{RNS} \in \hat{\Xi}^O_{CvRNS}$ then $\hat{\Omega}_{RNS} \in \hat{\Xi}_{CvRNS}$ but the converse is not true.

Theorem 3.7. *If $\hat{\Omega}_{RNS} \in \hat{\Xi}_{CxRNS}$ then $\hat{\Omega}^c_{RNS} \in \hat{\Xi}_{CvRNS}$.*

Proof. If $\hat{\Omega}_{RNS} \in \hat{\Xi}_{CxRNS}$ then for points $\hat{z}_1, \hat{z}_2, \hat{z}_3$ on $\overline{\hat{z}_1\hat{z}_2}$

$$\hat{\zeta}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}_3\right) \geq \min\left(\hat{\zeta}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right), \hat{\zeta}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right)\right), k \in [1, p]$$

so

$$\overline{\hat{\zeta}}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}_3\right) \leq 1 - \min\left(1 - \overline{\hat{\zeta}}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right), 1 - \overline{\hat{\zeta}}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right)\right), k \in [1, p] \tag{1}$$

now if

$$1 - \overline{\hat{\zeta}}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right) \leq 1 - \overline{\hat{\zeta}}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right)$$

then

$$\min\left(1 - \overline{\hat{\zeta}}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right), 1 - \overline{\hat{\zeta}}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right)\right) = 1 - \overline{\hat{\zeta}}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right)$$

and there from (1)

$$\overline{\hat{\zeta}}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}_3\right) \leq \overline{\hat{\zeta}}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right)$$

similarly if

$$1 - \overline{\hat{\zeta}}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right) \leq 1 - \overline{\hat{\zeta}}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right)$$

then

$$\min\left(1 - \overline{\hat{\zeta}}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right), 1 - \overline{\hat{\zeta}}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right)\right) = 1 - \overline{\hat{\zeta}}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right)$$

so from (1)

$$\overline{\hat{\zeta}}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}_3\right) \leq \overline{\hat{\zeta}}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right).$$

Hence

$$\overline{\hat{\zeta}}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}_3\right) \leq \max\left(\overline{\hat{\zeta}}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right), \overline{\hat{\zeta}}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right)\right), k \in [1, p].$$

Again

$$\hat{\vartheta}^l_{\hat{\Omega}_{RNS}}\left(\hat{z}_3\right) \leq \max\left(\hat{\vartheta}^l_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right), \hat{\vartheta}^l_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right)\right), l \in [1, s]$$

then

$$\overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_3) \geq 1 - \max\left(1 - \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_1), 1 - \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right), l \in [1, s] \tag{2}$$

now if

$$1 - \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_1) \geq 1 - \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_2)$$

then

$$\max\left(1 - \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_1), 1 - \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right) = 1 - \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_1)$$

and from (2)

$$\overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_3) \geq \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_1)$$

similarly if

$$1 - \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_2) \geq 1 - \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_1)$$

then

$$\max\left(1 - \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_1), 1 - \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right) = 1 - \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_2)$$

so from (2)

$$\overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_3) \geq \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_2).$$

Hence

$$\overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_3) \geq \min\left(\overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right), l \in [1, s].$$

Similarly

$$\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_3) \leq \max\left(\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\xi}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right), m \in [1, t]$$

so

$$\overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_3) \geq 1 - \max\left(1 - \overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_1), 1 - \overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right), m \in [1, t] \tag{3}$$

now if

$$1 - \overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_1) \geq 1 - \overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_2)$$

then

$$\max\left(1 - \overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_1), 1 - \overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right) = 1 - \overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_1)$$

and there from (3)

$$\overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_3) \geq \overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_1)$$

similarly if

$$1 - \overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_2) \geq 1 - \overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_1)$$

then

$$\max\left(1 - \overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_1), 1 - \overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right) = 1 - \overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_2)$$

so from (3)

$$\overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_3) \geq \overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_2).$$

Hence

$$\overline{\hat{\zeta}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_3) \geq \min\left(\overline{\hat{\zeta}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \overline{\hat{\zeta}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right), m \in [1, t]$$

consequently $\hat{\Omega}^{c}_{RNS} \in \hat{\Xi}_{CvRNS}$. □

Remark 3.8. If $\hat{\Omega}_{RNS} \in \hat{\Xi}^{O}_{CxRNS}$ then $\hat{\Omega}^{c}_{RNS} \in \hat{\Xi}^{O}_{CvRNS}$ and $\hat{\Omega}_{RNS} \in \hat{\Xi}_{CvRNS}$.

Theorem 3.9. *If $\hat{\Omega}_{RNS}, \hat{\Theta}_{RNS} \in \hat{\Xi}_{CxRNS}$ then $\hat{\Omega}_{RNS} \cup \hat{\Theta}_{RNS} \in \hat{\Xi}_{CxRNS}$.*

Proof. Let $\hat{\Omega}_{RNS}$ and $\hat{\Theta}_{RNS}$ be two convex RNSs and $\hat{\Psi}_{RNS} = \hat{\Omega}_{RNS} \cup \hat{\Theta}_{RNS}$ and the points $\hat{z}_1, \hat{z}_2, \hat{z}_3$ on $\overline{\hat{z}_1\hat{z}_2}$. Now

$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}_1) = \min\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_1)\right), k \in [1, p]$$
$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}_2) = \min\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_2), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_2)\right), k \in [1, p]$$
$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}_3) = \min\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_3), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_3)\right), k \in [1, p].$$

Now

$$\begin{aligned}
&\min\left(\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}_1), \hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}_2)\right) \\
&= \min\left(\min\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_1)\right), \min\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_2), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_2)\right)\right) \\
&= \min\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_1), \hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_2), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_2)\right) \qquad (4)
\end{aligned}$$

let

$$\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_3) \leq \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_3)$$

in (4) so that

$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}_3) = \hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_3)$$

as $\hat{\Omega}_{RNS}$ is convex RNS so

$$\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_3) \geq \min\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right)$$
$$\geq \min\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_1), \hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_2), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_2)\right)$$

i.e.

$$\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_3) = \hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}_3) \geq \min\left(\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}_1)\, \hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}_2)\right)$$

similarly for $\hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_3) \leq \hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_3)$ in equation (4) so that

$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}_3) = \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_3)$$

as $\hat{\Theta}_{RNS}$ is convex RNS so (4) becomes

$$\hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_3) \geq \min\left(\hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_1), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_2)\right)$$
$$\geq \min\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_1), \hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_2), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_2)\right)$$

i.e.

$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}_3\right) \geq \min\left(\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}_1\right)\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}_2\right)\right)$$

Again

$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}_1\right) = \max\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}_1\right)\right), l \in [1, s]$$
$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}_2\right) = \max\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}_2\right)\right), l \in [1, s]$$
$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}_3\right) = \max\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_3\right), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}_3\right)\right), l \in [1, s].$$

Now

$$\begin{aligned}
&\max\left(\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}_1\right), \hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}_2\right)\right) \\
&= \max\left(\max\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}_1\right)\right), \max\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}_2\right)\right)\right) \\
&= \max\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}_1\right), \hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}_2\right)\right) \qquad (5)
\end{aligned}$$

let

$$\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_3\right) \leq \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}_3\right)$$

in (5) so that

$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}_3\right) = \hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_3\right)$$

as $\hat{\Omega}_{RNS}$ is convex RNS so

$$\begin{aligned}
\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_3\right) &\leq \max\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right), \hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right)\right) \\
&\leq \max\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}_1\right), \hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}_2\right)\right)
\end{aligned}$$

i.e.

$$\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_3\right) = \hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}_3\right) \leq \max\left(\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}_1\right)\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}_2\right)\right)$$

similarly for $\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}_3\right) \geq \hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_3\right)$ in equation (5) so that

$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}_3\right) = \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}_3\right)$$

as $\hat{\Theta}_{RNS}$ is convex RNS so (5) becomes

$$\begin{aligned}
\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}_3\right) &\leq \max\left(\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}_1\right), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}_2\right)\right) \\
&\leq \max\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}_1\right), \hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}_2\right)\right)
\end{aligned}$$

i.e.

$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}_3\right) \leq \max\left(\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}_1\right)\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}_2\right)\right).$$

Similarly now

$$\hat{\xi}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}_1\right) = \max\left(\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right), \hat{\xi}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}_1\right)\right), m \in [1, t]$$
$$\hat{\xi}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}_2\right) = \max\left(\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right), \hat{\xi}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}_2\right)\right), m \in [1, t]$$
$$\hat{\xi}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}_3\right) = \max\left(\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}_3\right), \hat{\xi}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}_3\right)\right), m \in [1, t].$$

Now

$$\max\left(\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}(\hat{z}_1), \hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}(\hat{z}_2)\right)$$
$$= \max\left(\max\left(\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}(\hat{z}_1)\right), \max\left(\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_2), \hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}(\hat{z}_2)\right)\right)$$
$$= \max\left(\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}(\hat{z}_1), \hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_2), \hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}(\hat{z}_2)\right) \tag{6}$$

let

$$\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_3) \leq \hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}(\hat{z}_3)$$

in (6) so that

$$\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}(\hat{z}_3) = \hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_3)$$

as $\hat{\Omega}_{RNS}$ is convex RNS so

$$\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_3) \leq \max\left(\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right)$$
$$\leq \max\left(\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}(\hat{z}_1), \hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_2), \hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}(\hat{z}_2)\right)$$

i.e.

$$\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_3) = \hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}(\hat{z}_3) \leq \max\left(\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}(\hat{z}_1)\, \hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}(\hat{z}_2)\right)$$

similarly for $\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}(\hat{z}_3) \geq \hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_3)$ in equation (6) so that

$$\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}(\hat{z}_3) = \hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}(\hat{z}_3)$$

as $\hat{\Theta}_{RNS}$ is convex RNS so (6) becomes

$$\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}(\hat{z}_3) \leq \max\left(\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}(\hat{z}_1), \hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}(\hat{z}_2)\right)$$
$$\leq \max\left(\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}(\hat{z}_1), \hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_2), \hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}(\hat{z}_2)\right)$$

i.e.

$$\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}(\hat{z}_3) \leq \max\left(\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}(\hat{z}_1)\, \hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}(\hat{z}_2)\right).$$

□

Theorem 3.10. *If* $\hat{\Omega}_{RNS}, \hat{\Theta}_{RNS} \in \hat{\Xi}^{O}_{CxRNS}$ *then* $\hat{\Omega}_{RNS} \cup \hat{\Theta}_{RNS} \in \hat{\Xi}^{O}_{CxRNS}$ *and* $\hat{\Omega}_{RNS} \cup \hat{\Theta}_{RNS} \in \hat{\Xi}_{CxRNS}$.

Proof. Let $\hat{\Omega}_{RNS}$ and $\hat{\Theta}_{RNS}$ be two convex RNSs and $\hat{\Psi}_{RNS} = \hat{\Omega}_{RNS} \cup \hat{\Theta}_{RNS}$ and the points $\hat{z}'_1, \hat{z}'_2, \hat{z}'_3$ on $\overline{\hat{z}'_1\hat{z}'_2}\|$ axis.
Now

$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}'_1) = \min\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}'_1), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}'_1)\right), k \in [1, p]$$
$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}'_2) = \min\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}'_2), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}'_2)\right), k \in [1, p]$$
$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}'_3) = \min\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}'_3), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}'_3)\right), k \in [1, p].$$

Now

$$\min\left(\hat{\zeta}^k_{\hat{\Psi}_{RNS}}\left(\hat{z}'_1\right),\hat{\zeta}^k_{\hat{\Psi}_{RNS}}\left(\hat{z}'_2\right)\right)$$

$$=\min\left(\min\left(\hat{\zeta}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right),\hat{\zeta}^k_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right)\right),\min\left(\hat{\zeta}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right),\hat{\zeta}^k_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right)\right)$$

$$=\min\left(\hat{\zeta}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right),\hat{\zeta}^k_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right),\hat{\zeta}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right),\hat{\zeta}^k_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right) \tag{7}$$

let

$$\hat{\zeta}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right)\leq\hat{\zeta}^k_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)$$

in (7) so that

$$\hat{\zeta}^k_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right)=\hat{\zeta}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right)$$

as $\hat{\Omega}_{RNS}$ is ortho-convex RNS so

$$\hat{\zeta}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right)\geq\min\left(\hat{\zeta}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right),\hat{\zeta}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right)\right)$$

$$\geq\min\left(\hat{\zeta}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right),\hat{\zeta}^k_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right),\hat{\zeta}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right),\hat{\zeta}^k_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right)$$

i.e.

$$\hat{\zeta}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right)=\hat{\zeta}^k_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right)\geq\min\left(\hat{\zeta}^k_{\hat{\Psi}_{RNS}}\left(\hat{z}'_1\right)\hat{\zeta}^k_{\hat{\Psi}_{RNS}}\left(\hat{z}'_2\right)\right)$$

similarly for $\hat{\zeta}^k_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)\leq\hat{\zeta}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right)$ in equation (7) so that

$$\hat{\zeta}^k_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right)=\hat{\zeta}^k_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)$$

as $\hat{\Theta}_{RNS}$ is ortho-convex RNS so

$$\hat{\zeta}^k_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)\geq\min\left(\hat{\zeta}^k_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right),\hat{\zeta}^k_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right)$$

$$\geq\min\left(\hat{\zeta}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right),\hat{\zeta}^k_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right),\hat{\zeta}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right),\hat{\zeta}^k_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right)$$

i.e.

$$\hat{\zeta}^k_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)=\hat{\zeta}^k_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right)\geq\min\left(\hat{\zeta}^k_{\hat{\Psi}_{RNS}}\left(\hat{z}'_1\right)\hat{\zeta}^k_{\hat{\Psi}_{RNS}}\left(\hat{z}'_2\right)\right).$$

Again

$$\hat{\vartheta}^l_{\hat{\Psi}_{RNS}}\left(\hat{z}'_1\right)=\max\left(\hat{\vartheta}^l_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right),\hat{\vartheta}^l_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right)\right),l\in[1,s]$$

$$\hat{\vartheta}^l_{\hat{\Psi}_{RNS}}\left(\hat{z}'_2\right)=\max\left(\hat{\vartheta}^l_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right),\hat{\vartheta}^l_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right),l\in[1,s]$$

$$\hat{\vartheta}^l_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right)=\max\left(\hat{\vartheta}^l_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right),\hat{\vartheta}^l_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)\right),l\in[1,s].$$

Now

$$\max\left(\hat{\vartheta}^l_{\hat{\Psi}_{RNS}}\left(\hat{z}'_1\right),\hat{\vartheta}^l_{\hat{\Psi}_{RNS}}\left(\hat{z}'_2\right)\right)$$

$$=\max\left(\max\left(\hat{\vartheta}^l_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right),\hat{\vartheta}^l_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right)\right),\max\left(\hat{\vartheta}^l_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right),\hat{\vartheta}^l_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right)\right)$$

$$=\max\left(\hat{\vartheta}^l_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right),\hat{\vartheta}^l_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right),\hat{\vartheta}^l_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right),\hat{\vartheta}^l_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right) \tag{8}$$

let

$$\hat{\vartheta}^l_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right)\geq\hat{\vartheta}^l_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)$$

in (8) so that

$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right)=\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right)$$

as $\hat{\Omega}_{RNS}$ is ortho-convex RNS so

$$\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right)\leq\max\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right),\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right)\right)$$

$$\leq\max\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right),\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right),\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right),\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right)$$

i.e.

$$\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right)=\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right)\leq\max\left(\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_1\right)\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_2\right)\right)$$

similarly for $\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)\geq\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right)$ in equation (8) so that

$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right)=\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)$$

as $\hat{\Theta}_{RNS}$ is ortho-convex RNS so

$$\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)\leq\max\left(\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right),\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right)$$

$$\leq\max\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right),\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right),\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right),\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right)$$

i.e.

$$\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)=\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right)\leq\max\left(\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_1\right)\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_2\right)\right).$$

Similarly

$$\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_1\right)=\max\left(\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right),\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right)\right),m\in[1,t]$$

$$\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_2\right)=\max\left(\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right),\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right),m\in[1,t]$$

$$\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right)=\max\left(\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right),\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)\right),m\in[1,t].$$

Now

$$\max\left(\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_1\right),\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_2\right)\right)$$

$$=\max\left(\max\left(\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right),\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right)\right),\max\left(\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right),\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right)\right)$$

$$=\max\left(\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right),\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right),\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right),\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right)\tag{9}$$

let

$$\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right)\geq\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)$$

in (9) so that

$$\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right)=\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right)$$

as $\hat{\Omega}_{RNS}$ is ortho-convex RNS so

$$\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right)\leq\max\left(\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right),\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right)\right)$$

$$\leq\max\left(\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right),\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right),\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right),\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right)$$

i.e.

$$\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right) = \hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right) \leq \max\left(\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_1\right) \hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_2\right)\right)$$

similarly for $\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right) \geq \hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right)$ in equation (9) so that

$$\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right) = \hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)$$

as $\hat{\Theta}_{RNS}$ is ortho-convex RNS so

$$\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right) \leq \max\left(\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right), \hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right)$$

$$\leq \max\left(\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right), \hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right), \hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right), \hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right)$$

i.e.

$$\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right) = \hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right) \leq \max\left(\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_1\right) \hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_2\right)\right)$$

since every ortho-convex RNS is also convex RNS. Hence the proof. □

Remark 3.11. If $\hat{\Omega}^{\alpha}_{RNS} \in \hat{\Xi}_{CxRNS}$ then $\bigcup_{\alpha} \hat{\Omega}^{\alpha}_{RNS} \in \hat{\Xi}_{CxRNS}$.

Remark 3.12. If $\hat{\Omega}^{\alpha}_{RNS} \in \hat{\Xi}^{O}_{CxRNS}$ then $\bigcup_{\alpha} \hat{\Omega}^{\alpha}_{RNS} \in \hat{\Xi}^{O}_{CxRNS}$ and $\bigcup_{\alpha} \hat{\Omega}^{\alpha}_{RNS} \in \hat{\Xi}_{CxRNS}$.

Theorem 3.13. *If $\hat{\Omega}_{RNS} \in \hat{\Xi}_{CvRNS}$ then $\hat{\Omega}^{c}_{RNS} \in \hat{\Xi}_{CxRNS}$.*

Proof. Let $\hat{\Omega}_{RNS} \in \hat{\Xi}_{CvRNS}$ and the points $\hat{z}_1, \hat{z}_2, \hat{z}_3$ on $\overline{\hat{z}_1\hat{z}_2}$, then

$$\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}_3\right) \leq \max\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right), \hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right)\right), k \in [1, p]$$

so

$$\overline{\hat{\zeta}}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}_3\right) \geq 1 - \max\left(1 - \overline{\hat{\zeta}}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right), 1 - \overline{\hat{\zeta}}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right)\right), k \in [1, p] \tag{10}$$

now if

$$1 - \overline{\hat{\zeta}}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right) \leq 1 - \overline{\hat{\zeta}}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right)$$

then

$$\max\left(1 - \overline{\hat{\zeta}}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right), 1 - \overline{\hat{\zeta}}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right)\right) = 1 - \overline{\hat{\zeta}}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right)$$

and from (10)

$$\overline{\hat{\zeta}}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}_3\right) \geq \overline{\hat{\zeta}}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right)$$

similarly if

$$1 - \overline{\hat{\zeta}}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right) \leq 1 - \overline{\hat{\zeta}}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right)$$

then

$$\max\left(1 - \overline{\hat{\zeta}}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right), 1 - \overline{\hat{\zeta}}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}_2\right)\right) = 1 - \overline{\hat{\zeta}}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right)$$

so from (10)

$$\overline{\hat{\zeta}}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}_3\right) \geq \overline{\hat{\zeta}}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}_1\right).$$

Hence

$$\overline{\hat{\zeta}}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_3) \geq \min\left(\overline{\hat{\zeta}}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \overline{\hat{\zeta}}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right), k \in [1, p]$$

consequently $\hat{\Omega}^{c}_{RNS} \in \hat{\Xi}_{CxRNS}$.

Again

$$\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_3) \geq \min\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right), l \in [1, s]$$

so we have

$$\overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_3) \leq 1 - \min\left(1 - \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_1), 1 - \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right), l \in [1, s] \tag{11}$$

now if

$$1 - \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_1) \geq 1 - \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_2)$$

then

$$\min\left(1 - \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_1), 1 - \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right) = 1 - \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_2)$$

and there from (11)

$$\overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_3) \leq \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_2)$$

similarly if

$$1 - \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_2) \geq 1 - \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_1)$$

then

$$\min\left(1 - \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_1), 1 - \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right) = 1 - \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_1)$$

so from (11)

$$\overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_3) \leq \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_1).$$

Hence

$$\overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_3) \leq \max\left(\overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \overline{\hat{\vartheta}}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right), l \in [1, s].$$

Similarly

$$\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_3) \geq \min\left(\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\xi}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right), m \in [1, t]$$

so we have

$$\overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_3) \leq 1 - \min\left(1 - \overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_1), 1 - \overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right), m \in [1, t] \tag{12}$$

now if

$$1 - \overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_1) \geq 1 - \overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_2)$$

then

$$\min\left(1 - \overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_1), 1 - \overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right) = 1 - \overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_2)$$

and there from (12)

$$\overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_3) \leq \overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_2)$$

similarly if

$$1-\overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_2) \geq 1-\overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_1)$$

then

$$\min\left(1-\overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_1), 1-\overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right) = 1-\overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_1)$$

so from (12)

$$\overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_3) \leq \overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_1).$$

Hence

$$\overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_3) \leq \max\left(\overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \overline{\hat{\xi}}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right), m \in [1,t]$$

consequently $\hat{\Omega}^{c}_{RNS} \in \hat{\Xi}_{CxRNS}$. □

Remark 3.14. If $\hat{\Omega}_{RNS} \in \hat{\Xi}^{O}_{CvRNS}$ then $\hat{\Omega}^{c}_{RNS} \in \hat{\Xi}^{O}_{CxRNS}$ and $\hat{\Omega}_{RNS} \in \hat{\Xi}_{CxRNS}$.

Theorem 3.15. *If $\hat{\Omega}_{RNS}, \hat{\Theta}_{RNS} \in \hat{\Xi}_{CvRNS}$ then $\hat{\Omega}_{RNS} \cup \hat{\Theta}_{RNS} \in \hat{\Xi}_{CvRNS}$.*

Proof. Let $\hat{\Omega}_{RNS}, \hat{\Theta}_{RNS} \in \hat{\Xi}_{CvRNS}$, $\hat{\Psi}_{RNS} = \hat{\Omega}_{RNS} \cup \hat{\Theta}_{RNS}$ and the points $\hat{z}_1, \hat{z}_2, \hat{z}_3$ on $\overline{\hat{z}_1\hat{z}_2}$ now

$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}_1) = \max\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_1)\right), k \in [1,p]$$
$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}_2) = \max\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_2), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_2)\right), k \in [1,p]$$
$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}_3) = \max\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_3), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_3)\right), k \in [1,p]$$

now

$$\begin{aligned} &\max\left(\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}_1), \hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}_2)\right) \\ &= \max\left(\max\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_1)\right), \max\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_2), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_2)\right)\right) \\ &= \max\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_1), \hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_2), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_2)\right) \end{aligned} \tag{13}$$

let

$$\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_3) \geq \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_3)$$

in equation (13) so that

$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}_3) = \hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_3)$$

as $\hat{\Omega}_{RNS}$ is concave RNS so equation (13) becomes

$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}_3) = \hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_3) \leq \max\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right)$$

$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}_3) \leq \max\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_1), \hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\, \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_2)\right)$$

i.e.

$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}_3) \leq \max\left(\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}_1)\, \hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}_2)\right)$$

similarly for $\hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_3) \geq \hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_3)$,in equation (13) so that

$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}_3) = \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_3)$$

as $\hat{\Theta}_{RNS}$ is concave RNS so equation (13) becomes

$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}_3) = \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_3) \leq \max\left(\hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_1), \hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right)$$

$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}_3) \leq \max\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_1), \hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\,\hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}(\hat{z}_2)\right)$$

i.e.

$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}_3) \leq \max\left(\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}_1)\,\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}(\hat{z}_2)\right).$$

Now

$$\begin{aligned}
\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}(\hat{z}_1) &= \max\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}(\hat{z}_1)\right), l \in [1, s] \\
\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}(\hat{z}_2) &= \max\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_2), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}(\hat{z}_2)\right), l \in [1, s] \\
\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}(\hat{z}_3) &= \max\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_3), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}(\hat{z}_3)\right), l \in [1, s]
\end{aligned} \tag{14}$$

now

$$\begin{aligned}
&\max\left(\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}(\hat{z}_1), \hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}(\hat{z}_2)\right) \\
&= \max\left(\max\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}(\hat{z}_1)\right), \max\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_2), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}(\hat{z}_2)\right)\right) \\
&= \max\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}(\hat{z}_1), \hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_2), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}(\hat{z}_2)\right)
\end{aligned}$$

let

$$\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_3) \geq \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}(\hat{z}_3)$$

in equation (14) so that

$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}(\hat{z}_3) = \hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_3)$$

as $\hat{\Omega}_{RNS}$ is concave RNS so equation (14) becomes

$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}(\hat{z}_3) = \hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_3) \leq \max\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right)$$

$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}(\hat{z}_3) \leq \max\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}(\hat{z}_1), \hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\,\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}(\hat{z}_2)\right)$$

i.e.

$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}(\hat{z}_3) \leq \max\left(\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}(\hat{z}_1)\,\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}(\hat{z}_2)\right)$$

similarly for $\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}(\hat{z}_3) \geq \hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_3)$,in equation (14) so that

$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}(\hat{z}_3) = \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}(\hat{z}_3)$$

as $\hat{\Theta}_{RNS}$ is concave RNS so equation (14) becomes

$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}(\hat{z}_3) = \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}(\hat{z}_3) \leq \max\left(\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}(\hat{z}_1), \hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right)$$

$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}(\hat{z}_3) \leq \max\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}(\hat{z}_1), \hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\,\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}(\hat{z}_2)\right)$$

i.e.

$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}(\hat{z}_3) \leq \max\left(\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}(\hat{z}_1)\,\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}(\hat{z}_2)\right).$$

Similarly

$$\begin{aligned}
\hat{\xi}^{m}_{\hat{\Psi}_{RNS}}(\hat{z}_1) &= \max\left(\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\xi}^{m}_{\hat{\Theta}_{RNS}}(\hat{z}_1)\right), m \in [1,t] \\
\hat{\xi}^{m}_{\hat{\Psi}_{RNS}}(\hat{z}_2) &= \max\left(\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_2), \hat{\xi}^{m}_{\hat{\Theta}_{RNS}}(\hat{z}_2)\right), m \in [1,t] \qquad (15)\\
\hat{\xi}^{m}_{\hat{\Psi}_{RNS}}(\hat{z}_3) &= \max\left(\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_3), \hat{\xi}^{m}_{\hat{\Theta}_{RNS}}(\hat{z}_3)\right), m \in [1,t]
\end{aligned}$$

now

$$\begin{aligned}
&\max\left(\hat{\xi}^{m}_{\hat{\Psi}_{RNS}}(\hat{z}_1), \hat{\xi}^{m}_{\hat{\Psi}_{RNS}}(\hat{z}_2)\right) \\
&= \max\left(\max\left(\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\xi}^{m}_{\hat{\Theta}_{RNS}}(\hat{z}_1)\right), \max\left(\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_2), \hat{\xi}^{m}_{\hat{\Theta}_{RNS}}(\hat{z}_2)\right)\right) \\
&= \max\left(\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\xi}^{m}_{\hat{\Theta}_{RNS}}(\hat{z}_1), \hat{\xi}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_2), \hat{\xi}^{m}_{\hat{\Theta}_{RNS}}(\hat{z}_2)\right)
\end{aligned}$$

let

$$\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_3) \geq \hat{\xi}^{m}_{\hat{\Theta}_{RNS}}(\hat{z}_3)$$

in equation (15) so that

$$\hat{\xi}^{m}_{\hat{\Psi}_{RNS}}(\hat{z}_3) = \hat{\xi}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_3)$$

as $\hat{\Omega}_{RNS}$ is concave RNS so equation (15) becomes

$$\hat{\xi}^{m}_{\hat{\Psi}_{RNS}}(\hat{z}_3) = \hat{\xi}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_3) \leq \max\left(\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\xi}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right)$$

$$\hat{\xi}^{m}_{\hat{\Psi}_{RNS}}(\hat{z}_3) \leq \max\left(\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\xi}^{m}_{\hat{\Theta}_{RNS}}(\hat{z}_1), \hat{\xi}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\,\hat{\xi}^{m}_{\hat{\Theta}_{RNS}}(\hat{z}_2)\right)$$

i.e.

$$\hat{\xi}^{m}_{\hat{\Psi}_{RNS}}(\hat{z}_3) \leq \max\left(\hat{\xi}^{m}_{\hat{\Psi}_{RNS}}(\hat{z}_1)\,\hat{\xi}^{m}_{\hat{\Psi}_{RNS}}(\hat{z}_2)\right)$$

similarly for $\hat{\xi}^{m}_{\hat{\Theta}_{RNS}}(\hat{z}_3) \geq \hat{\xi}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_3)$,in equation (15) so that

$$\hat{\xi}^{m}_{\hat{\Psi}_{RNS}}(\hat{z}_3) = \hat{\xi}^{m}_{\hat{\Theta}_{RNS}}(\hat{z}_3)$$

as $\hat{\Theta}_{RNS}$ is concave RNS so equation (15) becomes

$$\hat{\xi}^{m}_{\hat{\Psi}_{RNS}}(\hat{z}_3) = \hat{\xi}^{m}_{\hat{\Theta}_{RNS}}(\hat{z}_3) \leq \max\left(\hat{\xi}^{m}_{\hat{\Theta}_{RNS}}(\hat{z}_1), \hat{\xi}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\right)$$

$$\hat{\xi}^{m}_{\hat{\Psi}_{RNS}}(\hat{z}_3) \leq \max\left(\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_1), \hat{\xi}^{m}_{\hat{\Theta}_{RNS}}(\hat{z}_1), \hat{\xi}^{m}_{\hat{\Omega}_{RNS}}(\hat{z}_2)\,\hat{\xi}^{m}_{\hat{\Theta}_{RNS}}(\hat{z}_2)\right)$$

i.e.

$$\hat{\xi}^{m}_{\hat{\Psi}_{RNS}}(\hat{z}_3) \leq \max\left(\hat{\xi}^{m}_{\hat{\Psi}_{RNS}}(\hat{z}_1)\,\hat{\xi}^{m}_{\hat{\Psi}_{RNS}}(\hat{z}_2)\right)$$

hence the proof. □

Theorem 3.16. *If $\hat{\Omega}_{RNS}, \hat{\Theta}_{RNS} \in \hat{\Xi}^{O}_{CvRNS}$ then $\hat{\Omega}_{RNS} \cup \hat{\Theta}_{RNS} \in \hat{\Xi}^{O}_{CvRNS}$ and $\hat{\Omega}_{RNS} \cup \hat{\Theta}_{RNS} \in \hat{\Xi}_{CvRNS}$.*

Proof. Let $\hat{\Omega}_{RNS}, \hat{\Theta}_{RNS} \in \hat{\Xi}^{O}_{CvRNS}$, $\hat{\Psi}_{RNS} = \hat{\Omega}_{RNS} \cup \hat{\Theta}_{RNS}$ and the points $\hat{z}'_1, \hat{z}'_2, \hat{z}'_3$ on $\overline{\hat{z}'_1\hat{z}'_2}$ so that $\overline{\hat{z}'_1\hat{z}'_2} \|$ axis.

Now

$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_1\right) = \max\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right)\right), k \in [1, p]$$
$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_2\right) = \max\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right), k \in [1, p]$$
$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right) = \max\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)\right), k \in [1, p]$$

now

$$\max\left(\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_1\right), \hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_2\right)\right)$$
$$= \max\left(\max\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right)\right), \max\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right)\right)$$
$$= \max\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right), \hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right) \tag{16}$$

let

$$\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right) \geq \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)$$

in (16) so that

$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right) = \hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right)$$

as $\hat{\Omega}_{RNS}$ is ortho-concave RNS so

$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right) = \hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right) \leq \max\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right), \hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right)\right)$$
$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right) \leq \max\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right), \hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right) \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right)$$

i.e.

$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right) \leq \max\left(\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_1\right) \hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_2\right)\right)$$

similarly for $\hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right) \geq \hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right)$,in equation (16) so that

$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right) = \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)$$

as $\hat{\Theta}_{RNS}$ is ortho-concave RNS so equation (16) becomes

$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right) = \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right) \leq \max\left(\hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right), \hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right)\right)$$
$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right) \leq \max\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right), \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right), \hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right) \hat{\zeta}^{k}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right)$$

i.e.

$$\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right) \leq \max\left(\hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_1\right) \hat{\zeta}^{k}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_2\right)\right).$$

Again

$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_1\right) = \min\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right)\right), l \in [1, s]$$
$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_2\right) = \min\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right), l \in [1, s]$$
$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right) = \min\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)\right), l \in [1, s]$$

now

$$\min\left(\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_1\right), \hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_2\right)\right)$$
$$= \min\left(\min\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right)\right), \min\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right)\right)$$
$$= \min\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right), \hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right) \tag{17}$$

let

$$\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right) \leq \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)$$

in (17) so that

$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right) = \hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right)$$

as $\hat{\Omega}_{RNS}$ is ortho-concave RNS so

$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right) = \hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right) \geq \min\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right), \hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right)\right)$$
$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right) \geq \min\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right), \hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right) \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right)$$

i.e.

$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right) \geq \min\left(\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_1\right) \hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_2\right)\right)$$

similarly for $\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right) \leq \hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right)$,in equation (17) so that

$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right) = \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)$$

as $\hat{\Theta}_{RNS}$ is ortho-concave RNS so equation (17) becomes

$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right) = \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right) \geq \min\left(\hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right), \hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right)\right)$$
$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right) \geq \min\left(\hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right), \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right), \hat{\vartheta}^{l}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right) \hat{\vartheta}^{l}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right)$$

i.e.

$$\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right) \geq \min\left(\hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_1\right) \hat{\vartheta}^{l}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_2\right)\right).$$

Similarly

$$\hat{\xi}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_1\right) = \min\left(\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right), \hat{\xi}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right)\right), m \in [1, t]$$
$$\hat{\xi}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_2\right) = \min\left(\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right), \hat{\xi}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right), m \in [1, t]$$
$$\hat{\xi}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right) = \min\left(\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right), \hat{\xi}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)\right), m \in [1, t]$$

now

$$\min\left(\hat{\xi}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_1\right), \hat{\xi}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_2\right)\right)$$
$$= \min\left(\min\left(\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right), \hat{\xi}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right)\right), \min\left(\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right), \hat{\xi}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right)\right)$$
$$= \min\left(\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right), \hat{\xi}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right), \hat{\xi}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right), \hat{\xi}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right) \tag{18}$$

let

$$\hat{\xi}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right) \leq \hat{\xi}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)$$

in (18) so that

$$\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right)=\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right)$$

as $\hat{\Omega}_{RNS}$ is ortho-concave RNS so

$$\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right)=\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right)\geq\min\left(\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right),\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right)\right)$$

$$\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right)\geq\min\left(\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right),\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right),\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right)\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right)$$

i.e.

$$\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right)\geq\min\left(\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_1\right)\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_2\right)\right)$$

similarly for $\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)\leq\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right)$,in equation (18) so that

$$\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right)=\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)$$

as $\hat{\zeta}_{\hat{\Theta}_{RNS}}$ is ortho-concave RNS so equation (18) becomes

$$\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right)=\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_3\right)\geq\min\left(\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right),\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right)\right)$$

$$\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right)\geq\min\left(\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right),\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_1\right),\hat{\zeta}^{m}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right)\hat{\zeta}^{m}_{\hat{\Theta}_{RNS}}\left(\hat{z}'_2\right)\right)$$

i.e.

$$\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_3\right)\geq\min\left(\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_1\right)\hat{\zeta}^{m}_{\hat{\Psi}_{RNS}}\left(\hat{z}'_2\right)\right).$$

Since every ortho-concave RNS is also concave RNS which leads to completion of proof. □

Remark 3.17. If $\hat{\Omega}^{\alpha}_{RNS}\in\hat{\Xi}_{CvRNS}$ then $\bigcup_{\alpha}\hat{\Omega}^{\alpha}_{RNS}\in\hat{\Xi}_{CvRNS}$.

Remark 3.18. If $\hat{\Omega}^{\alpha}_{RNS}\in\hat{\Xi}^{O}_{CvRNS}$ then $\bigcup_{\alpha}\hat{\Omega}^{\alpha}_{RNS}\in\hat{\Xi}^{O}_{CvRNS}$ and $\bigcup_{\alpha}\hat{\Omega}^{\alpha}_{RNS}\in\hat{\Xi}_{CvRNS}$.

Definition 3.19. If $\mathscr{L}$ be any line and p be any point on it with $\mathscr{L}_p\perp\mathscr{L}$ at $\hat{\Omega}_{RNS}$ then the inf projection of $\hat{\Omega}_{RNS}$, denoted by $\hat{\Omega}_{\mathscr{L}}$, is stated as a mapping $\hat{\psi}:\mathscr{L}\to\hat{X}$ such that for any $p\in\mathscr{L},\hat{\psi}(p)=\inf\left\{\hat{\Omega}_{RNS}(\hat{r}),\hat{r}\in\mathscr{L}_p\right\}$ where $\left\{\hat{\Omega}_{RNS}(\hat{r}),\hat{r}\in\mathscr{L}_p\right\}\subseteq\hat{X}$.

Definition 3.20. If $\mathscr{L}$ be any line and p be any point on it with $\mathscr{L}_p\perp\mathscr{L}$ at $\hat{\Omega}_{RNS}$ then the sup projection of $\hat{\Omega}_{RNS}$, denoted by $\hat{\Omega}_{\mathscr{L}}$, is stated as a mapping $\hat{\psi}:\mathscr{L}\to\hat{X}$ such that for any $p\in\mathscr{L},\hat{\psi}(p)=\sup\left\{\hat{\Omega}_{RNS}(\hat{r}),\hat{r}\in\mathscr{L}_p\right\}$ where $\left\{\hat{\Omega}_{RNS}(\hat{r}),\hat{r}\in\mathscr{L}_p\right\}\subseteq\hat{X}$.

Theorem 3.21. *If $\hat{\Omega}_{RNS}\in\hat{\Xi}_{CvRNS}$ then $\hat{\Omega}_{\mathscr{L}}\in\hat{\Xi}_{CvRNS}$.*

Proof. Let $\hat{z}_1,\hat{z}_2,\hat{z}_3$ are the points lying on $\mathscr{L}$ with $\hat{z}_3$ that is lying on $\overline{\hat{z}_1\hat{z}_2}$, for any $\hat{\varepsilon}>0$, let $\hat{z}'_1,\hat{z}'_2$ be the points lying on $\mathscr{L}_{\hat{z}_1}$ and $\mathscr{L}_{\hat{z}_2}$ with $\hat{\zeta}^{k}_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_1\right)>\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right)-\hat{\varepsilon}$ and $\hat{\zeta}^{k}_{\hat{\Omega}_{K}}\left(\hat{z}_2\right)>\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right)-\hat{\varepsilon}$. Let $\hat{z}'_3=\overline{\hat{z}'_1\hat{z}'_2}\cap\mathscr{L}_{\hat{z}_3}$. Since $\hat{\Omega}_{RNS}$ is concave and $\hat{z}'_3\in\overline{\hat{z}'_1\hat{z}'_2}$, then we have

$$\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right)\leq\max\left(\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right),\hat{\zeta}^{k}_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right)\right),k\in[1,p],$$

$$< \max\left(\hat{\zeta}^k_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_1\right)+\hat{\varepsilon}, \hat{\zeta}^k_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_2\right)+\hat{\varepsilon}\right)$$
$$= \max\left(\hat{\zeta}^k_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_1\right), \hat{\zeta}^k_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_2\right)\right)+\hat{\varepsilon}$$

but

$$\hat{\zeta}^k_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right) \geq \hat{\zeta}^k_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_3\right)$$

hence

$$\hat{\zeta}^k_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_3\right) < \max\left(\hat{\zeta}^k_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_1\right), \hat{\zeta}^k_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_2\right)\right)+\hat{\varepsilon}$$

as $\hat{\varepsilon} > 0$ is of arbitrary nature, therefore

$$\hat{\zeta}^k_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_3\right) \leq \max\left(\hat{\zeta}^k_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_1\right), \hat{\zeta}^k_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_2\right)\right).$$

Again

$$\hat{\vartheta}^l_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right) \geq \min\left(\hat{\vartheta}^l_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right), \hat{\vartheta}^l_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right)\right), l \in [1,s],$$
$$> \min\left(\hat{\vartheta}^l_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_1\right)+\hat{\varepsilon}, \hat{\vartheta}^l_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_2\right)+\hat{\varepsilon}\right)$$
$$= \min\left(\hat{\vartheta}^l_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_1\right), \hat{\vartheta}^l_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_2\right)\right)+\hat{\varepsilon}$$

but

$$\hat{\vartheta}^l_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right) \leq \hat{\vartheta}^l_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_3\right)$$

hence

$$\hat{\vartheta}^l_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_3\right) > \min\left(\hat{\vartheta}^l_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_1\right), \hat{\vartheta}^l_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_2\right)\right)+\hat{\varepsilon}$$

as $\hat{\varepsilon} > 0$ is of arbitrary nature, therefore

$$\hat{\vartheta}^l_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_3\right) \geq \min\left(\hat{\vartheta}^l_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_1\right), \hat{\vartheta}^l_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_2\right)\right).$$

Similarly

$$\hat{\xi}^m_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right) \geq \min\left(\hat{\xi}^m_{\hat{\Omega}_{RNS}}\left(\hat{z}'_1\right), \hat{\xi}^m_{\hat{\Omega}_{RNS}}\left(\hat{z}'_2\right)\right), m \in [1,t],$$
$$> \min\left(\hat{\xi}^m_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_1\right)+\hat{\varepsilon}, \hat{\xi}^m_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_2\right)+\hat{\varepsilon}\right)$$
$$= \min\left(\hat{\xi}^m_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_1\right), \hat{\xi}^m_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_2\right)\right)+\hat{\varepsilon}$$

but

$$\hat{\xi}^m_{\hat{\Omega}_{RNS}}\left(\hat{z}'_3\right) \leq \hat{\xi}^m_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_3\right)$$

hence

$$\hat{\xi}^m_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_3\right) > \min\left(\hat{\xi}^m_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_1\right), \hat{\xi}^m_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_2\right)\right)+\hat{\varepsilon}$$

as $\hat{\varepsilon} > 0$ is of arbitrary nature, therefore

$$\hat{\xi}^m_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_3\right) \geq \min\left(\hat{\xi}^m_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_1\right), \hat{\xi}^m_{\hat{\Omega}_{\mathscr{L}}}\left(\hat{z}_2\right)\right)$$

so $\hat{\Omega}_{\mathscr{L}}$ is concave. □

Remark 3.22. If $\hat{\Omega}_{RNS} \in \hat{\Xi}_{CxRNS}$ then $\hat{\Omega}_{\mathscr{L}} \in \hat{\Xi}_{CxRNS}$.

4. Conclusion

Through this research, the existing idea of NS is refined by categorizing its uncertain components into their respective multi-sub-grades. This refined idea is then integrated with the classical theory of convexity and concavity to make it applicable to solving optimization-related problems. Several useful axiomatic results are generalized with convex and concave RNS settings. It is observed that all classical results that are discussed in the paper, are quite valid for such settings. By taking into consideration the various kinds of convexity, the proposed model may be extended to generalize the results for them. Additionally, these results can also be utilized successfully for establishing various types of mathematical inequalities.

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New Control Chart Based On Neutrosophic Maxwell Distribution with Decision Making Applications

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Abstract: The neutrosophic approach is a potential area to provide a novel framework for dealing with uncertain data. This study aims to introduce the neutrosophic Maxwell distribution $(\widetilde{\mathrm{MD}})$ for dealing with imprecise data. The proposed notions are presented in such a manner that the proposed model may be used in a variety of circumstances involving indeterminate, ambiguous, and fuzzy data. The suggested distribution is particularly useful in statistical process control (SPC) for processing uncertain values in data collection. The existing formation of $\mathrm{V_{SQ}}$-chart is incapable of addressing uncertainty on the quality variables being investigated. The notion of neutrosophic $\mathrm{V_{SQ}}$-chart $(\widetilde{\mathrm{V}}_{\mathrm{SQ}})$ is developed based on suggested neutrosophic distribution. The parameters of the suggested $\widetilde{\mathrm{V}}_{\mathrm{SQ}}$-chart and other performance indicators, such as neutrosophic power curve $(\widetilde{\mathrm{PC}})$, neutrosophic characteristic curve $(\widetilde{\mathrm{CC}})$ and neutrosophic run length $(\widetilde{\mathrm{RL}})$ are established. The performance of the $\widetilde{\mathrm{V}}_{\mathrm{SQ}}$-chart under uncertain environment is also compared to the performance of the conventional model. The comparative findings depict that the proposed $\widetilde{\mathrm{V}}_{\mathrm{SQ}}$-chart outperforms in consideration of neutrosophic indicators. Finally, the implementation procedure for real data on the COVID-19 incubation period is explored to support the theoretical part of the proposed model.

Keywords: Neutrosophic probability; Maxwell distribution; Maxwell control chart; Simulation; Estimation

1. INTRODUCTION

Statistical process control (SPC) is a set of statistical methods for process improvement and quality control. SPC is applied to observe and control a process to reduce the possibility of rework [1]. The ability to work at maximum capacity is made possible by monitoring and controlling the process [2]. SPC is a process for determining whether or not produced products satisfy the criteria and then adjusting the process to generate the desired proportion of conforming items [3]. The control chart is one of the most well-known SPC tools for observing and reducing the variation in the process. Because of many inherent causes, normal variation occurrence in closely every manufactured object is the best possible phenomenon [4]. The SPC is a standard approach that uses statistical techniques for estimating fluctuations in production or manufacturing process parameters [5]. The role of a SPC is more significant in manufacturing industries [6]. This method is widely used to study the behaviour of processes and enhance their production [7]. SPC aims to detect irregularities in made items as early as feasible to stop the progression before defective products are made [8]. The Shewhart's model, developed by Walter A. Shewhart, is a popular predictive process tool that is simple to apply and comprehend [9]. The Shewhart control chart scheme is usually not recommended in service sectors and production operations where slight modifications can result in substantial monetary losses due to its ease of development and widespread use [10]. As a result, a chart of memory types that highly responsive to small shifts in study parameters. By contrast, most real-world systems can have uncertainties or indeterminacies [11]. Shewhart control charts cannot accurately identify a process if the process is ambiguous or essential quality characteristics are determined by human subjectivity [12]–[14]. As a result, problems are explained and modelled using fuzzy set theory. Research studies [15]–[17] reveal a simple application of fuzzy charts. On average, fuzzy-based control charts are more sensitive than traditional control charts [18]. The neutrosophic approach is a more general concept and provides a platform that combines a fuzzy concept set with the notion of a classical set [19]–[21]. The neutrosophic philosophy considers the existence of truth, false, and imprecise situations . The concept of neutrosophy is currently being used in various disciplines [22]. New application areas for SPC techniques are emerging, demanding further attention.

In a variety of real-world scenarios, the collected data may be ambiguous [23]. Various researchers use neutrosophical philosophy to address the problems of having incomplete data [24]–[29]. In the field of neutrosophic statistics, the traditional statistical methods have been comprehensive to address the management of data involving ambiguity. When the underlying data consists of incomplete, unclear, or uncertain data on quality characteristics, it is impossible to utilize a typical control chart technique. Numerous researchers such as [13], [16], [30], [31] have suggested statistical approaches that are linked with neutrosophic logic in the domains SPC [17]. When the premise of normality is seriously questioned, the use of commonly used control charts is far less appropriate [32]. The V_{SQ} is one of these approaches for dealing with nonnormality in quality data, which is best represented by the classical Maxwell model [33]. The Maxwell distribution is a statistical distribution that has sparked the interest of many scholars owing to its numerous practical applications [34], [35].

In this work, neutrosophic aspect of the Maxwell model with application domains in SPC is presented. The neutrosophic version of the V_{SQ}-chart that may handle the vague, incomplete or imprecise observations in underlying Maxwell quality characteristics is suggested.

The rest of the work is organized as follows: The notions of $\widetilde{MD}$ are first introduced in Section 2. Section 3 contains the proposed control chart based on $\widetilde{MD}$. The suggested neutrosophic design performance measure is provided in Section 4. Section 5 contains a comparative analysis of the $\widetilde{V}_{SQ}$-chart. An actual example of the useful execution of the suggested $\widetilde{V}_{SQ}$-chart is expounded in Section 6. Section 7 summarizes the key findings of the work.

2. STRUCTURE OF THE PROPOSED DISTRIBUTION

This section presents an overview of the suggested distribution and introduces it in a unified framework. The following definitions establish a connection between the proposed model and its applications in the neutrosophic framework.

Definition 1: The neutrosophic density function ($\widetilde{\mathrm{PDF}}$) and Distribution function ($\widetilde{\mathrm{CDF}}$) respectively of the $\widetilde{\mathrm{MD}}$ with fuzziness in the scale parameter $\tilde{\vartheta}$ are defined as:

$$f_N(t,\tilde{\vartheta}) = \sqrt{\left(\frac{2}{\pi}\right)}\tilde{\vartheta}^{-3}t^2e^{-\frac{t^2}{2\tilde{\vartheta}^2}}\ ;\ \tilde{\vartheta} > 0, t > 0 \tag{1}$$

$$F_N(t,\tilde{\vartheta}) = \left(\frac{2}{\sqrt{\pi}}\right)\gamma\left[\frac{3}{2}, \frac{t^2}{2\tilde{\vartheta}^2}\right];\ \tilde{\vartheta} > 0, t > 0 \tag{2}$$

where $\tilde{\vartheta} = [\vartheta_1, \vartheta_u]$ and the neutrosophic random variable T. In the framework of neutrosophic calculus, it is defined as the integral of the variable density over a specified range. The neutrosophic parameter $\tilde{\vartheta}$ denotes simply the scale factor whose different values result in a variety of neutrosophic curves of the proposed distribution. The graphs of $\widetilde{\mathrm{PDF}}$ and $\widetilde{\mathrm{CDF}}$ for a continuous random variable T with different neutrosophic values of the scale parameter are depicted in Figure 1 and Figure 2, respectively.

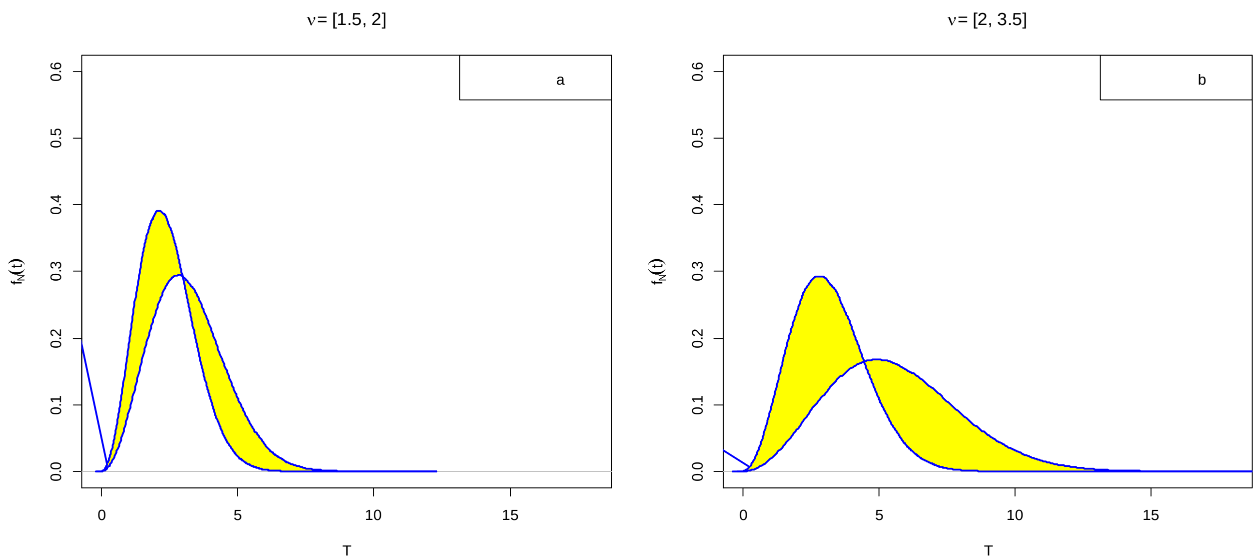

Figure 1. The $\widetilde{PDF}$ plots of the proposed model with (a) $\tilde{\vartheta} = [1.5, 2]$ and (b) $\tilde{\vartheta} = [2, 3.5]$

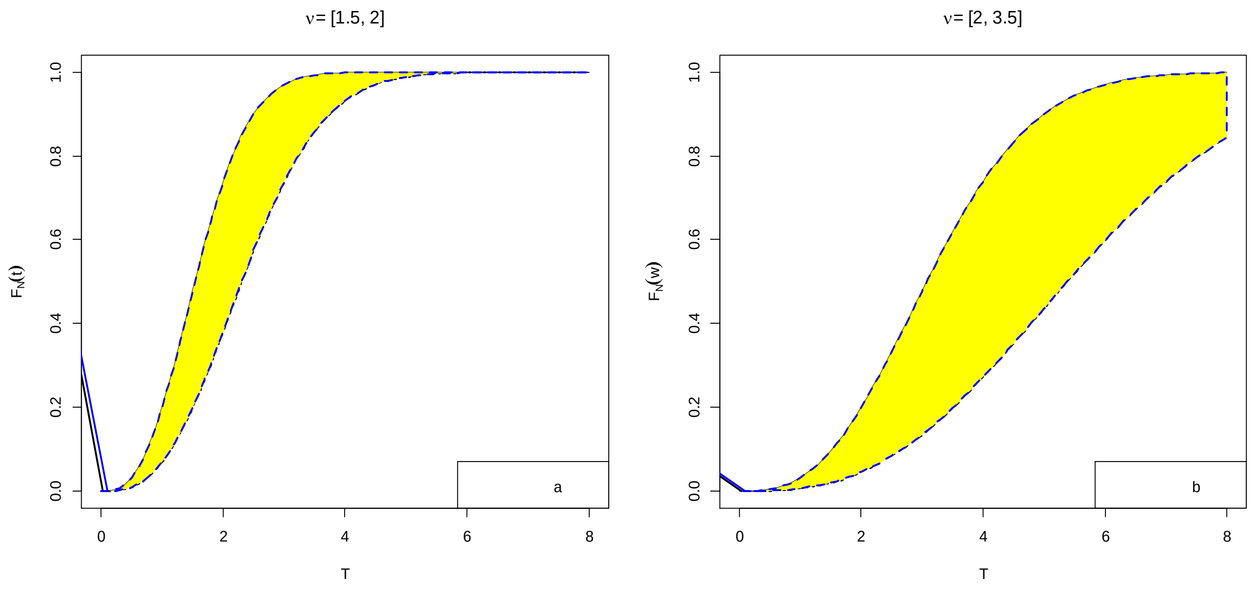

Figure 2. The $\widetilde{CDF}$ plots of the proposed model with (a) $\tilde{\vartheta} = [1.5, 2]$ and (b) $\tilde{\vartheta} = [2, 3.5]$

Figure 1 shows that the densities are asymmetric and skewed toward the right. In the neutrosophic framework, the density curve is represented by a thick layer rather than a single curve. The layer thickness (shaded region) corresponds to an indeterminacy part and total area under the sturdy curve equal to one due to completeness of $\widetilde{PDF}$. In addition, Figure 2 shows the overall behaviour of $\widetilde{CDF}$ which is right continuous and varies in the interval $[0,\ 1]$.

Definition 2 Mean and variance of the $\widetilde{\mathrm{MD}}$ are respectively given by

$$\tilde{\mu} = 2\tilde{\vartheta}\sqrt{\frac{2}{\pi}}, \text{ and } \tilde{\sigma}^2 = (3\pi - 8)\frac{\tilde{\vartheta}^2}{\pi}$$

Proof By definition

$$E(T) = \int_0^{\infty} t\ f_N(t)dt$$

$$= \int_0^{\infty} t\ [f_l(t), f_u(t)]dt$$

$$= \left[\int_0^{\infty} t\ f_l(t)dt, \int_0^{\infty} t\ f_u(t)dt\right]$$

$$= [2\tilde{\vartheta}_l\sqrt{\frac{2}{\pi}}\ ,\ 2\tilde{\vartheta}_u\sqrt{\frac{2}{\pi}}]$$

(3)

$= 2\check{\vartheta}\sqrt{\frac{2}{\pi}}$, is the required mean value of the random variable T

Now the second raw moment of the $\widetilde{\text{MD}}$ is given by:

$$E(T^2) = \int_0^\infty t^2\, f_N(t)dt$$

$$= \int_0^\infty t^2\ [f_l(t),\ f_u(t)]dt$$

$$= \left[\int_0^\infty t^2\, f_l(t)dt, \int_0^\infty t^2\, f_u(t)dt\right]$$

$$= [3\tilde{\vartheta}_l, \qquad 3\tilde{\vartheta}_u]$$

$$E(T^2) = 3\tilde{\vartheta}^2$$

Thus the variance becomes

Now $\sigma^2{}_N(t) = E(T^2) - (E(T))^2 = [3\tilde{\vartheta}_l,\ 3\tilde{\vartheta}_u] - ([2\tilde{\vartheta}_l\sqrt{\frac{2}{\pi}}\ ,\ 2\tilde{\vartheta}_u\sqrt{\frac{2}{\pi}}])^2$

After simplifying, we get

$$= \left[(3\pi - 8)\frac{\tilde{\vartheta}_l{}^2}{\pi}\ , (3\pi - 8)\frac{\tilde{\vartheta}_u{}^2}{\pi}\right] \qquad (4)$$

Further neutrosophic measures of the proposed can be derived in a similar way using the neutrosophic calculus.

The Maxwell distribution is extensively used to describe wind speed data, communications data in signals processing, modelling of wind speed data, lifetimes of different objects in reliability studies, and noise factor modelling in magnetic imaging and SPC. With particularly focus on SPC, designing of new $\tilde{V}_{SQ}$. The chart based on the neutrosophic version of the Maxell model is described in the next section.

3. CONSTRUCTION OF CONTROL CHART

Assume that the desired quality attribute is given by Y and that it follows the neutrosophic form of the Maxwell model as described in (1). In most real-world circumstances, the value of the neutrosophic parameter $\check{\vartheta}$ is rarely known and usually estimated by the maximum likelihood (ML) approach. Let $y_{1N}, y_{2N}, y_{3N} \dots \dots . y_{\tilde{m}N}$ be the observed interval values sample from $\widetilde{\mathrm{MD}}$ with density function $f_N(y, \tilde{\vartheta})$. Assume the parameter $\tilde{\vartheta}$ is unknown in the defined distribution, then $\prod_{i=1}^{\tilde{m}} f_N(y_i, \tilde{\vartheta})$ be the joint probability of the observed sample.

Taking the logarithm of the product $\prod_{i=1}^{\tilde{m}} \phi_N(y_i,\ \tilde{\vartheta})$ provides log-likelihood as:

$$\xi_N(y_{iN}, \tilde{\vartheta}) = \frac{\tilde{m}}{2}\log\left(\frac{2}{\pi}\right) - 3\tilde{m}log\tilde{\vartheta} + log\prod_{i=1}^{\tilde{m}} {y_{iN}}^2 - \frac{\sum_{i=1}^{\tilde{m}} {y_{iN}}^2}{2\tilde{\vartheta}^2} \quad (5)$$

where $\tilde{m} = [m_l, m_u]$ is the neutrosophic sample size which turns to classical sample size when $m_l = m_u = m$

The ML estimate of the unknown $\tilde{\vartheta}$ is the value that maximizes $\xi_N(y, \tilde{\vartheta})$ i.e.,

$$\hat{\tilde{\vartheta}} = \max\left(\xi_N(y_{iN}, \tilde{\vartheta})\right)$$

The ML estimates, namely $\hat{\lambda}_N$ can be obtained by using the neutrosophic calculus as:

$$\frac{\delta\xi_N(y,\tilde{\vartheta})}{\delta\tilde{\vartheta}} = [\frac{\delta\xi_l(y_{il},\tilde{\vartheta}_l)}{\delta\tilde{\vartheta}_u}, \frac{\delta\xi_u(y_{iu},\tilde{\vartheta}_u)}{\delta\tilde{\vartheta}_l}] \quad (6)$$

where $\xi_l(y, \tilde{\vartheta}_l) = \frac{m_l}{2}\log\left(\frac{2}{\pi}\right) - 3\tilde{m}log\tilde{\vartheta}_l + log\prod_{i=1}^{m_l} {y_{il}}^2 - \frac{\sum_{i=1}^{m_l} {y_{il}}^2}{2{\tilde{\vartheta}_l}^2}$

and

$$\xi_u(y, \tilde{\vartheta}_u) = \frac{m_u}{2}\log\left(\frac{2}{\pi}\right) - 3nlog\tilde{\vartheta}_u + log\prod_{i=1}^{m_u} {y_{iu}}^2 - \frac{\sum_{i=1}^{m_u} {y_{iu}}^2}{2{\tilde{\vartheta}_u}^2}.$$

Simplification of (6) provides:

$$\frac{\delta\xi_N(y,\tilde{\vartheta})}{\delta\tilde{\vartheta}} = [\frac{-3m_l}{\tilde{\vartheta}_l} + \frac{\sum_{i=1}^{m_l} {y_{il}}^2}{{\tilde{\vartheta}_l}^3}, \frac{-3m_u}{\tilde{\vartheta}_u} + \frac{\sum_{i=1}^{m_u} {y_{iu}}^2}{{\tilde{\vartheta}_u}^3}] \quad (7)$$

Equating (7) to $[0, 0]$ yields:

$$\left[\hat{\vartheta}_l, \hat{\vartheta}_u\right] = \left[\sqrt{\frac{\sum_{i=1}^{m_l} y_{il}^2}{3m_l}}, \sqrt{\frac{\sum_{i=1}^{m_u} y_{iu}^2}{3m_u}}\right] = \sqrt{\frac{\sum_{i=1}^{\tilde{m}} y_{iN}^2}{3\tilde{m}}}$$

Thus

$\widehat{\tilde{\vartheta}} = \sqrt{\frac{\sum_{i=1}^{\tilde{m}} y_{iN}^2}{3\tilde{m}}}$ is the required ML estimator for the neutrosophic parameter of $\widetilde{\text{MD}}$.

For structuring the parameters of proposed $\tilde{V}_{SQ}$-chart, we have to establish the distribution of the $\widehat{\tilde{\vartheta}}$-estiamator. The chi ($\chi$) random variable Z with 3-degree of freedom (df) is associated with the estimator $\widehat{\tilde{\vartheta}}$ as follows [31]:

$$\widehat{\tilde{\vartheta}} = \frac{\sigma}{\sqrt{3m}} Z \quad (8)$$

It is now assumed that uncertain values of σ and m are provided instead of accurate values. Under neutrosophic environment expression (8) can be written as follows:

$$\widehat{\tilde{\vartheta}} = \frac{\sigma_N}{\sqrt{3\tilde{m}}} \tilde{Z} \quad (9)$$

where $\sigma_N = [\sigma_l, \sigma_u]$, $\tilde{m} = [m_l, m_u]$ and $\tilde{Z}$ is the neutrosophic chi (χ_N) a random variable with $3\tilde{m}$ degree of freedom. The skewed curve is a collective term for the χ distribution. The density plot of the χ_N variable with neutrosophic df is displayed in Figure 3.

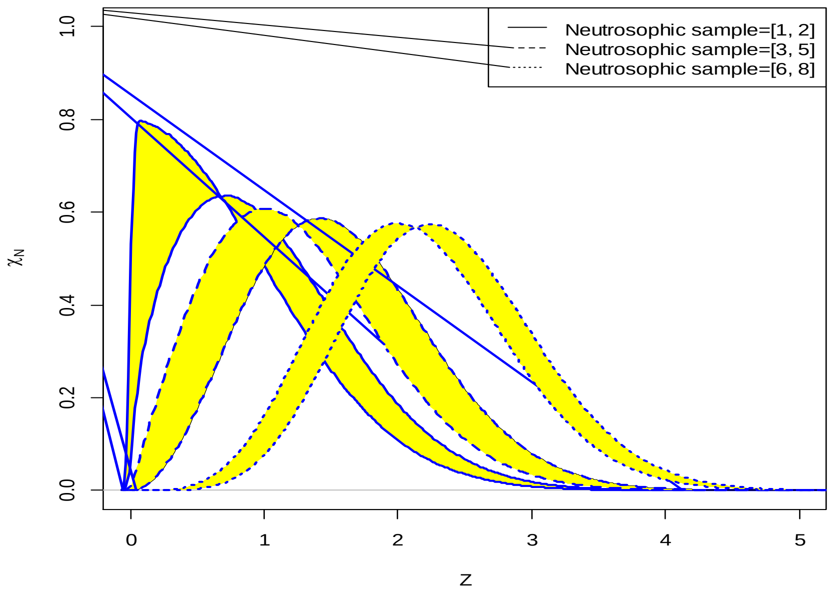

Figure 3. Probability curves of the χ_N random variable

Figure 3 is sketched to familiar with the neutrosophic form of the χ_N distribution for the case of various neutrosophic degrees of freedom. It is depicted from Figure 3 that distribution is skewed to the right for the lower degree of freedom. The distributional characteristics of the estimator $\widehat{\tilde{\vartheta}}$ using (9) can be established) as :

$$E\left(\widehat{\tilde{\vartheta}}\right) = \sigma_N \delta$$

$$V\left(\widehat{\tilde{\vartheta}}\right) = \sigma^2{}_N[1-\delta^2] \quad (10)$$

where $\delta = \sqrt{\left(\frac{2}{3\tilde{m}}\right)}\left(\frac{\Gamma\left(\frac{(3\tilde{m}+1)}{2}\right)}{\Gamma\left(\frac{3\tilde{m}}{2}\right)}\right)$ is an interval form of the neutrosophic constant and counts on $\tilde{m}$.

According to (10), the estimator $\widehat{\tilde{\vartheta}}$ is not the unbiased statistic of σ_N. For the analysis, suppose T samples with imprecise observations are available. For each sample batch, ML estimate of $\widehat{\tilde{\vartheta}}$ is obtained, then the mean of all collected sample groups will be:

$$\bar{\widehat{\tilde{\vartheta}}} = \frac{\sum_{i=1}^{T} \widehat{\tilde{\vartheta}_\iota}}{T} \quad (11)$$

Thus an unbiased estimator for σ_N can be developed as follows:

$$\hat{\sigma}_N = \frac{\bar{\widehat{\tilde{\vartheta}}}}{\delta} \quad (12)$$

Because the distribution of $\widehat{\tilde{\vartheta}}$ is highly skewed, particularly for smaller values of $\tilde{m}$, three-sigma limits are ordinarily inapplicable due to unequal tail sizes [36]. A common procedure in SPC is to use probability limits (PL) to address this issue. Since $\widehat{\tilde{\vartheta}}$ is followed by the χ_N distribution, its αt percentile is determined as:

$$F_{\chi_N}\left(\tilde{Z}\right) = \alpha \quad (13)$$

Using (8) in (13) yielded:

$$\widehat{\tilde{\vartheta}} = \frac{\sigma_N}{\sqrt{3\tilde{m}}} F_{\chi_N}{}^{-1}\left(\tilde{Z}\right) \quad (14)$$

As a result, the PL of the $\tilde{V}_{SQ}$-chart is constructed as follows:

$$upl_N = \frac{\sigma_N}{\sqrt{3\tilde{m}}} F_{\chi_N}{}^{-1}\left(1-\frac{\alpha}{2}\right) = \sigma_N \tilde{t}_1$$

$$lpl_N = \frac{\sigma_N}{\sqrt{3\tilde{m}}} F_{\chi_N}{}^{-1}\left(\frac{\alpha}{2}\right) = \sigma_N \tilde{t}_2 \tag{15}$$

where $\tilde{t}_1 = \frac{F_Y{}^{-1}\left(1-\frac{\alpha}{2}\right)}{\sqrt{3\tilde{m}}} = [t_{1l}, t_{1u}]$ and $\tilde{t_2} = \frac{F_Y{}^{-1}\left(\frac{\alpha}{2}\right)}{\sqrt{3\tilde{m}}} = [t_{2l}, t_{2u}]$ are neutrosophic values.

When the parameter defining the $\widetilde{\mathrm{MD}}$ distributed quality characteristic is not provided, it is derived using an estimator described in (10). Thus the estimated PL becomes:

$$\widehat{upl}_N = \frac{\hat{\sigma}_N}{\sqrt{3\tilde{m}}} F_{\chi_N}{}^{-1}\left(1 - \frac{\alpha}{2}\right) = \bar{\bar{\tilde{\vartheta}}}\tilde{t}_3$$

$$\widehat{lpl}_N = \frac{\hat{\sigma}_N}{\sqrt{3\tilde{m}}} F_{\chi_N}{}^{-1}\left(\frac{\alpha}{2}\right) = \bar{\bar{\tilde{\vartheta}}}\tilde{t}_4 \tag{16}$$

where $\tilde{t}_3 = \frac{F_Y{}^{-1}\left(1-\frac{\alpha}{2}\right)}{\delta\sqrt{3\tilde{m}}} = [t_{3l}, t_{3u}]$ and $\tilde{t}_4 = \frac{F_Y{}^{-1}\left(\frac{\alpha}{2}\right)}{\delta\sqrt{3\tilde{m}}} = [t_{4l}, t_{4u}]$

For a fixed of false alarm probability α and various values of m, the classic pair of crisp values $(\tilde{t}_1, \tilde{t}_2)$ and $(\tilde{t}_3, \tilde{t}_4)$ are easily computed and viewable in [31]. Three-sigma limits may be derived similarly but are not discussed in detail here due to the asymmetric form of the underlying statistic, particularly for the lower value of $\tilde{m}$.

4. PERFORMANCE METRICS

The performance measures applied in this study are explained in this section. The suggested control charts' performance is assessed using a variety of metrics, however the average neutrosophic run length $(\widetilde{\mathrm{ARL}})$ is the most frequently used and well accepted metric for analyzing neutrosophic control charts. The other related quantities such as neutrosophic power function $(\widetilde{\mathrm{PF}})$ and neutrosophic characteristics function $(\widetilde{\mathrm{CH}})$ are also described. The $\widetilde{\mathrm{PF}}$ and $\widetilde{\mathrm{CH}}$ functions are traditionally used to evaluate the sensitivity of the control chart to identify a sustained shift in key parameters. Whereas average number of neutrosophic points display on a control chart prior to the detection of an out-of-control signal is referred to as the $\widetilde{\mathrm{ARL}}$. In this concept, it has been considered that samples are taken at evenly spaced time intervals. The $\widetilde{\mathrm{ARL}}$ is actually the average value of the run-length distribution when the process is in-control (IC) and is usually denoted by $\widetilde{\mathrm{ARL}}_0$. On the other hand, when a shift occurs, the number of samples collected from that point onward is known as out-of-control (OC) run length $(\widetilde{\mathrm{ARL}}_1)$. Of course, the optimum circumstance for a given chart is for $\widetilde{\mathrm{ARL}}_0$ to be large and $\widetilde{\mathrm{ARL}}_1$ to be small. However, this is harder to establish, as it is with the Type-I and Type-II errors probabilities in the hypothesis test framework. As a remedy for this problem, the SPC literature employs an approach similar to hypothesis testing in which the $\widetilde{\mathrm{ARL}}_0$ value is fixed at a certain level and the $\widetilde{\mathrm{ARL}}_1$ value is reduced as much as feasible. To compute the value of $\widetilde{\mathrm{ARL}}_1$, the ability of $\tilde{\mathrm{V}}_{\mathrm{SQ}}$-chart of not detecting the shift is given by:

$$\beta_N = P[\mathrm{lpl}_N \le \tilde{\vartheta} \le \mathrm{upl}_N / H_1] \quad (17)$$

Further simplification of (17) yielded:

$$\beta = F_{\chi_N}\left(\theta\tilde{\vartheta}_1\sqrt{3\tilde{m}}\right) - F_{\chi_N}\left(\theta\tilde{\vartheta}_2\sqrt{3\tilde{m}}\right) \quad (18)$$

where θ is the shift constant that linked the IC parameter with OC parameter as:

$$\tilde{\vartheta}_1 = \theta\tilde{\vartheta}_0;\ \tilde{\vartheta}_0 = \left[\tilde{\vartheta}_{l0}, \tilde{\vartheta}_{u0}\right] \quad (19)$$

Thus $\widetilde{\mathrm{ARL}}_1$ can be defined as:

$$\widetilde{\mathrm{ARL}}_1 = \frac{1}{1 - F_{\chi_N}\left(\theta\tilde{t}_1\sqrt{3\tilde{m}}\right) + F_{\chi_N}\left(\theta\tilde{t}_2\sqrt{3\tilde{m}}\right)} = \frac{1}{1-\beta} \quad (20)$$

Note that the expression $1 - F_{\chi_N}\left(\theta\tilde{t}_1\sqrt{3\tilde{m}}\right) + F_{\chi_N}\left(\theta\tilde{t}_2\sqrt{3\tilde{m}}\right) = 1 - \beta$ establishes the $\widetilde{\mathrm{PF}}$ of the proposed chart and when θ becomes equal to 1, (20) provides $\widetilde{\mathrm{ARL}}_0$ i.e., mean of IC run length while the other values of θ i.e., $\theta \neq 1$, provides upwardly and downwardly shifts in the observed parameter of the proposed model. Now we compute the values of $\widetilde{\mathrm{ARL}}_1$ and $\left(\widetilde{\mathrm{PF}}\right)$ of the proposed chart for known value of process parameters. For this, we assume that $\tilde{m} = [3,5]$, crisp $\widetilde{\mathrm{ARL}}_0 = [370, 370]$ and upwardly shift in the observed parameter. In such case, computed $\widetilde{\mathrm{ARL}}_1$ and $\left(\widetilde{\mathrm{PF}}\right)$ are depicted in Figure 4 and Figure 5, respectively.

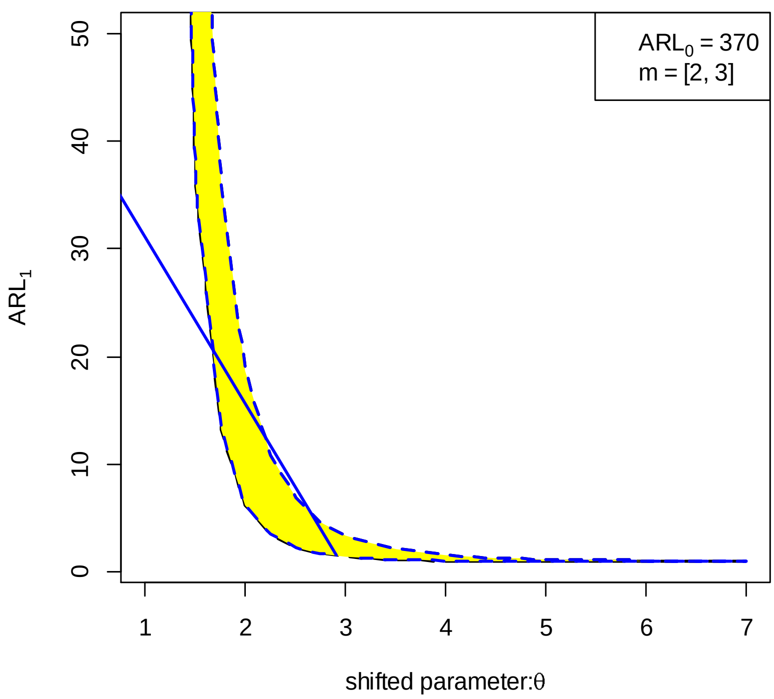

Figure 4 The $\widetilde{\mathrm{ARL}}_1$ curve of the proposed $\tilde{V}_{SQ}$-chart

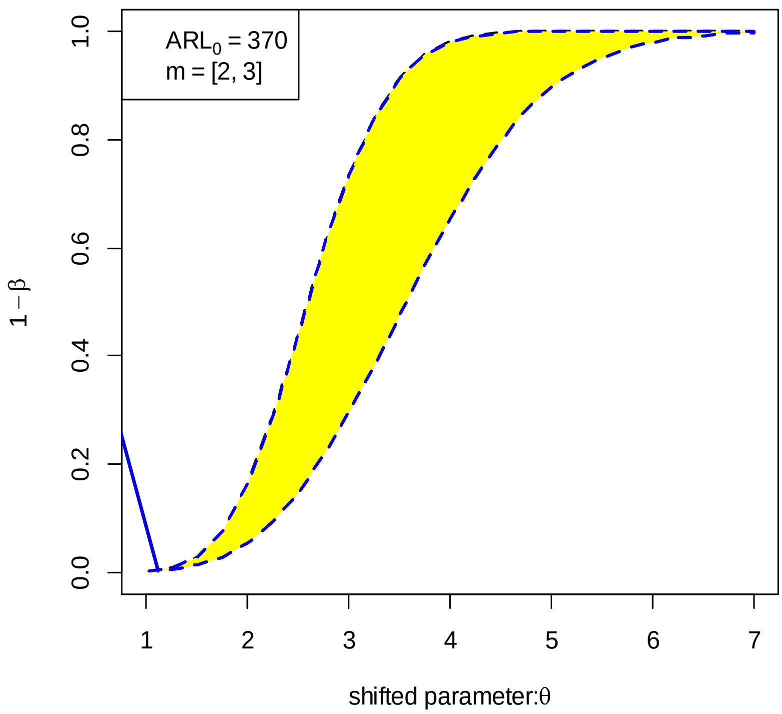

Figure 5 The $\widetilde{\mathrm{PF}}$ curve of the proposed $\tilde{\mathrm{V}}_{\mathrm{SQ}}$-chart

Figure 4 depicts the geometric shape of the RL distribution for a certain $\tilde{\mathrm{m}}$, several curves may be graphed similarly for various $\tilde{\mathrm{m}}$ values. It is clear from the graphs in Figure 4 that $\widetilde{\mathrm{ARL}}_1$ drops as θ or $\tilde{\mathrm{m}}$ grow. The graph in Figure 4 might be useful in determining the average interval number of samples required for a specific change in the study parameter. On the other hand, various changes in the observing parameter have been detected by the $\tilde{\mathrm{V}}_{\mathrm{SQ}}$-chart as revealed in Figure 5. For example, the shifted amounts of $3\tilde{\vartheta}_0$ and $5\tilde{\vartheta}_0$ have been detected by the proposed chart with interval powers equal to [0.23, 0.42] and [0.85, 0.95], respectively. Thus as a conventional the $\tilde{\mathrm{V}}_{\mathrm{SQ}}$-chart also detects the greater shifts with higher probabilities. In addition, we have provided the performance of $\tilde{\mathrm{V}}_{\mathrm{SQ}}$-chart in terms of $\widetilde{\mathrm{ARL}}_1$ in Table 1. The results in Table 1 are based on 10^5 simulations of each shift in the study parameter at a fixed benchmark value $\widetilde{\mathrm{ARL}}_0 = [370, 370]$.

Table 1 The estimated $\widetilde{\mathrm{ARL}}_1$of $\tilde{\mathrm{V}}_{\mathrm{SQ}}$-chart

	Sample size		
Shifting amount (θ)	◎2, 3◎	◎5, 7◎	◎9, 12◎
1.00	◎371.77,372.05◎	◎369.54,370.57◎	◎365.69,370.57◎
1.25	◎132.32,214.55◎	◎35.80,62.82◎	◎22.57, 35.80◎
1.50	◎35.88, 80.03◎	◎5.51, 11.62◎	◎3.27,5.51◎
1.75	◎13.23, 35.95◎	◎1.95, 3.81◎	◎1.36,1.95◎
2.00	◎6.18, 18.60◎	◎1.21, 1.91◎	◎1.04,1.21◎
2.25	◎3.47, 10.75◎	◎1.03, 1.30◎	◎1.00,1.03◎
2.50	◎2.27, 6.81◎	◎1.00, 1.09◎	◎1.00,1.00◎
2.75	◎1.67, 4.65◎	◎1.00, 1.02◎	◎1.00,1.00◎
3.00	◎1.36, 3.38◎	◎1.00,1.00◎	◎1.00,1.00◎
3.25	◎1.18, 2.60◎	◎1.00,1.00◎	◎1.00,1.00◎
3.50	◎1.18, 2.60◎	◎1.00,1.00◎	◎1.00,1.00◎
3.75	◎1.09,2.09◎	◎1.00,1.00◎	◎1.00,1.00◎
4.00	◎1.04, 1.75◎	◎◎1.00,1.00◎	◎1.00,1.00◎
4.25	◎1.01, 1.52◎	◎1.00,1.00◎	◎1.00,1.00◎
4.50	◎1.00, 1.36◎	◎1.00,1.00◎	◎1.00,1.00◎
4.75	◎1.00, 1.25◎	◎1.00,1.00◎	◎1.00,1.00◎
5.00	◎1.00, 1.17◎	◎1.00,1.00◎	◎1.00,1.00◎

Results in Table 1 show the performance of $\tilde{\mathrm{V}}_{\mathrm{SQ}}$-chart at various neutrosophic sample sizes. It looks that estimated $\widetilde{\mathrm{ARL}}_1$ is closer to the benchmark value of 370 when no shift occurred in the process parameter, i.e., $\theta = 1$. In contrast, for other values of θ, $\widetilde{\mathrm{ARL}}_1$ steadily decreases as expected with an increase in the shifted parameter.

5. COMPARATIVE STUDY

In this section, the performance of the suggested chart is compared to that of other existing model utilized to monitor the parameter of interest of the Maxwell model. It has been evaluated against an existing model of the V-chart in an indeterminate framework to observe how well $\tilde{\mathrm{V}}_{\mathrm{SQ}}$-chart performs. Various measures can be used for this comparison, although power curves are routinely employed in many research studies [37]–[39]. The equation (20) shows that power curve is a function of α, $\tilde{\mathrm{m}}$ and θ. Power curves are often used to show the connection between these parameters. The development of the power curve for the suggested model is also based on the distribution of the estimator $\hat{\tilde{\vartheta}}$. In estimating how large sample size is needed to detect an observable difference with a given probability, power curves can be helpful. In our case, the power of the $\tilde{\mathrm{V}}_{\mathrm{SQ}}$-chart is defined as if the computed $\hat{\tilde{\vartheta}}$ statistic surpasses the design limits for particular values of α and $\tilde{\mathrm{m}}$. To construct the power curve, assume that $\tilde{\vartheta}_0$ is an IC value of the observed process. Then $\widetilde{\mathrm{PF}}$ indicates the likelihood of detecting a shift to a new value, say $\tilde{\vartheta}_1$, where $\tilde{\vartheta}_1 = \theta\tilde{\vartheta}_0$ on the first sample after the shift. This approach is used to evaluate the neutrosophic power of the recommended chart and its counterpart for fixed values of $\widetilde{\mathrm{ARL}}_0$ and $\tilde{m}$ in Figure 6.

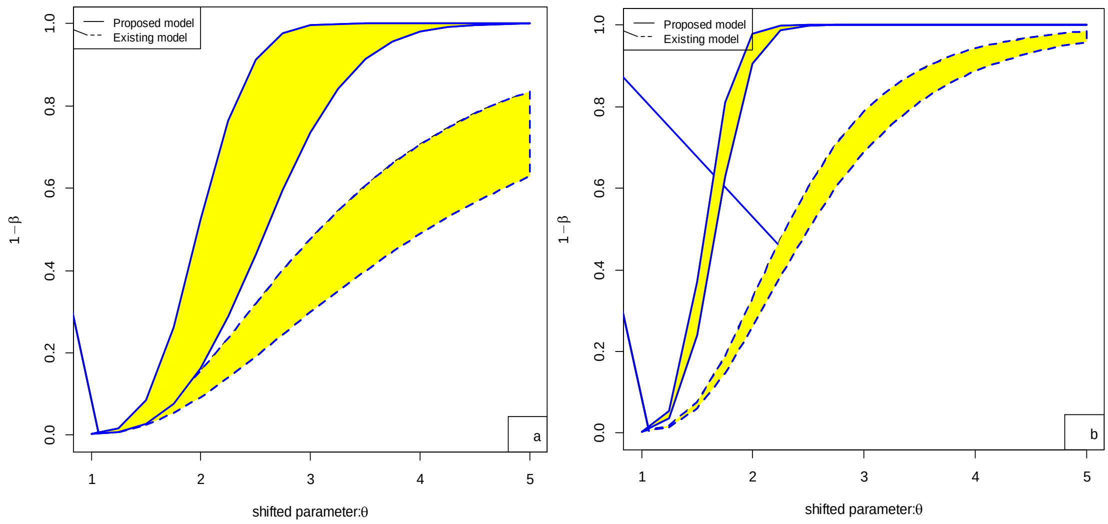

Figure 6 Power comparisons of $\tilde{V}_{SQ}$-chart at (a) $\widetilde{\mathrm{ARL}}_0 = [370, 370]$ and $\tilde{m} = [3,5]$, (b) $\widetilde{\mathrm{ARL}}_0 = [370, 370]$ and $\tilde{m} = [8,10]$

For examining Figure 6, it is observed that the suggested $\tilde{V}_{SQ}$-chart is particularly successful in identifying process changes even for small sample size. As an illustration, the power of $\tilde{V}_{SQ}$-chart and neutrosophic V-chart for detecting a shift of amount $3\tilde{\vartheta}_0$ are [0.75, 0.95] and [0.25, 0.40] respectively at $\tilde{m} = [3,5]$.Whereas the same comparison with higher probabilities hold for a larger sample size, i.e., $\tilde{m} = [8,10]$. Thus the proposed chart is deemed efficiently and highly sensitive for detecting the shift of different amounts in the studied parameter of the neutrosophic Maxwell process.

6. REAL-LIFE APPLICATION

In this section, a real-life example of the healthcare sector has been described to explain the theoretical framework of the proposed method. A patient's life or death is at stake in healthcare quality. To ensure patient satisfaction and safety, the healthcare system requires both investment and quality. Quality is a major concern for investors in this sector, which has seen a steady rise in investment. The assumption that the distribution of most medical data is normal is not accurate in most cases, so the customary assumption of normality approximation turns into inadequate for nonnormal data analysis. On the contrary, the techniques proposed in this work may effectively monitor and model the healthcare data. The capacity to reliably monitor the mean incubation time of COVID-19 and its variability in healthcare has become a major issue for the government, industry, the general public, and academics. We have applied $\widetilde{V}_{SQ}$-chart to COVID-19 mean incubation time data taken from the source [40] to examine the possible variability in incubation periods estimated from different studies. Being aware of the incubation period model while dealing with a point source pandemic enables statistical evaluation of exposure time. This information may also be used to evaluate hypotheses about whether the pandemic has come to an end by analyzing incubation time distribution during point-source epidemics. The incubation time, defined as the period between initial infection and illness manifestation, is an essential indicator for characterizing the spread of contagious diseases and developing quarantine policies [41]. It is important because reproduction numbers are often calculated using the mean incubation time, while quarantine durations are typically determined by using the maximum incubation period. The typical incubation time for COVID-19 varies widely, ranging from 3 days to 18 days [42]. As a result, it's impossible to quantify a precise quarantine period. Incubation periods have been found to vary widely in different research studies most likely because of the study population size and the estimating methodologies used. As a result, the mean incubation duration worldwide from the source is reported in Table 2 with uncertainties rather than precise figures in 12 subgroups.

Table 2: COVID-19 mean incubation period data with uncertainties

Sample batch	Mean incubation period values			
1	[7.81, 9.00]	[8.31, 9.16]	[4.48, 5.65]	[7.43, 8.51]
2	[4.95, 5.80]	[6.75, 7.62]	[5.05, 6.01]	[4.95, 5.22]
3	[6.49, 7.73]	[5.58, 6.45]	[5.55, 6.80]	[6.85, 7.18]
4	[3.99, 4.57]	[4.82, 5.04]	[6.58, 7.12]	[3.38, 4.45]
5	[8.94, 9.48]	[6.11, 7.60]	[5.36, 6.40]	[4.84, 5.05]
6	[5.35,6.63]	[9.48, 10.68]	[8.35, 9.27]	[7.93, 8.74]
7	[5.91, 6.16]	[9.90, 10.50]	[10.31, 11.37]	[8.32, 8.92]
8	[5.21, 5.98]	[5.88, 6.68]	[5.03, 5.31]	[3.83, 5.18]
9	[5.82, 6.90]	[6.01, 7.01]	[5.07, 6.14]	[2.52, 3.73]
10	[7.38, 8.18]	[5.80, 7.26]	[6.76, 7.57]	[6.52, 7.70]
11	[6.32, 6.86]	[5.33, 6.78]	[5.07,5.77]	[2.07, 3.65]
12	[7.42, 8.21]	[4.06, 5.92]	[4.17,5.48]	[3.72, 4.52]

Mean incubation time uncertainties are introduced to the technique devised in [13]. An informal graphical approach has shown that the Maxwell distribution is an acceptable model for representing the incubation time data since most actual data does not stray greatly from the theoretical red lines. The process data are skewed, as seen from the histogram and CDF plot in Figure 7 and Figure 8. As a result, the data may be further examined using the model that has been suggested.

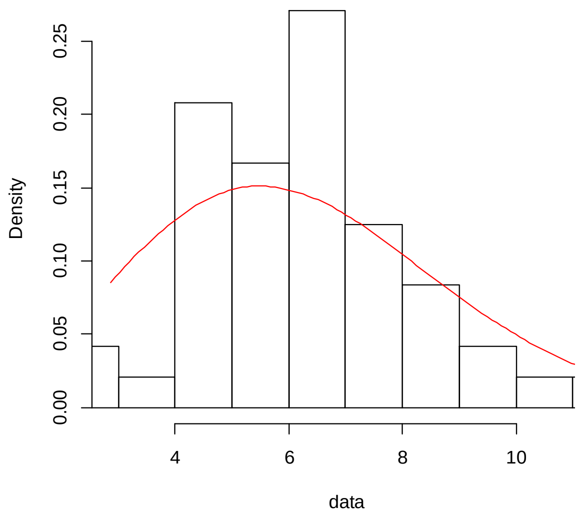

Figure 7. Histrogram of COVID-19 incubation period data

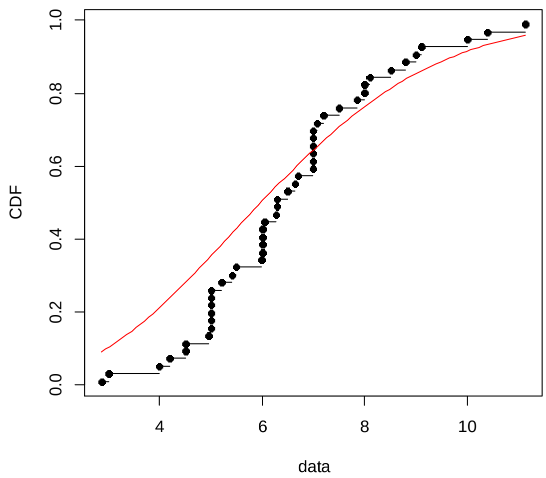

Figure 8. Theoretical and empirical CDF plots of incubation period data

By considering the individual values given in Table 2, the ML estimator $\hat{\hat{\vartheta}}$ can be obtained from each subgroup in Table 3:

Table 3: Neutrosophic estimates of the proposed model for each sample group

Sample batch	Neutrosphic estimator $\left(\widehat{\tilde{\vartheta}}\right)$
1	[4.14, 4.73]
2	[3.16, 3.59]
3	[3.54, 4.07]
4	[2.79, 3.12]
5	[3.75, 4.22]
6	[4.52, 5.16]
7	[5.06, 5.45]
8	[2.91, 3.36]
9	[2.92, 3.52]
10	[3.83, 4.44]
11	[2.86, 3.41]
12	[2.92, 3.56]

After finding ML estimate of $\widehat{\tilde{\vartheta}}$ from each sample batch, the mean of all collected sample groups from (10) can be obtained as:

$$\overline{\widehat{\tilde{\vartheta}}} = [3.54, 4.06]$$

The upper and lower probability limits for sample size 4 utilizing (16) thus can be obtained as:

$\widehat{\mathrm{upl}}_N = \max[5.90, 6.78]$ and $\widehat{\mathrm{lpl}}_N = \min[1.60, 1.80]$.

The proposed control chart based on these limits is depicted in Figure 9.

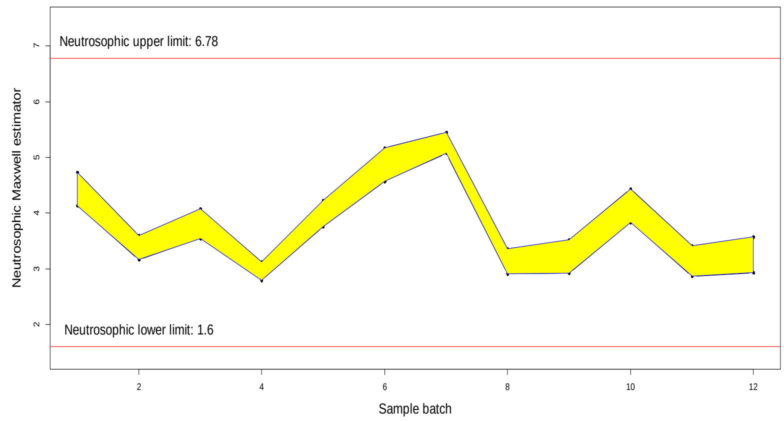

Figure 9. Control chart based on the proposed model

The depicted Maxwell estimator in Figure 9 exhibits a random tendency within the control limits. Thus, observable data-generating mechanisms may be inferred as a statistical control condition.

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7. CONCLUSIONS

The classical Maxwell model under the neutrosophic logic has been extended in this work. Several theoretical aspects of the proposed $\widetilde{\mathrm{MD}}$ distribution, such as its probability density function, characteristic function, and a few raw moments, are investigated. The theoretical framework of the suggested model, notably in domains of SPC, have been described for working data, including ambiguous, indeterminate, and imprecise observations on examined variables. Because of its suitability for dealing with ambiguous data in SPC applications, a new control chart based on the suggested $\widetilde{\mathrm{MD}}$ distribution has been developed. Some essential charting characteristics such as power curve $(\widetilde{\mathrm{PC}})$, the neutrosophic characteristic curve $(\widetilde{\mathrm{CC}})$ and neutrosophic run length $(\widetilde{\mathrm{RL}})$ of the proposed chart in terms of neutrosophic logic have been derived and validated through simulated data. A simulation study is carried out to demonstrate the theoretical results and the effectiveness of the $\widetilde{\mathrm{V}}_{\mathrm{SQ}}$-chart is evaluated to that of existing counterparts. Simulation results reveal that the proposed chart is deemed efficiently and have highly discriminatory power for detecting the shift of different amounts in the studied parameter of $\widetilde{\mathrm{MD}}$ distribution. Finally, the usefulness of the $\widetilde{\mathrm{V}}_{\mathrm{SQ}}$-chart has been described considering the real data example on the incubation period of COVID-19. Based on the results given in this study, neutrosophic extension may be designed for other statistical models in future work.

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A Neutrosophic based C-Means Approach for Improving Breast Cancer Clustering Performance

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Abstract: Breast cancer is among the most prevalent cancers, and early detection is crucial to successful treatment. One of the most crucial phases of breast cancer treatment is a correct diagnosis. Numerous studies exist about breast cancer classification in the literature. However, analyzing the cancer dataset in the context of clusterability for unsupervised modeling is rare. This work analyzes pointedly the breast cancer dataset clusterability via applying the widely used c-means clustering algorithm and its evolved versions fuzzy and neutrosophic ones. An in-depth comparative study is conducted utilizing a set of quantitative and qualitative clustering efficiency metrics. The study's outcomes divulge the presented neutrosophic c-means clustering superiority in segregating similar breast cancer instances into clusters.

Keywords: Breast cancer dataset clusterability; Fuzzy c-means clustering; Neutrosophic c-means clustering; t-SNE; Silhouette coefficient.

1. Introduction

One of the biggest problems in the healthcare system is cancer-related death. It ranks among the major causes of death among women [1]. More people have died from breast cancer than from any other disease, including tuberculosis and malaria.

Initial analysis of this condition can reduce the rate of mortality, which is on the rise [2]. Breast cancer is the sixth foremost reason of mortality globally, according to the Globocan 2020 data, and it is diagnosed in one out of every four women worldwide [3].

Making a precise diagnosis of malignancies is crucial. Most breast tumors are caused by benign (non-cancerous) alterations, however, if a benign tumor is assumed as a malignant one, it might have disastrous consequences. The most crucial actions to lowering breast cancer mortality are early detection and receiving state-of-the-art cancer therapy. Early-stage, mild breast cancer that hasn't spread can be treated successfully and quickly. Routine screening tests represent the most dependable method for identifying breast cancer in its earliest stages [4]. In an extraordinarily rich information environment, healthcare has extraordinarily little knowledge. Healthcare systems contain a vast amount of data, and it is crucial to find and establish connections with hidden data. The International Classification of Diseases (ICD) divided the foremost origins of death into five categories, with breast cancer being part of two of them [5]. According to a McKinsey report, the amount of data is increasing at a pace of 50% annually. Data science has now formally emerged as an especially important field. According to research, the phrase "data science" describes a systematic examination of the structure, properties, and evaluation of information along with the role that data play in society [6]. Statistics knowledge can be exploited from a diversity of areas, even though machine learning procedures are the most frequently used healthcare datasets.

A data analysis method called machine learning teaches a computer what results from various methods. The most popular machine learning algorithms are decision trees, k-means clustering, and neural networks [7].

The incidence of breast cancer among women, particularly those between the ages of 35 and 55, is rising because of the inhabitants of industrialized and developing nations changing their lifestyles from traditional to modern. By identifying breast tumors in their initial stages, it is possible to keep track of the prevalence of the illness [8]. Breast cancer screening methods include self- and professional breast exams, Magnetic Resonance Imaging (MRI), ultrasound, and mammography [9]. The mammogram, which includes the backdrop, the breast region, adipose tissue, breast masses, and microcalcifications with high intensities, is the result of the mammography procedure [10]. Radiologists may make mistakes or overlook crucial signs as the need for mammography processing increases because of weariness [11].

In [12]. The DCE-MRI enables a highly accurate follow-up for breast tumors. Fuzzy spatial clustering was used by Militello et al. To segment masses on DCE-MRI breast scans, and the results were superior to those of other traditional methods.

The Wisconsin Diagnostic Breast Cancer (WDBC) dataset, a highly well-known cancer dataset, was used as the basis for another cluster analysis work [13]. which incorporated a multidimensional data analysis. Because the multidimensionality of data has long been a barrier to data analysis this study hypothesized that a multidimensional data set must be projected into a lower dimensional space where it will inevitably lose some of its features to be displayed due to the limits of handling more than three spatial dimensions.

In [14], a new training dataset of breast cancer is produced using the modified k-means technique, which enhances the performance of the support vector machine model. A prediction model for breast cancer was developed using k-means and support vector machine. Using the updated k-means, a training dataset of the highest caliber was produced. Then, to group the cancerous instances of unidentified photographs, classification and accuracy were improved.

In [15], The R programming language, R visual studio, and Weka machine learning software have all been tried on the breast cancer dataset. Using various clustering algorithms were employed to examine the proper correlation in the Breast cancer dataset. In this unsupervised learning strategy, a pretrained model or label is not necessary.

The key contribution of this proposed methodology is as follows:

• Through the application of the widely known c-means clustering technique and its advanced versions fuzzy and neutrosophic ones, this work specifically investigates the clusterability of the breast cancer dataset.

• Using a collection of quantitative and qualitative clustering efficiency metrics, extensive comparative research is carried out. In terms of silhouette score, precision, and rand index, the suggested neutrosophic c-means clustering gets the best clustering performance.

Following are the last five portions of this study: Section 2 gives a review of materials and methods, Section 3 presents the metrics and results, and Section 4 presents the overall research conclusion.

2. Materials and methods

2.1 Dataset

The efficiency of the suggested model was evaluated using the WDBC datasets, which are breast cancer datasets [16]. Data from the University of Wisconsin Hospitals have previously been gathered. Each example was assigned a benign or malignant classification. The WDBC has 569 occurrences (about 62.7% benign and 37.3% malignant) and 32 significant patient features. A patient ID, 30 tumor-specific traits, and one class indicator are among these characteristics. The distinguishing features of the tumors of the patients were gathered using ten different elements, including texture, radius, area, perimeter, smoothness, concavity, compactness, concave spots, fractal dimension, and symmetry. These traits were generated from a breast lumps fine needle aspirate (FNA) picture. A set of 30 features was created by deciding the key, recognizing data for each image, such as mean, standard error, and the least or biggest standards of these features.

The dataset from Kaggle that included information about breast cancer. Thirty-two parameters make up the dataset. All the indicators can be used to categorize cancer, and if they have significantly high values, that could indicate the presence of malignant tissue. A number called ID serves as the first argument and is used for identification. The second factor is the diagnosis of membranes, which can be either malignant or benign depending on the tissue. The correct tissue diagnosis must be established for various cancer kinds if both membranes require various therapies. Following these two, a range between the center and a point on the perimeter is shown by estimated means, standard errors, and radius means. The estimated standard error is shown by radius se. The center of the projected range has the highest value of the radius worst. Knowing the distance between the center and the point is crucial since the size affects operation. With large tumors, surgery is not an option. The gray-scale values' standard deviation is represented by the texture mean. The estimated standard deviation of gray-scale values is represented by the texture se. Gray-scale values with the largest mean standard deviation are characterized as having the worst texture. Grayscale is frequently used to locate tumors, and the standard deviation is crucial to identifying data variation and explaining how to disperse the values. While the standard error of the mean indicates the core tumor expressed as perimeter se, the perimeter mean represents the mean value for the core tumor. The perimeter worst column displays the core tumor's maximum value. Area means, area se, and area worst point are identical to the previously mentioned mean of the cancer cell areas. Regional variations in the radius range are represented by smoothness mean, local variations in radius length are represented by smoothness se, and the biggest mean value is displayed as smoothness worst. The greatest mean value of the calculation is referred to as compactness worst. Compactness mean is a mean value of estimation of the perimeter and area. Compactness se is used to calculate the standard error of the mean. The severity of the concave regions of the shape is shown by the concavity mean, and the number of concave points in the shape is indicated by the concave points mean. Concavity se denotes the standard deviation of concave areas, whereas concave points se denotes the standard deviation of the shape's concave areas. The worst concavity and worst concave points represent the highest mean value. The fractal dimension means the calculated mean value for the coastline approximation, the fractal dimension se represents the standard error of the coastline approximation, and the fractal dimension worst represents the highest mean value [16].

Figure 1. Shows the distribution of thirty-one features for all 569 lesions using the Weka tool. Through the malignant and benign lesions, each feature was visualized to show how much affect the detection of diagnosis.

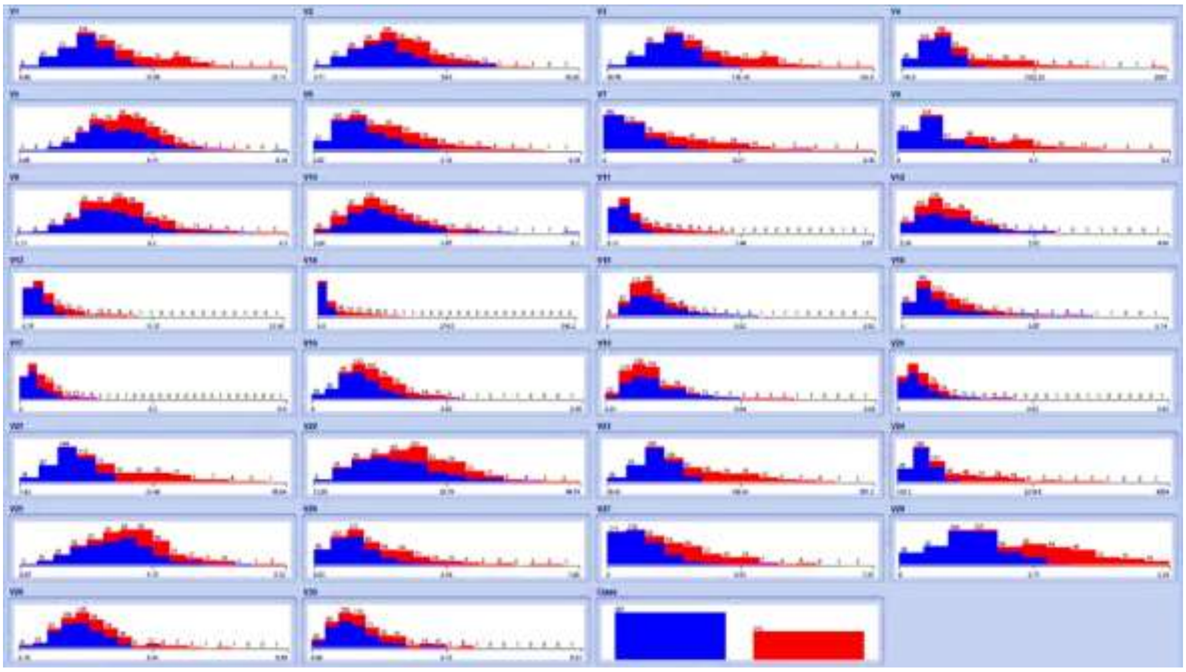


Figure 1 Distribution of dataset features.

The data is available for download in.csv format. Then, the CSV extension was updated to the Weka-compatible Attribute Relation Data Format (ARFF) extension. The data was then subjected to extensive preprocessing. There are 569 instances in the collection. The dataset is then further normalized using the min-max normalization approach in Weka software so that all feature values fall within the range [0, 1]. Being an unsupervised learning technique, clustering solely uses feature values. This indicates that the dataset's final column, the category label, is not normalized. We first eliminate the ID number. The Hopkins Statistic Index is then used to analyze the dataset to determine whether there is a strong propensity for clustering among the data points. Then, using Python programming language and Weka software tools, we apply several clustering techniques. Hopkins Statistic Index = 0.6809 shows the dataset is heavily clustered, according to our results.

2.2 K-means clustering

K-means is a clustering method that can group enormous volumes of data with a processing time that is both quicker and more efficient. The k-means algorithm, in contrast, has a flaw that is dependent on the initial value cluster that establishes the center. K-means clustering provides superior topical remedies as trial outcomes. However, the testing procedure calls for the data to be close together. In order to get a high degree of similarity among the cluster points, this can be divided into a number of clusters. The k-means algorithm is also multisided, according to (celebi et al. 2013) K-means are too straightforward to modify at each stage of the process because they are predicated on the conditions for iteration termination. They are also easy to measure in terms of distance. The first data point collection from each cluster's midway is crucial since the k-mean cluster is a local optimization [17]. The objectives of these adjustments are the best precision and the fastest convergence. If the initial point is selected from the cluster's midway, the k-mean cluster algorithm will also be limited to the optimum site. Additionally, a starting point for the k-mean clustering method will be chosen at random from the middle, up to style k. The initial centroid cluster, which is chosen at random, will have an impact on the total number of centroid cluster iterations. Therefore, by locating the centroid cluster in the high starting data points, it can be fixed to achieve higher execution.

Two familiar features of the K-means clustering technique. The first is that as a precondition parameter for clustering, it requires the usage of a specified cluster starting value, or "k centroid." However, in most cases, without prior knowledge, we are unable to determine the optimal initial number of clustering that a given data set can produce. Connecting each point to the closest cluster is the other feature.

2.3 The Fuzzy C-Means Clustering (FCM)

In their work, Dunn and Bezdek devised the fuzzy c-means method (FCM). Finding the optimal participation and clustering center to minimize the optimal solution is the main notion.

To set up the membership vector, the method must first decide on the number of clusters to create. After then, both the Center of Clustering and the Membership vector are regularly revised. Centers of various clusters and levels of membership may be produced when the optimal solution is smaller than some threshold.

These are some of the algorithm's drawbacks: Having a high degree of, sensitivity to, and depending on, the initial grouping. It is simple for the algorithm to become wedged in a local least if the starting cluster center is distant from the global optimum clustering center.

2.4 The Neutrosophic Sets

Smarandache introduced the neutrosophic concept, a generalization of previously expanded concepts, to overcome the shortcomings of conventional fuzzy clustering and enhance its capacity to manage and communicate unclear knowledge. When applied to fuzzy clustering, the neutrosophic theory is able not just to portray non-deterministic difficulties more accurately, but as well as provide solutions to those problems that remain open.

The central tenet of neutrosophic thought is the premise that every vantage point has some element of veracity, doubt, and fallacy. For this reason, the concepts of and were proposed as neutrosophic elements to signify the seriousness, ambiguity, and humorlessness of occurrences. True, indeterminate, and false outcomes are the names given to these agnostic components.

2.5 The Neutrosophic C-Means Clustering (NCM)

Conventional fuzzy clustering approaches in clustering algorithms can only explain the degree to which each group exists. It is challenging to distinguish which category a given sample belongs to and which divisions it joins, especially for the samples located in the border area among distinct groups. The neutrosophic c-means clustering method was introduced by Guo et al. To address these issues, which is an improvement on the FCM based on neutrosophic theory (NCM).

We propose a fresh special combination, A, which unites the determinant and indeterminate clusters. Let $A = C_j \cup B \cup R, j = 1, 2, \ldots, c, where\ C_j$ Is an indeterminate cluster, B refers to clusters near the edges, R relates to erratically sampled data, and is the union process. Clusters B and R both fall within the category of being agnostic. T indicates membership in the determinant cluster, I in the perimeter cluster, and F in the noisy set of data. With uncertainty in clustering in mind, we construct a new goal function and class membership as follows:

$$J(T,I,F,C) = \langle \sum_{i=1}^{n}\sum_{k=1}^{c}(w_1 T_{ik})^m \parallel x_i - v_k \parallel^2 + \sum_{i=1}^{n}\sum_{k=1}^{\binom{c}{2}}(w_2 I_{2ik})^m \parallel x_i - \overline{v_{2k}} \parallel^2 + \sum_{i=1}^{n}\sum_{k=1}^{\binom{c}{3}}(w_3 I_{3ik})^m \parallel x_i - \overline{v_{3k}} \parallel^2 + \sum_{i=1}^{n}\sum_{k=1}^{\binom{c}{4}}(w_4 I_{4ik})^m \parallel x_i - \overline{v_{4k}} \parallel^2 \rangle + \sum_{i=1}^{n}\sum_{k=1}^{\binom{c}{5}}(w_5 I_{5ik})^m \parallel x_i - \overline{v_{5k}} \parallel^2 + \sum_{i=1}^{n}\sum_{k=1}^{\binom{c}{c}}(w_c I_{cik})^m \parallel x_i - \overline{v_{ck}} \parallel^2 + \sum_{i=1}^{n}(\overline{w_{c+1}} F_i)^m \quad (1)$$

2.6 Hopkins statistic

The Hopkins statistic (Lawson and Jurs 1990) calculates the likelihood that a particular data set was produced by a uniform data distribution to evaluate the tendency of a data set to cluster [18]. In other words, it evaluates the data's spatial randomness.

Use the Hopkins score from clustered to estimate the likelihood of cluster formation before doing clustering. The outcome was two clusters, indicating the data is eligible for clustering. Unsupervised data has no notion of how many supposed clusters there are, therefore, assumptions range from two to six. Figure 2. Shows the Silhouette values vs. the number of clusters.

However, after clustering, the silhouette score used to measure cluster quality varied for each cluster. The formula is defined as follows:

$$\mathbf{H} = \frac{\sum_{i=1}^{n} \boldsymbol{y_i}}{\sum_{i=1}^{n} \boldsymbol{x_i} + \sum_{i=1}^{n} \boldsymbol{y_i}} \tag{2}$$

How should I interpret the Hopkins data?

In the case of a uniform distribution of D, $\sum_{i=1}^{n} \boldsymbol{y_i}$ and $\sum_{i=1}^{n} \boldsymbol{x_i}$, would be close to one another, and H would therefore be about 0.5. However, if clusters are present in D, the distances between manufactured points ($\sum_{i=1}^{n} \boldsymbol{y_i}$) would be expected to be much greater than those between genuine points ($\sum_{i=1}^{n} \boldsymbol{x_i}$), increasing the value of H.

Noting from figure 2. Through cluster numbers from two to six, we pick up the highest silhouette coefficient, which is determined by the number of two clusters, which suggests that two clusters are the optimum choice for data clustering. The average Silhouette Score plot of the number of clusters fluctuates between two and six and the highest silhouette value is 0.58, demonstrating that the breast cancer dataset is well matched to the given cluster when the cluster size is two.

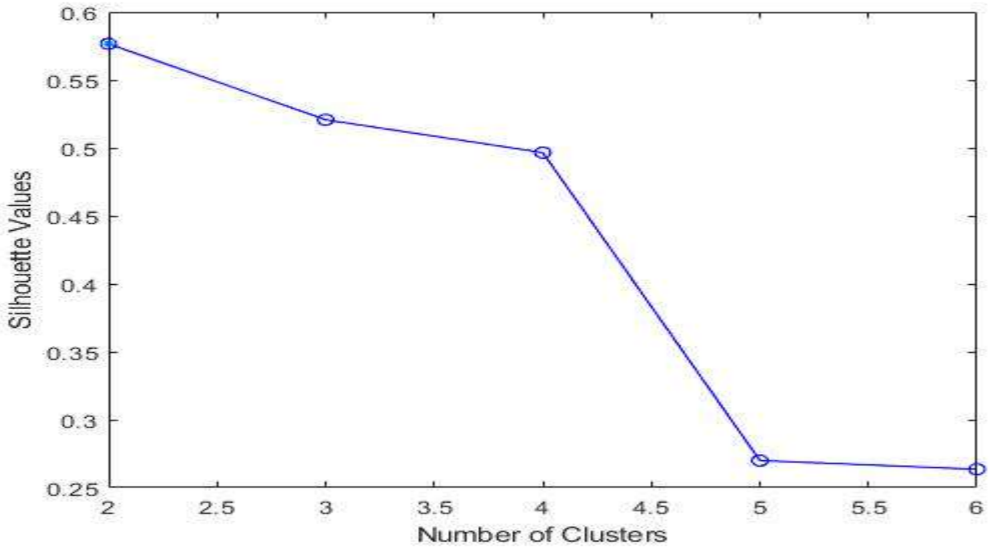


Figure 2. Silhouette values vs. the number of clusters.

2.7 Silhouette Score Analysis

Researchers may determine how closely related each observation is to the cluster to which it has been assigned about other clusters using silhouette analysis. For each observation in the data, this metric (silhouette width) runs from -1 to 1, and it can be interpreted as follows [19]:

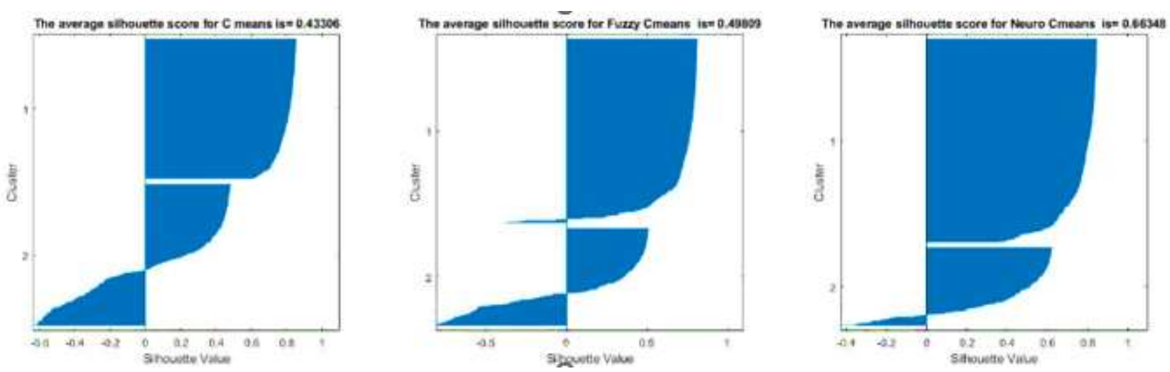
I) Values that are near 1 indicate that the allocated cluster is a good fit for the observation.

II) Values near 0 point to a possible borderline match between two groups of the observation.

III) Values near -1 point to the possibility that the observations were placed in the incorrect cluster.

In the study, we will use three well known clustering methods, investigating which one will be superior in detecting cancer cases for the aforementioned dataset. Applying k-means, fuzzy c-means, and neutrosophic c-means clustering methods.

In figure 3, We investigated two clusters of the provided data: a benign cluster and a malignant cluster. Clusters C1 and C2 are home to all 569 instances. The two clusters' average Silhouette values are 0.43 for the c-means cluster on the left, 0.5 for the fuzzy cluster in the middle, and 0.66 for the neutrosophic cluster on the right. When the Silhouette width has the highest value, which is the neutrosophic c-means in the outcomes from the three approaches, we can obtain the best clustering result. The silhouette score is shown in Table 1.



(a) (b) (c)

Figure 3. Silhouette Score for (a) k-means (left) , (b) fuzzy c-means (middle) , and (c) neutrosophic c-means clustering methods (right).

Table 1. The silhouette score of the three models.

Model 1	Silhouette score
K-Means	0.43306
Fuzzy c-means	0.49809
Neutrosophic c-means	0.66348

2.8 T-Distributed Stochastic Neighbor Embedding

T-Distributed Stochastic Neighbor Embedding (t-SNE) has emerged as a powerful standard for visualizing high-dimensional datasets in a variety of biological data sets, especially for large datasets. Using this method will help each class have a clearer image. T-SNE encompasses a variety of fields, including Bioinformatics, music analysis, computer security, and cancer biology. Similar to SNE, t-SNE chooses two distinct similarity measures for the two-dimensional embedding and the high-dimensional information. The objective of this stage is to produce a 2-dimensional embedding with a KL divergence between the vector of similarities between points in pairs over the entire dataset and the similarities between points in the encoding that is as little as possible. T-SNE is used to solve the nonconvex optimization problem utilizing gradient descent and random initialization.

Figure 4. Shows the three dimensions of T-SNE visualization (best viewed in color) for the four clustering methods actual clusters (right-bottom), c-means (left-upper), fuzzy c-means (right-upper), and neutrosophic c-means (left-bottom), respectively. By visualizing, it becomes evident that neutrosophic c- means is the best option because it is close to the actual clustering. C-means, on the other hand, is the clustering approach that is farthest from the actual means; as a result, fuzzy c-means is the second-closest method.

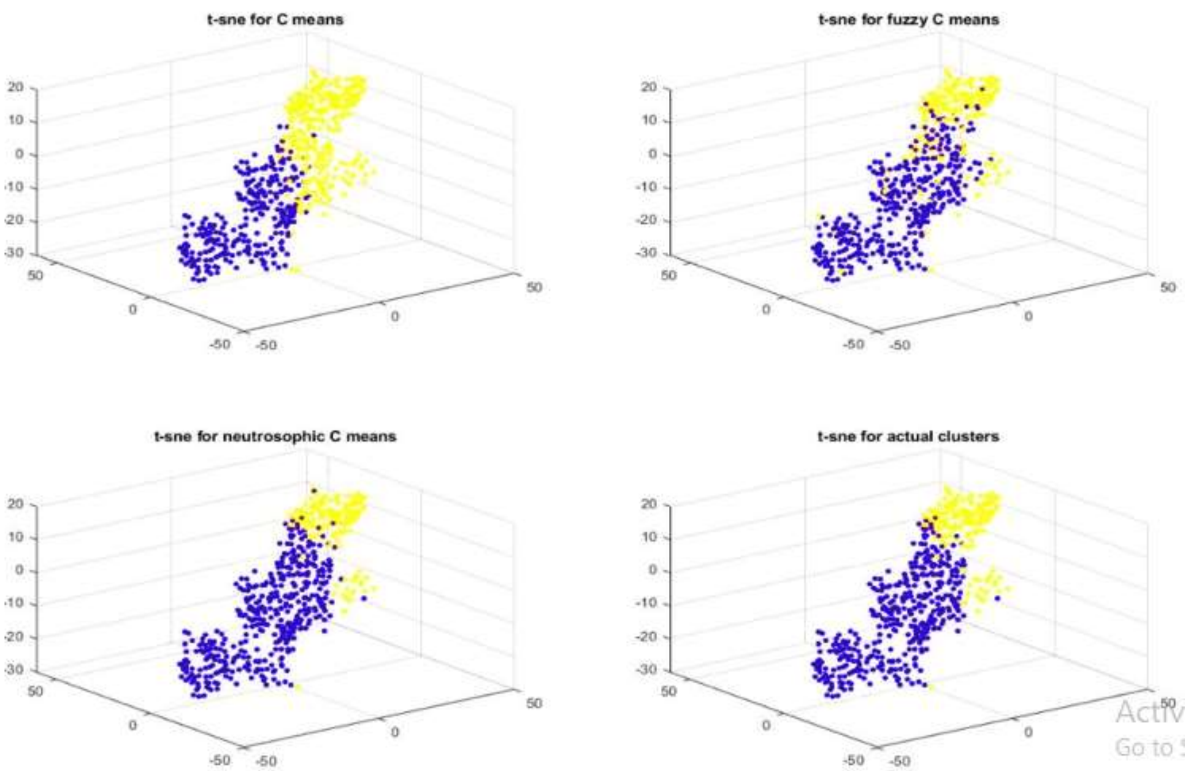


Figure 4. T-SNE graphs for c-means (left-upper), fuzzy c-means (right-upper), neutrosophic c-means (left-bottom), and actual clusters (right-bottom), (best viewed in color).

3. Results

3.1. Performance metrics*:*

Achieving high intra-cluster identity and low inter-cluster commonality is the primary focus of clustering methods (objects in the same cluster are more similar than the objects in different clusters).

In several of my investigations, the clustering methods failed to identify the optimal number of clusters. It has been shown that certain methods overestimate the size of clusters while others underestimate it. When the final class number matches the number of categories in the gold standard, we may use the typical criteria for analyzing recognition accuracy.

Equation. (3) depicts the clustering technique as a K x S matrix, where K is the expected number of clusters of the clustering method and S is the number of classes in the reference set.

Here, the element a_{ks} represents the entire number of objects that have been clustered into the k^{th} cluster and are of the sth class in the ideal distribution.

When K = S, the clustering method's estimated number of clusters exactly corresponds to the number of classes found in the reference data.

$$matrix\ k * s = \begin{matrix} k_1 \\ \dots \\ k_k \end{matrix} \begin{bmatrix} a_{11} & \cdots & a_{1s} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{ks} \end{bmatrix} \tag{3}$$

Precision

We find the benchmark class to which the most items have been allocated for each cluster. Following this, we take the sum of the largest number of items in every group and divide it by the whole number of grouped objects. precision is determined by calculating the resultant value by using K x S matrix, as seen in equation. (4).

$$precision = \frac{\sum_s \max_k a_{ks}}{\sum_k \sum_s \max_k a_{ks}} \tag{4}$$

Recall

We find the class where most items belong based on the gold standard. The complete list of grouped and unclustered items is then divided by the sum of the maximum number of objects in each gold standard class. Equation. (5) demonstrates the K x S matrix's role in deriving the recall (also called sensitivity). The number of items that are not in a cluster is denoted by U.

$$Recall = \frac{\sum_s \max_k a_{ks}}{\sum_k \sum_s \max_k a_{ks} + U} \tag{5}$$

F1-Score

According to equation. (6), the F1-score is determined by taking the mean of the accuracy and recall scores.

$$F1 = 2 * \frac{precision * recall}{precision + recall} \tag{6}$$

Rand Index

Two clustering strategies may be compared with one another using the Rand index.

The Rand Index, sometimes abbreviated as R, is determined using the following formula:

$$R = (a + b) / (nC2) \tag{7}$$

Where:

a: The frequency with which a given pair of items is assigned to the same cluster by two different techniques of clustering.

b: The frequency with which a given pair of items is found in different clusters when using two different clustering techniques.

nC2 is the count of all the non-matched pairings in a collection of n items.

3.2 results analysis

Applying the equations. (4,5,6) to compute the precision, recall, and f1 score. In the precision, the total clustered data is 569 and there is no unclustered data. In the c-means the maximum clustered data is 453, so the precision is computed by dividing the maximum clustered data by the total clustered data, the outcome is 0.796. Due to no unclustered data, the precision is equal to recall and f1 score. Table 2 shows all analyses of the precision and Rand Index.

Table 2. The overall performance analysis of the proposed model.

Model	Precision	Rand Index
C-Means	0.796	0.6748
FCM	0.8872	0.7919
NCM	0.9789	0.9586

Table 3. There are four predicted class data by the neutrosophic and fuzzy c-means clustering. In data 1, the fuzzy predicted class 0, neutrosophic predicted class 0, and the actual label is class 0, so the fuzzy and neutrosophic predicted this data truly. In data 2 the fuzzy predicted class 1, and the neutrosophic predicted class 1, also the actual data is class 1, so the fuzzy and neutrosophic predicted true labels. In data 3 the fuzzy predicted class 0, but the neutrosophic predicted class 1 and the actual labels are class 1, so the neutrosophic predicted true but the fuzzy predicted false. In data 4, the fuzzy predicted class 1, the neutrosophic predicted class 0, and the actual class is 0, so the neutrosophic predicted true and the fuzzy predicted false. Table 3. The neutrosophic predicted four true classes and the fuzzy predicted the two true classes and one false class.

Table 3. The predicted labels for Fuzzy and Neutrosophic vs. Actual label.

Data	Fuzzy Predicated Label	Neutrosophic Predicated Label	Actual Label
Data 1	0	0	0
Data 2	1	1	1
Data 3	0	1	1
Data 4	1	0	0

4. Conclusions

This paper analyzes the breast cancer dataset cluster ability via applying the widely used c-means clustering algorithm and its evolved versions fuzzy and neutrosophic ones. The conducted comparative study utilizes various metrics to fairly judge the breast cancer dataset clustering efficiency. The suggested neutrosophic c-means clustering achieves the highest clustering performance in terms of silhouette score, precision, and Rand index.

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The equations of neutrosophic straight line and neutrosophic circle

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Abstract: The purpose of this article is to study the equations of neutrosophic straight line and neutrosophic circle, where the neutrosophic point, general neutrosophic equation of a line, the equation of a neutrosophic straight line passing through two neutrosophic points and parallel and perpendicular neutrosophic lines are defined, in addition, four forms of the equations of neutrosophic circle were discussed. Where detailed examples are given to clarify each case.

Keywords: neutrosophic straight line; neutrosophic circle; the equations; polar; radius; center.

1. Introduction

As an alternative to the existing logics, Smarandache proposed the Neutrosophic Logic to represent a mathematical model of uncertainty, vagueness, ambiguity, imprecsion, undefined, unknown, incompleteness, inconsistency, redundancy, contradiction, where the concept of neutrosophy is a new branch of philosophy introduced by Smarandache [3-13]. He presented the definition of the standard form of neutrosophic real number and conditions for the division of two neutrosophic real numbers to exist, he defined the standard form of neutrosophic complex number, and found root index $n \geq 2$ of a neutrosophic real and complex number [2-4], studying the concept of the Neutrosophic probability [3-5], the Neutrosophic statistics [4][6], and professor Smarandache entered the concept of preliminary calculus of the differential and integral calculus, where he introduced for the first time the notions of neutrosophic mereo-limit, mereo-continuity, mereoderivative, and mereo-integral [1-8]. Madeleine Al- Taha presented results on single valued neutrosophic (weak) polygroups [9]. Edalatpanah proposed a new direct algorithm to solve the neutrosophic linear programming where the variables and right-hand side represented with triangular neutrosophic numbers [10]. Chakraborty used pentagonal neutrosophic number in networking problem, and Shortest Path Problem [11-12]. Y.Alhasan studied the concepts of neutrosophic complex numbers, the general exponential form of a neutrosophic complex, and the neutrosophic integrals and integration methods [7-14-18]. On the other hand, M.Abdel-Basset presented study in the science of neutrosophic about an approach of TOPSIS technique for developing supplier selection with group decision making under type-2 neutrosophic number [15]. Also, neutrosophic sets played an important role in applied science such as health care, industry, and optimization [16-17]. Giorgio Nordo, Arif Mehmood and Said Broumi studied single valued neutrosophic filters [19].

The paper is organized as follows. Section 1, provides an introduction, in which neutrosophic science review has given. Neutrosophic real number are discussed in Section 2. Section 3 frames the equations of neutrosophic straight line. the equations of neutrosophic circle were discussed in section 4. Finally, In Section 5 a conclusion to the paper is given.

2. Preliminaries

2.1. Neutrosophic Real Number [4]

Suppose that w is a neutrosophic number, then it takes the following standard form: $w = a + bI$ where a , b are real coefficients, and I represent indeterminacy, such $0.I = 0$ and $I^n = I$, for all positive integers n.

2.2. Division of neutrosophic real numbers [4]

Suppose that w_1, w_2 are two neutrosophic numbers, where:

$$w_1 = a_1 + b_1 I \ , \qquad w_2 = a_2 + b_2 I$$

To find $(a_1 + b_1 I) \div (a_2 + b_2 I)$, we can write:

$$\frac{a_1 + b_1 I}{a_2 + b_2 I} \equiv x + yI$$

where x and y are real unknowns.

$$a_1 + b_1 I \equiv (a_2 + b_2 I)(x + yI)$$

$$a_1 + b_1 I \equiv a_2 x + (b_2 x + a_2 y + b_2 y) I$$

by identifying the coefficients, we get:

$$a_1 = a_2 x$$

$$b_1 = b_2 x + (a_2 + b_{2)} y$$

We obtain unique one solution only, provided that:

$$\begin{vmatrix} a_2 & 0 \\ b_2 & a_2 + b_2 \end{vmatrix} \neq 0 \quad \Rightarrow \quad a_2(a_2 + b_2) \neq 0$$

Hence: $a_2 \neq 0 \quad and \ a_2 \neq -b_2$ are the conditions for the division of two neutrosophic real numbers to exist.

Then:

$$\frac{a_1 + b_1 I}{a_2 + b_2 I} = \frac{a_1}{a_2} + \frac{a_2 b_1 - a_1 b_2}{a_2(a_2 + b_2)} . I$$

2.3. Root index $n \geq 2$ of a neutrosophic real number [4]

1) Case: $n = 2$

Let $w = a + \mathrm{bI}$ be a neutrosophic real number, then:

$$\sqrt{a + \mathrm{bI}} = x + y.I$$

$$a + \mathrm{bI} \equiv (x + y.I)^2$$

$$a + bI \equiv x^2 + 2xy.I + y^2 I$$

by identifying the coefficients, we get:

$$x^2 = a$$

$$y^2 + 2xy = b$$

Hence $x = \pm\sqrt{a}$

$$y^2 \pm 2\sqrt{a}y - b = 0$$

By solving the second equation in respect to y we find:

$$y = \frac{\mp 2\sqrt{a} \pm \sqrt{4a + 4\mathrm{b}}}{2} = \mp\sqrt{a} \pm \sqrt{a + \mathrm{b}}$$

Then we fined four solutions of $\sqrt{a + \mathrm{bI}}$:

$$\sqrt{a + \mathrm{bI}} = \sqrt{a} + \left(-\sqrt{a} + \sqrt{a + \mathrm{b}}\right).I$$

Or: $$\sqrt{a + \mathrm{bI}} = \sqrt{a} - \left(-\sqrt{a} + \sqrt{a + \mathrm{b}}\right).I$$

Or: $$\sqrt{a + \mathrm{bI}} = -\sqrt{a} + \left(\sqrt{a} + \sqrt{a + \mathrm{b}}\right).I$$

Or: $$\sqrt{a + \mathrm{bI}} = -\sqrt{a} + \left(\sqrt{a} - \sqrt{a + \mathrm{b}}\right).I$$

particular case: $\sqrt{I} = \pm I$

2) Case: $n > 2$

$$\sqrt[n]{a + \mathrm{bI}} = x + y.I$$

$$a + \mathrm{bI} \equiv (x + y.I)^n$$

$$a + \mathrm{bI} \equiv x^n + \left(\sum_{k=0}^{n-1} C_n^k \, y^{n-k} \, x^k\right).I$$

$$x^n = a \;\Rightarrow\; \qquad x = \begin{cases} \sqrt[n]{a} & ; \; n \; odd \\ \pm\sqrt[n]{a} & ; \; n \; even \end{cases}$$

$$\sum_{k=0}^{n-1} C_n^k \, y^{n-k} \, a^{\frac{k}{n}} = b$$

Solve it in respect to y, we can distinguish two cases:
When the x and y solutions are real, we get neutrosophic real solutions,

When x and y solutions are complex, we get neutrosophic complex solutions.

3. The equations of neutrosophic straight line

3.1. Neutrosophic point

Definition3.1.1:

Let $x_A = x_a + x_b I$ and $y_A = y_a + y_b I$, where x_a, x_b, y_a, y_b are real numbers, while $I =$ Indeterminacy, then $A(x_A, y_A)$ represent the neutrosophic point.

3.2. General neutrosophic equation of a line

Definition3.2.1:

The general equation of a neutrosophic straight line is given by the following formula:

$$(a_0 + a_1 I\,)x + (b_0 + b_1 I)y + c_0 + c_1 I = 0$$

Where $a_0, b_0 \neq 0$ and $a_0, a_1, b_0, b_1, c_0, c_1$ are real numbers, while $I =$ indeterminacy.

Definition3.2.2:

Slope-intercept form of the equation of a neutrosophic straight line is given by the following formula:

$$y = (m_a + m_b I)x + p_a + p_b I$$

where m_a, m_b, p_a, p_b are real numbers, while I =indeterminacy.

Example3.1.1:

$$y = (3 + 2I)x + 2 - 4I$$

Definition3.2.3:

Equation of a neutrosophic straight line passing through two neutrosophic points, $A(x_1 + x_2 I, y_1 + y_2 I)$ *and* $B(\acute{x}_1 + \acute{x}_2 I, \acute{y}_1 + \acute{y}_2 I)$, is given by the following formula:

$$\frac{y - y_1 - y_2 I}{x - x_1 - x_2 I} = \frac{\acute{y}_1 + \acute{y}_2 I - y_1 - y_2 I}{\acute{x}_1 + \acute{x}_2 I - x_1 - x_2 I}$$

where $x_1, x_2, y_1, y_2, \acute{x}_1, \acute{x}_2, \acute{y}_1, \acute{y}_2$ are real numbers ($\acute{x}_1 \neq x_1$ *and not zero*), while $I =$ indeterminacy.

Example3.1.2:

Find the equation of a neutrosophic straight line passing through two neutrosophic points: $A(3 + 2I, 3I)$ *and* $B(7 - 3I, 5 + I)$

Solution:

$$\frac{y - 3I}{x - 3 - 2I} = \frac{5 + I - 3I}{7 - 3I - 3 - 2I}$$

$$\frac{y - 3I}{x - 3 - 2I} = \frac{5 - 2I}{4 - 5I}$$

$$\frac{y - 3I}{x - 3 - 2I} = \frac{5}{4} - \frac{17}{4}I$$

$$y - 3I = (x - 3 - 2I)\left(\frac{5}{4} - \frac{17}{4}I\right)$$

$$y - 3I = \left(\frac{5}{4} - \frac{17}{4}I\right)x - \frac{8}{4} + \frac{75}{4}I$$

$$y = \left(\frac{5}{4} - \frac{17}{4}I\right)x - \frac{8}{4} + \frac{75}{4}I + 3I$$

$$\Longrightarrow \qquad y = \left(\frac{5}{4} - \frac{17}{4}I\right)x - \frac{8}{4} + \frac{87}{4}I$$

Definition3.2.4:

Let $d_1, and\ d_2$ are two straight lines, we say $d_1, and\ d_2$ are parallel if their slopes are equal, and we say that they are perpendicular if the product of their slopes is -1 .

Example3.1.3:

$$d_1: \ y = (2 - 3I)x + 4 + 3I$$

$$d_2: \ y = (2 - 3I)x + 5 + 4I$$

$$m_{d_1} = m_{d_2} = 2 - 3I \quad \Longrightarrow d_1 \,//\, d_2$$

We can illustrate this by giving different values of I:

- $I = 0$, then:

  $d_1: \ y = 2x + 4$

  $d_2: \ y = 2x + 5$
- $I = 5$, then:

  $d_1: \ y = -7x + 13$

  $d_2: \ y = -7x + 17$

Example3.1.4:

$$d_1: \ y = (3 + 5I)x - 3 + I$$

$$d_2: \ y = \left(\frac{1}{-3-5I}\right)x + 7 - 2I$$

$$m_{d_1}.m_{d_2} = (3 + 5I)\left(\frac{1}{-3 - 5I}\right) = -1 \quad \Longrightarrow d_1 \perp d_2$$

4. The equations of neutrosophic circle

Definition4.1:

The standard equation of a neutrosophic circle: for point $p(x, y)$ to lie on a circle with center $c(h + h_1 I, k + k_1 I)$ and radius $r + r_1 I > 0$, the distance pc must be equal to radius $r + r_1 I$. Then, using the distance formula we get:

$$\overline{pc} = \sqrt{(x - h - h_1 I)^2 + (y - k - k_1 I)^2}$$

$$\overline{pc} = r + r_1 I$$

$$\sqrt{(x-h-h_1I)^2+(y-k-k_1I)^2}=r+r_1I$$

$$\left(\sqrt{(x-h-h_1I)^2+(y-k-k_1I)^2}\right)^2=(r+r_1I)^2$$

$$(x-h-h_1I)^2+(y-k-k_1I)^2=(r+r_1I)^2 \quad (1)$$

Example4.1:

$$(x-3-2I)^2+(y+4-3I)^2=4-3I$$

The center is $c(3+2I,-4+3I)$, we can find the radius as the following:

$$(r+r_1I)^2=4-3I$$

$$r+r_1I=\sqrt{4-3I}$$

Let's find $\sqrt{4-3I}$

$$\sqrt{4-3I}=r+r_1I$$

$$4-3I=r^2+2rr_1I+{r_1}^2I$$

$$4-3I=r^2+(2rr_1+{r_1}^2)I$$

then:

$$\begin{cases} r^2=4 \\ 2rr_1+{r_1}^2=-3 \end{cases}$$

$$\begin{cases} r=\pm 2 \\ r^2+2rr_1+3=0 \end{cases}$$

Find r_1:

When $r=2 \quad \Rightarrow {r_1}^2+2r_1+3=0$

$$(r_1+3)(r_1+1)=0 \quad \Rightarrow r_1=-3\,, r_1=-1$$

$$(2,-3),(2,-1)$$

When $r=-2 \quad \Rightarrow {r_1}^2-4r_1+3=0$

$$(r_1-3)(r_1-1)=0 \quad \Rightarrow r_1=3\,, r_1=1$$

$$(-2,3),(-2,1)$$

Hence:

$$r+r_1I=2-3I \quad ;I<\frac{2}{3}$$

Or $\quad r+r_1I=2-I \quad ;I<2$

Or $\quad r+r_1I=-2+3I \quad ;I>2/3$

Or $\quad r+r_1I=-2+I \quad ;I>2$

Definition4.2:

Equation of a neutrosophic circle when the centre is origin $O(0,0)$, it given by formula:

$$x^2 + y^2 = (r + r_1 I)^2$$

Example4.2:

$$x^2 + y^2 = 16 - 15I$$

The center is $O(0,0)$, we can find the radius as the following:

$$(r + r_1 I)^2 = 16 - 15I$$

$$r + r_1 I = \sqrt{16 - 15I}$$

Let's find $\sqrt{16 - 15I}$

$$\sqrt{16 - 15I} = r + r_1 I$$

$$16 - 15I = r^2 + 2rr_1 I + {r_1}^2 I$$

$$16 - 15I = r^2 + (2rr_1 + {r_1}^2)I$$

then:

$$\begin{cases} r^2 = 16 \\ 2rr_1 + {r_1}^2 = -15 \end{cases}$$

$$\begin{cases} r = \pm 4 \\ r^2 + 2rr_1 + 15 = 0 \end{cases}$$

Find r_1:
When $r = 4 \quad \Longrightarrow {r_1}^2 + 8r_1 + 15 = 0$

$$(r_1 + 3)(r_1 + 5) = 0 \quad \Rightarrow r_1 = -3 , r_1 = -5$$

$$(4, -3), (4, -5)$$

When $r = -4 \quad \Longrightarrow {r_1}^2 - 8r_1 + 15 = 0$

$$(r_1 - 3)(r_1 - 5) = 0 \quad \Rightarrow r_1 = 3 , r_1 = 5$$

$$(-4,3), (-4,5)$$

Hence:

$$r + r_1 I = 4 - 3I \quad ; I < \frac{4}{3}$$

Or $\quad r + r_1 I = 4 - 5I \quad ; I < 4/5$

Or $\quad r + r_1 I = -4 + 3I \quad ; I > 4/3$

Or $\quad r + r_1 I = -4 + 5I \quad ; I > 4/5$

Definition4.3:
The general equation of a neutrosophic circle given by formula:

$$x^2 + y^2 + (a + a_1 I)x + (b + b_1 I)y + c + c_1 I = 0$$

Adding $(a + a_1 I)^2 + (b + b_1 I)^2$ on both sides of the equation gives, we get:

$$x^2 + y^2 + (a + a_1 I)x + (b + b_1 I)y + (a + a_1 I)^2 + (b + b_1 I)^2 = (a + a_1 I)^2 + (b + b_1 I)^2 - c - c_1 I$$

$$x^2 + (a + a_1 I)x + (a + a_1 I)^2 + y^2 + (b + b_1 I)y + (b + b_1 I)^2 = (a + a_1 I)^2 + (b + b_1 I)^2 - c - c_1 I$$

$$\left(x + \frac{a + a_1 I}{2}\right)^2 + \left(y + \frac{b + b_1 I}{2}\right)^2 = \left(\frac{a + a_1 I}{2}\right)^2 + \left(\frac{b + b_1 I}{2}\right)^2 - c - c_1 I \quad (2)$$

Comparing (2) with (1), we find:

$$h + h_1 I = -\left(\frac{a + a_1 I}{2}\right)$$

$$k + k_1 I = -(\frac{b + b_1 I}{2})$$

$$(r + r_1 I)^2 = \left(\frac{a + a_1 I}{2}\right)^2 + \left(\frac{b + b_1 I}{2}\right)^2 - c - c_1 I$$

$$\Rightarrow \quad r + r_1 I = \sqrt{\left(\frac{a + a_1 I}{2}\right)^2 + \left(\frac{b + b_1 I}{2}\right)^2 - c - c_1 I} \quad > 0$$

Example4.3:
To find the standard equation, the center and radius of the following neutrosophic circle:

$$x^2 - 6x + y^2 - 6Iy + 2I = 0$$

we follow these steps:

$$h + h_1 I = -\left(\frac{a + a_1 I}{2}\right) = \frac{6}{2} = 3$$

$$k + k_1 I = -\left(\frac{b + b_1 I}{2}\right) = \frac{6I}{2} = 3I$$

$$(r + r_1 I)^2 = \left(\frac{a + a_1 I}{2}\right)^2 + \left(\frac{b + b_1 I}{2}\right)^2 - c - c_1 I$$

$$= 9 + 9I - 2I \ = 9 + 7I$$

hence:

$$(x - 3)^2 + (y - 3I)^2 = 9 + 7I$$

The center is $c(3,3I)$, we can find the radius as the following:

$$(r + r_1 I)^2 = 9 + 7I$$

$$r + r_1 I = \sqrt{9 + 7I}$$

Let's find $\sqrt{9 + 7I}$

$$\sqrt{9 + 7I} = r + r_1 I$$

$$9 + 7I = r^2 + 2rr_1 I + {r_1}^2 I$$

$$9 + 7I = r^2 + (2rr_1 + r_1{}^2)I$$

then:

$$\begin{cases} r^2 = 9 \\ 2rr_1 + r_1{}^2 = 7 \end{cases}$$

$$\begin{cases} r = \pm 3 \\ r^2 + 2rr_1 - 7 = 0 \end{cases}$$

find r_1:

when $r = 3 \quad \Rightarrow r_1{}^2 + 6r_1 - 7 = 0$

$$(r_1 + 7)(r_1 - 1) = 0 \quad \Rightarrow r_1 = -7 \,, r_1 = 1$$

$$(3, -7), (3,1)$$

when $r = -3 \quad \Rightarrow r_1{}^2 - 6r_1 - 7 = 0$

$$(r_1 - 7)(r_1 + 1) = 0 \quad \Rightarrow r_1 = 7 \,, r_1 = -1$$

$$(-3,7), (-3, -1)$$

hence:

$$r + r_1 I = 3 - 7I \quad ; I < \frac{3}{7}$$

Or $\quad r + r_1 I = 3 + I \quad ; I > -3$

Or $\quad r + r_1 I = -3 + 7I \quad ; I > 3/7$

Or $\quad r + r_1 I = -3 - I \quad ; I < -3$

4.1. Polar equation of a neutrosophic circle

The polar form of equation of a neutrosophic circle, with a center $S(\acute{r} + \acute{r}_1 I, \varphi + \varphi_1 I)$ and radius $R + R_1 I$, using the low of cosine:

$$(r + r_1 I\,)^2 + (\acute{r} + \acute{r}_1 I)^2 - 2(r + r_1 I\,)(\acute{r} + \acute{r}_1 I)\cos(\theta + \theta_1 I - \varphi - \varphi_1 I) = (\,R + R_1 I\,)^2$$

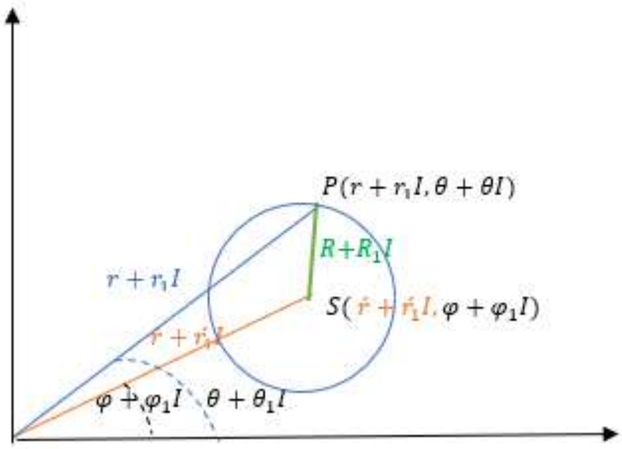

Figure 1

Note:

The polar equation of a neutrosophic circle, with radius $R+R_1I$ and a center on the polar axis running through the pole O (origin):

Since:

$$\cos(\theta + \theta_1 I) = \frac{r + r_1 I}{2(R+R_1 I)}$$

then:

$$r + r_1 I = 2(R+R_1 I)\cos(\theta + \theta_1 I)$$

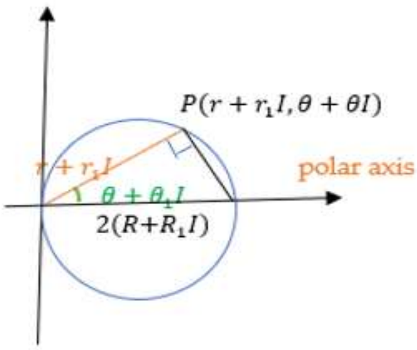

Figure 2

Example4.1.1:

Convert The polar equation of a neutrosophic circle:

$$r + r_1 I = (-4 + 6I)\cos(\theta + \theta_1 I)$$

into cartesian coordinatedes.

Solution:

$$r + r_1 I = (-4 + 6I)\cos(\theta + \theta_1 I)$$

$$(r + r_1 I)^2 = (-4 + 6I)(r + r_1 I)\cos(\theta + \theta_1 I)$$

by substitute in:

$$x^2 + y^2 = (r + r_1 I)^2$$

we get:

$$x^2 + y^2 = (-4 + 6I)(r + r_1 I)\cos(\theta + \theta_1 I)$$

we know:

$$x = (r + r_1 I)\cos(\theta + \theta_1 I)$$

then:

$$x^2 + y^2 = (-4 + 6I)x$$

$$x^2 - (-4 + 6I)x + y^2 = 0$$

$$h + h_1 I = -\left(\frac{a + a_1 I}{2}\right) = \frac{-4 + 6I}{2} = -2 + 3I$$

$$k + k_1 I = -\left(\frac{b + b_1 I}{2}\right) = \frac{0}{2} = 0 + 0I$$

$$(r + r_1 I)^2 = \left(\frac{a + a_1 I}{2}\right)^2 + \left(\frac{b + b_1 I}{2}\right)^2 - c - c_1 I$$

$$= (-2 + 3I\ )^2 + 0 + 0I - (0 + 0I) = 4 - 3I$$

hence:

$$(x + 2 - 3I)^2 + y^2 = 4 - 3I$$

The center is $c(-2 + 3I\ ,0 + 0I)$, we can find the radius as the following:

$$(r + r_1 I)^2 = 4 - 3I$$

$$r + r_1 I = \sqrt{4 - 3I}$$

let's find $\sqrt{4 - 3I}$

$$\sqrt{4 - 3I} = r + r_1 I$$

$$4 - 3I = r^2 + 2rr_1 I + {r_1}^2 I$$

$$4 - 3I = r^2 + (2rr_1 + {r_1}^2)I$$

then:

$$\begin{cases} r^2 = 4 \\ 2rr_1 + {r_1}^2 = -3 \end{cases}$$

$$\begin{cases} r = \pm 2 \\ r^2 + 2rr_1 + 3 = 0 \end{cases}$$

Find r_1:

When $r = 2 \quad \Rightarrow {r_1}^2 + 2r_1 + 3 = 0$

$$(r_1 + 3)(r_1 + 1) = 0 \quad \Rightarrow r_1 = -3\ , r_1 = -1$$

$$(2, -3), (2, -1)$$

When $r = -2 \quad \Rightarrow {r_1}^2 - 4r_1 + 3 = 0$

$$(r_1 - 3)(r_1 - 1) = 0 \quad \Rightarrow r_1 = 3\ , r_1 = 1$$

$$(-2,3), (-2,1)$$

Hence:

$$r + r_1 I = 2 - 3I \quad ; I < \frac{2}{3}$$

Or $\quad r + r_1 I = 2 - I \quad ; I < 2$

Or $\quad r + r_1 I = -2 + 3I \quad ; I > 2/3$

Or $\quad r + r_1 I = -2 + I \quad ; I > 2$

4. Conclusions

Geometry is important for many reasons. The world is overflowing with geometric shapes, and since geometric shapes surround us from every side, our understanding and appreciation of our world will be better if we learn something about geometry. This led us to introduce the concept of neutrosophic in geometry and to write this paper. The equations of the circle and the straight line in the neutrosophic field are defined. This paper is considered an introduction to the neutrosophic geometry.

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Neutrosophic DICOM Image Processing and its applications

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Abstract

Medical images are essential in contemporary medicine because they provide practicable entropy, which is used to diagnose medical conditions. It is useful to visualize abnormality in several parts of the body. Image segmentation in the medical has an important function in various applications in diagnosis systems. Researchers have become interested in segmentation algorithms as a result of Computed Tomography (CT) and Magnetic Resonance Imaging (MRI). The Region of Interest (ROI) extracts used in medical applications depend heavily on processes including cancer identification, bulk detection, and organ segmentation. Due to its capacity to deal with uncertainty and imprecision, Neutrosophic image processing (NIP) is a significant domain for uncertainty in medical image processing. Its methods in medicine demonstrate their transcendence. In the suggested work, the primary medical domains that NIP can create for image segmentation from DICOM pictures are highlighted. Due to the way it handles uncertain information, it has been found to be a better method.

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Keywords: Image processing, Neutrosophic image processing, Image segmentation, DICOM images.

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1. Introduction

Digital imaging is a vital role in medical image analysis in clinical theory therapy [18],[25], and [28]. Medical image classification has been thoroughly explained [4],[29]. outlined how crucial the problem of image segmentation is to image processing [16]. image segmentation techniques were explained [33]. The use of images has attracted the attention of several researchers [10], [11], [46], [41], [48] and [14]. image analysis was reviewed [45]. An explanation of an image segmentation pattern [12]. Some novel medical segmentation concepts are proposed in [2], [9], [13], and [22]. Explained image segmentation by threshold [39]. Segmentation is handled by region [52]. In this investigation, operators defined in the Neutrosophic theory will be applied for digital image processing. Neutrosophic is the branch of philosophy that studies everything related to neutralities. Along with the membership and non-membership function, it now also provides an indeterminacy membership function for the first time, allowing any one of them to exist independently of the others. Contradictions, inconsistencies, and ignorance in knowledge or information are modelled by indeterminacy. Explained filters in Neutrosophic image processing [42]. Studies on edge detection based on uniforms [18], [24] Using evolutionary algorithms and an enhanced Sobel operator, locate edges in photos [26]. Using hysteresis thresholds in thresholding techniques to detect edges [33]. There has been some investigation on the effectiveness of the Neutrosophic set approach filtering method for image denoising [39]. Grey picture extraction and segmentation using fuzzy logic [34]. [50]. ultrasonography breast image segmentation using the Neutrosophic approach. area merge approach using Newton-Raphson logic (49). [8] utilising ultrasound pictures for automated identification and categorization of breast cancer. [15]. Neutrosophic Sets: A New Approach for Improving Image Retrieval. [20],[35]. MRI denoising using the Wiener filtering nonlocal Neutrosophic set technique. [30],[19] innovative method for segmenting coloured images using fuzzy c-means and the Neutrosophic. [27] Modified Neutrosophic method for segmenting coloured images. [39] a DICOM image extraction type-2 fuzzy. [38] Using Type-2 Fuzzy Triangular Norms, find edges in a DICOM image.

Random noise throughout the process reduces the processing speed and quality of the MRI pictures. Denoising plays a crucial function in the earlier stage of picture processing. In Neutrosophic based noise reduction is MRI images converted to Neutrosophic sets. True, indeterminacy, and false in defined in Neutrosophic sets. The entropy is measured from indeterminacy. Image segmentation is considered for pattern identification [12].[3]. Proposed a new image segmentation in images on Neutrosophic histogram estimation. Neutrosophic set is high impact on deducing indeterminacy of uncertainty [6],[5],[7],[36],[44],[47] and [43]. After the development of Neutrosophic theory so many researchers concentrated on medical image processing [21],[17].[31] proposed breast lesion image segmentation from computed tomography. [32] introduced a contour model image segmentation. [1] Neutrosophic based liver tumor segmentation. [23] and [32] propose to introduce image processing through the Neutrosophic sets.

2.Methodology

Evaluation Metric

Cardinality for Neutrosophic images

If the image is being the pixel coordinate A(x, y), G(x,y) be the gray level pixel of A(x,y). $\mu_A(x)$ represent the membership function of the expert knowledge of the image. $I_A(x)$ is indeterminacy of the expert knowledge of the image and $F_A(x)$ is the non membership of the expert knowledge of the image. $\pi_A(x)$ is represent as hesitation value. If an image of size M x N pixel gray level L between 0 to L-1.

$$\pi_A(x) = 3 - \mu_A(x) - I_A(x) - F_A(x)$$

$\mu_A(g_{ij}), I_A(g_{ij})$ and $F_A(g_{ij})$ represent as the (i,j) the pixel of membership, indeterminacy and non membership function.

$\mu_A(g_{ij}) = \frac{g_{ij} - min g_{ij}}{g_{ij} - max g_{ij}}$ mingij,maxgij represent the gray level of images.

If N is a neutrosophic crisp set. The neutrosophic measure defines as

$$E(N) = \frac{1}{n}\sum_{i=1}^{n}\frac{Maxcount(E_i \cap E_i^c)}{Maxcount(E_i \cup E_i^c)}$$

Where n is the cardinal(E), Ei denotes a single element. The cardinality of E is given by

$$Maxcount(E) = \sum_{i=1}(\mu_E(x_i) + \pi_E(x_i))(E) + \sum_{i=1}(V_E(x_i) + \pi_E(x_i))$$

DICOM image is mapped in to Neutrosophic space, where the Neutrosophic space image N_I(x,y)◎◎T(x,y),I(x,y).F(x,y)◎.

Where T(x,y), I(x,y) and F(x,y) are the true ,indeterminate and false respectively on the image N.

$$T(x,y) = \frac{\bar{N}\,(i,j) - \bar{N}\,min}{\bar{N}\,max - \bar{N}\,min} \quad (1)$$

$$\bar{N}\,(x,y) = \frac{1}{w\,x\,w}\sum_{i=a-\frac{w}{2}}^{a+\frac{w}{2}}\sum_{j=b-\frac{w}{2}}^{b+\frac{w}{2}} N(i,j) \quad (2)$$

$$I\,(x,y) = \frac{\delta(x,y) - \delta min}{\delta max - \delta min} \quad (3)$$

$$\partial(x,y) = abs(N(x,y) - \bar{N}\,(x,y) \quad (4)$$

$$F(x,y) = 1 - T(x,y) \quad (5)$$

$$\text{Accuracy} = \frac{N_{\text{True Positive}} + N_{\text{True Negative}}}{N_{\text{True Positive}} + N_{\text{True Negative}} + N_{\text{False Positive}} + N_{\text{False Negative}}} \quad (6)$$

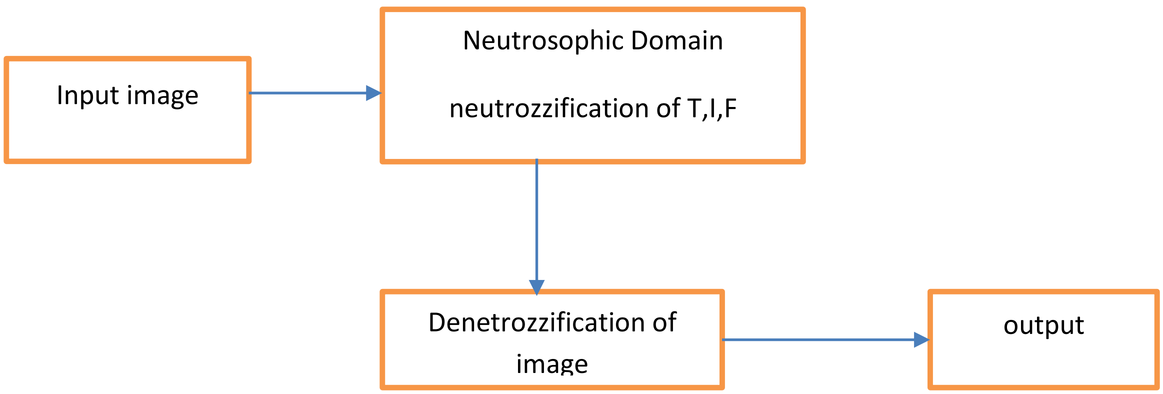
$$\text{Precision} = \frac{N_{\text{True Positive}}}{N_{\text{True Positive}} + N_{\text{False Positive}}} \quad (7)$$

$$\text{Harmonic mean} = \frac{2 \times N_{\text{True Positive}}}{2 \times N_{\text{True Positive}} + N_{\text{False Positive}} + N_{\text{False Negative}}} \quad (8)$$

True ,Indeterminacy and false entropies in Neutrosophic image are measured from entropy domain

$\bar{N}\,(x, y)$ is the local mean and $\partial(x, y)$ is the absolute value of difference between $N(x, y)$ and $\bar{N}\,(x, y)$. If the intensity have equal probability with uniformly distributed. Guo et al .,(2009)

3.Neutrosophic image processing



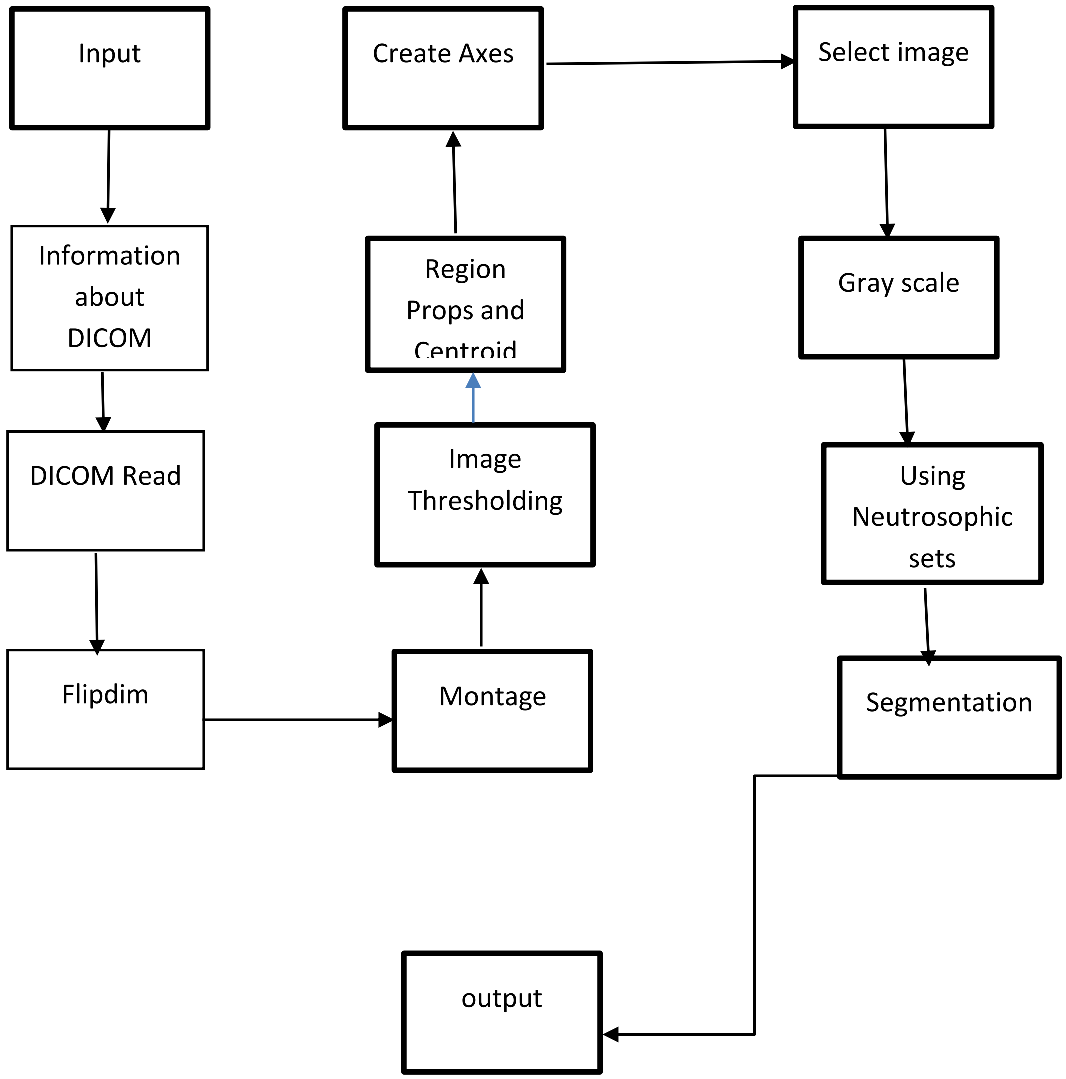
Fig(1) Neutrosophic image processing

The approved and standard data is called Digital Imaging and Communication in Medicine (DICM) .It is impossible to determine whether an edge is visible in a picture because most photographs lack sufficient brightness. Before the edge detection technique begins, edges may be improved. Opening and Closing, Maximum Erosion, and Minimum Dilation are morphological operations used in image processing (Idempotency).

4.Architecture of edge detection by Neutrosophic

Here the proposed design of the process of edge detection is described (Figure 2.)

Figure 2: Architecture of Edge Detection on DICOM image



5.Proposed edge detection algorithm

Step 1: Convert CT scans files to DICOM through filpdim

Step 2: Image convert to Gray Scale

Step 3: Do thresholding and region growing

Step 4: Convert RGB to green channel complement

Step 5: Give contrast limited adopting histogram equalization

Step 6: Use morphological operation

Step 7: Remove optic disc

Step 8: Use 2D medium filter and reduce the noise

Step 9: Remove background and image adjustment

Step 10: Do the segmentation using Neutrosophic sets

Step 11: Detect the edge

Step 12: End

6.Programme of DICOM image processing

```
g⊚imread('image.jpg');
g⊚rgb2gray(g);
g⊚double(g);
w⊚3;
for i⊚3:size(g,1)-2
for j⊚3:size(g,2)-2
s⊚0;
for m⊚i-round(w⊚2):i⊚round(w⊚2)
for n⊚j-round(w⊚2):j⊚round(w⊚2)
s⊚s⊚g(m,n);
end
end
g1(i,j)⊚s⊚(w⊚w);
segma(i,j)⊚abs(g(i,j)-g1(i,j));
end
end
g1min⊚min(min(g1));
g1max⊚max(max(g1));
segmamin⊚min(min(segma));
segmamax⊚max(max(segma));
for i⊚3:size(g,1)-2
for j⊚3:size(g,2)-2
T(i,j)⊚((g1(i,j)-g1min).⊚(g1max-g1min));
I(i,j)⊚((segma(i,j)-segmamin).⊚(segmamax-segmamin));
F(i,j)⊚1-T(i,j);
end
end
figure
subplot(3,1,1),imshow(T),title('T domain')
subplot(3,1,2),imshow(I),title('I-domain')
subplot(3,1,3),imshow(F),title('F-domain')
```

APPLICATION OF IMAGE PROCESSING

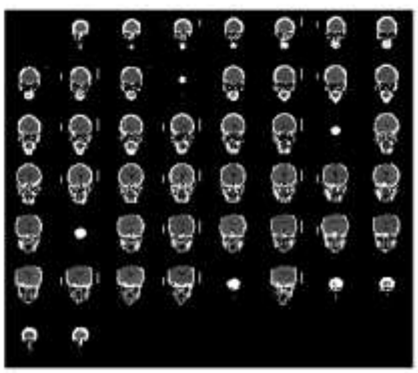
Image analysis is using MATLAB 2021b. The three-dimensional image in this instance is changed to a two-dimensional image. The image in Figure 1 was taken from a patient DICOM image as part of our experimental data collection.

For the purpose of the whole image in Figure 3, The data image was produced using computed tomography and is coloured in grayscale. The facial bone of a 50-year-old woman is mentioned in the study's description.Figure

3 displays a portion of the DICOM data collection. Convolution models for image segmentation can benefit from the usage of medical imaging. There are few data sets for medical picture segmentation.

Fig :3 DICOM Montage

S	Accuracy	sensitivity	specificity	Negative Rate	Predictive Value	score	precision	Harmonic mean
T	0.9668	0.1287	0.9838	0.0162	0.1394	0.1338	0.13944	0.13384321
I	0.1946	0.9583	0.048	0.952	0.162	0.2771	0.16197	0.27710843
F	0.3733	0.8333	0.3118	0.6882	0.1393	0.2387	0.13927	0.23866348
normal	0.987	0.037	0.9945	0.0055	0.0514	0.043	0.05139	0.04304932



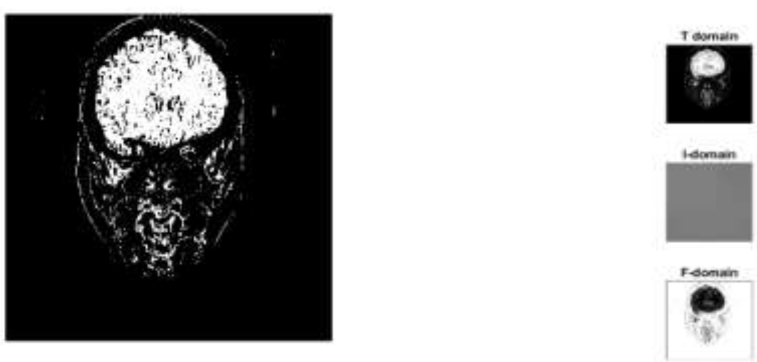


Fig :4 Image with Best view and Neutrosophic images

The optimum filter for extracting the image from DICOM data in the suggested system is found to be the 2D median filter. The experiment's classification result shows that the genuine membership image extraction accuracy is 97◎ , the sensitivity is 1◎ , the specification is 98◎ , the PPV is 12◎ , and the 12 harmonic mean of precision and sensitivity is 12◎ . In Table 1, the categorization outputs are displayed.

Table:1 Measures of the images.

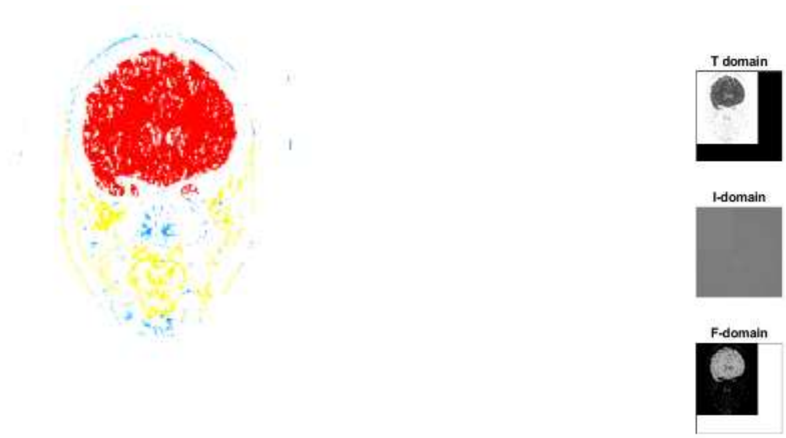


Fig :5 Thresholding Images and Neutrosophic images

Fig :6 Image segmentation and Neutrosophic images

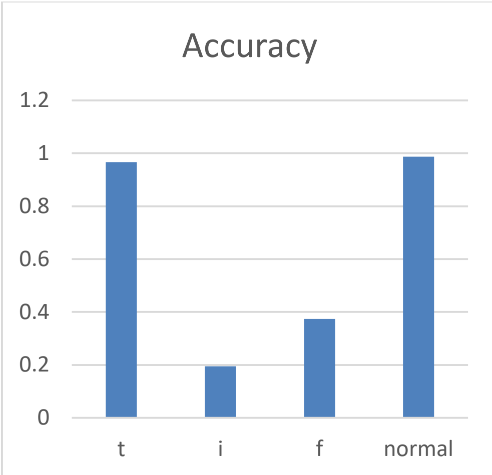


Fig. 7 Accuracy Analysis

Fig 7 shows that true membership is very nearest value from original images.

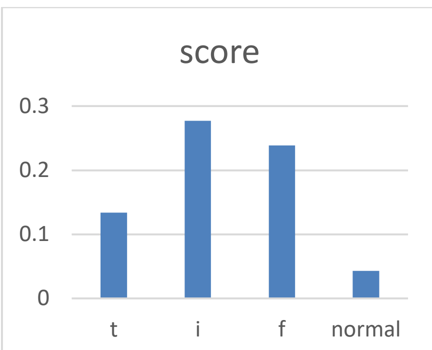


Fig. 8 Score Analysis

Fig 8 shows that true membership is very nearest value from original score images.

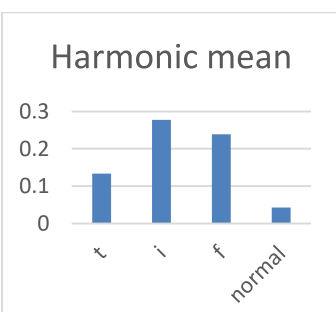


Fig. 9 Harmonic Analysis

Fig 9 shows that true membership is very nearest value from harmonic value of an images.

Conclusion

In Decision making using DICOM images involve vagueness, incompleteness, uncertainty and indeterminacy due to object orientation, staining degree and colors. NIP can achieve a better output in the vagueness of the images. NIP using three membership , it is effectively handled indeterminacy and uncertainty. NIP have impressive performance in DICOM image segmentation. NIP images transforming into Neutrosophic sets. Because of its imaging process, image's noise, inhomogeneity, and contrast, DICOM images play a crucial role in the diagnosis and treatment of brain cancers. For segmentation in these situations, neurosophic image processing is applied. This procedure seeks to make the image easier to depict as more significant and to determine or analyse. A patient's MRI's DICOM picture has undergone image segmentation. It has been noted that it requires extremely precise segmentation. Additionally, plithogenic conditions may be added to the process

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The Basics of Neutrosophic Simulation for Converting Random Numbers Associated with a Uniform Probability Distribution into Random Variables Follow an Exponential Distribution

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Abstract: When performing the simulation process, we encounter many systems that do not follow by their nature the uniform distribution adopted in the process of generating the random numbers necessary for the simulation process. Therefore, it was necessary to find a mechanism to convert the random numbers that follow the regular distribution over the period [0, 1] to random variables that follow the probability distribution that works on the system to be simulated. In classical logic, we use many techniques in the transformation process that results in random variables that follow irregular probability distributions. In this research, we used the inverse transformation technique, which is one of the most widely used techniques, especially for the probability distributions for which the inverse function of the cumulative distribution function can found. We applied this technique to generate neutrosophic random variables that follow an exponential distribution or a neutrosophic exponential distribution. This based on classical or neutrosophic random numbers that follow a regular distribution. We distinguished three cases according to the logic that each of the random numbers or the exponential distribution follows. We arrived at neutrosophic random variables that, when we use them in systems that operate according to an exponential distribution, such as queues and others, will provide us with a high degree of accuracy of results, and the reason for this is due to the indeterminacy provided by neutrosophic logic.

Keywords: Simulation - inverse transformation - uniform distribution - exponential distribution - neutrosophic exponential distribution - random numbers - random variables - neutrosophic logic.

1. **Introduction**

The generation of random variables that follow a certain distribution is the basis of the simulation. We can generate random events that simulate any real system by finding probability distributions that apply to the events and properties of that system, for example: "times between arrivals" in queues are random events that often follow an exponential distribution. There are several methods and algorithms for generating random variables from a given distribution [1,2,3].

To keep pace with the modern studies that emerged after the neutrosophic revolution, the logic laid down by the American mathematical philosopher Florentin Smarandache in 1995 [6,8,10,11,12,13,20] came as a generalization of the fuzzy logic and an extension of the theory of fuzzy sets presented by Lotfi Zadeh in 1965 [7]. As an extension of that logic, A. A. Salama presented the theory of classical neutrosophic sets as a generalization of the theory of classical sets and developed, introduced and formulated new concepts in the fields of mathematics, statistics, computer science and classical information systems through neutrosophic. Logic that studies the origin, nature and field of indeterminacy so that it takes into account every idea with its opposite (its negation) and with the spectrum of indeterminacy [4]. In addition, there were several achievements of many researchers in the field of neutrosophic [5,9,14,15,16,17,18,19,21,22,23,24,25,26]. It was necessary to work on transforming the random numbers that follow a neutrosophic uniform distribution into random variables that follow a neutrosophic exponential distribution. In this research, we present a study on the process of converting random numbers that follow a regular distribution over the period [0, 1] to random variables that follow an exponential distribution, based on the definition of regular and exponential distributions according to neutrosophic logic.

2. Experimental and Theoretical Part:

In view of the great importance that the exponential distribution has in most fields of science, and in order to obtain more accurate results when using it in a field. the researchers defined this distribution according to the neutrosophic logic. The logic that enables us to deal with all the cases that we can come across during the study. In previous research [28] entitled " Fundamentals of Neutrosophical Simulation for Generating Random Numbers Associated with Uniform Probability Distribution" we reached mathematical formulas that help us in generating neutrosophic random numbers that follow the uniform distribution on the period [0, 1]. In this paper, we have developed a mechanism to obtain the neutrosophic random variables that follow an exponential distribution. This based on the random numbers that follow the uniform distribution on the period [0, 1]. This done by using the inverse transformation of the cumulative distribution function. The study included all the cases that we need during the simulation process for the systems that operate according to the exponential distribution.

Previous studies: [1, 2, 3, 28]

If R_1, R_2 ... are a sequence of random numbers then R_i has a probability function defined as:

$$f_R(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Cumulative distribution function:

$$F_R(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

To generate x_1, x_2 ... observations of the random variable X. follow the distribution:

$$F(x) = P(X \le x), \quad -\infty < x < \infty$$

We use the sequence of random numbers R_1, R_2 …., and the cumulative distribution function for the random variable X. Then we apply the inverse transformation method. It is the most commonly used, especially for probability distributions in which $F^{-1}(x)$ can found. It based on matching:

$$F(X) = R \quad (1)$$

If the random variable X follows a classical exponential distribution.

Then the probability density function is:

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

The cumulative distribution function:

$$F(x) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

We substitute in the relation (⊚):

$$F(X) ⊚ R$$

$$1 - e^{-\lambda x} = R \quad x \geq 0$$

By solving the previous equation, it results in:

$$x = -\frac{1}{\lambda}\ln(1 - R) \qquad (**)$$

We call the equation (⊚⊚): the generator equation for the random variable that follows the exponential distribution. Are of the form:

$$X ⊚ F^{-1}(R)$$

Therefore, to obtain a sequence of observations, of the random variable X that follows the exponential distribution, we use the relationship $X = F^{-1}(R)$, and the sequence of random numbers R_1, R_2.... we write:

$$X_i = F^{-1}(R_i)$$

$$X_i = -\frac{1}{\lambda}\ln(1 - R_i) \quad ; \quad i = 1,2,..$$

It can be simplified to the form:

$$X_i = -\frac{1}{\lambda}\ln R_i$$

3. Results and Discussion

The current study: To generate random variables that follow an exponential distribution according to neutrosophic logic, we distinguish the following cases:

<u>First case</u>: the random numbers follow the neutrosophic uniform distribution on the period $[0+\varepsilon, 1+\varepsilon]$ and the exponential distribution in the classical form.

To generate random variables that follow the exponential distribution whose probability density function:

$$f(x) ⊚ \lambda e^{-\lambda x} \qquad x ⊚ 0$$

Cumulative Distribution Function:

$$F(x) ⊚ 1 - e^{-\lambda x} \qquad x \geq 0$$

Using the sequence of neutrosophic random numbers that follows the uniform distribution on the period $[0+\varepsilon, 1+\varepsilon]$, and which is given as $R_1 - \varepsilon$, $R_2 - \varepsilon$… , we apply the relationship (⊚):

$$F(x) \circledcirc R$$

In this case, we write:

$$F(x) \circledcirc R_i - \varepsilon \Rightarrow$$
$$1 - e^{-\lambda x} \circledcirc R_i - \varepsilon$$
$$e^{-\lambda x} \circledcirc 1 - (R_i - \varepsilon)$$
$$-\lambda x \circledcirc \ln(1 - (R_i - \varepsilon)) \Rightarrow$$
$$x = -\frac{\ln(1-(R_i-\varepsilon))}{\lambda} \qquad i = 1, 2\ldots$$

Accordingly, to obtain a sequence of observations of the random variable X using the random numbers that follow the neutrosophic uniform distribution on the period $[0+\varepsilon, 1+\varepsilon]$, which is given by the formula $R_i - \varepsilon$. We substitute, in the following relationship:

$$X_i = -\frac{\ln(1-(R_i-\varepsilon))}{\lambda} \qquad ; i = 1, 2\ldots$$

It can be simplified:

$$X_{Ni} = -\frac{\ln(R_i-\varepsilon)}{\lambda} \qquad ; i = 1, 2\ldots$$

<u>The second case: </u>classical random numbers and a neutrosophic exponential distribution.

Let's have a sequence of random numbers R_1, R_2 … that follows a uniform distribution on the period $[0, 1]$, and we want to generate random variables that follow a neutrosophic exponential distribution.

Probability density function of the neutrosophic exponential distribution ◎4◎:

$$f_N(x) = \lambda_N e^{-x\lambda_N} \qquad 0 \circledcirc x \circledcirc \infty$$

The cumulative distribution function given by:

$$NF(x) \circledcirc 1 - e^{-x\lambda_N}$$

Using the relation (◎):

$$NF(x) \circledcirc R \Rightarrow$$
$$1 - e^{-x\lambda_N} \circledcirc R \Rightarrow$$
$$e^{-x\lambda_N} \circledcirc 1 - R$$

$$x = -\frac{\ln(1-R)}{\lambda_N} \qquad \text{Or:} \qquad x = -\frac{\ln R}{\lambda_N}$$

Accordingly, to obtain a sequence of observations of the random variable X, "which follow the neutrosophic exponential distribution". Using the random numbers that follow the uniform distribution on the period $[0, 1]$, we substitute in the relationship:

$$X_{Ni} = -\frac{\ln R_i}{\lambda_N} \qquad i = 1, 2\ldots$$

<u>The third case:</u> the random numbers follow the neutrosophic uniform distribution and the neutrosophic exponential distribution.

To find the relationship through which we get: random variables that follow the neutrosophic exponential distribution starting from the sequence of neutrosophic random numbers that follow the regular distribution on the period $[0+\varepsilon, 1+\varepsilon]$, which are given as follows:

$R_1 - \varepsilon$, $R_2 - \varepsilon$, ….

We apply the relationship (⊚):

$$F(x) ⊚ R$$

$$1 - e^{x\lambda_N} ⊚ R - \varepsilon \Rightarrow$$

$$X = \frac{-1}{\lambda_N}\ln[1-(R-\varepsilon)]$$

or in the form:

$$X = \frac{-\ln(R-\varepsilon)}{\lambda_N}$$

Therefore, to obtain a sequence of observations of the random variable X that follows the neutrosophic exponential distribution using the random numbers that follow the neutrosophic uniform distribution on the period $[0+\varepsilon, 1+\varepsilon]$, we substitute in the relation:

$$X_{Ni} = \frac{-\ln(R_i-\varepsilon)}{\lambda_N} \qquad i = 1, 2\ldots$$

4. Application Example:

Suppose we have a system that operates according to an exponential distribution whose probability density function is $f(x) = 2e^{-2x} \;\; ; \;\; x \geq 0$. We want to conduct a neutrosophic simulation of this system. Where the indeterminate $\varepsilon = [0 , \;\; 0{,}03 \,]$. Here we need to generate neutrosophic random numbers. Therefore, we use one of the cases:

<u>First case:</u> The exponential distribution is classical, its probability density function is $f(x) = 2e^{-2x} \;\; ; \;\; x \geq 0$, and neutrosophic random numbers (We get it by one of the methods studied in the research ⊚28 ⊚).

In this example, we generate random numbers that follow the neutrosophic uniform distribution on the period $[0+\varepsilon, 1+\varepsilon]$, That is, we generate random numbers according to one of the known methods. Here we will use the "mean-squared" method, by taking the seed $R_\circ = 1276$. We get the random numbers:

$$R_1 = 0{,}6281 \quad , \quad R_2 = 0{,}4509 \quad , \quad R_3 = 0{,}3310 \quad , \quad R_4 = 0{,}095$$

By using the rule that we reached in previous research [28] to convert classical random numbers into random numbers that follow the neutrosophic uniform distribution on the period $[0+\varepsilon, 1+\varepsilon]$. In addition, take the given indeterminacy $\varepsilon = [0, \ 0{,}03]$. We get the neutrosophic random numbers:

$$R_{N\circ} = [0{,}0976, 0{,}1276] \quad , \quad R_{N1} = [0{,}5981, 0{,}6281] \quad , \quad R_{N2} = [0{,}4209, 0{,}4509]$$
$$R_{N3} = [0{,}3010, 0{,}3310] \quad , \quad R_{N4} = [0{,}0656, 0{,}0956]$$

Then we apply the following rule $X_{Ni} = -\dfrac{\ln(R_i - \varepsilon)}{\lambda} = \dfrac{\ln R_{Ni}}{\lambda} \quad ; \quad i = 0,1,2,3,4$

We get:

$$X_{N\circ} = -\frac{\ln R_{N\circ}}{\lambda} = -\frac{\ln[0{,}0976, 0{,}1276]}{2} = [1{,}0294, 1{,}16343]$$
$$X_{N1} = -\frac{\ln R_{N1}}{\lambda} = -\frac{\ln[0{,}5981, 0{,}6281]}{2} = [0{,}2325, 0{,}2570]$$
$$X_{N2} = -\frac{\ln R_{N2}}{\lambda} = -\frac{\ln[0{,}4209, 0{,}4509]}{2} = [0{,}3983, 0{,}4327]$$
$$X_{N3} = -\frac{\ln R_{N3}}{\lambda} = -\frac{\ln[0{,}3010, 0{,}3310]}{2} = [0{,}5528, 0{,}6003]$$
$$X_{N4} = -\frac{\ln R_{N4}}{\lambda} = -\frac{\ln[0{,}0656, 0{,}0956]}{2} = [1{,}1738, 1{,}3621]$$

It is sequence of neutrosophic random observations, which follow an exponential distribution.

<u>The second case:</u> The neutrosophic exponential distribution, its probability density function is $f(x) = [2, 2{,}03]e^{-[2, 2{,}03]x} \quad ; \quad x \geq 0$. Random numbers are classical.

To find the required neutrosophic random observations, we will take the random numbers that follow the uniform distribution on the period $[0, 1]$:

$$R_\circ = 1276 \quad , \quad R_1 = 0{,}6281 \quad , \quad R_2 = 0{,}4509$$
$$R_3 = 0{,}3310 \quad , \quad R_4 = 0{,}0951$$

Then we apply the rule: $X_{Ni} = -\dfrac{\ln R_i}{\lambda_N} \quad ; \quad i = 0,1,2,3,4$

We get:

$$X_{N\circ} = -\frac{\ln R_i}{\lambda_N} = -\frac{\ln 1276}{[2\,,\,2{,}03]} = [1{,}01421,\,1{,}0294]$$

$$X_{N1} = -\frac{\ln R_i}{\lambda_N} = -\frac{\ln 0{,}6281}{[2\,,\,2{,}03]} = [0{,}2290,\,0{,}2325]$$

$$X_{N2} = -\frac{\ln R_i}{\lambda_N} = -\frac{\ln 0{,}4509}{[2\,,\,2{,}03]} = [1{,}5045,\,0{,}3983]$$

$$X_{N3} = -\frac{\ln R_i}{\lambda_N} = -\frac{\ln 0{,}3310}{[2\,,\,2{,}03]} = [0{,}5446,\,0{,}5528]$$

$$X_{N4} = -\frac{\ln R_i}{\lambda_N} = -\frac{\ln 0{,}0951}{[2\,,\,2{,}03]} = [1{,}1590,\,1{,}1764]$$

It is sequence of neutrosophic random observations, which follow the neutrosophic exponential distribution.

The third case: The neutrosophic exponential distribution, its probability density function given by the following $f(x) = [2\,,\,2{,}03]e^{-[2\,,\,2{,}03]x}$; $x \geq 0$. The neutrosophic random numbers from the figure $R_i - \varepsilon$. To find neutrosophic random observations. We take the neutrosophic random numbers used in the first case and apply the following rule:

$$X_{Ni} = \frac{-\ln R_{Ni}}{\lambda_N} \quad ; \quad R_{Ni} = R_i - \varepsilon \quad ; \quad i = 0,1,2,3,4$$

We get:

$$X_{N\circ} = \frac{-\ln R_{N\circ}}{\lambda_N} = \frac{-\ln[0{,}0976,\,0{,}1276]}{[2\,,\,2{,}03]} = [1{,}01421,\,1{,}16343]$$

$$X_{N1} = \frac{-\ln R_{N1}}{\lambda_N} = \frac{-\ln[0{,}5981,\,0{,}6281]}{[2\,,\,2{,}03]} = [0{,}2291,\,0{,}2570]$$

$$X_{N2} = \frac{-\ln R_{N2}}{\lambda_N} = \frac{-\ln[0{,}4209,\,0{,}4509]}{[2\,,\,2{,}03]} = [0{,}3924,\,0{,}4327]$$

$$X_{N3} = \frac{-\ln R_{N3}}{\lambda_N} = \frac{-\ln[0{,}3010,\,0{,}3310]}{[2\,,\,2{,}03]} = [0{,}54465,\,0{,}6003]$$

$$X_{N4} = \frac{-\ln R_{N4}}{\lambda_N} = \frac{-\ln[0{,}0656,\,0{,}0956]}{[2\,,\,2{,}03]} = [1{,}1564,\,13621]$$

It is sequence of neutrosophic random observations, which follow the neutrosophic exponential distribution.

5. **Conclusions**:

Through the previous study, we found that, to generate a sequence of neutrosophic random observations that follow an exponential distribution, using a sequence of random numbers that follow a uniform distribution. We use one of the following cases, according to the case under study: The first case: neutrosophic random numbers, i.e. They follow the neutrosophic uniform distribution on the period $\left[0+\varepsilon\, ,1+\varepsilon\right]$, and the exponential distribution in the classical form.

The second case: random numbers that follow the uniform distribution on the period $\left[0\, ,1\right]$, and the neutrosophic exponential distribution.

The third case: the neutrosophic random numbers, i.e., they follow the neutrosophic uniform distribution on the period $\left[0+\varepsilon\, ,1+\varepsilon\right]$, and the neutrosophic exponential distribution.

By using techniques used in classical logic. In this paper, we used the inverse transformation technique. In addition, we found that for every random number (neutrosophic or classical) there is a random variable that follows the neutrosophic exponential distribution, which enables accurately simulate the systems that follow the exponential distribution. That is through the accuracy that neutrosophic logic provides us when studying any system according to its hypotheses.

In the near future, we are looking forward to preparing studies that will enable us to generate neutrosophic random variables that follow other probability distributions such as the Weibull distribution, the geometric distribution, and others.

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The Neutrosophic Limits

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Abstract: the purpose of this article is to study the neutrosophic limits, where the methods of neutrosophic factorization and neutrosophic rationalization were applied, useful theorems have been proven for facilitating the calculation of the neutrosophic limits. Also, the definition of a positive neutrosophic number was presented, and the necessary condition to find the square root of the neutrosophic number, in addition to studying some special limits and neutrosophic trigonometric limits. Where detailed examples were given to clarify each case.

Keywords: the neutrosophic limits; neutrosophic trigonometric limits; indeterminacy; method of neutrosophic factorization.

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1. Introduction

As an alternative to the existing logics, Smarandache proposed the neutrosophic logic to represent a mathematical model of uncertainty, vagueness, ambiguity, imprecision, undefined, unknown, incompleteness, inconsistency, redundancy, contradiction, where the concept of neutrosophy is a new branch of philosophy introduced by Smarandache [3-14]. He presented the definition of the standard form of neutrosophic real number [2-4], studying the concept of the Neutrosophic probability [3-6], the Neutrosophic statistics [4][7], and professor Smarandache entered the concept of preliminary calculus of the differential and integral calculus [1-9]. Madeleine Al- Taha presented results on single valued neutrosophic (weak) polygroups [10]. Edalatpanah proposed a new direct algorithm to solve the neutrosophic linear programming where the variables and right-hand side represented with triangular neutrosophic numbers [11]. Chakraborty used pentagonal neutrosophic number in networking problem, and Shortest Path Problem [12-13]. Y. Alhasan studied the concepts of neutrosophic complex numbers and the general exponential form of a neutrosophic complex [8][15]. On the other hand, M. Abdel-Basset presented study in the science of neutrosophic about an approach of TOPSIS technique for developing supplier selection with group decision making under type-2 neutrosophic number [16]. H. Khalid, F. Smarandache and A. Essa have presented a study on a neutrosophic binomial factorial theorem with their Refrains [5].

Paper consists of 5 sections. In 1st section, provides an introduction, in which neutrosophic science review has given. In 2nd section, some definitions, examples of neutrosophic real number and new theorems in neutrosophic limits are discussed. The 3rd section frames methods of neutrosophic factorization, neutrosophic rationalization, and neutrosophic trigonometric limits. The 4th section introduces the definition of a positive neutrosophic number, and the necessary condition to find the square root of the neutrosophic number, in addition to studying some special limits and neutrosophic trigonometric limits. In 5th section, a conclusion to the paper is given.

2. Preliminaries

2.1. Neutrosophic real number [4]

Suppose that w is a neutrosophic real number, then it takes the following standard form: $w = a + bI$ where a, b are real coefficients, and I represent indeterminacy, such $0.\mathrm{I} = 0$ and $\mathrm{I}^n = \mathrm{I}$, for all positive integers n.

2.2. Division of neutrosophic real numbers [4]

Suppose that w_1, w_2 are two neutrosophic numbers, where:

$$w_1 = a_1 + b_1 I \ , \qquad w_2 = a_2 + b_2 I$$

To find $(a_1 + b_1 I) \div (a_2 + b_2 I)$, we can write:

$$\frac{a_1 + b_1 I}{a_2 + b_2 I} \equiv x + yI$$

where x and y are real unknowns.

$$a_1 + b_1 I \equiv (a_2 + b_2 I)(x + yI)$$

$$a_1 + b_1 I \equiv a_2 x + (b_2 x + a_2 y + b_2 y)I$$

by identifying the coefficients, we get

$$a_1 = a_2 x$$

$$b_1 = b_2 x + (a_2 + b_{2)} y$$

We obtain unique one solution only, provided that:

$$\begin{vmatrix} a_2 & 0 \\ b_2 & a_2 + b_2 \end{vmatrix} \neq 0 \quad \Rightarrow \quad a_2(a_2 + b_2) \neq 0$$

Hence: $a_2 \neq 0$ *and* $a_2 \neq -b_2$ are the conditions for the division of two neutrosophic real numbers to exist.

Then:

$$\frac{a_1 + b_1 I}{a_2 + b_2 I} = \frac{a_1}{a_2} + \frac{a_2 b_1 - a_1 b_2}{a_2(a_2 + b_2)}.I$$

2.3 New theorems in neutrosophic limits [5]

Theorem2.3.1 (Binomial Factorial Theorem)

$$\lim_{x \to \infty} \left(I + \frac{1}{x} \right)^x = Ie$$

where I is the literal indeterminacy, $e = 2.7182828$.

Corollary 2.3.1

$$\lim_{x \to 0}(I + x)^{\frac{1}{x}} = Ie$$

Corollary 2.3.2

$$\lim_{x \to \infty}\left(I + \frac{k}{x}\right)^{x} = Ie^{k}$$

where $k > 0$ & $k \neq 0$, I is the literal indeterminacy.

Corollary 2.3.3

$$\lim_{x \to 0}\left(I + \frac{x}{k}\right)^{\frac{1}{x}} = \sqrt[k]{Ie}$$

where $k \neq 0$ & $k > 0$.

Theorem2.3.2

$$\lim_{x \to 0}\frac{(lna)[Ia^{x} - I]}{xlna + lnI} = \frac{lna}{1 + lnI}$$

where $a > 0, a \neq 0$.

Corollary 2.3.4

$$\lim_{x \to 0}\frac{Ia^{kx} - I}{x + \frac{lnI}{lna^{k}}} = \frac{k\ lna}{1 + lnI}$$

Corollary 2.3.5

$$\lim_{x \to 0}\frac{Ia^{x} - I}{x + lnI} = \frac{1}{1 + lnI}$$

Corollary 2.3.6

$$\lim_{x \to 0}\frac{Ia^{kx} - I}{x + \frac{lnI}{k}} = \frac{k}{1 + lnI}$$

Theorem2.3.3

$$\lim_{x \to 0}\frac{ln(I + kx)}{x} = k(1 + lnI)$$

Theorem2.3.4

Prove that, for any two real numbers a, b

$$\lim_{x \to 0}\frac{Ia^{x} - I}{Ib^{x} - I} = 1$$

where $a, b > 0$ & $a, b \neq 1$.

3. Method of neutrosophic factorization

Suppose $\frac{f(x,I)}{g(x,I)}$ is rational neutrosophic function, if $f(x,I), g(x,I)$ contains some common factors, then we can cancel out the common factors from the numerator and denominator and then put $x = a + bI$
where a , b are real coefficients, and I represent indeterminacy.

Example 3.1
Evaluate:

$$\lim_{x\to 2-3I} \frac{x-2+3I}{x^2-4+3I}$$

Solution:

$$\lim_{x\to 2-3I} \frac{x-2+3I}{x^2-4+3I} = \frac{0}{0}$$

Method1:

$$x^2-4+3I = (x-2+3I)(x+2-3I)$$

$$\lim_{x\to 2-3I} \frac{x-2+3I}{x^2-4+3I} = \lim_{x\to 2-3I} \frac{x-2+3I}{(x-2+3I)(x+2-3I)}$$

$$\lim_{x\to 2-3I} \frac{1}{x+2-3I} = \frac{1}{4-6I} = \frac{1}{4} - \frac{3}{4}I$$

Method2:
by using L'Hôpital's rule

$$\Rightarrow \quad \lim_{x\to 2-3I} \frac{x-2+3I}{x^2-4+3I} = \lim_{x\to 2-3I} \frac{1}{2x}$$

$$= \frac{1}{2(2-3I)} = \frac{1}{4-6I} = \frac{1}{4} - \frac{3}{4}I$$

3.1 The method of neutrosophic rationalization

Example 3.1.1
Evaluate:

$$\lim_{x\to 0+0I} \frac{\sqrt{1-(2+4I)x} - \sqrt{1+(2+4I)x}}{(1+3I)x}$$

Solution:

$$\lim_{x\to 0+0I} \frac{\sqrt{1-(2+4I)x} - \sqrt{1+(2+4I)x}}{(1+3I)x} = \frac{0}{0}$$

Method1:

$$\Rightarrow \quad \lim_{x\to 0+0I} \frac{\sqrt{1-(2+4I)x} - \sqrt{1+(2+4I)x}}{(1+3I)x}$$

$$= \lim_{x\to 0+0I} \frac{\left(\sqrt{1-(2+4I)x}-\sqrt{1+(2+4I)x}\right)\left(\sqrt{1-(2+4I)x}+\sqrt{1+(2+4I)x}\right)}{(1+3I)x\left(\sqrt{1-(2+4I)x}+\sqrt{1+(2+4I)x}\right)}$$

$$= \lim_{x\to 0+0I} \frac{1-(2+4I)x-(1+(2+4I)x)}{(1+3I)x\left(\sqrt{1-(2+4I)x}+\sqrt{1+(2+4I)x}\right)}$$

$$= \lim_{x\to 0+0I} \frac{-(4+8I)x}{(1+3I)x\left(\sqrt{1-(2+4I)x}+\sqrt{1+(2+4I)x}\right)}$$

$$= \lim_{x\to 0+0I} \frac{-4-8I}{(1+3I)\left(\sqrt{1-(2+4I)x}+\sqrt{1+(2+4I)x}\right)}$$

$$= \frac{-2-4I}{1+3I} = -2+\frac{1}{2}I$$

Method2:
by using L'Hôpital's rule

$$\Rightarrow \quad \lim_{x\to 0+0I} \frac{\sqrt{1-(2+4I)x}-\sqrt{1+(2+4I)x}}{(1+3I)x}$$

$$= \lim_{x\to 0+0I} \frac{\frac{-(2+4I)}{2\sqrt{1-(2+4I)x}}-\frac{(2+4I)}{2\sqrt{1+(2+4I)x}}}{1+3I}$$

$$= \frac{\frac{-(2+4I)}{2\sqrt{1-0}}-\frac{(2+4I)}{2\sqrt{1+0}}}{1+3I} = \frac{-1-2I-1-2I}{1+3I}$$

$$= \frac{-2-4I}{1+3I} = -2+\frac{1}{2}I$$

Example 3.1.2
Evaluate:

$$\lim_{x\to 5-I} \frac{1-\sqrt{x-4+I}}{x-5+I}$$

Solution:

$$\lim_{x\to 5-I} \frac{1-\sqrt{x-4+I}}{x-5+I} = \frac{0}{0}$$

Method1:

$$\Rightarrow \quad \lim_{x\to 5-I} \frac{\left(1-\sqrt{x-4+I}\right)\left(1+\sqrt{x-4+I}\right)}{(x-5+I)\left(1+\sqrt{x-4+I}\right)}$$

$$= \lim_{x\to 5-I} \frac{1-(x-4+I)}{(x-5+I)\left(1+\sqrt{x-4+I}\right)}$$

$$= \lim_{x \to 5-I} \frac{-x+5-I}{(x-5+I)\left(1+\sqrt{x-4+I}\right)}$$

$$= \lim_{x \to 5-I} \frac{-(x-5+I)}{(x-5+I)\left(1+\sqrt{x-4+I}\right)}$$

$$= \lim_{x \to 5-I} \frac{-1}{1+\sqrt{x-4+I}} = \frac{-1}{2}$$

Method2:
by using L'Hôpital's rule

$$\Rightarrow \quad \lim_{x \to 5-I} \frac{1-\sqrt{x-4+I}}{x-5+I}$$

$$= \lim_{x \to 5-I} \frac{\frac{-1}{2\sqrt{x-4+I}}}{1} = \lim_{x \to 5-I} \frac{-1}{2\sqrt{x-4+I}} = \frac{-1}{2}$$

Example 3.1.3

Evaluate:

$$\lim_{x \to a+bI} \frac{\sqrt{a+bI+2x}-\sqrt{3x}}{\sqrt{3a+3bI+x}-2\sqrt{x}}$$

Solution:

$$\lim_{x \to a+bI} \frac{\sqrt{a+bI+2x}-\sqrt{3x}}{\sqrt{3a+3bI+x}-2\sqrt{x}} = \frac{0}{0}$$

$$\Rightarrow \quad \lim_{x \to a+bI} \frac{\sqrt{a+bI+2x}-\sqrt{3x}}{\sqrt{3a+3bI+x}-2\sqrt{x}}$$

$$= \lim_{x \to a+bI} \frac{\left(\sqrt{a+bI+2x}-\sqrt{3x}\right)\left(\sqrt{a+bI+2x}+\sqrt{3x}\right)}{\left(\sqrt{3a+3bI+x}-2\sqrt{x}\right)\left(\sqrt{a+bI+2x}+\sqrt{3x}\right)}$$

$$= \lim_{x \to a+bI} \frac{a+bI+2x-3x}{\left(\sqrt{3a+3bI+x}-2\sqrt{x}\right)\left(\sqrt{a+bI+2x}+\sqrt{3x}\right)}$$

$$= \lim_{x \to a+bI} \frac{a+bI-x}{\left(\sqrt{3a+3bI+x}-2\sqrt{x}\right)\left(\sqrt{a+bI+2x}+\sqrt{3x}\right)} = \frac{0}{0}$$

$$\Rightarrow \quad \lim_{x \to a+bI} \frac{a+bI-x}{\left(\sqrt{3a+3bI+x}-2\sqrt{x}\right)\left(\sqrt{a+bI+2x}+\sqrt{3x}\right)} \frac{\sqrt{3a+3bI+x}+2\sqrt{x}}{\sqrt{3a+3bI+x}+2\sqrt{x}}$$

$$= \lim_{x \to a+bI} \frac{(a+bI-x)\left(\sqrt{3a+3bI+x}+2\sqrt{x}\right)}{(3a+3bI+x-4x)\left(\sqrt{a+bI+2x}+\sqrt{3x}\right)}$$

$$= \lim_{x \to a+bI} \frac{(a+bI-x)\left(\sqrt{3a+3bI+x}+2\sqrt{x}\right)}{3(a+bI-x)\left(\sqrt{a+bI+2x}+\sqrt{3x}\right)}$$

$$= \lim_{x \to a+bI} \frac{\sqrt{3a+3bI+x}+2\sqrt{x}}{3\left(\sqrt{a+bI+2x}+\sqrt{3x}\right)} = \frac{\sqrt{3a+3bI+a+bI}+2\sqrt{a+bI}}{3\left(\sqrt{a+bI+2(a+bI)}+\sqrt{3(a+bI)}\right)}$$

$$= \frac{\sqrt{4(a+bI)}+2\sqrt{a+bI}}{3\left(\sqrt{3(a+bI)}+\sqrt{3(a+bI)}\right)} = \frac{4\sqrt{a+bI}}{6\sqrt{3(a+bI)}}$$

$$= \frac{2\sqrt{a+bI}}{3\sqrt{3}\sqrt{a+bI}} = \frac{2}{3\sqrt{3}}$$

3.2 Neutrosophic trigonometric limits

1) $\lim_{x \to 0+0I} sin(a+bI)\,x = 0$

2) $\lim_{x \to 0+0I} cos(a+bI)\,x = 0$

3) $\lim_{x \to 0+0I} \frac{sin(a+bI)x}{x} = a+bI$

Proof (3):

Put $(a+bI)x = y \quad \Rightarrow \quad x = \frac{1}{a+bI} y$

When $x \longrightarrow 0+0I$ then: $y \longrightarrow 0+0I$

$$\Rightarrow \quad \lim_{x \to 0+0I} \frac{sin(a+bI)\,x}{x} = \lim_{y \to 0+0I} \frac{siny}{\frac{1}{a+bI} y} = (a+bI) \lim_{y \to 0+0I} \frac{siny}{y} = a+bI$$

4) $\lim_{x \to 0+0I} \frac{x}{sin(a+bI)x} = \frac{1}{a+bI} = \frac{1}{a} - \frac{b}{a(a+b)} I$

Where $a\,, b$ are real coefficients, $a_2 \neq 0$ $and\ a_2 \neq -b_2$, I represent indeterminacy.

Proof (4):

Put $(a+bI)x = y \quad \Rightarrow \quad x = \frac{1}{a+bI} y$

When $x \longrightarrow 0+0I$ then: $y \longrightarrow 0+0I$

$$\Rightarrow \quad \lim_{x \to 0+0I} \frac{x}{sin(a+bI)\,x} = \lim_{y \to 0+0I} \frac{\frac{1}{a+bI} y}{siny}$$

$$= \frac{1}{a+bI} \lim_{y \to 0+0I} \frac{siny}{y}$$

$$= \frac{1}{a+bI} = \frac{1}{a} - \frac{b}{a(a+b)} I$$

5) $\lim_{x \to 0+0I} \frac{tan(a+bI)x}{x} = a+bI$

6) $\lim\limits_{x\to 0+0I} \dfrac{x}{tan(a+bI)x} = \dfrac{1}{a+bI} = \dfrac{1}{a} - \dfrac{b}{a(a+b)} I$

Where a, b are real coefficients, $a_2 \neq 0$ $and\ a_2 \neq -b_2$, I represent indeterminacy.

We can prove 5 and 6 the same method in 3, 4.

Example 3.2.1

1) $\lim\limits_{x\to 0+0I} \dfrac{sin(5+4I)x}{(6-7I)x} = \dfrac{5+4I}{6-7I} \lim\limits_{x\to 0+0I} \dfrac{sin(5+4I)x}{(5+4I)x} = \dfrac{5+4I}{6-7I} = \dfrac{5}{6} - \dfrac{59}{6} I$

2) $\lim\limits_{x\to 0+0I} \dfrac{x}{sin(1+2I)x} = \dfrac{1}{1+2I} \lim\limits_{x\to 0+0I} \dfrac{(1+2I)x}{sin(1+2I)x} = \dfrac{1}{1+2I} = 1 - \dfrac{2}{3} I$

3) $\lim\limits_{x\to 0+0I} \dfrac{sin(3+4I)x}{tan(2-8I)x} = \lim\limits_{x\to 0+0I} \dfrac{\frac{sin(3+4I)x}{x}}{\frac{tan(2-8I)x}{x}} = \dfrac{\lim\limits_{x\to 0+0I} \frac{sin(3+4I)x}{x}}{\lim\limits_{x\to 0+0I} \frac{tan(2-8I)x}{x}} = \dfrac{3+4I}{2-8I} = \dfrac{3}{2} - \dfrac{8}{3} I$

4) $\lim\limits_{x\to 0} \dfrac{1-cos(10+4I)x}{x^2} = \lim\limits_{x\to 0+0I} \dfrac{2sin^2(5+2I)x}{x^2} = 2 \lim\limits_{x\to 0+0I} \left(\dfrac{sin(5+2I)x}{x}\right)^2 = 2(5+2I)^2 = 50 + 28I$

5) $\lim\limits_{x\to 0+0I} \dfrac{(3-5I)x - sin(2+1I)\,x}{(1-4I)x} = \lim\limits_{x\to 0+0I} \left(\dfrac{(3-5I)x}{(1-4I)x} - \dfrac{sin(2+1I)\,x}{(1-4I)x}\right)$

$= \lim\limits_{x\to 0+0I} \left(\dfrac{3-5I}{1-4I} - \dfrac{sin(2+1I)x}{(1-4I)x}\right) = \dfrac{3-5I}{1-4I} - \left(\dfrac{2+1I}{1-4I}\right) = 1 + \dfrac{2}{3} I$

4.

Deffinition4.1

Let the neutrosophic number $a + bI$, then $a + bI$ is positive neutrosophic number If it fulfills one of the following conditions:

1) $a > 0, b > 0\ and\ I > 0$
2) $a > 0, b < 0\ and\ I < 0$
3) $a < 0\ \ and \begin{cases} b > 0\ and\ \ I > \frac{a}{b} \\ b < 0\ and\ \ I < \frac{a}{b} \end{cases}$

Example 4.1

1) $5 + 3I,\ I > 0\ then: 5 + 3I > 0$
2) $1 - 3I,\ I < 0\ then: 1 - 3I > 0$
3) $-7 + 3I,\ I > \frac{7}{3}\ then: -7 + 3I > 0$
4) $-4 - I,\ I < -4\ then: -4 - I > 0$

Deffinition4.2

Let the neutrosophic real number $a + bI$, then $a + bI$ has a square root if it fulfills the following condition:

$$a \geq 0 \; and \; a + b \geq 0$$

Where:

$$\sqrt{a+bI} = \sqrt{a} + \left(-\sqrt{a} + \sqrt{a+b}\right)I$$

Or: $= \sqrt{a} - \left(\sqrt{a} + \sqrt{a+b}\right)I$

Or: $= -\sqrt{a} + \left(\sqrt{a} + \sqrt{a+b}\right)I$

Or: $= -\sqrt{a} + \left(\sqrt{a} - \sqrt{a+b}\right)I$

For fulfills the real square root condition $\sqrt{a} \; and \; \sqrt{a+b}$, the neutrosophic number $a + bI$ must fulfills the following condition:

$$a \geq 0 \; and \; a + b \geq 0$$

Example 4.2

1) $4 - 3I$, has a square root

2) $1 - 3I$, has not a square root because:

$$a + b = 1 - 3 = -2 < 0$$

3) $-4 + 2I$, has not a square root because:

$$a = -4 < 0 \; and \; a + b = -6 < 0$$

4.1 Some Special Limits

1) $\lim\limits_{x \to 0+0I} e^{(a+bI)x} = 1$

2) $\lim\limits_{x \to 0+0I} \dfrac{e^{(a+bI)x} - 1}{x} = a + bI$

Proof (2):

Put $(a + bI)x = y \quad \Rightarrow \quad x = \frac{1}{a+bI} y$

When $x \longrightarrow 0 + 0I$ then: $y \longrightarrow 0 + 0I$

$$\Rightarrow \quad \lim_{x \to 0+0I} \frac{e^{(a+bI)x} - 1}{x} = \lim_{y \to 0+0I} \frac{e^{y} - 1}{\frac{1}{a+bI} y}$$

$$= (a + bI) \lim_{y \to 0+0I} \frac{e^{y} - 1}{y} = (a + bI)(1) = a + bI$$

3) $\lim_{x\to 0+0I} \frac{ln(1+(a+bI)x)}{x} = a+bI$

Proof (3):

Put $(a+bI)x = y \quad \Rightarrow \quad x = \frac{1}{a+bI} y$

When $x \longrightarrow 0+0I$ then: $y \longrightarrow 0+0I$

$$\Rightarrow \quad \lim_{x\to 0+0I} \frac{ln(1+(a+bI)x)}{x} = \lim_{y\to 0+0I} \frac{ln(1+y)}{\frac{1}{a+bI} y}$$

$$= (a+bI) \lim_{y\to 0+0I} \frac{ln(1+y)}{y} = (a+bI)(1) = a+bI$$

4) $\lim_{x\to 0+0I} \frac{(a+bI)^x - 1}{x} = ln(a+bI) \quad ; \quad a+bI > 0$

Proof (4):

Put $(a+bI)^x - 1 = y \quad \Rightarrow \quad (a+bI)^x = y+1$

$$ln(a+bI)^x = ln(1+y)$$

$$x\, ln(a+bI) = ln(1+y)$$

$$x = \frac{1}{ln(a+bI)} ln(1+y)$$

When $x \longrightarrow 0+0I$ then: $y \longrightarrow 0+0I$

$$\Rightarrow \quad \lim_{x\to 0+0I} \frac{(a+bI)^x - 1}{x} = \lim_{x\to 0+0I} \frac{y}{\frac{1}{ln(a+bI)} ln(1+y)}$$

$$= ln(a+bI) \lim_{x\to 0+0I} \frac{y}{ln(1+y)} = ln(a+bI)\,(1) = ln(a+bI)$$

Corollary 4.1:

$$\lim_{x\to 0+0I} \frac{(a+bI)^x - 1}{(c+dI)^x - 1} = \frac{ln(a+bI)}{ln(c+dI)} \quad ; \quad a+bI > 0 \quad and \quad c+dI > 0$$

Proof:

$$\lim_{x\to 0+0I} \frac{\frac{(a+bI)^x - 1}{x}}{\frac{(c+dI)^x - 1}{x}} = \frac{\lim_{x\to 0+0I} \frac{(a+bI)^x - 1}{x}}{\lim_{x\to 0+0I} \frac{(c+dI)^x - 1}{x}} = \frac{ln(a+bI)}{ln(c+dI)}$$

Example 4.1.1

1) $\lim_{x\to 0+0I} e^{(4+5I)x} = 1$

2) $\lim_{x\to 0+0I} \frac{e^{(1+3I)x} - 1}{x} = 1+3I$

3) $\lim_{x\to 0+0I} \frac{(5+4I)^x - 1}{x} = ln(5+4I) \quad ; \ I > 0$

4) $\lim_{x\to 0+0I} \frac{(9-4I)^x - 1}{e^{(1+3I)x} - 1} \quad ; \quad I < 0$

$$\lim_{x\to 0+0I} \frac{(9-4I)^x - 1}{e^{(7+I)x} - 1} = \lim_{x\to 0+0I} \frac{\frac{(9-4I)^x - 1}{x}}{\frac{e^{(7+I)x} - 1}{x}}$$

$$= \frac{\lim_{x\to 0+0I} \frac{(9-4I)^x - 1}{x}}{\lim_{x\to 0+0I} \frac{e^{(7+I)x} - 1}{x}} = \frac{ln(9-4I)}{7+I} = \left(\frac{1}{7} - \frac{1}{56}I\right) ln(9-4I)$$

5) $\lim_{x\to 0+0I} \frac{(5-4I)^x - 1}{(6-2I)^x - 1} = \frac{ln(5-4I)}{ln(6-2I)} \quad ; \quad I < 0$

6) $\lim_{x\to 0+0I} \frac{(-7+3I)^x - 1}{x} = ln(-7+3I) \ ; \ -7+3I, \quad I > \frac{7}{3}$

7) $\lim_{x\to 0+0I} \frac{ln(1+(3+3I)x)}{x} = 3+3I$

8) $\lim_{x\to 0+0I} \frac{e^{(-4+2I)x} - 1}{\sin(1+2I)\,x} = \lim_{x\to 0+0I} \frac{\frac{e^{(-4+2I)x} - 1}{x}}{\frac{\sin(1+2I)\,x}{x}}$

$$= \lim_{x\to 0+0I} \frac{\frac{e^{(-4+2I)x} - 1}{x}}{\frac{sin(1+2I)x}{x}} = \frac{\lim_{x\to 0+0I} \frac{e^{(-4+2I)x} - 1}{x}}{\lim_{x\to 0+0I} \frac{sin(1+2I)x}{x}} = \frac{-4+2I}{1+2I} = -4 + \frac{10}{3}I$$

10) $\lim_{x\to 0+0I} \frac{(3+I)^x - (5+2I)^x}{x} \quad ; \ I > 0$

$$\lim_{x\to 0+0I} \frac{(3+I)^x - (5+2I)^x}{x} = \lim_{x\to 0+0I} \frac{(3+I)^x - (5+2I)^x - 1 + 1}{x}$$

$$= \lim_{x\to 0+0I} \frac{(3+I)^x - 1 - ((5+2I)^x - 1)}{x} = \lim_{x\to 0+0I} \frac{(3+I)^x - 1}{x} - \lim_{x\to 0+0I} \frac{(5+2I)^x - 1}{x}$$

$$= ln(3+I) - \ln(5+2I) = \ln\left(\frac{3+I}{5+2I}\right) = \ln\left(\frac{3}{5} - \frac{1}{35}I\right)$$

Theorem 4.1:

$$\lim_{x\to 0+0I} \frac{ln\big(I + (a+bI)x\big)}{x} = \big(a+bI\big)\,\big(1 + lnI\big)$$

Proof:

$$\lim_{x\to 0+0I} \frac{ln\big(I+(a+bI)x\big)}{x} = \lim_{x\to 0+0I} \frac{ln\big(I+(a+bI)x\big) - lnI + lnI}{x}$$

$$Let \quad y = ln(I+(a+bI)x) - lnI \quad \Rightarrow \quad y + lnI = ln(I+(a+bI)x)$$

$$e^{y+lnI} = e^{ln(I+(a+bI)x)} \quad \Rightarrow \quad e^y e^{lnI} = I + (a+bI)x$$

$$Ie^y - I = (a+bI)x \quad \Rightarrow \quad x = \frac{Ie^y - I}{a+bI}$$

$$y \longrightarrow 0+0I \quad as \quad x \longrightarrow 0+0I$$

$$\Rightarrow \qquad \lim_{x\to 0+0I} \frac{ln\big(I+(a+bI)x\big)}{x} = \lim_{y\to 0+0I} \frac{y+lnI}{\frac{Ie^y - I}{a+bI}}$$

$$= \lim_{y\to 0+0I} \frac{a+bI}{\frac{Ie^y - I}{y+lnI}} = \frac{a+bI}{\lim_{y\to 0+0I}\left(\frac{Ie^y - I}{y+lnI}\right)}$$

$$= \frac{a+bI}{\frac{1}{1+lnI}} = (a+bI)(1+lnI)$$

Example 4.3

1) $\lim_{x\to 0+0I} \frac{ln(I+(3+5I)x)}{x} = (3+5I)\,(1+lnI)$

2) $\lim_{x\to 0+0I} \frac{ln(1+(1+2I)x)}{ln(I+(6+4I)x)} = \lim_{x\to 0+0I} \frac{\frac{ln(1+(1+2I)x)}{x}}{\frac{ln(I+(6+4I)x)}{x}} = \frac{\lim_{x\to 0+0I} \frac{ln(1+(1+2I)x)}{x}}{\lim_{x\to 0+0I} \frac{ln(I+(6+4I)x)}{x}}$

$$= \frac{1+2I}{(6+4I)(1+lnI)} = \left(\frac{1}{6}+\frac{2}{15}\right)\frac{1}{1+lnI}$$

Theorem 4.2:

$$\lim_{x\to 0+0I} \frac{ln(a+bI)\,[I(a+bI)^x - I]}{x\ln(a+bI) - lnI} = \frac{ln(a+bI)}{1+lnI} \quad ; \; a+bI > 0$$

Proof:

Let $y = I(a+bI)^x - I \quad \Rightarrow \quad y + I = I(a+bI)^x$

$y \longrightarrow 0+0I \;\; as \;\; x \longrightarrow 0+0I$

$\Rightarrow \quad ln(y+I) = lnI + xln(a+bI) \quad \Rightarrow \quad x = \frac{ln(y+I) - lnI}{ln(a+bI)}$

Then:

$$\lim_{x\to 0+0I}\frac{ln(a+bI)\,[I(a+bI)^x-I]}{x\,ln(a+bI)-lnI}=\lim_{x\to 0+0I}\frac{I(a+bI)^x-I}{x-\dfrac{lnI}{ln(a+bI)}}=\lim_{y\to 0+0I}\frac{y}{\dfrac{\ln(y+I)-lnI}{ln(a+bI)}-\dfrac{lnI}{ln(a+bI)}}$$

$$\lim_{y\to 0+0I}\frac{ln(a+bI)}{\dfrac{ln(y+I)}{y}}=\frac{ln(a+bI)}{\lim\limits_{y\to 0+0I}\dfrac{ln(y+I)}{y}}=\frac{ln(a+bI)}{1+lnI}$$

Theorem 4.3:

$$\lim_{x\to 0+0I}\frac{I(a+bI)^x-I}{I(c+dI)^x-I}=1\ ;\quad a+bI>0\ \ and\ \ c+dI>0$$

Proof:

$$\lim_{x\to 0+0I}\frac{I(a+bI)^x-I}{I(c+dI)^x-I}=\lim_{x\to 0+0I}\frac{\dfrac{ln(a+bI)[I(a+bI)^x-I]}{x\,ln(a+bI)-lnI}}{\dfrac{ln(c+dI)[I(c+dI)^x-I]}{xln(c+dI)-lnI}}\cdot\frac{\dfrac{x\,ln(a+bI)-lnI}{ln(a+bI)}}{\dfrac{xln(c+dI)-lnI}{ln(c+dI)}}$$

$$=\frac{\lim\limits_{x\to 0+0I}\dfrac{ln(a+bI)[I(a+bI)^x-I]}{x\,ln(a+bI)-lnI}}{\lim\limits_{x\to 0+0I}\dfrac{ln(c+dI)[I(c+dI)^x-I]}{xln(c+dI)-lnI}}\cdot\frac{\lim\limits_{x\to 0+0I}(x\,ln(a+bI)-lnI)}{\lim\limits_{x\to 0+0I}(xln(c+dI)-lnI)}\cdot\frac{ln(c+dI)}{ln(a+bI)}$$

$$=\frac{\dfrac{ln(a+bI)}{1+lnI}}{\dfrac{ln(c+dI)}{1+lnI}}\cdot\frac{lnI}{lnI}\cdot\frac{ln(c+dI)}{ln(a+bI)}=1$$

Example 4.4

$$\lim_{x\to 0+0I}\frac{ln(5+3I)\,[I(5+3I)^x-I]}{x\ln(5+3I)-lnI}=\frac{ln(5+3I)}{1+lnI}\ ;\ I>0$$

Corollary 4.2:

$$\lim_{x\to\infty}\left[I+\frac{a}{x-b}\right]^x=I^{a+b}e^a\ ;\ a+b>0$$

Note: if $a+b=0$ then $I^{a+b}=I^0$ and if $a+b<0$ then $I^{a+b}=\frac{1}{I^{-(a+b)}}=\frac{1}{I}$ that is from forms of the indeterminate forms in neutrosophic calculus.

Proof:

$$y=\frac{a}{x-b}\ \Rightarrow\ xy-by=a\ \Rightarrow\ x=\frac{a}{y}+b$$

$$y\longrightarrow 0\ \ as\ \ x\longrightarrow\infty$$

$$\Rightarrow\ \lim_{x\to\infty}\left[I+\frac{a}{x-b}\right]^x=\lim_{y\to 0}[I+y]^{\frac{a}{y}+b}=\left(\lim_{y\to 0}[I+y]^{\frac{1}{y}}\right)^a.\lim_{y\to 0}[I+y]^b$$

$$=(Ie)^a.I^b=I^{a+b}e^a$$

Corollary 4.3:

$$\lim_{x\to\infty}\left[I+\frac{a}{x-b}\right]^{kx}=I^{k(a+b)}e^{ka}\ ;\ a+b>0\ \&\ k\neq 0$$

Proof:

$$y=\frac{a}{x-b}\quad\Rightarrow\quad xy-by=a\quad\Rightarrow\quad x=\frac{a}{y}+b$$

$$y\longrightarrow 0\ \ as\ \ x\longrightarrow\infty$$

$$\Rightarrow\quad\lim_{x\to\infty}\left[I+\frac{a}{x-b}\right]^{x}=\lim_{y\to 0}[I+y]^{k\left(\frac{a}{y}+b\right)}=\left(\lim_{y\to 0}[I+y]^{\frac{1}{y}}\right)^{ka}.\lim_{y\to 0}[I+y]^{kb}$$

$$=(Ie)^{ka}.I^{kb}\ =I^{k(a+b)}e^{ka}$$

Example 4.5

1) $\lim_{x\to\infty}\left[I+\frac{5}{x-4}\right]^{x}$

$$y=\frac{5}{x-4}\quad\Rightarrow\quad x=\frac{5}{y}+4$$

$$y\longrightarrow 0\ \ as\ \ x\longrightarrow\infty$$

$$\Rightarrow\quad\lim_{x\to\infty}\left[I+\frac{5}{x-4}\right]^{x}=\lim_{y\to 0}[I+y]^{\frac{5}{y}+4}=\left(\lim_{y\to 0}[I+y]^{\frac{1}{y}}\right)^{5}.\lim_{y\to 0}[I+y]^{4}\ =(Ie)^{5}.I^{4}\ =I^{9}e^{5}=Ie^{5}$$

2) $\lim_{x\to\infty}\left[I+\frac{1}{x-2}\right]^{\frac{x}{2}}$

$$y=\frac{1}{x-2}\quad\Rightarrow\quad x=\frac{1}{y}+2$$

$$y\longrightarrow 0\ \ as\ \ x\longrightarrow\infty$$

$$\Rightarrow\quad\lim_{x\to\infty}\left[I+\frac{1}{x-2}\right]^{\frac{x}{2}}=\lim_{y\to 0}[I+y]^{\frac{1}{2}\left(\frac{1}{y}+2\right)}=\left(\lim_{y\to 0}[I+y]^{\frac{1}{y}}\right)^{\frac{1}{2}}.\lim_{y\to 0}[I+y]$$

$$=(Ie)^{\frac{1}{2}}.I=\sqrt{Ie}\,I=\sqrt{I}\sqrt{e}\,I=\pm I\sqrt{e}$$

Where: $\sqrt{I}=\pm I$

5. Conclusions

Limits are one of the important principles of calculus. It is concerned with the study of derivation by studying the basic concepts of infinitesimal quantities. This led us to study the neutrosophic limits. Where the methods of neutrosophic factorization and neutrosophic rationalization were applied, in addition to introduce definition of the positive neutrosophic number, and the necessary condition to find the square root of the neutrosophic number. Also, studying some special limits and neutrosophic trigonometric limits. This paper is considered an introduction of the neutrosophic calculus.

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Fundamentals of Picture Fuzzy Hypersoft Set with Application

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Abstract. Theory of picture fuzzy soft set and generalized picture fuzzy soft sets(GPFSS) extended to picture fuzzy hypersoft sets (PFHSS) and generalized picture fuzzy hypersoft set (GPFHSS) respectively handle the uncertainties and multi-attribute values in the material during evaluation. The main focus of this research work is to initiate and learn new operations, along with properties and examples of PFHSS and GPFHSS. Several basic operations PFHSS are defined and also prove De Morgans laws for PFHSS. Furthermore, we construct an algorithm using GPFHSS and a new expectation score function for the positive value of the score function that is useful for ranking different MADM problems. We conclude from this study the proposed outlook used to manipulate the uncertainties and multi-attribute values decision-making problems.

Keywords: SS, HS, PFS, PFSS, GPFSS, PFHSS, GPFHSS.

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1. Introduction

Many researchers are intrigued by the Molodtsov [1] softset ($\mathbf{S}_s$) for specific applications in data analysis, cryptography, and distributed storage. Later, the work was expanded upon and some of its foundational ideas and set-theoretic operations were examined by Maji et al. [4] and Zou et al [5]. Picture fuzzy set was proposed by Couge [6]. Positive, neutral, and negative degree are the elements of PFS. Later Yang et al. [7] combine picture fuzzy set and soft set and introduce the new concept of picture fuzzy soft set. In 2018, the theory of the HS set was introduced by Smarandache [16]. It is an extension of a soft set. The basic operations of a HS such as HS containments, Zero HS, aggregation operators along with HS set relation, sub relation, complement relation, function, matrices, and operations on HS matrices discussed by Saeed et al. [17]. In 2020 Raman et al. [23] offers the concept of a hybrid HS set structure of FS, IFS and Neutrosophy sets. The concept of convexity and concavity on a HS set proposed by Rahman et al. [24] in 2020.

1.1. *Literature Review*

In 1965 [2], Zadeh's fuzzy set theory brought about a significant generalisation in mathematics. The membership function aids in the invention of the FS structure. By including a non-membership function, Atanassov [3] extended a fuzzy set structure to an intuitionistic fuzzy set in 1986. IFS lessens the challenges associated with dealing with fuzzy, uncertain, and incomplete information. A soft set theory for solving problems involving uncertainty and decision-making was developed by Molodtsov [1]. By fusing the ideas of soft sets and fuzzy sets, Maji et al. [4] produced fuzzy soft Sets. PFS was developed by Coung et al. [6] to deal with inconsistencies in real-world data. Favorable , neutraland negative degreemake up PFS.Voting is a PFS example, as is the process of conducting elections. To deal with ambiguity, Smarandache [16] developed a novel method. He made the soft to HS more general by breaking the function down into several decision functions. In 2019 Jaber et al. presented an algorithm with the help of extended intersection of GPFS and PFDWA for solving MADM problems.

NO	Structure	Authors	Year	Properties
01	Fuzzy Set	Zadeh [2]	1965	Each individual in universal set is assign to membership value between [0,1].
02	Intuitionistic Fuzzy Set	Atanassov [3]	1983	It describes the membership degree and non-membership degree of an element to a set.
03	Soft set	Molodtsov [1]	1999	It deals with uncertainy in parametric manner.
04	Fuzzy Soft set	Maji et al [4]	2001	Fuzzy values are assign to each power set of universal set.
05	Picture Fuzzy Set	Cuong [?]	2013	Handling issue of inconsistent information.
06	Picture Fuzzy Soft Set	Yang [28]	2015	combination of PFS and SS.
07	Hypersoft sets	Smarandache [16]	2018	Extension of SS.
08	Fuzzy Hypersoft Set	Yolcu et al	2021	Each element in power set have a fuzzy membership degree.
09	Intuitionistic Fuzzy Hypersoft Set	Yolcu et al [36]	2021	Combination of intuitionistic Fuzzy set and Hypersoft Set.
10	Generalized Picture Fuzzy Soft set	Jaber et al. [25]	2019	Hybrid modal of PFSS and PFS in which extra information is given in form of PFS in output for accuracy of results in decision making.

1.2. *Motivation*

The concept of IFS invented by Atanassov [40] has membership and nonmembership degrees. IFS was not playing a role in handling inconsistency-like voting problems. Overcomes such types of difficulties Cuong [[6], [**?**]] defined the notion of PFS and basic operations, opened a new area of research in decision-making problems. An important hybrid modal that generalized SS to PFSS discussed in [7] obtained effective outcomes in DM. In [29] generalized picture distance measure is used to investigate hidden knowledge from a mass of data sets. In 2017 Peng et al. [13] proposed an algorithm using distance measure between PFS.
In 2018 HSS defined by Smarandache [16] which is a generalization of SS by transforming the mapping into a multi-attribute mapping. Saeed et al. [17] gave an idea of fundamentals of HS like union, intersection, containment, null, and compliment. The concept of HS points introduced by Abbas et al. [18]. Yolcu et al. extend the idea of HS to FHS and IFHS. They also discussed the role of FHS and IFHS in decision-making problems. With the help of IFHS, an algorithm developed by Zulqernain et al. for the solution of MADM problems [22]. Rehman et al. [23] proposed the idea of HS with the complex fuzzy set also introduced the theory of concave and convex HS [24]. The main motivation of using Hypersoft Set (HSS) is that when the attributes are more than one and further bisected, the circumstance of a soft set cannot handle such types of cases. So, there is a worth requirement to define a new approach to solve these. Decision-making methods help experts to choose a suitable alternative by analyzing the effectiveness of the alternatives. Having motivation from the work in [25], we extend the existing theory of PFSS to PFHSS to make it adequate for multi-attribute valued function. All the new proposed operations and properties are equipped with illustrated examples.
In section 2 our center of attention are some basic definitions which are useful in this paper. Section 3, the concept of a PFHSS with its properties is presented. In Section 4 and 5, we present definition of GPFHSS and operation of GPFHSS understand by an example. In Section 6 and 7, an algorithm is examined and ranking the companies. Section 9 wind up the paper.

2. Preliminaries

In this section, we define basic definitions of IFS, PFS, SS, HS, PFSS, Score function and PFDWA.

In 1986 Atanassov [40] include non membership function in fuzzy set obtained IFS. It overcomes defects of fuzzy sets.

Definition 2.1. [40]
An IFS X on universe of discourse $X = \{x^{\varsigma}_{\kappa 1}, x^{\varsigma}_{\kappa 2}, ..., x^{\varsigma}_{\kappa n}\}$ is defined as :

$$\tilde{L}^{\varsigma}_{\kappa} = \{\langle \ddot{\mu}_{\tilde{L}^{\varsigma}_{\kappa}}(x^{\varsigma}_{\kappa i}), \ddot{\nu}_{\tilde{L}^{\varsigma}_{\kappa}}(x^{\varsigma}_{\kappa i})\rangle | x^{\varsigma}_{\kappa i} \in X\}$$

where $\ddot{\mu}_{\tilde{L}^{\varsigma}_{\kappa}}(x^{\varsigma}_{\kappa i}) : X \longrightarrow [0,1]$ denotes membership degree of $x^{\varsigma}_{\kappa i}$ in $\tilde{L}^{\varsigma}_{\kappa}$ $\ddot{\nu}_{\tilde{L}^{\varsigma}_{\kappa}}(x^{\varsigma}_{\kappa i}) : X \longrightarrow [0,1]$ is non membership degree of $x^{\varsigma}_{\kappa i}$ in $\tilde{L}^{\varsigma}_{\kappa}$ and $0 \leq \ddot{\mu}_{\tilde{L}^{\varsigma}_{\kappa}}(x^{\varsigma}_{\kappa i}) + \ddot{\nu}_{\tilde{L}^{\varsigma}_{\kappa}}(x^{\varsigma}_{\kappa i}) \leq 1$, $\forall x^{\varsigma}_{\kappa i} \in X$ $\pi_{\tilde{L}^{\varsigma}_{\kappa}}(x^{\varsigma}_{\kappa i}) = 1 - \ddot{\mu}_{\tilde{L}^{\varsigma}_{\kappa}}(x^{\varsigma}_{\kappa i}) - \ddot{\nu}_{\tilde{L}^{\varsigma}_{\kappa}}(x^{\varsigma}_{\kappa i})$ represent hesitancy degree of $x^{\varsigma}_{\kappa i}$ in $\tilde{L}^{\varsigma}_{\kappa}$, $\forall x^{\varsigma}_{\kappa i} \in X$, $0 \leq \pi_{\tilde{L}^{\varsigma}_{\kappa}}(x^{\varsigma}_{\kappa i}) \leq 1$

In 2013 coung [42] introduced PFS to solve inconsistent information in real life. The procedure of voting is a good example to understand the concept of PFS.

Definition 2.2. [?]

A PFS on universe of discourse $X = \{x^{\varsigma}_{\kappa 1}, x^{\varsigma}_{\kappa 2}, ..., x^{\varsigma}_{\kappa n}\}$ is defined as:

$$N^{\varsigma}_{\kappa 1} = \{\langle \ddot{\mu}_{N^{\varsigma}_{\kappa 1}}(x^{\varsigma}_{\kappa i}), \ddot{\eta}_{N^{\varsigma}_{\kappa 1}}(x^{\varsigma}_{\kappa i}), \ddot{\nu}_{N^{\varsigma}_{\kappa 1}}(x^{\varsigma}_{\kappa i})\rangle | x^{\varsigma}_{\kappa i} \in X\}$$

where $\ddot{\mu}_{N^{\varsigma}_{\kappa 1}}(x^{\varsigma}_{\kappa i}) : X \longrightarrow [0,1]$, $\ddot{\eta}_{N^{\varsigma}_{\kappa 1}}(x^{\varsigma}_{\kappa i}) : X \longrightarrow [0,1]$ and $\ddot{\nu}_{N^{\varsigma}_{\kappa 1}}(x^{\varsigma}_{\kappa i}) : X \longrightarrow [0,1]$ represent degree of membership, neutral and non membership function $x^{\varsigma}_{\kappa i}$ in $N^{\varsigma}_{\kappa 1}$ respectively. Also $0 \leq \ddot{\mu}_{N^{\varsigma}_{\kappa 1}}(x^{\varsigma}_{\kappa i}) + \ddot{\eta}_{N^{\varsigma}_{\kappa 1}}(x^{\varsigma}_{\kappa i}) + \ddot{\nu}_{N^{\varsigma}_{\kappa 1}}(x^{\varsigma}_{\kappa i}) \leq 1$, $\forall x^{\varsigma}_{\kappa i} \in X$.
$\rho_{N^{\varsigma}_{\kappa 1}}(x^{\varsigma}_{\kappa i}) = 1 - \ddot{\mu}_{N^{\varsigma}_{\kappa 1}}(x^{\varsigma}_{\kappa i}) - \ddot{\eta}_{N^{\varsigma}_{\kappa 1}}(x^{\varsigma}_{\kappa i}) - \ddot{\nu}_{N^{\varsigma}_{\kappa 1}}(x^{\varsigma}_{\kappa i})$ represent refusal membership degree of $x^{\varsigma}_{\kappa i}$ in $N^{\varsigma}_{\kappa 1}$, $\forall x^{\varsigma}_{\kappa i} \in X$ The set of all picture fuzzy subsets on universe of discourse $N^{\varsigma}_{\kappa 1}$ is denoted by PFSs(X).

Some basic operations of PFS is discussed as follows

Definition 2.3. [?]

The operations between two PFS $N^{\varsigma}_{\kappa 1} = \{\langle \ddot{\mu}_{N^{\varsigma}_{\kappa 1}}(g), \ddot{\eta}_{N^{\varsigma}_{\kappa 1}}(g), \ddot{\nu}_{N^{\varsigma}_{\kappa 1}}(g)\rangle | g \in X\}$ and $N^{\varsigma}_{\kappa 2} = \{\langle \ddot{\mu}_{N^{\varsigma}_{\kappa 2}}(g), \ddot{\eta}_{N^{\varsigma}_{\kappa 2}}(g), \ddot{\nu}_{N^{\varsigma}_{\kappa 2}}(g)\rangle | g \in X\}$ given as follows:

(i) $N^{\varsigma}_{\kappa 1} \subseteq N^{\varsigma}_{\kappa 2}$ iff $\ddot{\mu}_{N^{\varsigma}_{\kappa 1}}(g) \leq \ddot{\mu}_{N^{\varsigma}_{\kappa 2}}(g)$, $\ddot{\eta}_{N^{\varsigma}_{\kappa 1}}(g) \leq \ddot{\eta}_{N^{\varsigma}_{\kappa 2}}(g)$ and $\ddot{\nu}_{N^{\varsigma}_{\kappa 1}}(g) \geq \ddot{\nu}_{N^{\varsigma}_{\kappa 2}}(g)$
$N^{\varsigma}_{\kappa 1} = N^{\varsigma}_{\kappa 2}$ iff $N^{\varsigma}_{\kappa 1} \subseteq N^{\varsigma}_{\kappa 2}$ and $N^{\varsigma}_{\kappa 2} \subseteq N^{\varsigma}_{\kappa 1}$

(ii) $N^{\varsigma}_{\kappa 1} \cup N^{\varsigma}_{\kappa 2} = \{(g, (\ddot{\mu}_{N^{\varsigma}_{\kappa 1}}(g) \vee \ddot{\mu}_{N^{\varsigma}_{\kappa 2}}(g))), (g, (\ddot{\eta}_{N^{\varsigma}_{\kappa 1}}(g) \wedge \ddot{\eta}_{N^{\varsigma}_{\kappa 2}}(g))), (g, (\ddot{\nu}_{N^{\varsigma}_{\kappa 1}}(g) \wedge \ddot{\nu}_{N^{\varsigma}_{\kappa 2}}(g)))\}$

(iii) $N^{\varsigma}_{\kappa 1} \cap N^{\varsigma}_{\kappa 2} = \{(g, (\ddot{\mu}_{N^{\varsigma}_{\kappa 1}}(g) \wedge \ddot{\mu}_{N^{\varsigma}_{\kappa 2}}(g))), (g, (\ddot{\eta}_{N^{\varsigma}_{\kappa 1}}(g) \wedge \ddot{\eta}_{N^{\varsigma}_{\kappa 2}}(g))), (g, (\ddot{\nu}_{N^{\varsigma}_{\kappa 1}}(g) \vee \ddot{\nu}_{N^{\varsigma}_{\kappa 2}}(g)))\}$

(iv) let $N^{\varsigma}_{\kappa 1} = \{\langle \ddot{\mu}_{N^{\varsigma}_{\kappa 1}}(g), \ddot{\eta}_{N^{\varsigma}_{\kappa 1}}(g), \ddot{\nu}_{N^{\varsigma}_{\kappa 1}}(g)\rangle | g \in X\}$ then $N^{\varsigma c}_{\kappa 1} = \{\langle \ddot{\nu}_{N^{\varsigma}_{\kappa 1}}(g), \ddot{\eta}_{N^{\varsigma}_{\kappa 1}}(g), \ddot{\mu}_{N^{\varsigma}_{\kappa 1}}(g)\rangle | g \in X\}$

New scientific instrument SS, introduced by Molodtsov [1] in which parametrization helps to manage uncertainties.

Definition 2.4. [1]

A mapping $\mathcal{F} : \mathcal{A} \longrightarrow P(\mathcal{U})$

$(\mathcal{F}, \mathcal{A})$ is called a soft set over $\mathcal{U}$, where $\mathcal{A}$ is set of parameters.

In 2015 Yang et al [28] proposed PFSS which is combination of PFS and SS.

Definition 2.5. [28]

Let $\mathcal{E}$ is parametric set. Consider a function $\mathcal{F} : \mathcal{A} \longrightarrow PF(\mathcal{U})$, where $\mathcal{A} \subseteq \mathcal{E}$ and PF($\mathcal{U}$) is power set of PFS over $\mathcal{U}$ then pair $(\mathcal{F}, \mathcal{A})$ is representation of PFSS. .

In 2018 HSS defined by Smarandache [16] which is generalization of SS by transforming the mapping into a multi-attribute mapping.

Definition 2.6. [16]
Suppose n distinct attributes are $b_1, b_2, ..., b_n$, for $b \geq 1$, then for each attributes $\mathcal{Q}^{\varsigma}_{\kappa 1}, \mathcal{Q}^{\varsigma}_{\kappa 2}, ..., \mathcal{Q}^{\varsigma}_{\kappa n}$, with $\mathcal{Q}^{\varsigma}_{\kappa r} \bigcap \mathcal{Q}^{\varsigma}_{\kappa s} = \phi$, $i \neq j$, and r,s $\in \{1, 2, ..., n\}$ are corresponding attributes. The pair $(\ddot{\mathcal{H}}, \mathcal{Q}^{\varsigma}_{\kappa 1} \times \mathcal{Q}^{\varsigma}_{\kappa 2} \times ..... \times \mathcal{Q}^{\varsigma}_{\kappa n})$,
where $\ddot{\mathcal{H}} : \mathcal{Q}^{\varsigma}_{\kappa 1} \times \mathcal{Q}^{\varsigma}_{\kappa 2} \times ..... \times \mathcal{Q}^{\varsigma}_{\kappa n} \rightarrow P(\mathcal{U})$ represent Hypersoft Set over $\mathcal{U}$.

Chen and Tan [41] proposed an idea of score function which plays an important role to handle multicriteria fuzzy decision-making problems.

Definition 2.7. [41]
Let $\ddot{\mathcal{H}}(x^{\varsigma}_{\kappa i}) = \langle \mu(x^{\varsigma}_{\kappa i}), \eta(x^{\varsigma}_{\kappa i}), \nu(x^{\varsigma}_{\kappa i}) \rangle$ be PFSV. Υ and Λ denotes the score and accuracy functions respectively.

$$\Upsilon = \mu(x^{\varsigma}_{\kappa i}) - \nu(x^{\varsigma}_{\kappa i}) \quad \Upsilon \in [-1, 1]$$
$$\Lambda = \mu(x^{\varsigma}_{\kappa i}) + \eta(x^{\varsigma}_{\kappa i}) + \nu(x^{\varsigma}_{\kappa i}) \quad \Lambda \in [0, 1]$$

Jana et al [27] introduced Dombi aggregation operators in PFS sense for MADM problems.

Definition 2.8. [27]
let $\ddot{\mathcal{H}}(x^{\varsigma}_{\kappa i}) = \langle \mu(x^{\varsigma}_{\kappa i}), \eta(x^{\varsigma}_{\kappa i}), \nu(x^{\varsigma}_{\kappa i}) \rangle$ be PFSV then the function $P^n \rightarrow P$ is called PFDWA such that

$$PFDWA_{\mathcal{W}}(x^{\varsigma}_{\kappa 1}, x^{\varsigma}_{\kappa 2}, ...x^{\varsigma}_{\kappa n}) = \sum_{i=1}^{i=n} \mathcal{W}_i x^{\varsigma}_{\kappa i}$$

$$= (1 - \frac{1}{1+\{\sum_{i=1}^{i=n} \mathcal{W}_i(\frac{\mu(x^{\varsigma}_{\kappa i})}{1-\mu(x^{\varsigma}_{\kappa i})})^k\}^k}, \frac{1}{1+\{\sum_{i=1}^{i=n} \mathcal{W}_i(\frac{1-\eta(x^{\varsigma}_{\kappa i})}{\eta(x^{\varsigma}_{\kappa i})})^k\}^k}, \frac{1}{1+\{\sum_{i=1}^{i=n} \mathcal{W}_i(\frac{1-\nu(x^{\varsigma}_{\kappa i})}{\nu(x^{\varsigma}_{\kappa i})})^k\}^k})$$

For each $\mathcal{W}_i \geq 0$ and $\sum_{i=1}^{i=n} \mathcal{W}_i = 1$

3. **Picture Fuzzy Hypersoft Sets**

In [28] hybrid model of PFS and SS is defined. In this section we introduce PFHSS is an extension of PFSS which helps for paying crucial role in decision making for multi attribute characteristic.

Definition 3.1. Picture Fuzzy Hypersoft Sets
Suppose m disjoint attribute-valued sets are $p^a_{b\,1}, p^a_{b\,2}, p^a_{b\,3}, ..., p^a_{b\,m}$ then their corresponding m distinct attributes are $P^a_{b\,1}, P^a_{b\,2}, P^a_{b\,3}, ..., P^a_{b\,m}$ respectively and $P^a_b = P^a_{b\,1} \times P^a_{b\,2} \times P^a_{b\,3} \times .... \times P^a_{b\,m}$. A mapping is given by $\ddot{\mathcal{H}} : P^a_b \longrightarrow PF(\mathcal{U})$

$$\ddot{\mathcal{H}}(t^a_b) = \{\langle \mu_{\ddot{\mathcal{H}}(t^a_b)}(j^a_{b\,i}), \eta_{\ddot{\mathcal{H}}(t^a_b)}(j^a_{b\,i}), \nu_{\ddot{\mathcal{H}}(t^a_b)}(j^a_{b\,i}) \rangle | (j^a_{b\,i}) \in \mathcal{U}\} \quad for \;\; any \;\; t^a_b \in P^a_b$$

then pair $(\ddot{\mathcal{H}}, P_b^a)$ represent PFHSS.

Example 3.2. Consider $(\ddot{\mathcal{H}}, \mathcal{P})$ be PFHSS over $\mathcal{U}$. Let $\mathcal{U} = \{j_{b\,1}^a, j_{b\,2}^a, j_{b\,3}^a, j_{b\,4}^a\}$ be four schools any where in World. let $E = \{a_1, a_2, a_3, a_4\}$ where each a_i stands for Fee Structure , facilities , faculties and labs be the attributes values respectively, $\{A_1, A_2, A_3, A_4\}$ be attribute values against each a_i. let $A_1 = \{b_{11}$ = low, b_{12}= medium , b_{13} = expensive $\}$

$A_2 = \{b_{21}$ = playgrounds, b_{22}= library, b_{23} = cafeterias, b_{24} = bookshop, $b_{25} = \{b_{21}, b_{22}, b_{23}, b_{24}\}\}$

$A_3 = \{b_{31}$ = Science and Arts teacher, b_{32} = Oriental teacher, b_{33} = Physical education teacher, $b_{34} = \{b_{31}, b_{32}, b_{33}\}\}$

$A_4 = \{b_{41}$ = Science labs, b_{42} = Computer lab, $b_{43} = \{b_{41}, b_{42}\}\}$

then

$$\tilde{L}_\kappa^\varsigma = A_1 \times A_2 \times A_3 \times A_4$$

There are one hundred eighty outcomes but for simplicity we take only four outcomes.

$$\tilde{L}_\kappa^\varsigma = \left\{ \begin{array}{l} t_{b\,1}^a = (b_{11}, b_{21}, b_{31}, b_{41}), \quad t_{b\,2}^a = (b_{12}, b_{25}, b_{34}, b_{43}), \\ t_{b\,3}^a = (b_{12}, b_{22}, b_{31}), \quad t_{b\,4}^a = (b_{11}, b_{25}, b_{34}, b_{43}), \end{array} \right\}$$

$$(\ddot{\mathcal{H}}, \tilde{L}_\kappa^\varsigma) = \left\{ \begin{array}{l} \ddot{\mathcal{H}}(t_{b1}^a) = \{\langle 0.7, 0.1, 0.1\rangle / j_{b\,1}^a, \langle 0.3, 0.2, 0.4\rangle / j_{b\,2}^a, \langle 0.1, 0.5, 0.3\rangle / j_{b\,3}^a, \langle 0.4, 0.1, 0.3\rangle / j_{b\,4}^a\}, \\ \ddot{\mathcal{H}}(t_{b2}^a) = \{\langle 0.6, 0.1, 0.2\rangle / j_{b\,1}^a, \langle 0.4, 0.1, 0.3\rangle / j_{b\,2}^a, \langle 0.7, 0.1, 0.1\rangle / j_{b\,3}^a, \langle 0.2, 0.5, 0.2\rangle / j_{b\,4}^a, \}, \\ \ddot{\mathcal{H}}(t_{b3}^a) = \{\langle 0.2, 0.3, 0.5\rangle / j_{b\,1}^a, \langle 0.1, 0.1, 0.6\rangle / j_{b\,2}^a, \langle 0.2, 0.1, 0.7\rangle / j_{b\,3}^a, \langle 0.8, 0.1, 0.1\rangle / j_{b\,4}^a\}, \\ \ddot{\mathcal{H}}(t_{b4}^a) = \{\langle 0.3, 0.2, 0.4\rangle / j_{b\,1}^a, \langle 0.5, 0.1, 0.3\rangle / j_{b\,2}^a, \langle 0.1, 0.5, 0.3\rangle / j_{b\,3}^a, \langle 0.2, 0.1, 0.7\rangle / j_{b\,4}^a\}, \end{array} \right\}$$

The PFHSS is represented by Tab 1.

TABLE 1. Picture Fuzzy Hyper soft Set

$\mathcal{U}$	$t_{b\,1}^a$	$t_{b\,2}^a$	$t_{b\,3}^a$	$t_{b\,4}^a$
$j_{b\,1}^a$	$\langle 0.7, 0.1, 0.1\rangle$	$\langle 0.6, 0.1, 0.2\rangle$	$\langle 0.2, 0.3, 0.5\rangle$	$\langle 0.3, 0.2, 0.4\rangle$
$j_{b\,2}^a$	$\langle 0.3, 0.2, 0.4\rangle$	$\langle 0.4, 0.1, 0.3\rangle$	$\langle 0.1, 0.1, 0.6\rangle$	$\langle 0.5, 0.1, 0.3\rangle$
$j_{b\,3}^a$	$\langle 0.1, 0.5, 0.3\rangle$	$\langle 0.7, 0.1, 0.1\rangle$	$\langle 0.2, 0.1, 0.7\rangle$	$\langle 0.1, 0.5, 0.3\rangle$
$j_{b\,4}^a$	$\langle 0.4, 0.1, 0.3\rangle$	$\langle 0.2, 0.5, 0.2\rangle$	$\langle 0.8, 0.1, 0.1\rangle$	$\langle 0.2, 0.1, 0.7\rangle$

Definition 3.3.

Let $(\ddot{\mathcal{H}}_1, N_{\kappa 1}^\varsigma)$ and $(\ddot{\mathcal{H}}_2, N_{\kappa 2}^\varsigma)$ be two PFHSS then $(\ddot{\mathcal{H}}_1, N_{\kappa 1}^\varsigma) \subseteq (\ddot{\mathcal{H}}_2, N_{\kappa 2}^\varsigma)$ if $N_{\kappa 1}^\varsigma \subseteq N_{\kappa 2}^\varsigma$ and $\ddot{\mathcal{H}}_1(t_b^a) \subseteq \ddot{\mathcal{H}}_2(t_b^a)$ for all $t_b^a \in N_{\kappa 1}^\varsigma$

Example 3.4.

$$(\ddot{\mathcal{H}}_1, N^{\varsigma}_{\kappa 1}) = \left\{ \begin{array}{c} \ddot{\mathcal{H}}_1(t^a_{b\,1}) = \{\langle 0.3, 0.1, 0.3\rangle/j^a_{b\,1}, \langle 0.4, 0.2, 0.3\rangle/j^a_{b\,2}, \langle 0.1, 0.4, 0.3\rangle/j^a_{b\,3}, \langle 0.2, 0.1, 0.3\rangle/j^a_{b\,4}\}, \\ \ddot{\mathcal{H}}_1(t^a_{b\,3}) = \{\langle 0.3, 0.1, 0.4\rangle/j^a_{b\,1}, \langle 0.5, 0.1, 0.3\rangle/j^a_{b\,2}, \langle 0.1, 0.3, 0.5\rangle/j^a_{b\,3}, \langle 0.4, 0.1, 0.3\rangle/j^a_{b\,4}\}, \\ \ddot{\mathcal{H}}_1(t^a_{b\,4}) = \{\langle 0.2, 0.3, 0.4\rangle/j^a_{b\,1}, \langle 0.5, 0.1, 0.3\rangle/j^a_{b\,2}, \langle 0.1, 0.2, 0.6\rangle/j^a_{b\,3}, \langle 0.4, 0.1, 0.4\rangle/j^a_{b\,4}\} \end{array} \right\},$$

and

$$(\ddot{\mathcal{H}}_2, N^{\varsigma}_{\kappa 2}) = \left\{ \begin{array}{c} \ddot{\mathcal{H}}_2(t^a_{b\,1}) = \{\langle 0.5, 0.2, 0.1\rangle/j^a_{b\,1}, \langle 0.5, 0.3, 0.1\rangle/j^a_{b\,2}, \langle 0.3, 0.5, 0.1\rangle/j^a_{b\,3}, \langle 0.4, 0.3, 0.2\rangle/j^a_{b\,4}\}, \\ \ddot{\mathcal{H}}_2(t^a_{b\,3}) = \{\langle 0.5, 0.2, 0.2\rangle/j^a_{b\,1}, \langle 0.6, 0.2, 0.1\rangle/j^a_{b\,2}, \langle 0.3, 0.4, 0.1\rangle/j^a_{b\,3}, \langle 0.5, 0.2, 0.2\rangle/j^a_{b\,4}\}, \\ \ddot{\mathcal{H}}_2(t^a_{b\,4}) = \{\langle 0.4, 0.4, 0.1\rangle/j^a_{b\,1}, \langle 0.6, 0.2, 0.1\rangle/j^a_{b\,2}, \langle 0.5, 0.3, 0.2\rangle/j^a_{b\,3}, \langle 0.5, 0.3, 0.1\rangle/j^a_{b\,4}\} \end{array} \right\}.$$

be two PFHSS.

This implies that $(\ddot{\mathcal{H}}_1, N^{\varsigma}_{\kappa 1}) \subseteq (\ddot{\mathcal{H}}_2, N^{\varsigma}_{\kappa 2})$.

Definition 3.5.

The extended union of two PFHSS $(\ddot{\mathcal{H}}_1, N^{\varsigma}_{\kappa 1})$ and $(\ddot{\mathcal{H}}_2, N^{\varsigma}_{\kappa 2})$ is defined as $(\ddot{\mathcal{H}}_3, N^{\varsigma}_{\kappa 3}) = (\ddot{\mathcal{H}}_1, N^{\varsigma}_{\kappa 1}) \cup_e (\ddot{\mathcal{H}}_2, N^{\varsigma}_{\kappa 2})$, where $N^{\varsigma}_{\kappa 3} = N^{\varsigma}_{\kappa 1} \cup N^{\varsigma}_{\kappa 2}$ and for all $t^a_b \in N^{\varsigma}_{\kappa 3}$,

$$\ddot{\mathcal{H}}_3(t^a_b) = \left\{ \begin{array}{ll} \ddot{\mathcal{H}}_1(t^a_b), & if\ t^a_b \in N^{\varsigma}_{\kappa 1} \setminus N^{\varsigma}_{\kappa 2} \\ \ddot{\mathcal{H}}_2(t^a_b), & if\ t^a_b \in N^{\varsigma}_{\kappa 2} \setminus N^{\varsigma}_{\kappa 1} \\ \ddot{\mathcal{H}}_1(t^a_b) \cup \ddot{\mathcal{H}}_2(t^a_b) & if\ t^a_b \in N^{\varsigma}_{\kappa 1} \cap N^{\varsigma}_{\kappa 2} \end{array} \right\}$$

Example 3.6. Considering example 3.4, we have

$$(\ddot{\mathcal{H}}_3, C) = \left\{ \begin{array}{c} \ddot{\mathcal{H}}_3(t^a_{b\,1}) = \{\langle 0.5, 0.1, 0.1\rangle/j^a_{b\,1}, \langle 0.5, 0.2, 0.1\rangle/j^a_{b\,2}, \langle 0.3, 0.4, 0.1\rangle/j^a_{b\,3}, \langle 0.4, 0.1, 0.2\rangle/j^a_{b\,4}\}, \\ \ddot{\mathcal{H}}_3(t^a_{b\,3}) = \{\langle 0.5, 0.1, 0.2\rangle/j^a_{b\,1}, \langle 0.6, 0.1, 0.1\rangle/j^a_{b\,2}, \langle 0.3, 0.3, 0.1\rangle/j^a_{b\,3}, \langle 0.5, 0.1, 0.2\rangle/j^a_{b\,4}\}, \\ \ddot{\mathcal{H}}_3(t^a_{b\,4}) = \{\langle 0.4, 0.3, 0.1\rangle/j^a_{b\,1}, \langle 0.6, 0.1, 0.1\rangle/j^a_{b\,2}, \langle 0.5, 0.2, 0.2\rangle/j^a_{b\,3}, \langle 0.5, 0.1, 0.1\rangle/j^a_{b\,4}\} \end{array} \right\}.$$

Definition 3.7.

The extended intersection of $(\ddot{\mathcal{H}}_1, A)$ and $(\ddot{\mathcal{H}}_2, B)$ is defined as $(\ddot{\mathcal{H}}_4, C) = (\ddot{\mathcal{H}}_1, A) \cap_e (\ddot{\mathcal{H}}_2, B)$, where $C = A \cup B$ and for all $t^a_b \in C$,

$$\ddot{\mathcal{H}}_4(t^a_b) = \left\{ \begin{array}{ll} \ddot{\mathcal{H}}_1(t^a_b), & if\ t^a_b \in A \setminus B \\ \ddot{\mathcal{H}}_2(t^a_b), & if\ t^a_b \in B \setminus A \\ \ddot{\mathcal{H}}_1(t^a_b) \cap \ddot{\mathcal{H}}_2(t^a_b) & if\ t^a_b \in A \cap B \end{array} \right\}$$

Example 3.8. In example 3.4, we get

$$(\ddot{\mathcal{H}}_4, C) = \left\{ \begin{array}{c} \ddot{\mathcal{H}}_4(t^a_{b\,1}) = \{\langle 0.3, 0.1, 0.3\rangle/j^a_{b\,1}, \langle 0.4, 0.2, 0.3\rangle/j^a_{b\,2}, \langle 0.1, 0.4, 0.3\rangle/j^a_{b\,3}, \langle 0.2, 0.1, 0.3\rangle/j^a_{b\,4}\}, \\ \ddot{\mathcal{H}}_4(t^a_{b\,3}) = \{\langle 0.3, 0.1, 0.4\rangle/j^a_{b\,1}, \langle 0.5, 0.1, 0.3\rangle/j^a_{b\,2}, \langle 0.1, 0.3, 0.5\rangle/j^a_{b\,3}, \langle 0.4, 0.1, 0.3\rangle/j^a_{b\,4}\}, \\ \ddot{\mathcal{H}}_4(t^a_{b\,4}) = \{\langle 0.2, 0.3, 0.4\rangle/j^a_{b\,1}, \langle 0.5, 0.1, 0.3\rangle/j^a_{b\,2}, \langle 0.1, 0.2, 0.6\rangle/j^a_{b\,3}, \langle 0.4, 0.1, 0.4\rangle/j^a_{b\,4}\} \end{array} \right\}.$$

Definition 3.9. The restricted union of $(\ddot{\mathcal{H}}_1, N^{\varsigma}_{\kappa 1})$ and $(\ddot{\mathcal{H}}_2, N^{\varsigma}_{\kappa 2})$ is defined as $(\ddot{\mathcal{H}}_5, N^{\varsigma}_{\kappa 3}) = (\ddot{\mathcal{H}}_1, N^{\varsigma}_{\kappa 1}) \cup_r (\ddot{\mathcal{H}}_2, N^{\varsigma}_{\kappa 2})$, where $N^{\varsigma}_{\kappa 3} = N^{\varsigma}_{\kappa 1} \cap N^{\varsigma}_{\kappa 2} \neq \emptyset$ and for all $t^a_b \in N^{\varsigma}_{\kappa 3}$,

Example 3.10. From example 3.4, we get

$$(\ddot{\mathcal{H}}_3, C) = \left\{ \begin{array}{c} \ddot{\mathcal{H}}_3(t^a_{b\,1}) = \{\langle 0.5, 0.1, 0.1\rangle/j^a_{b\,1}, \langle 0.5, 0.2, 0.1\rangle/j^a_{b\,2}, \langle 0.3, 0.4, 0.1\rangle/j^a_{b\,3}, \langle 0.4, 0.1, 0.2\rangle/j^a_{b\,4}\}, \\ \ddot{\mathcal{H}}_3(t^a_{b\,3}) = \{\langle 0.5, 0.1, 0.2\rangle/j^a_{b\,1}, \langle 0.6, 0.1, 0.1\rangle/j^a_{b\,2}, \langle 0.3, 0.3, 0.1\rangle/j^a_{b\,3}, \langle 0.5, 0.1, 0.2\rangle/j^a_{b\,4}\}, \\ \ddot{\mathcal{H}}_3(t^a_{b\,4}) = \{\langle 0.4, 0.3, 0.1\rangle/j^a_{b\,1}, \langle 0.6, 0.1, 0.1\rangle/j^a_{b\,2}, \langle 0.5, 0.2, 0.2\rangle/j^a_{b\,3}, \langle 0.5, 0.1, 0.1\rangle/j^a_{b\,4}\} \end{array} \right\}.$$

Definition 3.11. The restricted intersection of $(\ddot{\mathcal{H}}_1, N^{\varsigma}_{\kappa 1})$ and $(\ddot{\mathcal{H}}_2, N^{\varsigma}_{\kappa 2})$ is defined as $(\ddot{\mathcal{H}}_5, N^{\varsigma}_{\kappa 3}) = (\ddot{\mathcal{H}}_1, N^{\varsigma}_{\kappa 1}) \cap_r (\ddot{\mathcal{H}}_2, N^{\varsigma}_{\kappa 2})$, where $N^{\varsigma}_{\kappa 3} = N^{\varsigma}_{\kappa 1} \cap N^{\varsigma}_{\kappa 2} \neq \emptyset$ and for all $t^a_b \in N^{\varsigma}_{\kappa 3}$,

Example 3.12. Example 3.4, implies that

$$(\ddot{\mathcal{H}}_4, C) = \left\{ \begin{array}{c} \ddot{\mathcal{H}}_4(t^a_{b\,1}) = \{\langle 0.3, 0.1, 0.3\rangle/j^a_{b\,1}, \langle 0.4, 0.2, 0.3\rangle/j^a_{b\,2}, \langle 0.1, 0.4, 0.3\rangle/j^a_{b\,3}, \langle 0.2, 0.1, 0.3\rangle/j^a_{b\,4}\}, \\ \ddot{\mathcal{H}}_4(t^a_{b\,3}) = \{\langle 0.3, 0.1, 0.4\rangle/j^a_{b\,1}, \langle 0.5, 0.1, 0.3\rangle/j^a_{b\,2}, \langle 0.1, 0.3, 0.5\rangle/j^a_{b\,3}, \langle 0.4, 0.1, 0.3\rangle/j^a_{b\,4}\}, \\ \ddot{\mathcal{H}}_4(t^a_{b\,4}) = \{\langle 0.2, 0.3, 0.4\rangle/j^a_{b\,1}, \langle 0.5, 0.1, 0.3\rangle/j^a_{b\,2}, \langle 0.1, 0.2, 0.6\rangle/j^a_{b\,3}, \langle 0.4, 0.1, 0.4\rangle/j^a_{b\,4}\} \end{array} \right\}.$$

Definition 3.13. If $(\ddot{\mathcal{H}}, \tilde{L}^{\varsigma}_{\kappa})$ be PFHSS then

$$(\ddot{\mathcal{H}}, P)' = \{\langle \nu_{\ddot{\mathcal{H}}(t^a_b)}(j^a_{b\,i}), \eta_{\ddot{\mathcal{H}}(t^a_b)}(j^a_{b\,i}), \mu_{\ddot{\mathcal{H}}(t^a_b)}(j^a_{b\,i})\rangle | (j^a_{b\,i}) \in \mathcal{U}\} \quad for \quad any \quad t^a_b \in \tilde{L}^{\varsigma}_{\kappa}$$

Example 3.14. If

$$(\ddot{\mathcal{H}}, N^{\varsigma}_{\kappa 1}) = \left\{ \ddot{\mathcal{H}}(t^a_b) = \{\langle 0.7, 0.1, 0.1\rangle/j^a_{b\,1}, \langle 0.2, 0.5, 0.2\rangle/j^a_{b\,3}, \langle 0.6, 0.1, 0.2\rangle/j^a_{b\,6}, \langle 0.4, 0.1, 0.3\rangle/j^a_{b\,8}\}, \right\}.$$

then

$$(\ddot{\mathcal{H}}, N^{\varsigma}_{\kappa 1})' = \left\{ \ddot{\mathcal{H}}(t^a_b) = \{\langle 0.1, 0.1, 0.7\rangle/j^a_{b\,1}, \langle 0.2, 0.5, 0.2\rangle/j^a_{b\,3}, \langle 0.2, 0.1, 0.6\rangle/j^a_{b\,6}, \langle 0.3, 0.1, 0.4\rangle/j^a_{b\,8}\}, \right\}.$$

Remark 3.15.

(i) $(\ddot{\mathcal{H}}, \tilde{L}^{\varsigma}_{\kappa}) \cup_e (\ddot{\mathcal{H}}, \tilde{L}^{\varsigma}_{\kappa}) = (\ddot{\mathcal{H}}, \tilde{L}^{\varsigma}_{\kappa}) \cup_r (\ddot{\mathcal{H}}, \tilde{L}^{\varsigma}_{\kappa}) = (\ddot{\mathcal{H}}, \tilde{L}^{\varsigma}_{\kappa})$

(ii) $(\ddot{\mathcal{H}}, \tilde{L}^{\varsigma}_{\kappa}) \cap_e (\ddot{\mathcal{H}}, \tilde{L}^{\varsigma}_{\kappa}) = (\ddot{\mathcal{H}}, \tilde{L}^{\varsigma}_{\kappa}) \cap_r (\ddot{\mathcal{H}}, \tilde{L}^{\varsigma}_{\kappa}) = (\ddot{\mathcal{H}}, \tilde{L}^{\varsigma}_{\kappa})$

Next, we check validity of the De Morgans laws in PFHSS with respect to extended, union and intersection.

Theorem 3.16. *If $(\ddot{\mathcal{H}}_1, N^{\varsigma}_{\kappa 1})$ and $(\ddot{\mathcal{H}}_2, N^{\varsigma}_{\kappa 2})$ be two PFHSS over $\mathcal{U}$. Then*

(i) $((\ddot{\mathcal{H}}_1, N^{\varsigma}_{\kappa 1}) \cup_e (\ddot{\mathcal{H}}_2, N^{\varsigma}_{\kappa 2}))' = (\ddot{\mathcal{H}}_1, N^{\varsigma}_{\kappa 1})' \cap_e (\ddot{\mathcal{H}}_2, N^{\varsigma}_{\kappa 2})'$

(ii) $((\ddot{\mathcal{H}}_1, N^{\varsigma}_{\kappa 1}) \cap_e (\ddot{\mathcal{H}}_2, N^{\varsigma}_{\kappa 2}))' = (\ddot{\mathcal{H}}_1, N^{\varsigma}_{\kappa 1})' \cup_e (\ddot{\mathcal{H}}_2, N^{\varsigma}_{\kappa 2})'$

Proof. (i) Since
$(\ddot{\mathcal{H}}_3, N^{\varsigma}_{\kappa 3}) = (\ddot{\mathcal{H}}_1, N^{\varsigma}_{\kappa 1}) \cup_e (\ddot{\mathcal{H}}_2, N^{\varsigma}_{\kappa 2})$, where $N^{\varsigma}_{\kappa 3} = N^{\varsigma}_{\kappa 1} \cup N^{\varsigma}_{\kappa 2}$

Then $(\ddot{\mathcal{H}}_3, N^{\varsigma}_{\kappa 3})' = ((\ddot{\mathcal{H}}_1, N^{\varsigma}_{\kappa 1}) \cup_e (\ddot{\mathcal{H}}_2, N^{\varsigma}_{\kappa 2}))'$

$$\ddot{\mathcal{H}}_3(t^a_b) = \left\{ \begin{array}{ll} \ddot{\mathcal{H}}_1(t^a_b), & if\ t^a_b \in N^{\varsigma}_{\kappa 1} \setminus N^{\varsigma}_{\kappa 2} \\ \ddot{\mathcal{H}}_2(t^a_b), & if\ t^a_b \in N^{\varsigma}_{\kappa 2} \setminus N^{\varsigma}_{\kappa 1} \\ \ddot{\mathcal{H}}_1(t^a_b) \cup \ddot{\mathcal{H}}_2(t^a_b) & if\ t^a_b \in N^{\varsigma}_{\kappa 1} \cap N^{\varsigma}_{\kappa 2} \end{array} \right\}$$

for all $t^a_b \in N^{\varsigma}_{\kappa 3}$ then

$$(\ddot{\mathcal{H}}_3(t^a_b))' = \left\{ \begin{array}{ll} (\ddot{\mathcal{H}}_1(t^a_b))', & if\ t^a_b \in N^{\varsigma}_{\kappa 1} \setminus N^{\varsigma}_{\kappa 2} \\ (\ddot{\mathcal{H}}_2(t^a_b))', & if\ t^a_b \in N^{\varsigma}_{\kappa 2} \setminus N^{\varsigma}_{\kappa 1} \\ (\ddot{\mathcal{H}}_1(t^a_b) \cup \ddot{\mathcal{H}}_2(t^a_b))' & if\ t^a_b \in N^{\varsigma}_{\kappa 1} \cap N^{\varsigma}_{\kappa 2} \end{array} \right\}$$

Since De Morgans laws hold in Picture Fuzzy Soft set

$$(\ddot{\mathcal{H}}_3(t^a_b))' = \left\{ \begin{array}{ll} (\ddot{\mathcal{H}}_1(t^a_b))', & if\ t^a_b \in N^{\varsigma}_{\kappa 1} \setminus N^{\varsigma}_{\kappa 2} \\ (\ddot{\mathcal{H}}_2(t^a_b))', & if\ t^a_b \in N^{\varsigma}_{\kappa 2} \setminus N^{\varsigma}_{\kappa 1} \\ (\ddot{\mathcal{H}}_1(t^a_b))' \cap (\ddot{\mathcal{H}}_2(t^a_b))' & if\ t^a_b \in N^{\varsigma}_{\kappa 1} \cap N^{\varsigma}_{\kappa 2} \end{array} \right\} \in (\ddot{\mathcal{H}}_3, N^{\varsigma}_{\kappa 3})'$$

Suppose $(\ddot{\mathcal{H}}_a, N^{\varsigma}_{\kappa 3}) = (\ddot{\mathcal{H}}_1, N^{\varsigma}_{\kappa 1})' \cap_e (\ddot{\mathcal{H}}_2, N^{\varsigma}_{\kappa 2})'$, where $N^{\varsigma}_{\kappa 3} = N^{\varsigma}_{\kappa 1} \cup N^{\varsigma}_{\kappa 2}$ for all $t^a_b \in N^{\varsigma}_{\kappa 3}$

$$(\ddot{\mathcal{H}}_a(t^a_b)) = \left\{ \begin{array}{ll} (\ddot{\mathcal{H}}_1(t^a_b))', & if\ t^a_b \in N^{\varsigma}_{\kappa 1} \setminus N^{\varsigma}_{\kappa 2} \\ (\ddot{\mathcal{H}}_2(t^a_b))', & if\ t^a_b \in N^{\varsigma}_{\kappa 2} \setminus N^{\varsigma}_{\kappa 1} \\ (\ddot{\mathcal{H}}_1(t^a_b))' \cap (\ddot{\mathcal{H}}_2(t^a_b))' & if\ t^a_b \in N^{\varsigma}_{\kappa 1} \cap N^{\varsigma}_{\kappa 2} \end{array} \right\} \in (\ddot{\mathcal{H}}_3, N^{\varsigma}_{\kappa 3})'$$

This Implies

$$((\ddot{\mathcal{H}}_1, N^{\varsigma}_{\kappa 1}) \cup_e (\ddot{\mathcal{H}}_2, N^{\varsigma}_{\kappa 2}))' = (\ddot{\mathcal{H}}_1, N^{\varsigma}_{\kappa 1})' \cap_e (\ddot{\mathcal{H}}_2, N^{\varsigma}_{\kappa 2})'$$

(ii) Since
$(\ddot{\mathcal{H}}_4, N^{\varsigma}_{\kappa 3}) = (\ddot{\mathcal{H}}_1, N^{\varsigma}_{\kappa 1}) \cap_e (\ddot{\mathcal{H}}_2, N^{\varsigma}_{\kappa 2})$, where $N^{\varsigma}_{\kappa 3} = N^{\varsigma}_{\kappa 1} \cup N^{\varsigma}_{\kappa 2}$
then $((\ddot{\mathcal{H}}_4, N^{\varsigma}_{\kappa 3}))' = ((\ddot{\mathcal{H}}_1, N^{\varsigma}_{\kappa 1}) \cap_e (\ddot{\mathcal{H}}_2, N^{\varsigma}_{\kappa 2}))'$,

$$\ddot{\mathcal{H}}_4(t^a_b) = \left\{ \begin{array}{ll} \ddot{\mathcal{H}}_1(t^a_b), & if\ t^a_b \in N^{\varsigma}_{\kappa 1} \setminus N^{\varsigma}_{\kappa 2} \\ \ddot{\mathcal{H}}_2(t^a_b), & if\ t^a_b \in N^{\varsigma}_{\kappa 2} \setminus N^{\varsigma}_{\kappa 1} \\ \ddot{\mathcal{H}}_1(t^a_b) \cap \ddot{\mathcal{H}}_2(t^a_b) & if\ t^a_b \in N^{\varsigma}_{\kappa 1} \cap N^{\varsigma}_{\kappa 2} \end{array} \right\}$$

for all $t^a_b \in N^{\varsigma}_{\kappa 3}$

$$(\ddot{\mathcal{H}}_4(t^a_b))' = \left\{ \begin{array}{ll} (\ddot{\mathcal{H}}_1(t^a_b))', & if\ t^a_b \in N^{\varsigma}_{\kappa 1} \setminus N^{\varsigma}_{\kappa 2} \\ (\ddot{\mathcal{H}}_2(t^a_b))', & if\ t^a_b \in N^{\varsigma}_{\kappa 2} \setminus N^{\varsigma}_{\kappa 1} \\ (\ddot{\mathcal{H}}_1(t^a_b) \cap \ddot{\mathcal{H}}_2(t^a_b))' & if\ t^a_b \in N^{\varsigma}_{\kappa 1} \cap N^{\varsigma}_{\kappa 2} \end{array} \right\}$$

As we know that

$$(\ddot{\mathcal{H}}_4(t^a_b))' = \left\{ \begin{array}{ll} (\ddot{\mathcal{H}}_1(t^a_b))', & if\ t^a_b \in N^{\varsigma}_{\kappa 1} \setminus N^{\varsigma}_{\kappa 2} \\ (\ddot{\mathcal{H}}_2(t^a_b))', & if\ t^a_b \in N^{\varsigma}_{\kappa 2} \setminus N^{\varsigma}_{\kappa 1} \\ (\ddot{\mathcal{H}}_1(t^a_b))' \cup (\ddot{\mathcal{H}}_2(t^a_b))' & if\ t^a_b \in N^{\varsigma}_{\kappa 1} \cap N^{\varsigma}_{\kappa 2} \end{array} \right\} \in (\ddot{\mathcal{H}}_4, N^{\varsigma}_{\kappa 3})'$$

Suppose $(\ddot{\mathcal{H}}_b, N^{\varsigma}_{\kappa 3}) = (\ddot{\mathcal{H}}_1, N^{\varsigma}_{\kappa 1})' \cup_e (\ddot{\mathcal{H}}_2, N^{\varsigma}_{\kappa 2})'$, where $N^{\varsigma}_{\kappa 3} = N^{\varsigma}_{\kappa 1} \cup N^{\varsigma}_{\kappa 2}$ for all $t^a_b \in N^{\varsigma}_{\kappa 3}$

$$(\ddot{\mathcal{H}}_b(t^a_b)) = \left\{ \begin{array}{ll} (\ddot{\mathcal{H}}_1(t^a_b))', & if\ t^a_b \in N^{\varsigma}_{\kappa 1} \setminus N^{\varsigma}_{\kappa 2} \\ (\ddot{\mathcal{H}}_2(t^a_b))', & if\ t^a_b \in N^{\varsigma}_{\kappa 2} \setminus N^{\varsigma}_{\kappa 1} \\ (\ddot{\mathcal{H}}_1(t^a_b))' \cup (\ddot{\mathcal{H}}_2(t^a_b))' & if\ t^a_b \in N^{\varsigma}_{\kappa 1} \cap N^{\varsigma}_{\kappa 2} \end{array} \right\} \in (\ddot{\mathcal{H}}_4, N^{\varsigma}_{\kappa 3})'$$

Hence

$$((\ddot{\mathcal{H}}_1, N^{\varsigma}_{\kappa 1}) \cap_e (\ddot{\mathcal{H}}_2, N^{\varsigma}_{\kappa 2}))' = (\ddot{\mathcal{H}}_1, N^{\varsigma}_{\kappa 1})' \cup_e (\ddot{\mathcal{H}}_2, N^{\varsigma}_{\kappa 2})'$$

□

We want to prove De Morgans laws for restricted union and restricted intersection in PFHSS.

Theorem 3.17.

(i) $((\ddot{\mathcal{H}}_1, N^{\varsigma}_{\kappa 1}) \cup_r (\ddot{\mathcal{H}}_2, N^{\varsigma}_{\kappa 2}))' = (\ddot{\mathcal{H}}_1, N^{\varsigma}_{\kappa 1})' \cap_r (\ddot{\mathcal{H}}_2, N^{\varsigma}_{\kappa 2})'$

(ii) $((\ddot{\mathcal{H}}_1, N^{\varsigma}_{\kappa 1}) \cap_r (\ddot{\mathcal{H}}_2, N^{\varsigma}_{\kappa 2}))' = (\ddot{\mathcal{H}}_1, N^{\varsigma}_{\kappa 1})' \cup_r (\ddot{\mathcal{H}}_2, N^{\varsigma}_{\kappa 2})'$

Proof. (i) Since $N^{\varsigma}_{\kappa 3} = N^{\varsigma}_{\kappa 1} \cap N^{\varsigma}_{\kappa 2} \neq \emptyset$ and
$(\ddot{\mathcal{H}}_5, N^{\varsigma}_{\kappa 3}) = (\ddot{\mathcal{H}}_1, N^{\varsigma}_{\kappa 1}) \cup_r (\ddot{\mathcal{H}}_2, N^{\varsigma}_{\kappa 2})$ then
$(\ddot{\mathcal{H}}_5, N^{\varsigma}_{\kappa 3})' = ((\ddot{\mathcal{H}}_1, N^{\varsigma}_{\kappa 1}) \cup_r (\ddot{\mathcal{H}}_2, N^{\varsigma}_{\kappa 2}))'$
for all $t^a_b \in N^{\varsigma}_{\kappa 3}$ $\ddot{\mathcal{H}}_5(t^a_b) = \ddot{\mathcal{H}}_1(t^a_b) \cup_r \ddot{\mathcal{H}}_2(t^a_b)$ since De morgan law hold in PFSS
Therefore $(\ddot{\mathcal{H}}_5(t^a_b))' = (\ddot{\mathcal{H}}_1(t^a_b))' \cap_r (\ddot{\mathcal{H}}_2(t^a_b))' \in (\ddot{\mathcal{H}}_5, N^{\varsigma}_{\kappa 3})'$
Suppose $(\ddot{\mathcal{H}}_c, N^{\varsigma}_{\kappa 3}) = (\ddot{\mathcal{H}}_1, N^{\varsigma}_{\kappa 1})' \cap_r (\ddot{\mathcal{H}}_2, N^{\varsigma}_{\kappa 2})'$
$\ddot{\mathcal{H}}_c(t^a_b) = (\ddot{\mathcal{H}}_1(t^a_b))' \cap_r (\ddot{\mathcal{H}}_2(t^a_b))'$ for all $t^a_b \in N^{\varsigma}_{\kappa 3}$ where $N^{\varsigma}_{\kappa 3} = N^{\varsigma}_{\kappa 1} \cup N^{\varsigma}_{\kappa 2}$
hence

$$((\ddot{\mathcal{H}}_1, A) \cup_r (\ddot{\mathcal{H}}_2, B))' = (\ddot{\mathcal{H}}_1, A)' \cap_r (\ddot{\mathcal{H}}_2, B)'$$

(ii) Straightforward □

4. Generalized Picture Fuzzy Hypersoft Sets

In this section, we describe an extension of PFHSS. It is a hybrid modal of PHSS and PFS known as generalized picture fuzzy hypersoft set (GPFHSS). GPFHSS has a character in decision-making exertion, when taking an important decision according to the given attributes it will minimize evaluation, and output will be in the form of PFS.

Definition 4.1. Generalized Picture Fuzzy Hypersoft Sets

Suppose $N^{\varsigma}_{\kappa 1}, N^{\varsigma}_{\kappa 2}, N^{\varsigma}_{\kappa 3}, ..., N^{\varsigma}_{\kappa m}$ be disjoint attribute-valued sets corresponding to m distinct attributes $p^a_{b1}, p^a_{b2}, p^a_{b3}, ..., p^a_{bm}$ respectively and $P = N^{\varsigma}_{\kappa 1} \times N^{\varsigma}_{\kappa 2} \times N^{\varsigma}_{\kappa 3} \times .... \times N^{\varsigma}_{\kappa m}$. A triplet $(\ddot{\mathcal{H}}, P, \Phi)$ is called a generalized picture fuzzy hypersoft set (GPFHSS), where Φ is a mapping given by $\Phi : P \longrightarrow \pounds(P)$. where $\pounds(P)$ is the set of all picture fuzzy hypersoft subsets of P and is called parametric picture fuzzy hypersoftset of GPFHSS.

Example 4.2. Considering example 3.4, $(\ddot{\mathcal{H}}, \tilde{L}^{\varsigma}_{\kappa})$ is PHSS and $\Phi = \{\langle 0.2, 0.1, 0.1\rangle/t^a_{b\,1}, \langle 0.5, 0.2, 0.1\rangle/t^a_{b\,2}, \langle 0.7, 0.2, 0.0\rangle/t^a_{b\,3}, \langle 0.6, 0.2, 0.1\rangle/t^a_{b\,4}\}$
where Φ is an extra viewpoint of a arbitrator on the general standard of work done to check out alternatives on the basis of given multi-attributes.

Table 2. GPFHSS

$\mathcal{U}$	$t^a_{b\,1}$	$t^a_{b\,2}$	$t^a_{b\,3}$	$t^a_{b\,4}$
$j^a_{b\,1}$	$\langle 0.7, 0.1, 0.1\rangle$	$\langle 0.4, 0.2, 0.1\rangle$	$\langle 0.2, 0.3, 0.5\rangle$	$\langle 0.6, 0.2, 0.1\rangle$
$j^a_{b\,2}$	$\langle 0.3, 0.2, 0.4\rangle$	$\langle 0.5, 0.2, 0.1\rangle$	$\langle 0.6, 0.1, 0.2\rangle$	$\langle 0.3, 0.2, 0.4\rangle$
$j^a_{b\,3}$	$\langle 0.5, 0.3, 0.2\rangle$	$\langle 0.7, 0.1, 0.1\rangle$	$\langle 0.2, 0.1, 0.7\rangle$	$\langle 0.6, 0.2, 0.1\rangle$
$j^a_{b\,4}$	$\langle 0.6, 0.1, 0.2$	$\langle 0.5, 0.3, 0.1\rangle$	$\langle 0.4, 0.1, 0.1\rangle$	$\langle 0.5, 0.1, 0.2\rangle$
Φ	$\langle 0.2, 0.1, 0.1\rangle$	$\langle 0.5, 0.2, 0.1\rangle$	$\langle 0.7, 0.2, 0.0\rangle$	$\langle 0.6, 0.2, 0.1\rangle$

Definition 4.3. let $\nabla_1 = (\ddot{\mathcal{H}}_1, N^{\varsigma}_{\kappa 1}, \Phi_1)$ and $\nabla_2 = (\ddot{\mathcal{H}}_2, N^{\varsigma}_{\kappa 2}, \Phi_2)$ be two GPFHSS over $\mathcal{U}$. Then $\nabla_1 \subseteq \nabla_2$ if

(i) $(\ddot{\mathcal{H}}_1, N^{\varsigma}_{\kappa 1}) \subseteq (\ddot{\mathcal{H}}_2, N^{\varsigma}_{\kappa 2})$

(ii) $\langle \mu_{\Phi_1(t^a_b)}(j^a_{b\,i}) \leq \langle \mu_{\Phi_2(t^a_b)}(j^a_{b\,i})$, $\langle \eta_{\Phi_1(t^a_b)}(j^a_{b\,i}) \leq \langle \eta_{\Phi_2(t^a_b)}(j^a_{b\,i})$, $\langle \nu_{\Phi_1(t^a_b)}(j^a_{b\,i}) \geq \langle\langle \nu_{\Phi_2(t^a_b)}(j^a_{b\,i})$

Definition 4.4. Two GPFHSS $\nabla_1 = (\ddot{\mathcal{H}}_1, N^{\varsigma}_{\kappa 1}, \Phi_1)$ and $\nabla_2 = (\ddot{\mathcal{H}}_2, N^{\varsigma}_{\kappa 2}, \Phi_2)$ over $\mathcal{U}$ are said to be equal if $\ddot{\mathcal{H}}_1 = \ddot{\mathcal{H}}_2$, $N^{\varsigma}_{\kappa 1} = N^{\varsigma}_{\kappa 2}$ and $\Phi_1 = \Phi_2$.

Definition 4.5. let $\nabla_1 = (\ddot{\mathcal{H}}_1, N^{\varsigma}_{\kappa 1}, \Phi_1)$ be GPFHSS then the complement of $\nabla_1 =$ is denoted as $(\nabla_1)^c$ and defined as
$(\nabla_1)^c = (\ddot{\mathcal{H}}_1, N^{\varsigma}_{\kappa 1}, \Phi_1)^c = (\ddot{\mathcal{H}}_5, N^{\varsigma}_{\kappa 5}, \Phi_3)$ where $(\ddot{\mathcal{H}}_5, N^{\varsigma}_{\kappa 5})$ is compliment of $(\ddot{\mathcal{H}}_1, N^{\varsigma}_{\kappa 1})$ and Φ_3 is compliment of Φ_1.

5. Basic Operation of Generalized Picture Fuzzy Hypersoft Sets

In this section, we introduce some basic operations for GPFHSS.

Definition 5.1. The extended union of two GPFHSS $\nabla_1 = (\ddot{\mathcal{H}}_1, N^{\varsigma}_{\kappa 1}, \Phi_1)$ and $\nabla_2 = (\ddot{\mathcal{H}}_2, N^{\varsigma}_{\kappa 2}, \Phi_2)$ over $\mathcal{U}$ is denoted by
$\nabla_3 = (\ddot{\mathcal{H}}_6, N^{\varsigma}_{\kappa 6}, \Phi_4) = (\ddot{\mathcal{H}}_1, N^{\varsigma}_{\kappa 1}, \Phi_1) \cup_e (\ddot{\mathcal{H}}_2, N^{\varsigma}_{\kappa 2}, \Phi_2)$ and defined as

(i) $(\ddot{\mathcal{H}}_6, N^{\varsigma}_{\kappa 6}) = (\ddot{\mathcal{H}}_1, N^{\varsigma}_{\kappa 1}) \cup_e (\ddot{\mathcal{H}}_2, N^{\varsigma}_{\kappa 2})$ where $N^{\varsigma}_{\kappa 6} = N^{\varsigma}_{\kappa 1} \cup N^{\varsigma}_{\kappa 2}$

(ii)

$$\mu_{\Phi_4}(t_b^a) = \left\{ \begin{array}{ll} \mu_{\Phi_1}(t_b^a), & if\ t_b^a \in N_{\kappa 1}^{\varsigma} \setminus N_{\kappa 2}^{\varsigma} \\ \mu_{\Phi_2}(t_b^a), & if\ t_b^a \in N_{\kappa 2}^{\varsigma} \setminus N_{\kappa 1}^{\varsigma} \\ max(\mu_{\Phi_1}(t_b^a)\mu_{\Phi_2}(t_b^a)) & if\ t_b^a \in N_{\kappa 1}^{\varsigma} \cap N_{\kappa 2}^{\varsigma} \end{array} \right\}$$

(iii)

$$\eta_{\Phi_4}(t_b^a) = \left\{ \begin{array}{ll} \eta_{\Phi_1}(t_b^a), & if\ t_b^a \in N_{\kappa 1}^{\varsigma} \setminus N_{\kappa 2}^{\varsigma} \\ \eta_{\Phi_2}(t_b^a), & if\ t_b^a \in N_{\kappa 2}^{\varsigma} \setminus N_{\kappa 1}^{\varsigma} \\ min(\eta_{\Phi_1}(t_b^a)\eta_{\Phi_2}(t_b^a)) & if\ t_b^a \in N_{\kappa 1}^{\varsigma} \cap N_{\kappa 2}^{\varsigma} \end{array} \right\}$$

(iv)

$$\nu_{\Phi_4}(t_b^a) = \left\{ \begin{array}{ll} \nu_{\Phi_1}(t_b^a), & if\ t_b^a \in N_{\kappa 1}^{\varsigma} \setminus N_{\kappa 2}^{\varsigma} \\ \nu_{\Phi_2}(t_b^a), & if\ t_b^a \in N_{\kappa 2}^{\varsigma} \setminus N_{\kappa 1}^{\varsigma} \\ min(\nu_{\Phi_1}(t_b^a)\nu_{\Phi_2}(t_b^a)) & if\ t_b^a \in N_{\kappa 1}^{\varsigma} \cap N_{\kappa 2}^{\varsigma} \end{array} \right\}$$

Definition 5.2. The extended intersection of two GPFHSS $\nabla_1 = (\ddot{\mathcal{H}}_1, N_{\kappa 1}^{\varsigma}, \Phi_1)$ and $\nabla_2 = (\ddot{\mathcal{H}}_2, N_{\kappa 2}^{\varsigma}, \Phi_2)$ over $\mathcal{U}$ is denoted by
$\nabla_3 = (\ddot{\mathcal{H}}_6, N_{\kappa 6}^{\varsigma}, \Phi_4) = (\ddot{\mathcal{H}}_1, N_{\kappa 1}^{\varsigma}, \Phi_1) \cap_e (\ddot{\mathcal{H}}_2, N_{\kappa 2}^{\varsigma}, \Phi_2)$ and defined as

(i) $(\ddot{\mathcal{H}}_6, N_{\kappa 6}^{\varsigma}) = (\ddot{\mathcal{H}}_1, N_{\kappa 1}^{\varsigma}) \cap_e (\ddot{\mathcal{H}}_2, N_{\kappa 2}^{\varsigma})$ where $N_{\kappa 6}^{\varsigma} = N_{\kappa 1}^{\varsigma} \cap N_{\kappa 2}^{\varsigma}$

(ii)

$$\mu_{\Phi_4}(t_b^a) = \left\{ \begin{array}{ll} \mu_{\Phi_1}(t_b^a), & if\ t_b^a \in N_{\kappa 1}^{\varsigma} \setminus N_{\kappa 2}^{\varsigma} \\ \mu_{\Phi_2}(t_b^a), & if\ t_b^a \in N_{\kappa 2}^{\varsigma} \setminus N_{\kappa 1}^{\varsigma} \\ min(\mu_{\Phi_1}(t_b^a)\mu_{\Phi_2}(t_b^a)) & if\ t_b^a \in N_{\kappa 1}^{\varsigma} \cap N_{\kappa 2}^{\varsigma} \end{array} \right\}$$

(iii)

$$\eta_{\Phi_4}(t_b^a) = \left\{ \begin{array}{ll} \eta_{\Phi_1}(t_b^a), & if\ t_b^a \in N_{\kappa 1}^{\varsigma} \setminus N_{\kappa 2}^{\varsigma} \\ \eta_{\Phi_2}(t_b^a), & if\ t_b^a \in N_{\kappa 2}^{\varsigma} \setminus N_{\kappa 1}^{\varsigma} \\ min(\eta_{\Phi_1}(t_b^a)\eta_{\Phi_2}(t_b^a)) & if\ t_b^a \in N_{\kappa 1}^{\varsigma} \cap N_{\kappa 2}^{\varsigma} \end{array} \right\}$$

(iv)

$$\nu_{\Phi_4}(t_b^a) = \left\{ \begin{array}{ll} \nu_{\Phi_1}(t_b^a), & if\ t_b^a \in N_{\kappa 1}^{\varsigma} \setminus N_{\kappa 2}^{\varsigma} \\ \nu_{\Phi_2}(t_b^a), & if\ t_b^a \in N_{\kappa 2}^{\varsigma} \setminus N_{\kappa 1}^{\varsigma} \\ max(\nu_{\Phi_1}(t_b^a)\nu_{\Phi_2}(t_b^a)) & if\ t_b^a \in N_{\kappa 1}^{\varsigma} \cap N_{\kappa 2}^{\varsigma} \end{array} \right\}$$

Definition 5.3. The restricted union of Two GPFHSS $\nabla_1 = (\ddot{\mathcal{H}}_1, N_{\kappa 1}^{\varsigma}, \Phi_1)$ and $\nabla_2 = (\ddot{\mathcal{H}}_2, N_{\kappa 2}^{\varsigma}, \Phi_2)$ over $\mathcal{U}$ is defined as
$\nabla_5 = (\ddot{\mathcal{H}}_8, N_{\kappa 8}^{\varsigma}, \Phi_6) = (\ddot{\mathcal{H}}_1, N_{\kappa 1}^{\varsigma}, \Phi_1) \cup_r (\ddot{\mathcal{H}}_2, N_{\kappa 2}^{\varsigma}, \Phi_2)$ such that

(i) $(\ddot{\mathcal{H}}_8, N_{\kappa 8}^{\varsigma}) = (\ddot{\mathcal{H}}_1, N_{\kappa 1}^{\varsigma}) \cup_r (\ddot{\mathcal{H}}_1, N_{\kappa 1}^{\varsigma})$

(ii) $\mu_{\Phi_8}(t_b^a) = max(\mu_{\Phi_1}(t_b^a), \mu_{\Phi_2}(t_b^a))$, $\eta_{\Phi_8}(t_b^a) = min(\eta_{\Phi_1}(t_b^a), \eta_{\Phi_2}(t_b^a))$ and $\nu_{\Phi_8}(t_b^a) = min(\nu_{\Phi_1}(t_b^a), \nu_{\Phi_2}(t_b^a))$

Definition 5.4. The restricted intersection of two GPFHSS $\nabla_1 = (\ddot{\mathcal{H}}_1, N_{\kappa 1}^{\varsigma}, \Phi_1)$ and $\nabla_2 = (\ddot{\mathcal{H}}_2, N_{\kappa 2}^{\varsigma}, \Phi_2)$ over $\mathcal{U}$ is defined as
$\nabla_6 = (\ddot{\mathcal{H}}_9, N_{\kappa 9}^{\varsigma}, \Phi_7) = (\ddot{\mathcal{H}}_1, N_{\kappa 1}^{\varsigma}, \Phi_1) \cap_r (\ddot{\mathcal{H}}_2, N_{\kappa 2}^{\varsigma}, \Phi_2)$ such that

(i) $(\ddot{\mathcal{H}}_9, N^{\varsigma}_{\kappa 9}) = (\ddot{\mathcal{H}}_1, N^{\varsigma}_{\kappa 1}) \cap_r (\ddot{\mathcal{H}}_1, N^{\varsigma}_{\kappa 1})$

(ii) $\mu_{\Phi_9}(t^a_b) = min(\mu_{\Phi_1}(t^a_b), \mu_{\Phi_2}(t^a_b))$, $\eta_{\Phi_9}(t^a_b) = min(\eta_{\Phi_1}(t^a_b), \eta_{\Phi_2}(t^a_b))$ and $\nu_{\Phi_9}(t^a_b) = max(\nu_{\Phi_1}(t^a_b), \nu_{\Phi_2}(t^a_b))$

Example 5.5. Suppose $\mathcal{U} = \{c^{\varsigma}_{\kappa 1}, c^{\varsigma}_{\kappa 2}, c^{\varsigma}_{\kappa 3}, c^{\varsigma}_{\kappa 4}\}$ be four hospital. . Let $E = \{e_1, e_2, e_3, e_4\}$ stand for organ transplant services, medical and surgical specialties and support services whose corresponding attribute values are $\{A_1, A_2, A_3\}$ respectively. let $A_1 = \{b_{11}=$ liver transplant,$b_{12} =$ kidney transplant,$b_{13}=$ corneal transplant,$b_{14} = \{b_{11}, b_{12}, b_{13}\}\}$
$A_2 = \{b_{21} =$ medical and surgical clinics, $b_{22} =$ emergency services, $b_{23} =$ diagnostic services, $b_{24} = \{b_{21}, b_{22}, b_{23}\}\}$
$A_3 = \{b_{31} =$ Pharmecy $\}$
then

$$\tilde{L}^{\varsigma}_{\kappa} = A_1 \times A_2 \times A_3$$

$$\tilde{L}^{\varsigma}_{\kappa} = \left\{ \begin{array}{ll} t^a_{b1} = (b_{11}, b_{21}, b_{31},), & t^a_{b2} = (b_{14}, b_{24}, b_{31}), \\ t^a_{b3} = (b_{12}, b_{22}), & t^a_{b4} = (b_{12}, b_{31}), \end{array} \right\}$$

$$(\ddot{\mathcal{H}}_1, \tilde{L}^{\varsigma}_{\kappa}) = \left\{ \begin{array}{l} \ddot{\mathcal{H}}(t^a_{b1}) = \{\langle 0.7, 0.1, 0.1\rangle/c^{\varsigma}_{\kappa 1}, \langle 0.3, 0.2, 0.4\rangle/c^{\varsigma}_{\kappa 2}, \langle 0.1, 0.5, 0.3\rangle/c^{\varsigma}_{\kappa 3}, \langle 0.4, 0.1, 0.3\rangle/c^{\varsigma}_{\kappa 4}\}, \\ \ddot{\mathcal{H}}(t^a_{b2}) = \{\langle 0.7, 0.1, 0.1\rangle/c^{\varsigma}_{\kappa 4}, \langle 0.2, 0.5, 0.2\rangle/c^{\varsigma}_{\kappa 1}, \langle 0.6, 0.1, 0.2\rangle/c^{\varsigma}_{\kappa 2}, \langle 0.4, 0.1, 0.3\rangle/c^{\varsigma}_{\kappa 3}\}, \\ \ddot{\mathcal{H}}(t^a_{b3}) = \{\langle 0.2, 0.3, 0.5\rangle/c^{\varsigma}_{\kappa 1}, \langle 0.2, 0.1, 0.7\rangle/c^{\varsigma}_{\kappa 4}, \langle 0.1, 0.1, 0.6\rangle/c^{\varsigma}_{\kappa 2}, \langle 0.8, 0.1, 0.1\rangle/c^{\varsigma}_{\kappa 3}\}, \\ \ddot{\mathcal{H}}(t^a_{b4}) = \{\langle 0.3, 0.2, 0.4\rangle/c^{\varsigma}_{\kappa 3}, \langle 0.5, 0.1, 0.3\rangle/c^{\varsigma}_{\kappa 2}, \langle 0.1, 0.5, 0.3\rangle/c^{\varsigma}_{\kappa 4}, \langle 0.2, 0.1, 0.7\rangle/c^{\varsigma}_{\kappa 1}\}, \end{array} \right\}$$

In addition, Φ_1 is the PPFS which is given by
$\Phi_1 = \{\langle 0.5, 0.1, 0.2\rangle$, $\langle 0.8, 0.2, 0.0\rangle$, $\langle 0.5, 0.2, 0.2\rangle$, $\langle 0.9, 0.0, 0.0\rangle\}$ Tabular representation of $\nabla_1 = (\ddot{\mathcal{H}}_1, N^{\varsigma}_{\kappa 1}, \Phi_1)$ is given in Table

Table 3. $\nabla_1 = (\ddot{\mathcal{H}}_1, N^{\varsigma}_{\kappa 1}, \Phi_1)$

$\mathcal{U}$	t^a_{b1}	t^a_{b2}	t^a_{b3}	t^a_{b4}
$c^{\varsigma}_{\kappa 1}$	$\langle 0.7, 0.1, 0.1\rangle$	$\langle 0.2, 0.5, 0.2\rangle$	$\langle 0.2, 0.3, 0.5\rangle$	$\langle 0.2, 0.1, 0.7\rangle$
$c^{\varsigma}_{\kappa 2}$	$\langle 0.3, 0.2, 0.4\rangle$	$\langle 0.6, 0.1, 0.2\rangle$	$\langle 0.1, 0.1, 0.6\rangle$	$\langle 0.5, 0.1, 0.3\rangle$
$c^{\varsigma}_{\kappa 3}$	$\langle 0.1, 0.5, 0.3\rangle$	$\langle 0.4, 0.1, 0.3\rangle$	$\langle 0.8, 0.1, 0.1\rangle$	$\langle 0.3, 0.2, 0.4\rangle$
$c^{\varsigma}_{\kappa 4}$	$\langle 0.4, 0.1, 0.3\rangle$	$\langle 0.7, 0.1, 0.1\rangle$	$\langle 0.2, 0.1, 0.7\rangle$	$\langle 0.1, 0.5, 0.3\rangle$
Φ_1	$\langle 0.5, 0.1, 0.2\rangle$	$\langle 0.8, 0.2, 0.0\rangle$	$\langle 0.5, 0.2, 0.2\rangle$	$\langle 0.9, 0.0, 0.0\rangle$

$$(\ddot{\mathcal{H}}_2, \tilde{L}^{\varsigma}_{\kappa}) = \left\{ \begin{array}{l} \ddot{\mathcal{H}}(t^a_{b1}) = \{\langle 0.6, 0.2, 0.1\rangle/c^{\varsigma}_{\kappa 1}, \langle 0.3, 0.3, 0.4\rangle/c^{\varsigma}_{\kappa 2}, \langle 0.1, 0.2, 0.4\rangle/c^{\varsigma}_{\kappa 3}, \langle 0.5, 0.1, 0.3\rangle/c^{\varsigma}_{\kappa 4}\}, \\ \ddot{\mathcal{H}}(t^a_{b2}) = \{\langle 0.7, 0.0, 0.2\rangle/c^{\varsigma}_{\kappa 4}, \langle 0.2, 0.1, 0.3\rangle/c^{\varsigma}_{\kappa 1}, \langle 0.8, 0.1, 0.1\rangle/c^{\varsigma}_{\kappa 2}, \langle 0.6, 0.1, 0.3\rangle/c^{\varsigma}_{\kappa 3}\}, \\ \ddot{\mathcal{H}}(t^a_{b3}) = \{\langle 0.5, 0.1, 0.2\rangle/c^{\varsigma}_{\kappa 1}, \langle 0.2, 0.1, 0.1\rangle/c^{\varsigma}_{\kappa 4}, \langle 0.4, 0.1, 0.1\rangle/c^{\varsigma}_{\kappa 2}, \langle 0.9, 0.1, 0.0\rangle/c^{\varsigma}_{\kappa 3}\}, \\ \ddot{\mathcal{H}}(t^a_{b4}) = \{\langle 0.7, 0.2, 0.1\rangle/c^{\varsigma}_{\kappa 3}, \langle 0.6, 0.1, 0.3\rangle/c^{\varsigma}_{\kappa 2}, \langle 0.5, 0.3, 0.2\rangle/c^{\varsigma}_{\kappa 4}, \langle 0.2, 0.1, 0.6\rangle/c^{\varsigma}_{\kappa 1}\}, \end{array} \right\}$$

In addition, Φ_2 is the PPFS which is given by
$\Phi_2 = \{\langle 0.7, 0.1, 0.2\rangle, \langle 0.5, 0.2, 0.1\rangle, \langle 0.6, 0.2, 0.2\rangle, \langle 0.8, 0.1, 0.0\rangle\}$ Whose tabular representation is given in Table 4

TABLE 4. $\nabla_2 = (\ddot{\mathcal{H}}_2, N^{\varsigma}_{\kappa 2}, \Phi_2)$

$\mathcal{U}$	t^a_{b1}	t^a_{b2}	t^a_{b3}	t^a_{b4}
$c^{\varsigma}_{\kappa 1}$	$\langle 0.6, 0.2, 0.1\rangle$	$\langle 0.2, 0.1, 0.3\rangle$	$\langle 0.5, 0.1, 0.2\rangle$	$\langle 0.2, 0.1, 0.6\rangle$
$c^{\varsigma}_{\kappa 2}$	$\langle 0.3, 0.3, 0.4\rangle$	$\langle 0.8, 0.1, 0.1\rangle$	$\langle 0.4, 0.1, 0.1\rangle$	$\langle 0.6, 0.1, 0.3\rangle$
$c^{\varsigma}_{\kappa 3}$	$\langle 0.1, 0.2, 0.4\rangle$	$\langle 0.6, 0.1, 0.3\rangle$	$\langle 0.9, 0.1, 0.0\rangle$	$\langle 0.7, 0.2, 0.1\rangle$
$c^{\varsigma}_{\kappa 4}$	$\langle 0.5, 0.1, 0.3\rangle$	$\langle 0.7, 0.0, 0.2\rangle$	$\langle 0.2, 0.1, 0.1\rangle$	$\langle 0.5, 0.3, 0.2\rangle$
Φ_2	$\langle 0.7, 0.1, 0.2\rangle$	$\langle 0.5, 0.2, 0.1\rangle$	$\langle 0.6, 0.2, 0.2\rangle$	$\langle 0.8, 0.1, 0.0\rangle$

TABLE 5. Intersection of GPFHSSs

$\mathcal{U}$	t^a_{b1}	t^a_{b2}	t^a_{b3}	t^a_{b4}
$c^{\varsigma}_{\kappa 1}$	$\langle 0.7, 0.1, 0.1\rangle$	$\langle 0.2, 0.1, 0.2\rangle$	$\langle 0.5, 0.1, 0.2\rangle$	$\langle 0.2, 0.1, 0.6\rangle$
$c^{\varsigma}_{\kappa 2}$	$\langle 0.3, 0.2, 0.4\rangle$	$\langle 0.8, 0.1, 0.1\rangle$	$\langle 0.4, 0.1, 0.1\rangle$	$\langle 0.6, 0.1, 0.3\rangle$
$c^{\varsigma}_{\kappa 3}$	$\langle 0.1, 0.2, 0.3\rangle$	$\langle 0.6, 0.1, 0.3\rangle$	$\langle 0.9, 0.1, 0.0\rangle$	$\langle 0.7, 0.2, 0.1\rangle$
$c^{\varsigma}_{\kappa 4}$	$\langle 0.5, 0.1, 0.3\rangle$	$\langle 0.7, 0.0, 0.1\rangle$	$\langle 0.2, 0.1, 0.1\rangle$	$\langle 0.5, 0.3, 0.2\rangle$
Φ_3	$\langle 0.7, 0.1, 0.2\rangle$	$\langle 0.5, 0.2, 0.0\rangle$	$\langle 0.6, 0.2, 0.2\rangle$	$\langle 0.9, 0.0, 0.0\rangle$

In this section of paper we introduce, an expectation score function and an algorithm for interpreting MADM problems.

Definition 5.6. let $\ddot{\mathcal{H}}(t^a_b) = \langle \mu_{\ddot{\mathcal{H}}(t^a_b)}(x^{\varsigma}_{\kappa i}), \eta_{\ddot{\mathcal{H}}(t^a_b)}(x^{\varsigma}_{\kappa i}), \nu_{\ddot{\mathcal{H}}(t^a_b)}(x^{\varsigma}_{\kappa i})\rangle$ be PFHSV then the expectation score function is define as

$$\mathcal{S}(\ddot{\mathcal{H}}(t^a_b)) = \frac{2+\mu_{\ddot{\mathcal{H}}(t^a_b)}(x^{\varsigma}_{\kappa i})+\eta_{\ddot{\mathcal{H}}(t^a_b)}(x^{\varsigma}_{\kappa i})-\nu_{\ddot{\mathcal{H}}(t^a_b)}(x^{\varsigma}_{\kappa i})}{3} \qquad \mathcal{S}(\ddot{\mathcal{H}}(t^a_b)) \quad \in [0,1]$$

Definition 5.7. The Weight vector $\mathcal{W}(\ddot{\mathcal{H}}(t^a_b))$ is defined as

$$\mathcal{W}(\ddot{\mathcal{H}}(t^a_b)) = \frac{\mathcal{S}(\ddot{\mathcal{H}}(t^a_b))}{m}$$

where $m = \sum \mathcal{S}(\ddot{\mathcal{H}}(t^a_b)$

6. Algorithim

$\nabla_1 \leftarrow$ First GPFHSS

$\nabla_2 \leftarrow$ Second GPFHSS

$\nabla_1 \cap_e \nabla_2 \leftarrow$ Extended intersection of First and Second GPFHSS

$\mathcal{S}(\ddot{\mathcal{H}}(t^a_b)) \leftarrow$ Compute expected sore function

$\mathcal{W}(\ddot{\mathcal{H}}(t^a_b)) \leftarrow$ Compute weight vector

DAPFDV $\leftarrow$Compute Dombi aggregated picture fuzzy decision values

$\mathcal{SF} \leftarrow$ Compute Score Function

Rank $\leftarrow$ Maximum value of score function is greater.

7. Case Study: Construction Company Problem

A firm want to construct a labor colony for their worker, major qualities to look for completion of their project are qualities, services, skilled team and equipments. Consider $\mathcal{U} = \{c^{\varsigma}_{\kappa 1}, c^{\varsigma}_{\kappa 2}, c^{\varsigma}_{\kappa 3}, c^{\varsigma}_{\kappa 4}, c^{\varsigma}_{\kappa 5}\}$ be be five construction company.
Let $E = \{e_1, e_2, e_3, e_4\}$ stand for qualities, services, skilled team and equipments whose corresponding attribute values are $\{A_1, A_2, A_3, A_4\}$ respectively. let $A_1 = \{b_{11}$ = Credentials, b_{12} = Experience, b_{13} = Goodwill and Reputation, $b_{14} = \{b_{11}, b_{12}, b_{13}\}\}$
$A_2 = \{b_{21}$ = New construction, b_{22} = Repair, b_{23} = demolition, $b_{24} = \{b_{21}, b_{22}, b_{23}\}\}$
$A_3 = \{b_{31}$ = builders and architects, b_{31} = civil engineers $\}$
$A_4 = \{b_{41}$ = Modern equipments $\}$

$$\tilde{L}^{\varsigma}_{\kappa} = \left\{ \begin{array}{ll} t^a_{b1} = (b_{11}, b_{21}, b_{31}), & t^a_{b2} = (b_{14}, b_{24}, b_{31}, b_{41}), \\ t^a_{b3} = (b_{12}, b_{22}), & t^a_{b4} = (b_{12}, b_{31}), \\ t^a_{b5} = (b_{11}, b_{22}, b_{31}, b_{42}), & t^a_{b6} = (b_{11}, b_{23}), \end{array} \right\}$$

The CEO makes two groups of members of administration of firm to do the evaluation.. The set of attributes $N^{\varsigma}_{\kappa 1} = \{t^a_{b1}, t^a_{b2}, t^a_{b3}, t^a_{b4}\}$ observed by first group and second group monitoring attributes value $N^{\varsigma}_{\kappa 2} = \{t^a_{b2}, t^a_{b4}, t^a_{b5}, t^a_{b6}\}$. Two GPFHSS $(\ddot{\mathcal{H}}_1, N^{\varsigma}_{\kappa 1}, \Phi_3)$ and $(\ddot{\mathcal{H}}_2, N^{\varsigma}_{\kappa 2}, \Phi_4)$ are given in table.
Step: 01

Table 6. GPFHSS 1

$\mathcal{U}$	t^a_{b1}	t^a_{b2}	t^a_{b3}	t^a_{b4}
$c^{\varsigma}_{\kappa 1}$	$\langle 0.5, 0.1, 0.1\rangle$	$\langle 0.7, 0.1, 0.2\rangle$	$\langle 0.3, 0.1, 0.3\rangle$	$\langle 0.3, 0.1, 0.2\rangle$
$c^{\varsigma}_{\kappa 2}$	$\langle 0.3, 0.2, 0.4\rangle$	$\langle 0.6, 0.1, 0.1\rangle$	$\langle 0.5, 0.1, 0.2\rangle$	$\langle 0.8, 0.0, 0.2\rangle$
$c^{\varsigma}_{\kappa 3}$	$\langle 0.7, 0.1, 0.1\rangle$	$\langle 0.4, 0.1, 0.3\rangle$	$\langle 0.9, 0.1, 0.0\rangle$	$\langle 0.5, 0.2, 0.1\rangle$
$c^{\varsigma}_{\kappa 4}$	$\langle 0.6, 0.1, 0.3\rangle$	$\langle 0.8, 0.1, 0.1\rangle$	$\langle 0.4, 0.1, 0.1\rangle$	$\langle 0.4, 0.3, 0.3\rangle$
$c^{\varsigma}_{\kappa 5}$	$\langle 0.4, 0.1, 0.2\rangle$	$\langle 0.3, 0.1, 0.5\rangle$	$\langle 0.3, 0.1, 0.4\rangle$	$\langle 0.7, 0.1, 0.2\rangle$
Φ_3	$\langle 0.8, 0.0, 0.1\rangle$	$\langle 0.6, 0.1, 0.1\rangle$	$\langle 0.1, 0.2, 0.7\rangle$	$\langle 0.3, 0.1, 0.5\rangle$

Step: 02 Calculate intersection of GPFHSS 1 and GPFHSS 2.

Table 7. GPFHSS 2

$\mathcal{U}$	t^a_{b2}	t^a_{b4}	t^a_{b5}	t^a_{b6}
$c^s_{\kappa 1}$	$\langle 0.8, 0.1, 0.1\rangle$	$\langle 0.5, 0.3, 0.2\rangle$	$\langle 0.4, 0.1, 0.2\rangle$	$\langle 0.3, 0.1, 0.3\rangle$
$c^s_{\kappa 2}$	$\langle 0.6, 0.1, 0.2\rangle$	$\langle 0.5, 0.0, 0.3\rangle$	$\langle 0.2, 0.1, 0.3\rangle$	$\langle 0.7, 0.1, 0.2\rangle$
$c^s_{\kappa 3}$	$\langle 0.4, 0.1, 0.3\rangle$	$\langle 0.8, 0.1, 0.0\rangle$	$\langle 0.9, 0.0, 0.1\rangle$	$\langle 0.5, 0.1, 0.1\rangle$
$c^s_{\kappa 4}$	$\langle 0.3, 0.3, 0.1\rangle$	$\langle 0.4, 0.2, 0.0\rangle$	$\langle 0.5, 0.2, 0.1\rangle$	$\langle 0.6, 0.2, 0.1\rangle$
$c^s_{\kappa 5}$	$\langle 0.5, 0.1, 0.3\rangle$	$\langle 0.7, 0.0, 0.1\rangle$	$\langle 0.2, 0.1, 0.1\rangle$	$\langle 0.5, 0.3, 0.2\rangle$
Φ_4	$\langle 0.8, 0.1, 0.1\rangle$	$\langle 0.1, 0.2, 0.7\rangle$	$\langle 0.6, 0.2, 0.2\rangle$	$\langle 0.3, 0.1, 0.5\rangle$

Table 8. Intersection of GPFHSS 1 and GPFHSS 2

$\mathcal{U}$	t^a_{b1}	t^a_{b2}	t^a_{b3}	t^a_{b4}	t^a_{b5}	t^a_{b6}
$c^s_{\kappa 1}$	$\langle 0.5, 0.1, 0.1\rangle$	$\langle 0.7, 0.1, 0.2\rangle$	$\langle 0.3, 0.1, 0.3\rangle$	$\langle 0.3, 0.1, 0.2\rangle$	$\langle 0.4, 0.1, 0.2\rangle$	$\langle 0.3, 0.1, 0.3\rangle$
$c^s_{\kappa 2}$	$\langle 0.3, 0.2, 0.4\rangle$	$\langle 0.6, 0.1, 0.2\rangle$	$\langle 0.5, 0.1, 0.2\rangle$	$\langle 0.5, 0.0, 0.3\rangle$	$\langle 0.2, 0.1, 0.3\rangle$	$\langle 0.7, 0.1, 0.2\rangle$
$c^s_{\kappa 3}$	$\langle 0.7, 0.1, 0.1\rangle$	$\langle 0.4, 0.1, 0.3\rangle$	$\langle 0.9, 0.1, 0.0\rangle$	$\langle 0.5, 0.1, 0.1\rangle$	$\langle 0.9, 0.0, 0.1\rangle$	$\langle 0.5, 0.1, 0.1\rangle$
$c^s_{\kappa 4}$	$\langle 0.6, 0.1, 0.3\rangle$	$\langle 0.3, 0.1, 0.1\rangle$	$\langle 0.4, 0.1, 0.1\rangle$	$\langle 0.4, 0.2, 0.3\rangle$	$\langle 0.5, 0.2, 0.1\rangle$	$\langle 0.6, 0.2, 0.1\rangle$
$c^s_{\kappa 5}$	$\langle 0.4, 0.1, 0.2\rangle$	$\langle 0.3, 0.1, 0.5\rangle$	$\langle 0.3, 0.1, 0.4\rangle$	$\langle 0.7, 0.0, 0.2\rangle$	$\langle 0.2, 0.1, 0.1\rangle$	$\langle 0.5, 0.3, 0.2\rangle$
Φ_4	$\langle 0.8, 0.0, 0.1\rangle$	$\langle 0.1, 0.2, 0.7\rangle$	$\langle 0.1, 0.2, 0.7\rangle$	$\langle 0.1, 0.1, 0.7\rangle$	$\langle 0.6, 0.2, 0.2\rangle$	$\langle 0.3, 0.1, 0.5\rangle$

Step:03 Calculate value of expectation score function using 5.6 and weight vector by 5.7 shown in Tab 9

Table 9. Expectation Score Function and Weight vector

$\mathcal{U}$	t^a_{b1}	t^a_{b2}	t^a_{b3}	t^a_{b4}	t^a_{b5}	t^a_{b6}
Φ_4	$\langle 0.8, 0.0, 0.1\rangle$	$\langle 0.1, 0.2, 0.7\rangle$	$\langle 0.1, 0.2, 0.7\rangle$	$\langle 0.1, 0.1, 0.7\rangle$	$\langle 0.6, 0.2, 0.2\rangle$	$\langle 0.3, 0.1, 0.5\rangle$
$\mathcal{S}(\ddot{\mathcal{H}}(t^a_b))$	0.9333	0.5333	0.5333	0.5	0.8666	0.6333
$\mathcal{W}(\ddot{\mathcal{H}}(t^a_b))$	0.2333	0.1333	0.1333	0.125	0.217	0.158

Step: 04 Calculate Dombi aggregated picture fuzzy decision values (DAPFDVs) for k = 1 by 2.8 and score function using 2.7. The DAPFDVs can be calculated as:

From above table

$c^s_{\kappa 2} \prec c^s_{\kappa 5} \prec c^s_{\kappa 1} \prec c^s_{\kappa 4} \prec c^s_{\kappa 3}$

$c^s_{\kappa 3}$ is suitable construction company for labor colony construction.

8. Comparison

The algorithm proposed by Jaber et al. [25] face challenges to deal with the MADM problem where attributes of the alternates have their corresponding sub-attributes. To overcome such

Table 10. DAPFDVs and Score Function

$\mathcal{U}$	$DAPFDVs$	$\mathcal{SF}$
$c^{\varsigma}_{\kappa 1}$	$\langle 0.4645, 0.1000, 0.1760\rangle$	$\langle 0.2885\rangle$
$c^{\varsigma}_{\kappa 2}$	$\langle 0.4952, 0.0000, 0.2599\rangle$	$\langle 0.2353\rangle$
$c^{\varsigma}_{\kappa 3}$	$\langle 0.8027, 0.0000, 0.0000\rangle$	$\langle 0.8027\rangle$
$c^{\varsigma}_{\kappa 4}$	$\langle 0.5081, 0.1333, 0.1313\rangle$	$\langle 0.3768\rangle$
$c^{\varsigma}_{\kappa 5}$	$\langle 0.4362, 0.0000, 0.1868\rangle$	$\langle 0.2494\rangle$

difficulties we developed an extension of GPFSS by changing the mapping to a multi-attribute mapping. With the help of GPFHSS, we define an algorithm, which plays an important role to study the picture fuzzy and hypersoft environment.

9. Conclusions

In this article, we introduce PFHSS and GPFHSS. We defined some operations of PFHSS and GPFHSS, also proved De Morgans laws for these operations. For the solution of MADM problems, we construct an algorithm by using the extended intersection of GPFHSS information and also we introduced a new expectation score function to find the value of the weight vector. With the help of the weight vector and new expectation score function, we calculate DAPFDVs and score function. Then we rank the construction companies which are given in the example of a case study of the construction of labor colonies by using ascending values of the score function. After comparison with prior proposed techniques and arise it to be more generalized and productive to deal with multi-attribute classifications.

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Study on Neutrosophic Priority Discipline Queuing Model

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Abstract. In this paper, the priority disciplined queuing models are investigated under neutrosophic environment. It develops and optimizes a model with non-preemptive priorities system, denoted by $NM/NM/1$. It is a queuing model where the arrivals follow a Poisson process, service times are exponentially distributed and there is only one server whose arrival rate and service rate are represented in terms of single valued trapezoidal Neutrosophic number (SVTNN). Using (α, β, γ)-cut approach and Zadehs extension principle, the Neutrosophic queuing model is reduced to a crisp model and results are discussed. An illustrative example is provided to understand the analytical procedure developed in this paper.

Keywords: Neutrosophic set; single value trapezoidal Neutrosophic number; Neutrosophic Markov chain; priority queue.

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1. Introduction

Basic queueing systems involve organized queues where the arrival rate of customers is in an order and waiting discipline is ensured. But in real life situations most of the queuing models require priority discipline as most urgent work has to be given preference. Priority queueing models are useful in a variety of different applications. In priority queues customers are served based on their service priorities. The high-priority customers with high urgency are served first and the lower priority customers are served with less urgency. In communication engineering, priority queues are used to study networks with differentiated levels of quality of service. Steady state distribution of single server priority queue was developed by Miller [1]. Prade [2] dealt with fuzzy service time and fuzzy service rule in a queuing problem with application. Li et al. [3] investigated two fuzzy queues denoted by $M/F/1$ and $FM/FM/1$ whose interarrival time and service rate are fuzzified. Negi et al. [4] discussed analytical and simulation results of fuzzy and probability approaches of traditional queuing models. Maria et al. [5] developed two fuzzy queueing models with priority-discipline both with non-pre-emptive priorities system and pre-emptive priorities system. Varadharajan et al. [6] analysed fuzzy priority discipline queue models using a parametric programming approach.

Kalpana et al. [7] investigated the performance measures for non-pre-emptive priority fuzzy queues. Usha et al. [8] made an interpretation of a non-pre-emptive priority queueing system in fuzzy environment with asymmetrical service rates. Aarthi et al. [9] analyzed the performance of a non-pre-emptive intuitionistic fuzzy queuing model. Khudr Al-Kridi et al. [10] discussed the performance measures of $FM/FM/1$ queueing model where both arrival and departure rates are fuzzy numbers Kumuthavalli et al. [11] focused on developing a neutrosophic probability for solving queue operation in the real standard domain.

Fariborz Jolai et al. [12] presented a new formulation for the problem of fuzzy priority assignment and buffer control. Mohamed Bisher Zeina [13] provided Neutrosophic Littles Formulas which is a main tool in queueing systems problems under neutrosophic environment. Also he [14, 15] discussed about Erlang service queueing model under neutrosophic environment. Heba Rashad et al. [16] discussed the performance measures of $NM/NM/1, NM/NM/s$, and $NM/NM/1/b$ queueing models. Zhivko Tomov et al. [17] proposed generalized net models of different queueing disciplines in queueing systems. Buckley [18, 19] dealt fuzzy queue model using possibility theory. Many researchers [20, 21], have shown light over Intuitionistic fuzzy queueing models.

Florentin Smarandache [22] introduced Neutrosophic set as an generalization of Intutionistic fuzzy set developed by Atanassov [23] which is a powerful tool to deal with ambiguity compared to fuzzy set proposed by Zadeh [24] as it considers membership, indeterminacy and non-membership degree of an object simultaneously. Also Florentin Smarandache [25, 26] has explored various concepts such as Neutrosophic measure, Neutrosophic logic, Neutrosophic probability etc.,. Wang et al. [27] discussed about operations and properties of single valued Neutrosophic set (SVNS). Later applications involving SVNS are considered by many researchers [28, 29]. This paper aims at investigating a single server queuing model with priority discipline involving SVNS. A comparison table 1 of existing queueing model under uncertainty is discussed below.

Table 1. Comparison with the existing queueing model

Author	Queueing model	Uncertainty used	Methodology
Prade, H. M (1980)	General queuing model	Fuzzy sets	Zadehs extension principle
Li, R. J. et al. (1989)	General queuing model	Fuzzy sets	Zadehs extension principle
Negi, D. S. et al. (1992)	General queuing model	Fuzzy sets	-cut approach
Khudr Al-Kridi et al. (2018)	General queuing model	Fuzzy sets	Zadehs extension principle
Zhivko Tomov (2019)	General queuing model	Intuitionistic fuzzy set	Generalized Net models
Kumuthavalli et al. (2017)	General queuing model	Neutrosophic sets	Zadehs Exclusion Principle
Mohamed Bisher Zeina (2020)	General queuing model	Neutrosophic sets	Neutrosophic Littles Formulas
Mohamed Bisher Zeina (2020)	Erlang service queueing model	Neutrosophic sets	Neutrosophic statistical interval method
Maria Jose Pardoa et al. (2007)	Priority queues	Fuzzy sets	Zadehs extension principle
Varadharajan et al. (2018)	Priority queues	Fuzzy sets	α-cut approach
Kalpana et al. (2018)	Priority queues	Fuzzy sets	LR method
Usha Prameela et al.(2021)	Priority queue	Fuzzy sets	α-cut approach
Aarthi et al. (2022)	Priority queue	Intuitionistic Fuzzy sets	Ranking method
Fariborz Jolai et al. (2016)	Multi objective priority queue	Fuzzy sets	Fuzzy Data Envelopment Analysis
Heba Rashad et al. (2021)	General queueing model	Neutrosophic sets	Neutrosophic Littles Formulas
Proposed	Priority model	Neutrosophic sets	(α, β, γ)-cut

In this paper, we explored the neutrosophic queueing model and its application. To the best of the authors knowledge, none of the previous works has addressed the neutrosophic decision-making regarding prioritization and queue selection of service-needing people in disaster aftermath. The main contributions of the study include:

(1) The innovative concept of priority queuing model under neutrosophic sets is introduced.

(2) Formulation of $NM/NM/1$ queue with priority model is proposed.

(3) Also, a numerical example is discussed to show the effectiveness of the proposed queueing model.

(4) To make the decision maker understand the solution graphical representation are provided.

In Section 2, we discusses the Neutrosophic preliminaries. Section 3 briefly discussed the neutrosophic queueing model. In section 4, numerical illustration are solved for showing performance measures of neutrosophic in queueing model and Section 5 presents the conclusion, and future work.

2. Preliminaries

Definition 2.1. [26] A neutrosophic set N is given as

$$N = \{(s, \mathcal{T}_{\mathcal{A}}(s), \mathcal{I}_{\mathcal{A}}(s), \mathcal{F}_{\mathcal{A}}(s))/s \in s\}$$

where $\mathcal{T}_{\mathcal{A}}(s), \mathcal{I}_{\mathcal{A}}(s), \mathcal{F}_{\mathcal{A}}(s) : s \rightarrow]0^-, 1^+[$ are the degree of truth, ondeterminancy and falsity such that $0^- \leq \sup \mathcal{T}_{\mathcal{A}}(s) + \sup \mathcal{I}_{\mathcal{A}}(s) + \sup \mathcal{F}_{\mathcal{A}}(s) \leq 3^+$.

Definition 2.2. [26] A single valued neutrosophic set (SVNS) N in s is stated as

$$N = \{(s, \mathcal{T}_{\mathcal{A}}(s), \mathcal{I}_{\mathcal{A}}(s), \mathcal{F}_{\mathcal{A}}(s))/s \in s\}$$

where $\mathcal{T}_{\mathcal{A}}(s), \mathcal{I}_{\mathcal{A}}(s), \mathcal{F}_{\mathcal{A}}(s) \in [0, 1]$ and $0 \leq \sup \mathcal{T}_{\mathcal{A}}(s) + \sup \mathcal{I}_{\mathcal{A}}(s) + \sup \mathcal{F}_{\mathcal{A}}(s) \leq 3$.

Definition 2.3. [25] Let $(\nu\Omega, NF, NP)$ be a neutrosophic probability space, where $\nu\Omega$ is a neutrosophic sample space, NF is a neutrosophic event space, and NP is a neutrosophic probability measure.The following neutrosophic probability axioms are as follows
(i) The neutrosophic probability of an event A
$NP(A) = \left(ch(A), ch(indeterm_A), ch(\bar{A})\right)$,
where $ch(A) \geq 0, ch(indeterm_A) \geq 0, ch(\bar{A}) \geq 0$, for any $A \in NF$; with the notations that $indeterm(A)$ means indeterminacy related to event A and $\bar{A}$ is the complement event of A (the $antiA$ event).
(ii) The neutrosophic probability of the sample space is between -0 and 3^+.
$NP(\nu\Omega) = \left(\sum\limits_{x\in\nu\Omega} ch(x), ch(indeterm_{\nu\Omega}), ch(anti\nu\Omega)\right)$,
where $-0 \leq \sum\limits_{x\in\nu\Omega} ch(x), ch(indeterm_{\nu\Omega}), ch(anti\nu\Omega) \leq 3^+$,
with the notation $indeterm_{\nu\Omega}$ means total indeterminacy that may occur in the neutrosophic sample space. For the classical complete (normalized) sample space, $ch(anti\nu\Omega) = 0$, but for incomplete sample space $ch(anti\nu\Omega) > 0$.
(iii) The neutrosophic σ-additivity is defined as

$NP\left(A_1 \cup A_2 \cup \ldots\right) = \left(\sum_{j=1}^{\infty} ch(A_j), ch(indeterm_{A_1 \cup A_2 \cup \ldots}), ch(\overline{A_1 \cup A_2 \cup \ldots})\right),$
where $A_1, A_2, \ldots$ is a countable sequence of disjoint neutrosophic events.

Definition 2.4. [25] A random variable (r.v) which have an indeterminate outcome is said to be neutrosophic r.v.

A neutrosophic stochastic process is a collection of neutrosophic r.v which represents the evolution over time of some neutrosophic random values.

Definition 2.5. [25] A neutrosophic stochastic process $\{X(n) : n \in \mathbb{N}\}$ is said to be a neutrosophic Markov chain if it satisfies the Markov property:
$P\left(X_{n+1} = j/X_n = i, X_{n-1} = k, \ldots X_0 = m\right) = P\left(X_{n+1} = j/X_n = i\right)$
where i, j, k establish the state space S of the process.
Here $\widetilde{P}_{ij} = P\left(X_{n+1} = j/X_n = i\right)$ are called the neutrosophic probabilities of moving from state i to state j in one step. Hence $\widetilde{P}_{ij} = \left(\mathcal{T}_{\widetilde{P}_{ij}}, \mathcal{I}_{\widetilde{P}_{ij}}, \mathcal{F}_{\widetilde{P}_{ij}}\right)$, where $\mathcal{T}_{\widetilde{P}_{ij}} (\mathcal{I}_{\widetilde{P}_{ij}}, \mathcal{F}_{\widetilde{P}_{ij}})$ is the truth (indeterminate, falsity) membership of the transition from state i to state j. The matrix $P = \widetilde{P}_{ij}$ is called the neutrosophic transition probability matrix.

Definition 2.6. [30] A single valued trapezoidal neutrosophic number (SVTNN) $\mathcal{A}$ is defined as follows

$$\mathcal{T}_{\mathcal{A}}(s) = \begin{cases} \dfrac{s^{\mathcal{T}} - t_1^{\mathcal{T}}}{t_2^{\mathcal{T}} - t_1^{\mathcal{T}}} & \text{for } t_1^{\mathcal{T}} \leq s^{\mathcal{T}} \leq t_2^{\mathcal{T}} \\ 1 & \text{for } t_2^{\mathcal{T}} \leq s^{\mathcal{T}} \leq t_3^{\mathcal{T}} \\ \dfrac{t_4^{\mathcal{T}} - s^{\mathcal{T}}}{t_4^{\mathcal{T}} - t_3^{\mathcal{T}}} & \text{for } t_3^{\mathcal{T}} \leq s^{\mathcal{T}} \leq t_4^{\mathcal{T}} \\ 0 & \text{otherwise} \end{cases}$$

where $t_1^{\mathcal{T}} \leq t_2^{\mathcal{T}} \leq t_3^{\mathcal{T}} \leq t_4^{\mathcal{T}}$.

$$\mathcal{I}_{\mathcal{A}}(s) = \begin{cases} \dfrac{t_2^{\mathcal{I}} - s^{\mathcal{I}}}{t_2^{\mathcal{I}} - t_1^{\mathcal{I}}} & \text{for } t_1^{\mathcal{I}} \leq s^{\mathcal{I}} \leq t_2^{\mathcal{I}} \\ 1 & \text{for } t_2^{\mathcal{I}} \leq s^{\mathcal{I}} \leq t_3^{\mathcal{I}} \\ \dfrac{t_4^{\mathcal{I}} - s^{\mathcal{I}}}{t_4^{\mathcal{I}} - t_3^{\mathcal{I}}} & \text{for } t_3^{\mathcal{I}} \leq s^{\mathcal{I}} \leq t_4^{\mathcal{I}} \\ 1 & \text{otherwise} \end{cases}$$

where $t_1^{\mathcal{I}} \leq t_2^{\mathcal{I}} \leq t_3^{\mathcal{I}} \leq t_4^{\mathcal{I}}$.

$$\mathcal{F}_{\mathcal{A}}(s) = \begin{cases} \dfrac{t_2^{\mathcal{F}} - s^{\mathcal{F}}}{t_2^{\mathcal{F}} - t_1^{\mathcal{F}}} & \text{for } t_1^{\mathcal{F}} \leq s^{\mathcal{F}} \leq t_2^{\mathcal{F}} \\ 1 & \text{for } t_2^{\mathcal{F}} \leq s^{\mathcal{F}} \leq t_3^{\mathcal{F}} \\ \dfrac{t_4^{\mathcal{F}} - s^{\mathcal{F}}}{t_4^{\mathcal{F}} - t_3^{\mathcal{F}}} & \text{for } t_3^{\mathcal{F}} \leq s^{\mathcal{F}} \leq t_4^{\mathcal{F}} \\ 1 & \text{otherwise} \end{cases}$$

where $t_1^{\mathcal{F}} \leq t_2^{\mathcal{F}} \leq t_3^{\mathcal{F}} \leq t_4^{\mathcal{F}}$.

Definition 2.7. [30]
(α, β, γ)-cut of a TSVNN is defined as follows:
$\mathcal{A}_{\alpha,\beta,\gamma} = [A_1(\alpha), A_2(\alpha)]\,; \left[A_1^{'}(\beta), A_2^{'}(\beta)\right]; \left[A_1^{''}(\gamma), A_2^{''}(\gamma)\right], 0 \leq \alpha+\beta+\gamma \leq 3$, where

$$[A_1(\alpha), A_2(\alpha)] = \left[\left(t_1^{\mathcal{T}} + \alpha(t_2^{\mathcal{T}} - t_1^{\mathcal{T}})\right), \left(t_4^{\mathcal{T}} - \alpha(t_4^{\mathcal{T}} - t_3^{\mathcal{T}})\right)\right],$$
$$\left[A_1^{'}(\beta), A_2^{'}(\beta)\right] = \left[\left(t_2^{\mathcal{I}} - \beta(t_2^{\mathcal{I}} - t_1^{\mathcal{I}})\right), \left(t_3^{\mathcal{I}} + \beta(t_4^{\mathcal{I}} - t_3^{\mathcal{I}})\right)\right],$$
$$\left[A_1^{''}(\gamma), A_2^{''}(\gamma)\right] = \left[\left(t_2^{\mathcal{F}} - \gamma(t_2^{\mathcal{F}} - t_1^{\mathcal{F}})\right), \left(t_3^{\mathcal{F}} + \gamma(t_4^{\mathcal{F}} - t_3^{\mathcal{F}})\right)\right].$$

Definition 2.8. [32] Let $[r_1, r_2]$ and $[r_3, r_4]$ be two closed and bounded real intervals. If $*$ denotes addition, substraction, multiplication or division, then $[r_1, r_2] * [r_3, r_4] = [\alpha, \beta]$.
For division, it is assumed that $0 \notin [r_3, r_4]$. With basic operations, is developed as follows :

i . $[r_1, r_2] + [r_3, r_4] = [r_1 + r_3, r_2 + r_4]$
ii . $[r_1, r_2] - [r_3, r_4] = [r_1 - r_4, r_2 - r_3]$
iii . $[r_1, r_2] . [r_3, r_4] = [min\{r_1r_3, r_1, r_4, r_2r_3, r_2r_4\}, max\{r_1r_3, r_1, r_4, r_2r_3, r_2r_4\}]$
iv . $\dfrac{[r_1, r_2]}{[r_3, r_4]} = \left[min\left\{\dfrac{r_1}{r_3}, \dfrac{r_1}{r_4}, \dfrac{r_2}{r_3}, \dfrac{r_2}{r_4}\right\}, max\left\{\dfrac{r_1}{r_3}, \dfrac{r_1}{r_4}, \dfrac{r_2}{r_3}, \dfrac{r_2}{r_4}\right\}\right]$

3. The Neutrosophic Queueing Model

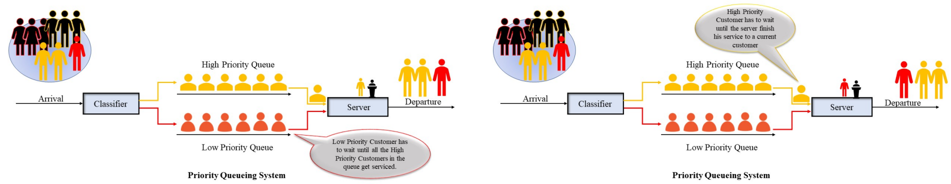
In this section, we analyze a single server queue with priority in neutrosophic environment.

3.1. Classical $M/M/1$ queue with priority queue

We consider a single server queue with priority. Assume that there are two arrival stream of customers called higher priority and low priority customers and they follow different Poisson process with parameters λ_1 and λ_2 respectively. A single server provides service to these customers and the service time follows exponential distribution with rate μ. The higher priority customers have the right to be served ahead of the others without preemption. The system capacity is infinite and within the priority group the first come first served rule is applied. Some system performance are

- Average queue length of higher priority: $L_{q_1} = \dfrac{\rho.\lambda_1}{\mu - \lambda_1}$
- Average queue length of low priority: $L_{q_2} = \dfrac{\rho.\lambda_2}{(1-\rho)(\mu - \lambda_1)}$
- Average waiting time of higher priority queue: $W_{q_1} = \dfrac{\rho}{\mu - \lambda_1}$
- Average waiting time of low priority queue: $W_{q_2} = \dfrac{\rho}{(\mu - \lambda)(\mu - \lambda_1)}$

where $\lambda = \lambda_1 + \lambda_2$ and traffic intensity $(\rho) = \dfrac{\lambda}{\mu}$.
An $M/M/1$ priority queue with infinite capacity as depicted in figure 1.



(A) Higher priority customers in service (B) Low priority customers in service

FIGURE 1. $M/M/1$ queue with priority queue

3.2. Formulation of $NM/NM/1$ queue with priority model

Consider a single server $NM/NM/1$ queueing system with priority. The neutrosophic interarrival times $\widetilde{A}_i, i = 1, 2$ of units in the first and second priority, neutrosophic service time $\widetilde{S}$ are approximately known and are represented by the follows

$$\widetilde{A_i} = \left\{ \left(a, \mathcal{T}_{\widetilde{A}_i}(a), \mathcal{I}_{\widetilde{A}_i}(a), \mathcal{F}_{\widetilde{A}_i}(a) \right) / a \in X \right\};\; i = 1, 2$$

$$\widetilde{S} = \left\{ \left(s, \mathcal{T}_{\widetilde{S}}(s), \mathcal{I}_{\widetilde{S}}(s), \mathcal{F}_{\widetilde{S}}(s) \right) / s \in Y \right\}$$

where X and Y are crisp universal sets of the neutrosophic interarrival times and neutrosophic service time and $\mu_{\widetilde{A}_i}(a); i = 1, 2, \mathcal{T}_{\widetilde{S}}(s)$ are the corresponding membership functions. The (α, β, γ)-cut of $\widetilde{A}_i; i = 1, 2$ and $\widetilde{S}$ are

$$A_i(\alpha, \beta, \gamma) = \left\{ a \in X / \mathcal{T}_{\widetilde{A}_i}(a) \geq \alpha, \mathcal{I}_{\widetilde{A}_i}(a) \leq \beta, \mathcal{F}_{\widetilde{A}_i}(a) \leq \gamma \right\}; i = 1, 2$$

$$S(\alpha, \beta, \gamma) = \left\{ s \in Y / \mathcal{T}_{\widetilde{S}}(s) \geq \alpha, \mathcal{I}_{\widetilde{S}}(s) \leq \beta, \mathcal{F}_{\widetilde{S}}(s) \leq \beta \right\}$$

where the $A_i(\alpha, \beta, \gamma)$ and $S(\alpha, \beta, \gamma)$ are the crisp subsets of X and Y respectively. Using (α, β, γ)-cuts, the Neutrosophic interarrival times and Neutrosophic service time can be represented by different levels of confidence intervals. Consequently, a Neutrosophic queue can be reduced to a family of crisp queues with different (α, β, γ)-cuts $\{A_i(\alpha, \beta, \gamma) : 0 < \alpha \leq 1, 0 \leq \beta < 1, 0 \leq \gamma < 1\}$ and
$\{S(\alpha, \beta, \gamma) : 0 < \alpha \leq 1, 0 \leq \beta < 1, 0 \leq \gamma < 1\}$.

In this paper, we proposed queueing model with both interarrival times $\widetilde{A}_i, i = 1, 2$ and service time $\widetilde{S}$ are represented as SVTNN. Denote confidence intervals of $\widetilde{A}_i$ and $\widetilde{S}$ by $[l_{\widetilde{A}_i(\alpha,\beta,\gamma)}, u_{\widetilde{A}_i(\alpha,\beta,\gamma)}]$ and $[l_{\widetilde{S}(\alpha,\beta,\gamma)}, u_{\widetilde{S}(\alpha,\beta,\gamma)}]$.
Let us denote the performance measure by $p(\widetilde{A_i}, \widetilde{S})$ and the truth membership function, the indeterminacy membership function and the falsity membership function of $p(\widetilde{A_i}, \widetilde{S})$ can be defined using Zadeh's extension principle [31, 32], as:

$$\mathcal{T}_{p(\widetilde{A_i}, \widetilde{S})}(z) = \sup \left\{ min_{a \in X, a' \in Y} (\mu_{\widetilde{A}_i(a)}, \mathcal{T}_{\widetilde{S}(a')}) : z = p(a, a') \right\}$$

$$\mathcal{I}_{p(\widetilde{A_i},\widetilde{S})}(z) = \inf\left\{min_{a\in X,a'\in Y}(\mu_{\widetilde{A_i}(a)},\mathcal{T}_{\widetilde{S}(a')}) : z = p(a,a')\right\}$$

and

$$\mathcal{F}_{p(\widetilde{A_i},\widetilde{S})}(z) = \inf\left\{min_{a\in X,a'\in Y}(\mu_{\widetilde{A_i}(a)},\mathcal{T}_{\widetilde{S}(a')}) : z = p(a,a')\right\}$$

We can find the lower and upper bounds of the (α,β,γ) cuts of $\widetilde{A_i},\widetilde{S}$ as follows:

$$l_{p(\alpha,\beta,\gamma)} = \min p(a,a') \text{ such that } l_{\widetilde{A}_i(\alpha,\beta,\gamma)} \leq a \leq u_{\widetilde{A}_i(\alpha,\beta,\gamma)}, l_{\widetilde{S}(\alpha,\beta,\gamma)} \leq a' \leq u_{\widetilde{S}(\alpha,\beta,\gamma)} \quad (1)$$

$$u_{p(\alpha,\beta,\gamma)} = \max p(a,a') \text{ such that } l_{\widetilde{A}_i(\alpha,\beta,\gamma)} \leq a \leq u_{\widetilde{A}_i(\alpha,\beta,\gamma)}, l_{\widetilde{S}(\alpha,\beta,\gamma)} \leq a' \leq u_{\widetilde{S}(\alpha,\beta,\gamma)} \quad (2)$$

provided $a \in \widetilde{A}_i(\alpha,\beta,\gamma)$ and $a' \in \widetilde{S}(\alpha,\beta,\gamma)$.

If both $l_{p(\alpha,\beta,\gamma)}$ and $u_{p(\alpha,\beta,\gamma)}$ are invertible with respect to (α,β,γ) then the left shape function $L_{\mathcal{T}}(z) = \left(l_{p(\alpha,\beta,\gamma)}\right)^{-1}$ and the right shape function $R_{\mathcal{T}}(z) = \left(u_{p(\alpha,\beta,\gamma)}\right)^{-1}$ can be obtained, from which the truth membership function $\mu_{p(\widetilde{A_i},\widetilde{S})}(z)$ is given by

$$\mathcal{T}_{p(\widetilde{A_i},\widetilde{S})}(z) = \begin{cases} L_{\mathcal{T}}(z); & z_1^{\mathcal{T}} \leq z \leq z_2^{\mathcal{T}} \\ R_{\mathcal{T}}(z); & z_3^{\mathcal{T}} \leq z \leq z_4^{\mathcal{T}} \\ 0; & \text{otherwise} \end{cases}$$

where $z_1^{\mathcal{T}} \leq z \leq z_4^{\mathcal{T}}$ and $L_{\mathcal{T}}(z_1^{\mathcal{T}}) = R_{\mathcal{T}}(z_4^{\mathcal{T}}) = 0$ for the SVTNN.

Similarly the indeterminacy membership function $\eta_{p(\widetilde{A_i},\widetilde{S})}(z)$ and the falsity membership function $\nu_{p(\widetilde{A_i},\widetilde{S})}(z)$,are derived as follows

$$\mathcal{I}_{p(\widetilde{A_i},\widetilde{S})}(z) = \begin{cases} L_{\mathcal{I}}(z); & z_1^{\mathcal{I}} \leq z \leq z_2 v \\ R_{\mathcal{I}}(z); & z_3^{\mathcal{I}} \leq z \leq z_4^{\mathcal{I}} \\ 0; & \text{otherwise} \end{cases}$$

where $z_1^{\mathcal{I}} \leq z \leq z_4^{\mathcal{I}}$ and $L_{\mathcal{I}}(z_1^{\mathcal{I}}) = R_{\mathcal{I}}(z_4^{\mathcal{I}}) = 0$ for the SVTNN.

$$\mathcal{F}_{p(\widetilde{A_i},\widetilde{S})}(z) = \begin{cases} L_{\mathcal{F}}(z); & z_1^{\mathcal{F}} \leq z \leq z_2^{\mathcal{F}} \\ R_{\mathcal{F}}(z); & z_3^{\mathcal{F}} \leq z \leq z_4^{\mathcal{F}} \\ 0; & \text{otherwise} \end{cases}$$

where $z_1^{\mathcal{F}} \leq z \leq z_4^{\mathcal{F}}$ and $L_{\mathcal{F}}(z_1^{\mathcal{F}}) = R_{\mathcal{F}}(z_4^{\mathcal{F}}) = 0$ for the SVTNN.

The proposed $NM/NM/1$ queue with priority can be reduced it to classical $M/M/1$ queue with priority by using the concept of (α,β,γ)-cut approach.

4. Numerical Illustration

In this section, we present a numerical example to explain the proposed $NM/NM/1$ queueing model with priority.

Let the arrival rates of first and second priority with the same service rate are represented by SVTNN $\widetilde{A}_1 = \langle(3,4,5,6)\ (2,5,8,11)\ (2,4,6,8)\rangle$

$\widetilde{A}_2 = \langle(4,5,6,7)\ (3,4,5,6)\ (6,6,7,8)\rangle$ and

$\widetilde{S} = \langle(16,17,18,19)\ (18,20,22,24)\ (17,19,21,23)\rangle$ per hour respectively.

The (α,β,γ)-cut of $\widetilde{A}_i, i=1,2; \widetilde{S}$ are
$\widetilde{A}_1 = \langle [3+\alpha, 6-\alpha], [5-3\beta, 8+3\beta], [4-2\gamma, 6+2\gamma] \rangle$,
$\widetilde{A}_2 = \langle [4+\alpha, 7-\alpha], [4-\beta, 5+\beta], [6-\gamma, 7+\gamma] \rangle$ and
$\widetilde{S} = \langle [16+\alpha, 19-\alpha], [20-2\beta, 22-2\beta], [19-2\gamma, 21+2\gamma] \rangle$
From equations (1) and (2) the parametric programming problems are formulated to derive the membership function $\overline{L}_{q_1}, \overline{L}_{q_2}, \overline{W}_{q_1}$ and $\overline{W}_{q_2}$. They are calculated as follows.
The performance functions of (i) $\overline{L}_{q_1}$- average queue length of higher priority (ii) $\overline{L}_{q_2}$- average queue length of low priority (iii) $\overline{W}_{q_1}$ -average waiting time in higher priority queue (iv) $\overline{W}_{q_2}$ - average waiting time in low priority queue are derived from the respective parametric programs. These differ only in their objective functions and are listed below.

$$\begin{aligned} l_{L_{q_1}(\alpha)} &= \min\left\{\frac{e_1(e_1+e_2)}{e_3(e_3-e_1)}\right\}, u_{L_{q_1}(\alpha)} = \max\left\{\frac{e_1(e_1+e_2)}{e_3(e_3-e_1)}\right\} \\ \text{such that } & 3+\alpha < e_1 < 6-\alpha \\ & 4+\alpha < e_2 < 7-\alpha \\ & 16+\alpha < e_3 < 19-\alpha \end{aligned} \tag{3}$$

where $0 < \alpha \leq 1$. $l_{L_{q_1}(\alpha)}$ is found when e_1 and e_2 approach their lower bounds (l.b) and e_3 approaches its upper bound (u.b) and also $u_{L_{q_1}(\alpha)}$ is found when e_1 and e_2 approach their u.b's and e_3 approaches its l.b. Consequently the optimal solution for (3) are

$$l_{L_{q_1}(\alpha)} = \frac{21+13\alpha+2\alpha^2}{304-54\alpha+2\alpha^2} \text{ and } u_{L_{q_1}(\alpha)} = \frac{78-25\alpha+2\alpha^2}{160+42\alpha+2\alpha^2}$$

The truth membership function is

$$\mathcal{T}_{\overline{L}_{q_1}}(z) = \begin{cases} L_{\mathcal{T}}(z); & \left[l_{L_{q_1}(\alpha)}\right]_{\alpha=0} \leq z \leq \left[l_{L_{q_1}(\alpha)}\right]_{\alpha=1} \\ R_{\mathcal{T}}(z); & \left[u_{L_{q_1}(\alpha)}\right]_{\alpha=1} \leq z \leq \left[u_{L_{q_1}(\alpha)}\right]_{\alpha=0} \\ 0; & \text{otherwise} \end{cases}$$

which is estimated as

$$\mathcal{T}_{\overline{L}_{q_1}}(z) = \begin{cases} \dfrac{(54z+13)-(484z^2+4004z-1)^{\frac{1}{2}}}{2(2z-2)}; & 0.07 \leq z \leq 0.14 \\ \dfrac{-(42z+25)+(484z^2+4004z-1)^{\frac{1}{2}}}{2(2z-2)}; & 0.27 \leq z \leq 0.49 \\ 0; & \text{otherwise} \end{cases}$$

$$\begin{aligned} l_{L_{q_1}(\beta)} &= \min\left\{\frac{e_1(e_1+e_2)}{e_3(e_3-e_1)}\right\}, u_{L_{q_1}(\beta)} = \max\left\{\frac{e_1(e_1+e_2)}{e_3(e_3-e_1)}\right\} \\ \text{such that } & 5-3\beta < e_1 < 8+3\beta \\ & 4-\beta < e_2 < 5+\beta \\ & 20-2\beta < e_3 < 22+2\beta \end{aligned} \tag{4}$$

where $0 < \beta \leq 1$. $l_{L_{q_1}(\beta)}$ is found when e_1 and e_2 approach their l.b's and e_3 approaches its u.b. and also $u_{L_{q_1}(\beta)}$ is found when e_1 and e_2 approach their u.b's and e_3 approaches its l.b. Consequently the optimal solution for (4) is

$$l_{L_{q_1}(\beta)} = \frac{45 - 47\beta + 12\beta^2}{374 + 144\beta + 10\beta^2} \text{ and } u_{L_{q_1}(\beta)} = \frac{104 + 71\beta + 12\beta^2}{240 - 124\beta + 10\beta^2}$$

The indeterminacy membership function is

$$\mathcal{I}_{\overline{L}_{q_1}}(z) = \begin{cases} L_{\mathcal{I}}(z); & \left[l_{L_{q_1}(\beta)}\right]_{\beta=1} \leq z \leq \left[l_{L_{q_1}(\beta)}\right]_{\beta=0} \\ R_{\mathcal{I}}(z); & \left[u_{L_{q_1}(\beta)}\right]_{\beta=0} \leq z \leq \left[u_{L_{q_1}(\beta)}\right]_{\beta=1} \\ 0; & \text{otherwise} \end{cases}$$

which is estimated as

$$\mathcal{I}_{\overline{L}_{q_1}}(z) = \begin{cases} \dfrac{-(144z + 47) + (5776z^2 + 33288z + 49)^{\frac{1}{2}}}{2(10z - 12)}; & 0.02 \leq z \leq 0.12 \\ \dfrac{(124z + 71) - (5776z^2 + 33288z + 49)^{\frac{1}{2}}}{2(10z - 12)}; & 0.43 \leq z \leq 1.48 \\ 0; & \text{otherwise} \end{cases}$$

$$\begin{aligned} l_{L_{q_1}(\gamma)} &= \min\left\{\frac{e_1(e_1 + e_2)}{e_3(e_3 - e_1)}\right\}, u_{L_{q_1}(\gamma)} = \max\left\{\frac{e_1(e_1 + e_2)}{e_3(e_3 - e_1)}\right\} \\ \text{such that } & 4 - 2\gamma < e_1 < 6 + 2\gamma \\ & 6 - \gamma < e_2 < 7 + \gamma \\ & 19 - 2\gamma < e_3 < 21 + 2\gamma \end{aligned} \tag{5}$$

where $0 < \gamma \leq 1$. $l_{L_{q_1}(\gamma)}$ is found when e_1 and e_2 approach their l.b's and e_3 approaches its u.b. and also $u_{L_{q_1}(\gamma)}$ is found when e_1 and e_2 approach their u.b's and e_3 approaches its l.b. Consequently the optimal solution for (5) is

$$l_{L_{q_1}(\gamma)} = \frac{40 - 32\gamma + 6\gamma^2}{357 + 118\gamma + 8\gamma^2} \text{ and } u_{L_{q_1}(\gamma)} = \frac{78 + 44\gamma + 6\gamma^2}{247 - 102\gamma + 8\gamma^2} \tag{6}$$

The falsity membership function is

$$\mathcal{F}_{\overline{L}_{q_1}}(z) = \begin{cases} L_{\mathcal{F}}(z); & \left[l_{L_{q_1}(\gamma)}\right]_{\gamma=1} \leq z \leq \left[l_{L_{q_1}(\gamma)}\right]_{\gamma=0} \\ R_{\mathcal{F}}(z); & \left[u_{L_{q_1}(\gamma)}\right]_{\gamma=0} \leq z \leq \left[u_{L_{q_1}(\gamma)}\right]_{\gamma=1} \\ 0; & \text{otherwise} \end{cases}$$

which is estimated as

$$\mathcal{F}_{\overline{L}_{q_1}}(z) = \begin{cases} \dfrac{-(118z+32)+(2500z^2+17400z+64)^{\frac{1}{2}}}{2(8z-6)}; & 0.03 \le z \le 0.11 \\ \dfrac{(102z+44)-(2500z^2+17400z+64)^{\frac{1}{2}}}{2(8z-6)}; & 0.32 \le z \le 0.83 \\ 0; & \text{otherwise} \end{cases}$$

For different values of $\alpha, \beta, \gamma \in [0,1]$, the average queue length of higher priority $\overline{L}_{q_1}$ is calculated and given in table 2. Also a graphical interpolation of truth, Indeterminacy and falsity of average queue length of higher priority is shown in figure 2.

TABLE 2. $\overline{L}_{q_1}$

α	$l_{L_{q_1(\alpha)}}$	$u_{L_{q_1(\alpha)}}$	β, γ	$l_{L_{q_1(\beta)}}$	$u_{L_{q_1(\beta)}}$	$l_{L_{q_1(\gamma)}}$	$u_{L_{q_1(\gamma)}}$
0.0	0.06908	0.48750	1.0	0.12032	0.43333	0.11204	0.31579
0.1	0.07474	0.45987	0.9	0.10404	0.48845	0.09992	0.34811
0.2	0.08074	0.43376	0.8	0.08948	0.55046	0.08884	0.38357
0.3	0.08709	0.40908	0.7	0.07649	0.62042	0.07870	0.42253
0.4	0.09380	0.38573	0.6	0.06491	0.69958	0.06945	0.46539
0.5	0.10090	0.36364	0.5	0.05463	0.78947	0.06100	0.51263
0.6	0.10840	0.34273	0.4	0.04552	0.89196	0.05331	0.56477
0.7	0.11633	0.32293	0.3	0.03748	1.00936	0.04631	0.62244
0.8	0.12469	0.30419	0.2	0.03043	1.14457	0.03995	0.68637
0.9	0.13353	0.28643	0.1	0.02427	1.30125	0.03419	0.75742
1.0	0.14286	0.26961	0.0	0.01894	1.48413	0.02899	0.83660

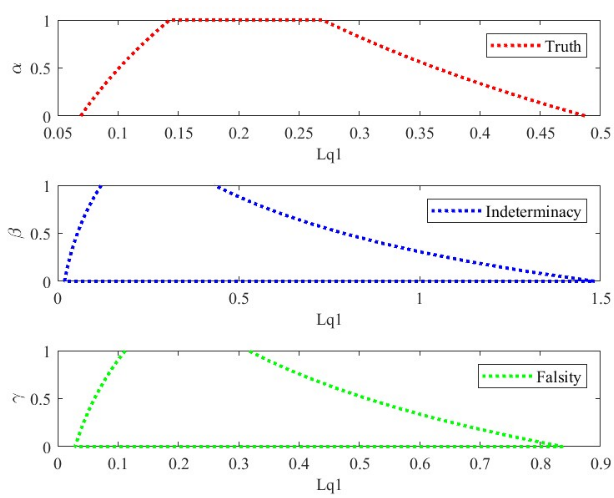


FIGURE 2. Average queue length of higher priority

Similarly the performance functions of $\overline{L}_{q2}$ is derived from the respective parametric programs. These differ only in their objective functions and are listed below.

$$l_{L_{q_2}(\alpha)} = \min\left\{\frac{e_2(e_1+e_2)}{[e_3-(e_1+e_2)](e_3-e_1)}\right\} \quad (7)$$

and

$$u_{L_{q_2}(\alpha)} = \max\left\{\frac{e_2(e_1+e_2)}{[e_3-(e_1+e_2)](e_3-e_1)}\right\} \quad (8)$$

The objective functions given through the equations (7) and (8) with the constraints given with the equation (3) yield the following results:

$$l_{L_{q_2}(\alpha)} = \frac{28+15\alpha+2\alpha^2}{192-72\alpha+6\alpha^2};\ u_{L_{q_2}(\alpha)} = \frac{91-27\alpha+2\alpha^2}{30+36\alpha+6\alpha^2}$$

$$\mathcal{T}_{\overline{L}_{q_2}}(z) = \begin{cases} \dfrac{(72z+15)-(576z^2+4368z+1)^{\frac{1}{2}}}{2(6z-2)}; & 0.14 \le z \le 0.36 \\ \dfrac{-(36z+27)+(576z^2+4368z+1)^{\frac{1}{2}}}{2(6z-2)}; & 0.92 \le z \le 3.03 \\ 0; & \text{otherwise} \end{cases}$$

Similarly the performance functions of $\overline{L}_{q2}$ is derived from the respective parametric programs. These differ only in their objective functions and are listed below.

$$l_{L_{q_2}(\beta)} = \min\left\{\frac{e_2(e_1+e_2)}{[e_3-(e_1+e_2)](e_3-e_1)}\right\} \quad (9)$$

and

$$u_{L_{q_2}(\beta)} = \max\left\{\frac{e_2(e_1+e_2)}{[e_3-(e_1+e_2)](e_3-e_1)}\right\} \quad (10)$$

The objective functions given through the equations (9) and (10) with the constraints given with the equations (4) yield the following results:

$$l_{L_{q_2}(\beta)} = \frac{36-25\beta+4\beta^2}{221+167\beta+30\beta^2};\ u_{L_{q_2}(\beta)} = \frac{65+33\beta+4\beta^2}{84-107\beta+30\beta^2}$$

$$\mathcal{I}_{\overline{L}_{q_2}}(z) = \begin{cases} \dfrac{-(167z+25)+(1369z^2+16206z+49)^{\frac{1}{2}}}{2(30z-4)}; & 0.04 \le z \le 0.16 \\ \dfrac{(107z+33)-(1369z^2+16206z+49)^{\frac{1}{2}}}{2(30z-4)}; & 0.77 \le z \le 14.57 \\ 0; & \text{otherwise} \end{cases}$$

Similarly the performance functions of $\overline{L}_{q2}$ is derived from the respective parametric programs. These differ only in their objective functions and are listed below.

$$l_{L_{q_2}(\gamma)} = \min\left\{\frac{e_2(e_1+e_2)}{[e_3-(e_1+e_2)](e_3-e_1)}\right\} \quad (11)$$

and

$$l_{L_{q2}(\gamma)} = \max\left\{\frac{e_2(e_1+e_2)}{[e_3-(e_1+e_2)](e_3-e_1)}\right\} \tag{12}$$

The objective functions given through the equations (11) and (12) with the constraints given with the equations (5) yield the following results:

$$l_{L_{q2}(\gamma)} = \frac{60-28\gamma+3\gamma^2}{187+129\gamma+20\gamma^2};\ u_{L_{q2}(\gamma)} = \frac{91+34\gamma+3\gamma^2}{78-89\gamma+20\gamma^2}$$

$$\mathcal{F}_{\overline{L}_{q2}}(z) = \begin{cases} \dfrac{-(129z+28)+(1681z^2+14268z+64)^{\frac{1}{2}}}{2(20z-3)}; & 0.1 \le z \le 0.32 \\ \dfrac{(89z+34)-(1681z^2+14268z+64)^{\frac{1}{2}}}{2(20z-3)}; & 1.17 \le z \le 14.22 \\ 0; & \text{otherwise} \end{cases}$$

For different values of $\alpha, \beta, \gamma \in [0,1]$, the average queue length of low priority $\overline{L}_{q2}$ is calculated and given in table 3. Also a graphical interpolation of truth, Indeterminacy and falsity of average queue length of low priority is shown in figure 3.

Table 3. $\overline{L}_{q2}$

α	$l_{L_{q2(\alpha)}}$	$u_{L_{q2(\alpha)}}$	β,γ	$l_{L_{q2(\beta)}}$	$u_{l_{q2(\beta)}}$	$l_{L_{q2(\gamma)}}$	$u_{l_{q2(\gamma)}}$
0.0	0.14583	3.03333	1.0	0.16290	0.77381	0.32086	1.16667
0.1	0.15969	2.62389	0.9	0.14092	0.92853	0.28601	1.36263
0.2	0.17476	2.28846	0.8	0.12191	1.12476	0.25524	1.60525
0.3	0.19118	2.00968	0.7	0.10541	1.37839	0.22800	1.91092
0.4	0.20906	1.77513	0.6	0.09105	1.71391	0.20380	2.30439
0.5	0.22857	1.57576	0.5	0.07853	2.17105	0.18226	2.82468
0.6	0.24987	1.40476	0.4	0.06759	2.81830	0.16303	3.53711
0.7	0.27314	1.25697	0.3	0.05803	3.78403	0.14584	4.55961
0.8	0.29861	1.12835	0.2	0.04965	5.33864	0.13043	6.12857
0.9	0.32652	1.01576	0.1	0.04232	8.16167	0.11660	8.79646
1.0	0.35714	0.91667	0.0	0.03589	14.57144	0.10417	14.22223

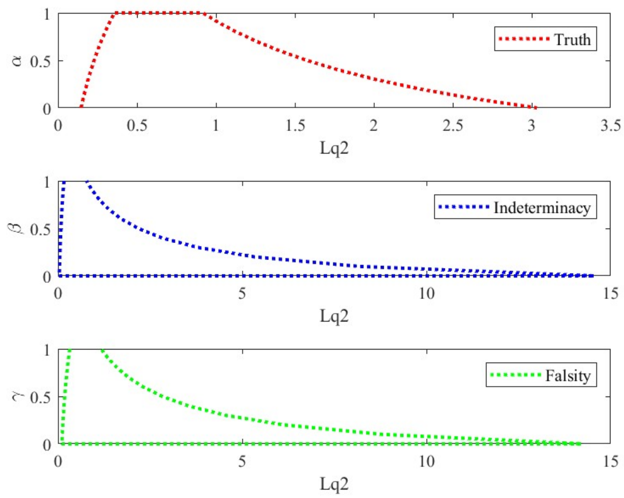


FIGURE 3. Average queue length of low priority

Similarly the performance functions of $\overline{W}_{q_1}$ is derived from the respective parametric programs. These differ only in their objective functions and are listed below.

$$u_{W_{q_1}(\alpha)} = \min\left\{\frac{(e_1+e_2)}{e_3(e_3-e_1)}\right\} \tag{13}$$

and

$$u_{W_{q_1}(\alpha)} = \max\left\{\frac{(e_1+e_2)}{e_3(e_3-e_1)}\right\} \tag{14}$$

The objective functions given through the equations (13) and (14) with the constraints given with the equations (3) yield the following results:

$$l_{W_{q_1}(\alpha)} = \frac{7+2\alpha}{304-54\alpha+2\alpha^2};\ u_{W_{q_1}(\alpha)} = \frac{13-2\alpha}{160+42\alpha+2\alpha^2}$$

$$\mathcal{T}_{\overline{W}_{q_1}}(z) = \begin{cases} \dfrac{(54z+2)-(484z^2+272z+4)^{\frac{1}{2}}}{4z}; & 0.02 \leq z \leq 0.04 \\ \dfrac{-(42z+2)+(484z^2+272z+4)^{\frac{1}{2}}}{4z}; & 0.05 \leq z \leq 0.08 \\ 0; & \text{otherwise} \end{cases}$$

Similarly the performance functions of $\overline{W}_{q_1}$ is derived from the respective parametric programs. These differ only in their objective functions and are listed below.

$$l_{W_{q_1}(\beta)} = \min\left\{\frac{(e_1+e_2)}{e_3(e_3-e_1)}\right\} \tag{15}$$

and

$$u_{W_{q_1}(\beta)} = \max\left\{\frac{(e_1+e_2)}{e_3(e_3-e_1)}\right\} \tag{16}$$

The objective functions given through the equations (15) and (16) with the constraints given with the equations (3) yield the following results:

$$l_{W_{q_1}(\beta)} = \frac{9-4\beta}{374+144\beta+10\beta^2};\ u_{W_{q_1}(\beta)} = \frac{13+4\beta}{240+124\beta+10\beta^2}$$

$$\mathcal{I}_{\overline{W}_{q_1}}(z) = \begin{cases} \dfrac{-(144z+4)+(5776z^2+1512z+16)^{\frac{1}{2}}}{20z}; & 0.009 \leq z \leq 0.02 \\ \dfrac{(124z+4)-(5776z^2+1512z+16)^{\frac{1}{2}}}{20z}; & 0.05 \leq z \leq 0.13 \\ 0; & \text{otherwise} \end{cases}$$

Similarly the performance functions of $\overline{W}_{q_1}$ is derived from the respective parametric programs. These differ only in their objective functions and are listed below.

$$l_{W_{q_1}(\gamma)} = \min\left\{\frac{(e_1+e_2)}{e_3(e_3-e_1)}\right\} \tag{17}$$

and

$$u_{W_{q_1}(\gamma)} = \max\left\{\frac{(e_1+e_2)}{e_3(e_3-e_1)}\right\} \tag{18}$$

The objective functions given through the equations (17) and (18) with the constraints given with the equations (5) yield the following results:

$$l_{W_{q_1}(\gamma)} = \frac{10-3\gamma}{357+118\gamma+8\gamma^2};\ u_{W_{q_1}(\gamma)} = \frac{13+3\gamma}{247-102\gamma+8\gamma^2}$$

$$\mathcal{F}_{\overline{W}_{q_1}}(z) = \begin{cases} \dfrac{-(118z+3)+(2500z^2+1028z+9)^{\frac{1}{2}}}{16z}; & 0.01 \leq z \leq 0.03 \\ \dfrac{(102z+3)-(2500z^2+1028z+9)^{\frac{1}{2}}}{16z}; & 0.05 \leq z \leq 0.11 \\ 0; & \text{otherwise} \end{cases}$$

For different values of $\alpha, \beta, \gamma \in [0,1]$, the average waiting time in the higher priority queue $\overline{W}_{q_1}$ is calculated and given in table 4. Also a graphical interpolation of truth, Indeterminacy and falsity of average waiting time in the higher priority queue is shown in figure 4.

TABLE 4. $\overline{W}_{q_1}$

α	$l_{W_{q_1(\alpha)}}$	$u_{W_{q_1(\alpha)}}$	β,γ	$l_{W_{q_1(\beta)}}$	$u_{W_{q_1(\beta)}}$	$l_{W_{q_1(\gamma)}}$	$u_{W_{q_1(\gamma)}}$
0.0	0.02303	0.08125	1.0	0.02406	0.05417	0.02801	0.05263
0.1	0.02411	0.07794	0.9	0.02214	0.05885	0.02630	0.05615
0.2	0.02523	0.07479	0.8	0.02034	0.06401	0.02468	0.05993
0.3	0.02639	0.07177	0.7	0.01866	0.06971	0.02315	0.06402
0.4	0.02759	0.06888	0.6	0.01708	0.07604	0.02170	0.06844
0.5	0.02883	0.06612	0.5	0.01561	0.08310	0.02033	0.07323
0.6	0.03011	0.06347	0.4	0.01422	0.09102	0.01904	0.07844
0.7	0.03144	0.06093	0.3	0.01292	0.09994	0.01781	0.08411
0.8	0.03281	0.05850	0.2	0.01170	0.11005	0.01665	0.09031
0.9	0.03424	0.05616	0.1	0.01055	0.12161	0.01554	0.09711
1.0	0.03571	0.05392	0.0	0.00947	0.13492	0.01449	0.10458

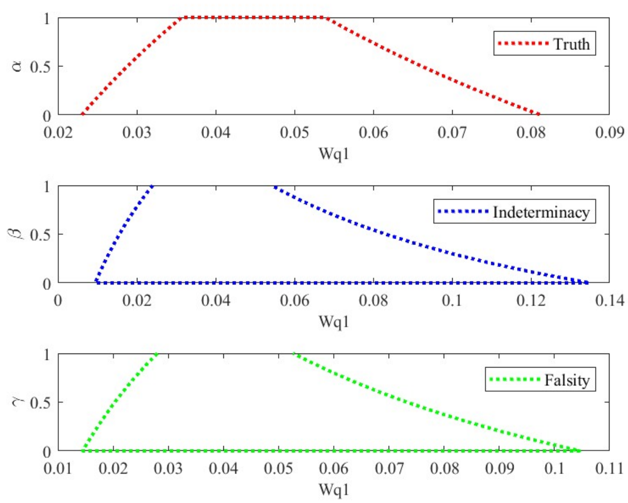


FIGURE 4. Average waiting time in the higher priority queue

Similarly the performance functions of $\overline{W}_{q_2}$ is derived from the respective parametric programs. These differ only in their objective functions and are listed below.

$$l_{W_{q_2}(\alpha)} = \min\left\{\frac{(e_1+e_2)}{[e_3-(e_1+e_2)](e_3-e_1)}\right\} \tag{19}$$

and

$$u_{W_{q_2}(\alpha)} = \max\left\{\frac{(e_1+e_2)}{[e_3-(e_1+e_2)](e_3-e_1)}\right\} \tag{20}$$

The objective functions given through the equations (19) and (20) with the constraints given with the equations (3) yield the following results:

$$l_{W_{q_2}(\alpha)} = \frac{7+2\alpha}{192-72\alpha+6\alpha^2};\ u_{W_{q_2}(\alpha)} = \frac{13-2\alpha}{30+36\alpha+6\alpha^2}$$

$$\mathcal{T}_{\overline{W}_{q_2}}(z) = \begin{cases} \dfrac{(72z+2) - (576z^2+456z+4)^{\frac{1}{2}}}{12z}; & 0.04 \leq z \leq 0.07 \\ \dfrac{-(36z+2) + (576z^2+456z+4)^{\frac{1}{2}}}{12z}; & 0.15 \leq z \leq 0.43 \\ 0; & \text{otherwise} \end{cases}$$

Similarly the performance functions of $\overline{W}_{q_2}$ is derived from the respective parametric programs. These differ only in their objective functions and are listed below.

$$l_{W_{q_2}(\beta)} = \min \left\{ \frac{(e_1+e_2)}{[e_3-(e_1+e_2)](e_3-e_1)} \right\} \tag{21}$$

and

$$u_{W_{q_2}(\beta)} = \max \left\{ \frac{(e_1+e_2)}{[e_3-(e_1+e_2)](e_3-e_1)} \right\} \tag{22}$$

The objective functions given through the equations (21) and (22) with the constraints given with the equations (3) yield the following results:

$$l_{W_{q_2}(\beta)} = \frac{9-4\beta}{221+167\beta+30\beta^2};\ u_{W_{q_2}(\beta)} = \frac{13+4\beta}{84-107\beta+30\beta^2}$$

$$\mathcal{I}_{\overline{W}_{q_2}}(z) = \begin{cases} \dfrac{-(167z+4) + (1369z^2+2416z+16)^{\frac{1}{2}}}{60z}; & 0.01 \leq z \leq 0.04 \\ \dfrac{(107z+4) - (1369z^2+2416z+16)^{\frac{1}{2}}}{60z}; & 0.15 \leq z \leq 2.43 \\ 0; & \text{otherwise} \end{cases}$$

Similarly the performance functions of $\overline{W}_{q_2}$ is derived from the respective parametric programs. These differ only in their objective functions and are listed below.

$$l_{W_{q_2}(\gamma)} = \min \left\{ \frac{(e_1+e_2)}{[e_3-(e_1+e_2)](e_3-e_1)} \right\} \tag{23}$$

and

$$u_{W_{q_2}(\gamma)} = \max \left\{ \frac{(e_1+e_2)}{[e_3-(e_1+e_2)](e_3-e_1)} \right\} \tag{24}$$

The objective functions given through the equations (23) and (24) with the constraints given with the equations (5) yield the following results:

$$l_{W_{q_2}(\gamma)} = \frac{10-3\gamma}{187+129\gamma+20\gamma^2};\ u_{W_{q_2}(\gamma)} = \frac{13+3\gamma}{78-89\gamma+20\gamma^2}$$

$$\mathcal{F}_{\overline{W}_{q_2}}(z) = \begin{cases} \dfrac{-(129z+3) + (1681z^2+1574z+9)^{\frac{1}{2}}}{40z}; & 0.02 \leq z \leq 0.05 \\ \dfrac{(89z+3) - (1681z^2+1574z+9)^{\frac{1}{2}}}{40z}; & 0.17 \leq z \leq 1.78 \\ 0; & \text{otherwise} \end{cases}$$

For different values of $\alpha, \beta, \gamma \in [0,1]$, the average waiting time in the low priority queue $\overline{W}_{q2}$ is calculated and given in table 5. Also a graphical interpolation of truth, Indeterminacy and falsity of average waiting time in the low priority queue is shown in figure 5.

TABLE 5. $\overline{W}_{q2}$

α	$l_{W_{q2(\alpha)}}$	$u_{W_{q2(\alpha)}}$	β, γ	$l_{W_{q2(\beta)}}$	$u_{W_{q2(\beta)}}$	$l_{W_{q2(\gamma)}}$	$u_{W_{q2(\gamma)}}$
0.0	0.03646	0.43333	1.0	0.04072	0.15476	0.05348	0.16667
0.1	0.03895	0.38027	0.9	0.03613	0.18207	0.04848	0.19192
0.2	0.04161	0.33654	0.8	0.03208	0.21630	0.04401	0.22295
0.3	0.04446	0.29995	0.7	0.02849	0.26007	0.04000	0.26177
0.4	0.04751	0.26896	0.6	0.02529	0.31739	0.03639	0.31140
0.5	0.05079	0.24242	0.5	0.02244	0.39474	0.03314	0.37662
0.6	0.05432	0.21949	0.4	0.01988	0.50327	0.03019	0.46541
0.7	0.05812	0.19952	0.3	0.01758	0.663870	0.02752	0.59216
0.8	0.06221	0.18199	0.2	0.01552	0.92045	0.02508	0.78571
0.9	0.06664	0.16652	0.1	0.01365	1.38333	0.02286	1.11348
1.0	0.07143	0.15278	0.0	0.01196	2.42857	0.02083	1.77778

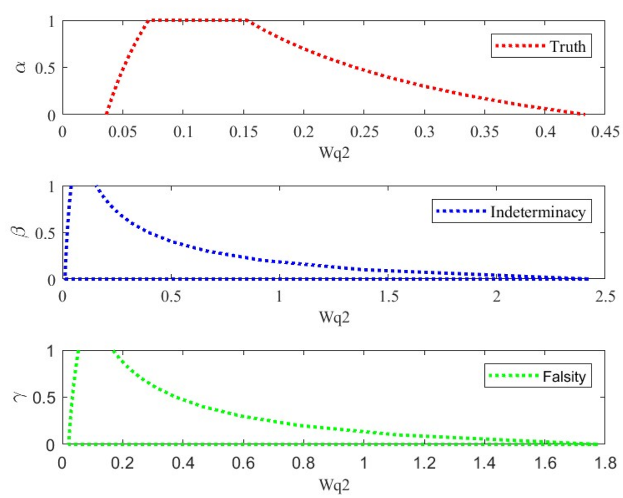


FIGURE 5. Average waiting time in the low priority queue

5. Conclusion

Priority queueing models are useful in real world problems such as emergency cases in hospital medical treatment, communication networks etc. The parameters for queueing decision models can be known imprecisely and hence the performance measurements of the system can be dealt in neutrosophic environment. This paper, proposes a single server queuing model with priority discipline and its characteristics. The service time and arrival time of proposed model are expressed in terms of single valued trapezoidal Neutrosophic number. An illustrative example is provided to show the performance measures of the proposed model which are constructed using truth, indeterminacy and falsity membership degree of SVTNN. In future, this queueing model can extended multi objective priority queuing model. The extensions of neutrosophic sets such as pythogorean and Fermatean neutrosophic sets can used in the proposed model to explore its new aspects.

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A Framework of Type-2 Neutrosophic for Requirements Prioritization

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Abstract

Addressing the relative importance and urgency of different requirements to cope with the limited resources of projects such as budget and time is called Requirements Prioritization (RP), and it is a crucial step in the project management process, it involves several stakeholders deciding between multiple requirements based on several criteria, which is a multi-criteria decision making (MCDM). Different organizations use different requirements prioritization methods depending on the scope and level of the project. But the challenge arises when the number of requirements is large, and multiple stakeholders with conflicting goals are involved, which makes it hard to get consensus on the project direction. Another challenge in the prioritization process is that the judgement of the different stakeholders can be vague and imprecise, making it difficult to be represented in exact numbers. Therefore, this paper presents a MCDM framework based on the type-2 neutrosophic numbers (T2NNs) for the prioritization of requirements using T2NNs Decision making trial and evaluation laboratory (DEMATEL) and T2NNs technique for order of preference by similarity to ideal solution (TOPSIS). T2NNs are used to deal with the uncertainty and vagueness in stakeholders' preferences. The initial step of the proposed RP framework is to identify the relevant stakeholders, the goals, and the requirements. Second, we use the T2NN-based DEMATEL method to compute and rank the criteria importance. Then the T2NN-based TOPSIS is used to rank the requirements. Finally, the applicability of the proposed framework is demonstrated with the help of a numeric case study.

Keywords

Requirements Prioritization; Requirements Selection; MCDM; DEMETAL; TOPSIS; T2NN.

1.Introduction

The ultimate goal of any project, system or service is to meet the users and stakeholders needs and expectations, by effectively identifying the requirements and using them as a guide in the project development process. But in most projects, there are more requirements than we can address within the projects constrains. Thus, it becomes a major challenge for user experience designers, product managers and business analysts during the initial phase of project development to find out the list of requirements or features to develop and prioritize some requirements to be implemented immediately and some to be reserved for a later release while still producing a system that meets the essential needs of users and stakeholders. [1]. Requirements prioritization (RP) is the process of addressing the relative importance and urgency of different requirements to cope with the limited resources of projects such as budget and time, so prioritization of requirements is a way of maximizing the benefits from finite resources allotted to a particular iteration or release of a project. Requirements prioritization is an essential aspect of software release planning. The requirements that make the top of this list are given top priority, and the work for these requirements takes precedence over others. Prioritization is an essential and ongoing process during any product development process as it is the only way to deal with competing demands from stakeholders, clients, end users for limited resources.

However, Requirements prioritization is a daunting task. Different criteria of software requirements must be considered when prioritizing requirements, such as dependency, cost–value, risk, and other criteria [2]. And this only gets even more complicated when stakeholders with different levels of expertise, understanding, and opinions are involved, so the prioritized requirements would need to align with different goals such as business, user and technical goals which can often be conflicting. That's why making an informed decision on what to prioritize can be challenging.

Many useful methods have been successfully developed to execute the RP process, including MoSCoW analysis, Ranking Method, Value-Oriented Prioritization (VOP), Planning Game, Weighted Sum Method, Impact-Effort Matrix, Feasibility-Desirability-Viability Scorecard, RICE method, Kano model, NUF test, Analytical Hierarchy Process (AHP), Minimal spanning tree, Cumulative Voting (CV), Multi-factor matrix etc. In the systematic literature review on requirements prioritization techniques by [3], 40 techniques have been identified for requirements prioritization from 2009 to 2017. The choice of the appropriate RP technique depends on the scope and level of the project as some techniques are too simple to deal with large number of requirements, conflicted goals, and multiple decision makers.

Requirements prioritization is a multi-criteria decision making (MCDM) problem whose objective is to prioritize the requirements on the basis of different criteria. As the prioritization process includes the judgement and preferences of different stakeholders which can be vague, imprecise, and difficult to represent in exact numbers (like most of the prioritization techniques), stakeholders may then use linguistic terminologies instead of exact numbers to specify their preferences. Therefore, this paper focuses on implementing a type-2 neutrosophic framework to prioritize requirements by considering the different criteria as well as uncertainty and vagueness in stakeholders' preferences by using the neutrosophic approach which is a promising method to deal with uncertainty. The initial step of the proposed RP framework is to identify the relevant stakeholders, their goals, and requirements. Second, we use a technique called T2NN-based DEMATEL to compute and rank the criteria importance, and we use another technique called T2NN-based TOPSIS to rank the requirements.

The remainder of the paper is organized as follows. Technical background and literature review in Section 2. Section 3 presents the proposed framework methodology. Section 4 presents a numeric case study to demonstrate the applicability of the proposed framework. Finally, we conclude this paper in Section 5.

2. Technical background and literature review

In this section, we give a quick overview of requirements prioritization methods then a literature review of previous work.

2.1 Concepts and terminologies

Business requirements:

These are the requirements related to what the business wants to achieve from the project, they define the business needs and the success criteria. Business requirements include Project timeline and scope, Branding rules, marketing, sales, customer services, Competitors, and Stakeholder expectations. [4]

User requirements:

User requirements gathering is a process used to understand what typical users will need from a service or a product which is about to be designed, it involves understanding the needs, goals, and expectations of the users to identify a list of requirements, features, and functionality the new service must have. This helps to ensure that the product or service meets the user's needs and expectations. This process answers questions like: Who are the target users, and what are their needs and pain points? What usability or accessibility issues that designers need to consider?

Technical requirements:

Technical requirements are related to how the project will be implemented, they answer key technical questions and address technical limitations, and they fall into two categories:

Functional (FRs): Outlines the product's specifications, technical capabilities, and limitations.

Non-functional (NFRs): Describes the product's performance, such as usability, performance, data integrity, and maintenance.

2.2 Requirements prioritization techniques:

There are several methods to assess the priorities of requirements, [5], for clarity, we classify them into three categories: visual plots, Scoreboards, and comparison-based methods.

Visual Plots:

Visual plots techniques are a quick, flexible, collaborative, and simplified approach for prioritization, they can work with large number of features involving different stakeholders. But their simplicity can have a downside when we need a more structured approach for decision making. Some of the techniques are:

A. Impact-Effort or Value-Complexity Matrix

This is a four-quadrant prioritization technique that prioritizes the requirements regarding their impact and the effort needed to implement them [1]. requirements that have high impact but need low effort, are done right away, on the other hand, requirements having a low impact, but high effort are not worth it. Requirements that have high impact but need high effort too, are strategic and defensible. And lastly, the requirements that need low effort, and have low impact are kept for later in case they become needed.

B. MoSCoW analysis

MoSCoW analysis was created by Dai Clegg and is used in many Agile frameworks. It breaks requirements into four groups: Must Have, Should Have, Could Have, and Will Not Have. Must have requirements represents the mandatory requirements that are vital to the product or project. Should have, represents requirements that support core functionality and are important to the project or context, but the project or product will still work without them. Could have, refers to requirements that are not essential, but wanted and nice to have. Will not have, are requirements that are not needed. They don't present enough value and can be left out. [6]

C. Eisenhower decision matrix

This technique by Steven Covey [7] breaks requirements into four groups: DO, Schedule, Delegate, Don't do. based on their urgency and importance. Urgent refers to requirements that need immediate action. Failing to address an urgent requirement often results in clear consequences. And Important refers to the requirements that contribute to the long-term goals and require planning and careful action.

D. Kano Model

This technique by Dr. Noriaki Kano 1984 prioritizes requirements based on the degree they are likely to satisfy and delight the end user, by weighing a high satisfactory feature against its implementation investment to determine whether to include it in the product roadmap. It clusters the requirements into five categories: Basic features, Performance features, Excitement features, Indifferent features, and Dissatisfaction features. [8]

Scoreboards or weighted sum methods (WSM):

Sometimes features are complicated and need to be prioritized with more detail than a simple visual plot can do. In this case, the scoreboards methods are a great way to score priorities, scoreboards or score matrix can be customized according to the specific needs and criteria of the different stakeholders involved, each criteria can be assigned a relative weight representing its importance [9]. Some of the famous scoreboard techniques:

E. Feasibility, Desirability, and Viability Scorecard (FDV)

FDV was invented by IDEO in the early 2000s [6], it ranks requirements based on feasibility, desirability, and viability. Feasibility refers to the degree to which the requirement can be technically feasible. Desirability refers to the degree to which the user desires the feature. Finally, viability relates to the benefit the feature will bring to the business. A matrix is made with rows representing each of the features and columns representing the three categories, then each of the stakeholders assigns a score to each of the feature regarding each category on an importance scale from 1 to 10, then, a total score is calculated, and the features are ranked.

F. NUF test

Similar to the feasibility, desirability and viability scoreboard, this technique developed by Dave Gray [10] prioritizes requirements based on three criteria: New, Useful, Usable. New refers to the degree to which the feature is new and innovative. Useful refers to how useful a feature is in solving and addressing the user. Feasible assesses the features in terms of the resources and effort needed to get implemented.

G. RICE Method

This prioritization framework developed by Intercom [11] considers four factors: Reach, Impact, Confidence, and Effort to prioritize which features to implement. Each feature has a score calculated by multiplying Reach (the number of users affected by the feature) by Impact (the value the feature has on users) and Confidence (how valid these estimates are). Then dividing the resulting number by Effort (the effort it will take to implement the feature).

Comparison based methods:

The comparison-based prioritization techniques can lead to the most accurate results [12] [13] , but as the requirements list gets bigger, these methods become more complex and time consuming to implement. Two of the most popular comparison methods are:

H. AHP:

This feature prioritization method is used to identify the most important features of a product or service based on multiple objectives. All possible pairs of features are compared, to determine the relative importance of each feature. Usually, this is done with a scale from 1 to 9 where 1 represents equal importance and 9 represents that the feature is a lot more important.

AHP is considered the most promising prioritization method in comparison with other methods, as it yields the most trustworthy results due to the comparison redundancies that makes it less sensitive to judgment errors, it provides consistency check and the results are based on a ratio scale to compare the requirements instead of an ordinal scale, which is more meaningful, thus the priority distance between the requirements is given [14], [15]. But to come up with a prioritization, many comparisons have to be made, which requires a lot of time and effort and can be a challenge task for User Experience (UX) teams with limited resources. [16]

I. Bubble Sorting

This technique is based on the comparison of two requirements and swapping the one with the most importance to have more priority than the other one. The comparison is carried out until the last item is prioritized and sorted. [12] [17]. This method can be time consuming when the feature list is large and challenging when different stakeholders are involved.

Each of these prioritization techniques has its own strengths and weaknesses, but a common limitation is that none of these methods considers the interdependency between the criteria when weighing them, also, it's not an easy process to define requirements in numeric values as in the scoreboards' methods, instead it's more meaningful to use linguistic terms.

2.3 Requirements Prioritization studies:

Different MCDM techniques have been used for the prioritization and selection of requirements i.e., Analytic Hierarchy Process (AHP), TOPSIS, etc. The previous studies on MCDM for requirements prioritization can be classified into single methods studies and combined methods, either using crisp or fuzzy values in the prioritization process.

[18] Presented a prioritization method using fuzzy AHP to assess the goal-oriented requirements elicitation process, this method used binary sort to get the prioritized list of requirements, this method was demonstrated on a case study of ten functional requirements, three criteria, for the prioritization of requirements and ten stakeholders' and five Decision makers (DM) participating to prioritize the requirements. [19] proposed a fuzzy based MoSCoW method for software requirements prioritization, they applied their proposed method to prioritize the requirements of Library Management System (LMS), using the "goal-oriented requirements elicitation process" (GOREP) to determine the ten functional requirements and using three non-functional requirements as the criteria. This study didn't include multiple stakeholders' opinions in the prioritization. [20] proposed a prioritization method combining Planning Games (PG) and analytical hierarchy process (AHP) techniques. The proposed method was applied on a Library Management System case study. This method reduced the number of pairwise comparisons from 105 to 31 for the same number of FRS and NFRs. Another study by [21] used the fuzzy TOPSIS method to rank 10 FRs functional requirements of an Institute Examination System (IES), based on 3 NFRs, by five decision makers. [22] Proposed a combined method of fuzzy AHP and fuzzy TOPSIS for requirements prioritization. This method was applied for the selection of the requirements of Institute Examination Systems, where 16 FRs were identified, 3 NFRs as the criteria and 4 DMs. Fuzzy AHP was used for computing the requirements weights and Fuzzy TOPSIS was used to compute the ranking. [23] proposed another combined method using MoScoW and AHP, this technique has combined the benefits of both MOSCOW and AHP. It performs categorization of 21 requirements using MOSCOW and then ranking using AHP, using AHP in MOSCOW reduced the number of comparisons from 210 to 45. [24] conducted a comparative study between fuzzy AHP and fuzzy TOPSIS for software requitements selection as they're the most used methods in this domain, the results of their study stated that both fuzzy AHP and fuzzy TOPSIS methods produce the same set of functional requirements, but AHP causes the rank reversal issue; unlike TOPSIS. Fuzzy TOPSIS requires less judgment by decision makers compared to fuzzy AHP and there is no limit in the FRs and NFRs when using fuzzy TOPSIS, on the other hand, the fuzzy AHP is limited to the number of FRs and NFRs as it requires large number of comparisons to be made.

3. Proposed methodology

In this study, the T2NNs are used to rank the requirements by using the DEMATEL[25] and TOPSIS[26] methods. The DEMATEL method is used to compute the weights of criteria taking into account the interdependency between them, and the TOPSIS method is used to rank the requirements. This section shows the steps of the proposed methodology. A summary of the proposed methodology is depicted in figure 1.

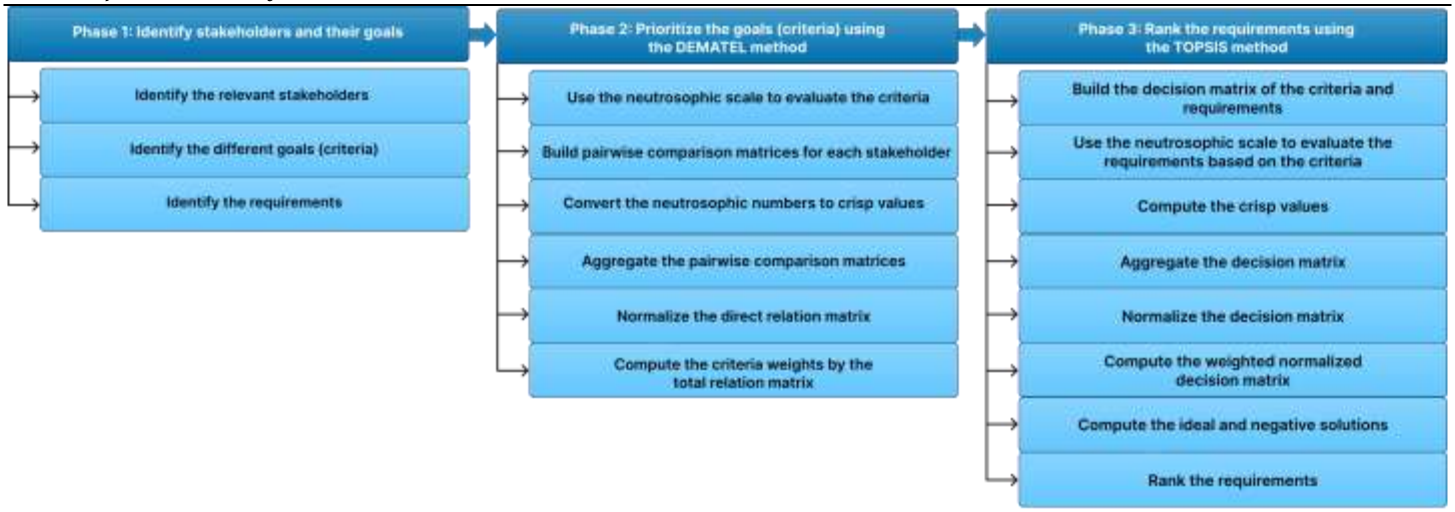

Fig 1. The proposed methodology.

Phase 1: Identify the relevant stakeholders and their goals

Step 1: Identify the relevant stakeholders to take part on the prioritization process.

Step 2: Identify the different goals (Criteria).

Step 3: Identify the requirements.

Phase 2: Compute the weights of the criteria using DEMATEL method

The T2NN-based DEMATEL addresses the vagueness and uncertainty in the stakeholders' judgements, using the indeterminacy degree, DEMATEL is helpful in handling interrelated problems, as all criteria fall into two categories: cause and effect, making it a perfect choice for computing the weights of the criteria as in most cases the different criteria are interrelated and affect one another, i.e., customer satisfaction can cause higher revenues.

Step 4: Use the neutrosophic scale to evaluate the different criteria.

Step 5: Build the pairwise comparison matrix for each stakeholder.

Step 6: Convert the neutrosophic numbers to crisp values [27].

Step 7: Aggregate the pairwise comparison matrix by the average method to obtain direct relation matrix.

Step 8: Normalize the direct relation matrix [25].

$$N = \frac{1}{\max\limits_{1\le i\le n} \sum_{j=1}^{n} x_{ij}} \tag{1}$$

Where $i = 1,2,3 \ldots\ldots m\ (alternatives); j = 1,2,3 \ldots.\ n\ (criteria)$

Step 9: Compute the total relation matrix as:

$$R = N \times (I - N)^{-1} \tag{2}$$

Step 10: Compute the weights of criteria by the total relation matrix.

Phase 3: Rank the requirements by the TOPSIS method

In T2NN-based TOPSIS, the set of requirements are scored against the set of criteria using linguistic terms for each criterion. Each criteria have a direction of preference based on whether more or less of that criterion is preferred. This makes the T2NN-based TOPSIS a good choice for requirements prioritization as it simulates the real prioritization process where we score a set of requirements against cost/value criteria.

Step 11: Build the decision matrix of the criteria and requirements.

Step 12: Use the neutrosophic scale to evaluate the requirements based on the criteria.

Step 13: Compute the crisp values [27].

Step 14: Aggregate the decision matrix.

Step 15: Normalize the decision matrix [26].

$$N_{ij} = \frac{x_{ij}}{\sum_{i=1}^{m} x_{ij}^2} \quad (3)$$

Step 16: Compute the weighted normalized decision matrix as:

$$WN_{ij} = N_{ij} \times W_j \quad (4)$$

Step 17: Compute the ideal and negative solution

$$P_i^+ = \sqrt{\sum_{j=1}^{n} \left(WN_{ij} - \max WN_{ij}\right)^2} \quad (5)$$

$$P_i^- = \sqrt{\sum_{j=1}^{n} \left(WN_{ij} - \min WN_{ij}\right)^2} \quad (6)$$

Step 18: Rank the requirements (alternatives) by the highest value of S as:

$$S_i = \frac{P_i^-}{P_i^- + P_i^+} \quad (7)$$

4 Numeric case study

The aim of this section is to apply the steps of the proposed framework and show the results of the DEMATEL and TOPSIS methods for the prioritization and selection of the requirements of an online banking system (OBS) [2]. We take five main online banking requirements as an example, they'll be referred to as OBSR1, OBSR2, etc. And five NFRs as the criteria, they'll be referred to as OBSC1, OBSC2, etc.

Phase 1: Identify the stakeholders and their goals

Step 1: Three stakeholders were chosen to evaluate the criteria and requirements.

Step 2: Five criteria were identified as the project's priorities, namely speed, integrity, security, customer satisfaction, and services.

Phase 2: Compute the weights of the criteria using DEMATEL method

Step 3: T2NNs were used by stakeholders to evaluate the criteria [27].

Step 4: The pairwise comparison matrix for the five criteria were constructed using linguistic terms, by each of the three stakeholders.

Step 5: The linguistic terms were converted to T2NNs then into crisp values by the score function [27].

Step 6: The direct relation matrix is computed by the aggregation matrix as in table 1.

Step 7: The normalized matrix is built by Eq. (1) as in table 2.

Step 8: Using Eq. (2) the total relation matrix is built as shown in table 3.

Step 9: The weights of criteria were computed by the relation matrix as in Fig. 2. From Fig. 2 we can see that security is the most important criteria followed by speed, and the criteria with the lowest weight being services.

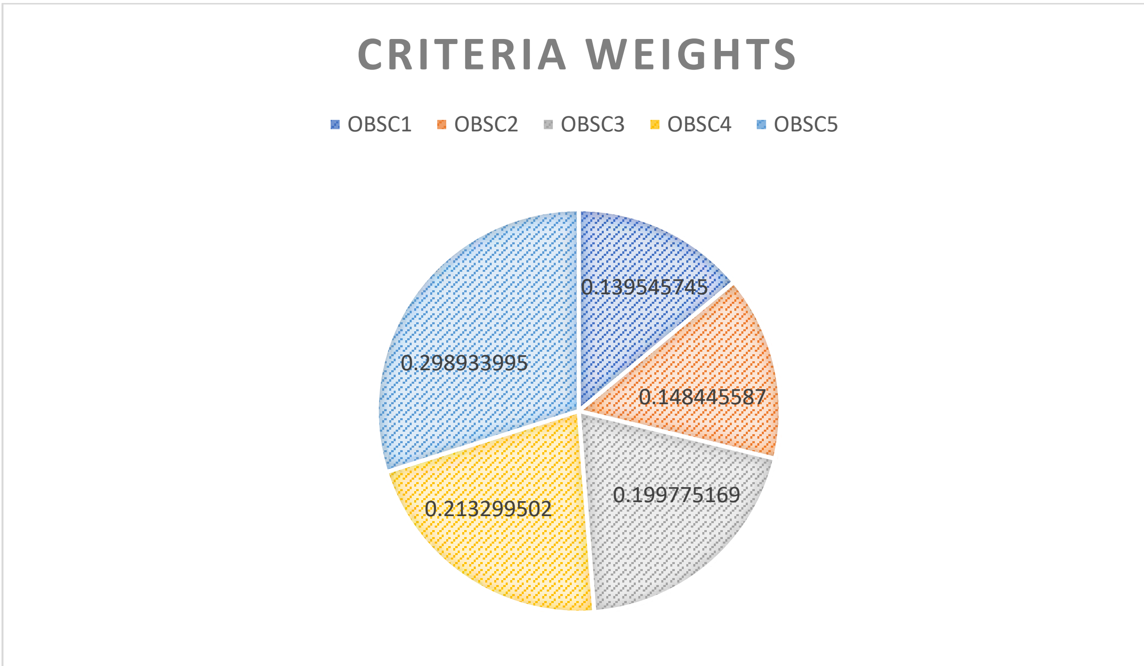

Fig 2. The weights of criteria.

Table 1. Aggregated pairwise comparison matrix

	OBSC1	OBSC2	OBSC3	OBSC4	OBSC5
OBSC1	1	0.513333	0.663333	0.686667	0.536667
OBSC2	2.415825	1	0.62	0.606667	0.576667
OBSC3	2.411569	3.79283	1	0.493333	0.78
OBSC4	1.477273	1.847643	2.713805	1	0.55
OBSC5	2.55635	2.238366	1.282051	2.020202	1

Table 2. Normalized pairwise comparison matrix

	OBSC1	OBSC2	OBSC3	OBSC4	OBSC5
OBSC1	0.391183	0.135343	0.244429	0.3399	0.536667
OBSC2	0.945029	0.263655	0.228462	0.3003	0.576667
OBSC3	0.943364	1	0.368486	0.2442	0.78
OBSC4	0.577884	0.487141	1	0.495	0.55
OBSC5	1	0.590157	0.472418	1	1

Table 3. Total relation matrix

	OBSC1	OBSC2	OBSC3	OBSC4	OBSC5
OBSC1	-0.37166	-0.27894	-0.14724	-0.11183	-0.15032
OBSC2	-0.03338	-0.36486	-0.29251	-0.21853	-0.21831
OBSC3	-0.20654	0.027849	-0.52124	-0.50664	-0.31093
OBSC4	-0.51107	-0.10905	0.093692	-0.51986	-0.57395
OBSC5	-0.72436	-0.42023	-0.37067	-0.1451	-0.61035

Phase 3: Rank the requirements by the TOPSIS method

Step 10: The decision matrix for evaluating the requirements against the criteria were built by each stakeholder using linguistic terms, then converted to T2NNs then to crisp values. Then the aggregated decision matrix was calculated as in table 4.

Step 11: The aggregated decision matrix was normalized by Eq. (3) as in table 5.

Step 12: The weights of criteria were multiplied by the normalization matrix by using Eq. (4) as in table 6.

Step 13: Then the ideal and negative solutions were computed by Eqs. (5,6).

Step 14: To get the requirements ranks, we use the values of S computed by Eq. (7), the requirement with the highest value of S being the most important one etc., as shown in Fig. 3. From Fig. 3 we see that requirement 1 is the most important requirement, followed by requirement 3, and the least important requirement is requirement 5.

Fig 3. The rank of requirements

Table 4. Aggregated decision matrix

	OBSC1	OBSC2	OBSC3	OBSC4	OBSC5
OBSR1	0.6	0.513333	0.663333	0.686667	0.53
OBSR2	0.48	0.436667	0.62	0.62	0.58
OBSR3	0.28	0.32	0.72	0.476667	0.786667
OBSR4	0.472333	0.35	0.32	0.833333	0.52
OBSR5	0.28	0.366667	0.286667	0.433333	0.826667

Table 5. Normalized decision matrix

	OBSC1	OBSC2	OBSC3	OBSC4	OBSC5
OBSR1	0.609114	0.56913	0.536731	0.489809	0.358147
OBSR2	0.487291	0.48413	0.501669	0.442255	0.391935
OBSR3	0.284253	0.354783	0.582583	0.340013	0.53159
OBSR4	0.479508	0.388043	0.258926	0.594428	0.35139
OBSR5	0.284253	0.406522	0.231954	0.309103	0.558619

Table 6. Weighted normalized decision matrix

	OBSC1	OBSC2	OBSC3	OBSC4	OBSC5
OBSR1	0.084999	0.084485	0.107226	0.104476	0.107062
OBSR2	0.067999	0.071867	0.100221	0.094333	0.117163
OBSR3	0.039666	0.052666	0.116386	0.072525	0.15891
OBSR4	0.066913	0.057603	0.051727	0.126791	0.105042
OBSR5	0.039666	0.060346	0.046339	0.065931	0.16699

Conclusions

In this work, we present a new framework for requirements prioritization using the DEMATEL and TOPSIS methods under the neutrosophic environment. The DEMATEL method is used to compute the criteria weights, while the TOPSIS method was later used to rank the requirements based on the identified criteria. The proposed framework was explained using a numeric case study of an OBS, where three stakeholders were chosen to participate in the RP process, five criteria and five requirements were selected to be used as an example. The proposed framework has shown few interesting advantages over previous methods, The DEMATEL method used in the framework addresses the interdependency between the different criteria, as some criteria can influence and cause other criteria. The TOPSIS method used requires few stakeholders' judgements compared to other method such as AHP, making it the perfect choice for dealing with large number of requirements, it's also more meaningful and easier for stakeholders as it simulates the basic prioritization matrix where a set of requirements are evaluated against a set of criteria. The TOPSIS method also avoids the rank reversal issue, thus, making the proposed framework more dynamic. The neutrosophic approach used in this framework addresses the imprecision and vagueness in the stakeholders' judgements, making it possible for stakeholders to use linguistic terms instead of numbers and scales which can be understood differently by everyone, which can drive inaccurate results. For future research, we plan to test this framework on a large project to further validate its results.

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An Efficient Neutrosophic Technique for Uncertain Multi Objective Transportation Problem

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Abstract. It can be difficult to figure out how to satisfy customers' ever rising demands and keep up one's market competitiveness while containing controllable costs. Inefficiencies in the supply chain network are thus discovered by our investigation. Finding the best allocation order for products from diverse sources going to numerous destinations is the primary objective. Moreover, The information that is readily available is typically not clear-cut in real-world circumstances. So, it gives rise to the uncertain transportation problem. With the aim of helping the decision maker to have the suitable transportation plan with real suitation, in this paper, a solution procedure for multi objective transportation problem involving uncertainvariables has been studied under neutrosophic environment. A chance constraint model is constructed foruncertain multi objective transportation problem and then a neutrosophic compromise approach is used toobtain the pareto optimal solution for the problem. As neutrosophic sets are built with truth, indeterminacyand falsity membership functions, they are capable to help the decision maker in this complex transportation model. A numerical example has been reported to demonstrate the efficiency of the proposed approach towardsthe best compromise solution and a comparison study has been made with the existing methods.

Keywords: Multi objective transportation problem; Chance constraint programming; Neutrosophic set theory.

1. Introduction

In the real world, transportation planning decision problems play a vital role in logistics and supply chain management with diverse challenges to be addressed. A transportation planning problem involves a large number of factors such as shipment, distance, delivery time; transportation cost etc and are defined on the basis of quantitative evaluation. More often than not, the market scenario keeps varying and posing challenges, because of which various objective functions are needed related to a transportation problem. For example maximizing the profit of the transportation, minimizing the transportation cost and toll tax etc. Since the cost parameters of various objectives of the transportation problem are not related to each other, these are considered as conflicting and commensurable model of the multi objective transportation problem (MOTP). In the present-day scenario, most of the transportation planning decisions is made under uncertain environment due to many unpredictable factors. Traditional methods failed to capture the decision maker's ambiguities and are non-effective to solve these complex ill-defined models. Many researchers had developed different stochastic, fuzzy and uncertain models to solve complex uncertain transportation engineering problems.

In this paper, we've proposed a solution procedure for multi-objective transportation problem whose parameters are all uncertain variables. Motivated by neutrosophic sets studied by Smarandache [19] which provides a general structure to deal with uncertainty, a compromise solution to the proposed model is obtained. The term "neutrosophy"means the knowledge of neutral thought and considers that all elements can be represented by three degrees namely-truth, falsity, indeterminacy which lie between 0 and 1. Since its establishment by Smarandache [25], some attention has been developed for optimization aspects [20]. Rizk M [21] proposed an algorithm based upon MOTP under neutrosophic environment. Since neutrosophic models effectively assist the decision-maker by incorporating satisfaction, satisfaction to some degree, and dissatisfaction of objective functions in determining the best compromise solution. we have applied the neutrosophic technique for the first time to the MOTP whose parameters are uncertain normal variables.

The rest of the paper is structured as follows. Section 2 contains the existing research papers related to the proposed work. In section 3, weve reviewed the preliminaries of uncertainty theory. In section 4, the mathematical model of uncertain multi objective transportation model is introduced. Deterministic multi objective transportation model, uncertain MOTP model and chance constraint programming model are presented in the subsections 4.1,4.2 and 4.3 respectively. In section 5, a neutrosophic compromise programming approach is introduced and we presented the preliminaries of neutrosophic set. In subsection 5.1, neutrosophic decision making is explained and in subsection 5.2, an algorithm to solve uncertain MOTP is presented. A numerical example has been given in section 6, to understand the applicability of the proposed model and compared with a existing approach. The result and discussion, Implications, and the conclusion have been presented in Section 7,8 and 9 respectively.

2. **Literature Review**

The basic study of the transportation problem (TP) was carried over by Hitchcock [1] and Koopmans [2] played a significant role in its development. Abdelaziz et al [3] had proposed a compromise chance constraint programming model (CCCP) for multi-objective stochastic programming portfolio models.

Aouni et al [4], for the stochastic goal programming model, explicitly introduced the decision-makers preferences adapted chance-constrained-program. A fuzzy multi-objective programming (FMOP) vendor selection model was developed by Wu et al [5]. Bit et al [6] presented an approach to multicriteria decision making transportation problem under fuzzy environment. Zimmerman [7], using fuzzy set theory, solved the multi-objective transportation problem by considering suitable membership functions. A fuzzy goal programming approach to determine an optimal compromise solution for the multi-objective transportation problem by assuming that each objective function has a fuzzy goal was proposed by Zangiabadi and Maleki [8]. Gupta et al [28] proposed a model for the probabilistic fuzzy goal multi-objective supply chain network (PFG-MOSCN) and discussed the solution procedure for the same.

Although fuzzy set theory proposed by Zadeh [9] is widely applied in many uncertain models, it could not handle human uncertainty in some contexts involving incomplete information. As an attempt to deal with such indeterminacies, Liu founded uncertainty theory [10, 11]. Nowadays, uncertainty theory is considered as a mathematical branch for modeling belief degrees and has been adopted in many mathematical models like uncertain programming, uncertain logic, uncertain graph, uncertain statistics and uncertain finance [12–14]. The belief degree of an uncertain event to happen is measured by uncertain measure. The usage of random uncertain variable and chance measure was also introduced by Liu [15]. Post that, he also presented uncertain random programming to model optimization problems containing more than one random variable. Gao [16], in his paper, newly proposed certain properties based on continuously uncertain measures. Seyyed Mojtaba Chasence [17] introduced uncertain linear fractional programming problem and also presented three methods for conversion of uncertain optimization problem into an equivalent deterministic problem. Liu [18] provided a new uncertain multi objective programming and introduced uncertain goal programming as a compromised method to solve multi-objective programming with the uncertain variables, considering the operational law of uncertain variables through inverse uncertainty distribution. Gupta et al [29] formulated the model of an Uncertain multi-objective capacitated transportation problem with mixed constraints. Latter, Srikant Gupta et al [30] proposed the procedure for solving multi-objective capacitated transportation problem under an uncertain environment. S Das et al [39] presented a solution procedure for solving fully fuzzy linear programming problems whose parameters are considered as the trapezoidal fuzzy number. Utilising the aggregate ranking function, Sapan Kumar Das [40] constructed a new framework for neutrosophic integer programming problems involving triangular neutrosophic numbers. SK Das's [41] studied a transportation problem involving pentagonal Neutrosophic numbers where in the supply, demand, and cost of transportation were all ambiguous . Constraints under neutrosophic environment Das et al [42] proposed the solution procedure for solving the Linear Programming Problems with Mixed . Motivated by the above said works, we have proposed the solution procedure for solving the uncertain MOTP by using the neutosophic techniques.

Table 1. Comparision between existing transportation models with proposed model

Author	Nature of the objective		Environment	Methodology Used
	Single	Multiple		
Lakhveer et al [31]	×	✓	Crisp	Using the weighted approach
Subhakantra Dash et al [32]	✓	×	Rough	Using the uncertainity distribution
Bharati et al [33]	×	✓	Interval valued intuitionistic fuzzy sets	Based on extended Yager's function Interval valued intuitionistic fuzzy sets
Haiying Guo et al [34]	✓	×	Uncertainty theory	Using the simplex method
Thamaraiselvi [35]	✓	×	Neutrosophic	The arithmetic operations on single valued neutrosophic trapezoidal numbers areemployed
RizkM.Rizk Allah [36]	×	✓	Neutrosophic	Using Neutrosophic compromise programming approach
Somnath maity [37]	✓	×	Type-2 fuzzy	Using fuzzy number approximation
Deshabrata Roy Mahapatra [38]	×	✓	stochastic	Using fuzzy goal programming
Proposed Model	✓	✓	Uncertainty theory	Using **BOTH** uncertainty theory and Neutrosophic method

The current research on the transportation issue is presented in Table 1. We compared the transportation problems on the basis of the numbers of objectives and the various types of environments. To the best of our knowledge, no one has investigated a multi-objective transportation problem with the simultaneous goals of maximization of profit, minimization of toll tax, and minimization of transportation cost in both neutrosophic and uncertain environment. We have used both the methods to bring the level of indeterminancy down to the maximum.

3. Preliminaries

The concepts and definitions which will be used in the subsequent discussions has been presented in the section.

Definition 3.1. [13] [10] Let $\mathcal{L}$ be a σ - algebra of collection of events Λ of a universal set Γ. A set function $\mathcal{M}$ is said to be uncertain measure defined on the σ - algebra where $\mathcal{M}\{\Lambda\}$ indicate the belief degree with which we believe that the event will happen; It satisfies the following axioms:

(1) Normality Axiom: For the universal set Γ, we have $\mathcal{M}\{\Gamma\} = 1$.
(2) Duality Axiom: For any event Γ, we have $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^C\} = 1$.
(3) Subadditivity Axiom: For every countable sequence of events $\Lambda_1, \Lambda_2, \cdots$, we have $\mathcal{M}\{\bigcup_{i=1}^{\infty} \Lambda_i\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}$.
(4) Product Axiom: Let $(\Gamma_i, \mathcal{L}_i, \mathcal{M}_i)$ be uncertainty spaces for $i = 1, 2, 3, \cdots$ The product uncertain measure is an uncertain measure holds $\mathcal{M}\{\Pi_{i=1}^{\infty} \wedge_i\} = \wedge_{i=1}^{\infty} \mathcal{M}\{\wedge_i\}$ where $\wedge_i \in \mathcal{L}_i$ for $i = 1, 2, 3, \cdots \infty$.

Definition 3.2. [10] A function $\xi : (\Gamma, \mathcal{L}, \mathcal{M}) \rightarrow \mathcal{R}$ is said to be an uncertain variable such that $\{\xi \in B\} = \{\gamma \in \Gamma / \xi(\gamma) \in B\}$ is an event for any Borel set B of real numbers.

Definition 3.3. [10] An uncertain variable ξ defined on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ is said to be non- negative if $\mathcal{M}\{\xi < 0\} = 0$ and positive if $\mathcal{M}\{\xi \leq 0\} = 0$.

Definition 3.4. [10] The uncertainty distribution $\phi(x)$ of an uncertain variable ξ for any real number x is defined by $\phi(x) = \mathcal{M}\{\xi \leq x\}$.

Definition 3.5. Let $\phi(x)$ be the regular uncertainty distribution of an uncertain variable ξ. Then $\phi^{-1}(\alpha)$ is called inverse uncertainty distribution of ξ and it exists on (0, 1).

Definition 3.6. [10] The uncertain variable ξ_i $(i = 1, 2, 3, \cdots n)$ are said to be independent if

$$\mathcal{M}\left\{\bigcap_{i=1}^{n} (\xi_i \in B_i)\right\} = \wedge_{i=1}^{n} \mathcal{M}(\xi_i \in B_i) \tag{1}$$

where $B_i (i = 1, 2, 3, \cdots n)$ are called Borel sets of real numbers.

Theorem 3.7. *Let ξ be an uncertain variable with regular uncertain distribution function ψ. Then its α - optimistic value and α - pessimistic values are*

$$\xi_{\sup}(\alpha) = \psi^{-1}(1-\alpha),\ \xi_{\inf}(\alpha) = \psi^{-1}(\alpha) \tag{2}$$

Theorem 3.8. *[11] The regular uncertainty distributions of independent uncertain variables $\xi_i(i = 1, 2, 3, \cdots n)$ are $\phi_i(i = 1, 2, 3, \cdots n)$ respectively. If the function $f(x_1, x_2, \cdots, x_n)$ is strictly increasing and strictly decreasing with respect to $x_1, x_2, \cdots, x_m$ and $x_{m+1}, x_{m+2}, \cdots, x_n$ respectively then the uncertain variable $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ has an inverse uncertainty distribution*

$$\begin{aligned}\psi^{-1}(\alpha) =& f(\phi_1^{-1}(\alpha), \phi_2^{-1}(\alpha), \cdots, \phi_m^{-1}(\alpha),\\ & \phi_{m+1}^{-1}(1-\alpha), \phi_{m+2}^{-1}(1-\alpha), \cdots, \phi_n^{-1}(1-\alpha))\end{aligned} \tag{3}$$

Definition 3.9. [10] The expected value of uncertain variable ξ is given by

$$E(\xi) = \int_0^{\infty} \mathcal{M}\{\xi \geq x\}dx - \int_{-\infty}^{0} \mathcal{M}\{\xi \leq x\}dx \tag{4}$$

This is valid only if at least one of the integral is finite.

Theorem 3.10. *[22] Let $\phi_i(i = 1, 2, 3, \cdots n)$ be regular uncertainty distributions of independent $\xi_i(i = 1, 2, 3, \cdots n)$ with respectively. If the function $f(x_1, x_2, \cdots, x_n)$ is strictly increasing and strictly decreasing w.r.to*
$x_1, x_2, \cdots, x_m$ and $x_{m+1}, x_{m+2}, \cdots, x_n$ respectively, then

$$\begin{aligned}E(\xi) = \int_0^1 & f(\phi_1^{-1}(\alpha), \cdots, \phi_m^{-1}(\alpha),\\ & \phi_{m+1}^{-1}(1-\alpha), \cdots, \phi_n^{-1}(1-\alpha))d\alpha\end{aligned} \tag{5}$$

From the above theorem, we know that

$$E(\xi) = \int_0^1 \phi^{-1}(\alpha)d\alpha \tag{6}$$

where ξ is an uncertain variable with regular uncertainty distribution Φ.

Definition 3.11. [10] A linear uncertain variable ξ is defined as

$$\phi(x) = \begin{cases} 0 & \text{if}\, x \leq 1 \\ \dfrac{x-l}{m-l} & \text{if}\, l \leq x \leq m \\ 1 & \text{if}\, x \geq m \end{cases} \tag{7}$$

represented by $L(l, m)$, where l and $m \in R$ with $l < m$.
The inverse distribution function of a linear uncertain variable $L(l, m)$ is given by

$$\phi^{-1}(\alpha) = (1-\alpha)l + \alpha m \tag{8}$$

and its expected value is given by

$$E(\xi) = \frac{l+m}{2} \tag{9}$$

Definition 3.12. [10] The distribution function of a normal uncertain variable is

$$\phi(x) = \left(1 + exp^{\left(\frac{\pi(\mu - x)}{\sigma\sqrt{3}}\right)}\right)^{-1}, x \geq 0 \tag{10}$$

and it is denoted as $N(\mu, \sigma); \mu, \sigma \in R$ with $\sigma > 0$.
The inverse uncertainty distribution and the expected value of $N(\mu, \sigma)$ is defined as follows

$$\phi^{-1}(\alpha) = \mu + \frac{\sigma\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha} \tag{11}$$

$$E(\xi) = \mu \tag{12}$$

4. Uncertain Multi objective transportation model

In this section, we introduce the mathematical formulation of uncertain multi objective transportation problem (UMOTP). For the formulation of UMOTP, the following assumptions such as indexes, decision variables and parameters are considered as follows.

i	index for origins
j	index for destinations
k	index for objective function
x_{ij}	quantity transported from i^{th} origin to j^{th} destination
$\mathcal{Z}_k$	k^{th} objective function
c_{ij}^k	the unit cost of transportation from i^{th} origin to j^{th} destination for the k^{th} objective function

a_i	total amount of product available at orgin i
b_j	total demand of the product at destination j
$Z_k(\boldsymbol{x}:\boldsymbol{\xi})$	k^{th} objective function with uncertain variable
ξ_{ij}^k	uncertain cost coefficient of the k^{th} objective
γ_i	uncertain availability at origin i
η_j	uncertain capacity of destination j
α	confidence level for objective function, $\alpha \in (0,1)$
α_i	confidence level for availability constraint, $\alpha_i \in (0,1)$
β_j	confidence level for destination constraint, $\beta_j \in (0,1)$
ψ^k	regular uncertainty distribution for the independent uncertain variable ξ^k
ψ_{ij}^k	regular uncertainty distribution for the independent uncertain variable ξ_{ij}^k
ϕ_i	regular uncertainty distribution for the independent uncertain variable γ_i
θ_j	regular uncertainty distribution for the independent uncertain variable η_j
N	neutrosophic set
X	space of objects
T_N	truth membership function
I_N	indeterminacy membership function
F_N	falsity membership function
t_k, s_k	predetermined numbers in (0,1).
U_k	upper bound of the k^{th} objective
L_k	lower bound of the k^{th} objective
D_N	neutrosophic decision set
G_k	neutrosophic goal
C_i	neutrosophic constraint
$\lambda_T, \lambda_I, \lambda_F$	auxiliary parameters

4.1. *Deterministic model of Multi objective transportation problem*

The mathematical formulation of deterministic multi objective transportation problem is

$$\begin{aligned} \text{Min}\, Z_k(x) &= \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}^k x_{ij}\,(k=1,2,\cdots,K) \\ \text{subject to} &\sum_{j=1}^{n} x_{ij} \leq a_i,\, i=1,2,\cdots,m \\ &\sum_{i=1}^{m} x_{ij} \geq b_j,\, j=1,2,\cdots,n \\ &x_{ij} \geq 0, \forall i,j \end{aligned} \tag{13}$$

Here $c_{ij}^k, a_i, (i = 1,2,\cdots,m)$ and $b_j, (j = 1,2,\cdots,n)$ are the cost, supply and demand parameters of multi objective transportation problem respectively which are represented by crisp numbers. Without loss of generality, it may be considered that $a_i \geq 0, \forall i, b_j \geq 0, \forall j$ and $c_{ij}^k \geq 0, \forall k$ and $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$.

4.2. *Mathematical model for uncertain multi objective transportation problem*

In real life scenario, planning is made in prior before the transportation process. But many uncertain factors like road conditions, climate changes, changes in sales due to attitude of customers, operate parallelly, making demand, supply and transportation cost remain uncertain. Hence, cost, supply and demand parameters c_{ij}^k, a_i and b_j respectively are considered as uncertain variables and are represented by ξ_{ij}^k, γ_i and η_j.
Then the mathematical model for uncertain multi objective transportation problem is defined as

$$\begin{aligned} \text{Min}\, Z_k(x;\xi) &= \sum_{i=1}^{m}\sum_{j=1}^{n} \xi_{ij}^k x_{ij}\,(k=1,2,\cdots,K) \\ \text{subject to} &\sum_{j=1}^{n} x_{ij} \leq \gamma_i,\, i=1,2,\cdots,m \\ &\sum_{i=1}^{m} x_{ij} \geq \eta_j,\, j=1,2,\cdots,n \\ &x_{ij} \geq 0, \forall i,j \end{aligned} \tag{14}$$

As we cannot deal with uncertain environment directly, we have to convert(14) into an equivalent deterministic model by using expected value model or chance constrained model or taking confidence level on the constraint functions and expected value on the objective function. As chance constraint programming model provides most suitable solutions [23], we make use of the chance constraint model for uncertain multi objective transportation problem as shown below.

4.3. *Chance constraint model of UMOTP*

Let α be the predetermined confidence level with $\alpha \in (0,1)$. The decision maker aims to get a smallest value $\tilde{f}$ such that uncertain variable $Z_k(\boldsymbol{x} : \boldsymbol{\xi}) \leq \tilde{f}$ with the predetermined confidence level α.

Definition 4.1. The solution vector $\boldsymbol{x} = (x_{ij}) \geq 0$ is a feasible solution of the model (14), if it holds the below constraints.

$$\mathcal{M}\left\{\sum_{i=1}^{m}\sum_{j=1}^{n}\xi_{ij}^{k}x_{ij} \leq \tilde{f}\right\} \geq \alpha,\ k = 1, 2, \cdots, K \tag{15}$$

$$\mathcal{M}\left\{\sum_{j=1}^{n}x_{ij} \leq \gamma_i\right\} \geq \alpha_i,\ i = 1, 2, \cdots, m \tag{16}$$

$$\mathcal{M}\left\{\sum_{i=1}^{m}x_{ij} \geq \eta_j\right\} \geq \beta_j,\ j = 1, 2, \cdots, n \tag{17}$$

Definition 4.2. A feasible solution x^* is said to be pareto optimal solution of the model (14) if there exists no other feasible solution $\boldsymbol{x}$ such that

$$\begin{gathered}\text{Min}\left\{\tilde{f}/\mathcal{M}\left\{Z_k(\boldsymbol{x} : \boldsymbol{\xi}) \leq \tilde{f}\right\} \geq \alpha\right\} \leq \text{Min}\left\{\tilde{f}/\mathcal{M}\left\{Z_k(\boldsymbol{x^*} : \boldsymbol{\xi}) \leq \tilde{f}\right\} \geq \alpha\right\} \\ \forall k = 1, 2, \cdots, K\end{gathered} \tag{18}$$

Definition 4.3.

$$\begin{gathered}\text{Min}\left\{\tilde{f}/\mathcal{M}\left\{Z_k(\boldsymbol{x} : \boldsymbol{\xi}) \leq \tilde{f}\right\} \geq \alpha\right\} < \text{Min}\left\{\tilde{f}/\mathcal{M}\left\{Z_k(\boldsymbol{x^*} : \boldsymbol{\xi}) \leq \tilde{f}\right\} \geq \alpha\right\} \\ \text{for atleast one}\, k = 1, 2, \cdots, K\end{gathered} \tag{19}$$

The chance constraint programming model of UMOTP can be constructed as follows

$$\begin{aligned}&\text{Min}\tilde{f} \\ &\text{subject to} \\ &\mathcal{M}\left\{\sum_{i=1}^{m}\sum_{j=1}^{n}\xi_{ij}^{k}\boldsymbol{x}_{ij} \leq \tilde{f}\right\} \geq \alpha,\ k = 1, 2, \cdots, K \\ &\mathcal{M}\left\{\sum_{j=1}^{n}\boldsymbol{x}_{ij} \leq \gamma_i\right\} \geq \alpha_i \\ &\mathcal{M}\left\{\sum_{i=1}^{m}\boldsymbol{x}_{ij} \geq \eta_j\right\} \geq \beta_j \\ &\boldsymbol{x}_{ij} \geq 0, i = 1, 2, \cdots, m, j = 1, 2, \cdots, n\end{aligned} \tag{20}$$

Here, the confidence levels $\alpha, \alpha_i, \beta_j$ are predetermined from the interval (0,1).

Definition 4.4 (Pareto optimal solution)**.** Pareto optimal solution is defined as a set of 'non-inferior' solutions in the objective space defining a boundary beyond which none of the objectives can be improved without sacrificing at least one of the other objectives.

Theorem 4.5. *Suppose that $\xi_{ij}^k, \gamma_i, \eta_j$ are independent uncertain variables with regular uncertainty distribution $\psi_{ij}^k, \phi_i, \theta_j$ respectively. The equivalent deterministic model of chance constraint model is*

$$\begin{aligned} & Min\, Z_k^* = \sum_{i=1}^{m}\sum_{j=1}^{n}(\psi_{ij}^k)^{-1}(\alpha)\; x_{ij}\,(k=1,2,\cdots,K) \\ & subject\ to \\ & \sum_{j=1}^{n} x_{ij} \le (\phi_i)^{-1}(1-\alpha_i),\; i=1,2,\cdots,m \\ & \sum_{i=1}^{m} x_{ij} \ge (\theta_j)^{-1}(\beta_j),\; j=1,2,\cdots,n \\ & x_{ij} \ge 0, i=1,2,\cdots,m, j=1,2,\cdots,n \end{aligned} \tag{21}$$

Proof:
Assume that uncertainty variable $\xi_k = \sum_{i=1}^{m}\sum_{j=1}^{n}(\xi_{ij}^k)x_{ij}$ has distribution function ψ_k.
Let $f(y_{11}, y_{12}, \cdots, y_{mn}) = y_{11}x_{11} + y_{12}x_{12} + \cdots + y_{mn}x_{mn}$
It is clear that this function is strictly increasing with respect to $y_{11}, y_{12}, \cdots, y_{mn}$ then by the theorem (3.8), the uncertain variable $\boldsymbol{\xi}_k$ has an inverse uncertainty distribution.

$$(\psi_k)^{-1}(\alpha) = \sum_{j=1}^{n}\sum_{i=1}^{m}(\psi_{ij}^k)^{-1}(\alpha)x_{ij}$$

So, we have

$$\begin{aligned} & \mathcal{M}\left\{\sum_{j=1}^{n}\sum_{i=1}^{m}(\xi_{ij})^k x_{ij} \le \tilde{f}\right\} \ge \alpha \\ & \Leftrightarrow \psi^k(\tilde{f}) \ge \alpha \\ & \Leftrightarrow (\psi^k)^{-1}(\alpha) \le \tilde{f} \\ & (i.e.) \sum_{j=1}^{n}\sum_{i=1}^{m}(\psi_{ij}^k)^{-1}(\alpha)x_{ij} \le \tilde{f} \end{aligned}$$

For the constraints, we have

$$\begin{aligned}
&\mathcal{M}\left\{\sum_{j=1}^{n} x_{ij} \leq \gamma_i\right\} \geq \alpha_i \\
&\Leftrightarrow \mathcal{M}\left\{\sum_{j=1}^{n} x_{ij} - \gamma_i \leq 0\right\} \geq \alpha_i \\
&\Leftrightarrow \sum_{j=1}^{n} x_{ij} - (\varphi_i)^{-1}(1-\alpha) \leq 0 \\
&\Leftrightarrow \sum_{j=1}^{n} x_{ij} \leq (\varphi_i)^{-1}(1-\alpha_i)
\end{aligned}$$

Similarly $\mathcal{M}\left\{\sum_{i=1}^{m} x_{ij} \geq \eta_j\right\} \geq \beta_j$ is equivalent to
$\sum_{i=1}^{m} x_{ij} \geq (\theta_j)^{-1}(\beta_j)$, $j = 1, 2, \cdots, n$.
Hence the theorem is proved.

Corollary 4.6. *Let $x_{ij}, i = 1, 2, \cdots, m, j = 1, 2, \cdots, n$ be the non negative decision variable and $\xi_k, k = 1, 2, \cdots, K$ are independently uncertain variables with expected values $e_{ij}, i = 1, 2, \cdots, m, j = 1, 2, \cdots, n$ and the variances $\sigma_{ij}^2, i = 1, 2, \cdots, m, j = 1, 2, \cdots, n$ respectively. If ξ be a normal uncertain variable $N(e, \sigma)$, then for any $\alpha \in (0, 1)$, the model (21) can be converted into the following model.*

$$\begin{aligned}
&\text{Min}\,(e_{ij})_k + \frac{(\sigma_{ij})_k\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha},\ k = 1, 2, \cdots, K \\
&\text{subject to} \sum_{j=1}^{n} x_{ij} \leq e_i + \frac{\sigma_i\sqrt{3}}{\pi}\ln\frac{1-\alpha_i}{\alpha_i}\ i = 1, 2, \cdots, m \\
&\sum_{i=1}^{m} x_{ij} \geq e_j^* + \frac{\sigma_j^*\sqrt{3}}{\pi}\ln\frac{\beta_j}{1-\beta_j},\ j = 1, 2, \cdots, n \\
&x_{ij} \geq 0, i = 1, 2, \cdots, m, j = 1, 2, \cdots, n
\end{aligned} \tag{22}$$

5. Neutrosophic compromise programming approach

In this section first we introduce some basic definitions of neutrosophic set theory and then we will discuss about neutrosophic compromise programming approach.

Definition 5.1. A neutrosophic set N defined in the universal set X is characterized by truth membership function $T_N(x)$, indeterminacy membership function $I_N(x)$ and a falsity membership function $F_N(x)$ and is denoted by

$$N = \{\langle x, T_N(x), I_N(x), F_N(x)\rangle\,|x \in X\} \tag{23}$$

where $T_N(x), I_N(x), F_N(x)$ are real standard or non standard subsets belonging to]0- ,1+[. Also the membership grades of truth, indeterminacy and falsity are the functions from X to]0- ,1+[. Also we have $0^- \leq \sup T_N(x) + \sup I_N(x) + \sup F_N(x) \leq 3^+$ as there is no restriction on the sum of $T_N(x), I_N(x)$ & $F_N(x)$.
Wang [24] introduced Single valued Neutrosophic set (SVNS) in engineering problem as it is computationally more comfortable.

Definition 5.2. [24] A single valued neutrosophic set N defined on X is expressed as
$N = \{\langle x, T_N(x), I_N(x), F_N(x)\rangle \,|x \in X\}$ where
$T_N(x), I_N(x), F_N(x) \in [0, 1], \forall x \in X$ and
$0 \leq T_N(x), I_N(x), F_N(x) \leq 3$. Clearly, SVNS is subset of neutrosophic set.

Definition 5.3. [25] Let P and Q are the two Single Valued Netuosophic Sets (SVNSs). Then their union also a SVNS and their membership functions are given by

$$T_{P\cup Q}(x) = \text{Max}\{T_P(x), T_Q(x)\};$$
$$I_{P\cup Q}(x) = \text{Max}\{I_P(x), I_Q(x)\};$$
$$F_{P\cup Q}(x) = \text{Min}\{F_P(x), F_Q(x)\}$$

Definition 5.4. [25] Let P and Q are SVNS, then their intersection also a SVNS with the following membership functions

$$T_{P\cap Q}(x) = \text{Min}\{T_P(x), T_Q(x)\};$$
$$I_{P\cap Q}(x) = \text{Min}\{I_P(x), I_Q(x)\};$$
$$F_{P\cap Q}(x) = \text{Max}\{F_P(x), F_Q(x)\}$$

Definition 5.5. The complement of the neutrosophic set N is denoted by $c(N)$ and is defined by $T_{c(N)}(x) = F_N(x), I_{c(N)}(x) = 1 - I_N(x),\ F_{c(N)}(x) = T_N(x),\ \forall x \in X$

5.1. *Neutrosophic Decision making*

In this section, a neutrosophic approach to solve a deterministic model (21) is presented. Indeterminacy part present in the optimization problem considered, is handled by neutrosophic programming approach as it simultaneously maximizes the degree of satisfaction(truth) and the degree of dissatisfaction(falsity) and minimizes the degree of satisfaction to some extent(Indeterminacy) of neutrosophic decis ion [21, 26]. A conjunction of neutrosophic goal G_k and neutrosophic constraint C_i is the neutrosophic decision set D_N, that is,

$$D_N = \left(\bigcap_{k=1}^{K} G_k\right)\left(\bigcap_{i=1}^{m} C_i\right)$$
$$= \{\langle x, T_D(x), I_D(x), F_D(x)\rangle \,|x \in X\}$$

where

$$T_D(x) = \min \left\{ \begin{array}{l} T_{G_1}(x), T_{G_2}(x), \cdots, T_{G_k}(x); \\ T_{C_1}(x), T_{C_2}(x), \cdots, T_{C_m}(x); \end{array} \right\}, x \in X$$

$$I_D(x) = \min \left\{ \begin{array}{l} I_{G_1}(x), I_{G_2}(x), \cdots, I_{G_k}(x); \\ I_{C_1}(x), I_{C_2}(x), \cdots, I_{C_m}(x); \end{array} \right\}, x \in X$$

$$F_D(x) = \max \left\{ \begin{array}{l} F_{G_1}(x), F_{G_2}(x), \cdots, F_{G_k}(x); \\ F_{C_1}(x), F_{C_2}(x), \cdots, F_{C_m}(x); \end{array} \right\}, x \in X$$

where $T_D(x), I_D(x), F_D(x)$ are truth, indeterminacy and falsity membership functions respectively of neutrosophic decision set D_N. To formulate the membership function for the deterministic model (21) for the uncertain MOTP, the upper bound U_k and lower bound L_k for each objective function is calculated. By solving K objective function individually subject to the constraints we obtained k solutions $\boldsymbol{x_1, x_2, \cdots, x_K}$.

To find the bounds for each objective function, these K solutions are substituted in each objective function.

$$\begin{aligned} &(i.e.)\, U_k = \max\{F_k(\boldsymbol{x_1}), F_k(\boldsymbol{x_2}), \cdots, F_k(\boldsymbol{x_K})\} \\ &\text{and } L_k = \min\{F_k(\boldsymbol{x_1}), F_k(\boldsymbol{x_2}), \cdots, F_k(\boldsymbol{x_K})\} \end{aligned} \tag{24}$$

Hence, the upper and lower bounds for truth, falsity and indeterminacy membership function are given by

$$\left. \begin{aligned} &U_k^T = U_k, L_k^T = L_k \\ &U_k^F = U_k^T, L_k^F = L_k^T + t_k(U_k^T - L_k^T) \\ &U_k^I = L_k^T + s_k(U_k^T - L_k^T), L_k^I = L_k^T \end{aligned} \right\} \tag{25}$$

where t_k, s_k are predetermined real numbers in (0,1).

Using the above upper and lower bounds, the membership functions of truth, indeterminacy and falsity of model (21) can be interpreted as follows:

$$T_k(Z_k^*(x)) = \begin{cases} 1 & \text{if } Z_k^*(x) < L_k^T \\ \dfrac{U_k^T - Z_k^*(x)}{U_k^T - L_k^T} & \text{if } L_k^T \leq Z_k^*(x) \leq U_k^T \\ 0 & \text{if } Z_k^*(x) > U_k^T \end{cases} \tag{26}$$

$$I_k(Z_k^*(x)) = \begin{cases} 1 & \text{if } Z_k^*(x) < L_k^I \\ \dfrac{U_k^I - Z_k^*(x)}{U_k^I - L_k^I} & \text{if } L_k^I \leq Z_k^*(x) \leq U_k^I \\ 0 & \text{if } Z_k^*(x) > U_k^I \end{cases} \tag{27}$$

$$F_k(Z_k^*(x)) = \begin{cases} 1 & \text{if } Z_k^*(x) > U_k^F \\ \dfrac{Z_k^F - L_k^*(x)}{U_k^F - L_k^F} & \text{if } L_k^F \leq Z_k^*(x) \leq U_k^F \\ 0 & \text{if } Z_k^*(x) < L_k^F \end{cases} \tag{28}$$

where $U_k^{(.)} \neq L_k^{(.)}$ for all objectives. The value of this membership function is set to one, if $U_k^{(.)} = L_k^{(.)}$. Following the Bellman and Zadeh [26], the neutroshopic optimization model of (21) can be stated as follows

$$\begin{aligned}
&\text{Max } \min_k \{T_k(Z_k^*(x))\} : k = 1, 2, \cdots, K \\
&\text{Min } \max_k \{F_k(Z_k^*(x))\} : k = 1, 2, \cdots, K \\
&\text{Max } \min_k \{I_k(Z_k^*(x))\} : k = 1, 2, \cdots, K
\end{aligned}$$

where

$$\begin{aligned}
&\text{Min } Z_k^*(x) = \sum_{i=1}^{m}\sum_{j=1}^{n} (\psi_{ij}^k)^{-1}(\alpha) x_{ij},\, k = 1, 2, \cdots, K \\
&\text{subject to} \\
&\sum_{j=1}^{n} x_{ij} \leq (\varphi_i)^{-1}(1-\alpha_i)\, i = 1, 2, \cdots, m \\
&\sum_{i=1}^{m} x_j \geq (\theta_j)^{-1}(\beta_j),\, j = 1, 2, \cdots, n \\
&x_{ij} \geq 0, i = 1, 2, \cdots, m, j = 1, 2, \cdots, n
\end{aligned} \tag{29}$$

By using the auxiliary parameters, the above problem can be transformed as

$$\begin{aligned}
&\text{Max}\, \lambda_T \\
&\text{Max}\, \lambda_I \\
&\text{Min}\, \lambda_F \\
&\text{subject to} \\
&T_{z_k}(x) \geq \lambda_T,\, I_{z_k}(x) \geq \lambda_I,\, F_{z_k}(x) \leq \lambda_F \\
&\sum_{j=1}^{n} x_{ij} \leq (\varphi_i)^{-1}(1-\alpha_i)\, i = 1, 2, \cdots, m \\
&\sum_{i=1}^{m} x_{ij} \geq (\theta_j)^{-1}(\beta_j),\, j = 1, 2, \cdots, n \\
&x_{ij} \geq 0, i = 1, 2, \cdots, m, j = 1, 2, \cdots, n \\
&\lambda_T \geq \lambda_I, \lambda_T \geq \lambda_F, \lambda_T + \lambda_I + \lambda_F \leq 3, \lambda_T, \lambda_I, \lambda_F \in [0, 1]
\end{aligned} \tag{30}$$

The simplified model of uncertain MOTP (21) can be represented as follows:

$$\begin{aligned}
&\text{Max}\,\lambda_T - \lambda_F + \lambda_I \\
&\text{subject to} \\
&\sum_{j=1}^{n} x_{ij} \leq (\varphi_i)^{-1}(1-\alpha_i),\ i = 1, 2, \cdots, m \\
&\sum_{i=1}^{m} x_{ij} \geq (\theta_j)^{-1}(\beta_j),\ j = 1, 2, \cdots, n \\
&x_{ij} \geq 0, i = 1, 2, \cdots, m, j = 1, 2, \cdots, n \\
&Z_k^*(x) + (U_k^T - L_k^T)\lambda_T \leq U_k^T \\
&Z_k^*(x) + (U_k^I - L_k^I)\lambda_I \leq U_k^I \\
&Z_k^*(x) - (U_k^F - L_k^F)\lambda_F \leq L_k^F \\
&\lambda_T \geq \lambda_I, \lambda_T \geq \lambda_F, \lambda_T + \lambda_I + \lambda_F \leq 3, \\
&\lambda_T, \lambda_I, \lambda_F \in [0, 1]
\end{aligned} \tag{31}$$

5.2. *Algorithm for solving uncertain MOTP under Neutrosophic environment*

In this section, the algorithm for solving uncertain MOTP under neutosophic environment to obtain the pareto optimal solution is presented.

Step 1: Convert the Uncertain MOTP (14) into a deterministic model by using chance constraint model (21).

Step 2: Solve each objective function individually subject to the constraints.

Let $\boldsymbol{x_1, x_2, \cdots, x_K}$ represent the respective ideal solutions for k objective transportation problems. If all k objectives have same solutions $\boldsymbol{x_1 = x_2 = \cdots = x_K} = \{x_{ij}\}_{i,j=1}^{m,n}$ choose one of them as optimal compromise solution, otherwise go to step 3.

Step 3: Calculate the lower and upper bounds for all objectives functions

$$\begin{aligned}
&U_1 = \text{Max}\,\{F_1(x_1), \ldots, F_1(x_k)\} \\
&U_2 = \text{Max}\,\{F_2(x_1), \ldots, F_2(x_k)\} \\
&\vdots \\
&U_k = \text{Max}\,\{F_k(x_1), \ldots, F_k(x_k)\} \\
&L_1 = \text{Min}\,\{F_1(x_1), \ldots, F_1(x_k)\} \\
&\vdots \\
&L_k = \text{Min}\,\{F_k(x_1), \ldots, F_k(x_k)\}
\end{aligned} \tag{32}$$

Step 4: Define the truth, indeterminacy and falsity membership functions of the objective functions and constraints using equations (26), (27), (28).

Step 5: Formulate the neutrosophic compromise programming model for given the uncertain MOTP using the model (31) and solve it for Pareto optimal solution.

6. Illustrative example

Illustrative example from Gurupada et al [27] is considered to demonstrate the proposed approach where all the multi objective functions parameters are considered to be uncertain. The decision maker aims to distribute the product from three sources namely M_1, M_2, M_3 to 4 destinations namely C_1, C_2, C_3 and C_4 in the planning process he likes to optimize the following objective function as

* Minimize the transportation cost (Z_1)
* Minimize the toll tax (Z_2)
* Maximize the profit (Z_3)

Table 2. Transportation cost C^1_{ij} (in $) and loss of time (in week)

	C_1	C_2	C_3	C_4
M_1	(20, .1)	(18, .1)	(22, .1)	(24 , .1)
M_2	(10, 0)	(12, .2)	(15, 0)	(13 , 0)
M_3	(22, 0)	(20, .1)	(24, 1)	(23, .15)

Table 3. Toll tax cost C^2_{ij} (in $) for transportation goods

	C_1	C_2	C_3	C_4
M_1	5	6	4	3
M_2	6	5	5	4
M_3	9	8	8	10

Table 4. Cost parameters C^3_{ij} related to profit (in $) and loss of time (in week).

	C_1	C_2	C_3	C_4
M_1	(3, 0.1)	(3.5, 0.1)	(2.5, 0.1)	(5 ,0.1)
M_2	(3, 0)	(6, 0.2)	(4, 0)	(4 ,0)
M_3	(4, 0)	(3, 0.1)	(4, 1)	(5, 0.15)

The supply parameters a_1, a_2 and a_3 of mines M_1, M_2 and M_3 the demand parameters b_1, b_2, b_3 and b_4 of cities C_1, C_2, C_3 and C_4 follow normal distribution $N(e_i^1, \sigma_i^1)$, for $i = 1, 2, 3$ and $N(e_j^2, \sigma_j^2)$, for $j = 1, 2, 3, 4$ respectively. The data for supply a_i and demand $b_j, \forall i, j$ are presented in table 4 and 5.

TABLE 5. Uncertain supply parameters a_i.

M_1	M_2	M_3
(55, 4)	(60, 5)	(70, 4)

TABLE 6. Uncertain demand parameters b_j.

C_1	C_2	C_3	C_4
(40, 3)	(36, 4)	(35, 5)	(40, 3)

Step 1:

Assume the confidence level as $\alpha = 0.9, \alpha_i = 0.9$ and $\beta_j = 0.9$ for all $i = 1, 2, 3$ and $j = 1, 2, 3, 4$. By using the theorem (4.5), the equivalent deterministic model of the problem is

Min Z_1=Min $Z_1^* = 20.1x_{11} + 18.1x_{12} + 22.1x_{13} + 24.1x_{14} + 10x_{21}$

$+12.2x_{22} + 15x_{23} + 13x_{24} + 22x_{31} + 20.1x_{32} + 25.2x_{33} + 23.2x_{34}$

Min Z_2=Min $Z_2^* = 5x_{11} + 6x_{12} + 4x_{13} + 3x_{14} + 6x_{21} + 5x_{22}$

$+5x_{23} + 4x_{24} + 9x_{31} + 8x_{32} + 8x_{33} + 10x_{34}$

Max Z_3=Min $Z_3^* = -3.1x_{11} - 3.6x_{12} - 2.6x_{13} - 5.1x_{14} - 3x_{21}$

$-6.2x_{22} - 4x_{23} - 4x_{24} - 4x_{31} - 3.1x_{32} - 5.2x_{33} - 5.2x_{34}$

Subject to

$x_{11} + x_{12} + x_{13} + x_{14} + x_{15} = 50.2$

$x_{21} + x_{22} + x_{23} + x_{24} + x_{25} = 53.9$

$x_{31} + x_{32} + x_{33} + x_{34} + x_{35} = 65.2$

$x_{11} + x_{21} + x_{31} = 43.6$

$x_{12} + x_{22} + x_{32} = 40.8$

$x_{13} + x_{23} + x_{33} = 41.1$

$x_{14} + x_{24} + x_{34} = 43.6$

$x_{15} + x_{25} + x_{35} = 0.2$

Step 2: Solving the above objective functions individually, we get

$\boldsymbol{x_1} = (0, 9.1, 41.1, 0, 0, 43.6, 0, 0, 10.3, 0, 0, 31.7, 0, 33.3, 0.2)$

$\boldsymbol{x_2} = (0, 0, 6.6, 43.6, 0, 0, 19.4, 34.5, 0, 0, 43.6, 21.4, 0, 0, 0, 2)$

$\boldsymbol{x_3} = (6.6, 0, 0, 43.6, 0, 12.9, 40.8, 0, 0, 0.2, 24.1, 0, 41.1, 0, 0)$

Clearly $\boldsymbol{x_1 \neq x_2 \neq x_3}$.

Step 3: By using the above solutions, we have

$\boldsymbol{Z_1^*(x_1)} = 3052.65, \boldsymbol{Z_1^*(x_2)} = 3340.14, \boldsymbol{Z_1^*(x_3)} = 3376.1$

$\boldsymbol{Z_2^*(x_1)} = 1108.4, \boldsymbol{Z_2^*(x_2)} = 990.3, \boldsymbol{Z_2^*(x_3)} = 990.9$

$\boldsymbol{Z_3^*(x_1)} = -583.05, \boldsymbol{Z_3^*(x_2)} = -738.54, \boldsymbol{Z_3^*(x_3)} = -844.6$

The upper and lower bounds of each objective functions are as follows:

$U_{Z_1^*} = 3376.1, L_{Z_1^*} = 3052.65, U_{Z_2^*} = 1108.4,$

$L_{Z_2^*} = 990.3, U_{Z_3^*} = -583.05, L_{Z_3^*} = -844.6$

Step 4: Formulate the membership functions of the given objectives using the equations (26), (27) and (28).

For Z_1^*:

$U^T_{Z_1^*} = 3376.1, L^T_{Z_1^*} = 3052.65$

$U^F_{Z_1^*} = 3376.1, L^F_{Z_1^*} = 3052.65 + 323.45t_1$

$U^I_{Z_1^*} = 3052.65 + 323.45s_1, L^I_{Z_1^*} = 3052.65$

$$T_1(Z_1^*(x)) = \begin{cases} 1 & \text{if } Z_1^*(x) < 3052.65 \\ \dfrac{3376.1 - Z_1^*(x)}{3376.1 - 3052.65} & \text{if } 3052.65 \leq Z_1^*(x) \leq 3376.1 \\ 0 & \text{if } Z_1^*(x) > 3376.1 \end{cases}$$

$$I_1(Z_1^*(x)) = \begin{cases} 1 & \text{if } Z_1^*(x) < 3052.65 \\ \dfrac{3052.65 + 323.45s_1 - Z_1^*(x)}{323.45s_1} & \text{if } 3052.65 \leq Z_1^*(x) \leq 3052.65 + 323.45s_1 \\ 0 & \text{if } Z_1^*(x) > 3052.65 + 323.45s_1 \end{cases}$$

$$F_1(Z_1^*(x)) = \begin{cases} 1 & \text{if } Z_1^*(x) > 3376.1 \\ \dfrac{Z_1^*(x) - 3052.65 - 323.45t_1}{323.45 - 323.45t_1} & \text{if } 3052.65 + 323.45t_1 \leq Z_1^*(x) \leq 3376.1 \\ 0 & \text{if } Z_1^*(x) < 3052.65 + t_1(323.45) \end{cases}$$

For Z_2^*:

$U^T_{Z_2^*} = 1108.4, L^T_{Z_2^*} = 990.3$

$U^F_{Z_2^*} = 1108.4, L^F_{Z_2^*} = 990.3 + 118.1t_2$

$U^I_{Z_2^*} = 990.3 + 118.1s_2, L^I_{Z_2^*} = 990.3$

$$T_2(Z_2^*(x)) = \begin{cases} 1 & \text{if } Z_2^*(x) < 990.3 \\ \dfrac{1108.4 - Z_2^*(x)}{118.1} & \text{if } 990.3 \leq Z_2^*(x) \leq 1108.4 \\ 0 & \text{if } Z_2^*(x) > 1108.4 \end{cases}$$

$$I_2(Z_2^*(x)) = \begin{cases} 1 & \text{if } Z_2^*(x) < 990.3 \\ \dfrac{990.3 + 118.1s_2 - Z_2^*(x)}{118.1s_2} & \text{if } 990.3 \leq Z_2^*(x) \leq 990.3 + 118.1s_2 \\ 0 & \text{if } Z_2^*(x) > 990.3 + 118.1s_2 \end{cases}$$

$$F_2(Z_2^*(x)) = \begin{cases} 1 & \text{if } Z_2^*(x) > 1108.4 \\ \dfrac{Z_2^*(x) - 990.3 - 118.1t_2}{118.1 - 118.1t_2} & \text{if } 990.3 + 118.1t_2 \leq Z_2^*(x) \leq 1108.4 \\ 0 & \text{if } Z_2^*(x) < 990.3 + 118.1t_2 \end{cases}$$

For Z_3^*:

$U^T_{Z_3^*} = -583.05, L^T_{Z_3^*} = -844.6$

$U^F_{Z_3^*} = -583.05, L^F_{Z_3^*} = -844.6 + 261.55t_3$

$U^I_{Z_3^*} = -844.6 + 261.55s_3, L^I_{Z_3^*} = -844.6$

$$T_3(Z_3^*(x)) = \begin{cases} 1 & \text{if } Z_3^*(x) < -844.6 \\ \dfrac{-583.05 - Z_3^*(x)}{261.55} & \text{if } -844.6 \leq Z_3^*(x) \leq -583.05 \\ 0 & \text{if } Z_3^*(x) > -583.05 \end{cases}$$

$$I_3(Z_3^*(x)) = \begin{cases} 1 & \text{if } Z_3^*(x) < -844.6 \\ \dfrac{-844.6 + 261.55s_3 - Z_3^*(x)}{261.55s_3} & \text{if } -844.6 \leq Z_3^*(x) \leq -844.6 + 261.55s_3 \\ 0 & \text{if } Z_3^*(x) > -844.6 + 261.55s_3 \end{cases}$$

$$F_3(Z_3^*(x)) = \begin{cases} 1 & \text{if } Z_3^*(x) > -583.05 \\ \dfrac{Z_3^*(x) + 844.6 - 261.55t_3}{261.55 - 261.55t_3} & \text{if } -844.6 + 261.55t_3 \leq Z_3^*(x) \leq -583.05 \\ 0 & \text{if } Z_3^*(x) < -844.6 + 261.55t_3 \end{cases}$$

Step 5: The neutrosophic compromise programming model for given the uncertain MOTP using the model (31) is

$$\text{Max } \lambda_T - \lambda_F + \lambda_I$$

subject to

$$x_{11} + x_{12} + x_{13} + x_{14} + x_{15} = 50.2$$

$$x_{21} + x_{22} + x_{23} + x_{24} + x_{25} = 53.9$$

$$x_{31} + x_{32} + x_{33} + x_{34} + x_{35} = 65.2$$

$$x_{11} + x_{21} + x_{31} = 43.6$$

$$x_{12} + x_{22} + x_{32} = 40.8$$

$$x_{13} + x_{23} + x_{33} = 41.1$$

$$x_{14} + x_{24} + x_{34} = 43.6$$

$$x_{15} + x_{25} + x_{35} = 0.2$$

$20.1x_{11} + 18.1x_{12} + 22.1x_{13} + 24.1x_{14} + 10x_{21} + 12.2x_{22} + 15x_{23} + 13x_{24} + 22x_{31} + 20.1x_{32} + 25.2x_{33} + 23.2x_{34} + 233.45\lambda_T \leq 3376.1$

$5x_{11}+6x_{12}+4x_{13}+3x_{14}+6x_{21}+5x_{22}+5x_{23}+4x_{24} +9x_{31}+8x_{32}+8x_{33}+10x_{34}+118.1\lambda_T \leq 1108.4$

$-3.1x_{11} - 3.6x_{12} - 2.6x_{13} - 5.1x_{14} - 3x_{21} - 6.2x_{22} - 4x_{23} - 4x_{24} - 4x_{31} - 3.1x_{32} - 5.2x_{33} - 5.2x_{34}$

$+261.55\lambda_T \leq -583.05$

$20.1x_{11} + 18.1x_{12} + 22.1x_{13} + 24.1x_{14} + 10x_{21} + 12.2x_{22} + 15x_{23} + 13x_{24} + 22x_{31} + 20.1x_{32} + 25.2x_{33} + 23.2x_{34} + 323.45t_1(\lambda_T - 1) \leq 3052.65$

$5x_{11}+6x_{12}+4x_{13}+3x_{14}+6x_{21}+5x_{22}+5x_{23}+4x_{24} +9x_{31}+8x_{32}+8x_{33}+10x_{34}+118.1t_2(\lambda_T-1) \leq 990.3$

$-3.1x_{11} - 3.6x_{12} - 2.6x_{13} - 5.1x_{14} - 3x_{21} - 6.2x_{22} - 4x_{23} - 4x_{24} - 4x_{31} - 3.1x_{32} - 5.2x_{33} - 5.2x_{34} + 261.55t_3(\lambda_T - 1) \leq -844.6$

$20.1x_{11} + 18.1x_{12} + 22.1x_{13} + 24.1x_{14} + 10x_{21} + 12.2x_{22} + 15x_{23} + 13x_{24} + 22x_{31} + 20.1x_{32} + 25.2x_{33} + 23.2x_{34} + (\lambda_F - 1)(3052.65 + 323.45s_1) - 3376.1\lambda_F \leq 0$

$5x_{11} + 6x_{12} + 4x_{13} + 3x_{14} + 6x_{21} + 5x_{22} + 5x_{23} + 4x_{24} + 9x_{31} + 8x_{32} + 8x_{33} + 10x_{34} + (\lambda_F - 1)(990.3 + 118.1s_2) - 1108.4\lambda_F \leq 0$

$-3.1x_{11} - 3.6x_{12} - 2.6x_{13} - 5.1x_{14} - 3x_{21} - 6.2x_{22} - 4x_{23} - 4x_{24} - 4x_{31} - 3.1x_{32} - 5.2x_{33} - 5.2x_{34} + (\lambda_F - 1)(-844.6 + 261.55s_3) + 583.05\lambda_F \leq 0$

$\lambda_T \geq \lambda_I, \lambda_T \geq \lambda_F, \lambda_T + \lambda_F + \lambda_I \leq 3, \lambda_T \leq 1,\ \lambda_I \leq 1, \lambda_F \leq 1$

$0 \leq t_1, s_1 \leq 323.5, 0 \leq t_2, s_2 \leq 118.1, 0 \leq t_3,\ s_3 \leq 261.55, \lambda_T, \lambda_F, \lambda_I \in [0, 1]$

solving the above model by using the LINGO (17.0) software, we get

$\lambda_T = 0.523, \lambda_F = 0, \lambda_I = 0.52,$

$x_{11} = 21.1, x_{12} = 28.1, x_{14} = 0.9, x_{22} = 11.2,$

$x_{24} = 42.6, x_{31} = 22, x_{32} = 1.4, x_{33} = 41.1, x_{35} = 0.2$

$t_1 = 1, t_2 = 1.2, t_3 = 0.9,$

$s_1 = 1.2, s_2 = 1.2, s_3 = 0.47,$

$Z_1 = 3192.71, Z_2 = 1041.2, Z_3 = 717.06.$

Table 7 illustrates the comparison between the results obtained from Fuzzy Multi Choice goal programming method and the proposed method. Table 8 provides the comparison study of solution obtained by fuzzy goal programming method and proposed method.

In Gurupada et al [27] work, wherein he proved that Fuzzy multi choice goal programming was more efficient in providing an optimal solution than by employing goal programming and revised multi choice goal programming approach. Contrasting to his work in the proposed method, the decision maker need not fix the goals of the objective function using any of the existing techniques, to get a better optimal value for the objective function. In short, we have overcome the difficulty of the decision maker to fix the objective value goal.

Clearly it can be seen that by using neutrosophic compromise programming approach, we obtained an improvised pareto optimal solution. As in table 8, we can observe that the proposed method yields a more minimal value for transportation cost and a considerable increase in profit. As neutrosophic programming explores the indeterminacy part of a optimization problem, it helps the decision maker to get better results.

Table 7. Comparison between the pareto optimal solution of the existing and the proposed method.

Method	Pareto-optimal solution
Fuzzy Multi Choice goal programming method [27]	$x_{11} = 3.12$, $x_{12} = 0$, $x_{13} = 18.95$, $x_{14} = 29.10$, $x_{21} = 11.26$, $x_{22} = 25.07$, $x_{23} = 4.36$, $x_{24} = 14.54$, $x_{31} = 29.26$, $x_{32} = 15.42$, $x_{33} = 17.74$, $x_{34} = 0$
Proposed method	$x_{11} = 21.1$, $x_{12} = 28.1$, $x_{14} = 0.9$, $x_{22} = 11.2$, $x_{24} = 42.6$, $x_{31} = 22$, $x_{32} = 1.4$, $x_{33} = 41.1$, $x_{35} = 0.2$

Table 8. The comparison between the existing and the proposed method.

Method	Min Z_1	Min Z_2	Max Z_3
Fuzzy Multi Choice goal programming method [27]	3400	980.13	650
Proposed method	3192.71	1041.2	717.06

7. Result and Discussion

In our work, we have obtained the compromise solution of the Uncertain MOTP using the neutrosphic technique.

Table 7 illustrates the comparison between the results obtained from Fuzzy Multi Choice goalprogramming method and the proposed method. Table 8 provides the comparison study ofsolution obtained by fuzzy goal programming method and proposed method.In Gurupada et al [27] work, wherein he proved that Fuzzy multi choice goal programmingwas more efficient in providing an optimal solution than by employing goal programming andrevised multi choice goal programming approach. Contrasting to his work in the proposedmethod, the decision maker need not fix the goals of the objective function using any of theexisting techniques, to get a better optimal value for the objective function. In short, we haveovercome the difficulty of the decision maker to fix the objective value goal.Clearly it can be seen that by using neutrosophic compromise programming approach, weobtained an improvised pareto optimal solution. As in Table 8, we can observe that the proposed method yields a more minimal value for transportation cost and a considerable increasein profit. As neutrosophic programming explores the indeterminacy part of a optimizationproblem, it helps the decision maker to get better results.

8. Implications

This paper used the neutrosophic approach to discuss the uncertain MOTP. The literature review section includes studies that are comparable to these ones. According to the author's knowledge, no research has been done on applying the neutrosophic method to solve the uncertain MOTP. The method for solving uncertain MOTP utilizing the neutrosophic technique has been provided in the suggested work to close the aforementioned research gap. The efficiency of the proposed work has been demonstrated by comparing Gurupata's [27]'s work. It has been explained that the suggested work will assist the decision maker to have the suitable and desired transportation plan.

9. Conclusion

In this work, a procedure to solve multi objective transportation problem with uncertainvariables is studied under neutrosophic environment. The uncertain MOTP is converted into an equivalent chance constraint deterministic model with the use of operational law of uncertain variables. Then using neutrosophic compromise programming approach the best compromise solution is obtained. Since the solution searches of UMOTP based on different membership function such as truth, indeterminacy and falsity, it allows the decision maker to know about the various functions and provides more practicable and reasonable compromise solution. More info It has been established that, in order to obtain a better optimal value for the objective function, the decision maker does not need to fix the goals of the objective function using any of the available strategies. In other words, we have succeeded in fixing the decision-difficulty maker's with regard to the objective value aim. A numerical example had been considered and obtained the compromise solution and is tabulated in Table 8. It is evident that we were able to achieve an improvised pareto optimum solution by applying the neutrosophic compromise programming approach.

Conflicts of Interest: The authors confirm that there are no known conflicts of interest associated with this publication.

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An intelligent fuzzy parameterized MADM-approach to optimal selection of electronic appliances based on neutrosophic hypersoft expert set

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Abstract. When compared to its extension, the hypersoft set, which deals with discontinuous attribute-valued sets corresponding to different attributes, the soft set only works with a single set of attributes. Numerous scholars created models based on soft sets to address issues in a variety of domains, including decision-making and medical diagnostics. However, these models only take into account one expert, which causes numerous issues for users, particularly when creating questions. We provide a fuzzy parameterized neutrosophic hypersoft expert set to eliminate this mismatch. In addition to addressing the issue of dealing with a single expert, this approach also addresses the problem of soft sets not being adequate for discontinuous attribute-valued sets corresponding to different attributes. The notion of fuzzy parameterized neutrosophic hypersoft expert sets, which combines fuzzy parameterized neutrosophic sets and hypersoft expert sets, is first introduced in this work. Examples are provided to help illustrate some key fundamental concepts, aggregation operations and results. A decision-making application is shown at the end to demonstrate the viability of the suggested theory.

Keywords: Soft set; Soft expert set; Neutrosophic set; Hypersoft set; Fuzzy parameterized neutrosophic hypersoft expert set.

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1. **Introduction**

For a correct description of an object in an ambiguous and uncertain environment, we sometimes consider both the truth membership and the falsity membership in professional systems, belief systems, and information systems. The neutrosophic set was defined by Smarandache [1–3] as a generalisation of classical sets, fuzzy sets, and intuitionistic fuzzy sets. Membership functions are used to define fuzzy sets [4], while membership and nonmembership functions describe intuitionistic fuzzy sets [5], which are used to solve problems involving imprecise, ambiguous, and inconsistent data. The neutrosophic set has numerous applications in a variety of disciplines, including topology, control theory, databases, and medical diagnosis.

Truth membership, indeterminacy membership, and false membership are all wholly independent in the neutrosophic set, where the indeterminacy is clearly quantified. The neutrosophic set and set-theoretic view operators need to be described from a scientific or technical perspective. If not, it will be challenging to apply in actual applications. Therefore, Wang et al. [6] described the set-theoretic operations and various properties of single-valued neutrosophic sets (SVNS). In both theories and applications, work on neutrosophic sets (NS) and their hybrid structures has made rapid progress.advanced quickly [7].

In his conceptualization of soft set theory, Molodtsov [8] described it as a brand-new family of parameterized subsets of the universe of discourse. Different structures of convexity (concavity) on an s-set were introduced by Rahman et al. [9,10]. They explored the different convexity and concavity characteristics in the context of fs-set, s-set, and hypersoft set (an extension of s-set) settings with some altered findings. As a parametrization technique to handle uncertainty, Maji et al. [11] developed fuzzy soft set. This idea has been expanded upon and used in other domains by scholars [12]. Soft expert set (SE-set) and fuzzy soft expert set (FSE-set) are concepts developed by Alkhazaleh et al. [19, 20]. They talked about how they could be used in decision-making. Convexity-cum-concavity on SE-set was conceptualised by Ihsan et al. [21], who also highlighted some of its characteristics. The convexity on the FSE-set was once more gestated and its specific qualities were elucidated by Ihsan et al. [22]. In their conceptualization of intuitionistic fuzzy soft expert sets, Broumi et al. [23] presented their use in decision-making.

Through the substitution of a multi-attribute valued function for a single attribute-valued function in 2018, Smarandache [24] extended soft set to hypersoft set. Saeed et al. [25] developed the idea and covered the principles of the hypersoft set, including its relation, sub relation, complement relation, function, matrices, and operations on hypersoft matrices, as well as its hypersoft subset, complement, and non hypersoft set. Mujahid et al. [25] discussed hypersoft points in several fuzzy-like environments. Complex hypersoft set was defined by Rahman et al. [27], who also created its hybrids with the complex fuzzy set, complex intuitionistic fuzzy set, and complex neutrosophic set. The principles, such as subset, equal sets, null set, absolute set, etc., as well as the theoretic operations, such as complement, union, intersection, etc., were also covered. Convexity and concavity were theorised on a hypersoft set by Rahman et al. [28], who also provided their pictorial representations and examples to illustrate them. Rahman et al. [33] created the preliminary HS-set structure and provided an application for the optimal chemical material choice in DM. Rahman et al. [34] developed a novel method for studying neutrosophic hypersoft graphs and discussed some of its characteristics. The aggregation operations of complex FHS-set were first employed in DMPs by Rahman et al. (AUR5). The structure of interval-valued complex FHS-set was also devised by them. The bijective HS-set was conceptualised by Rahman et al. [36] and its uses in DMPs were covered. In order to know the opinions of various experts in various models when attributed sets are further divided into disjoint attribute valued sets, Ihsan et al. [37] generalised the HS-set to hypersoft expert set (HSE-set). The fuzzy hypersoft expert set (FHSE-set) was conceptualised by Ihsan et al. [38], who also used the proposed technique to demonstrate how DMPs were used.

Çağman et al. [39] applied a significant degree to the parameters and conceptualised the fuzzy parameterized soft set (FPS-set). In order to create hybrids of the fuzzy parameterized soft expert set (FPSE-set) for use in DMPs, Bashir et al. [40] combined the structures of fuzzy parameterized with SE-set. By converting a single set of attributes into several disjoint attribute valued sets, Rahman et al. [41] enhanced the work of fuzzy parameterized soft set to fuzzy parameterized hypersoft set and examined the applications in DMPs. A novel structure for the fuzzy parameterized neutrosophic hypersoft expert set is required by the literature. New ideas on the fuzzy parameterized neutrosophic hypersoft expert set are created as a result. Figure 1 shows how the rest of the paper is organised.

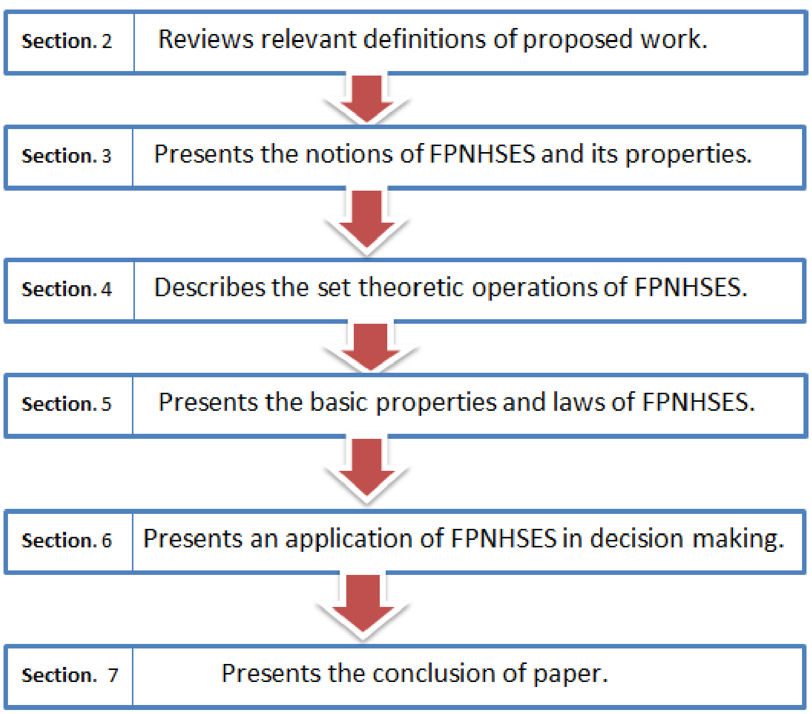

Figure 1. Organization of the paper

2. **Preliminaries**

This section provides definitions and explanations of several key terminology and concepts connected to the primary study.

Definition 2.1. [13] A neutrosophic set $\mathfrak{N}$ in $\triangledown$ is defined by $\mathfrak{N} = \{< \curlywedge, (\mathsf{T}_{\mathfrak{N}}(\curlywedge), \mathsf{I}_{\mathfrak{N}}(\curlywedge), F_{\mathfrak{N}}(\curlywedge)) >: \curlywedge \in \mathsf{E}, \mathsf{T}_{\mathfrak{N}}, \mathsf{I}_{\mathfrak{N}}, F_{\mathfrak{N}} \in]^{-}0, 1^{+}[\}$ where $\mathsf{T}_{\mathfrak{N}}, \mathsf{I}_{\mathfrak{N}}, F_{\mathfrak{N}}$ are truth, indeterminacy, and falsity membership function and $0^{-} \leq \mathsf{T}_{\mathfrak{N}}(\curlywedge), \mathsf{I}_{\mathfrak{N}}(\curlywedge), F_{\mathfrak{N}}(\curlywedge) \leq 3^{+}$.

Definition 2.2. [10] Let $\mathfrak{Q}$ and $\mathfrak{P}$ are two neutrosophic sets such that
$\mathfrak{Q} = \{< \curlywedge, (\mathsf{T}_{\mathfrak{Q}}(\curlywedge), \mathsf{I}_{\mathfrak{Q}}(\curlywedge), F_{\mathfrak{Q}}(\curlywedge)) >: \curlywedge \in \mathsf{E}, \mathsf{T}_{\mathfrak{Q}}, \mathsf{I}_{\mathfrak{Q}}, F_{\mathfrak{Q}} \in]^{-}0, 1^{+}[\}$,
$\mathfrak{P} = \{< \curlywedge, (\mathsf{T}_{\mathfrak{P}}(\curlywedge), \mathsf{I}_{\mathfrak{P}}(\curlywedge), F_{\mathfrak{P}}(\curlywedge)) >: \curlywedge \in \mathsf{E}, \mathsf{T}_{\mathfrak{P}}, \mathsf{I}_{\mathfrak{P}}, F_{\mathfrak{P}} \in]^{-}0, 1^{+}[\}$,
then, the following operations between two neutrosophic sets can be defined like subset, complement, union and intersection:

(1) Neutrosophic set $\mathfrak{Q}$ is a subset of another Neutrosophic set $\mathfrak{P}$ if
$\mathsf{T}_{\mathfrak{Q}}(\curlywedge) \leq \mathsf{T}_{\mathfrak{P}}(\curlywedge)$, $\mathsf{I}_{\mathfrak{Q}} \geq \mathsf{I}_{\mathfrak{P}}$, $F_{\mathfrak{Q}}(\curlywedge) \leq F_{\mathfrak{P}}(\curlywedge)$.
(2) The compliment of neutrosophic set $\mathfrak{Q}$ is defined as
$\mathfrak{Q}^{c} = \{< \curlywedge, (\mathsf{T}_{\mathfrak{Q}}(\curlywedge), 1 - \mathsf{I}_{\mathfrak{Q}}(\curlywedge), F_{\mathfrak{Q}}(\curlywedge)) >: \curlywedge \in \mathsf{E}, \mathsf{T}_{\mathfrak{Q}}, \mathsf{I}_{\mathfrak{Q}}, F_{\mathfrak{Q}} \in]^{-}0, 1^{+}[\}$,
(3) The union of neutrosophic sets between $\mathfrak{Q}$ and $\mathfrak{P}$ is defined by
$max(\mathsf{T}_{\mathfrak{Q}}(\curlywedge), \mathsf{T}_{\mathfrak{P}}(\curlywedge)$, $min(\mathsf{I}_{\mathfrak{Q}}(\curlywedge), \mathsf{I}_{\mathfrak{P}}(\curlywedge))$, $min(F_{\mathfrak{Q}}(\curlywedge), F_{\mathfrak{P}}(\curlywedge))$,
(4) The intersection of neutrosophic sets between $\mathfrak{Q}$ and $\mathfrak{P}$ is defined by
$min(\mathsf{T}_{\mathfrak{Q}}(\curlywedge), \mathsf{T}_{\mathfrak{P}}(\curlywedge)$, $max(\mathsf{I}_{\mathfrak{Q}}(\curlywedge), \mathsf{I}_{\mathfrak{P}}(\curlywedge))$, $max(F_{\mathfrak{Q}}(\curlywedge), F_{\mathfrak{P}}(\curlywedge))$.

Definition 2.3. [23] Let $\mathcal{I}$ represents set of specialists(experts) and set of parameters is denoted by $\mathcal{L}$, $\mathcal{O} = \mathcal{L} \times \mathcal{I} \times \mathcal{U}$ with $\mathcal{S} \subseteq \mathcal{O}$. While $\mathcal{U}$ represents a set of conclusions i.e, $\mathcal{U}$= $\{0 = disagree, 1 = agree\}$ and $\hat{\triangle}$ represents the universe with power set $\mathsf{P}(\hat{\triangle})$ and $\mathbb{I}$ =$[0, 1]$. A $\mathcal{FPSVNSE}$-set can be described as a pair $(\mathsf{g}_{\wedge}, \mathcal{R})$ with $\mathsf{g}_{\wedge}$ is $\mathsf{g}_{\wedge} : \mathcal{R} \to \mathsf{P}(\hat{\triangle})$ such that $\mathsf{P}(\hat{\triangle})$ is going to use for collection of all SVN subsets of $\hat{\triangle}$ and $\mathcal{R} \subseteq \mathcal{O}$.

Definition 2.4. [24] An agree $\mathcal{FPSVNSE}$-set can be defined as a subset of $\mathcal{FPSVNSE}$-set and shown as: $(\mathsf{g}_{\wedge}, \mathcal{R})^{1} = \{\mathsf{g}_{\wedge}(\ddot{\mathrm{v}}) : \ddot{\mathrm{v}} \in \mathcal{L} \times \mathcal{I} \times 1\}$.

Definition 2.5. [24] An disagree $\mathcal{FPSVNSE}$-set can be defined as a subset of $\mathcal{FPSVNSE}$-set and shown as: $(\mathsf{g}_{\wedge}, \mathcal{R})^{0} = \{\mathsf{g}_{\wedge}(\ddot{\mathrm{v}}) : \ddot{\mathrm{v}} \in \mathcal{L} \times \mathcal{I} \times 0\}$.

Definition 2.6. [27] Considering disjoint sets $\underline{\mathrm{H}}_1, \underline{\mathrm{H}}_2, \underline{\mathrm{H}}_3, ..., \underline{\mathrm{H}}_w$ as a corresponding attribute values for w different characteristics $\underline{\mathrm{h}}_1, \underline{\mathrm{h}}_2, \underline{\mathrm{h}}_3, ..., \underline{\mathrm{h}}_w$. Then hypersoft set can be considered as a pair $(Æ, \Upsilon)$, where $\Upsilon = \underline{\mathrm{H}}_1 \times \underline{\mathrm{H}}_2 \times \mathrm{H}_3 \times ..... \times \underline{\mathrm{H}}_m$ and $Æ : \Upsilon \to \mathsf{P}(\Delta)$.

3. Fuzzy Parameterized Neutrosophic Hypersoft Expert Set ($\mathcal{FPNHSE}$-set)

Fuzzy parameterized single valued neutrosophic soft expert set, an existing idea, has been used to build fuzzy parameterized neutrosophic hypersoft expert set in this part. Here, several fundamental qualities are shown.

Definition 3.1. Fuzzy Parameterized Neutrosophic Hypersoft Expert Set A fuzzy parameterized neutrosophic hypersoft expert set $\Psi_{\mathcal{F}}$ over $\hat{\triangle}$ is defined as
$\Psi_{\mathcal{F}} = \left\{ \left(\left(\frac{\hat{q}}{\mu_{\mathcal{F}}(\hat{q})}, \hat{E}_i, \hat{O}_i \right), \frac{\hat{\delta}}{\psi_{\mathcal{F}}(\hat{\delta})} \right); \forall \hat{q} \in \mathcal{Q}, \hat{E}_i \in \mathcal{I}, \hat{O}_i \in \mathcal{U}, \hat{\delta} \in \hat{\triangle} \right\}$ where

(1) $\mu_{\mathcal{F}} : \check{\mathcal{J}} \to \mathsf{FP}(\hat{\triangle})$
(2) $\psi_{\mathcal{F}} : \check{\mathcal{J}} \to \mathsf{NP}(\hat{\triangle})$ is called approximate function of $\mathcal{FPNHSE}$-set
(3) $\check{\mathcal{J}} \subseteq \mathcal{H} = \mathcal{L} \times \mathcal{I} \times \mathcal{U}$ with $\mathcal{S} \subseteq \mathcal{O}$.
(4) where $\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, ..., \mathcal{Q}_r$ are different sets of parameter corresponding to r different parameters $q_1, q_2, q_3, ..., q_r$.
(5) $\mathcal{I}$ be a set of specialists (operators)
(6) $\mathcal{U}$ be a set of conclusions.

Example 3.2. Suppose that a college chain is searching for a construction company to upgrade the college building with globalisation and requires certain specialists(experts) to evaluate its working. Let $\Delta = \{\eta_1, \eta_2, \eta_3, \eta_4\}$ be a set of companies and $\mathcal{G}_1 = \{p_{11}, p_{12}\}$, $\mathcal{G}_2 = \{p_{21}, p_{22}\}$, $\mathcal{G}_3 = \{p_{31}, p_{32}\}$ be disjoint attributive sets for distinct attributes p_1= cheap, p_2= standard, p_3= cooperative. Now $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2 \times \mathcal{G}_3$

$$\mathcal{G} = \left\{ \begin{array}{c} \mho_1/0.2/0.2 = (p_{11}, p_{21}, p_{31}), \mho_2/0.3 = (p_{11}, p_{21}, p_{32}), \\ \mho_3/0.4 = (p_{11}, p_{22}, p_{31}), \mho_4/0.5 = (p_{11}, p_{22}, p_{32}), \\ \mho_5/0.6 = (p_{12}, p_{21}, p_{31}), \mho_6/0.7 = (p_{12}, p_{21}, p_{32}), \\ \mho_7/0.8 = (p_{12}, p_{22}, p_{31}), \mho_8/0.9 = (p_{12}, p_{22}, p_{32}) \end{array} \right\}$$

Now $\mathcal{H} = \mathcal{G} \times \mathcal{D} \times \mathbb{C}$

$$\mathcal{H} = \left\{ \begin{array}{c} (\mho_1/0.2, c, 0), (\mho_1/0.2, c, 1), (\mho_1/0.2, d, 0), (\mho_1/0.2, d, 1), (\mho_1/0.2, e, 0), (\mho_1/0.2, e, 1), \\ (\mho_2/0.3, c, 0), (\mho_2/0.3, c, 1), (\mho_2/0.3, d, 0), (\mho_2/0.3, d, 1), (\mho_2/0.3, e, 0), (\mho_2/0.3, e, 1), \\ (\mho_3/0.4, c, 0), (\mho_3/0.4, c, 1), (\mho_3/0.4, d, 0), (\mho_3/0.4, d, 1), (\mho_3/0.4, e, 0), (\mho_3/0.4, e, 1), \\ (\mho_4/0.5, c, 0), (\mho_4/0.5, c, 1), (\mho_4/0.5, d, 0), (\mho_4/0.5, d, 1), (\mho_4/0.5, e, 0), (\mho_4/0.5, e, 1), \\ (\mho_5/0.6, c, 0), (\mho_5/0.6, c, 1), (\mho_5/0.6, d, 0), (\mho_5/0.6, d, 1), (\mho_5/0.6, e, 0), (\mho_5/0.6, e, 1), \\ (\mho_6/0.7, c, 0), (\mho_6/0.7, c, 1), (\mho_6/0.7, d, 0), (\mho_6/0.7, d, 1), (\mho_6/0.7, e, 0), (\mho_6/0.7, e, 1), \\ (\mho_7/0.8, c, 0), (\mho_7/0.8, c, 1), (\mho_7/0.8, d, 0), (\mho_7/0.8, d, 1), (\mho_7/0.8, e, 0), (\mho_7/0.8, e, 1), \\ (\mho_8/0.9, c, 0), (\mho_8/0.9, c, 1), (\mho_8/0.9, d, 0), (\mho_8/0.9, d, 1), (\mho_8/0.9, e, 0), (\mho_8/0.9, e, 1) \end{array} \right\}$$

let

$$\mathbb{Q} = \left\{ \begin{array}{l} (\mho_1/0.2, c, 0), (\mho_1/0.2, c, 1), (\mho_1/0.2, d, 0), (\mho_1/0.2, d, 1), (\mho_1/0.2, e, 0), (\mho_1/0.2, e, 1), \\ (\mho_2/0.3, c, 0), (\mho_2/0.3, c, 1), (\mho_2/0.3, d, 0), (\mho_2/0.3, d, 1), (\mho_2/0.3, e, 0), (\mho_2/0.3, e, 1), \\ (\mho_3/0.4, c, 0), (\mho_3/0.4, c, 1), (\mho_3/0.4, d, 0), (\mho_3/0.4, d, 1), (\mho_3/0.4, e, 0), (\mho_3/0.4, e, 1), \end{array} \right\}$$

be a subset of $\mathcal{H}$ and $\mathcal{D} = \{c, d, e,\}$ be a set of specialists.

Following survey depicts choices of three specialists:

$$\hbar_1 = \hbar(\mho_1/0.2, c, 1) = \left\{ \frac{\eta_1}{<0.2,0.5,0.4>}, \frac{\eta_2}{<0.7,0.2,0.5>}, \frac{\eta_3}{<0.5,0.4,0.6>}, \frac{\eta_4}{<0.1,0.3,0.6>} \right\},$$
$$\hbar_2 = \hbar(\mho_1/0.2, d, 1) = \left\{ \frac{\eta_1}{<0.4,0.2,0.3>}, \frac{\eta_2}{<0.8,0.1,0.5>}, \frac{\eta_3}{<0.4,0.5,0.6>}, \frac{\eta_4}{<0.2,0.5,0.3>} \right\},$$
$$\hbar_3 = \hbar(\mho_1/0.2, e, 1) = \left\{ \frac{\eta_1}{<0.7,0.2,0.3>}, \frac{\eta_2}{<0.5,0.3,0.6>}, \frac{\eta_3}{<0.6,0.3,0.7>}, \frac{\eta_4}{<0.3,0.5,0.6>} \right\},$$
$$\hbar_4 = \hbar(\mho_2/0.3, c, 1) = \left\{ \frac{\eta_1}{<0.9,0.1,0.3>}, \frac{\eta_2}{<0.4,0.5,0.4>}, \frac{\eta_3}{<0.7,0.2,0.6>}, \frac{\eta_4}{<0.3,0.4,0.8>} \right\},$$
$$\hbar_5 = \hbar(\mho_2/0.3, d, 1) = \left\{ \frac{\eta_1}{<0.4,0.5,0.6>}, \frac{\eta_2}{<0.8,0.1,0.7>}, \frac{\eta_3}{<0.3,0.6,0.5>}, \frac{\eta_4}{<0.2,0.6,0.7>} \right\},$$
$$\hbar_6 = \hbar(\mho_2/0.3, e, 1) = \left\{ \frac{\eta_1}{<0.5,0.4,0.7>}, \frac{\eta_2}{<0.3,0.6,0.4>}, \frac{\eta_3}{<0.6,0.2,0.5>}, \frac{\eta_4}{<0.8,0.1,0.6>} \right\},$$
$$\hbar_7 = \hbar(\mho_3/0.4, c, 1) = \left\{ \frac{\eta_1}{<0.2,0.7,0.5>}, \frac{\eta_2}{<0.9,0.1,0.4>}, \frac{\eta_3}{<0.4,0.5,0.7>}, \frac{\eta_4}{<0.5,0.4,0.8>} \right\},$$
$$\hbar_8 = \hbar(\mho_3/0.4, d, 1) = \left\{ \frac{\eta_1}{<0.4,0.3,0.2>}, \frac{\eta_2}{<0.6,0.3,0.1>}, \frac{\eta_3}{<0.7,0.2,0.3>}, \frac{\eta_4}{<0.9,0.1,0.4>} \right\},$$
$$\hbar_9 = \hbar(\mho_3/0.4, e, 1) = \left\{ \frac{\eta_1}{<0.7,0.2,0.6>}, \frac{\eta_2}{<0.3,0.5,0.7>}, \frac{\eta_3}{<0.5,0.4,0.5>}, \frac{\eta_4}{<0.2,0.7,0.8>} \right\},$$
$$\hbar_{10} = \hbar(\mho_1/0.2, c, 0) = \left\{ \frac{\eta_1}{<0.3,0.2,0.1>}, \frac{\eta_2}{<0.2,0.4,0.5>}, \frac{\eta_3}{<0.4,0.5,0.8>}, \frac{\eta_4}{<0.1,0.8,0.3>} \right\},$$
$$\hbar_{11} = \hbar(\mho_1/0.2, d, 0) = \left\{ \frac{\eta_1}{<0.1,0.8,0.4>}, \frac{\eta_2}{<0.9,0.1,0.2>}, \frac{\eta_3}{<0.6,0.3,0.4>}, \frac{\eta_4}{<0.2,0.7,0.5>} \right\},$$
$$\hbar_{12} = \hbar(\mho_1/0.2, e, 0) = \left\{ \frac{\eta_1}{<0.2,0.7,0.5>}, \frac{\eta_2}{<0.1,0.8,0.6>}, \frac{\eta_3}{<0.3,0.5,0.7>}, \frac{\eta_4}{<0.5,0.4,0.6>} \right\},$$
$$\hbar_{13} = \hbar(\mho_2/0.3, c, 0) = \left\{ \frac{\eta_1}{<0.8,0.1,0.6>}, \frac{\eta_2}{<0.3,0.6,0.7>}, \frac{\eta_3}{<0.5,0.4,0.8>}, \frac{\eta_4}{<0.7,0.2,0.9>} \right\},$$
$$\hbar_{14} = \hbar(\mho_2/0.3, d, 0) = \left\{ \frac{\eta_1}{<0.7,0.2,0.5>}, \frac{\eta_2}{<0.2,0.6,0.4>}, \frac{\eta_3}{<0.9,0.1,0.6>}, \frac{\eta_4}{<0.4,0.5,0.7>} \right\},$$
$$\hbar_{15} = \hbar(\mho_2/0.3, e, 0) = \left\{ \frac{\eta_1}{<0.6,0.2,0.5>}, \frac{\eta_2}{<0.7,0.2,0.4>}, \frac{\eta_3}{<0.3,0.5,0.4>}, \frac{\eta_4}{<0.2,0.7,0.6>} \right\},$$
$$\hbar_{16} = \hbar(\mho_3/0.4, c, 0) = \left\{ \frac{\eta_1}{<0.1,0.7,0.5>}, \frac{\eta_2}{<0.4,0.5,0.7>}, \frac{\eta_3}{<0.7,0.2,0.9>}, \frac{\eta_4}{<0.8,0.2,0.4>} \right\},$$
$$\hbar_{17} = \hbar(\mho_3/0.4, d, 0) = \left\{ \frac{\eta_1}{<0.2,0.7,0.4>}, \frac{\eta_2}{<0.9,0.1,0.6>}, \frac{\eta_3}{<0.8,0.2,0.4>}, \frac{\eta_4}{<0.3,0.5,0.7>} \right\},$$
$$\hbar_{18} = \hbar(\mho_3/0.4, e, 0) = \left\{ \frac{\eta_1}{<0.5,0.4,0.2>}, \frac{\eta_2}{<0.3,0.6,0.1>}, \frac{\eta_3}{<0.6,0.3,0.2>}, \frac{\eta_4}{<0.1,0.8,0.3>} \right\}.$$

The FPNHSES can be described as

$(\hbar, \mathbb{Q}) =$

$$
\left\{\begin{array}{l}
\left((\mho_1/0.2, c, 1), \left\{ \frac{\eta_1}{<0.2,0.5,0.4>}, \frac{\eta_2}{<0.7,0.2,0.5>}, \frac{\eta_3}{<0.5,0.4,0.6>}, \frac{\eta_4}{<0.1,0.3,0.6>} \right\} \right), \\
\left((\mho_1/0.2, d, 1), \left\{ \frac{\eta_1}{<0.4,0.2,0.3>}, \frac{\eta_2}{<0.8,0.1,0.5>}, \frac{\eta_3}{<0.4,0.5,0.6>}, \frac{\eta_4}{<0.2,0.5,0.3>} \right\} \right), \\
\left((\mho_1/0.2, e, 1), \left\{ \frac{\eta_1}{<0.7,0.2,0.3>}, \frac{\eta_2}{<0.5,0.3,0.6>}, \frac{\eta_3}{<0.6,0.3,0.7>}, \frac{\eta_4}{<0.3,0.5,0.6>} \right\} \right), \\
\left((\mho_2/0.3, c, 1), \left\{ \frac{\eta_1}{<0.9,0.1,0.3>}, \frac{\eta_2}{<0.4,0.5,0.4>}, \frac{\eta_3}{<0.7,0.2,0.6>}, \frac{\eta_4}{<0.3,0.4,0.8>} \right\} \right), \\
\left((\mho_2/0.3, d, 1), \left\{ \frac{\eta_1}{<0.4,0.5,0.6>}, \frac{\eta_2}{<0.8,0.1,0.7>}, \frac{\eta_3}{<0.3,0.6,0.5>}, \frac{\eta_4}{<0.2,0.6,0.7>} \right\} \right), \\
\left((\mho_2/0.3, e, 1), \left\{ \frac{\eta_1}{<0.5,0.4,0.7>}, \frac{\eta_2}{<0.3,0.6,0.4>}, \frac{\eta_3}{<0.6,0.2,0.5>}, \frac{\eta_4}{<0.8,0.1,0.6>} \right\} \right), \\
\left((\mho_3/0.4, c, 1), \left\{ \frac{\eta_1}{<0.2,0.7,0.5>}, \frac{\eta_2}{<0.9,0.1,0.4>}, \frac{\eta_3}{<0.4,0.5,0.7>}, \frac{\eta_4}{<0.5,0.4,0.8>} \right\} \right), \\
\left((\mho_3/0.4, d, 1), \left\{ \frac{\eta_1}{<0.4,0.3,0.2>}, \frac{\eta_2}{<0.6,0.3,0.1>}, \frac{\eta_3}{<0.7,0.2,0.3>}, \frac{\eta_4}{<0.9,0.1,0.4>} \right\} \right), \\
\left((\mho_3/0.4, e, 1), \left\{ \frac{\eta_1}{<0.7,0.2,0.6>}, \frac{\eta_2}{<0.3,0.5,0.7>}, \frac{\eta_3}{<0.5,0.4,0.5>}, \frac{\eta_4}{<0.2,0.7,0.8>} \right\} \right), \\
\left((\mho_1/0.2, c, 0), \left\{ \frac{\eta_1}{<0.3,0.2,0.1>}, \frac{\eta_2}{<0.2,0.4,0.5>}, \frac{\eta_3}{<0.4,0.5,0.8>}, \frac{\eta_4}{<0.1,0.8,0.3>} \right\} \right), \\
\left((\mho_1/0.2, d, 0), \left\{ \frac{\eta_1}{<0.1,0.8,0.4>}, \frac{\eta_2}{<0.9,0.1,0.2>}, \frac{\eta_3}{<0.6,0.3,0.4>}, \frac{\eta_4}{<0.2,0.7,0.5>} \right\} \right), \\
\left((\mho_1/0.2, e, 0), \left\{ \frac{\eta_1}{<0.2,0.7,0.5>}, \frac{\eta_2}{<0.1,0.8,0.6>}, \frac{\eta_3}{<0.3,0.5,0.7>}, \frac{\eta_4}{<0.5,0.4,0.6>} \right\} \right), \\
\left((\mho_2/0.3, c, 0), \left\{ \frac{\eta_1}{<0.8,0.1,0.6>}, \frac{\eta_2}{<0.3,0.6,0.7>}, \frac{\eta_3}{<0.5,0.4,0.8>}, \frac{\eta_4}{<0.7,0.2,0.9>} \right\} \right), \\
\left((\mho_2/0.3, d, 0), \left\{ \frac{\eta_1}{<0.7,0.2,0.5>}, \frac{\eta_2}{<0.2,0.6,0.4>}, \frac{\eta_3}{<0.9,0.1,0.6>}, \frac{\eta_4}{<0.4,0.5,0.7>} \right\} \right), \\
\left((\mho_2/0.3, e, 0), \left\{ \frac{\eta_1}{<0.6,0.2,0.5>}, \frac{\eta_2}{<0.7,0.2,0.4>}, \frac{\eta_3}{<0.3,0.5,0.4>}, \frac{\eta_4}{<0.2,0.7,0.6>} \right\} \right), \\
\left((\mho_3/0.4, c, 0), \left\{ \frac{\eta_1}{<0.1,0.7,0.5>}, \frac{\eta_2}{<0.4,0.5,0.7>}, \frac{\eta_3}{<0.7,0.2,0.9>}, \frac{\eta_4}{<0.8,0.2,0.4>} \right\} \right), \\
\left((\mho_3/0.4, d, 0), \left\{ \frac{\eta_1}{<0.2,0.7,0.4>}, \frac{\eta_2}{<0.9,0.1,0.6>}, \frac{\eta_3}{<0.8,0.2,0.4>}, \frac{\eta_4}{<0.3,0.5,0.7>} \right\} \right), \\
\left((\mho_3/0.4, e, 0), \left\{ \frac{\eta_1}{<0.5,0.4,0.2>}, \frac{\eta_2}{<0.3,0.6,0.1>}, \frac{\eta_3}{<0.6,0.3,0.2>}, \frac{\eta_4}{<0.1,0.8,0.3>} \right\} \right)
\end{array}\right\}.
$$

Definition 3.3. A FPNHSES $(\hbar_1, \mathbb{Q})$ is said to be FPNHSE subset of $(\hbar_2, \mathbb{P})$ over Δ, if

(i) $\mathbb{Q} \subseteq \mathbb{P}$,

(ii) $\forall\ \gamma \in \mathbb{Q}, \hbar_1(\gamma) \subseteq \hbar_2(\gamma)$ and shown by $(\hbar_1, \mathbb{Q}) \subseteq (\hbar_2, \mathbb{P})$.

Whereas $(\hbar_2, \mathbb{P})$ is said to be FPNHSE-superset of $(\hbar_1, \mathbb{Q})$.

Example 3.4. Considering Example 3.2, suppose

$$
\mathbb{Q}_1 = \left\{ \begin{array}{l} (\mho_1/0.2, c, 1), (\mho_3/0.4, c, 0), (\mho_1/0.2, d, 1), (\mho_3/0.4, d, 1), \\ (\mho_3/0.4, d, 0), (\mho_1/0.2, e, 0), (\mho_3/0.4, e, 1) \end{array} \right\}
$$

$$
\mathbb{Q}_2 = \left\{ \begin{array}{l} (\mho_1/0.2, c, 1), (\mho_3/0.4, c, 0), (\mho_3/0.4, c, 1), (\mho_1/0.2, d, 1), (\mho_3/0.4, d, 1), \\ (\mho_1/0.2, d, 0), (\mho_3/0.4, d, 0), (\mho_1/0.2, e, 0), (\mho_3/0.4, e, 1), (\mho_1/0.2, e, 1) \end{array} \right\}
$$

It is clear that $\mathbb{Q}_1 \subset \mathbb{Q}_2$.

Suppose $(\hbar_1, \mathbb{Q}_1)$ and $(\hbar_2, \mathbb{Q}_2)$ be defined as following

$$(\hbar_1, \mathbb{Q}_1) = \left\{ \begin{array}{l} \left((\mho_1/0.2, c, 1), \left\{ \frac{\eta_1}{<0.1,0.6,0.7>}, \frac{\eta_2}{<0.6,0.5,0.8>}, \frac{\eta_3}{<0.4,0.6,0.9>}, \frac{\eta_4}{<0.1,0.8,0.6>} \right\}\right), \\ \left((\mho_1/0.2, d, 1), \left\{ \frac{\eta_1}{<0.3,0.4,0.5>}, \frac{\eta_2}{<0.6,0.4,0.6>}, \frac{\eta_3}{<0.2,0.5,0.7>}, \frac{\eta_4}{<0.1,0.5,0.6>} \right\}\right), \\ \left((\mho_3/0.4, d, 1), \left\{ \frac{\eta_1}{<0.2,0.6,0.4>}, \frac{\eta_2}{<0.5,0.4,0.7>}, \frac{\eta_3}{<0.6,0.5,0.8>}, \frac{\eta_4}{<0.8,0.6,0.4>} \right\}\right), \\ \left((\mho_3/0.4, e, 1), \left\{ \frac{\eta_1}{<0.6,0.4,0.3>}, \frac{\eta_2}{<0.2,0.7,0.6>}, \frac{\eta_3}{<0.4,0.5,0.3>}, \frac{\eta_4}{<0.1,0.7,0.4>} \right\}\right), \\ \left((\mho_1/0.2, e, 0), \left\{ \frac{\eta_1}{<0.1,0.6,0.3>}, \frac{\eta_2}{<0.1,0.7,0.4>}, \frac{\eta_3}{<0.2,0.7,0.6>}, \frac{\eta_4}{<0.1,0.6,0.7>} \right\}\right), \\ \left((\mho_3/0.4, c, 0), \left\{ \frac{\eta_1}{<0.1,0.8,0.6>}, \frac{\eta_2}{<0.3,0.6,0.5>}, \frac{\eta_3}{<0.6,0.3,0.4>}, \frac{\eta_4}{<0.7,0.2,0.6>} \right\}\right), \\ \left((\mho_3/0.4, d, 0), \left\{ \frac{\eta_1}{<0.1,0.7,0.4>}, \frac{\eta_2}{<0.6,0.3,0.6>}, \frac{\eta_3}{<0.7,0.2,0.5>}, \frac{\eta_4}{<0.2,0.7,0.4>} \right\}\right) \end{array} \right\}$$

$$(\hbar_2, \mathbb{Q}_2) = \left\{ \begin{array}{l} \left((\mho_1/0.2, c, 1), \left\{ \frac{\eta_1}{<0.2,0.3,0.6>}, \frac{\eta_2}{<0.7,0.4,0.7>}, \frac{\eta_3}{<0.5,0.4,0.8>}, \frac{\eta_4}{<0.2,0.4,0.5>} \right\}\right), \\ \left((\mho_1/0.2, d, 1), \left\{ \frac{\eta_1}{<0.4,0.3,0.4>}, \frac{\eta_2}{<0.8,0.3,0.5>}, \frac{\eta_3}{<0.4,0.3,0.6>}, \frac{\eta_4}{<0.2,0.6,0.5>} \right\}\right), \\ \left((\mho_3/0.4, c, 1), \left\{ \frac{\eta_1}{<0.1,0.3,0.4>}, \frac{\eta_2}{<0.9,0.1,0.3>}, \frac{\eta_3}{<0.4,0.5,0.4>}, \frac{\eta_4}{<0.5,0.3,0.4>} \right\}\right), \\ \left((\mho_3/0.4, d, 1), \left\{ \frac{\eta_1}{<0.4,0.2,0.3>}, \frac{\eta_2}{<0.6,0.3,0.6>}, \frac{\eta_3}{<0.7,0.4,0.5>}, \frac{\eta_4}{<0.9,0.5,0.2} \right\}\right), \\ \left((\mho_1/0.2, e, 1), \left\{ \frac{\eta_1}{<0.7,0.2,0.4>}, \frac{\eta_2}{<0.5,0.2,0.6>}, \frac{\eta_3}{<0.6,0.2,0.7>}, \frac{\eta_4}{<0.3,0.5,0.8} \right\}\right), \\ \left((\mho_3/0.4, e, 1), \left\{ \frac{\eta_1}{<0.7,0.3,0.1>}, \frac{\eta_2}{<0.3,0.5,0.4>}, \frac{\eta_3}{<0.5,0.4,0.2>}, \frac{\eta_4}{<0.2,0.6,0.3>} \right\}\right), \\ \left((\mho_1/0.2, e, 0), \left\{ \frac{\eta_1}{<0.2,0.5,0.1>}, \frac{\eta_2}{<0.2,0.6,0.3>}, \frac{\eta_3}{<0.3,0.5,0.4>}, \frac{\eta_4}{<0.5,0.3,0.5>} \right\}\right), \\ \left((\mho_1/0.2, d, 0), \left\{ \frac{\eta_1}{<0.1,0.6,0.4>}, \frac{\eta_2}{<0.9,0.1,0.6>}, \frac{\eta_3}{<0.6,0.3,0.8>}, \frac{\eta_4}{<0.2,0.6,0.8>} \right\}\right), \\ \left((\mho_3/0.4, c, 0), \left\{ \frac{\eta_1}{<0.2,0.7,0.4>}, \frac{\eta_2}{<0.4,0.5,0.3>}, \frac{\eta_3}{<0.7,0.2,0.1>}, \frac{\eta_4}{<0.8,0.1,0.5>} \right\}\right), \\ \left((\mho_3/0.4, d, 0), \left\{ \frac{\eta_1}{<0.2,0.5,0.1>}, \frac{\eta_2}{<0.7,0.2,0.3>}, \frac{\eta_3}{<0.8,0.2,0.4>}, \frac{\eta_4}{<0.3,0.5,0.2>} \right\}\right) \end{array} \right\}$$

which shows that $(\hbar_1, \mathbb{Q}_1) \subseteq (\hbar_2, \mathbb{Q}_2)$.

Definition 3.5. Two $\mathcal{FPNHSE}$-sets $(\hbar_1, \mathbb{Q}_1)$ and $(\hbar_2, \mathbb{Q}_2)$ over Δ are said to be equal if $(\hbar_1, \mathbb{Q}_1)$ is a $\mathcal{FPNHSE}$-subset of $(\hbar_2, \mathbb{Q}_2)$ and $(\hbar_2, \mathbb{Q}_2)$ is a $\mathcal{FPNHSE}$-subset of $(\hbar_1, \mathbb{Q}_1)$.

Definition 3.6. The complement of a $\mathcal{FPNHSE}$-set $(\hbar, \mathbb{Q})$, denoted by $(\hbar, \mathbb{Q})^c$, is defined by $(\hbar, \mathbb{Q})^c = \tilde{c}(\hbar(\sigma)) \ \forall \ \sigma \in \Delta$ while $\tilde{c}$ is a NF complement.

Example 3.7. Taking complement of $\mathcal{FPNHSE}$-set determined in 3.2, we have

$$(\hbar,\mathbb{Q})^c=\left\{\begin{array}{l}
\{(\zeta_1/0.2,c,1),\{\frac{\eta_1}{<0.4,0.5,0.2>},\frac{\eta_2}{<0.5,0.8,0.7>},\frac{\eta_3}{<0.6,0.6,0.5>},\frac{\eta_4}{<0.6,0.7,0.1>}\}\},\\
\{(\mho_1/0.2,d,1),\{\frac{\eta_1}{<0.3,0.8,0.6>},\frac{\eta_2}{<0.5,0.9,0.8>},\frac{\eta_3}{<0.6,0.5,0.4>},\frac{\eta_4}{<0.3,0.5,0.2>}\}\},\\
((\mho_1/0.2,e,1),\{\frac{\eta_1}{<0.3,0.8,0.7>},\frac{\eta_2}{<0.6,0.7,0.5>},\frac{\eta_3}{<0.7,0.7,0.6>},\frac{\eta_4}{<0.6,0.5,0.3>}\}),\\
((\mho_2/0.3,c,1),\{\frac{\eta_1}{<0.3,0.9,0.9>},\frac{\eta_2}{<0.4,0.5,0.4>},\frac{\eta_3}{<0.6,0.8,0.7>},\frac{\eta_4}{<0.8,0.6,0.3>}\}),\\
((\mho_2/0.3,d,1),\{\frac{\eta_1}{<0.6,0.5,0.4>},\frac{\eta_2}{<0.7,0.9,0.8>},\frac{\eta_3}{<0.5,0.4,0.3>},\frac{\eta_4}{<0.7,0.4,0.2>}\}),\\
((\mho_2/0.3,e,1),\{\frac{\eta_1}{<0.7,0.6,0.5>},\frac{\eta_2}{<0.4,0.4,0.3>},\frac{\eta_3}{<0.5,0.8,0.6>},\frac{\eta_4}{<0.6,0.9,0.8>}\}),\\
((\mho_3/0.4,c,1),\{\frac{\eta_1}{<0.5,0.3,0.2>},\frac{\eta_2}{<0.4,0.9,0.9>},\frac{\eta_3}{<0.7,0.5,0.4>},\frac{\eta_4}{<0.8,0.6,0.5>}\}),\\
((\mho_3/0.4,d,1),\{\frac{\eta_1}{<0.2,0.7,0.4>},\frac{\eta_2}{<0.1,0.7,0.6>},\frac{\eta_3}{<0.3,0.8,0.7>},\frac{\eta_4}{<0.4,0.9,0.9>}\}),\\
((\mho_3/0.4,e,1),\{\frac{\eta_1}{<0.6,0.8,0.7>},\frac{\eta_2}{<0.7,0.5,0.3>},\frac{\eta_3}{<0.5,0.6,0.5>},\frac{\eta_4}{<0.8,0.3,0.2>}\}),\\
((\mho_1/0.2,c,0),\{\frac{\eta_1}{<0.1,0.8,0.3>},\frac{\eta_2}{<0.5,0.6,0.2>},\frac{\eta_3}{<0.8,0.5,0.4>},\frac{\eta_4}{<0.3,0.2,0.1>}\}),\\
((\mho_1/0.2,d,0),\{\frac{\eta_1}{<0.4,0.2,0.1>},\frac{\eta_2}{<0.2,0.9,0.9>},\frac{\eta_3}{<0.4,0.7,0.6>},\frac{\eta_4}{<0.5,0.3,0.2>}\}),\\
((\mho_1/0.2,e,0),\{\frac{\eta_1}{<0.5,0.3,0.2>},\frac{\eta_2}{<0.6,0.2,0.1>},\frac{\eta_3}{<0.7,0.5,0.3>},\frac{\eta_4}{<0.6,0.6,0.5>}\}),\\
((\mho_2/0.3,c,0),\{\frac{\eta_1}{<0.6,0.9,0.8>},\frac{\eta_2}{<0.7,0.4,0.3>},\frac{\eta_3}{<0.8,0.6,0.5>},\frac{\eta_4}{<0.9,0.8,0.7>}\}),\\
((\mho_2/0.3,d,0),\{\frac{\eta_1}{<0.5,0.8,0.7>},\frac{\eta_2}{<0.4,0.4,0.2>},\frac{\eta_3}{<0.6,0.9,0.9>},\frac{\eta_4}{<0.7,0.5,0.4>}\}),\\
((\mho_2/0.3,e,0),\{\frac{\eta_1}{<0.5,0.8,0.6>},\frac{\eta_2}{<0.4,0.8,0.7>},\frac{\eta_3}{<0.4,0.5,0.3>},\frac{\eta_4}{<0.6,0.3,0.2>}\}),\\
((\mho_3/0.4,c,0),\{\frac{\eta_1}{<0.5,0.3,0.1>},\frac{\eta_2}{<0.7,0.5,0.3>},\frac{\eta_3}{<0.9,0.8,0.8>},\frac{\eta_4}{<0.4,0.8,0.8>}\}),\\
((\mho_3/0.4,d,0),\{\frac{\eta_1}{<0.4,0.3,0.2>},\frac{\eta_2}{<0.6,0.9,0.9>},\frac{\eta_3}{<0.4,0.8,0.8>},\frac{\eta_4}{<0.7,0.5,0.3>}\}),\\
((\mho_3/0.4,e,0),\{\frac{\eta_1}{<0.2,0.6,0.5>},\frac{\eta_2}{<0.1,0.4,0.3>},\frac{\eta_3}{<0.2,0.7,0.6>},\frac{\eta_4}{<0.3,0.2,0.1>}\})
\end{array}\right\}.$$

Definition 3.8. An agree-$\mathcal{FPNHSE}$-set $(\hbar,\mathbb{Q})_{ag}$ over Δ, is a $\mathcal{FPNHSE}$-subset of $(\hbar,\mathbb{Q})$ and is characterized as $(\hbar,\mathbb{Q})_{ag}=\{\hbar_{ag}(\sigma):\sigma\in \mathrm{G}\times \mathrm{D}\times\{1\}\}$.

Example 3.9. Finding agree-$\mathcal{FPNHSE}$-set determined in 3.2, we get

$$(\hbar,\mathbb{Q})=\left\{\begin{array}{l}
((\mho_1/0.2,c,1),\{\frac{\eta_1}{<0.2,0.5,0.4>},\frac{\eta_2}{<0.7,0.2,0.5>},\frac{\eta_3}{<0.5,0.4,0.6>},\frac{\eta_4}{<0.1,0.3,0.6>}\}),\\
((\mho_1/0.2,d,1),\{\frac{\eta_1}{<0.4,0.2,0.3>},\frac{\eta_2}{<0.8,0.1,0.5>},\frac{\eta_3}{<0.4,0.5,0.6>},\frac{\eta_4}{<0.2,0.5,0.3>}\}),\\
((\mho_1/0.2,e,1),\{\frac{\eta_1}{<0.7,0.2,0.3>},\frac{\eta_2}{<0.5,0.3,0.6>},\frac{\eta_3}{<0.6,0.3,0.7>},\frac{\eta_4}{<0.3,0.5,0.6>}\}),\\
((\mho_2/0.3,c,1),\{\frac{\eta_1}{<0.9,0.1,0.3>},\frac{\eta_2}{<0.4,0.5,0.4>},\frac{\eta_3}{<0.7,0.2,0.6>},\frac{\eta_4}{<0.3,0.4,0.8>}\}),\\
((\mho_2/0.3,d,1),\{\frac{\eta_1}{<0.4,0.5,0.6>},\frac{\eta_2}{<0.8,0.1,0.7>},\frac{\eta_3}{<0.3,0.6,0.5>},\frac{\eta_4}{<0.2,0.6,0.7>}\}),\\
((\mho_2/0.3,e,1),\{\frac{\eta_1}{<0.5,0.4,0.7>},\frac{\eta_2}{<0.3,0.6,0.4>},\frac{\eta_3}{<0.6,0.2,0.5>},\frac{\eta_4}{<0.8,0.1,0.6>}\}),\\
((\mho_3/0.4,c,1),\{\frac{\eta_1}{<0.2,0.7,0.5>},\frac{\eta_2}{<0.9,0.1,0.4>},\frac{\eta_3}{<0.4,0.5,0.7>},\frac{\eta_4}{<0.5,0.4,0.8>}\}),\\
((\mho_3/0.4,d,1),\{\frac{\eta_1}{<0.4,0.3,0.2>},\frac{\eta_2}{<0.6,0.3,0.1>},\frac{\eta_3}{<0.7,0.2,0.3>},\frac{\eta_4}{<0.9,0.1,0.4>}\}),\\
((\mho_3/0.4,e,1),\{\frac{\eta_1}{<0.7,0.2,0.6>},\frac{\eta_2}{<0.3,0.5,0.7>},\frac{\eta_3}{<0.5,0.4,0.5>},\frac{\eta_4}{<0.2,0.7,0.8>}\})
\end{array}\right\}.$$

Definition 3.10. A disagree-$\mathcal{FPNHSE}$-set $(\hbar,\mathbb{Q})_{dag}$ over Δ, is a $\mathcal{FPNHSE}$-subset of $(\hbar,\mathbb{Q})$ and is characterized as $(\hbar,\mathbb{Q})_{dag}=\{\hbar_{dag}(\sigma):\sigma\in\mathbb{G}\times\mathcal{D}\times\{0\}\}$.

Example 3.11. Getting disagree-$\mathcal{FPNHSE}$-set determined in 3.2,

$$(\hbar,\mathbb{Q})=\left\{\begin{array}{l}((\mho_1/0.2,c,0),\{\frac{\eta_1}{<0.3,0.2,0.1>},\frac{\eta_2}{<0.2,0.4,0.5>},\frac{\eta_3}{<0.4,0.5,0.8>},\frac{\eta_4}{<0.1,0.8,0.3>}\}),\\((\mho_1/0.2,d,0),\{\frac{\eta_1}{<0.1,0.8,0.4>},\frac{\eta_2}{<0.9,0.1,0.2>},\frac{\eta_3}{<0.6,0.3,0.4>},\frac{\eta_4}{<0.2,0.7,0.5>}\}),\\((\mho_1/0.2,e,0),\{\frac{\eta_1}{<0.2,0.7,0.5>},\frac{\eta_2}{<0.1,0.8,0.6>},\frac{\eta_3}{<0.3,0.5,0.7>},\frac{\eta_4}{<0.5,0.4,0.6>}\}),\\((\mho_2/0.3,c,0),\{\frac{\eta_1}{<0.8,0.1,0.6>},\frac{\eta_2}{<0.3,0.6,0.7>},\frac{\eta_3}{<0.5,0.4,0.8>},\frac{\eta_4}{<0.7,0.2,0.9>}\}),\\((\mho_2/0.3,d,0),\{\frac{\eta_1}{<0.7,0.2,0.5>},\frac{\eta_2}{<0.2,0.6,0.4>},\frac{\eta_3}{<0.9,0.1,0.6>},\frac{\eta_4}{<0.4,0.5,0.7>}\}),\\((\mho_2/0.3,e,0),\{\frac{\eta_1}{<0.6,0.2,0.5>},\frac{\eta_2}{<0.7,0.2,0.4>},\frac{\eta_3}{<0.3,0.5,0.4>},\frac{\eta_4}{<0.2,0.7,0.6>}\}),\\((\mho_3/0.4,c,0),\{\frac{\eta_1}{<0.1,0.7,0.5>},\frac{\eta_2}{<0.4,0.5,0.7>},\frac{\eta_3}{<0.7,0.2,0.9>},\frac{\eta_4}{<0.8,0.2,0.4>}\}),\\((\mho_3/0.4,d,0),\{\frac{\eta_1}{<0.2,0.7,0.4>},\frac{\eta_2}{<0.9,0.1,0.6>},\frac{\eta_3}{<0.8,0.2,0.4>},\frac{\eta_4}{<0.3,0.5,0.7>}\}),\\((\mho_3/0.4,e,0),\{\frac{\eta_1}{<0.5,0.4,0.2>},\frac{\eta_2}{<0.3,0.6,0.1>},\frac{\eta_3}{<0.6,0.3,0.2>},\frac{\eta_4}{<0.1,0.8,0.3>}\})\end{array}\right\}.$$

Definition 3.12. A $\mathcal{FPNHSE}$-set $(\hbar_1,\mathbb{Q}_1)$ is called a relative null $\mathcal{FPNHSE}$-set w.r.t $\mathbb{Q}_1\subset\mathbb{Q}$, denoted by $(\hbar_1,\mathbb{Q}_1)$, if $\hbar_1(g)=\emptyset,\forall g\in\mathbb{Q}_1$.

Example 3.13. Taking the concept of Example 3.2, if

$$(\hbar_1,\mathbb{Q}_1)=\{((\eta_1,c,1),\emptyset),((\eta_2,d,1),\emptyset),((\eta_3,e,1),\emptyset)\}$$

.

Definition 3.14. A $\mathcal{FPNHSE}$-set $(\hbar_2,\mathbb{Q}_2)$ is called a relative whole $\mathcal{FPNHSE}$-set w.r.t $\mathbb{Q}_2\subset\mathbb{Q}$, denoted by $(\hbar_2,\mathbb{Q}_2)_\Delta$, if $\hbar_1(g)=\Delta$, $\forall$ g $\in\mathbb{Q}_2$.

Example 3.15. Taking the concept of Example 3.2, if

$$(\hbar_2,\mathbb{Q}_2)_\Delta=\{((\eta_1,c,1),\Delta),((\eta_2,d,1),\Delta),((\eta_3,e,1),\Delta)\}$$

where $\mathbb{Q}_2\subseteq\mathbb{Q}$.

Definition 3.16. A $\mathcal{FPNHSE}$-set $(\hbar,\mathbb{Q})$ is called absolute whole $\mathcal{FPNHSE}$-set denoted by $(\hbar,\mathbb{Q})_\Delta$, if $\hbar(g)=\Delta$, $\forall$ g $\in\mathbb{Q}$.

Example 3.17. Using Example 3.2, if

$$(\Psi,\mathbb{S})_\Delta=\left\{\begin{array}{l}(\mho_1/0.2,c,1),\Delta),(\mho_1/0.2,d,1),\Delta),(\mho_1/0.2,e,1),\Delta),(\mho_3/0.4,c,1),\Delta),\\(\mho_3/0.4,d,1),\Delta),(\mho_3/0.4,e,1),\Delta),(\mho_5/0.6,c,1),\Delta),(\mho_5/0.6,d,1),\Delta),\\(\mho_5/0.6,e,1),\Delta),(\mho_1/0.2,c,0),\Delta),(\mho_1/0.2,d,0),\Delta),(\mho_1/0.2,e,0),\Delta),\\(\mho_3/0.4,c,0),\Delta),(\mho_3/0.4,d,0),\Delta),(\mho_3/0.4,e,0),\Delta),(\mho_5/0.6,c,0),\Delta),\\(\mho_5/0.6,d,0),\Delta),(\mho_5/0.6,e,0),\Delta)\end{array}\right\}$$

Proposition 3.18. *Suppose* $(\hbar_1,\mathbb{Q}_1)_\Delta$, $(\hbar_2,\mathbb{Q}_2)_\Delta$, $(\hbar_3,\mathbb{Q}_3)_\Delta$, *be three $\mathcal{FPNHSE}$-sets over* Δ, *then*

(1) $(\hbar_1,\mathbb{Q}_1)\subset(\hbar_2,\mathbb{Q}_2)_\Delta$,

(2) $(\hbar_1,\mathbb{Q}_1)_\hbar\subset(\hbar_1,\mathbb{Q}_1)$,

(3) $(\hbar_1, \mathbb{Q}_1) \subset (\hbar_1, \mathbb{Q}_1)$,

(4) *If* $(\hbar_1, \mathbb{Q}_1) \subset (\hbar_2, \mathbb{Q}_2)$, $(\hbar_2, \mathbb{Q}_2) \subset (\hbar_3, \mathbb{Q}_3)$, *then* $(\hbar_1, \mathbb{Q}_1) \subset (\hbar_3, \mathbb{S}_3)$,

(5) *If* $(\hbar_1, \mathbb{Q}_1) = (\hbar_2, \mathbb{Q}_2)$, $(\hbar_2, \mathbb{Q}_2) = (\hbar_3, \mathbb{Q}_3)$, *then* $(\hbar_1, \mathbb{Q}_1) = (\hbar_3, \mathbb{Q}_3)$.

Proposition 3.19. *If* $(\hbar, \mathbb{Q})$ *is a* $\mathcal{FPNHSE}$*-set over* Δ*, then*

(1) $((\hbar, \mathbb{Q})^c)^c = (\hbar, \mathbb{Q})$

(2) $(\hbar, \mathbb{Q})^c_{ag} = (\hbar, \mathbb{Q})_{dag}$

(3) $(\hbar, \mathbb{Q})^c_{dag} = (\hbar, \mathbb{Q})_{ag}$.

4. Set Theoretic Operations of $\mathcal{FPNHSE}$-set

In this portion, some set theoretic operations are presented with detailed examples.

Definition 4.1. The union of $\mathcal{FPNHSE}$-sets $(\hbar_1, \mathbb{Q})$ and $(\hbar_2, \mathbb{R})$ over Δ is $(\hbar_3, \mathbb{L})$ with $\mathbb{L} = \mathbb{Q} \cup \mathbb{R}$, defined as

$$\hbar_3(\sigma) = \begin{cases} \hbar_1(\sigma) & ; \sigma \in \mathbb{Q} - \mathbb{R} \\ \hbar_2(\sigma) & ; \sigma \in \mathbb{R} - \mathbb{Q} \\ \cup(\hbar_1(\sigma), \hbar_2(\sigma)) & ; \sigma \in \mathbb{Q} \cap \mathbb{R} \end{cases}$$

where

$\cup(\hbar_1(\sigma), \hbar_2(\sigma)) = \{< u, max(\hbar_1(\sigma), \hbar_2(\sigma)), min(\hbar_1(\sigma), (\hbar_2(\sigma)), min(\hbar_1(\sigma), (\hbar_2(\sigma))\} >: u \in \Delta\}$.

Example 4.2. Using Example 3.2, with two sets

$$\mathbb{Q}_1 = \left\{ (\mho_1/0.2, c, 1), (\mho_1/0.2, d, 1), (\mho_3/0.4, d, 1) \right\}$$

$$\mathbb{Q}_2 = \left\{ (\mho_1/0.2, c, 1), (\mho_3/0.4, c, 1), (\mho_1/0.2, d, 1), (\mho_3/0.4, d, 1) \right\}.$$

Suppose $(\hbar_1, \mathbb{Q}_1)$ and $(\hbar_2, \mathbb{Q}_2)$ over Δ are two $\mathcal{FPNHSE}$-sets such that

$$(\hbar_1, \mathbb{Q}_1) = \left\{ \begin{array}{l} ((\mho_1/0.2, c, 1), \{\frac{\eta_1}{<0.1,0.6,0.4>}, \frac{\eta_2}{<0.6,0.3,0.2>}, \frac{\eta_3}{<0.4,0.5,0.1>}, \frac{\eta_4}{<0.1,0.8,0.5>}\}), \\ ((\mho_1/0.2, d, 1), \{\frac{\eta_1}{<0.3,0.4,0.5>}, \frac{\eta_2}{<0.6,0.2,0.3>}, \frac{\eta_3}{<0.2,0.5,0.6>}, \frac{\eta_4}{<0.1,0.5,0.3>}\}), \\ ((\mho_3/0.4, d, 1), \{\frac{\eta_1}{<0.2,0.6,0.7>}, \frac{\eta_2}{<0.5,0.2,0.3>}, \frac{\eta_3}{<0.6,0.3,0.5>}, \frac{\eta_4}{<0.8,0.1,0.9>}\}) \end{array} \right\}$$

$$(\hbar_2, \mathbb{Q}_2) = \left\{ \begin{array}{l} ((\mho_1/0.2, c, 1), \{\frac{\eta_1}{<0.2,0.3,0.4>}, \frac{\eta_2}{<0.7,0.4,0.5>}, \frac{\eta_3}{<0.5,0.4,0.6>}, \frac{\eta_4}{<0.2,0.4,0.7>}\}), \\ ((\mho_1/0.2, d, 1), \{\frac{\eta_1}{<0.4,0.3,0.8>}, \frac{\eta_2}{<0.8,0.3,0.5>}, \frac{\eta_3}{<0.4,0.3,0.5>}, \frac{\eta_4}{<0.2,0.6,0.7>}\}), \\ ((\mho_3/0.4, c, 1), \{\frac{\eta_1}{<0.1,0.3,0.6>}, \frac{\eta_2}{<0.9,0.1,0.7>}, \frac{\eta_3}{<0.4,0.5,0.8>}, \frac{\eta_4}{<0.5,0.3,0.5>}\}), \\ ((\mho_3/0.4, d, 1), \{\frac{\eta_1}{<0.4,0.2,0.3>}, \frac{\eta_2}{<0.6,0.3,0.5>}, \frac{\eta_3}{<0.7,0.4,0.5>}, \frac{\eta_4}{<0.9,0.5,0.7}\}), \end{array} \right\}$$

Then $(\hbar_1, \mathbb{Q}_1) \cup (\hbar_2, \mathbb{Q}_2) = (\hbar_3, \mathbb{Q}_3)$

$$(\hbar_3, \mathbb{Q}_3) = \left\{ \begin{array}{l} ((\mho_1/0.2, c, 1), \{\frac{\eta_1}{<0.2,0.3,0.4>}, \frac{\eta_2}{<0.7,0.3,0.5>}, \frac{\eta_3}{<0.5,0.4,0.1>}, \frac{\eta_4}{<0.2,0.4,0.5>}\}), \\ ((\mho_1/0.2, d, 1), \{\frac{\eta_1}{<0.4,0.3,0.5>}, \frac{\eta_2}{<0.8,0.2,0.3>}, \frac{\eta_3}{<0.4,0.3,0.5>}, \frac{\eta_4}{<0.2,0.4,0.3>}\}), \\ ((\mho_3/0.4, c, 1), \{\frac{\eta_1}{<0.1,0.30,0.6>}, \frac{\eta_2}{<0.9,0.10,0.7>}, \frac{\eta_3}{<0.4,0.5,0.8>}, \frac{\eta_4}{<0.5,0.3,0.5>}\}), \\ ((\mho_3/0.4, d, 1), \{\frac{\eta_1}{<0.4,0.2,0.3>}, \frac{\eta_2}{<0.6,0.2,0.3>}, \frac{\eta_3}{<0.7,0.3,0.5>}, \frac{\eta_4}{<0.9,0.1,0.7}\}) \end{array} \right\}.$$

Definition 4.3. Restricted Union of two fuzzy parameterized neutrosophic hypersoft expert sets $(\hbar_1, \mathbb{Q}_1)$, $(\hbar_2, \mathbb{Q}_2)$ over Δ is $(\hbar_3, \mathbb{L})$ with $\mathbb{L} = \mathbb{Q}_1 \cap \mathbb{Q}_2$, defined as
$\hbar_3(\sigma) = \hbar_1(\sigma) \cup_{\mathbb{R}} \hbar_2(\sigma)$ for $\sigma \in \mathbb{Q}_1 \cap \mathbb{Q}_2$.

Example 4.4. Taking Example 3.2, with two sets
$\mathbb{Q}_1 = \left\{ (\mho_1/0.2, c, 1), (\mho_1/0.2, d, 1), (\mho_3/0.4, d, 1) \right\}$
$\mathbb{Q}_2 = \left\{ (\mho_1/0.2, c, 1), (\mho_3/0.4, c, 1), (\mho_1/0.2, d, 1), (\mho_3/0.4, d, 1) \right\}$
Suppose $(\hbar_1, \mathbb{Q}_1)$ and $(\hbar_2, \mathbb{Q}_2)$ over Δ are two $\mathcal{FPNHSE}$-sets such that

$$(\hbar_1, \mathbb{Q}_1) = \left\{ \begin{array}{l} \left((\mho_1/0.2, c, 1), \left\{\frac{\eta_1}{<0.1,0.6,0.4>}, \frac{\eta_2}{<0.6,0.3,0.2>}, \frac{\eta_3}{<0.4,0.5,0.1>}, \frac{\eta_4}{<0.1,0.8,0.5>}\right\}\right), \\ \left((\mho_1/0.2, d, 1), \left\{\frac{\eta_1}{<0.3,0.4,0.5>}, \frac{\eta_2}{<0.6,0.2,0.3>}, \frac{\eta_3}{<0.2,0.5,0.6>}, \frac{\eta_4}{<0.1,0.5,0.3>}\right\}\right), \\ \left((\mho_3/0.4, d, 1), \left\{\frac{\eta_1}{<0.2,0.6,0.7>}, \frac{\eta_2}{<0.5,0.2,0.3>}, \frac{\eta_3}{<0.6,0.3,0.5>}, \frac{\eta_4}{<0.8,0.1,0.9>}\right\}\right) \end{array} \right\}$$

$$(\hbar_2, \mathbb{Q}_2) = \left\{ \begin{array}{l} \left((\mho_1/0.2, c, 1), \left\{\frac{\eta_1}{<0.2,0.3,0.4>}, \frac{\eta_2}{<0.7,0.4,0.5>}, \frac{\eta_3}{<0.5,0.4,0.6>}, \frac{\eta_4}{<0.2,0.4,0.7>}\right\}\right), \\ \left((\mho_1/0.2, d, 1), \left\{\frac{\eta_1}{<0.4,0.3,0.8>}, \frac{\eta_2}{<0.8,0.3,0.5>}, \frac{\eta_3}{<0.4,0.3,0.5>}, \frac{\eta_4}{<0.2,0.6,0.7>}\right\}\right), \\ \left((\mho_3/0.4, c, 1), \left\{\frac{\eta_1}{<0.1,0.3,0.6>}, \frac{\eta_2}{<0.9,0.1,0.7>}, \frac{\eta_3}{<0.4,0.5,0.8>}, \frac{\eta_4}{<0.5,0.3,0.5>}\right\}\right), \\ \left((\mho_3/0.4, d, 1), \left\{\frac{\eta_1}{<0.4,0.2,0.3>}, \frac{\eta_2}{<0.6,0.3,0.5>}, \frac{\eta_3}{<0.7,0.4,0.5>}, \frac{\eta_4}{<0.9,0.5,0.7}\right\}\right), \end{array} \right\}$$

Then $(\hbar_1, \mathbb{Q}_1) \cup_{\mathbb{R}} (\hbar_2, \mathbb{Q}_2) = (\hbar_3, \mathbb{L})$

$$(\hbar_3, \mathbb{L}) = \left\{ \begin{array}{l} \left((\mho_1/0.2, c, 1), \left\{\frac{\eta_1}{<0.2,0.3,0.4>}, \frac{\eta_2}{<0.7,0.3,0.5>}, \frac{\eta_3}{<0.5,0.4,0.1>}, \frac{\eta_4}{<0.2,0.4,0.5>}\right\}\right), \\ \left((\mho_1/0.2, d, 1), \left\{\frac{\eta_1}{<0.4,0.3,0.5>}, \frac{\eta_2}{<0.8,0.2,0.3>}, \frac{\eta_3}{<0.4,0.3,0.5>}, \frac{\eta_4}{<0.2,0.4,0.3>}\right\}\right), \\ \left((\mho_3/0.4, d, 1), \left\{\frac{\eta_1}{<0.4,0.2,0.3>}, \frac{\eta_2}{<0.6,0.2,0.3>}, \frac{\eta_3}{<0.7,0.3,0.5>}, \frac{\eta_4}{<0.9,0.1,0.7}\right\}\right) \end{array} \right\}.$$

Proposition 4.5. *If* $(\hbar_1, \mathbb{Q}_1)$,$(\hbar_2, \mathbb{Q}_2)$ *and* $(\hbar_3, \mathbb{Q}_3)$ *are three* $\mathcal{FPNHSE}$*-sets over* Δ*, then*

(1) $(\hbar_1, \mathbb{Q}_1) \cup (\hbar_2, \mathbb{Q}_2) = (\hbar_2, \mathbb{Q}_2) \cup (\hbar_1, \mathbb{Q}_1)$,
(2) $((\hbar_1, \mathbb{Q}_1) \cup (\hbar_2, \mathbb{Q}_2)) \cup (\hbar_3, \mathbb{Q}_3) = (\hbar_1, \mathbb{Q}_1) \cup ((\hbar_2, \mathbb{Q}_2) \cup (\hbar_3, N_3))$,
(3) $(\hbar, \mathbb{Q}) \cup \Phi = (\hbar, \mathbb{Q})$.

Definition 4.6. The intersection of $\mathcal{FPNHSE}$-sets $(\hbar_1, \mathbb{Q})$ and $(\hbar_2, \mathbb{R})$ over Δ is $(\hbar_3, \mathbb{L})$ with $\mathbb{L} = \mathbb{Q} \cap \mathbb{R}$, defined as

$$\hbar_3(\sigma) = \left\{ \begin{array}{ll} \hbar_1(\sigma) & ; \sigma \in \mathbb{Q} - \mathbb{R} \\ \hbar_2(\sigma) & ; \sigma \in \mathbb{R} - \mathbb{Q} \\ \cap(\hbar_1(\sigma), \hbar_2(\sigma)) & ; \sigma \in \mathbb{Q} \cap \mathbb{R} \end{array} \right.$$

where

$$\cap(\hbar_1(\sigma), \hbar_2(\sigma)) = \{< u, min(\hbar_1(\sigma), \hbar_2(\sigma)), max(\hbar_1(\sigma), (\hbar_2(\sigma)), max(\hbar_1(\sigma), (\hbar_2(\sigma))\} >: u \in \Delta\}.$$

Example 4.7. Using Example 3.2, with two sets
$\mathbb{Q}_1 = \left\{ (\mho_1/0.2, c, 1), (\mho_1/0.2, d, 1), (\mho_3/0.4, d, 1) \right\}$

$\mathbb{Q}_2 = \Big\{ (\mho_1/0.2, c, 1), (\mho_3/0.4, c, 1), (\mho_1/0.2, d, 1), (\mho_3/0.4, d, 1) \Big\}$

Suppose $(\hbar_1, \mathbb{Q}_1)$ and $(\hbar_2, \mathbb{Q}_2)$ over Δ are two $\mathcal{FPNHSE}$-sets such that

$$(\hbar_1, \mathbb{Q}_1) = \left\{ \begin{array}{l} \left((\mho_1/0.2, c, 1), \left\{\frac{\eta_1}{<0.1,0.6,0.4>}, \frac{\eta_2}{<0.6,0.3,0.2>}, \frac{\eta_3}{<0.4,0.5,0.1>}, \frac{\eta_4}{<0.1,0.8,0.5>}\right\}\right), \\ \left((\mho_1/0.2, d, 1), \left\{\frac{\eta_1}{<0.3,0.4,0.5>}, \frac{\eta_2}{<0.6,0.2,0.3>}, \frac{\eta_3}{<0.2,0.5,0.6>}, \frac{\eta_4}{<0.1,0.5,0.3>}\right\}\right), \\ \left((\mho_3/0.4, d, 1), \left\{\frac{\eta_1}{<0.2,0.6,0.7>}, \frac{\eta_2}{<0.5,0.2,0.3>}, \frac{\eta_3}{<0.6,0.3,0.5>}, \frac{\eta_4}{<0.8,0.1,0.9>}\right\}\right) \end{array} \right\}$$

$$(\hbar_2, \mathbb{Q}_2) = \left\{ \begin{array}{l} \left((\mho_1/0.2, c, 1), \left\{\frac{\eta_1}{<0.2,0.3,0.4>}, \frac{\eta_2}{<0.7,0.4,0.5>}, \frac{\eta_3}{<0.5,0.4,0.6>}, \frac{\eta_4}{<0.2,0.4,0.7>}\right\}\right), \\ \left((\mho_1/0.2, d, 1), \left\{\frac{\eta_1}{<0.4,0.3,0.8>}, \frac{\eta_2}{<0.8,0.3,0.5>}, \frac{\eta_3}{<0.4,0.3,0.5>}, \frac{\eta_4}{<0.2,0.6,0.7>}\right\}\right), \\ \left((\mho_3/0.4, c, 1), \left\{\frac{\eta_1}{<0.1,0.3,0.6>}, \frac{\eta_2}{<0.9,0.1,0.7>}, \frac{\eta_3}{<0.4,0.5,0.8>}, \frac{\eta_4}{<0.5,0.3,0.5>}\right\}\right), \\ \left((\mho_3/0.4, d, 1), \left\{\frac{\eta_1}{<0.4,0.2,0.3>}, \frac{\eta_2}{<0.6,0.3,0.5>}, \frac{\eta_3}{<0.7,0.4,0.5>}, \frac{\eta_4}{<0.9,0.5,0.7}\right\}\right), \end{array} \right\}$$

Then $(\hbar_1, \mathbb{Q}_1) \cap (\hbar_2, \mathbb{Q}_2) = (\hbar_3, \mathbb{Q}_3)$

$$(\hbar_3, \mathbb{Q}_3) = \left\{ \begin{array}{l} \left((\mho_1/0.2, c, 1), \left\{\frac{\eta_1}{<0.1,0.6,0.4>}, \frac{\eta_2}{<0.6,0.3,0.2>}, \frac{\eta_3}{<0.4,0.5,0.6>}, \frac{\eta_4}{<0.2,0.8,0.7>}\right\}\right), \\ \left((\mho_1/0.2, d, 1), \left\{\frac{\eta_1}{<0.4,0.4,0.8>}, \frac{\eta_2}{<0.6,0.3,0.5>}, \frac{\eta_3}{<0.2,0.5,0.6>}, \frac{\eta_4}{<0.1,0.6,0.7>}\right\}\right), \\ \left((\mho_3/0.4, d, 1), \left\{\frac{\eta_1}{<0.2,0.6,0.7>}, \frac{\eta_2}{<0.5,0.3,0.5>}, \frac{\eta_3}{<0.6,0.4,0.5>}, \frac{\eta_4}{<0.8,0.5,0.9}\right\}\right) \end{array} \right\}.$$

Definition 4.8. Extended intersection of $(\hbar_1, \mathbb{S})$ and $(\hbar_2, \mathbb{R})$ over Δ is $(\hbar_3, \mathbb{L})$ with $\mathbb{L} = \mathbb{S} \cup \mathbb{R}$, defined as

$$\hbar_3(\sigma) = \begin{cases} \hbar_1(\sigma) & ; \sigma \in \mathbb{S} - \mathbb{R} \\ \hbar_2(\sigma) & ; \sigma \in \mathbb{R} - \mathbb{S} \\ \hbar_1(\sigma) \cap \hbar_2(\sigma) & ; \sigma \in \mathbb{S} \cap \mathbb{R}. \end{cases}$$

Example 4.9. Reconsidering Example 3.2, consider the following two sets

$$\mathbb{Q}_1 = \left\{ \begin{array}{l} (\mho_1/0.2, c, 1), (\mho_3/0.4, c, 0), (\mho_1/0.2, d, 1), (\mho_3/0.4, d, 1), \\ (\mho_3/0.4, d, 0), (\mho_1/0.2, e, 0), (\mho_3/0.4, e, 1) \end{array} \right\}$$

$$\mathbb{Q}_2 = \left\{ \begin{array}{l} (\mho_1/0.2, c, 1), (\mho_3/0.4, c, 0), (\mho_3/0.4, c, 1), (\mho_1/0.2, d, 1), \\ (\mho_3/0.4, d, 1), (\mho_1/0.2, d, 0), (\mho_3/0.4, d, 0), (\mho_1/0.2, e, 0), \\ (\mho_3/0.4, e, 1), (\mho_1/0.2, e, 1) \end{array} \right\}$$

Suppose $(\hbar_1, \mathbb{Q}_1)$ and $(\hbar_2, \mathbb{Q}_2)$ over Δ are two $\mathcal{FPNHSE}$-sets such that

$$(\hbar_1, \mathbb{Q}_1) = \left\{ \begin{array}{l} \left((\mho_1/0.2, c, 1), \left\{\frac{\eta_1}{<0.1,0.6,0.4>}, \frac{\eta_2}{<0.6,0.3,0.2>}, \frac{\eta_3}{<0.4,0.5,0.1>}, \frac{\eta_4}{<0.1,0.8,0.5>}\right\}\right), \\ \left((\mho_1/0.2, d, 1), \left\{\frac{\eta_1}{<0.3,0.4,0.5>}, \frac{\eta_2}{<0.6,0.2,0.3>}, \frac{\eta_3}{<0.2,0.5,0.6>}, \frac{\eta_4}{<0.1,0.5,0.3>}\right\}\right), \\ \left((\mho_3/0.4, d, 1), \left\{\frac{\eta_1}{<0.2,0.6,0.7>}, \frac{\eta_2}{<0.5,0.2,0.3>}, \frac{\eta_3}{<0.6,0.3,0.5>}, \frac{\eta_4}{<0.8,0.1,0.9>}\right\}\right), \\ \left((\mho_3/0.4, e, 1), \left\{\frac{\eta_1}{<0.6,0.2,0.4>}, \frac{\eta_2}{<0.2,0.7,0.6>}, \frac{\eta_3}{<0.4,0.3,0.5>}, \frac{\eta_4}{<0.1,0.5,0.4>}\right\}\right), \\ \left((\mho_1/0.2, e, 0), \left\{\frac{\eta_1}{<0.1,0.3,0.5>}, \frac{\eta_2}{<0.1,0.7,0.6>}, \frac{\eta_3}{<0.2,0.7,0.4>}, \frac{\eta_4}{<0.4,0.6,0.8>}\right\}\right), \\ \left((\mho_3/0.4, c, 0), \left\{\frac{\eta_1}{<0.1,0.6,0.9>}, \frac{\eta_2}{<0.3,0.6,0.7>}, \frac{\eta_3}{<0.6,0.1,0.2>}, \frac{\eta_4}{<0.7,0.2,0.3>}\right\}\right), \\ \left((\mho_3/0.4, d, 0), \left\{\frac{\eta_1}{<0.1,0.7,0.3>}, \frac{\eta_2}{<0.8,0.1,0.2>}, \frac{\eta_3}{<0.7,0.2,0.4>}, \frac{\eta_4}{<0.2,0.7,0.6>}\right\}\right) \end{array} \right\}$$

$$(\hbar_2,\mathbb{Q}_2)=\left\{\begin{array}{l}
\left((\mho_1/0.2,c,1),\left\{\frac{\eta_1}{<0.2,0.3,0.4>},\frac{\eta_2}{<0.7,0.4,0.5>},\frac{\eta_3}{<0.5,0.4,0.6>},\frac{\eta_4}{<0.2,0.4,0.7>}\right\}\right),\\
\left((\mho_1/0.2,d,1),\left\{\frac{\eta_1}{<0.4,0.3,0.8>},\frac{\eta_2}{<0.8,0.3,0.5>},\frac{\eta_3}{<0.4,0.3,0.5>},\frac{\eta_4}{<0.2,0.6,0.7>}\right\}\right),\\
\left((\mho_3/0.4,c,1),\left\{\frac{\eta_1}{<0.1,0.3,0.6>},\frac{\eta_2}{<0.9,0.1,0.7>},\frac{\eta_3}{<0.4,0.5,0.8>},\frac{\eta_4}{<0.5,0.3,0.5>}\right\}\right),\\
\left((\mho_3/0.4,d,1),\left\{\frac{\eta_1}{<0.4,0.2,0.3>},\frac{\eta_2}{<0.6,0.3,0.5>},\frac{\eta_3}{<0.7,0.4,0.5>},\frac{\eta_4}{<0.9,0.5,0.7}\right\}\right),\\
\left((\mho_1/0.2,e,1),\left\{\frac{\eta_1}{<0.7,0.2,0.3>},\frac{\eta_2}{<0.5,0.2,0.4>},\frac{\eta_3}{<0.6,0.2,0.4>},\frac{\eta_4}{<0.3,0.5,0.6}\right\}\right),\\
\left((\mho_3/0.4,e,1),\left\{\frac{\eta_1}{<0.7,0.3,0.7>},\frac{\eta_2}{<0.3,0.5,0.6>},\frac{\eta_3}{<0.5,0.4,0.3>},\frac{\eta_4}{<0.2,0.6,0.4>}\right\}\right),\\
\left((\mho_1/0.2,e,0),\left\{\frac{\eta_1}{<0.2,0.5,0.4>},\frac{\eta_2}{<0.2,0.6,0.3>},\frac{\eta_3}{<0.3,0.5,0.6>},\frac{\eta_4}{<0.5,0.3,0.7>}\right\}\right),\\
\left((\mho_1/0.2,d,0),\left\{\frac{\eta_1}{<0.1,0.6,0.3>},\frac{\eta_2}{<0.9,0.1,0.2>},\frac{\eta_3}{<0.6,0.3,0.4>},\frac{\eta_4}{<0.2,0.6,0.3>}\right\}\right),\\
\left((\mho_3/0.4,c,0),\left\{\frac{\eta_1}{<0.2,0.7,0.5>},\frac{\eta_2}{<0.4,0.5,0.6>},\frac{\eta_3}{<0.7,0.2,0.3>},\frac{\eta_4}{<0.8,0.1,0.4>}\right\}\right),\\
\left((\mho_3/0.4,d,0),\left\{\frac{\eta_1}{<0.2,0.5,0.4>},\frac{\eta_2}{<0.7,0.2,0.3>},\frac{\eta_3}{<0.8,0.2,0.6>},\frac{\eta_4}{<0.3,0.5,0.7>}\right\}\right)
\end{array}\right\}$$

Then $(\hbar_1,\mathbb{Q}_1)\cap_E(\hbar_2,\mathbb{Q}_2)=(\hbar_3,\mathbb{L})$

$$(\hbar_3,\mathbb{L})=\left\{\begin{array}{l}
\left((\mho_1/0.2,c,1),\left\{\frac{\eta_1}{<0.1,0.6,0.4>},\frac{\eta_2}{<0.6,0.4,0.5>},\frac{\eta_3}{<0.4,0.5,0.6>},\frac{\eta_4}{<0.1,0.8,0.7>}\right\}\right),\\
\left((\mho_1/0.2,d,1),\left\{\frac{\eta_1}{<0.3,0.4,0.8>},\frac{\eta_2}{<0.6,0.3,0.5>},\frac{\eta_3}{<0.2,0.5,0.6>},\frac{\eta_4}{<0.1,0.6,0.7>}\right\}\right),\\
\left((\mho_3/0.4,d,1),\left\{\frac{\eta_1}{<0.2,0.6,0.7>},\frac{\eta_2}{<0.5,0.4,0.5>},\frac{\eta_3}{<0.6,0.4,0.5>},\frac{\eta_4}{<0.8,0.1,0.7>}\right\}\right),\\
\left((\mho_3/0.4,e,1),\left\{\frac{\eta_1}{<0.6,0.3,0.7>},\frac{\eta_2}{<0.2,0.7,0.6>},\frac{\eta_3}{<0.4,0.4,0.5>},\frac{\eta_4}{<0.1,0.6,0.4>}\right\}\right),\\
\left((\mho_1/0.2,e,0),\left\{\frac{\eta_1}{<0.1,0.5,0.5>},\frac{\eta_2}{<0.1,0.6,0.6>},\frac{\eta_3}{<0.2,0.7,0.6>},\frac{\eta_4}{<0.4,0.6,0.8>}\right\}\right),\\
\left((\mho_3/0.4,c,0),\left\{\frac{\eta_1}{<0.1,0.7,0.9>},\frac{\eta_2}{<0.3,0.6,0.7>},\frac{\eta_3}{<0.6,0.2,0.3>},\frac{\eta_4}{<0.7,0.2,0.4>}\right\}\right),\\
\left((\mho_3/0.4,d,0),\left\{\frac{\eta_1}{<0.1,0.7,0.4>},\frac{\eta_2}{<0.8,0.2,0.3>},\frac{\eta_3}{<0.7,0.2,0.6>},\frac{\eta_4}{<0.2,0.7,0.7>}\right\}\right)\\
\left((\mho_1/0.2,d,0),\left\{\frac{\eta_1}{<0.1,0.6,0.3>},\frac{\eta_2}{<0.9,0.1,0.2>},\frac{\eta_3}{<0.6,0.3,0.4>},\frac{\eta_4}{<0.2,0.6,0.3>}\right\}\right),\\
\left((\mho_1/0.2,e,0),\left\{\frac{\eta_1}{<0.2,0.5,0.4>},\frac{\eta_2}{<0.2,0.6,0.3>},\frac{\eta_3}{<0.3,0.5,0.6>},\frac{\eta_4}{<0.5,0.3,0.7>}\right\}\right),\\
\left((\mho_3/0.4,c,1),\left\{\frac{\eta_1}{<0.1,0.3,0.6>},\frac{\eta_2}{<0.9,0.1,0.7>},\frac{\eta_3}{<0.4,0.5,0.8>},\frac{\eta_4}{<0.5,0.3,0.5>}\right\}\right)
\end{array}\right\}.$$

Proposition 4.10. *If* $(\hbar_1,\mathbb{Q}_1)$,$(\hbar_2,\mathbb{Q}_2)$ *and* $(\hbar_3,\mathbb{Q}_3)$ *are three* $\mathcal{FPNHSE}$*-sets over* Δ*, then*

(1) $(\hbar_1,\mathbb{Q}_1)\cap(\hbar_2,\mathbb{Q}_2)=(\hbar_2,\mathbb{Q}_2)\cap(\hbar_1,\mathbb{Q}_1)$,

(2) $((\hbar_1,\mathbb{Q}_1)\cap(\hbar_2,\mathbb{Q}_2))\cap(\hbar_3,\mathbb{Q}_3)=(\hbar_1,\mathbb{Q}_1)\cap((\hbar_2,\mathbb{Q}_2)\cap(\hbar_3,\mathbb{Q}_3))$,

(3) $(\hbar,\mathbb{Q})\cap\phi=\phi$.

Proposition 4.11. *If* $(\hbar_1,\mathbb{Q}_1)$,$(\hbar_2,\mathbb{Q}_2)$ *and* $(\hbar_3,\mathbb{Q}_3)$ *are three* $\mathcal{FPNHSE}$*-sets over* Δ*, then*

(1) $(\hbar_1,\mathbb{Q}_1)\cup((\hbar_2,\mathbb{Q}_2)\cap(\hbar_3,\mathbb{Q}_3))=((\hbar_1,\mathbb{Q}_1)\cup((\hbar_2,\mathbb{Q}_2))\cap((\hbar_1,\mathbb{Q}_1)\cup(\hbar_3,\mathbb{Q}_3))$,

(2) $(\hbar_1,\mathbb{Q}_1)\cap((\hbar_2,\mathbb{Q}_2)\cup(\hbar_3,\mathbb{Q}_3))=((\hbar_1,\mathbb{Q}_1)\cap((\hbar_2,\mathbb{Q}_2))\cup((\hbar_1,\mathbb{Q}_1)\cap(\hbar_3,\mathbb{Q}_3))$.

Definition 4.12. If $(\hbar_1,\mathbb{Q}_1)$ and $(\hbar_2,\mathbb{Q}_2)$ are two $\mathcal{FPNHSE}$-sets over Δ then $(\hbar_1,\mathbb{Q}_1)$ AND $(\hbar_2,\mathbb{Q}_2)$ denoted by $(\hbar_1,\mathbb{Q}_1)\wedge(\hbar_2,\mathbb{Q}_2)$ is defined by $(\hbar_1,\mathbb{Q}_1)\wedge(\hbar_2,\mathbb{Q}_2)=(\hbar_3,\mathbb{Q}_1\times\mathbb{Q}_2)$, while $\hbar_3(\sigma,\gamma)=\hbar_1(\sigma)\cap\hbar_2(\gamma),\forall(\sigma,\gamma)\in\mathbb{Q}_1\times\mathbb{Q}_2$.

Example 4.13. Retaking Example 3.2, let two sets
$\mathbb{Q}_1=\left\{(\mho_1/0.2,c,1),(\mho_1/0.2,d,1),(\mho_3/0.4,c,0)\right\}$
$\mathbb{Q}_2=\left\{(\mho_1/0.2,c,0),(\mho_3/0.4,c,1)\right\}$

Suppose $(\hbar_1, \mathbb{Q}_1)$ and $(\hbar_2, \mathbb{Q}_2)$ over Δ are two $\mathcal{FPNHSE}$-sets such that

$$(\hbar_1, \mathbb{Q}_1) = \left\{ \begin{array}{l} ((\mho_1/0.2, c, 1), \{\frac{\eta_1}{<0.1,0.6,0.4>}, \frac{\eta_2}{<0.6,0.4,0.5>}, \frac{\eta_3}{<0.4,0.5,0.6>}, \frac{\eta_4}{<0.1,0.8,0.7>}\}), \\ ((\mho_1/0.2, d, 1), \{\frac{\eta_1}{<0.3,0.4,0.8>}, \frac{\eta_2}{<0.6,0.3,0.5>}, \frac{\eta_3}{<0.2,0.5,0.6>}, \frac{\eta_4}{<0.1,0.6,0.7>}\}), \\ ((\mho_3/0.4, c, 0), \{\frac{\eta_1}{<0.1,0.6,0.9>}, \frac{\eta_2}{<0.3,0.6,0.7>}, \frac{\eta_3}{<0.6,0.1,0.2>}, \frac{\eta_4}{<0.7,0.2,0.3>}\}), \end{array} \right\}$$

$$(\hbar_2, \mathbb{Q}_2) = \left\{ \begin{array}{l} ((\mho_1/0.2, c, 0), \{\frac{\eta_1}{<0.2,0.1,0.3>}, \frac{\eta_2}{<0.7,0.2,0.4>}, \frac{\eta_3}{<0.5,0.2,0.5>}, \frac{\eta_4}{<0.2,0.3,0.6>}\}), \\ ((\mho_3/0.4, c, 1), \{\frac{\eta_1}{<0.1,0.5,0.6>}, \frac{\eta_2}{<0.4,0.2,0.5>}, \frac{\eta_3}{<0.7,0.1,0.2>}, \frac{\eta_4}{<0.8,0.1,0.4>}\}), \end{array} \right\}$$

Then $(\hbar_1, \mathbb{Q}_1) \wedge (\hbar_2, \mathbb{Q}_2) = (\hbar_3, \mathbb{Q}_1 \times \mathbb{Q}_2)$

$$\left\{ \begin{array}{l} (((\mho_1/0.2, c, 1), (\mho_1/0.2, c, 0)), \{\frac{\eta_1}{<0.1,0.35,0.4>}, \frac{\eta_2}{<0.6,0.30,0.5>}, \frac{\eta_3}{<0.4,0.35,0.6>}, \frac{\eta_4}{<0.1,0.55,0.7>}\}), \\ (((\mho_1/0.2, d, 1), (\mho_1/0.2, c, 0)), \{\frac{\eta_1}{<0.2,0.25,0.8>}, \frac{\eta_2}{<0.6,0.25,0.5>}, \frac{\eta_3}{<0.2,0.35,0.6>}, \frac{\eta_4}{<0.1,0.45,0.7>}\}), \\ (((\mho_1/0.2, d, 1), (\mho_3/0.4, c, 1)), \{\frac{\eta_1}{<0.1,0.45,0.8>}, \frac{\eta_2}{<0.4,0.25,0.5>}, \frac{\eta_3}{<0.2,0.30,0.6>}, \frac{\eta_4}{<0.1,0.35,0.7>}\}), \\ (((\mho_1/0.2, c, 1), (\mho_3/0.4, c, 1)), \{\frac{\eta_1}{<0.1,0.55,0.6>}, \frac{\eta_2}{<0.4,0.30,0.5>}, \frac{\eta_3}{<0.4,0.30,0.6>}, \frac{\eta_4}{<0.1,0.45,0.7>}\}), \\ (((\mho_3/0.4, c, 0), (\mho_1/0.2, c, 0)), \{\frac{\eta_1}{<0.1,0.35,0.9>}, \frac{\eta_2}{<0.3,0.40,0.7>}, \frac{\eta_3}{<0.5,0.15,0.5>}, \frac{\eta_4}{<0.2,0.25,0.6>}\}), \\ (((\mho_3/0.4, c, 0), (\mho_3/0.4, c, 1)), \{\frac{\eta_1}{<0.1,0.55,0.9>}, \frac{\eta_2}{<0.3,0.40,0.7>}, \frac{\eta_3}{<0.6,0.10,0.2>}, \frac{\eta_4}{<0.7,0.15,0.4>}\}). \end{array} \right\}$$

Definition 4.14. If $(\hbar_1, \mathbb{Q}_1)$ and $(\hbar_2, \mathbb{Q}_2)$ are two $\mathcal{FPNHSE}$-sets over Δ, then $(\hbar_1, \mathbb{Q}_1)$ OR $(\hbar_2, \mathbb{Q}_2)$ denoted by $(\hbar_1, \mathbb{Q}_1) \vee (\hbar_2, \mathbb{Q}_2)$ is defined by $(\hbar_1, \mathbb{Q}_1) \vee (\hbar_2, \mathbb{Q}_2) = (\hbar_3, \mathbb{Q}_1 \times \mathbb{Q}_2)$, while $\hbar_3(\delta, \gamma) = \hbar_1(\delta) \cup \hbar_2(\gamma), \forall(\delta, \gamma) \in \mathbb{Q}_1 \times \mathbb{Q}_2$.

Example 4.15. Reconsidering Example 3.2, suppose the following sets

$\mathbb{Q}_1 = \left\{ (\mho_1/0.2, c, 1), (\mho_1/0.2, d, 1), (\mho_3/0.4, c, 0) \right\}$

$\mathbb{Q}_2 = \left\{ (\mho_1/0.2, c, 0), (\mho_3/0.4, c, 1) \right\}$

Suppose $(\hbar_1, \mathbb{Q}_1)$ and $(\hbar_2, \mathbb{Q}_2)$ over Δ are two $\mathcal{FPNHSE}$-sets such that

$$(\hbar_1, \mathbb{Q}_1) = \left\{ \begin{array}{l} ((\mho_1/0.2, c, 1), \{\frac{\eta_1}{<0.1,0.6,0.4>}, \frac{\eta_2}{<0.6,0.4,0.5>}, \frac{\eta_3}{<0.4,0.5,0.6>}, \frac{\eta_4}{<0.1,0.8,0.7>}\}), \\ ((\mho_1/0.2, d, 1), \{\frac{\eta_1}{<0.3,0.4,0.8>}, \frac{\eta_2}{<0.6,0.3,0.5>}, \frac{\eta_3}{<0.2,0.5,0.6>}, \frac{\eta_4}{<0.1,0.6,0.7>}\}), \\ ((\mho_3/0.4, c, 0), \{\frac{\eta_1}{<0.1,0.6,0.9>}, \frac{\eta_2}{<0.3,0.6,0.7>}, \frac{\eta_3}{<0.6,0.1,0.2>}, \frac{\eta_4}{<0.7,0.2,0.3>}\}) \end{array} \right\}$$

$$(\hbar_2, \mathbb{Q}_2) = \left\{ \begin{array}{l} ((\mho_1/0.2, c, 0), \{\frac{\eta_1}{<0.2,0.1,0.3>}, \frac{\eta_2}{<0.7,0.2,0.4>}, \frac{\eta_3}{<0.5,0.2,0.5>}, \frac{\eta_4}{<0.2,0.3,0.6>}\}), \\ ((\mho_3/0.4, c, 1), \{\frac{\eta_1}{<0.1,0.5,0.6>}, \frac{\eta_2}{<0.4,0.2,0.5>}, \frac{\eta_3}{<0.7,0.1,0.2>}, \frac{\eta_4}{<0.8,0.1,0.4>}\}). \end{array} \right\}$$

Then $(\hbar_3, \mathbb{Q}_3) \vee (\hbar_2, \mathbb{Q}_2) = (\hbar_3, \mathbb{Q}_1 \times \mathbb{Q}_2)$

$$\left\{ \begin{array}{l} (((\mho_1/0.2, c, 1), (\mho_1/0.2, c, 0)), \{\frac{\eta_1}{<0.2,0.35,0.3>}, \frac{\eta_2}{0<0.7,0.30,0.4>}, \frac{\eta_3}{<0.5,0.35,0.5>}, \frac{\eta_4}{<0.2,0.55,0.6>}\}), \\ (((\mho_1/0.2, d, 1), (\mho_1/0.2, c, 0)), \{\frac{\eta_1}{<0.3,0.25,0.3>}, \frac{\eta_2}{<0.7,0.25,0.4>}, \frac{\eta_3}{<0.5,0.35,0.5>}, \frac{\eta_4}{<0.2,0.45,0.6>}\}), \\ (((\mho_1/0.2, d, 1), (\mho_3/0.4, c, 1)), \{\frac{\eta_1}{<0.3,0.45,0.6>}, \frac{\eta_2}{<0.6,0.25,0.5>}, \frac{\eta_3}{<0.7,0.30,0.2>}, \frac{\eta_4}{<0.8,0.35,0.4>}\}), \\ (((\mho_1/0.2, c, 1), (\mho_3/0.4, c, 1)), \{\frac{\eta_1}{<0.1,0.55,0.4>}, \frac{\eta_2}{<0.6,0.30,0.5>}, \frac{\eta_3}{<0.7,0.30,0.2>}, \frac{\eta_4}{<0.8,0.45,0.4>}\}), \\ (((\mho_3/0.4, c, 0), (\mho_1/0.2, c, 0)), \{\frac{\eta_1}{<0.2,0.35,0.3>}, \frac{\eta_2}{<0.7,0.40,0.4>}, \frac{\eta_3}{<0.6,0.15,0.2>}, \frac{\eta_4}{<0.7,0.25,0.3>}\}), \\ (((\mho_3/0.4, c, 0), (\mho_3/0.4, c, 1)), \{\frac{\eta_1}{<0.1,0.55,0.6>}, \frac{\eta_2}{<0.4,0.40,0.5>}, \frac{\eta_3}{<0.7,0.10,0.2>}, \frac{\eta_4}{<0.8,0.15,0.4>}\}). \end{array} \right\}$$

Proposition 4.16. *If* $(\hbar_1, \mathbb{Q}_1)$,$(\hbar_2, \mathbb{Q}_2)$ *and* $(\hbar_3, \mathbb{Q}_3)$ *are three* $\mathcal{FPNHSE}$*-sets over* Δ*, then*

(1) $((\hbar_1, \mathbb{Q}_1) \wedge (\hbar_2, \mathbb{Q}_2))^c = ((\hbar_1, \mathbb{Q}_1))^c \vee ((\hbar_2, \mathbb{Q}_2))^c$

(2) $((\hbar_1, \mathbb{Q}_1) \vee (\hbar_2, \mathbb{Q}_2))^c = ((\hbar_1, \mathbb{Q}_1))^c \wedge ((\hbar_2, \mathbb{Q}_2))^c$

Proposition 4.17. *If* $(\hbar_1, \mathbb{Q}_1)$,$(\hbar_2, \mathbb{Q}_2)$ *and* $(\hbar_3, \mathbb{Q}_3)$ *are three* $\mathcal{FPNHSE}$*-sets over* Δ*, then*

(1) $((\hbar_1, \mathbb{Q}_1) \wedge (\hbar_2, \mathbb{Q}_2)) \wedge (\hbar_3, \mathbb{Q}_3) = (\hbar_1, \mathbb{Q}_1) \wedge ((\hbar_2, \mathbb{Q}_2) \wedge (\hbar_3, \mathbb{Q}_3))$

(2) $((\hbar_1, \mathbb{Q}_1) \vee (\hbar_2, \mathbb{Q}_2)) \vee (\hbar_3, \mathbb{Q}_3) = (\hbar_1, \mathbb{Q}_1) \vee ((\hbar_2, \mathbb{Q}_2) \vee (\hbar_3, \mathbb{Q}_3))$

(3) $(\hbar_1, \mathbb{Q}_1) \vee ((\hbar_2, \mathbb{Q}_2) \wedge (\hbar_3, \mathbb{Q}_3) = ((\hbar_1, \mathbb{Q}_1) \vee ((\hbar_2, \mathbb{Q}_2)) \wedge ((\hbar_1, \mathbb{Q}_1) \vee (\hbar_3, \mathbb{Q}_3))$

(4) $(\hbar_1, \mathbb{Q}_1) \wedge ((\hbar_2, \mathbb{Q}_2) \vee (\hbar_3, \mathbb{Q}_3)) = ((\hbar_1, \mathbb{Q}_1) \wedge ((\hbar_2, \mathbb{Q}_2)) \vee ((\hbar_1, \mathbb{Q}_1) \wedge (\hbar_3, \mathbb{Q}_3))$.

5. An Application to Fuzzy Parameterized Neutrosophic Hypersoft Expert Set

In this section, an application of $\mathcal{FPNHSE}$-set theory with a proposed algorithm in a decision-making problem, is presented.

Statement of the problem

The procurement of an electronic equipment has evolved in the product selection scenario into a difficult issue for a person and an organisation. For the usage of his family, Mr. Bay is looking for an LED TV. He has never purchased it before. He solicits assistance from his buddies who may have knowledge on where to buy such a device. Consider the following while buying this device in light of their friends' experiences:

(1) **Screen Resolution**: The screen goal of a LED TV is the number of pixels in each aspect that the TV can show locally. Higher-goal screens permit you to see all the more fine subtleties in your beloved substance.

(2) **Refresh Rate**: Assuming you're on the lookout for a LED TV, you've likely heard a great deal about "speed." When promotions and audits talk about how quick a LED TV is, they allude to the showcase's invigorate rate or how regularly it changes the image. TV and motion pictures don't show natural movement, even handfuls, and many casings each second, similar to a reel of film or a colossal flipbook. The quicker the LED TV, the more casings it shows each second.

(3) **Warranty**: Service agreement for your TV or TV covering all assembling absconds, programming issues, and electrical glitches or breakdowns. The maintenance agreement for TV or TV begins following your producer or OEM guarantee lapses.

(4) **Ports**: Somewhere around four ports ought to be accessible, including USB, HDMI, sound/video, and VGA. Additionally, ensure it upholds every hard circle and pen drive to play recordings.

(5) **Screen Size**: There is a broad scope of Tv sizes accessible in the market to choose from. The right TV size gives you a vivid review experience. Looking for an ideal space from room, lounge, and nearness to the TV screen. Contrast TV stands and divider mounted set up to track down a suitable spot in the space to put the TV set. The right screen size and distance give you immersive survey encounters.

Proposed Algorithm : Selection of LED TV

▷ **Start**:

▷ **Construction**:

———1. Construct $\mathcal{FPNHSE}$-set (ξ, K)

▷ **Computation**:

———2. Determine Agree-$\mathcal{FPNHSE}$-set and Disagree-$\mathcal{FPNHSE}$-set.

———3. Calculation of Values of $T_(0_i) - I_(0_i) - F_(0_i)$ for each $0_i \in \Omega$.

———4. Calculate the the highest numerical grade for Agree and Disagree-$\mathcal{FPNHSE}$-sets.

———5. Determine the score of each element $0_i \in \Omega$ for Agree and Disagree-$\mathcal{FPNHSE}$-sets.

———-6. Determine the score difference for each element $c_i \in \Omega$.

▷ **Output**:

———7. Compute n, for which M= max j_i to decide the best solution of the problem.

▷ **End**:

Step-1

Let four categories of LED TV forming the universe of discourse $\Omega = \{t_1, t_2, t_3, t_4\}$ and $X = \{E_1 = Henry, E_2 = John, E_3 = Watson\}$ be a set of experts for this purchase. The following are the attribute-valued sets for prescribed attributes:

$W_1 = ScreenResolution = \{w_1 = 1280 \times 720pixels, w_2 = 1920 \times 1080pixels\}$

$W_2 = RefreshRate = \{w_3 = 60Hz, w_4 = 120Hz\}$, $W_3 = Warranty = \{w_5 = 4years, w_6 = 5years\}$, $W_4 = Ports = \{w_7 = 4, w_8 = 5\}$, $W_5 = ScreenSize = \{w_9 = 24inch, w_{10} = 32inch\}$ and then $W =_1 \times W_2 \times W_3 \times W_4 \times W_5$

$$W = \left\{ \begin{array}{l} (w_1, w_3, w_5, w_7, w_9), (w_1, w_3, w_5, w_7, w_{10}), (w_1, w_3, w_5, w_8, w_9), (w_1, w_3, w_5, w_8, w_{10}), \\ (w_1, w_3, w_6, w_7, w_9), (w_1, w_3, w_6, w_7, w_{10}), (w_1, w_3, w_6, w_8, w_9), (w_1, w_3, w_6, w_8, w_{10}), \\ (w_1, w_4, w_5, w_7, w_9), (w_1, w_4, w_5, w_7, w_{10}), (w_1, w_4, w_5, w_8, w_9), (w_1, w_4, w_5, w_8, w_{10}), \\ (w_1, w_4, w_6, w_7, w_9), (w_1, w_4, w_6, w_7, w_{10}), (w_1, w_4, w_6, w_8, w_9), (w_1, w_4, w_6, w_8, w_{10}), \\ (w_2, w_3, w_5, w_7, w_9), (w_2, w_3, w_5, w_7, w_{10}), (w_2, w_3, w_5, w_8, w_9), (w_2, w_3, w_5, w_8, w_{10}), \\ (w_2, w_3, w_6, w_7, w_9), (w_2, w_3, w_6, w_7, w_{10}), (w_2, w_3, w_6, w_8, w_9), (w_2, w_3, w_6, w_8, w_{10}), \\ (w_2, w_4, w_5, w_7, w_9), (w_2, w_4, w_5, w_7, w_{10}), (w_2, w_4, w_5, w_8, w_9), (w_2, w_4, w_5, w_8, w_{10}), \\ (w_2, w_4, w_6, w_7, w_9), (w_2, w_4, w_6, w_7, w_{10}), (w_2, w_4, w_6, w_8, w_9), (w_2, w_4, w_6, w_8, w_{10}) \end{array} \right\}$$

Now take $K \subseteq N$ as

$K = \{k_1/0.2 = (w_1, w_3, w_5, w_7, w_9), k_2/0.3 = (w_1, w_3, w_6, w_7, w_{10}), k_3/0.4 = (w_1, w_4, w_6, w_8, w_9), k_4/0.5 = (w_2, w_3, w_6, w_8, w_9), k_5/0.6 = (w_2, w_4, w_6, w_7, w_{10})\}$

$$(\xi, K)_1 = \left\{ \begin{array}{l}
\left((k_1, E_1, 1), \left\{ \frac{c_1}{<0.9,0.1,0.7>}, \frac{c_2}{<0.5,0.2,0.1>}, \frac{c_3}{<0.6,0.2,0.4>}, \frac{c_4}{<0.7,0.1,0.2>} \right\}\right), \\
\left((k_1, E_2, 1), \left\{ \frac{c_1}{<0.8,0.2,0.4>}, \frac{c_2}{<0.9,0.2,0.3>}, \frac{c_3}{<0.6,0.2,0.3>}, \frac{c_4}{<0.9,0.1,0.5>} \right\}\right), \\
\left((k_1, E_3, 1), \left\{ \frac{c_1}{<0.7,0.3,0.1>}, \frac{c_2}{<0.4,0.1,0.2>}, \frac{c_3}{<0.6,0.1,0.4>}, \frac{c_4}{<0.8,0.6,0.1>} \right\}\right), \\
\left((k_2, E_1, 1), \left\{ \frac{c_1}{<0.8,0.4,0.2>}, \frac{c_2}{<0.4,0.2,0.1>}, \frac{c_3}{<0.7,0.1,0.5>}, \frac{c_4}{<0.5,0.2,0.1>} \right\}\right), \\
\left((k_2, E_2, 1), \left\{ \frac{c_1}{<0.5,0.1,0.3>}, \frac{c_2}{<0.9,0.4,0.4>}, \frac{c_3}{<0.7,0.4,0.1>}, \frac{c_4}{<0.9,0.2,0.3>} \right\}\right), \\
\left((k_2, E_3, 1), \left\{ \frac{c_1}{<0.7,0.3,0.1>}, \frac{c_2}{<0.8,0.2,0.4>}, \frac{c_3}{<0.9,0.2,0.5>}, \frac{c_4}{<0.8,0.2,0.1>} \right\}\right), \\
\left((k_3, E_1, 1), \left\{ \frac{c_1}{<0.9,0.4,0.2>}, \frac{c_2}{<0.7,0.2,0.3>}, \frac{c_3}{<0.8,0.2,0.5>}, \frac{c_4}{<0.7,0.2,0.4>} \right\}\right), \\
\left((k_3, E_2, 1), \left\{ \frac{c_1}{<0.8,0.3,0.4>}, \frac{c_2}{<0.9,0.2,0.5>}, \frac{c_3}{<0.6,0.2,0.3>}, \frac{c_4}{<0.9,0.2,0.3>} \right\}\right), \\
\left((k_3, E_3, 1), \left\{ \frac{c_1}{<0.7,0.4,0.1>}, \frac{c_2}{<0.6,0.3,0.2>}, \frac{c_3}{<0.8,0.2,0.3>}, \frac{c_4}{<0.7,0.1,0.4>} \right\}\right), \\
\left((k_4, E_1, 1), \left\{ \frac{c_1}{<0.9,0.1,0.7>}, \frac{c_2}{<0.8,0.3,0.1>}, \frac{c_3}{<0.7,0.1,0.4>}, \frac{c_4}{<0.6,0.3,0.2>} \right\}\right), \\
\left((k_4, E_2, 1), \left\{ \frac{c_1}{<0.8,0.1,0.4>}, \frac{c_2}{<0.4,0.3,0.1>}, \frac{c_3}{<0.8,0.3,0.1>}, \frac{c_4}{<0.7,0.2,0.1>} \right\}\right), \\
\left((k_4, E_3, 1), \left\{ \frac{c_1}{<0.6,0.2,0.4>}, \frac{c_2}{<0.1,0.3,0.2>}, \frac{c_3}{<0.9,0.5,0.3>}, \frac{c_4}{<0.6,0.1,0.3>} \right\}\right), \\
\left((k_5, E_1, 1), \left\{ \frac{c_1}{<0.6,0.3,0.2>}, \frac{c_2}{<0.8,0.2,0.4>}, \frac{c_3}{<0.3,0.2,.01>}, \frac{c_4}{<0.7,0.1,0.4>} \right\}\right), \\
\left((k_5, E_2, 1), \left\{ \frac{c_1}{<0.5,0.3,0.2>}, \frac{c_2}{<0.6,0.2,0.3>}, \frac{c_3}{<0.4,0.3,0.1>}, \frac{c_4}{<0.6,0.1,0.2>} \right\}\right), \\
\left((k_5, E_3, 1), \left\{ \frac{c_1}{<0.4,0.3,0.1>}, \frac{c_2}{<0.6,0.1,0.2>}, \frac{c_3}{<0.3,0.1,0.2>}, \frac{c_4}{<0.9,0.4,0.2>} \right\}\right)
\end{array} \right\}.$$

and

$$(\xi, K)_0 = \left\{ \begin{array}{l}
\left((k_1, E_1, 0), \left\{ \frac{c_1}{<0.6,0.3,0.1>}, \frac{c_2}{<0.9,0.1,0.2>}, \frac{c_3}{<0.8,0.2,0.3>}, \frac{c_4}{<0.6,0.1,0.4>} \right\}\right), \\
\left((k_1, E_2, 0) \left\{ \frac{c_1}{<0.7,0.2,0.3>}, \frac{c_2}{<0.6,0.3,0.2>}, \frac{c_3}{<0.7,0.2,0.1>}, \frac{c_4}{<0.8,0.2,0.1>} \right\}\right), \\
\left((k_1, E_3, 0), \left\{ \frac{c_1}{<0.8,0.3,0.4>}, \frac{c_2}{<0.6,0.2,0.1>}, \frac{c_3}{<0.7,0.4,0.2>}, \frac{c_4}{<0.9,0.4,0.4>} \right\}\right), \\
\left((k_2, E_1, 0), \left\{ \frac{c_1}{<0.9,0.4,0.2>}, \frac{c_2}{<0.9,0.3,0.3>}, \frac{c_3}{<0.8,0.1,0.4>}, \frac{c_4}{<0.8,0.2,0.2>} \right\}\right), \\
\left((k_2, E_2, 0), \left\{ \frac{c_1}{<0.9,0.5,0.1>}, \frac{c_2}{<0.7,0.2,0.3>}, \frac{c_3}{<0.6,0.3,0.2>}, \frac{c_4}{<0.5,0.3,0.2>} \right\}\right), \\
\left((k_2, E_3, 0), \left\{ \frac{c_1}{<0.7,0.2,0.1>}, \frac{c_2}{<0.8,0.5,0.2>}, \frac{c_3}{<0.7,0.1,0.2>}, \frac{c_4}{<0.8,0.4,0.3>} \right\}\right), \\
\left((k_3, E_1, 0), \left\{ \frac{c_1}{<0.9,0.1,0.6>}, \frac{c_2}{<0.7,0.2,0.1>}, \frac{c_3}{<0.9,0.1,0.5>}, \frac{c_4}{<0.8,0.2,0.1>} \right\}\right), \\
\left((k_3, E_2, 0), \left\{ \frac{c_1}{<0.8,0.2,0.1>}, \frac{c_2}{<0.9,0.2,0.1>}, \frac{c_3}{<0.9,0.6,0.1>}, \frac{c_4}{<0.9,0.2,0.1>} \right\}\right), \\
\left((k_3, E_3, 0), \left\{ \frac{c_1}{<0.6,0.2,0.4>}, \frac{c_2}{<0.6,0.1,0.3>}, \frac{c_3}{<0.9,0.4,0.1>}, \frac{c_4}{<0.7,0.4,0.2>} \right\}\right), \\
\left((k_4, E_1, 0), \left\{ \frac{c_1}{<0.6,0.3,0.2>}, \frac{c_2}{<0.7,0.5,0.1>}, \frac{c_3}{<0.8,0.2,0.1>}, \frac{c_4}{<0.9,0.4,0.1>} \right\}\right), \\
\left((k_4, E_2, 0), \left\{ \frac{c_1}{<0.5,0.2,0.1>}, \frac{c_2}{<0.5,0.2,0.1>}, \frac{c_3}{<0.3,0.1,0.0>}, \frac{c_4}{<0.6,0.2,0.1>} \right\}\right), \\
\left((k_4, E_3, 0), \left\{ \frac{c_1}{<0.7,0.1,0.3>}, \frac{c_2}{<0.7,0.2,0.1>}, \frac{c_3}{<0.6,0.1,0.4>}, \frac{c_4}{<0.9,0.1,0.6>} \right\}\right), \\
\left((k_5, E_1, 0), \left\{ \frac{c_1}{<0.9,0.5,0.1>}, \frac{c_2}{<0.6,0.2,0.1>}, \frac{c_3}{<0.9,0.1,0.3>}, \frac{c_4}{<0.9,0.2,0.5>} \right\}\right), \\
\left((k_5, E_2, 0), \left\{ \frac{c_1}{<0.8,0.6,0.1>}, \frac{c_2}{<0.9,0.2,0.1>}, \frac{c_3}{<0.8,0.5,0.2>}, \frac{c_4}{<0.7,0.3,0.2>} \right\}\right), \\
\left((k_5, E_3, 0), \left\{ \frac{c_1}{<0.8,0.1,0.3>}, \frac{c_2}{<0.5,0.2,0.4>}, \frac{c_3}{<0.8,0.1,0.7>}, \frac{c_4}{<0.6,0.1,0.3>} \right\}\right)
\end{array} \right\}$$

are $\mathcal{FPNHSE}$-sets.

Step-2

Table 1 and Table 2 represent the values of $\mathsf{T}_(c_i)$-$\mathsf{I}_(c_i)$-$F_(c_i)$.

Step-(2)

Grade values of agree and disagree $\mathcal{FPNHSE}$-sets have been represented in Table 3 and Table 4 respectively.

Step-(3-5)

Table 1. Agree-$\mathcal{FPNHSE}$-set

C	c_1	c_2	c_3	c_4
$(k_1, E_1, 1)$	0.1	0.3	0.0	0.4
$(k_1, E_2, 1)$	0.2	0.4	0.1	0.3
$(k_1, E_3, 1)$	0.4	0.1	0.1	0.1
$(k_2, E_1, 1)$	0.2	0.1	0.1	0.3
$(k_2, E_2, 1)$	0.2	0.1	0.2	0.4
$(k_2, E_3, 1)$	0.3	0.2	0.2	0.2
$(k_3, E_1, 1)$	0.3	0.2	0.1	0.1
$(k_3, E_2, 1)$	0.1	0.2	0.3	0.2
$(k_3, E_3, 1)$	0.2	0.1	0.3	0.2
$(k_4, E_1, 1)$	0.1	0.4	0.2	0.1
$(k_4, E_2, 1)$	0.3	0.0	0.2	0.4
$(k_4, E_3, 1)$	0.0	0.0	0.1	0.2
$(k_5, E_1, 1)$	0.1	0.2	0.0	0.3
$(k_5, E_2, 1)$	0.0	0.1	0.0	0.3
$(k_5, E_3, 1)$	0.0	0.4	0.0	0.3

Table 2. Disagree-$\mathcal{FPNHSE}$-set

C	c_1	c_2	c_3	c_4
$(k_1, E_1, 0)$	0.4	0.6	0.3	0.2
$(k_1, E_2, 0)$	0.2	0.1	0.4	0.5
$(k_1, E_3, 0)$	0.1	0.1	0.1	0.4
$(k_2, E_1, 0)$	0.3	0.3	0.3	0.4
$(k_2, E_2, 0)$	0.3	0.2	0.1	0.0
$(k_2, E_3, 0)$	0.3	0.1	0.4	0.1
$(k_3, E_1, 0)$	0.2	0.4	0.3	0.5
$(k_3, E_2, 0)$	0.5	0.4	0.2	0.6
$(k_3, E_3, 0)$	0.0	0.4	0.2	0.3
$(k_4, E_1, 0)$	0.1	0.1	0.5	0.4
$(k_4, E_2, 0)$	0.1	0.2	0.0	0.3
$(k_4, E_3, 0)$	0.4	0.1	0.1	0.2
$(k_5, E_1, 0)$	0.1	0.3	0.5	0.2
$(k_5, E_2, 0)$	0.1	0.6	0.1	0.2
$(k_5, E_3, 0)$	0.5	0.1	0.0	0.4

The difference of scores of agree and disagree-$\mathcal{FPNHSE}$-sets have been shown in Table 5. The scores for agree-$\mathcal{FPNHSE}$-set are :

$S(c_1) = 0.6$, $S(c_2) = 1.3$, $S(c_3) = 0.6$ and $S(c_4) = 2.0$

whereas scores for disagree-$\mathcal{FPNHSE}$-set are:

Table 3. Numerical Grades of agree $\mathcal{FPNHSE}$-set

Pairs	c_i	Highest Numerical Grade
$(k_1, E_1, 1)$	c_4	0.4
$(k_1, E_2, 1)$	c_2	0.4
$(k_1, E_3, 1)$	c_1	0.4
$(k_2, E_1, 1)$	c_2	0.1
$(k_2, E_2, 1)$	c_4	0.4
$(k_2, E_3, 1)$	c_1	0.3
$(k_3, E_1, 1)$	c_1	0.3
$(k_3, E_2, 1)$	c_3	0.3
$(k_3, E_3, 1)$	c_3	0.3
$(k_4, E_1, 1)$	c_2	0.4
$(k_4, E_2, 1)$	c_4	0.4
$(k_4, E_3, 1)$	c_4	0.2
$(k_5, E_1, 1)$	c_4	0.3
$(k_5, E_2, 1)$	c_4	0.3
$(k_5, E_3, 1)$	c_2	0.4

Table 4. Numerical Grades of disagree $\mathcal{FPNHSE}$-set

Pairs	c_i	Highest Numerical Grade
$(k_1, E_1, 0)$	c_2	0.6
$(k_1, E_2, 0)$	c_3	0.4
$(k_1, E_3, 0)$	c_4	0.4
$(k_2, E_1, 0)$	c_4	0.4
$(k_2, E_2, 0)$	c_1	0.4
$(k_2, E_3, 0)$	c_3	0.4
$(k_3, E_1, 0)$	c_4	0.5
$(k_3, E_2, 0)$	c_4	0.6
$(k_3, E_3, 0)$	c_2	0.4
$(k_4, E_1, 0)$	c_3	0.5
$(k_4, E_2, 0)$	c_4	0.3
$(k_4, E_3, 0)$	c_1	0.4
$(k_5, E_1, 0)$	c_3	0.5
$(k_5, E_2, 0)$	c_2	0.6
$(k_5, E_3, 0)$	c_1	0.5

$S(c_1) = 1.3$, $S(c_2) = 1.6$, $S(c_3) = 1.8$ and $S(c_4) = 2.2$.

Step-6; Decision

As from above result, c_4 is preferred to be best and have been mentioned in Figure 2.

TABLE 5. Numerical values of $j_i = G_i - H_i$

G_i	H_i	$j_i = G_i - H_i$
$S(c_1) = 0.6$	$S(c_1) = 1.3$	-0.7
$S(c_2) = 1.3$	$S(c_2) = 1.6$	-0.3
$S(c_3) = 0.6$	$S(c_3) = 1.8$	-1.2
$S(c_4) = 2.0$	$S(c_4) = 2.2$	0.2

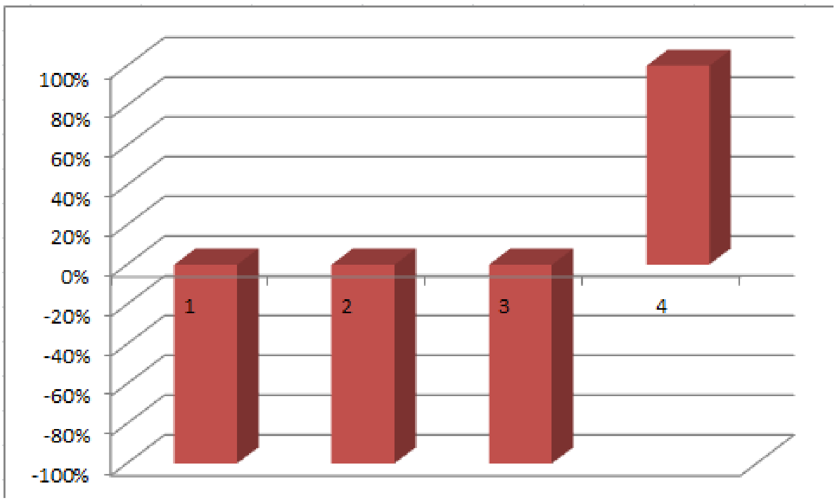

FIGURE 2. Ranking of Alternatives for Algorithm

6. Conclusions

The foundations of the fuzzy parameterized neutrosophic hypersoft expert set are developed in this study, along with certain generalisations of theoretical operations like union, intersection, complement, AND, and OR. With specific instances, some fundamental concepts like exclusion, contradiction, and laws are explored. These concepts include idempotent, absorption, domination, identity, associative, and distributive laws. In the end, an algorithm is created to describe how the decision-making problem is solved. This new work inspires further advancements of related research and practical applications while providing an exceptional expansion to existing theories for handling ambiguity, untruth, and truthness.

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Single-Valued Neutrosophic Covering-Based Rough Set Model Over Two Universes and Its Application in MCDM

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Abstract: This article aims to propose a new type of single-valued neutrosophic(SVN) covering-based rough sets over two universes by using Wang's single-valued neutrosophic covering rough sets. Wang's model is based on one universe but the proposed model is based on two universes and thus the new model gives a new perspective for decision-making on uncertain problems. First, we define SVN β-neighborhood, which is considered as a mapping from the universe to the set of SVN sets in another universe and study its properties. Then we investigate the properties of the new type of SVN covering-based rough set model over two universes. Also, we give a necessary and sufficient condition under which two SVN β-coverings generate the same SVN covering lower and the upper approximation. In addition, we also present the matrix representation of SVN covering lower and upper approximation operators over two universes for solving real-life-based multi-criteria decision-making problems.

Keywords: SVN sets; SVN β-neighborhood; SVN covering-based rough set; MCDM

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1. Introduction

The discovery of fuzzy set (FS), introduced by Zadeh[1] has been regarded as the finest discovery that is utilized to solve vague and uncertain information with an aid of a membership function. The FS concept provides a new perspective for the decision-makers to address the issues that cannot be tackled by using traditional mathematical tools. Due to the novelty of FS, it can be employed in various practical applications given in [2-6]. To realize the importance of the non-membership value along with the membership value of an attribute in a universe, Atanassov[7] introduced the intuitionistic fuzzy set(IFS). To handle more complexity that arises in various real uncertain decision-making problems; the FS concept has been further extended by introducing interval-valued fuzzy set [8], interval-valued intuitionistic fuzzy set [9], picture fuzzy set [10], spherical fuzzy set [11], hesitant fuzzy set [12], Pythagorean fuzzy set [13], etc.

In our previous discussion, we were mainly concerned with fuzzy sets and their various extensions to address uncertain and vague information. But, all these types of fuzzy sets are not capable to model the indeterminate information present in human cognition. To fill up this gap, Smarandache[14] introduced the notion of neutrosophy as a new branch of philosophy. Later on, he introduced the neutrosophic set (NS) [15] as an extension of IFS. In NS, every object in the universe is characterized by the three membership functions called the truth-membership function (T_A), indeterminate-membership function (I_A), and the falsity membership function (F_A) with a restriction $T_A, I_A, F_A : X \to \left]^{-}0, 1^{+}\right[$ and $^{-}0 \le T_A(x) + I_A(x) + F_A(x) \le 3^{+}$.

Later on, Wang et al.[16] introduced the single-valued neutrosophic set(SVNS) as an instance of NS. For handling decision-making problems under a neutrosophic environment, the decision-makers face problems while deciding due to the involvement of non-standard unit intervals. So, SVNS is introduced where the non-standard unit interval is replaced by a standard unit interval. The concept of SVNS has been extensively used in numerous decision-making problems (see the references [17-21]). Also, we would like to discuss some other important topics that are useful for the further development of the proposed study as follows: fixed point results in orthogonal neutrosophic metric spaces [22]; contractive and weakly compatible mappings in neutrosophic metric spaces are utilized in solving nonlinear differential equations [23]; some new aspects of fixed point theory under the intuitionistic fuzzy set and neutrosophic set [24]; fuzzy b-metric like spaces [25]; new aspects in fuzzy fixed point theory [26]; pentagonal controlled fuzzy metric spaces and its application [27].

In 1982, the Polish mathematician Pawlak [28] proposed another useful mathematical tool known as the rough set(RS) theory. Like FS, RS is another kind of generalization of a classical set. In RS, every subset of the universe is characterized by lower and upper approximations (see [29]). Also, in RS, the concept of equivalence classes is the key issue to form two approximations. It can be useful in discovering the hidden data, modeling information systems, eliminating the redundant data, and applied in data analysis, pattern recognition, data mining, intelligent systems, medical diagnosis, machine learning, and many more (see the references [30-34]). RS deals with crisp approximation space. But, we encounter some information system that contains fuzzy characteristics. To cope with such an issue, a rough set is combined with different types of fuzzy sets and obtain new hybrid structures and their associated applications are as follows: rough fuzzy sets and soft rough sets [35]; intuitionistic fuzzy rough sets and their topological properties [36]; interval-valued intuitionistic rough set [37]; generalized interval-valued fuzzy rough set and its decision-making approach [38], etc. Furthermore, with the combination of rough set and neutrosophic set, many theories and their practical implications are proposed in [39-46].

Pawlak's rough set model is based on partition or equivalence relation. There exist many applications in real life where the notion of an equivalence relation is restrictive. To overcome such difficulties, Yiyu et al.[47] introduced the covering-based rough set model as an extension of classical RS. In [48], Kong et al. proposed the covering-based fuzzy rough sets and their properties. By introducing the fuzzy β-covering and fuzzy β-neighborhood, Ma [49] presented two types of fuzzy covering RS models. Zhang et al.[50] introduced fuzzy β-covering (I,T) fuzzy rough set model and its application in MADM problem. In [51], Zhang et al. defined the TOPSIS-WAA method built upon a covering-based fuzzy rough set. Zhou et al.[52] defined three types of fuzzy covering-based RS models. Furthermore, Yang et al. introduced some types of covering based rough sets [53]; Deer et al. investigated the properties and interrelationships of fuzzy covering-based rough set models [54]; fuzzy information system based covering based rough sets are prosed in [55]; fuzzy covering-based rough set on two different universes and its application is successfully executed by Yang [56]; Zhan et al. [57] initiated the PROMETHEE EDAS method via covering-based variable precision fuzzy rough sets [58]; MADM method under the hesitant fuzzy β -covering rough sets setting is successfully applied in [59]; TOPSIS method for MADM via covering-based spherical fuzzy rough set model is given in [60]. For further extension of the hybrid covering-based rough set model, Zhan et al.[61] defined covering-based intuitionistic fuzzy rough sets and their application in the MADM problem; two types of intuitionistic fuzzy covering rough sets in the MCGDM problem are defined by Wang et al.[62]; two types of single-valued neutrosophic rough sets and their decision-making approach are proposed in [63]; in [64], Wang et al. introduced a new type of single-valued neutrosophic covering rough set model. Some more recent works are based on neutrosophic covering rough set model proposed in [65-68].

The objectives of this paper are furnished below:

- The purpose of this article is to propose a single-valued neutrosophic covering-based rough set model over two universes by using Wang's approach given in **[63].**
- Construction of SVN β -neighborhood operators on two universes and study their properties.
- Construction of SVN β -covering lower and upper approximation operators over two universes.
- Matrix representation of SVN β -neighborhood and SVN β -covering lower and upper approximation operators over two universes and studied some propositions on them.
- A new type of MCDM problem is solved under the proposed study with the help of an algorithm.

To visualize the effectiveness of the proposed study over the existing theories, see the following Fig 1.

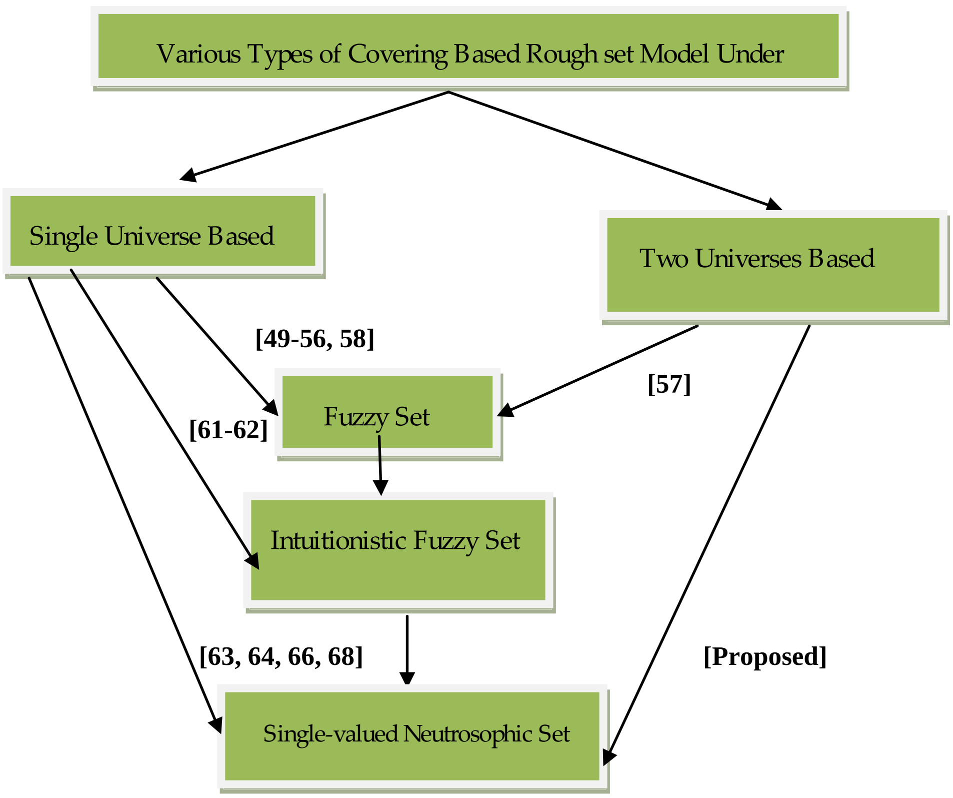

Fig 1. A brief diagrammatic presentation of the proposed study

2. Preliminaries

In this section, we give some basic concepts that are useful for the proposed study.

Definition 2.1 [29] Let ú be a universal set and R be an equivalence relation on ú. Then the pair (ú, R) is called a Pawlak approximation space. Let $\Upsilon_1, \Upsilon_2, ..., \Upsilon_n$ are the equivalence classes generated by R. Therefore, R generates a partition ú/R $= \{\Upsilon_1, \Upsilon_2, ..., \Upsilon_n\}$ on ú.

Definition 2.2 [69] Let ú be a universal set and C be a family of non-empty subsets of ú. If $\bigcup$ C = ú, then C is known as a covering of ú. Also, the pair (ú, C) is called a covering approximation space.

Definition 2.3 [16] A single-valued neutrosophic set (SVNS) A defined on X is an object of the form given below:

$\mathrm{A} = \left\{ \left\langle \varepsilon, T_{\mathrm{A}}(\varepsilon), I_{\mathrm{A}}(\varepsilon), F_{\mathrm{A}}(\varepsilon) \right\rangle : \varepsilon \in \mathrm{X} \right\}$, where $T_{\mathrm{A}}(\varepsilon)$ is the degree of truth-membership, $I_{\mathrm{A}}(\varepsilon)$ is the degree of indeterminacy-membership, and $F_{\mathrm{A}}(\varepsilon)$ is the degree of falsity-membership such that $T_{\mathrm{A}}(\varepsilon), I_{\mathrm{A}}(\varepsilon), F_{\mathrm{A}}(\varepsilon) \in [0,1]$ and $0 \le T_{\mathrm{A}}(\varepsilon) + I_{\mathrm{A}}(\varepsilon) + F_{\mathrm{A}}(\varepsilon) \le 3$ for all $\varepsilon \in \mathrm{X}$. The family of SVNS over X is denoted by $\mathrm{I}^{SVNS(\mathrm{X})}$.

Definition 2.4 [63] Let $\mathrm{I}^{SVNS(\mathrm{X})}$ denotes the family of SVNS in X and $\beta = \langle p, q, r \rangle$ be a SVN number. Then, for $\tilde{\mathrm{M}} = \{M_1, M_2, ..., M_k\}$ with $M_j \in \mathrm{I}^{SVNS(\mathrm{X})}$ $(j = 1, 2, .., k)$, a SVN β-covering of X, if for all $\varepsilon \in \mathrm{X}$, there exists $M_j \in \tilde{\mathrm{M}}$ such that $M_j(\varepsilon) \ge \beta$. The pair ($\mathrm{X}, \tilde{\mathrm{M}}$) is called a SVN β-covering approximation space.

Definition 2.5 [63] Let $\tilde{\mathrm{M}}$ be a SVN β-covering of X, where $\tilde{\mathrm{M}} = \{M_1, M_2, ..., M_k\}$. For any $\varepsilon \in \mathrm{X}$, the SVN β-neighborhood $\tilde{\tilde{\mathrm{N}}}_{\varepsilon}^{\beta}$ of ε induced by $\tilde{\mathrm{M}}$ can be defined as

$$\tilde{\tilde{\mathrm{N}}}_{\varepsilon}^{\beta} = \bigcap \left\{ M_j \in \tilde{\mathrm{M}} : M_j(\varepsilon) \ge \beta \right\}$$

It is to be noted that $M_j(\varepsilon) \ge \beta \Rightarrow T_{M_j}(\varepsilon) \ge p, I_{M_j}(\varepsilon) \le q$ and $F_{M_j}(\varepsilon) \le r$, where $\beta = \langle p, q, r \rangle$ is a SVN number.

3. Construction of SVN β-covering Approximation Space over Two Universes

In this section, we first introduce the notion of SVN β-neighborhood, and then we define a type of SVN covering-based rough set model over two universes. Here $\Gamma(X, Y)$ denotes the family of all mappings from X to Y.

Definition 3.1 Let $\overset{\prec}{G}$ be a SVN β-covering of the universe X where $\overset{\prec}{G}=\left\{G_1,G_2,G_3,......,G_p\right\}$. For any $x\in X$, the SVN β-neighborhood $\overset{\approx}{\mathrm{N}}{}_x^{\beta}$ of x induced by $\overset{\prec}{G}$ can be defined as:

$$\overset{\approx}{\mathrm{N}}{}_x^{\beta}=\bigcap\left\{G_i\in\overset{\prec}{G}:G_i\left(x\right)\geq\beta,i=1,2,..,p\right\}.$$

By introducing $\overset{\approx}{\mathrm{N}}{}_x^{\beta}:X\rightarrow SVN(Y)$, we define a new type of SVN covering-based rough set model over two universes.

By Wang's **[63]** approach, If $\overset{\prec}{G}$ be a SVN β-covering of the universe X for some $\beta=\left(\mu,\nu,\gamma\right)$ where $\mu,\nu,\gamma\in\left[0,1\right]$ such that $\mu+\nu+\gamma\leq 3$, we do not sure that $f\left(\overset{\prec}{G}\right)$ is a SVN β-covering over Y. To support this claim we give an example in the following:

Example 3.2 Let $X=\left\{x_1,x_2,x_3,x_4\right\}$ and $Y=\left\{y_1,y_2,y_3\right\}$ be two universal sets and $f\in\Gamma\left(X,Y\right)$ such that $f\left(x_1\right)=f\left(x_2\right)=y_1$, and $f\left(x_3\right)=f\left(x_4\right)=y_2$. Let $\overset{\prec}{G}=\left\{G_1,G_2\right\}$, where

$$G_1=\left(\frac{\langle 0.3,0.2,0.2\rangle}{x_1},\frac{\langle 0.4,0.1,0.3\rangle}{x_2},\frac{\langle 0.3,0.1,0.3\rangle}{x_3},\frac{\langle 0.3,0.3,0.4\rangle}{x_4}\right)$$

$$G_2=\left(\frac{\langle 0.4,0.3,0.4\rangle}{x_1},\frac{\langle 0.3,0.4,0.3\rangle}{x_2},\frac{\langle 0.4,0.2,0.4\rangle}{x_3},\frac{\langle 0.3,0.3,0.4\rangle}{x_4}\right)$$

It is to be noted that $\overset{\prec}{G}$ is a SVN β-covering of X, where $\beta=\left(0.3,0.3,0.4\right)$.

Now, $f\left(G_1\right)\left(y_1\right)=\bigcup\limits_{x\in f^{-1}(y_1)}G_1\left(x\right)=G_1\left(x_1\right)\vee G_1\left(x_2\right)=\langle 0.4,0.1,0.2\rangle$

$$f\left(G_1\right)\left(y_2\right)=\bigcup_{x\in f^{-1}(y_2)}G_1\left(x\right)=G_1\left(x_3\right)\vee G_1\left(x_4\right)=\langle 0.3,0.1,0.3\rangle$$

$$f\left(G_1\right)\left(y_3\right)=\bigcup_{x\in f^{-1}(y_3)}G_1\left(x\right)=0$$

$$f\left(G_1\right)=\left(\frac{\langle 0.4,0.1,0.2\rangle}{y_1},\frac{\langle 0.3,0.1,0.3\rangle}{y_2}\right)$$

Similarly, $f\left(G_2\right)\left(y_1\right)=\bigcup\limits_{x\in f^{-1}(y_1)}G_2\left(x\right)=G_2\left(x_1\right)\vee G_2\left(x_2\right)=\langle 0.4,0.3,0.3\rangle$

$$f(G_2)(y_2)=\bigcup_{x\in f^{-1}(y_2)} G_2(x)=G_2(x_3)\vee G_2(x_4)=\langle 0.4,0.2,0.4\rangle$$

$$f(G_2)(y_3)=\bigcup_{x\in f^{-1}(y_3)} G_2(x)=0$$

$$f(G_2)=\left(\frac{\langle 0.4,0.3,0.3\rangle}{y_1},\frac{\langle 0.4,0.2,0.4\rangle}{y_2}\right)$$

Therefore, $\{f(G_1),f(G_2)\}$ is not a SVN β-covering of Y.

Based on the above example, a natural question arises that under what condition $f\left(\overset{\prec}{G}\right)$ is a SVN β-covering of Y for which $\overset{\prec}{G}$ is a SVN β-covering of X. For further investigation we discuss the following:

Proposition 3.3 Let X and Y be two universes and $\beta=(\mu,\nu,\gamma)$ where $\mu,\nu,\gamma\in[0,1]$ such that $\mu+\nu+\gamma\le 3$ and the family of all surjective mappings from X to Y be denoted by $Sur(X,Y)$, where $f\in Sur(X,Y)$. Then we consider the following:

(1) If $\overset{\prec}{G}$ is a SVN β-covering of X, then $f\left(\overset{\prec}{G}\right)$ is a SVN β-covering of Y.

(2) If $\overset{\prec}{H}$ is a SVN β-covering of Y if and only if $f^{-1}\left(\overset{\prec}{H}\right)$ is a SVN β-covering of X.

The converse of the Proposition (1) does not hold. To hold the converse of Proposition (1), we give the following necessary condition:

Theorem 3.4 Let $f:X\to Y$ be a bijection from X to Y, $\beta=(\mu,\nu,\gamma)$ where $\mu,\nu,\gamma\in[0,1]$ such that $\mu+\nu+\gamma\le 3$ and $\overset{\prec}{G}$ be a family of SVN sets on X. Then $\overset{\prec}{G}$ is a SVN β-covering of X iff $f\left(\overset{\prec}{G}\right)$ is also a SVN β-covering of Y.

Definition 3.5 Let X and Y be two non-empty finite universal sets and $f\in Sur(X,Y)$. Let $\overset{\prec}{G}=\{G_1,G_2,....,G_m\}$ be a family of SVN β-covering for some $\beta=(\mu,\nu,\gamma)$. For all $x\in X$, we define the SVN β-neighborhood $\overset{\approx}{\mathrm{N}}{}_x^{\beta}$ as:

$$\overset{\approx}{\mathrm{N}}{}_x^{\beta}=\bigcap\{f(G_i):G_i(x)\ge\beta\}$$

In [50], the SVN β-neighborhood of $x \in X$ was defined in $\left(X, \overset{\prec}{G}\right)$ but in the proposed study, we define this in $\left(X, Y, \overset{\prec}{G}\right)$. For better understanding, we consider the following example:

Example 3.6 Let $f: X \rightarrow Y$ be a surjection, where $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $Y = \{y_1, y_2, y_3, y_4\}$ and

$$f(x) = \begin{cases} y_1, x \in \{x_1, x_3\} \\ y_2, x \in \{x_2, x_5\} \\ y_3, x = x_4 \\ y_4, x = x_6 \end{cases}$$

Let, $\overset{\prec}{G} = \{G_1, G_2, G_3, G_4\}$ be a SVN set over X, where

$$G_1 = \left\{ \frac{\langle 0.3, 0.5, 0.4 \rangle}{x_1}, \frac{\langle 0.4, 0.2, 0.6 \rangle}{x_2}, \frac{\langle 0.2, 0.4, 0.5 \rangle}{x_3}, \frac{\langle 0.3, 0.5, 0.6 \rangle}{x_4}, \frac{\langle 0.5, 0.6, 0.3 \rangle}{x_5}, \frac{\langle 0.3, 0.4, 0.2 \rangle}{x_6} \right\}$$

$$G_2 = \left\{ \frac{\langle 0.4, 0.3, 0.2 \rangle}{x_1}, \frac{\langle 0.5, 0.3, 0.4 \rangle}{x_2}, \frac{\langle 0.3, 0.3, 0.5 \rangle}{x_3}, \frac{\langle 0.3, 0.4, 0.5 \rangle}{x_4}, \frac{\langle 0.6, 0.3, 0.2 \rangle}{x_5}, \frac{\langle 0.5, 0.3, 0.3 \rangle}{x_6} \right\}$$

$$G_3 = \left\{ \frac{\langle 0.5, 0.2, 0.3 \rangle}{x_1}, \frac{\langle 0.4, 0.3, 0.4 \rangle}{x_2}, \frac{\langle 0.6, 0.5, 0.3 \rangle}{x_3}, \frac{\langle 0.4, 0.2, 0.1 \rangle}{x_4}, \frac{\langle 0.4, 0.5, 0.6 \rangle}{x_5}, \frac{\langle 0.4, 0.5, 0.3 \rangle}{x_6} \right\}$$

$$G_4 = \left\{ \frac{\langle 0.4, 0.3, 0.5 \rangle}{x_1}, \frac{\langle 0.5, 0.3, 0.4 \rangle}{x_2}, \frac{\langle 0.6, 0.4, 0.4 \rangle}{x_3}, \frac{\langle 0.3, 0.4, 0.5 \rangle}{x_4}, \frac{\langle 0.5, 0.5, 0.4 \rangle}{x_5}, \frac{\langle 0.3, 0.4, 0.2 \rangle}{x_6} \right\}$$

$$f(G_1) = \left\{ \frac{\langle 0.3, 0.4, 0.4 \rangle}{y_1}, \frac{\langle 0.5, 0.2, 0.3 \rangle}{y_2}, \frac{\langle 0.3, 0.5, 0.6 \rangle}{y_3}, \frac{\langle 0.3, 0.4, 0.2 \rangle}{y_4} \right\}$$

$$f(G_2) = \left\{ \frac{\langle 0.4, 0.3, 0.2 \rangle}{y_1}, \frac{\langle 0.6, 0.3, 0.2 \rangle}{y_2}, \frac{\langle 0.3, 0.4, 0.5 \rangle}{y_3}, \frac{\langle 0.5, 0.3, 0.3 \rangle}{y_4} \right\}$$

$$f(G_3) = \left\{ \frac{\langle 0.6, 0.2, 0.3 \rangle}{y_1}, \frac{\langle 0.4, 0.3, 0.4 \rangle}{y_2}, \frac{\langle 0.4, 0.2, 0.1 \rangle}{y_3}, \frac{\langle 0.4, 0.5, 0.3 \rangle}{y_4} \right\}$$

$$f(G_4)=\left\{\frac{\langle 0.6,0.3,0.4\rangle}{y_1},\frac{\langle 0.5,0.3,0.4\rangle}{y_2},\frac{\langle 0.3,0.4,0.5\rangle}{y_3},\frac{\langle 0.3,0.4,0.2\rangle}{y_4}\right\}$$

For $\beta=(0.2,0.6,0.6)$, $\overset{\prec}{G}$ and $f\left(\overset{\prec}{G}\right)$ are SVN β-coverings over X and Y respectively.

Suppose $\beta=(0.3,0.4,0.3)$, then we calculate the SVN β-neighborhood $\overset{\approx}{\mathrm{N}}{}_x^{\beta}$ for each $x\in X$ as follows:

$$\overset{\approx}{\mathrm{N}}{}_{x_1}^{(0.3,0.4,0.3)}=\frac{\langle 0.4,0.3,0.2\rangle}{y_1}\cap\frac{\langle 0.6,0.2,0.3\rangle}{y_1}=\frac{\langle 0.4,0.3,0.3\rangle}{y_1}$$

$$\overset{\approx}{\mathrm{N}}{}_{x_2}^{(0.3,0.4,0.3)}=\varnothing,\ \overset{\approx}{\mathrm{N}}{}_{x_3}^{(0.3,0.4,0.3)}=\varnothing,\ \overset{\approx}{\mathrm{N}}{}_{x_4}^{(0.3,0.4,0.3)}=\frac{\langle 0.4,0.2,0.1\rangle}{y_3},\ \overset{\approx}{\mathrm{N}}{}_{x_5}^{(0.3,0.4,0.3)}=\varnothing$$

$$\overset{\approx}{\mathrm{N}}{}_{x_6}^{(0.3,0.4,0.3)}=\frac{\langle 0.3,0.4,0.2\rangle}{y_4}\cap\frac{\langle 0.5,0.3,0.3\rangle}{y_4}\cap\frac{\langle 0.3,0.4,0.2\rangle}{y_4}=\frac{\langle 0.3,0.4,0.3\rangle}{y_4}$$

Proposition 3.7 Let $f\in Sur(X,Y)$ and $\overset{\prec}{G}=\{G_1,G_2,G_3,\ldots\ldots,G_m\}$ be a SVN β-covering for some $\beta=(\mu,\nu,\gamma)$. Then we consider the following properties:

(1) $\overset{\approx}{\mathrm{N}}{}_x^{\beta}(f(x))\geq\beta$ for each $x\in X$.

(2) Let f be injective and for all $x,y,z\in X$, if $\overset{\approx}{\mathrm{N}}{}_x^{\beta}(f(y))\geq\beta$ and $\overset{\approx}{\mathrm{N}}{}_y^{\beta}(f(z))\geq\beta$, then $\overset{\approx}{\mathrm{N}}{}_x^{\beta}(f(z))\geq\beta$.

(3) For each $\beta=(\mu,\nu,\gamma)$, where $\mu,\nu,\gamma\in[0,1]$ and $\mu+\nu+\gamma\leq 3$, we can write the following:

$$f(G_i)\supseteq\bigcup\left\{\overset{\approx}{\mathrm{N}}{}_x^{\beta}:G_i(x)\geq\beta,\ x\in X\ \text{and}\ \mathrm{i}⊚1,2,..,\mathrm{m}\right\}$$

(4) If $0<\beta_1\leq\beta_2\leq\beta$, then $\overset{\approx}{\mathrm{N}}{}_x^{\beta_1}\subseteq\overset{\approx}{\mathrm{N}}{}_x^{\beta_2}$ for all $x\in X$.

Proof. (1) For each $x\in X$,

$$\overset{\approx}{\mathrm{N}}{}_x^{\beta}(f(x))⊚\left(\bigcap_{G_i(x)\geq\beta}f(G_i)\right)(f(x))⊚\bigwedge_{G_i(x)\geq\beta}f(G_i)(f(x))⊚\bigwedge_{G_i(x)\geq\beta}\left(\bigvee_{x^{⊚}\in f^{-1}(f(x))}G_i(x^{⊚})\right)$$

$$\geq\bigwedge_{G_i(x)\geq\beta}G_i(x)\geq\beta$$

(2) We have $\overset{\approx\ \beta}{\mathrm{N}}_x\left(f\left(y\right)\right) \geq \underset{G_i(x)\geq\beta}{\wedge} G_i\left(y\right)\geq\beta$ and $\overset{\approx\ \beta}{\mathrm{N}}_y\left(f\left(z\right)\right) \geq \underset{G_i(y)\geq\beta}{\wedge} G_i\left(z\right)\geq\beta$. Then for each $i=1,2,..,m$, $G_i(x)\geq\beta \Rightarrow G_i(y)\geq\beta$ and $G_i(y)\geq\beta \Rightarrow G_i(z)\geq\beta$. Thus, $\overset{\approx\ \beta}{\mathrm{N}}_x\left(f\left(z\right)\right) = \underset{G_i(x)\geq\beta}{\wedge} G_i\left(z\right)\geq\beta$.

(3) By definition **3.5,** $G_i(x)\geq\beta \Rightarrow \overset{\approx\ \beta}{\mathrm{N}}_x \subseteq f\left(G_i\right)$.

Then $f\left(G_i\right)\supseteq\bigcup\left\{\overset{\approx\ \beta}{\mathrm{N}}_x : G_i\left(x\right)\geq\beta, x\in X \text{ and i⊚1,2,..,m}\right\}$

(4) For each $x\in X$, $\beta_1\leq\beta_2\Rightarrow\left\{f\left(G_i\right):G_i\left(x\right)\geq\beta_1\right\}\supseteq\left\{f\left(G_i\right):G_i\left(x\right)\geq\beta_2\right\}$.

Then, $\overset{\approx\ \beta_1}{\mathrm{N}}_x = \bigcap\left\{f\left(G_i\right):G_i\left(x\right)\geq\beta_1\right\}\subseteq\bigcap\left\{f\left(G_i\right):G_i\left(x\right)\geq\beta_2\right\} = \overset{\approx\ \beta_2}{\mathrm{N}}_x$ for all $x\in X$.

Proposition 3.8 Let X and Y be two finite universes, $f\in Sur\left(X,Y\right)$, f be injective and $\overset{\prec}{G}=\left\{G_1,G_2,G_3,.....,G_m\right\}$ be a SVN β -covering for some $\beta=\left(\mu,\nu,\gamma\right)$. For all $x,y\in X$, $\overset{\approx\ \beta}{\mathrm{N}}_x\left(f\left(y\right)\right)\geq\beta$ iff $\overset{\approx\ \beta}{\mathrm{N}}_y \subseteq \overset{\approx\ \beta}{\mathrm{N}}_x$. So $\overset{\approx\ \beta}{\mathrm{N}}_x\left(f\left(y\right)\right)\geq\beta$ and $\overset{\approx\ \beta}{\mathrm{N}}_y\left(f\left(x\right)\right)\geq\beta$ if and only if $\overset{\approx\ \beta}{\mathrm{N}}_y = \overset{\approx\ \beta}{\mathrm{N}}_x$.

Proof. $(\Rightarrow)$ $\overset{\approx\ \beta}{\mathrm{N}}_x\left(f\left(y\right)\right) ⊚ \left(\bigcap_{G_i(x)\geq\beta} f\left(G_i\right)\right)\left(y\right) ⊚ \underset{G_i(x)\geq\beta}{\wedge} f\left(G_i\right)\left(f\left(y\right)\right)\geq\beta$.

We have, $\left\{f\left(G_i\right)\in f\left(\overset{\prec}{G}\right):G_i\left(x\right)\geq\beta\right\}\subseteq\left\{f\left(G_i\right)\in f\left(\overset{\prec}{G}\right):G_i\left(y\right)\geq\beta\right\}$.

Since, $\overset{\approx\ \beta}{\mathrm{N}}_x\left(f\left(y\right)\right)\geq\beta$,

$$T_{\overset{\approx\ \beta}{\mathrm{N}}_x\left(f(y)\right)} = T\bigcap_{\substack{T_{G_i(x)\geq\mu}\\ I_{G_i(x)\leq\nu}\\ F_{G_i(x)\leq\gamma}}} f\left(G_i\right)f\left(y\right) = \underset{\substack{T_{G_i(x)\geq\mu}\\ I_{G_i(x)\leq\nu}\\ F_{G_i(x)\leq\gamma}}}{\wedge} T_{f(G_i)f(y)} \geq\mu ,$$

$$I_{\overset{\approx\ \beta}{\mathrm{N}}_x\left(f(y)\right)} = I\bigcap_{\substack{T_{G_i(x)\geq\mu}\\ I_{G_i(x)\leq\nu}\\ F_{G_i(x)\leq\gamma}}} f\left(G_i\right)f\left(y\right) = \underset{\substack{T_{G_i(x)\geq\mu}\\ I_{G_i(x)\leq\nu}\\ F_{G_i(x)\leq\gamma}}}{\vee} I_{f(G_i)f(y)} \leq\nu ,$$

and

$$F_{\overset{\approx\ \beta}{\mathrm{N}}_x\left(f(y)\right)} = F\bigcap_{\substack{T_{G_i(x)\geq\mu}\\ I_{G_i(x)\leq\nu}\\ F_{G_i(x)\leq\gamma}}} f\left(G_i\right)f\left(y\right) = \underset{\substack{T_{G_i(x)\geq\mu}\\ I_{G_i(x)\leq\nu}\\ F_{G_i(x)\leq\gamma}}}{\vee} F_{f(G_i)f(y)} \leq\gamma$$

Again, for $z\in X$,

$$T_{\overset{\approx}{N}_x^{\beta}(f(z))} = \bigwedge_{\substack{T_{G_i(x)}\geq\mu \\ I_{G_i(x)}\leq\nu \\ F_{G_i(x)}\leq\gamma}} T_{f(G_i)f(z)} \geq \bigwedge_{\substack{T_{G_i(y)}\geq\mu \\ I_{G_i(y)}\leq\nu \\ F_{G_i(y)}\leq\gamma}} T_{f(G_i)f(z)} = T_{\overset{\approx}{N}_y^{\beta}(f(z))}$$

$$I_{\overset{\approx}{N}_x^{\beta}(f(z))} = \bigvee_{\substack{T_{G_i(x)}\geq\mu \\ I_{G_i(x)}\leq\nu \\ F_{G_i(x)}\leq\gamma}} I_{f(G_i)f(z)} \leq \bigvee_{\substack{T_{G_i(y)}\geq\mu \\ I_{G_i(y)}\leq\nu \\ F_{G_i(y)}\leq\gamma}} I_{f(G_i)f(z)} = I_{\overset{\approx}{N}_y^{\beta}(f(z))}$$

$$F_{\overset{\approx}{N}_x^{\beta}(f(z))} = \bigvee_{\substack{T_{G_i(x)}\geq\mu \\ I_{G_i(x)}\leq\nu \\ F_{G_i(x)}\leq\gamma}} F_{f(G_i)f(z)} \leq \bigvee_{\substack{T_{G_i(y)}\geq\mu \\ I_{G_i(y)}\leq\nu \\ F_{G_i(y)}\leq\gamma}} F_{f(G_i)f(z)} = F_{\overset{\approx}{N}_y^{\beta}(f(z))}$$

Therefore, for $z \in X$, $\overset{\approx}{N}_y^{\beta} \subseteq \overset{\approx}{N}_x^{\beta}$.

$(\Leftarrow)$: For any $x, y \in X$, we have $\overset{\approx}{N}_y^{\beta} \subseteq \overset{\approx}{N}_x^{\beta}$,

$T_{\overset{\approx}{N}_x^{\beta}(f(y))} \geq T_{\overset{\approx}{N}_y^{\beta}(f(y))} \geq \mu$, $I_{\overset{\approx}{N}_x^{\beta}(f(z))} \leq I_{\overset{\approx}{N}_y^{\beta}(f(y))} \leq \nu$, $F_{\overset{\approx}{N}_x^{\beta}(f(y))} \leq F_{\overset{\approx}{N}_y^{\beta}(f(y))} \leq \gamma$

Therefore, $\overset{\approx}{N}_x^{\beta}(f(y)) \geq \beta$.

Proposition 3.9 Let $f \in Sur(X,Y)$, f be injective and $\overset{\prec}{G} = \{G_1, G_2, G_3, \ldots.., G_m\}$ be a SVN β-covering for some $\beta = (\mu, \nu, \gamma)$. For all $x, y, z \in X$, if $f(x) \in \overset{\approx}{N}_y^{\beta}$ and $f(y) \in \overset{\approx}{N}_z^{\beta}$, then $f(x) \in \overset{\approx}{N}_z^{\beta}$.

Proof. For all $x, y, z \in X$,

$f(x) \in \overset{\approx}{N}_y^{\beta} \Leftrightarrow \overset{\approx}{N}_y^{\beta}(f(x)) \geq \beta \Leftrightarrow \overset{\approx}{N}_x^{\beta} \subseteq \overset{\approx}{N}_y^{\beta}$ and $f(y) \in \overset{\approx}{N}_z^{\beta} \Leftrightarrow \overset{\approx}{N}_z^{\beta}(f(y)) \geq \beta \Leftrightarrow \overset{\approx}{N}_y^{\beta} \subseteq \overset{\approx}{N}_z^{\beta}$.

Then $\overset{\approx}{N}_x^{\beta} \subseteq \overset{\approx}{N}_y^{\beta} \subseteq \overset{\approx}{N}_z^{\beta}$ and $\beta \leq \overset{\approx}{N}_x^{\beta}(f(x)) \leq \overset{\approx}{N}_z^{\beta}(f(x))$. Therefore, $f(x) \in \overset{\approx}{N}_z^{\beta}$. This completes the proof.

4. Construction of Single-valued neutrosophic covering based approximation operators over Two Universes

In this section, a new type of SVN covering based rough set model over two universes for neutrosophic subsets is defined and its properties are explored:

Definition 4.1 Let X and Y be two non-empty finite universes, $f \in sur(X,Y)$ and $\overset{\prec}{G}$ be a SVN β-covering on X for some $\beta = (\mu, \nu, \gamma)$. For each $A \in f(Y)$, we define the SVN covering lower approximation $\overset{\prec}{G}_{\circledcirc}(A)$ and SVN covering upper approximation $\overset{\prec}{G}^{\circledcirc}(A)$ as follows:

$$\overset{\prec}{G}_{\circledcirc}(A)(x)=\left\{\left\langle x, \underset{y\in Y}{\wedge}\left[\left(1-T_{\tilde{\tilde{N}}_x^{\beta}(f(y))}\right)\vee T_{A(y)}\right], \underset{y\in Y}{\vee}\left[I_{\tilde{\tilde{N}}_x^{\beta}(f(y))}\wedge I_{A(y)}\right], \underset{y\in Y}{\vee}\left[F_{\tilde{\tilde{N}}_x^{\beta}(f(y))}\wedge F_{A(y)}\right]\right\rangle : x\in X\right\}$$

$$\overset{\prec}{G}^{\circledcirc}(A)(x)=\left\{\left\langle x, \underset{y\in Y}{\vee}\left[T_{\tilde{\tilde{N}}_x^{\beta}(f(y))}\wedge T_{A(y)}\right], \underset{y\in Y}{\wedge}\left[\left(1-I_{\tilde{\tilde{N}}_x^{\beta}(f(y))}\right)\vee I_{A(y)}\right], \underset{y\in Y}{\wedge}\left[\left(1-F_{\tilde{\tilde{N}}_x^{\beta}(f(y))}\right)\vee F_{A(y)}\right]\right\rangle : x\in X\right\}$$

If $\overset{\prec}{G}_{\circledcirc}(A)\neq\overset{\prec}{G}^{\circledcirc}(A)$, then A is called the SVN covering based rough set.

Example 4.1.1 Let us consider the two finite universes as $X=\{x_1,x_2,x_3,x_4,x_5\}$,

$Y=\{y_1,y_2,y_3\}$ and $f: X\to Y,\ f(x)=\begin{cases} y_1, x\in\{x_1,x_2\} \\ y_2, x\in\{x_3,x_4\} \\ y_3, x=x_5 \end{cases}$.

Let $\overset{\prec}{G}=\{G_1,G_2,G_3,G_4\}$, where

$$G_1=\left\{\frac{\langle 0.3,0.4,0.5\rangle}{x_1},\frac{\langle 0.6,0.7,0.4\rangle}{x_2},\frac{\langle 0.4,0.5,0.6\rangle}{x_3},\frac{\langle 0.8,0.4,0.7\rangle}{x_4},\frac{\langle 0.5,0.5,0.6\rangle}{x_5}\right\}$$

$$G_2=\left\{\frac{\langle 0.2,0.6,0.4\rangle}{x_1},\frac{\langle 0.5,0.4,0.4\rangle}{x_2},\frac{\langle 0.6,0.3,0.7\rangle}{x_3},\frac{\langle 0.5,0.6,0.4\rangle}{x_4},\frac{\langle 0.3,0.7,0.5\rangle}{x_5}\right\}$$

$$G_3=\left\{\frac{\langle 0.7,0.6,0.4\rangle}{x_1},\frac{\langle 0.5,0.4,0.6\rangle}{x_2},\frac{\langle 0.6,0.4,0.5\rangle}{x_3},\frac{\langle 0.7,0.5,0.6\rangle}{x_4},\frac{\langle 0.3,0.6,0.5\rangle}{x_5}\right\}$$

$$G_4=\left\{\frac{\langle 0.6,0.5,0.6\rangle}{x_1},\frac{\langle 0.5,0.6,0.7\rangle}{x_2},\frac{\langle 0.6,0.4,0.5\rangle}{x_3},\frac{\langle 0.4,0.6,0.5\rangle}{x_4},\frac{\langle 0.5,0.6,0.4\rangle}{x_5}\right\}$$

$$f(G_1)=\left\{\frac{\langle 0.6,0.4,0.4\rangle}{y_1},\frac{\langle 0.8,0.4,0.6\rangle}{y_2},\frac{\langle 0.5,0.5,0.6\rangle}{y_3}\right\}$$

$$f(G_2)=\left\{\frac{\langle 0.5,0.4,0.4\rangle}{y_1},\frac{\langle 0.6,0.3,0.4\rangle}{y_2},\frac{\langle 0.3,0.7,0.5\rangle}{y_3}\right\}$$

$$f(G_3)=\left\{\frac{\langle 0.7,0.4,0.4\rangle}{y_1},\frac{\langle 0.7,0.4,0.5\rangle}{y_2},\frac{\langle 0.3,0.6,0.5\rangle}{y_3}\right\}$$

$$f(G_4)=\left\{\frac{\langle 0.6,0.5,0.6\rangle}{y_1},\frac{\langle 0.6,0.4,0.5\rangle}{y_2},\frac{\langle 0.5,0.6,0.4\rangle}{y_3}\right\}$$

Clearly, for $\beta=\langle 0.2,0.7,0.7\rangle$, $\overset{\prec}{G}$ and $f\left(\overset{\prec}{G}\right)$ are SVN β-coverings of X and Y respectively.

$$\overset{\approx}{\mathrm{N}}_{x_1}^{(0.2,0.7,0.7)}=\frac{\langle 0.6,0.4,0.4\rangle}{y_1}\cap\frac{\langle 0.5,0.4,0.4\rangle}{y_1}\cap\frac{\langle 0.7,0.4,0.4\rangle}{y_1}\cap\frac{\langle 0.6,0.5,0.6\rangle}{y_1}=\frac{\langle 0.5,0.5,0.6\rangle}{y_1}$$

$$\overset{\approx}{\mathrm{N}}_{x_2}^{(0.2,0.7,0.7)}=\frac{\langle 0.6,0.4,0.4\rangle}{y_1}\cap\frac{\langle 0.5,0.4,0.4\rangle}{y_1}\cap\frac{\langle 0.7,0.4,0.4\rangle}{y_1}\cap\frac{\langle 0.6,0.5,0.6\rangle}{y_1}=\frac{\langle 0.5,0.5,0.6\rangle}{y_1}$$

$$\overset{\approx}{\mathrm{N}}_{x_3}^{(0.2,0.7,0.7)}=\frac{\langle 0.8,0.4,0.6\rangle}{y_2}\cap\frac{\langle 0.6,0.3,0.4\rangle}{y_2}\cap\frac{\langle 0.7,0.4,0.5\rangle}{y_2}\cap\frac{\langle 0.6,0.4,0.5\rangle}{y_2}=\frac{\langle 0.6,0.4,0.6\rangle}{y_2}$$

$$\overset{\approx}{\mathrm{N}}_{x_4}^{(0.2,0.7,0.7)}=\frac{\langle 0.8,0.4,0.6\rangle}{y_2}\cap\frac{\langle 0.6,0.3,0.4\rangle}{y_2}\cap\frac{\langle 0.7,0.4,0.5\rangle}{y_2}\cap\frac{\langle 0.6,0.4,0.5\rangle}{y_2}=\frac{\langle 0.6,0.4,0.6\rangle}{y_2}$$

$$\overset{\approx}{\mathrm{N}}_{x_5}^{(0.2,0.7,0.7)}=\frac{\langle 0.5,0.5,0.6\rangle}{y_3}\cap\frac{\langle 0.3,0.7,0.5\rangle}{y_3}\cap\frac{\langle 0.3,0.6,0.5\rangle}{y_3}\cap\frac{\langle 0.5,0.6,0.4\rangle}{y_3}=\frac{\langle 0.3,0.7,0.6\rangle}{y_3}$$

For $A=\left\{\frac{\langle 0.3,0.5,0.6\rangle}{y_1},\frac{\langle 0.4,0.5,0.6\rangle}{y_2},\frac{\langle 0.6,0.4,0.3\rangle}{y_3}\right\}$

$$\overset{\prec}{G}_{\circledcirc}(A)(x)=\left\{\frac{\langle 0.5,0.5,0.6\rangle}{x_1},\frac{\langle 0.5,0.5,0.6\rangle}{x_2},\frac{\langle 0.4,0.4,0.6\rangle}{x_3},\frac{\langle 0.4,0.4,0.6\rangle}{x_4},\frac{\langle 0.7,0.5,0.6\rangle}{x_5}\right\}$$

$$\overset{\prec}{G}^{\circledcirc}(A)(x)=\left\{\frac{\langle 0.5,0.5,0.4\rangle}{x_1},\frac{\langle 0.5,0.5,0.4\rangle}{x_2},\frac{\langle 0.6,0.6,0.4\rangle}{x_3},\frac{\langle 0.6,0.6,0.4\rangle}{x_4},\frac{\langle 0.3,0.4,0.4\rangle}{x_5}\right\}$$

Thus, $\overset{\prec}{G}_{\circledcirc}(A)\neq\overset{\prec}{G}^{\circledcirc}(A)$

Some properties of SVN covering-based rough set model over two universes can be presented through the following proposition:

Proposition 4.2 Let X and Y be two non-empty finite universes and $f\in Sur(X,Y)$. Let $\overset{\prec}{G}=\{G_1,G_2,G_3,.....,G_m\}$ be a SVN β-covering on X for some $\beta=(\mu,\nu,\gamma)$, where $\mu,\nu,\gamma\in[0,1]$ and $\mu+\nu+\gamma\leq 3$. For each $A,B\in\Gamma(X,Y)$, we have the following statements:

(1) $\overset{\prec}{G}_{\circledcirc}(Y)=X$, $\overset{\prec}{G}^{\circledcirc}(\varnothing)=\varnothing$

$$(2)\overset{\prec}{G}_{\circledcirc}\left(A^{c}\right)=\left(\overset{\prec}{G}^{\circledcirc}(A)\right)^{c},\overset{\prec}{G}^{\circledcirc}\left(A^{c}\right)=\left(\overset{\prec}{G}_{\circledcirc}(A)\right)^{c}$$

$$(3)\overset{\prec}{G}_{\circledcirc}(A\cap B)=\overset{\prec}{G}_{\circledcirc}(A)\cap\overset{\prec}{G}_{\circledcirc}(B),\overset{\prec}{G}^{\circledcirc}(A\cup B)=\overset{\prec}{G}^{\circledcirc}(A)\cup\overset{\prec}{G}^{\circledcirc}(B)$$

$$(4) If\ A\subseteq B, then\ \overset{\prec}{G}_{\circledcirc}(A)\subseteq\overset{\prec}{G}_{\circledcirc}(B),\overset{\prec}{G}^{\circledcirc}(A)\subseteq\overset{\prec}{G}^{\circledcirc}(B)$$

$$(5)\overset{\prec}{G}_{\circledcirc}(A\cup B)\supseteq\overset{\prec}{G}_{\circledcirc}(A)\cup\overset{\prec}{G}_{\circledcirc}(B),\overset{\prec}{G}^{\circledcirc}(A\cap B)\subseteq\overset{\prec}{G}^{\circledcirc}(A)\cap\overset{\prec}{G}^{\circledcirc}(B)$$

$$(6) For\ each\ x\in X, if\ 1-T_{\overset{\approx}{N}^{\beta}_{x}(f(y))}\le T_{A(y)}\le T_{\overset{\approx}{N}^{\beta}_{x}(f(y))}, I_{\overset{\approx}{N}^{\beta}_{x}(f(y))}\le I_{A(y)}\le 1-I_{\overset{\approx}{N}^{\beta}_{x}(f(y))}\ and$$

$$F_{\overset{\approx}{N}^{\beta}_{x}(f(y))}\le F_{A(y)}\le 1-F_{\overset{\approx}{N}^{\beta}_{x}(f(y))}\ for\ all\ y\in Y, then\ \overset{\prec}{G}_{\circledcirc}(A)\subseteq\overset{\prec}{G}^{\circledcirc}(A)$$

Proof.

(1) For each $x\in X$,

$$\overset{\prec}{G}_{\circledcirc}(Y)(x)=$$

$$\left\{\left\langle x,\underset{y\in Y}{\wedge}\left[\left(1-T_{\overset{\approx}{N}^{\beta}_{x}(f(y))}\right)\vee T_{Y(y)}\right],\underset{y\in Y}{\vee}\left[I_{\overset{\approx}{N}^{\beta}_{x}(f(y))}\wedge I_{Y(y)}\right],\underset{y\in Y}{\vee}\left[F_{\overset{\approx}{N}^{\beta}_{x}(f(y))}\wedge F_{Y(y)}\right]\right\rangle: x\in X\right\}=X(x)$$

$$\overset{\prec}{G}^{\circledcirc}(\varnothing)(x)=\left\{\left\langle x,\underset{y\in Y}{\vee}\left[T_{\overset{\approx}{N}^{\beta}_{x}(f(y))}\wedge T_{\varnothing(y)}\right],\underset{y\in Y}{\wedge}\left[\left(1-I_{\overset{\approx}{N}^{\beta}_{x}(f(y))}\right)\vee I_{\varnothing(y)}\right],\underset{y\in Y}{\wedge}\left[\left(1-F_{\overset{\approx}{N}^{\beta}_{x}(f(y))}\right)\vee F_{\varnothing(y)}\right]\right\rangle: x\in X\right\}=\varnothing(x)$$

Hence

$$\overset{\prec}{G}_{\circledcirc}(Y)=X,\overset{\prec}{G}^{\circledcirc}(\varnothing)=\varnothing$$

(2) For each $x\in X$,

$$\overset{\prec}{G}_{\circledcirc}\left(A^{c}\right)(x)=\left\{\left\langle x,\underset{y\in Y}{\wedge}\left[\left(1-T_{\overset{\approx}{N}^{\beta}_{x}(f(y))}\right)\vee T_{A^{c}(y)}\right],\underset{y\in Y}{\vee}\left[I_{\overset{\approx}{N}^{\beta}_{x}(f(y))}\wedge I_{A^{c}(y)}\right],\underset{y\in Y}{\vee}\left[F_{\overset{\approx}{N}^{\beta}_{x}(f(y))}\wedge F_{A^{c}(y)}\right]\right\rangle: x\in X\right\}$$

$$=1-\left\{\left\langle x,\underset{y\in Y}{\vee}\left[T_{\overset{\approx}{N}^{\beta}_{x}(f(y))}\wedge T_{A(y)}\right],\underset{y\in Y}{\wedge}\left[\left(1-I_{\overset{\approx}{N}^{\beta}_{x}(f(y))}\right)\vee I_{A(y)}\right],\underset{y\in Y}{\wedge}\left[\left(1-F_{\overset{\approx}{N}^{\beta}_{x}(f(y))}\right)\vee F_{A(y)}\right]\right\rangle: x\in X\right\}$$

$$=1-\overset{\prec}{G}^{\circledcirc}(A)(x)$$

$$=\left(\overset{\prec}{G}^{\circledcirc}(A)\right)^{c}(x)$$

Similarly,

$$\overset{\prec}{G}^{\circledcirc}\left(A^{c}\right)(x)=\left(\overset{\prec}{G}_{\circledcirc}(A)\right)^{c}(x)$$

Then, $\overset{\prec}{G}_{\circledcirc}\left(A^{c}\right)=\left(\overset{\prec}{G}^{\circledcirc}(A)\right)^{c}, \overset{\prec}{G}^{\circledcirc}\left(A^{c}\right)=\left(\overset{\prec}{G}_{\circledcirc}(A)\right)^{c}$

(3) For each $x \in X$,

$$\overset{\prec}{G}_{\circledcirc}(A \cap B)(x)=$$

$$\left\{\left\langle x, \bigwedge_{y \in Y}\left[\left(1-T_{\tilde{\tilde{N}}_x^{\beta}(f(y))}\right) \vee T_{A \cap B(y)}\right], \bigvee_{y \in Y}\left[I_{\tilde{\tilde{N}}_x^{\beta}(f(y))} \wedge I_{A \cap B(y)}\right], \bigvee_{y \in Y}\left[F_{\tilde{\tilde{N}}_x^{\beta}(f(y))} \wedge F_{A \cap B(y)}\right]\right\rangle: x \in X\right\}$$

$$=\left\{\left\langle\begin{array}{l} x, \bigwedge_{y \in Y}\left[\left(\left(1-T_{\tilde{\tilde{N}}_x^{\beta}(f(y))}\right) \vee T_{A(y)}\right) \wedge\left(\left(1-T_{\tilde{\tilde{N}}_x^{\beta}(f(y))}\right) \vee T_{B(y)}\right)\right], \\ \bigvee_{y \in Y}\left[\left(I_{\tilde{\tilde{N}}_x^{\beta}(f(y))} \wedge I_{A(y)}\right) \wedge\left(I_{\tilde{\tilde{N}}_x^{\beta}(f(y))} \wedge I_{B(y)}\right)\right], \bigvee_{y \in Y}\left[\left(F_{\tilde{\tilde{N}}_x^{\beta}(f(y))} \wedge F_{A(y)}\right) \wedge\left(F_{\tilde{\tilde{N}}_x^{\beta}(f(y))} \wedge F_{B(y)}\right)\right]\end{array}\right\rangle: x \in X\right\}$$

$$\circledcirc \overset{\prec}{G}_{\circledcirc}(A)(x) \cap \overset{\prec}{G}_{\circledcirc}(B)(x)$$

and

$$\overset{\prec}{G}^{\circledcirc}(A \cup B)(x)=\left\{\left\langle\begin{array}{l} x, \bigvee_{y \in Y}\left[T_{\tilde{\tilde{N}}_x^{\beta}(f(y))} \wedge T_{A \cup B(y)}\right], \bigwedge_{y \in Y}\left[\left(1-I_{\tilde{\tilde{N}}_x^{\beta}(f(y))}\right) \vee I_{A \cup B(y)}\right], \\ \bigwedge_{y \in Y}\left[\left(1-F_{\tilde{\tilde{N}}_x^{\beta}(f(y))}\right) \vee F_{A \cup B(y)}\right]\end{array}\right\rangle: x \in X\right\}$$

$$=\left\{\left\langle\begin{array}{l} x, \bigvee_{y \in Y}\left[\left(T_{\tilde{\tilde{N}}_x^{\beta}(f(y))} \wedge T_{A(y)}\right) \vee\left(T_{\tilde{\tilde{N}}_x^{\beta}(f(y))} \wedge T_{B(y)}\right)\right], \\ \bigwedge_{y \in Y}\left[\left(\left(1-I_{\tilde{\tilde{N}}_x^{\beta}(f(y))}\right) \vee I_{A(y)}\right) \vee\left(\left(1-I_{\tilde{\tilde{N}}_x^{\beta}(f(y))}\right) \vee I_{B(y)}\right)\right], \\ \bigwedge_{y \in Y}\left[\left(\left(1-F_{\tilde{\tilde{N}}_x^{\beta}(f(y))}\right) \vee F_{A(y)}\right) \vee\left(\left(1-F_{\tilde{\tilde{N}}_x^{\beta}(f(y))}\right) \vee F_{B(y)}\right)\right]\end{array}\right\rangle: x \in X\right\}$$

$$=\overset{\prec}{G}^{\circledcirc}(A)(x) \cup \overset{\prec}{G}^{\circledcirc}(B)(x)$$

Then, $\overset{\prec}{G}_{\circledcirc}(A \cap B)=\overset{\prec}{G}_{\circledcirc}(A) \cap \overset{\prec}{G}_{\circledcirc}(B), \overset{\prec}{G}^{\circledcirc}(A \cup B)=\overset{\prec}{G}^{\circledcirc}(A) \cup \overset{\prec}{G}^{\circledcirc}(B)$

(4) *If* $A \subseteq B$, *then* $A(Y) \leq B(Y)$ for each $y \in Y$. For every $x \in X$, we have

$$\overset{\prec}{G}_{\circledcirc}(A)(x)=\left\{\left\langle x, \underset{y\in Y}{\wedge}\left[\left(1-T_{\tilde{\tilde{N}}_x^\beta(f(y))}\right)\vee T_{A(y)}\right], \underset{y\in Y}{\vee}\left[I_{\tilde{\tilde{N}}_x^\beta(f(y))}\wedge I_{A(y)}\right], \underset{y\in Y}{\vee}\left[F_{\tilde{\tilde{N}}_x^\beta(f(y))}\wedge F_{A(y)}\right]\right\rangle : x\in X\right\}$$

$$\leq\left\{\left\langle x, \underset{y\in Y}{\wedge}\left[\left(1-T_{\tilde{\tilde{N}}_x^\beta(f(y))}\right)\vee T_{B(y)}\right], \underset{y\in Y}{\vee}\left[I_{\tilde{\tilde{N}}_x^\beta(f(y))}\wedge I_{B(y)}\right], \underset{y\in Y}{\vee}\left[F_{\tilde{\tilde{N}}_x^\beta(f(y))}\wedge F_{B(y)}\right]\right\rangle : x\in X\right\}$$

$$=\overset{\prec}{G}_{\circledcirc}(B)(x)$$

and

$$\overset{\prec}{G}^{\circledcirc}(A)(x)=\left\{\left\langle x, \underset{y\in Y}{\vee}\left[T_{\tilde{\tilde{N}}_x^\beta(f(y))}\wedge T_{A(y)}\right], \underset{y\in Y}{\wedge}\left[\left(1-I_{\tilde{\tilde{N}}_x^\beta(f(y))}\right)\vee I_{A(y)}\right], \underset{y\in Y}{\wedge}\left[\left(1-F_{\tilde{\tilde{N}}_x^\beta(f(y))}\right)\vee F_{A(y)}\right]\right\rangle : x\in X\right\}$$

$$\leq\left\{\left\langle x, \underset{y\in Y}{\vee}\left[T_{\tilde{\tilde{N}}_x^\beta(f(y))}\wedge T_{B(y)}\right], \underset{y\in Y}{\wedge}\left[\left(1-I_{\tilde{\tilde{N}}_x^\beta(f(y))}\right)\vee I_{B(y)}\right], \underset{y\in Y}{\wedge}\left[\left(1-F_{\tilde{\tilde{N}}_x^\beta(f(y))}\right)\vee F_{B(y)}\right]\right\rangle : x\in X\right\}$$

$$=\overset{\prec}{G}^{\circledcirc}(B)(x)$$

Then, $\overset{\prec}{G}_{\circledcirc}(A)\subseteq\overset{\prec}{G}_{\circledcirc}(B), \overset{\prec}{G}^{\circledcirc}(A)\subseteq\overset{\prec}{G}^{\circledcirc}(B)$.

(5) Since $A\subseteq A\bigcup B, B\subseteq A\bigcup B,\ A\bigcap B\subseteq A$ and $A\bigcap B\subseteq B$, then from (4) we can write

$\overset{\prec}{G}_{\circledcirc}(A)\subseteq\overset{\prec}{G}_{\circledcirc}(A\bigcup B)$, $\overset{\prec}{G}_{\circledcirc}(B)\subseteq\overset{\prec}{G}_{\circledcirc}(A\bigcup B)$, $\overset{\prec}{G}^{\circledcirc}(A\bigcap B)\subseteq\overset{\prec}{G}^{\circledcirc}(A)$ and $\overset{\prec}{G}^{\circledcirc}(A\bigcap B)\subseteq\overset{\prec}{G}^{\circledcirc}(B)$.

Therefore, $\overset{\prec}{G}_{\circledcirc}(A\bigcup B)\supseteq\overset{\prec}{G}_{\circledcirc}(A)\bigcup\overset{\prec}{G}_{\circledcirc}(B), \overset{\prec}{G}^{\circledcirc}(A\bigcap B)\subseteq\overset{\prec}{G}^{\circledcirc}(A)\bigcap\overset{\prec}{G}^{\circledcirc}(B)$.

(6) This proof is obvious.

Proposition 4.3 Let X and Y be two non-empty finite universes and $f\in Sur(X,Y)$. Again, let $\overset{\prec}{G}=\{G_1,G_2,...,G_p\}$ be a SVN β-covering on X for some $\beta=(\mu,\nu,\gamma)$, where $\mu,\nu,\gamma\in[0,1]$ and $\mu+\nu+\gamma\leq 3$. For $M\in Ƒ(Y)$, where $Ƒ(Y)$ denotes the family of all subsets of Y and $\lambda\in[0,1]$, we have the following results:

(1) $\overset{\prec}{G}^{\circledcirc}(M\cap\lambda_Y)=\overset{\prec}{G}^{\circledcirc}(M)\bigcap\lambda_X$

(2) $\overset{\prec}{G}_{\circledcirc}(M\cup\lambda_Y)=\overset{\prec}{G}_{\circledcirc}(M)\bigcup\lambda_X$

Proof. (1) For any $x\in X$,

$$\overset{\prec}{G}^{\circledcirc}\left(M\cap\lambda_Y\right)(x) \qquad \circledcirc$$

$$\left\langle \underset{y\in Y}{\vee}\left[T_{\overset{\approx}{N}^{\beta}_x(y)}\wedge T_{M\cap\lambda_Y(y)}\right], \underset{y\in Y}{\wedge}\left[\left(1-I_{\overset{\approx}{N}^{\beta}_x(y)}\right)\vee I_{M\cap\lambda_Y(y)}\right], \underset{y\in Y}{\wedge}\left[\left(1-F_{\overset{\approx}{N}^{\beta}_x(y)}\right)\vee F_{M\cap\lambda_Y(y)}\right]\right\rangle$$

$$=\left\langle \underset{y\in Y}{\vee}\left[T_{\overset{\approx}{N}^{\beta}_x(y)}\wedge T_{M(y)}\wedge\lambda\right], \underset{y\in Y}{\wedge}\left[\left(1-I_{\overset{\approx}{N}^{\beta}_x(y)}\right)\vee I_{M(y)}\wedge\lambda\right], \underset{y\in Y}{\wedge}\left[\left(1-F_{\overset{\approx}{N}^{\beta}_x(y)}\right)\vee F_{M(y)}\wedge\lambda\right]\right\rangle$$

$$=\left\langle \begin{array}{l} \left(\left[\underset{y\in Y}{\vee}\left(T_{\overset{\approx}{N}^{\beta}_x(y)}\wedge T_{M(y)}\right)\right]\wedge\left[\underset{y\in Y}{\vee}\left(T_{\overset{\approx}{N}^{\beta}_x(y)}\wedge\lambda\right)\right]\right), \left(\left[\underset{y\in Y}{\wedge}\left(\left(1-I_{\overset{\approx}{N}^{\beta}_x(y)}\right)\vee I_{M(y)}\right)\right]\wedge\left[\underset{y\in Y}{\wedge}\left(\left(1-I_{\overset{\approx}{N}^{\beta}_x(y)}\right)\vee\lambda\right)\right]\right), \\ \left(\left[\underset{y\in Y}{\wedge}\left(\left(1-F_{\overset{\approx}{N}^{\beta}_x(y)}\right)\vee F_{M(y)}\right)\right]\wedge\left[\underset{y\in Y}{\wedge}\left(\left(1-F_{\overset{\approx}{N}^{\beta}_x(y)}\right)\vee\lambda\right)\right]\right) \end{array}\right\rangle$$

$$=\overset{\prec}{G}^{\circledcirc}\left(M\right)(x)\cap\lambda$$

Thus, $\overset{\prec}{G}^{\circledcirc}\left(M\cap\lambda_Y\right)=\overset{\prec}{G}^{\circledcirc}\left(M\right)\cap\lambda_X$

(2) For every $z\in X$,

$$\overset{\prec}{G}_{\circledcirc}\left(M\cup\lambda_Y\right)(z)=$$

$$\left\langle \underset{z\in Y}{\wedge}\left[\left(1-T_{\overset{\approx}{N}^{\beta}_x(z)}\right)\vee T_{M\cup\lambda_Y(z)}\right], \underset{z\in Y}{\vee}\left[I_{\overset{\approx}{N}^{\beta}_x(z)}\wedge I_{M\cup\lambda_Y(z)}\right], \underset{z\in Y}{\vee}\left[F_{\overset{\approx}{N}^{\beta}_x(z)}\wedge F_{M\cup\lambda_Y(z)}\right]\right\rangle$$

$$=\left\langle \underset{z\in Y}{\wedge}\left[\left(1-T_{\overset{\approx}{N}^{\beta}_x(z)}\right)\vee T_{M(z)}\vee\lambda\right], \underset{z\in Y}{\vee}\left[I_{\overset{\approx}{N}^{\beta}_x(z)}\wedge I_{M(z)}\vee\lambda\right], \underset{z\in Y}{\vee}\left[F_{\overset{\approx}{N}^{\beta}_x(z)}\wedge F_{M(z)}\vee\lambda\right]\right\rangle$$

$$=\left\langle \begin{array}{l} \left(\left[\underset{z\in Y}{\wedge}\left(\left(1-T_{\overset{\approx}{N}^{\beta}_x(z)}\right)\vee T_{M(z)}\right)\right]\vee\left[\underset{z\in Y}{\wedge}\left(\left(1-T_{\overset{\approx}{N}^{\beta}_x(z)}\right)\vee\lambda\right)\right]\right), \left(\left[\underset{z\in Y}{\vee}\left(I_{\overset{\approx}{N}^{\beta}_x(z)}\wedge I_{M(z)}\right)\right]\vee\left[\underset{z\in Y}{\vee}\left(I_{\overset{\approx}{N}^{\beta}_x(z)}\wedge\lambda\right)\right]\right), \\ \left(\left[\underset{z\in Y}{\vee}\left(F_{\overset{\approx}{N}^{\beta}_x(z)}\wedge F_{M(z)}\right)\right]\vee\left[\underset{z\in Y}{\vee}\left(F_{\overset{\approx}{N}^{\beta}_x(z)}\wedge\lambda\right)\right]\right) \end{array}\right\rangle$$

$$=\overset{\prec}{G}_{\circledcirc}\left(M\right)(z)\cup\lambda$$

Hence, $\overset{\prec}{G}_{\circledcirc}\left(M\cup\lambda_Y\right)=\overset{\prec}{G}_{\circledcirc}\left(M\right)\cup\lambda_X$

5. Matrix Representation of SVN Covering-Based Approximation Operators

In this section, we have investigated the matrix representations of SVN covering-based lower and upper approximation operators and performed some matrix operations on them. Also, the algorithmic representation helps to calculate the matrix operations through the computer.

Definition 5.1 Let $P=\left(p_{ij}\right)_{m\times n}=\left(T_{p_{ij}},I_{p_{ij}},F_{p_{ij}}\right)_{m\times n}$ and $Q=\left(q_{jk}\right)=\left(T_{q_{jk}},I_{q_{jk}},F_{q_{jk}}\right)_{n\times l}$ be two SVN matrices. Then, we perform the following two operations on $P=\left(p_{ij}\right)_{m\times n}$ and $Q=\left(q_{jk}\right)_{n\times l}$ as follows:

$$P\Delta Q=\left(r_{ik}\right)_{m\times l}=\left\langle \overset{n}{\underset{j=1}{\vee}}\left(T_{p_{ij}}\wedge T_{q_{jk}}\right), \overset{n}{\underset{j=1}{\wedge}}\left(\left(1-I_{p_{ij}}\right)\vee I_{q_{jk}}\right), \overset{n}{\underset{j=1}{\wedge}}\left(\left(1-F_{p_{ij}}\right)\vee F_{q_{jk}}\right)\right\rangle \quad , \quad i=1,2,..,m \quad ;$$

$$j=1,2,..,1$$

$$P\nabla Q=\left(s_{ik}\right)_{m\times l}=\left\langle \overset{n}{\underset{j=1}{\wedge}}\left(\left(1-T_{p_{ij}}\right)\vee T_{q_{jk}}\right), \overset{n}{\underset{j=1}{\vee}}\left(I_{p_{ij}}\wedge I_{q_{jk}}\right), \overset{n}{\underset{j=1}{\vee}}\left(F_{p_{ij}}\wedge F_{q_{jk}}\right)\right\rangle \quad , \quad i=1,2,..,m \quad ;$$

$$j=1,2,..,1$$

Definition 5.2 Let $X=\left\{x_1,x_2,...,x_m\right\}$ and $Y=\left\{y_1,\mathrm{y}_2,...,\mathrm{y}_n\right\}$ be two finite universal sets and $f\in Sur(X,Y)$. Then the Boolean matrix under SVN environment is denoted by $Z_f=\left(z_{ij}\right)_{m\times n}$, where

$$z_{ij}=\begin{cases}\langle 1,0,0\rangle, \; when \; f\left(x_i\right)=y_j\\ (0,1,1), \; when \; f\left(x_i\right)\neq y_j\end{cases}$$

Definition 5.3 Let $X=\left\{x_1,x_2,...,x_m\right\}$ be a non-empty finite universe and $\overset{\prec}{G}=\left\{G_1,G_2,...,G_n\right\}$ be a SVN β -covering on X for some $\beta=\left(\mu,\nu,\gamma\right)$, where $\mu,\nu,\gamma\in[0,1]$ and $\mu+\nu+\gamma\leq 3$. Then $Z_{\overset{\prec}{G}}=\left(G_j\left(x_i\right)\right)_{m\times n}$ is a matrix representation of $\overset{\prec}{G}$. Also, the Boolean matrix $Z_\beta=\left(t_{ij}\right)_{m\times n}$ is called a SVN covering based β -matrix representation of $\overset{\prec}{G}$, where

$$t_{ij}=\begin{cases}\langle 1,0,0\rangle, \; when \; G_j\left(x_i\right)\geq\beta\\ (0,1,1), \quad otherwise\end{cases}$$

Example 5.3.1 Let $X=\left\{x_1,x_2,x_3,x_4,x_5\right\}$ and $Y=\left\{y_1,\mathrm{y}_2,\mathrm{y}_3\right\}$ be two non-empty finite universes and $f:X\to Y$, where $f\left(x\right)=\begin{cases}y_1, \; x\in\left\{x_1,x_4\right\}\\ y_2, \; x\in\left\{x_2\right\}\\ y_3, \; x\in\left\{x_3,x_5\right\}\end{cases}$.

Let $\overset{\prec}{G}=\left\{G_1,G_2,G_3\right\}$, where

$$G_1=\left\{\frac{\langle 0.3,0.6,0.5\rangle}{x_1},\frac{\langle 0.4,0.6,0.4\rangle}{x_2},\frac{\langle 0.6,0.3,0.5\rangle}{x_3},\frac{\langle 0.7,0.5,0.6\rangle}{x_4},\frac{\langle 0.4,0.3,0.2\rangle}{x_5}\right\}$$

$$G_2=\left\{\frac{\langle 0.4,0.6,0.3\rangle}{x_1},\frac{\langle 0.5,0.8,0.5\rangle}{x_2},\frac{\langle 0.6,0.4,0.3\rangle}{x_3},\frac{\langle 0.7,0.5,0.6\rangle}{x_4},\frac{\langle 0.7,0.3,0.4\rangle}{x_5}\right\}$$

$$G_3=\left\{\frac{\langle 0.2,0.3,0.4\rangle}{x_1},\frac{\langle 0.5,0.3,0.6\rangle}{x_2},\frac{\langle 0.6,0.4,0.5\rangle}{x_3},\frac{\langle 0.4,0.6,0.5\rangle}{x_4},\frac{\langle 0.3,0.4,0.5\rangle}{x_5}\right\}$$

For $\beta=(0.2,0.9,0.7)$, $\overset{\prec}{G}$ is a SVN β-covering of X.

Now,

$$Z_f=\begin{matrix} \\ x_1\\ x_2\\ x_3\\ x_4\\ x_5\end{matrix}\begin{bmatrix} y_1 & y_2 & y_3\\ (1,0,0) & (0,1,1) & (0,1,1)\\ (0,1,1) & (1,0,0) & (0,1,1)\\ (0,1,1) & (0,1,1) & (1,0,0)\\ (1,0,0) & (0,1,1) & (0,1,1)\\ (0,1,1) & (0,1,1) & (1,0,0)\end{bmatrix},\quad Z_{\overset{\prec}{G}}=\begin{matrix} \\ x_1\\ x_2\\ x_3\\ x_4\\ x_5\end{matrix}\begin{bmatrix} G_1 & G_2 & G_3\\ (0.3,0.6,0.5) & (0.4,0.6,0.3) & (0.2,0.3,0.4)\\ (0.4,0.6,0.4) & (0.5,0.8,0.5) & (0.5,0.3,0.6)\\ (0.6,0.3,0.5) & (0.6,0.4,0.3) & (0.6,0.4,0.5)\\ (0.7,0.5,0.6) & (0.7,0.5,0.6) & (0.4,0.6,0.5)\\ (0.4,0.3,0.2) & (0.7,0.3,0.4) & (0.3,0.4,0.5)\end{bmatrix}$$

For $\beta=(0.4,0.5,0.6)$

$$Z_{(0.4,0.5,0.6)}=\begin{matrix} \\ x_1\\ x_2\\ x_3\\ x_4\\ x_5\end{matrix}\begin{bmatrix} G_1 & G_2 & G_3\\ (0,1,1) & (0,1,1) & (0,1,1)\\ (0,1,1) & (0,1,1) & (1,0,0)\\ (1,0,0) & (1,0,0) & (1,0,0)\\ (1,0,0) & (1,0,0) & (0,1,1)\\ (1,0,0) & (1,0,0) & (0,1,1)\end{bmatrix}$$

Proposition 5.4 Let X and Y be two non-empty finite universes, and $f\in Sur(X,Y)$. Let $\overset{\prec}{G}$ be a SVN β -covering on X for some $\beta=(\mu,\nu,\gamma)$, where $\mu,\nu,\gamma\in[0,1]$ and $\mu+\nu+\gamma\le 3$. Then $(Z_f)^T \,\Delta\, Z_{\overset{\prec}{G}}=Z_{f\left(\overset{\prec}{G}\right)}$.

Proof. The proof is simple and straight forward.

Example 5.4.1 Considering the example **5.3.1**, we have

$$Z_{f\left(\overset{\prec}{G}\right)} = \begin{bmatrix} (1,0,0) & (0,1,1) & (0,1,1) & (1,0,0) & (0,1,1) \\ (0,1,1) & (1,0,0) & (0,1,1) & (0,1,1) & (0,1,1) \\ (0,1,1) & (0,1,1) & (1,0,0) & (0,1,1) & (1,0,0) \end{bmatrix} \Delta \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} \begin{bmatrix} \overset{G_1}{(0.3,0.6,0.5)} & \overset{G_2}{(0.4,0.6,0.3)} & \overset{G_3}{(0.2,0.3,0.4)} \\ (0.4,0.6,0.4) & (0.5,0.8,0.5) & (0.5,0.3,0.6) \\ (0.6,0.3,0.5) & (0.6,0.4,0.3) & (0.6,0.4,0.5) \\ (0.7,0.5,0.6) & (0.7,0.5,0.6) & (0.4,0.6,0.5) \\ (0.4,0.3,0.2) & (0.7,0.3,0.4) & (0.3,0.4,0.5) \end{bmatrix}$$

$$\circledcirc$$

$$\begin{bmatrix} \langle \vee(0.3,0,0,0.7,0), \wedge(1,0.6,0.3,1,0.3), \wedge(1,0.4,0.5,1,0.2)\rangle & \langle \vee(0.4,0,0,0.7,0), \wedge(1,0.8,0.4,1,0.3), \wedge(1,0.5,0.3,1,0.4)\rangle & \langle \vee(0.2,0,0,0.4,0), \wedge(1,0.3,0.4,1,0.4), \wedge(1,0.4,0.5,1,0.5)\rangle \\ \langle \vee(0,0.4,0,0,0), \wedge(0.6,1,0.3,0.5,0.3), \wedge(0.5,1,0.5,0.6,0.2)\rangle & \langle \vee(0,0.5,0,0,0), \wedge(0.6,1,0.4,0.5,0.3), \wedge(0.3,1,0.3,0.6,0.4)\rangle & \langle \vee(0,0.5,0,0,0), \wedge(0.3,1,0.4,0.6,0.4), \wedge(0.4,1,0.5,0.5,0.5)\rangle \\ \langle \vee(0,0,0.6,0,0.4), \wedge(0.6,0.6,1,0.5,0.1), \wedge(0.5,0.4,1,0.6,1)\rangle & \langle \vee(0,0,0.6,0,0.7), \wedge(0.6,0.8,1,0.5,1), \wedge(0.3,0.5,1,0.6,1)\rangle & \langle \vee(0,0,0.6,0,0.3), \wedge(0.3,0.3,1,0.6,1), \wedge(0.4,0.6,1,0.5,1)\rangle \end{bmatrix}$$

$$= \begin{bmatrix} \langle 0.7,0.3,0.2\rangle & \langle 0.7,0.3,0.4\rangle & \langle 0.4,0.3,0.4\rangle \\ \langle 0.4,0.3,0.2\rangle & \langle 0.5,0.3,0.3\rangle & \langle 0.5,0.3,0.4\rangle \\ \langle 0.6,0.5,0.4\rangle & \langle 0.7,0.5,0.3\rangle & \langle 0.6,0.3,0.4\rangle \end{bmatrix}$$

Proposition 5.5 Let $X = \{x_1, x_2, ..., x_m\}$ and $Y = \{y_1, y_2, ..., y_n\}$ be two non-empty finite universal sets $f \in Sur(X,Y)$ and $\overset{\prec}{G} = \{G_1, G_2, ..., G_l\}$ be a SVN β-covering on X for some $\beta = (\mu, \nu, \gamma)$, where $\mu, \nu, \gamma \in [0,1]$ and $\mu + \nu + \gamma \leq 3$. If Z_β be a β-matrix representation of $\overset{\prec}{G}$, Z_f be a matrix representation of f, and $Z_{\overset{\prec}{G}}$ be a matrix representation of $\overset{\prec}{G}$, then

$$Z_\beta \nabla \left(\left(Z_f \right)^T \Delta\, Z_{\overset{\prec}{G}} \right)^T = \left(\overset{\approx\,\beta}{\mathrm{N}}_{x_i}\left(y_j\right) \right)_{m\times n}.$$

Proof. This proof is simple and obvious.

Example 5.5.1 With reference to example **5.3.1** and the continuation of example **5.4.1,** we have

$$\left(\overset{\approx\,\beta}{\mathrm{N}}_{x_i}\left(y_j\right) \right) \circledcirc \begin{bmatrix} (0,1,1) & (0,1,1) & (0,1,1) \\ (0,1,1) & (0,1,1) & (1,0,0) \\ (1,0,0) & (1,0,0) & (1,0,0) \\ (1,0,0) & (1,0,0) & (0,1,1) \\ (1,0,0) & (1,0,0) & (0,1,1) \end{bmatrix} \nabla \begin{bmatrix} \langle 0.7,0.3,0.2\rangle & \langle 0.4,0.3,0.2\rangle & \langle 0.6,0.5,0.4\rangle \\ \langle 0.7,0.3,0.4\rangle & \langle 0.5,0.3,0.3\rangle & \langle 0.7,0.5,0.3\rangle \\ \langle 0.4,0.3,0.4\rangle & \langle 0.5,0.3,0.4\rangle & \langle 0.6,0.3,0.4\rangle \end{bmatrix}$$

$$=\begin{bmatrix}\langle 1,0.3,0.4\rangle & \langle 1,0.5,0.4\rangle & \langle 1,0.5,0.4\rangle\\ \langle 0.4,0.3,0.4\rangle & \langle 0.5,0.3,0.4\rangle & \langle 0.6,0.5,0.4\rangle\\ \langle 0.4,0,0\rangle & \langle 0.4,0,0\rangle & \langle 0.6,0,0\rangle\\ \langle 0.7,0.3,0.4\rangle & \langle 0.4,0.3,0.4\rangle & \langle 0.6,0.3,0.4\rangle\\ \langle 0.7,0.3,0.4\rangle & \langle 0.4,0.3,0.4\rangle & \langle 0.6,0.3,0.4\rangle\end{bmatrix}$$

Proposition 5.6 Let $X=\{x_1,x_2,...,x_m\}$ and $Y=\{y_1,y_2,...,y_n\}$ be two non-empty finite universal sets $f\in Sur(X,Y)$ and $\overset{\prec}{G}=\{G_1,G_2,...,G_l\}$ be a SVN β-covering on X for some $\beta=(\mu,\nu,\gamma)$, where $\mu,\nu,\gamma\in[0,1]$ and $\mu+\nu+\gamma\le 3$. If Z_β be a β-matrix representation of $\overset{\prec}{G}$, Z_f be a matrix representation of f, and $Z_{\overset{\prec}{G}}$ be a matrix representation of $\overset{\prec}{G}$, then for each $X\in \acute{F}(Y)$, we have

$$\overset{\prec}{G}_{\circledcirc}(X)=\left(Z_\beta\nabla\left(\left(Z_f\right)^T\Delta Z_{\overset{\prec}{G}}\right)^T\right)\nabla Z_X \quad , \quad \overset{\prec}{G}^{\circledcirc}(X)=\left(Z_\beta\nabla\left(\left(Z_f\right)^T\Delta Z_{\overset{\prec}{G}}\right)^T\right)\Delta Z_X,$$

where $Z_X=\left(X\left(y_i\right)\right)_{1\times n}$.

Proof. It is obvious.

Example 5.6.1 In a continuation of example **5.5.1**, we can obtain $\overset{\prec}{G}_{\circledcirc}(X)$ and $\overset{\prec}{G}^{\circledcirc}(X)$ as follows:

$$\overset{\prec}{G}_{\circledcirc}(X)=\begin{bmatrix}\langle 1,0.3,0.4\rangle & \langle 1,0.5,0.4\rangle & \langle 1,0.5,0.4\rangle\\ \langle 0.4,0.3,0.4\rangle & \langle 0.5,0.3,0.4\rangle & \langle 0.6,0.5,0.4\rangle\\ \langle 0.4,0,0\rangle & \langle 0.4,0,0\rangle & \langle 0.6,0,0\rangle\\ \langle 0.7,0.3,0.4\rangle & \langle 0.4,0.3,0.4\rangle & \langle 0.6,0.3,0.4\rangle\\ \langle 0.7,0.3,0.4\rangle & \langle 0.4,0.3,0.4\rangle & \langle 0.6,0.3,0.4\rangle\end{bmatrix}\nabla\begin{bmatrix}\langle 0.3,0.4,0.5\rangle\\ \langle 0.2,0.5,0.6\rangle\\ \langle 0.5,0.7,0.4\rangle\end{bmatrix}$$

$$=\begin{bmatrix}\langle 0.2,0.5,0.4\rangle\\ \langle 0.5,0.5,0.4\rangle\\ \langle 0.5,0,0\rangle\\ \langle 0.3,0.3,0.4\rangle\\ \langle 0.3,0.3,0.4\rangle\end{bmatrix}$$

$$\overset{\prec}{G}^{\circledcirc}(X)=\begin{bmatrix}\langle 1,0.3,0.4\rangle & \langle 1,0.5,0.4\rangle & \langle 1,0.5,0.4\rangle \\ \langle 0.4,0.3,0.4\rangle & \langle 0.5,0.3,0.4\rangle & \langle 0.6,0.5,0.4\rangle \\ \langle 0.4,0,0\rangle & \langle 0.4,0,0\rangle & \langle 0.6,0,0\rangle \\ \langle 0.7,0.3,0.4\rangle & \langle 0.4,0.3,0.4\rangle & \langle 0.6,0.3,0.4\rangle \\ \langle 0.7,0.3,0.4\rangle & \langle 0.4,0.3,0.4\rangle & \langle 0.6,0.3,0.4\rangle \end{bmatrix}\Delta\begin{bmatrix}\langle 0.3,0.4,0.5\rangle \\ \langle 0.2,0.5,0.6\rangle \\ \langle 0.5,0.7,0.4\rangle\end{bmatrix}$$

$$=\begin{bmatrix}\langle 0.5,0.5,0.6\rangle \\ \langle 0.5,0.7,0.6\rangle \\ \langle 0.5,1,1\rangle \\ \langle 0.5,0.7,0.6\rangle \\ \langle 0.5,0.7,0.6\rangle\end{bmatrix}$$

Clearly, $\overset{\prec}{G}_{\circledcirc}(X)\neq\overset{\prec}{G}^{\circledcirc}(X)$

6. Application of SVN Covering Based Rough Set Model over Two Universes in MCDM Problem

Multiple-criteria decision-making (MCDM) is a scientific approach that is useful to evaluate an optimal alternative under certain criteria or attributes. It is taken care of while evaluating the multiple conflicting criteria. Due to the uncertainty involved in many decision-making problems, makes the decision model more complex, and to overcome such type and reach a better decision, we need to consider a multiple-criteria model that provides a better option for the decision-makers to select the best option. Over the years, a variety of methods and approaches are developed to implement MCDM in many fields to enhance the decision-making approach. According to the traditional approach to MCDM, we select the best alternative according to the attribute values. But in modern MCDM methods, the selection of the best alternative is done according to the profit/loss type attribute values. So, the modern MCDM approaches are more flexible and powerful than the traditional approaches. The MCDM methods include TOPSIS, DEA, AHP, ANP, MULTIMOORA, etc.

In this section, we put forward an attempt to initiate a new approach to MCDM problems based on SVN covering-based rough set over two universes. For this, we describe the following MCDM problem:

Let $X=\{x_1,x_2,...,x_m\}$ be the set of m patients and $Y=\{y_1,y_2,....,y_n\}$ be the set of n diseases. Again, let $\overset{\prec}{G}=\{G_1,G_2,...,G_l\}$ be the set of diagnosis set, it is also known as SVN β -covering on X for some $\beta=(\mu,\nu,\gamma)$, where $\mu,\nu,\gamma\in[0,1]$ and $\mu+\nu+\gamma\leq 3$. Let $f\in sur(X,Y)$ such that $f(x_i)=y_j, where\ i=1,2,...,m\ and\ j=1,2,...,n$. We claim that f partitions X into n classes. Therefore, to identify the disease of patients through diagnosis, the set of doctors (experts) specifies a suitable diagnosis scores line according to the symptoms of all the patients. For this, we set a suitable $\beta=\left(\vee_{i=1}^{m}\left(\wedge_{j=1}^{n}T_{\overset{\prec}{G}_j(x_i)}\right),\wedge_{i=1}^{m}\left(\vee_{j=1}^{n}I_{\overset{\prec}{G}_j(x_i)}\right),\wedge_{i=1}^{m}\left(\vee_{j=1}^{n}F_{\overset{\prec}{G}_j(x_i)}\right)\right)$. It can be easily verified that

$\overset{\prec}{G}$ is a SVN β -covering on X. Afterward, we obtain $f\left(\overset{\prec}{G}_k\right)\left(y_j\right)=\left(\bigvee_{(x_i\in X)\wedge\left(x_i\in f^{-1}\left(y_j\right)\right)} T_{\overset{\prec}{G}_k(x_i)}, \bigwedge_{(x_i\in X)\vee\left(x_i\in f^{-1}\left(y_j\right)\right)} I_{\overset{\prec}{G}_k(x_i)}, \bigwedge_{(x_i\in X)\vee\left(x_i\in f^{-1}\left(y_j\right)\right)} F_{\overset{\prec}{G}_k(x_i)}\right)$ which denotes the degree of criteria $\overset{\prec}{G}_k$ to the diseases y_j. Also, $\overset{\approx}{\mathrm{N}}{}^{\beta}_{x_i}\left(y_j\right)=\bigwedge_{\overset{\prec}{G}_k(x)\geq\beta} f\left(\overset{\prec}{G}_k\right)\left(y_j\right)$ denotes the possibility of the patient x_i having a disease y_j.

Moreover, for a given criterion M over a SVN set of the universe Y, the SVN covering-based lower approximation $\overset{\prec}{G}_{\circledcirc}(M)$ of M denotes the neighborhood degree of M and $\overset{\approx}{\mathrm{N}}{}^{\beta}_{x_i}$. And the SVN covering-based upper approximation $\overset{\prec}{G}{}^{\circledcirc}(M)$ of M denotes the degree of intersection of M and $\overset{\approx}{\mathrm{N}}{}^{\beta}_{x_i}$. If $\overset{\prec}{G}_{\circledcirc}(M)(x_i)<\beta$ and $\overset{\prec}{G}{}^{\circledcirc}(M)(x_i)<\beta$, then the patient x_i does not satisfied with the attribute M. Otherwise, if $\overset{\prec}{G}_{\circledcirc}(M)(x_i)\geq\beta$ and $\overset{\prec}{G}{}^{\circledcirc}(M)(x_i)\geq\beta$, then the patient x_i satisfies the criteria.

To implement the MCDM process, we consider the following steps:

<u>Input:</u> Assuming the SVN information system $\left(X, Y, \overset{\prec}{G}\right)$ over two universes for MCDM problem, $f\in Sur(X,Y)$ and a criteria value $\beta=(\mu,\nu,\gamma)$, where $\mu,\nu,\gamma\in[0,1]$ and $\mu+\nu+\gamma\leq 3$.

<u>Computations:</u>

Step 1: Construct a SVN covering-based rough set model over two universes.

Step2: Calculate the SVN covering-based lower approximation $\overset{\prec}{G}_{\circledcirc}(M)$ and the SVN covering-based upper approximation $\overset{\prec}{G}{}^{\circledcirc}(M)$ for the criterion M (defined by the SVN set of the universe Y) provided by the hospital.

Step 3: If $\overset{\prec}{G}_{\circledcirc}(M)(x_i)\vee\overset{\prec}{G}{}^{\circledcirc}(M)(x_i)<\beta$, then the patient x_i cannot be diagnosed to detect the disease y_j under the critical value β.

Step 4: If $\overset{\prec}{G}_{\circledcirc}(M)(x_i)\vee\overset{\prec}{G}{}^{\circledcirc}(M)(x_i)\geq\beta$, then the patient x_i be diagnosed to detect the disease y_j under the critical value β.

Step 5: Rank the alternatives to select the patient who needs a diagnosis to detect a certain disease.

Output: Ranking orders of all the alternatives.

7. Conclusions and Future Scope

The notion of a single-valued neutrosophic β-covering set is introduced by Wang et al.[63] which makes a connection between a single-valued neutrosophic set and a covering-based rough set. Using this concept, in this paper, a new type of SVN covering-based rough set model over two universes is developed. We also introduce SVN β-covering rough set model over two universes with an aid of SVN β-neighborhood and studied some of its properties. Furthermore, we have presented the matrix representations of the SVN covering-based lower and upper approximation operators. Finally, we give a method for MCDM under the SVN β-covering-based lower and upper approximation operators over two universes.

In the future, to handle more critical decision-making problems, we can extend the proposed model by replacing the SVN covering information with the refined single-valued neutrosophic(RSVN) and quadripartitioned single-valued neutrosophic(QSVN) covering information and use them to develop TOPSIS, AHP, MULTIMOORA method in the MADM, MCDM, MCGDM, MAGDM problems. Topology and Entropy-based study in the same setting can also be possible to develop soon. To handle the parametric information, we can add the flavor of the soft set and hypersoft set in the present study to make it more flexible to encounter the uncertain information in a sophisticated way.

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Number of Neutrosophic Topological Spaces on Finite Set with $\mathfrak{k} \leq 4$ Open Sets

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Abstract. In this paper, the number of neutrosophic topological spaces having two, three, and four open sets are computed for a finite set $\mathbb{X}^{NT}$ whose membership values lies in $\mathbb{M}^{NT}$. Further, the number of neutrosophic bitopological spaces and neutrosophic tritopological spaces having $\mathfrak{k}(\mathfrak{k} = 2, 3, 4)$ neutrosophic open sets on finite sets are computed.

Keywords: : Neutrosophic Set; Neutrosophic Topology; Two Open Set; Three Open Set; Four Open Set.

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1. **Introduction**

Finding the number of topologies in a set is an interesting task. Many authors have done their work in this field. Krishnamurty [1] obtained a sharper bound namely $2^{\mathfrak{n}(\mathfrak{n}-1)}$ for the number of distinct topologies. Sharp [2] shows that only discrete topology has cardinal greater than $\frac{3}{4}2^{\mathfrak{n}}$ and derived bounds for the cardinality of topologies which are connected, non-connected, non-$\mathscr{T}_0$, and some more. After obtaining all non-homeomorphic topologies with $\mathfrak{n}$ points and $> \frac{7}{16}2^{\mathfrak{n}}$ open sets, Stanley [3] also determined which of these are $\mathscr{T}_0$. The concept of partial chain topologies supported Kamel [4] to formulate a special case for computing the number of chain topologies and maximal elements with natural generalization. Ragnarsson *et al.* [5], have also studied obtainable sizes of topologies on a finite set. Benoumhani [6] computed the number of topologies having $2, 3, \ldots, 12$-open sets, and also $\mathscr{T}_0$ topologies having $\mathfrak{n}+4$, $\mathfrak{n}+5$, and $\mathfrak{n}+6$ open sets. These results are extended in [7].

Later on, Benoumhani *et al.* [8] extended their work to fuzzy topological spaces (FTS). They computed the number of FTS having $2, 3, 4$, and 5-open sets and certain cases, where the number of open sets is large. Basumatary *et al.* [9] discussed the number of fuzzy bitopological spaces and gave some formulae.

After the generalization of the fuzzy set [10] from crisp set and intuitionistic fuzzy set [11], Smarandache discovered the concept of the neutrosophic set by combining the fuzzy set and intuitionistic fuzzy set. Since the introduction of the NS (Neutrosophic set) by Smarandache [12], several authors have contributed their work in science and technology by taking NS as a tool. Wang [13] studied single-valued NSs in multiset and multistructure. Salama *et al.* [14] studied the neutrosophic topological spaces (NTS). Lupiáñez [15–18] investigated NTS. Mwchahary *et al.* [19] studied neutrosophic bitopological space (NBTS). Devi *et al.* [20] and Ozturk *et al.* [21] also discussed NBTS. Kelly [22] and Kovar [23] introduced the notion of bitopological space and tritopological space respectively. The neutrosophic crisp tri-topological spaces are studied by Al-Hamido *et al.* [24].

Ishtiaq *et al.* [25, 26] studied fixed-point results in orthogonal neutrosophic metric spaces and also certain new aspects in fuzzy fixed-point theory. Ali *et al.* [27] discussed solving nonlinear fractional differential equations for contractive and weakly compatible mappings in neutrosophic metric spaces. Hussain *et al.* [28] worked on some new aspects of the intuitionistic fuzzy and neutrosophic fixed point theory. Javed *et al.* [29] studied the fuzzy b-metric-like spaces. Hussain *et al.* [30] studied the pentagonal controlled fuzzy metric spaces with an application to dynamic market equilibrium.

From the literature survey, it is observed that generally finding the number of topologies (NoTs) for a set is not an easy task. Because of this current authors started research work in this area. This article discusses formulae for calculating the NNTSs (number of NTSs) with $2, 3$, or 4-open sets, as well as the NNBTSs (number of NBTSs) and NNTRSs (number of neutrosophic tritopological spaces) with the same number of open sets in topologies.

Let $\mathbb{X}^{NT}$ be a non-empty finite set, $\mathbb{M}^{NT}$ be the finite totally ordered set with $|\mathbb{M}^{NT}| = \mathfrak{m} \geq 2$ and $\mathscr{N}_{\mathbb{X}}^{\mathscr{T}}$ be a set that contains all the neutrosophic subsets (NSubs) of $\mathbb{X}^{NT}$ with membership values in $\mathbb{M}^{NT}$.

Note that in this paper $\mathscr{T}_{\mathbb{X}}^{NT}(\mathfrak{n}, \mathfrak{m}, \mathfrak{k})$ denotes NNTSs on $\mathbb{X}^{NT}$ with $|\mathbb{X}^{NT}| = \mathfrak{n}$ and $\mathfrak{k}$-open sets, $(\mathscr{T}_{i}^{NT}, \mathscr{T}_{j}^{NT})_{\mathbb{X}}^{NT}(\mathfrak{n}, \mathfrak{m}, \mathfrak{k})$ and $(\mathscr{T}_{i}^{NT}, \mathscr{T}_{j}^{NT}, \mathscr{T}_{k}^{NT})_{\mathbb{X}}^{NT}(\mathfrak{n}, \mathfrak{m}, \mathfrak{k})$ denotes NNBTSs and NNTRSs respectively on $\mathbb{X}^{NT}$ consisting $\mathfrak{k}$-open sets in topologies at a time where $\mathfrak{n}, \mathfrak{m}, \mathfrak{k} \in \mathbb{N}$, $\mathfrak{n} \geq 1, \mathfrak{m} \geq 2$ and $\mathfrak{k} \geq 2$.

2. Preliminaries

Definition 2.1. [14] On a universe of discourse $\mathbb{X}^{NT}$ a NS $\mathfrak{U}^{NT}$ is defined as $\mathfrak{U}^{NT} = \langle \frac{u}{(T_{\mathfrak{U}}^{NT}(u), I_{\mathfrak{U}}^{NT}(u), F_{\mathfrak{U}}^{NT}(u))} : u \in \mathbb{X}^{NT} \rangle$, where $T_{\mathfrak{U}}^{NT}, I_{\mathfrak{U}}^{NT}, F_{\mathfrak{U}}^{NT} : \mathbb{X}^{NT} \to]^{-}0, 1^{+}[$. Here $^{-}0 \leq T_{\mathfrak{U}}^{NT}(u) + I_{\mathfrak{U}}^{NT}(u) + F_{\mathfrak{U}}^{NT}(u) \leq 3^{+}$; $T_{\mathfrak{U}}^{NT}(u)$ represents degree of membership function, $I_{\mathfrak{U}}^{NT}(u)$ degree of indeterminacy and $F_{\mathfrak{U}}^{NT}(u)$ degree of non-membership function.

Definition 2.2. [14,15] Let $\mathscr{T}^{NT} \subseteq \mathscr{N}_{\mathbb{X}}^{NT}$ then $\mathscr{T}^{NT}$ is called a neutrosophic topology (NT) on $\mathbb{X}^{NT}$ if

- $0^{NT}, 1^{NT} \in \mathscr{T}^{NT}$
- $\mathfrak{U}_1^{NT} \cap \mathfrak{U}_2^{NT} \in \mathscr{T}^{NT}$ for any $\mathfrak{U}_1^{NT}, \mathfrak{U}_2^{NT} \in \mathscr{T}^{NT}$.
- $\cup \mathfrak{U}_i^{NT} \in \mathscr{T}^{NT}$, for arbitrary family $\{\mathfrak{U}_i^{NT} : i \in \mathbb{I}\} \in \mathscr{T}^{NT}$.

The pair $(\mathbb{X}^{NT}, \mathscr{T}^{NT})$ is called NTS and any NS in $\mathscr{T}^{NT}$ is called NOS (neutrosophic open set) in $\mathbb{X}^{NT}$.

Definition 2.3. [19] Let $\mathscr{T}_1^{NT}$ and $\mathscr{T}_2^{NT}$ be the two NTs on $\mathbb{X}^{NT}$. Then $(\mathbb{X}^{NT}, \mathscr{T}_1^{NT}, \mathscr{T}_2^{NT})$ is called a NBTS.

Example 2.4. If $\mathbb{X}^{NT} = \{u, v, w\}$ and if $\mathscr{T}_1^{NT} = \{0^{NT}, 1^{NT}, \mathfrak{U}_1^{NT}\}$ and $\mathscr{T}_2^{NT} = \{0^{NT}, 1^{NT}, \mathfrak{U}_2^{NT}\}$, where

$$\mathfrak{U}_1^{NT} = \langle \frac{u}{(0.7,0.1,0.5)}, \frac{v}{(0.5,0.2,0.3)}, \frac{w}{(0.3,0.4,0.4)} \rangle, \mathfrak{U}_2^{NT} = \langle \frac{u}{(0.2,0.5,0.1)}, \frac{v}{(0.1,0.2,0.3)}, \frac{w}{(0.6,0.3,0.5)} \rangle.$$

Then $(\mathbb{X}^{NT}, \mathscr{T}_1^{NT})$ and $(\mathbb{X}^{NT}, \mathscr{T}_2^{NT})$ form NTS. Therefore, $(\mathbb{X}^{NT}, \mathscr{T}_1^{NT}, \mathscr{T}_2^{NT})$ is a NBTS.

Definition 2.5. [31] Let $\mathscr{T}_1^{NT}, \mathscr{T}_2^{NT}$ and $\mathscr{T}_3^{NT}$ be the three NTs on $\mathbb{X}^{NT}$. Then $(\mathbb{X}^{NT}, \mathscr{T}_1^{NT}, \mathscr{T}_2^{NT}, \mathscr{T}_3^{NT})$ is called a neutrosophic tritopological space (NTRS).

Example 2.6. If $\mathbb{X}^{NT} = \{u, v, w\}$ and consider $\mathscr{T}_1^{NT} = \{0^{NT}, 1^{NT}, \mathfrak{U}_1^{NT}\}$, $\mathscr{T}_2^{NT} = \{0^{NT}, 1^{NT}, \mathfrak{U}_2^{NT}\}$ and $\mathscr{T}_3^{NT} = \{0^{NT}, 1^{NT}, \mathfrak{U}_3^{NT}\}$.

Here, $\mathfrak{U}_1^{NT} = \langle \frac{u}{(0.7,0.1,0.5)}, \frac{v}{(0.5,0.2,0.3)}, \frac{w}{(0.3,0.6,0.2)} \rangle$, $\mathfrak{U}_2^{NT} = \langle \frac{u}{(0.6,0.5,0.3)}, \frac{v}{(0.7,0,0.2)}, \frac{w}{(0.8,0.1,0.1)} \rangle$,

$$\mathfrak{U}_3^{NT} = \langle \frac{u}{(0.5,0.2,0.3)}, \frac{v}{(0.2,0.1,0.2)}, \frac{w}{(0.1,0,0.1)} \rangle.$$

Then $(\mathbb{X}^{NT}, \mathscr{T}_1^{NT})$, $(\mathbb{X}^{NT}, \mathscr{T}_2^{NT})$ and $(\mathbb{X}^{NT}, \mathscr{T}_3^{NT})$ form NTS.

Therefore $(\mathbb{X}^{NT}, \mathscr{T}_1^{NT}, \mathscr{T}_2^{NT}, \mathscr{T}_3^{NT})$ is a NTRS. In this case, $(\mathbb{X}^{NT}, \mathscr{T}_1^{NT}, \mathscr{T}_2^{NT}, \mathscr{T}_3^{NT})$ is a NTRS having 3-NOS in each of the topologies.

3. Results on NNTS

Proposition 3.1. *The NNTs (Number of Neutrosophic Topologies) on $\mathbb{X}^{NT}$, whose membership values lies in $\mathbb{M}^{NT}$, is finite if and only if both $\mathbb{X}^{NT}$ and $\mathbb{M}^{NT}$ are finite.*

Result 3.2. *The NNTSs having 2-NOS is one i.e., $\mathscr{T}_{\mathbb{X}}^{NT}(\mathfrak{n}, \mathfrak{m}, 2) = 1$.*

The NT having 2-open set is the indiscrete NT which is $\mathscr{T}_1^{NT} = \{0^{NT}, 1^{NT}\}$.

Result 3.3. *The NNTs having 3-NOS is $\mathfrak{m}^{\mathfrak{n}} - 2$ i.e., $\mathscr{T}_{\mathbb{X}}^{NT}(\mathfrak{n}, \mathfrak{m}, 3) = \mathfrak{m}^{\mathfrak{n}} - 2$.*

These NTs necessarily consists of a chain containing $0^{NT}, 1^{NT}$ and any one NSub of $\mathbb{X}^{NT}$. In this case NTs are in the chain, of the form $0^{NT} \subseteq \mathfrak{U}_1^{NT} \subseteq 1^{NT}, \mathfrak{U}_1^{NT}$ is any NSub of $\mathbb{X}^{NT}$.

Example 3.4. Let $\mathbb{X}^{NT} = \{u, v\}$ and $\mathbb{M}^{NT} = \{(0,1,1),(0.6,0.1,0.2),(1,0,0)\}$. It is seen that, $|\mathbb{X}^{NT}| = \mathfrak{n} = 2$, $|\mathbb{M}^{NT}| = \mathfrak{m} = 3$.
Then number of elements in $N^{\mathscr{T}}_{\mathbb{X}}$ i.e., $|N^{\mathscr{T}}_{\mathbb{X}}| = 3^2 = 9$. These are
0^{NT}, 1^{NT}, $\mathfrak{U}_1^{NT} = \langle \frac{u}{(0,1,1)}, \frac{v}{(0.6,0.1,0.2)} \rangle$, $\mathfrak{U}_2^{NT} = \langle \frac{u}{(0,1,1)}, \frac{v}{(1,0,0)} \rangle$, $\mathfrak{U}_3^{NT} = \langle \frac{u}{(0.6,0.1,0.2)}, \frac{v}{(0,1,1)} \rangle$,
$\mathfrak{U}_4^{NT} \langle \frac{u}{(0.6,0.1,0.2)}, \frac{v}{(0.6,0.1,0.2)} \rangle$, $\mathfrak{U}_5^{NT} = \langle \frac{u}{(0.6,0.1,0.2)}, \frac{v}{(1,0,0)} \rangle$, $\mathfrak{U}_6^{NT} = \langle \frac{u}{(1,0,0)}, \frac{v}{(0,1,1)} \rangle$,
$\mathfrak{U}_7^{NT} = \langle \frac{u}{((1,0,0)}, \frac{v}{(0.6,0.1,0.2)} \rangle$.
So, $\mathscr{T}^{NT}_{\mathbb{X}}(2,3,3) = 3^2 - 2 = 7$.
The NTs having 3-open sets are:

$$\mathscr{T}_1^{NT} = \{0^{NT}, 1^{NT}, \mathfrak{U}_1^{NT}\},\ \mathscr{T}_2^{NT} = \{0^{NT}, 1^{NT}, \mathfrak{U}_2^{NT}\},\ \mathscr{T}_3^{NT} = \{0^{NT}, 1^{NT}, \mathfrak{U}_3^{NT}\},$$
$$\mathscr{T}_4^{NT} = \{0^{NT}, 1^{NT}, \mathfrak{U}_4^{NT}\},\ \mathscr{T}_5^{NT} = \{0^{NT}, 1^{NT}, \mathfrak{U}_5^{NT}\},\ \mathscr{T}_6^{NT} = \{0^{NT}, 1^{NT}, \mathfrak{U}_6^{NT}\},$$
$$\mathscr{T}_7^{NT} = \{0^{NT}, 1^{NT}, \mathfrak{U}_7^{NT}\}.$$

Result 3.5. *An arbitrary NT with 4-NOSs is an NT consisting of $1^{NT}, 0^{NT}$ and other two NSubs. These NSubs are either chain of 2-elements or anti-chain of 2-elements having 1^{NT} and 0^{NT} as union and intersection respectively.*

Theorem 3.6. *In $\hat{\mathscr{N}}^{\mathscr{T}}_{\mathbb{X}} = \mathscr{N}^{\mathscr{T}}_{\mathbb{X}} - \{0^{NT}, 1^{NT}\}$, the number of chains (NCs) of length 2 is obtained by*

$$c_2(\mathscr{N}^{\mathscr{T}}_{\mathbb{X}}) = \binom{\mathfrak{m}+1}{2}^{\mathfrak{n}} - 3\mathfrak{m}^{\mathfrak{n}} + 3.$$

Corollary 3.7. *In $\mathscr{N}^{\mathscr{T}}_{\mathbb{X}}$, the NCs of length 4 having both 0^{NT} and 1^{NT} is same as $c_2(\mathscr{N}^{\mathscr{T}}_{\mathbb{X}})$.*

Lemma 3.8. *In $\mathscr{N}^{\mathscr{T}}_{\mathbb{X}}$, the number of anti-chains (NACs) of size 2 (having 2-elements) with 1^{NT} as union and 0^{NT} as intersection is $2^{\mathfrak{n}-1} - 1$.*

Corollary 3.9. *The NAC NTs of $\mathscr{N}^{\mathscr{T}}_{\mathbb{X}}$ consisting of 4-open set is $2^{\mathfrak{n}-1} - 1$.*

Theorem 3.10. *The NNTs in $\mathscr{N}^{\mathscr{T}}_{\mathbb{X}}$ with 4-NOSss is*

$$\mathscr{T}^{NT}_{\mathbb{X}}(\mathfrak{n}, \mathfrak{m}, 4) = \left(\frac{\mathfrak{m}(\mathfrak{m}+1)}{2}\right)^{\mathfrak{n}} - 3\mathfrak{m}^{\mathfrak{n}} + 2^{\mathfrak{n}-1} + 2.$$

Follow Cor. 3.7 and Cor. 3.9 for the prove of theorem.

Example 3.11. Let, $\mathbb{X}^{NT} = \{u, v\}$ and $\mathbb{M}^{NT} = \{(0,1,1),(0.1,0.3,0.8),(1,0,0)\}$. Therefore $|\mathscr{N}^{\mathscr{T}}_{\mathbb{X}}| = 3^2 = 9$. These NSubs are

$0^{NT} = \langle \frac{u}{(0,1,1)}, \frac{v}{(0,1,1)} \rangle$, $1^{NT} = \langle \frac{u}{(1,0,0)}, \frac{v}{(1,0,0)} \rangle$, $\mathfrak{U}_1^{NT} = \langle \frac{u}{(0,1,1)}, \frac{v}{(0.1,0.3,0.8)} \rangle$,
$\mathfrak{U}_2^{NT} = \langle \frac{u}{(0,1,1)}, \frac{v}{(1,0,0)} \rangle$, $\mathfrak{U}_3^{NT} = \langle \frac{u}{(0.1,0.3,0.8)}, \frac{v}{(0,1,1)} \rangle$, $\mathfrak{U}_4^{NT} = \langle \frac{u}{(0.1,0.3,0.8)}, \frac{v}{(0.1,0.3,0.8)} \rangle$,
$\mathfrak{U}_5^{NT} = \langle \frac{u}{(0.1,0.3,0.8)}, \frac{v}{(1,0,0)} \rangle$, $\mathfrak{U}_6^{NT} = \langle \frac{u}{(1,0,0)}, \frac{v}{(0,1,1)} \rangle$, $\mathfrak{U}_7^{NT} = \langle \frac{u}{(1,0,0)}, \frac{v}{(0.1,0.3,0.8)} \rangle$.

In this case, $\mathfrak{n} = 2$, $\mathfrak{m} = 3$,
Therefore, $\mathscr{T}^{NT}_{\mathbb{X}}(2,3,4) = \left(\frac{3(3+1)}{2}\right)^2 - 3.3^2 + 2^{2-1} + 2 = 6^2 - 23 = 13.$
These NTs with 4-NOSs are

$$\mathscr{T}_1^{NT} = \{0^{NT}, 1^{NT}, \mathfrak{U}_1^{NT}, \mathfrak{U}_2^{NT}\},\qquad \mathscr{T}_2^{NT} = \{0^{NT}, 1^{NT}, \mathfrak{U}_1^{NT}, \mathfrak{U}_4^{NT}\},$$

$\mathscr{T}_3^{NT} = \{0^{NT}, 1^{NT}, \mathfrak{U}_1^{NT}, \mathfrak{U}_5^{NT}\}$, $\mathscr{T}_4^{NT} = \{0^{NT}, 1^{NT}, \mathfrak{U}_1^{NT}, \mathfrak{U}_7^{NT}\}$,
$\mathscr{T}_5^{NT} = \{0^{NT}, 1^{NT}, \mathfrak{U}_2^{NT}, \mathfrak{U}_5^{NT}\}$, $\mathscr{T}_6^{NT} = \{0^{NT}, 1^{NT}, \mathfrak{U}_2^{NT}, \mathfrak{U}_6^{NT}\}$,
$\mathscr{T}_7^{NT} = \{0^{NT}, 1^{NT}, \mathfrak{U}_3^{NT}, \mathfrak{U}_4^{NT}\}$, $\mathscr{T}_8^{NT} = \{0^{NT}, 1^{NT}, \mathfrak{U}_3^{NT}, \mathfrak{U}_5^{NT}\}$,
$\mathscr{T}_9^{NT} = \{0^{NT}, 1^{NT}, \mathfrak{U}_3^{NT}, \mathfrak{U}_6^{NT}\}$, $\mathscr{T}_{10}^{NT} = \{0^{NT}, 1^{NT}, \mathfrak{U}_3^{NT}, \mathfrak{U}_7^{NT}\}$,
$\mathscr{T}_{11}^{NT} = \{0^{NT}, 1^{NT}, \mathfrak{U}_4^{NT}, \mathfrak{U}_5^{NT}\}$, $\mathscr{T}_{12}^{NT} = \{0^{NT}, 1^{NT}, \mathfrak{U}_4^{NT}, \mathfrak{U}_7^{NT}\}$,
$\mathscr{T}_{13}^{NT} = \{0^{NT}, 1^{NT}, \mathfrak{U}_6^{NT}, \mathfrak{U}_7^{NT}\}$.

Here, the only anti-chain NTs in $\mathscr{N}_{\mathbb{X}}^{\mathscr{T}}$ is $\mathscr{T}_6^{NT}$ with 0^{NT} and 1^{NT} as intersection and union respectively.

4. Results on NNBTS

In this section, the NBTS having 3-NOSs in both NTs and the NBTS having 3-NOSs in both NTs without repetition means NBTS of the form $(\mathbb{X}^{NT}, \mathscr{T}_i^{NT}, \mathscr{T}_j^{NT})$, where $\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT}$ are identical or non-identical topologies, and non-identical topologies having 3-NOSs respectively. A similar meaning is used for 4-NOSs.

Result 4.1. *In $\mathscr{N}_{\mathbb{X}}^{\mathscr{T}}$, the NNBTS with two NOSs in both the NTs is*

$$(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT})_{\mathbb{X}}^{NT}(\mathfrak{n}, \mathfrak{m}, 2) = 1.$$

From Result 3.2, $\mathscr{T}_{\mathbb{X}}^{\mathscr{T}}(\mathfrak{n}, \mathfrak{m}, 2) = 1$, which is the indiscrete topology $\mathscr{T}_1^{NT} = \{0^{NT}, 1^{NT}\}$. Hence, NBTS with 2-NOSs is only one i.e., $(\mathbb{X}^{NT}, \mathscr{T}_1^{NT}, \mathscr{T}_1^{NT})$.

Result 4.2. *In $\mathscr{N}_{\mathbb{X}}^{\mathscr{T}}$, the NNBTSs having 3-NOSs in both NTs is*

$$(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT})_{\mathbb{X}}^{NT}(\mathfrak{n}, \mathfrak{m}, 3) = \binom{\mathscr{T}_{\mathbb{X}}^{NT}(\mathfrak{n},\mathfrak{m},3)+1}{2} = \frac{\mathfrak{m}^{2\mathfrak{n}} - 3\mathfrak{m}^{\mathfrak{n}} + 2}{2}.$$

Example 4.3. Example 3.4 gives $\mathscr{T}_{\mathbb{X}}^{NT}(2,3,3) = 7$.
Therefore,$(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT})_{\mathbb{X}}^{NT}(2,3,3) = \binom{\mathscr{T}_{\mathbb{X}}^{NT}(2,3,3)+1}{2} = 28$.
Then, these NBTSs are
$(\mathbb{X}^{NT}, \mathscr{T}_1^{NT}, \mathscr{T}_1^{NT})$, $(\mathbb{X}^{NT}, \mathscr{T}_1^{NT}, \mathscr{T}_2^{NT})$, $(\mathbb{X}^{NT}, \mathscr{T}_1^{NT}, \mathscr{T}_3^{NT})$, $(\mathbb{X}^{NT}, \mathscr{T}_1^{NT}, \mathscr{T}_4^{NT})$,
$(\mathbb{X}^{NT}, \mathscr{T}_1^{NT}, \mathscr{T}_5^{NT})$, $(\mathbb{X}^{NT}, \mathscr{T}_1^{NT}, \mathscr{T}_6^{NT})$, $(\mathbb{X}^{NT}, \mathscr{T}_1^{NT}, \mathscr{T}_7^{NT})$,
$(\mathbb{X}^{NT}, \mathscr{T}_2^{NT}, \mathscr{T}_2^{NT})$, $(\mathbb{X}^{NT}, \mathscr{T}_2^{NT}, \mathscr{T}_3^{NT})$, $(\mathbb{X}^{NT}, \mathscr{T}_2^{NT}, \mathscr{T}_4^{NT})$, $(\mathbb{X}^{NT}, \mathscr{T}_2^{NT}, \mathscr{T}_5^{NT})$,
$(\mathbb{X}^{NT}, \mathscr{T}_2^{NT}, \mathscr{T}_6^{NT})$, $(\mathbb{X}^{NT}, \mathscr{T}_2^{NT}, \mathscr{T}_7^{NT})$,
$(\mathbb{X}^{NT}, \mathscr{T}_3^{NT}, \mathscr{T}_3^{NT})$, $(\mathbb{X}^{NT}, \mathscr{T}_3^{NT}, \mathscr{T}_4^{NT})$, $(\mathbb{X}^{NT}, \mathscr{T}_3^{NT}, \mathscr{T}_5^{NT})$, $(\mathbb{X}^{NT}, \mathscr{T}_3^{NT}, \mathscr{T}_6^{NT})$,
$(\mathbb{X}^{NT}, \mathscr{T}_3^{NT}, \mathscr{T}_7^{NT})$,
$(\mathbb{X}^{NT}, \mathscr{T}_4^{NT}, \mathscr{T}_4^{NT})$, $(\mathbb{X}^{NT}, \mathscr{T}_4^{NT}, \mathscr{T}_5^{NT})$, $(\mathbb{X}^{NT}, \mathscr{T}_4^{NT}, \mathscr{T}_6^{NT})$, $(\mathbb{X}^{NT}, \mathscr{T}_4^{NT}, \mathscr{T}_7^{NT})$,
$(\mathbb{X}^{NT}, \mathscr{T}_5^{NT}, \mathscr{T}_5^{NT})$, $(\mathbb{X}^{NT}, \mathscr{T}_5^{NT}, \mathscr{T}_6^{NT})$, $(\mathbb{X}^{NT}, \mathscr{T}_5^{NT}, \mathscr{T}_7^{NT})$,
$(\mathbb{X}^{NT}, \mathscr{T}_6^{NT}, \mathscr{T}_6^{NT})$, $(\mathbb{X}^{NT}, \mathscr{T}_6^{NT}, \mathscr{T}_7^{NT})$,
$(\mathbb{X}^{NT}, \mathscr{T}_7^{NT}, \mathscr{T}_7^{NT})$.

Result 4.4. *In $\mathscr{N}_{\mathbb{X}}^{\mathscr{T}}$, the NNBTSs having 3-NOSs in both NTs without repetition is*

$$(\mathcal{T}_{i}^{NT}, \mathcal{T}_{j}^{NT})_{\mathbb{X}}^{NT}(\mathfrak{n}, \mathfrak{m}, 3) = \binom{\mathcal{T}_{\mathbb{X}}^{NT}(\mathfrak{n},\mathfrak{m},3)}{2}.$$

Example 4.5. Following Example 3.4 and Result 4.4., the number of NBTSs without repetition is

$$21 = \binom{\mathcal{T}_{\mathbb{X}}^{NT}(2,3,3)}{2} = \binom{7}{2}.$$

Result 4.6. *The NNBTSs in* $\mathcal{N}_{\mathbb{X}}^{\mathcal{T}}$, *consisting* 4*-NOSs in both the NT is*

$$(\mathcal{T}_{i}^{NT}, \mathcal{T}_{j}^{NT})_{\mathbb{X}}^{NT}(\mathfrak{n}, \mathfrak{m}, 4) = \binom{\mathcal{T}_{\mathbb{X}}^{NT}(\mathfrak{n},\mathfrak{m},4)+1}{2}.$$

Example 4.7. Let $\mathbb{X}^{NT} = \{u, v\}$ and $\mathbb{M}^{NT} = \{(0,1,1), (0.1, 0.3, 0.8), (1,0,0)\}$.
Then, $\mathcal{T}_{\mathbb{X}}^{NT}(2,3,4) = 13$.
and the NNBTSs is

$$(\mathcal{T}_{i}^{NT}, \mathcal{T}_{j}^{NT})_{\mathbb{X}}^{NT}(2,3,4) = \binom{\mathcal{T}_{\mathbb{X}}^{NT}(2,3,4)+1}{2} = 91.$$

These NBTSs are
$(\mathbb{X}^{NT}, \mathcal{T}_1^{NT}, \mathcal{T}_1^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_1^{NT}, \mathcal{T}_2^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_1^{NT}, \mathcal{T}_3^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_1^{NT}, \mathcal{T}_4^{NT})$,
$(\mathbb{X}^{NT}, \mathcal{T}_1^{NT}, \mathcal{T}_5^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_1^{NT}, \mathcal{T}_6^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_1^{NT}, \mathcal{T}_7^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_1^{NT}, \mathcal{T}_8^{NT})$,
$(\mathbb{X}^{NT}, \mathcal{T}_1^{NT}, \mathcal{T}_9^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_1^{NT}, \mathcal{T}_{10}^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_1^{NT}, \mathcal{T}_{11}^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_1^{NT}, \mathcal{T}_{12}^{NT})$,
$(\mathbb{X}^{NT}, \mathcal{T}_1^{NT}, \mathcal{T}_{13}^{NT})$,
$(\mathbb{X}^{NT}, \mathcal{T}_2^{NT}, \mathcal{T}_2^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_2^{NT}, \mathcal{T}_3^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_2^{NT}, \mathcal{T}_4^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_2^{NT}, \mathcal{T}_5^{NT})$,
$(\mathbb{X}^{NT}, \mathcal{T}_2^{NT}, \mathcal{T}_6^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_2^{NT}, \mathcal{T}_7^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_2^{NT}, \mathcal{T}_8^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_2^{NT}, \mathcal{T}_9^{NT})$,
$(\mathbb{X}^{NT}, \mathcal{T}_2^{NT}, \mathcal{T}_{10}^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_2^{NT}, \mathcal{T}_{11}^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_2^{NT}, \mathcal{T}_{12}^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_2^{NT}, \mathcal{T}_{13}^{NT})$,
$(\mathbb{X}^{NT}, \mathcal{T}_3^{NT}, \mathcal{T}_3^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_3^{NT}, \mathcal{T}_4^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_3^{NT}, \mathcal{T}_5^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_3^{NT}, \mathcal{T}_6^{NT})$,
$(\mathbb{X}^{NT}, \mathcal{T}_3^{NT}, \mathcal{T}_7^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_3^{NT}, \mathcal{T}_8^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_3^{NT}, \mathcal{T}_9^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_3^{NT}, \mathcal{T}_{10}^{NT})$,
$(\mathbb{X}^{NT}, \mathcal{T}_3^{NT}, \mathcal{T}_{11}^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_3^{NT}, \mathcal{T}_{12}^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_3^{NT}, \mathcal{T}_{13}^{NT})$,
$(\mathbb{X}^{NT}, \mathcal{T}_4^{NT}, \mathcal{T}_4^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_4^{NT}, \mathcal{T}_5^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_4^{NT}, \mathcal{T}_6^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_4^{NT}, \mathcal{T}_7^{NT})$,
$(\mathbb{X}^{NT}, \mathcal{T}_4^{NT}, \mathcal{T}_8^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_4^{NT}, \mathcal{T}_9^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_4^{NT}, \mathcal{T}_{10}^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_4^{NT}, \mathcal{T}_{11}^{NT})$,
$(\mathbb{X}^{NT}, \mathcal{T}_4^{NT}, \mathcal{T}_{12}^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_4^{NT}, \mathcal{T}_{13}^{NT})$,
$(\mathbb{X}^{NT}, \mathcal{T}_5^{NT}, \mathcal{T}_5^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_5^{NT}, \mathcal{T}_6^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_5^{NT}, \mathcal{T}_7^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_5^{NT}, \mathcal{T}_8^{NT})$,
$(\mathbb{X}^{NT}, \mathcal{T}_5^{NT}, \mathcal{T}_9^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_5^{NT}, \mathcal{T}_{10}^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_5^{NT}, \mathcal{T}_{11}^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_5^{NT}, \mathcal{T}_{12}^{NT})$,
$(\mathbb{X}^{NT}, \mathcal{T}_5^{NT}, \mathcal{T}_{13}^{NT})$,
$(\mathbb{X}^{NT}, \mathcal{T}_6^{NT}, \mathcal{T}_6^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_6^{NT}, \mathcal{T}_7^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_6^{NT}, \mathcal{T}_8^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_6^{NT}, \mathcal{T}_9^{NT})$,
$(\mathbb{X}^{NT}, \mathcal{T}_6^{NT}, \mathcal{T}_{10}^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_6^{NT}, \mathcal{T}_{11}^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_6^{NT}, \mathcal{T}_{12}^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_6^{NT}, \mathcal{T}_{13}^{NT})$,
$(\mathbb{X}^{NT}, \mathcal{T}_7^{NT}, \mathcal{T}_7^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_7^{NT}, \mathcal{T}_8^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_7^{NT}, \mathcal{T}_9^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_7^{NT}, \mathcal{T}_{10}^{NT})$,
$(\mathbb{X}^{NT}, \mathcal{T}_7^{NT}, \mathcal{T}_{11}^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_7^{NT}, \mathcal{T}_{12}^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_7^{NT}, \mathcal{T}_{13}^{NT})$,
$(\mathbb{X}^{NT}, \mathcal{T}_8^{NT}, \mathcal{T}_8^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_8^{NT}, \mathcal{T}_9^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_8^{NT}, \mathcal{T}_{10}^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_8^{NT}, \mathcal{T}_{11}^{NT})$,
$(\mathbb{X}^{NT}, \mathcal{T}_8^{NT}, \mathcal{T}_{12}^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_8^{NT}, \mathcal{T}_{13}^{NT})$,
$(\mathbb{X}^{NT}, \mathcal{T}_9^{NT}, \mathcal{T}_9^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_9^{NT}, \mathcal{T}_{10}^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_9^{NT}, \mathcal{T}_{11}^{NT})$, $(\mathbb{X}^{NT}, \mathcal{T}_9^{NT}, \mathcal{T}_{12}^{NT})$,
$(\mathbb{X}^{NT}, \mathcal{T}_9^{NT}, \mathcal{T}_{13}^{NT})$,

$(\mathbb{X}^{NT}, \mathscr{T}_{10}^{NT}, \mathscr{T}_{10}^{NT})$, $(\mathbb{X}^{NT}, \mathscr{T}_{10}^{NT}, \mathscr{T}_{11}^{NT})$, $(\mathbb{X}^{NT}, \mathscr{T}_{10}^{NT}, \mathscr{T}_{12}^{NT})$, $(\mathbb{X}^{NT}, \mathscr{T}_{10}^{NT}, \mathscr{T}_{13}^{NT})$,
$(\mathbb{X}^{NT}, \mathscr{T}_{11}^{NT}, \mathscr{T}_{11}^{NT})$, $(\mathbb{X}^{NT}, \mathscr{T}_{11}^{NT}, \mathscr{T}_{12}^{NT})$, $(\mathbb{X}^{NT}, \mathscr{T}_{11}^{NT}, \mathscr{T}_{13}^{NT})$,
$(\mathbb{X}^{NT}, \mathscr{T}_{12}^{NT}, \mathscr{T}_{12}^{NT})$, $(\mathbb{X}^{NT}, \mathscr{T}_{12}^{NT}, \mathscr{T}_{13}^{NT})$,
$(\mathbb{X}^{NT}, \mathscr{T}_{13}^{NT}, \mathscr{T}_{13}^{NT})$.

Result 4.8. *In $\mathscr{N}_{\mathbb{X}}^{\mathscr{T}}$, the NNBTSs having 4-NOSs in both NTs without repetition is*

$$(\mathscr{T}_{i}^{NT}, \mathscr{T}_{j}^{NT})_{\mathbb{X}}^{NT}(\mathfrak{n}, \mathfrak{m}, 4) = \binom{\mathscr{T}_{\mathbb{X}}^{NT}(\mathfrak{n}, \mathfrak{m}, 4)}{2}.$$

Example 4.9. Following Example 3.11 and result 4.8, the number of NBTSs without repetition is $78 = \binom{\mathscr{T}_{\mathbb{X}}^{NT}(2,3,4)}{2} = \binom{13}{2}$.

5. Results on NNTRS

In this section, the NTRS having 3-NOS in three NTs and the NTRS having 3-NOS in three NTs without repetition means NTRS of the form $(\mathbb{X}^{NT}, \mathscr{T}_{i}^{NT}, \mathscr{T}_{j}^{NT}, \mathscr{T}_{k}^{NT})$ where $\mathscr{T}_{i}^{NT}, \mathscr{T}_{j}^{NT}, \mathscr{T}_{k}^{NT}$ are identical or non-identical topologies and non-identical topologies having 3-NOS respectively. A similar meaning is used for 4-NOS.

Result 5.1. *In $\mathscr{N}_{\mathbb{X}}^{\mathscr{T}}$ the NNTRS consisting 2-NOSs in three NT is*

$$(\mathscr{T}_{i}^{NT}, \mathscr{T}_{j}^{NT}, \mathscr{T}_{k}^{NT})_{\mathbb{X}}^{NT}(\mathfrak{n}, \mathfrak{m}, 2) = 1.$$

In this case NT with 2-NOSs is the indiscrete one i.e., $\mathscr{T}_{1}^{NT} = \{0^{NT}, 1^{NT}\}$. Therefore, NNTRS with 2-NOSs is exactly one, namely $(\mathbb{X}^{NT}, \mathscr{T}_{1}^{NT}, \mathscr{T}_{1}^{NT}, \mathscr{T}_{1}^{NT})$.

Result 5.2. *The NNTRSs consisting 3-NOSs in all three NT in $\mathscr{N}_{\mathbb{X}}^{\mathscr{T}}$ is*

$$(\mathscr{T}_{i}^{NT}, \mathscr{T}_{j}^{NT}, \mathscr{T}_{k}^{NT})_{\mathbb{X}}^{NT}(\mathfrak{n}, \mathfrak{m}, 3) = \binom{\mathscr{T}_{\mathbb{X}}^{NT}(\mathfrak{n}, \mathfrak{m}, 3)+2}{3}.$$

Example 5.3. Example 3.4 implies $(\mathscr{T}_{\mathbb{X}}^{NT}(2,3,3) = 7$.
Therefore, $(\mathscr{T}_{i}^{NT}, \mathscr{T}_{j}^{NT}, \mathscr{T}_{k}^{NT})_{\mathbb{X}}^{NT}(2,3,3) = \binom{\mathscr{T}_{\mathbb{X}}^{NT}(2,3,3)+2}{3} = \frac{9\times 8\times 7}{6} = 84$.

Result 5.4. *The NNTRSs consisting 3-NOSs in all three NT without repetition in $\mathscr{N}_{\mathbb{X}}^{\mathscr{T}}$ is*

$$(\mathscr{T}_{i}^{NT}, \mathscr{T}_{j}^{NT}, \mathscr{T}_{k}^{NT})_{\mathbb{X}}^{NT}(\mathfrak{n}, \mathfrak{m}, 3) = \binom{\mathscr{T}_{\mathbb{X}}^{NT}(\mathfrak{n}, \mathfrak{m}, 3)}{3}.$$

Example 5.5. From Example 3.4, $\mathscr{T}_{\mathbb{X}}^{NT}(2,3,3) = 7$. In this case, the NTRSs having 3-NOSs in three NTs without repetition are

$$(\mathbb{X}^{NT}, \mathscr{T}_{1}^{NT}, \mathscr{T}_{2}^{NT}, \mathscr{T}_{3}^{NT}),\ (\mathbb{X}^{NT}, \mathscr{T}_{1}^{NT}, \mathscr{T}_{2}^{NT}, \mathscr{T}_{4}^{NT}),\ (\mathbb{X}^{NT}, \mathscr{T}_{1}^{NT}, \mathscr{T}_{2}^{NT}, \mathscr{T}_{5}^{NT}),$$
$$(\mathbb{X}^{NT}, \mathscr{T}_{1}^{NT}, \mathscr{T}_{2}^{NT}, \mathscr{T}_{6}^{NT}),\ (\mathbb{X}^{NT}, \mathscr{T}_{1}^{NT}, \mathscr{T}_{2}^{NT}, \mathscr{T}_{7}^{NT}),$$
$$(\mathbb{X}^{NT}, \mathscr{T}_{1}^{NT}, \mathscr{T}_{3}^{NT}, \mathscr{T}_{4}^{NT}),\ (\mathbb{X}^{NT}, \mathscr{T}_{1}^{NT}, \mathscr{T}_{3}^{NT}, \mathscr{T}_{5}^{NT}),\ (\mathbb{X}^{NT}, \mathscr{T}_{1}^{NT}, \mathscr{T}_{3}^{NT}, \mathscr{T}_{6}^{NT}),$$
$$(\mathbb{X}^{NT}, \mathscr{T}_{1}^{NT}, \mathscr{T}_{3}^{NT}, \mathscr{T}_{7}^{NT}),$$
$$(\mathbb{X}^{NT}, \mathscr{T}_{1}^{NT}, \mathscr{T}_{4}^{NT}, \mathscr{T}_{5}^{NT}),\ (\mathbb{X}^{NT}, \mathscr{T}_{1}^{NT}, \mathscr{T}_{4}^{NT}, \mathscr{T}_{6}^{NT}),\ (\mathbb{X}^{NT}, \mathscr{T}_{1}^{NT}, \mathscr{T}_{4}^{NT}, \mathscr{T}_{7}^{NT}),$$
$$(\mathbb{X}^{NT}, \mathscr{T}_{1}^{NT}, \mathscr{T}_{5}^{NT}, \mathscr{T}_{6}^{NT}),\ (\mathbb{X}^{NT}, \mathscr{T}_{1}^{NT}, \mathscr{T}_{5}^{NT}, \mathscr{T}_{7}^{NT}),$$
$$(\mathbb{X}^{NT}, \mathscr{T}_{1}^{NT}, \mathscr{T}_{6}^{NT}, \mathscr{T}_{7}^{NT}),$$

$$(\mathbb{X}^{NT}, \mathscr{T}_2^{NT}, \mathscr{T}_3^{NT}, \mathscr{T}_4^{NT}),\ (\mathbb{X}^{NT}, \mathscr{T}_2^{NT}, \mathscr{T}_3^{NT}, \mathscr{T}_5^{NT}),\ (\mathbb{X}^{NT}, \mathscr{T}_2^{NT}, \mathscr{T}_3^{NT}, \mathscr{T}_6^{NT}),$$
$$(\mathbb{X}^{NT}, \mathscr{T}_2^{NT}, \mathscr{T}_3^{NT}, \mathscr{T}_7^{NT}),$$
$$(\mathbb{X}^{NT}, \mathscr{T}_2^{NT}, \mathscr{T}_4^{NT}, \mathscr{T}_5^{NT}),\ (\mathbb{X}^{NT}, \mathscr{T}_2^{NT}, \mathscr{T}_4^{NT}, \mathscr{T}_6^{NT}),\ (\mathbb{X}^{NT}, \mathscr{T}_2^{NT}, \mathscr{T}_4^{NT}, \mathscr{T}_7^{NT}),$$
$$(\mathbb{X}^{NT}, \mathscr{T}_2^{NT}, \mathscr{T}_5^{NT}, \mathscr{T}_6^{NT}),\ (\mathbb{X}^{NT}, \mathscr{T}_2^{NT}, \mathscr{T}_5^{NT}, \mathscr{T}_7^{NT}),$$
$$(\mathbb{X}^{NT}, \mathscr{T}_2^{NT}, \mathscr{T}_6^{NT}, \mathscr{T}_7^{NT}),$$
$$(\mathbb{X}^{NT}, \mathscr{T}_3^{NT}, \mathscr{T}_4^{NT}, \mathscr{T}_5^{NT}),\ (\mathbb{X}^{NT}, \mathscr{T}_3^{NT}, \mathscr{T}_4^{NT}, \mathscr{T}_6^{NT}),\ (\mathbb{X}^{NT}, \mathscr{T}_3^{NT}, \mathscr{T}_4^{NT}, \mathscr{T}_7^{NT}),$$
$$(\mathbb{X}^{NT}, \mathscr{T}_3^{NT}, \mathscr{T}_5^{NT}, \mathscr{T}_6^{NT}),\ (\mathbb{X}^{NT}, \mathscr{T}_3^{NT}, \mathscr{T}_5^{NT}, \mathscr{T}_7^{NT}),$$
$$(\mathbb{X}^{NT}, \mathscr{T}_3^{NT}, \mathscr{T}_6^{NT}, \mathscr{T}_7^{NT}),$$
$$(\mathbb{X}^{NT}, \mathscr{T}_4^{NT}, \mathscr{T}_5^{NT}, \mathscr{T}_6^{NT}),\ (\mathbb{X}^{NT}, \mathscr{T}_4^{NT}, \mathscr{T}_5^{NT}, \mathscr{T}_7^{NT}),$$
$$(\mathbb{X}^{NT}, \mathscr{T}_4^{NT}, \mathscr{T}_6^{NT}, \mathscr{T}_7^{NT}).$$
$$(\mathbb{X}^{NT}, \mathscr{T}_5^{NT}, \mathscr{T}_6^{NT}, \mathscr{T}_7^{NT}).$$

Therefore, the NNTRSs consisting 3-NOSs in all three NTs without repetition is

$$(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT}, \mathscr{T}_k^{NT})_{\mathbb{X}}^{NT}(2,3,3) = 35 = \binom{\mathscr{T}_{\mathbb{X}}^{NT}(2,3,3)}{3} = \binom{7}{3}.$$

Result 5.6. $(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT}, \mathscr{T}_k^{NT})_{\mathbb{X}}^{NT}(\mathfrak{n}, \mathfrak{m}, 3) = \frac{\mathfrak{m}^{\mathfrak{n}}}{3}(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT})_{\mathbb{X}}^{NT}(\mathfrak{n}, \mathfrak{m}, 3)$.

Example 5.7. From Example 4.3 and 5.3, we have,

$$(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT})_{\mathbb{X}}^{NT}(2,3,3) = 28 \text{ and } (\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT}, \mathscr{T}_k^{NT})_{\mathbb{X}}^{NT}(2,3,3) = 84.$$

Therefore $\frac{3^2}{3} \times (\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT})_{\mathbb{X}}^{NT}(2,3,3) = \frac{3^2}{3} \times 28 = 84 = (\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT}, \mathscr{T}_k^{NT})_{\mathbb{X}}^{NT}(2,3,3)$.

Result 5.8. *In $\mathscr{N}_{\mathbb{X}}^{\mathscr{T}}$, the NNTRSs consisting 4-NOSs in three NTs is*

$$(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT}, \mathscr{T}_k^{NT})_{\mathbb{X}}^{NT}(\mathfrak{n}, \mathfrak{m}, 4) = \binom{\mathscr{T}_{\mathbb{X}}^{NT}(\mathfrak{n},\mathfrak{m},4)+2}{3}.$$

Example 5.9. Example 3.11 implies,

$$\mathscr{T}_{\mathbb{X}}^{NT}(2,3,4) = 13.$$

Then the NNTRS having 4-NOSs is

$$(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT}, \mathscr{T}_k^{NT})_{\mathbb{X}}^{NT}(2,3,4) = \binom{\mathscr{T}_{\mathbb{X}}^{NT}(2,3,4)+2}{3} = \frac{13(13+1)(13+2)}{6} = 455.$$

Result 5.10. *The NNTRSs consisting 4-NOSs in all three NT without repetition in $\mathscr{N}_{\mathbb{X}}^{\mathscr{T}}$ is*

$$(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT}, \mathscr{T}_k^{NT})_{\mathbb{X}}^{NT}(\mathfrak{n}, \mathfrak{m}, 4) = \binom{\mathscr{T}_{\mathbb{X}}^{NT}(\mathfrak{n},\mathfrak{m},4)}{3}.$$

Example 5.11. From Example 3.11, $\mathscr{T}_{\mathbb{X}}^{NT}(2,3,4) = 13$. Following Example 5.5 and result 5.10, the NNTRSs consisting 4-NOSs in all three NT without repetition is $(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT}, \mathscr{T}_k^{NT})_{\mathbb{X}}^{NT}(2,3,4) = 286$.

Result 5.12. $(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT}, \mathscr{T}_k^{NT})_{\mathbb{X}}^{NT}(\mathfrak{n}, \mathfrak{m}, 4) = \frac{(\mathscr{T}_{\mathbb{X}}^{NT}(\mathfrak{n},\mathfrak{m},4)+2)}{3}(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT})_{\mathbb{X}}^{NT}(\mathfrak{n}, \mathfrak{m}, 4)$.

Example 5.13. From Examples 3.11, 4.7 and 5.9, we have

$$\mathscr{T}_{\mathbb{X}}^{NT}(2,3,4) = 13,\ (\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT})_{\mathbb{X}}^{NT}(2,3,4) = 91 \text{ and } (\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT}, \mathscr{T}_k^{NT})_{\mathbb{X}}^{NT}(2,3,4) = 455.$$

Therefore,

$$\frac{(\mathscr{T}_{\mathbb{X}}^{NT}(2,3,4)+2)}{3}(\mathscr{T}_{\imath}^{NT},\mathscr{T}_{\jmath}^{NT})_{\mathbb{X}}^{NT}(2,3,4)=\frac{13+2}{3}\times 91=455=(\mathscr{T}_{\imath}^{NT},\mathscr{T}_{\jmath}^{NT},\mathscr{T}_{\Bbbk}^{NT})_{\mathbb{X}}^{NT}(2,3,4).$$

6. Effective of the proposed method

The formula for giving the number of topologies $T(\mathfrak{n})$ is still not obtained for a finite set $\mathbb{X}$ having $\mathfrak{n}$ elements. If $\mathfrak{n}$ is small, then we can compute it by hand. But the difficulty increases when $\mathfrak{n}$ becomes large. Studying this particular area is also a highly valued part of the topology, and this is one of the fascinating and challenging research areas. Note that the explicit formula for finding the number of topologies is undetermined till now. This paper is towards the formulae for finding the number of neutrosophic topological spaces having $2, 3, 4$-open sets, the number of neutrosophic bitopological spaces, and tritopological spaces having the same number of open sets in topologies.

7. Conclusions

In this paper, the NNTSs consisting of small NOSs i.e., $2, 3$, and 4-open sets are computed. Moreover, the NNBTSs and NNTRSs are computed. It is also observed that formulae for finding NNTSs, NNBTSs, and NNTRSs are interrelated. Hope this work will help in further study of NNTSs with greater open sets. In the future, the NNBTSs having k, l-open sets and the NNTRSs having k, l, m-open sets can be found where $k \neq l \neq m$. Moreover, we aim to extend our work to study the existence of NNTSs in the topological group.

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Homomorphism and Isomorphism of Neutrosophic Over Topologized Graphs

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Abstract: In this paper introduce the homomorphism, isomorphism, weak isomorphism and co-weak isomorphism of Neutrosophic over topologized graphs. Some properties of isomorphism are introduced. The isomorphism of Neutrosophic over topologized graphs equivalnce relation, weak isomorphism of Neutrosophic over topologized graphs partial order relation and complement of Neutrosophic over topologized graphs also derived here.

Keywords: Neutrosophic over topologized graphs, homomorphsim,isomorphism, weak isomorphism and co-weak isomorphism

1 Introduction

In 1965 Zadeh [12] was invent the idea of a fuzzy set as a mathematical frame work for representing vagueness and imprecise information. Rosenfield (1975) introduced the notion of fuzzy graph [10]. Fuzzy graphs have numerous applications in diverse parts of science and engineering like broad cost communications producing, social network. Attanassov introduced the concept of intuitionstic fuzzy set as a gerealization of fuzzy sets [2]. Many researchers established and studied about fuzzy graphs and Intuitionstic fuzzy graphs in[1].Neutrosophic set proposed by Smarandache [11,13,14] is a powerful tool for dealing incomplete , inconsistance, imprecision, uncertain, false and inderminate problems in the real world whenever the fuzzy and intuitionstic fuzzy approches fail in such type of situation. Also he extended the neutrosophic set respectively to Neutrosophic Overset when some neutrosophic component is > 1, to Neutrosophic Underset when some neutrosophic component is < 0, and to Neutrosophic Offset when some neutrosophic components are off the interval [0, 1], i.e. some neutrosophic component $>$ 1 and other neutrosophic component < 0. since our real-world has numerous examples and applications of over-/under-/off-neutrosophic components[4,5,6]. Later, Narmada Devi [15,16,17,18,19,20,21,22,23] worked on new type of Neutrosophic over, Neutrosophic off graph and minimal domination via Neutrosophic over graph and Neutrosophic over topologized graph. In this paper, we introduce the notion of homomorphism and isomorphism between Neutrosophic over topologized graphs.

2 Preliminiaries

Definition 2.1. **[3]** A topologized graph is a topological space $\mathscr{X}$ such that

(i) every singleton is open or closed

(ii) $\forall x \in \mathscr{X}$, $|\eth(x)| \leq 2$, since $\eth(x)$ is denoted by the boundary of a point x.

Definition 2.2. [16] A single-valued *neutrosophic over set* A is defined as $A = (X, \langle \mathscr{T}(x), \mathscr{I}(x), \mathscr{F}(x) \rangle), x \in X$ such that there exist some elements in A that have atleast one neutrosophic component that is > 1 and no element has neutrosophic components that are < 0 and $\mathscr{T}(x), \mathscr{I}(x), \mathscr{F}(x) \in [0, \Omega]$, where Ω is called overlimit such that $0 < 1 < \Omega$.

Definition 2.3. [16] A *Neutrosophic over graph* $G = (P, Q)$ on a crisp graph G^* where P is an *neutrosophic vertex over set* on V and Q is a *neutrosophic edge over set* on E respectively such that

(i) $\mathscr{T}_Q(mn) \leq [\mathscr{T}_P(m) \wedge \mathscr{T}_P(n)]]$

(ii) $\mathscr{I}_Q(mn) \leq [\mathscr{I}_P(m) \wedge \mathscr{I}_P(n)]]$

(iii) $\mathscr{F}_Q(mn) \geq [\mathscr{F}_P(m) \vee \mathscr{F}_P(n)]]$ for every $mn \in E \subseteq V \times V$.

3 Homomorphism of Neutrosophic over Topologized Graphs

Definition 3.1. Let $G = (A, B)$ be a Neutrosophic over topologized graph [In short Neutrosophic over top graph] . The order of G denoted by $O(G)$ is defined as $O(G) = (O_T(G), O_I(G), O_F(G))$, where
$O_T(G) = \sum_{v \in V} T_A(v)$ denotes the $T-$order of G,
$O_I(G) = \sum_{v \in V} I_A(v)$ denotes the $I-$order of G,
$O_F(G) = \sum_{v \in V} F_A(v)$ denotes the $F-$order of G.

Definition 3.2. Let $G = (A, B)$ be a Neutrosophic over top graph. The size of G denoted by $S(G)$ is defined as $S(G) = (S_T(G), S_I(G), S_F(G))$, where
$S_T(G) = \sum_{v_i \neq v_j} T_B(v_i, v_j)$ denotes the $T-$size of G,
$S_I(G) = \sum_{v_i \neq v_j} I_B(v_i, v_j)$ denotes the $I-$size of G,
$S_F(G) = \sum_{v_i \neq v_j} F_B(v_i, v_j)$ denotes the $F-$size of G,

Definition 3.3. Let G_1 and G_2 be the Neutrosophic over top graphs. A homomorphism $f : G_1 \to G_2$ is a map $f : V_1 \to V_2$ which satisfies the following conditions
(a) $T_{A_1}(x_1) = T_{A_2}(f(x_1)), I_{A_1}(x_1) = I_{A_2}(f(x_1)), F_{A_1}(x_1) = F_{A_2}(f(x_1))$
(b) $T_{B_1}(x_1y_1) = T_{B_2}(f(x_1)f(y_1)), I_{B_1}(x_1y_1) = I_{B_2}(f(x_1)f(y_1)), F_{B_1}(x_1y_1) = F_{B_2}(f(x_1)f(y_1)),$
$\forall x_1 \in V_1, x_1y_1 \in E_1$

Definition 3.4. Let G_1 and G_2 be the Neutrosophic over top graphs. Isomorphism $f : G_1 \to G_2$ is a map $f : V_1 \to V_2$ which is a bijective mapping that satisfies the following conditions
(i) $T_{A_1}(x_1) = T_{A_2}(f(x_1)), I_{A_1}(x_1) = I_{A_2}(f(x_1)), F_{A_1}(x_1) = F_{A_2}(f(x_1)),$
(ii) $T_{B_1}(x_1y_1) = T_{B_2}(f(x_1)f(y_1)), I_{B_1}(x_1y_1) = I_{B_2}(f(x_1)f(y_1)), F_{B_1}(x_1y_1) = F_{B_2}(f(x_1)f(y_1)),$
$\forall x_1 \in V_1, x_1y_1 \in E_1$

Definition 3.5. Let G_1 and G_2 be the Neutrosophic over top graphs. Then a weak isomorphism $f : G_1 \to G_2$ is bijective mapping $f : V_1 \to V_2$ which satisfies the following conditions

(1) f is homomorphism

(2) $T_{A_1}(x_1) = T_{A_2}(f(x_1))$, $I_{A_1}(x_1) = I_{A_2}(f(x_1))$ and $F_{A_1}(x_1) = F_{A_2}(f(x_1)), \forall x_1 \in V_1$.

Thus, a weak isomorphism preserves the weights of the vertex but not necessarily the weight of the edges.

Theorem 3.1. For any two isomorphic Neutrosophic over top graphs their order and size are same.

Proof: If $f : G_1 \to G_2$ is an isomorphism between the Neutrosophic over top graphs G_1 and G_2 with the underlying sets V_1 and V_2 respectively.
(i) $T_{A_1}(x_1) = T_{A_2}(f(x_1)), I_{A_1}(x_1) = I_{A_2}(f(x_1))$ and $F_{A_1}(x_1) = F_{A_2}(f(x_1)), \forall x_1 \in V_1$
(ii) $T_{B_1}(x_1y_1) = T_{B_2}(f(x_1)f(y_1)), I_{B_1}(x_1y_1) \ = I_{B_2}(f(x_1)f(y_1)), F_{B_1}(x_1y_1) = F_{B_2}(f(x_1)f(y_1)), \ \ \forall x_1y_1 \in E_1$

$$
\begin{aligned}
(i) \text{ order of } G &= (O_T(G_1), O_I(G_1), O_F(G_1)) \\
&= \left(\sum_{x\in V} T_A(x), \sum_{x\in V} I_A(x), \sum_{x\in V} F_A(x)\right) \\
&= \left(\sum_{x,y\in E} T_A(f(x)), \sum_{x,y\in E} I_A(f(x)), \sum_{x,y\in E} F_A(f(x))\right) \\
&= (O_T(G_2), O_I(G_2), O_F(G_2)) \\
&= O(G_2)
\end{aligned}
$$

$$
\begin{aligned}
(ii)\ S &= (S_T(G_1), S_I(G_1), S_F(G_1)) \\
&= \left(\sum_{x_1\in V_1} T_{B_1}(x_1y_1), \sum_{x_1\in V_1} I_{B_1}(x_1y_1), \sum_{x_1\in V_1} F_{B_1}(x_1y_1)\right) \\
&= \left(\sum_{x_1,y_1\in E_1} T_{B_2}(f(x_1), f(y_1)), \sum_{x_1,y_1\in E_1} I_{B_2}(f(x_1), f(y_1)), \sum_{x_1,y_1\in E_1} F_{B_2}(f(x_1), f(y_1))\right) \\
&= (S_T(G_2), S_I(G_2), S_F(G_2)) \\
&= S(G_2)
\end{aligned}
$$

Hence the theorem.

Theorem 3.2. Isomorphism between Neutrosophic over top graphs is an equivalence relation.
Proof: Let $G = (A, B)$, $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be Neutrosophic over top graphs with underlying sets V, V_1 and V_2 respectively.

(i) Reflexive:
Consider the identity map $f : V \to V$ such that $f(v) = v, \forall v \in V$.

This f is a bijective map satisfying $T_A(v_i) = T_A(f(v_i)), I_A(v_i) \ = I_A(f(v_i)), F_A(v_i) = F_A(f(v_i)), \forall v_i \in V$
$T_B(v_i, v_j) = T_A(f(v_i), f(v_j)), I_A(v_i, v_j) \ = I_A(f(v_i), f(v_j)), F_A(v_i, v_j) = F_A(f(v_i), f(v_j)), \forall v_i, v_j \in V$
Hence f is an isomorphism of the Neutrosophic over top graph to itself. $\therefore$ It satisfies reflexive relation.

(ii) Symmetric:
Let $f : V \to V_1$ be an isomorphism of G onto G_1, then f is a bijective map such that $f(v) = v_1, v \in V$ satisfying

$$\begin{aligned} T_A(v) &= T_A(f(v)) \\ I_A(v) &= I_A(f(v)) \\ F_A(v) &= F_A(f(v)), \forall v \in V \\ T_B(v_i, v_j) &= T_{B_1}(f(v_i), f(v_j)) \\ I_B(v_i, v_j) &= I_{B_1}(f(v_i), f(v_j)) \\ F_B(v_i, v_j) &= F_{B_1}(f(v_i), f(v_j)), \forall v_i, v_j \in V \end{aligned} \tag{3.1}$$

As f is bijective, by equation (3.1)

$$f^1(v_1) = v, \forall v_1 \in V_1 \tag{3.2}$$

using (3.2) in (3.1), we get

$$\begin{aligned} T_A(f^{-1}(v_1)) &= T_{A_1}(v_1) \\ I_A(f^{-1}(v_1)) &= I_{A_1}(v_1) \\ F_A(f^{-1}(v_1)) &= F_{A_1}(v_1), \forall v_1 \in V_1 \\ T_B(f^{-1}(v_i), f^{-1}(v_j)) &= T_{B_1}(v_{i1}), f(v_j 1)) \\ I_B(f^{-1}(v_i), f^{-1}(v_j)) &= I_{B_1}(v_{i1}), f(v_j 1)) \\ F_B(f^{-1}(v_i), f^{-1}(v_j)) &= F_{B_1}(v_{i1}), f(v_j 1)), \forall v_{i1}, v_{j1} \in V_1 \end{aligned} \tag{3.3}$$

Hence we get a 1-1, onto map $f^{-1} : V_1 \in V$, which is an isomorphism from G_1 to G

$$\text{i.e., } G \cong G' \implies G' \cong G$$

$\therefore$ It satisfies symmetric property.

(iii) Transitive:
Let $f : V \to V_1$ and $g : V_1 \to V_2$ be isomorphisms of the Neutrosophic over top graphs G onto G_1 and G_1 onto G_2 respectively.

Then $g \circ f$ is a 1-1 onto map from $V \to V_2$ where

$$(g \circ f)(v) = g(f(v)), \forall v \in V$$

As $f : V \to V_1$ is an isomorphism

$$f(v) = v_1, v \in V \tag{3.4}$$

$$\begin{aligned} T_A(v) &= T_{A_1}(f(v)) \\ I_A(v) &= I_{A_1}(f(v)) \\ F_A(v) &= F_{A_1}(f(v)), \forall v \in V \\ T_B(v_i, v_j) &= T_{B_1}(f(v_i), f(v_j)) \\ I_B(v_i, v_j) &= I_{B_1}(f(v_i), f(v_j)) \\ F_B(v_i, v_j) &= F_{B_1}(f(v_i), f(v_j)), \forall v_i, v_j \in V \end{aligned} \tag{3.5}$$

using equation (3.4) in equation (3.5), we have

$$\begin{aligned} T_A(v)) &= T_{A_1}(v_1) \\ I_A(v) &= I_{A_1}(v_1) \\ F_A(v)) &= F_{A_1}(v_1), \forall v \in V \end{aligned} \tag{3.6}$$

$$\begin{aligned} T_B(v_i, v_j) &= T_{B_1}(f(v_{1i}), f(v_{1j})) \\ I_B(v_i, v_j) &= I_{B_1}(f(v_{1i}), f(v_{1j})) \\ F_B(v_i, v_j) &= F_{B_1}(f(v_{1i}), f(v_{1j})), \forall v_i, v_j \in V \end{aligned} \tag{3.7}$$

As $g : V_1 \to V_2$ is an isomorphisms

$$g(v_1) = v_2, v_1 \in V_1 \tag{3.8}$$

$$\begin{aligned} T_{A1}(v_1)) &= T_{A_2}(g(v_1)) \\ I_{A1}(v_1)) &= I_{A_2}(g(v_1)) \\ F_{A1}(v_1)) &= F_{A_2}(g(v_1)), \forall v_1 \in V_1 \end{aligned} \tag{3.9}$$

$$\begin{aligned} T_{B_1}(v_{1i}, v_{1j}) &= T_{B_2}(g(v_{1i}), f(v_{1j})) \\ I_{B_1}(v_{1i}, v_{1j}) &= I_{B_2}(g(v_{1i}), f(v_{1j})) \\ F_{B_1}(v_{1i}, v_{1j}) &= F_{B_2}(g(v_{1i}), f(v_{1j})), \forall v_i, v_j \in V_1 \end{aligned} \tag{3.10}$$

Equations (3.5), (3.7) and (3.10) implies

$$\begin{aligned} &T_A(v) = T_{A_2}(g(v_1)) = T_{A_2}(g(f(v))) \\ &I_A(v) = I_{A_2}(g(f(v))) \\ &F_A(v) = F_{A_2}(g(v_1)) \end{aligned} \tag{3.11}$$

Equations (3.5), (3.8) and (3.11) implies

$$\begin{aligned} &T_B(v_i, v_j) = T_{B_2}(g(v_{1i}), g(v_{1j})) = T_{B_2}(g(f(v_i)), g(f(v_j))) \\ &I_B(v_i, v_j) = I_{B_2}(g(f(v_i)), g(f(v_j))) \\ &F_B(v_i, v_j) = F_{B_2}(g(f(v_i)), g(f(v_j))) \end{aligned} \tag{3.12}$$

Equations (3.12) and (3.13) implies

$g \circ f$ is an isomorphism between G and G'' is $G \cong G''$ i.e., isomorphism between Neutrosophic over top graphs is an equivalence relation.

Theorem 3.3. Weak isomorphism between Neutrosophic over top graphs satisfies the partial order relation.

Proof: Let $G = (A, B)$, $G' = (A', B')$, $G'' = (A'', B'')$ be Neutrosophic over top graphs with underlying sets V, V' and V'' respectively.

(i) Reflexive:

Consider the identity map $h : V \to V$ such that $h(v) = v, \forall v \in V$.

This h is a bijective map satisfying

$$\begin{aligned} T_A(v_i) &= T_A(h(v_i)) \\ I_A(v_i) &= I_A(h(v_i)) \\ F_A(v_i) &= F_A(h(v_i)), \forall v_i \in V \end{aligned}$$

$$\begin{aligned} T_B(v_i, v_j) &\leq T_B(h(v_i), h(v_j)) \\ I_B(v_i, v_j) &\leq I_B(h(v_i), h(v_j)) \\ F_B(v_i, v_j) &\leq F_B(h(v_i), h(v_j)), \forall v_i, v_j \in V \end{aligned}$$

Hence h is a weak isomorphism of the Neutrosophic over top graph to itself.

$\therefore$ it satisfies reflexive relation.

(ii) Anti symmetric:

Let h be a weak isomorphism between G and G' and g be a weak isomorphism between G' and G.

i.e., $h : V \to V'$ is a bijective map such that $h(v) = v'$, $v \in V$ satisfying

$$\begin{aligned} T_A(v) &= T_{A'}(h(v)) \\ I_A(v) &= I_{A'}(h(v)) \\ F_A(v) &= F_{A'}(h(v)), \forall v \in V \end{aligned}$$

$$\begin{aligned} T_B(v_i, v_j) &\leq T_{B'}(h(v_i), h(v_j)) \\ I_B(v_i, v_j) &\leq I_{B'}(h(v_i), h(v_j)) \\ F_B(v_i, v_j) &\leq F_{B'}(h(v_i), h(v_j)), \forall v_i, v_j \in V \end{aligned} \tag{3.13}$$

and $g : V' \to V$ is a bijective map satisfying

$$\begin{aligned} T_{A'}(v') &= T_A(g(v')) \\ I_{A'}(v') &= I_A(g(v')) \\ F_{A'}(v') &= F_A(g(v')), \forall v' \in V' \end{aligned}$$

$$\begin{aligned} T_{B'}(v'_i, v'_j) &\leq T_B(h(v'_i), h(v'_j)) \\ I_{B'}(v'_i, v'_j) &\leq I_B(h(v'_i), h(v'_j)) \\ F_{B'}(v'_i, v'_j) &\leq F_B(h(v'_i), h(v'_j)), \forall v'_i, v'_j \in V' \end{aligned} \tag{3.14}$$

The inequalities (3.13) and (3.14) hold good on the finite sets $V \; V'$ only when G and G' have the same number of edges and the corresponding edges have same weights.

Hence G and G' are identical.

(iii) Transitive:

Let $h : V \to V^{'}$ and $g : V^{'} \to V^{''}$ be weak isomorphism of the Neutrosophic over top graphs G onto $G^{'}$ and $G^{'}$ onto $G^{''}$ respectively.

Then $g \circ h$ is a 1-1, onto map from $V \to V^{''}$ where $(g \circ h)(v) = g(h(v)), v \in V$.

As $h : V \to V^{'}$ is a weak isomorphism $h(v) = v^{'}, v \in V$.

$$\begin{aligned} T_A(v) &= T_{A'}(h(v)) \\ I_A(v) &= I_{A'}(h(v)) \\ F_A(v) &= F_{A'}(h(v)), \forall v \in V \end{aligned}$$

$$\begin{aligned} T_B(v_i, v_j) &\leq T_{B'}(h(v_i), h(v_j)) \\ I_B(v_i, v_j) &\leq I_{B'}(h(v_i), h(v_j)) \\ F_B(v_i, v_j) &\leq F_{B'}(h(v_i), h(v_j)), \forall v_i, v_j \in V \end{aligned} \tag{3.15}$$

As $g : V^{'} \to V^{''}$ is a weak isomorphism $g(v^{'}) = v^{''}, \forall v^{'} \in V^{'}$.

$$\begin{aligned} T_{A'}(v^{'}) &= T_{A''}(h(v^{'})) \\ I_{A'}(v^{'}) &= I_{A''}(h(v^{'})) \\ F_{A'}(v^{'}) &= F_{A''}(h(v^{'})), \forall v^{'} \in V \end{aligned}$$

$$\begin{aligned} T_{B'}(v_i, v_j) &\leq T_{B''}(h(v_i^{'}), h(v_j^{'})) \\ I_{B'}(v_i, v_j) &\leq I_{B''}(h(v_i^{'}), h(v_j^{'})) \\ F_{B'}(v_i, v_j) &\leq F_{B''}(h(v_i^{'}), h(v_j^{'})), \forall v_i^{'}, v_j^{'} \in V^{'} \end{aligned} \tag{3.16}$$

Equation (3.15) and (3.16) implies

$$\begin{aligned} T_A(v) &= T_{A''}(g(v^{'})) = T_{A''}(g(h(v))) \\ I_A(v) &= I_{A''}(g(v^{'})) = I_{A''}(g(h(v))) \\ F_A(v) &= F_{A''}(g(v^{'})) = F_{A''}(g(h(v))), \forall v^{'} \in V \end{aligned} \tag{3.17}$$

$$\begin{aligned} T_B(v_i, v_j) &\leq T_{B''}(g(v_i^{'}), g(v_j^{'})) = T_{B''}(g(h(v_i)), g(h(v_j))) \\ I_B(v_i, v_j) &\leq I_{B''}(g(v_i^{'}), g(v_j^{'})) = I_{B''}(g(h(v_i)), g(h(v_j))) \\ F_B(v_i, v_j) &\leq F_{B''}(g(v_i^{'}), g(v_j^{'})) = F_{B''}(g(h(v_i)), g(h(v_j))) \end{aligned} \tag{3.18}$$

Equations (3.17) & (3.18) implies $g \circ h$ is a weak isomorphism between G & $G^{''}$.

i.e., weak isomorphism satisfies transitity.

(i), (ii) & (iii) implies weak isomorphism between Neutrosophic over top graphs is partial relation.

4 Isomorphic Neutrosophic Over topologized graphs and their complements

Definition 4.1. The complement of a Neutrosophic over top graph $G = (A, B)$ is a Neutrosophic over top graph $\overline{G} = (\overline{A}, \overline{B})$, where

(1) $\overline{V} = V$

(2) $\overline{T}_A(v_i) = T_A(v_i)$
$\overline{I}_A(v_i) = I_A(v_i)$
$\overline{F}_A(v_i) = F_A(v_i)$

(3) $\overline{T}_B(v_i, v_j) = \begin{cases} \min[T_A(v_i), T_A(v_j)], & \text{if } T_B(v_i, v_j) \\ \min[T_A(v_i), T_A(v_j)] - T_B(v_i, v_j), & \text{if } T_B(v_i, v_j) > 0 \end{cases}$

$\overline{I}_B(v_i, v_j) = \begin{cases} \min[I_A(v_i), I_A(v_j)], & \text{if } I_B(v_i, v_j) \\ \min[I_A(v_i), I_A(v_j)] - I_B(v_i, v_j), & \text{if } I_B(v_i, v_j) > 0 \end{cases}$

$\overline{F}_B(v_i, v_j) = \begin{cases} \max[F_A(v_i), F_A(v_j)], & \text{if } F_B(v_i, v_j) \\ \max[F(v_i), F_A(v_j)] - F_B(v_i, v_j), & \text{if } F_B(v_i, v_j) > 0 \end{cases}$

$\forall v_i, v_j \in V$

Example 4.1. Consider a Neutrosophic over top graph $G = (A, B)$ on the non-empty set $V = \{v_1, v_2, v_3, v_4\}$ $E = \{v_1v_2, v_2v_3, v_3v_4, v_4v_1\}$. Neutrosophic over top graph $G = (A, B)$ and complement Neutrosophic over top graph $\overline{G} = (\overline{A}, \overline{B})$.

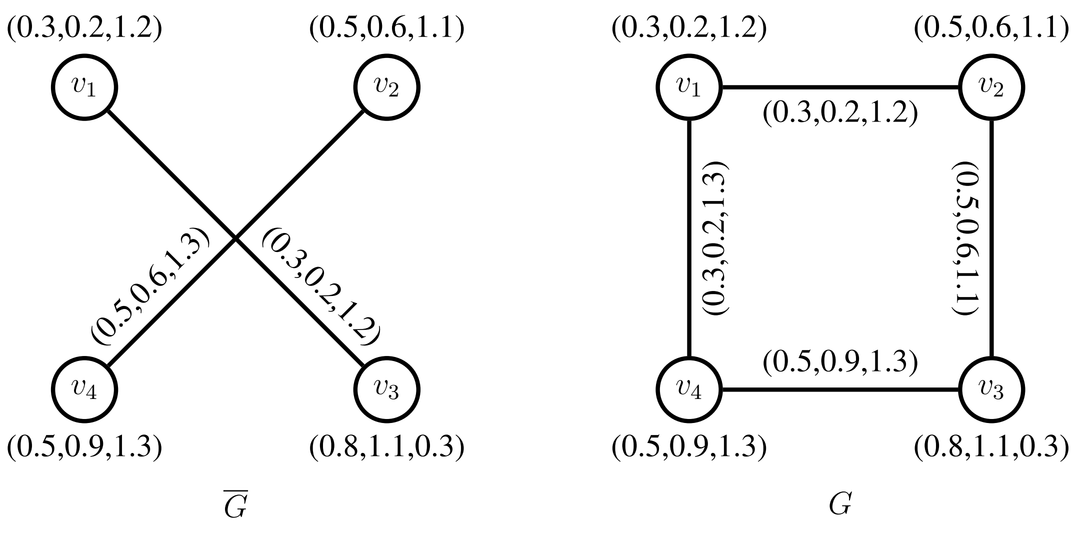


Figure 1: Neutrosophic over top graph G and its complement $\overline{G}$

Theorem 4.1. If two Neutrosophic over top graphs are isomorphic then their complements are isomorphic.

Proof: Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be the two Neutrosophic over top graphs. Assume $G_1 \cong G_2$. There exists a bijective map $h : V_1 \to V_2$ satisfying

$$T_{A_1}(v) = T_{A_2}(h(v))$$

$$\begin{aligned} I_{A_1}(v) &= I_{A_2}(h(v)) \\ F_{A_1}(v) &= F_{A_2}(h(v)), \forall v \in V \\ T_{B_1}(v_i, v_j) &= T_{B_2}(h(v_i), h(v_j)) \\ I_{B_1}(v_i, v_j) &= I_{B_2}(h(v_i), h(v_j)) \\ F_{B_1}(v_i, v_j) &= F_{B_2}(h(v_i), h(v_j)), \forall v_i, v_j \in V \end{aligned}$$

By definition

$$\begin{aligned} \overline{T}_{B_1}(v_i, v_j) &= \min[T_{A_1}(v_i), T_{A_1}(v_j)] - T_{B_1}(v_i, v_j) \\ &= \min[T_{A_2}(h(v_i)), T_{A_2}(h(v_j))] - T_{B_2}(h(v_i), h(v_j)) \\ &= \overline{T}_{B_2}(h(v_i), h(v_j)), \forall v_i, v_j \in V \\ \overline{I}_{B_1}(v_i, v_j) &= \overline{I}_{B_2}(h(v_i), h(v_j)), \forall v_i, v_j \in V \\ \overline{F}_{B_1}(v_i, v_j) &= \overline{F}_{B_2}(h(v_i), h(v_j)), \forall v_i, v_j \in V \end{aligned}$$

Hence $\overline{G}_1 \cong \overline{G}_2$.

Theorem 4.2. If $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ are weak isomorphic, then $\bar{G}_2 = (\bar{A}_2, \bar{B}_2)$ and $\bar{G}_1 = (\bar{A}_1, \bar{B}_1)$ are also weak isomorphic.

Proof: If h is a weak isomorphic between G_1 & G_2 then $h : V_1 \to V_2$ is a one-one-onto mapping and $h(v) = v_1, v \in V$

$$\begin{aligned} T_{A_1}(v) &= T_{A_2}(h(v)) \\ I_{A_1}(v) &= I_{A_2}(h(v)) \\ F_{A_1}(v) &= F_{A_2}(h(v)), \forall v \in V \end{aligned} \tag{4.1}$$

$$\begin{aligned} T_{B_1}(v_i, v_j) &= T_{B_2}(h(v_i), h(v_j)) \\ I_{B_1}(v_i, v_j) &= I_{B_2}(h(v_i), h(v_j)) \\ F_{B_1}(v_i, v_j) &= F_{B_2}(h(v_i), h(v_j)), \forall v_i, v_j \in V \end{aligned} \tag{4.2}$$

Since $h^{-1} : V_2 \to V_1$ is also one-one and onto for every v in V_2 there is a $v \in V_1$ such that $h^{-1}(v_1) = V$.
By equation number (4.2),

we have

$$\begin{aligned} T_{A_2}(v) &= T_{A_1}(h^{-1}(v)) \\ I_{A_2}(v) &= I_{A_1}(h^{-1}(v)) \\ F_{A_2}(v) &= F_{A_1}(h^{-1}(v)), \forall v \in V \end{aligned} \tag{4.3}$$

$$\begin{aligned} T_{B_1}(v_i, v_j) &= \min[T_{A_1}(v_i), T_{A_1}(v_j)] - T_{B_1}(v_i, v_j) \\ &= \min[T_{A_2}(h(v_i)), T_{A_2}(h(v_j))] - T_{B_2}(h(v_i), h(v_j)) \\ &= \bar{T}_{B_2}(h(v_i), h(v_j))_{,i}, v_j \in V \end{aligned}$$

Similarly

$$\begin{aligned} \bar{I}_{B_1}(v_i, v_j) &= \bar{I}_{B_2}(h(v_i), h(v_j))_{,i}, v_j \in V \\ \bar{F}_{B_1}(v_i, v_j) &= \bar{F}_{B_2}(h(v_i), h(v_j))_{,i}, v_j \in V \end{aligned}$$

Definition 4.2. Let G_1 and G_2 be Neutrosophic over top graphs. A co-weak isomorphism $f : G_1 \to G_2$ is bijective mapping $f : V_1 \to V_2$ which satisfies the following conditions

(i) f is homomorphism

(ii) $T_{B_1}(x_1y_1) = T_{B_2}(f(x_1), f(y_1))$
$I_{B_1}(x_1y_1) = I_{B_2}(f(x_1), f(y_1))$
$F_{B_1}(x_1y_1) = F_{B_2}(f(x_1), f(y_1))$

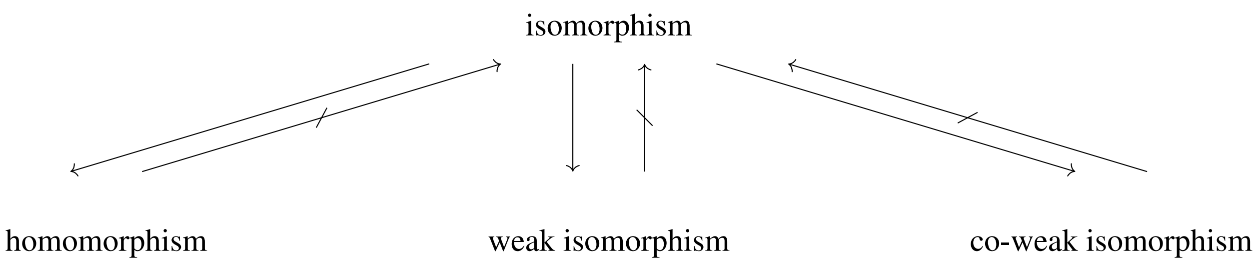
Thus, a co-weak isomorphism preserves the weight of the arcs but not necessarily the weights of the nodes.

Remark 4.1. 1. If $G_1 = G_2 = G$, then the homomorphism f over itself is called an endomorphism. An isomorphism f over G is called an automorphism.

2. If $G_1 = G_2$, then the weak and co-weak isomorphism actually become isomorphic.

3. If $V_1 \to V_2$ is a bijective map, then $f^{-1} : V_1 \to V_2$ is also a bijective map.

Remark 4.2. The interrelationship among Neutrosophic over top graphs as given below



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Multi-Valued Multi-Polar Neutrosophic Sets with an application in Multi-Criteria Decision-Making

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Abstract. This research directs to obtain optimum fuzzy soft constants through Bonferroni mean and TOPSIS with the initial data represented in terms of multi-valued m-polar neutrosophic soft set. Multi-valued m-polar neutrosophic soft set is defined in this paper, which is the generalization of m-polar neutrosophic soft set, obtained by combining it with multi-valued neutrosophic soft set. Optimum fuzzy soft constants play a fundamental role for the construction of the system of differential equations which helps to observe the experts' future attitudes. Sometimes experts feel a requirement to rethink their choices or decisions due to the observation of others' choice especially when others choose different alternatives. After the individual decisions of experts, an analysis of experts' attitudes is produced by using phase portraits and line graphs of the system of differential equations. This analysis can also be provided by using system of differential equations with fuzzy initial conditions. To find the multi-valued m-polar neutrosophic Bonferroni mean, some basic operations on the elements of the defined set are introduced. An illustrative example is given where a system of two differential equations is developed for attitude analysis of two persons with independent variable t.

Keywords: Multi valued neutrosophic set; Multi polar neutrosophic set; Bonferroni mean; Fuzzy soft differential equations.

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1. **Introduction**

Fuzzy sets have been conveniently utilized to deal with a plethora of problems regarding to uncertainties since when it was introduced by Zadeh [10]. It allocates each element of a set with a membership degree in the real standard $[0, 1]$. Intuitionistic fuzzy set (IFS) was introduced by Atanassov [11] which generalizes the concept of fuzzy set and handles some complicated fuzzy information in multi-criteria decision making (MCDM). IFS determines the membership and non-membership degrees for each element of a set. The concept of IFS was extended by Atanassov and Gargov [12] to interval-valued intuitionistic fuzzy set (IVIFS) which was applied for MCDM methods by several authors [13–18]. Despite of a number of research achievements on IFS, there is a need of indeterminate information. Smarandache [1] proposed an indeterminacy membership function which leads to the neutrosophic set (NS). NS generalizes the fuzzy set and IFS. Hesitant fuzzy set (HFS) was defined by Torra [20] which is identified by a function h_A on a universe U that returns a subset of $[0, 1]$. Many extensions of fuzzy set were further extended by combining with hesitant fuzzy set to Interval-valued hesitant fuzzy set (IVHFS) [25], hesitant fuzzy soft set (HFSS) [21], Interval-valued hesitant fuzzy soft set (IVHFSS) [22], dual hesitant fuzzy set (DHFS) [23], dual hesitant fuzzy soft set (DHFSS) [24], interval-valued dual hesitant fuzzy set (IVDHFS) [26] and some others. All these extended representations of hesitant fuzzy set have a substantial amount of research work for MCDM [27–29, 49]. Single valued neutrosophic set (SVNS) is an NS for which membership function, indeterminacy function and falsity function assign a single value from the interval $[0, 1]$ for each element of a set [2,45]. Interval-valued neutrosophic set (IVNS) involves the functions (membership, indeterminacy, non-membership) assigning the intervals from the interval $[0, 1]$ for each element [43]. NS has remarkably contributed in MCDM [44,47], and recently in TOPSIS [46]. Sometimes, decision makers hesitate to assign a single value to membership, non-membership or indeterminacy functions. They may suggest two or more values to these functions. HFS, IVHFS, DHFS and multi-valued neutrosophic set (MVNS) [3] facilitates those problems.

Bipolar fuzzy set [50] is an extension of a fuzzy set whose membership degree ranges from -1 to 1, It represents the double-sided uncertainties (e.g. positive-negative, yes-no, gains-loses,bright-dim, effect-side effect, etc.). These two sides are reciprocally related. Some bipolar representations with their applications have been done by different authors [30–33]. Chen *et al.* [34] presented a multi-polar fuzzy set which is an abstraction of a bipolar fuzzy set. They also explained some real world problems involving multi-agent, multi-attribute, multi-object and multi-index information. Deli *et al.* [5] defined multi-polar neutrosophic soft set and Saeed *et al.* [9] presented some operations on this set.

Bonferroni mean (BM) and geometric Bonferroni mean (WBM) are the aggregation operators which generalize arithmetic mean and geometric mean respectively [35]. BM and WBM represent the interrelationships between the arguments of individuals and have some properties discussed by Yager [36], Xu and Yager [37] and Xia *et al.* [39]. Multi-valued neutrosophic Bonferroni mean (MVNBM) was defined by Liu *et al.* [3] and some of its applications in multiple attribute group decision-making are also presented. Hesitant fuzzy Bonferroni mean (HFBM) was defined by Zhu *et al.* [38] which facilitates to calculate BM for hesitant fuzzy elements.

Beg *et al.* [41] utilized HFBM to analyze the human attitude by developing fuzzy soft differential equations. This investigation along with some others [42, 48] guides us to think about the changes in attitudes or experts' interpersonal influences after the decisions.

In this paper multi-valued m-polar neutrosophic set (MVmNS) is defined by combining the multi-valued neutrosophic set (MVNS) and m-polar neutrosophic set (mNS). Then operational laws are defined for its elements which lead to formulate multi-valued m-polar neutrosophic Bonferroni mean (MVmNBM) and multi-valued m-polar neutrosophic weighted Bonferroni mean (MVmNWBM) operators which are the extension of multi-valued neutrosophic Bonferroni mean (MVNBM) and multi-valued neutrosophic weighted Bonferroni mean (MVNWBM) operators [3] respectively. Then by utilizing the score values of MVmNWBM and coefficients of relative closeness obtained through TOPSIS for each alternative, a system of fuzzy soft differential equations is constructed to observe the change in experts' attitudes. Another contribution of this research work is the utilization of system of differential equations with fuzzy initial conditions.

2. Preliminaries

2.1. *Neutrosophic Set*

Neutrosophy is a branch of Philosophy and a basis of neutrosophic set. Neutrosophy considers a unit "A" in relation to "anti-A" and "neither A nor anti-A". Smarandache presented the neutrosophic set with some applications [1].

Definition 2.1. [2] Let Z be a universal set. A single valued neutrosophic set (SVNS) X is defined as:

$X = \{z, (T_X(z), I_X(z), F_X(z)) : z \in Z\}$,

where, $T_X(z)$, $I_X(z)$ and $F_X(z)$ are three real values in $[0, 1]$, denoting the truth-membership degree, the indeterminacy-membership degree and the falsity-membership degree of the element $z \in Z$ to the set X respectively, satisfying

$0 \leq T_X(z) + I_X(z) + F_X(z) \leq 3$

for all $z \in Z$.

Definition 2.2. [3] Let Z be a universal set. An MVNS X is defined as:

$X = \{z, (\widetilde{T}_X(z), \widetilde{I}_X(z), \widetilde{F}_X(z)) : z \in Z\}$,

where, $\widetilde{T}_X(z)$, $\widetilde{I}_X(z)$ and $\widetilde{F}_X(z)$ are three collections of discrete real values in $[0, 1]$, denoting the truth-membership degree, the indeterminacy-membership degree and the falsity-membership degree of the element $z \in Z$ to the set X respectively, satisfying

$0 \leq \gamma, \mu, \varphi \leq 1$, $0 \leq \gamma^+ + \mu^+ + \varphi^+ \leq 3$ and $\gamma \in \widetilde{T}_X(z)$, $\gamma^+ \in \sup \widetilde{T}_X(z)$, $\mu \in \widetilde{I}_X(z)$, $\mu^+ \in \sup \widetilde{I}_X(z)$, $\varphi \in \widetilde{F}_X(z)$, $\varphi^+ \in \sup \widetilde{F}_X(z)$.

An element $\widetilde{n}$ of an MVNS X can have the following expression:
$\widetilde{n} = \left(\widetilde{T}_X(z), \widetilde{I}_X(z), \widetilde{F}_X(z)\right)$ for some $z \in Z$, where
$\widetilde{T}_X(z) = \{\gamma, \gamma \in [0,1]\}$,
$\widetilde{I}_X(z) = \{\mu, \mu \in [0,1]\}$,
$\widetilde{F}_X(z) = \{\varphi, \varphi \in [0,1]\}$.

Definition 2.3. [3] Let $\widetilde{n}_1 = \left(\widetilde{T}_1, \widetilde{I}_1, \widetilde{F}_1\right)$ and $\widetilde{n}_2 = \left(\widetilde{T}_2, \widetilde{I}_2, \widetilde{F}_2\right)$ be two elements of an MVNS, then their operational laws are defined as follows:

(1) $\widetilde{n}_1 \oplus \widetilde{n}_2 = \left(\widetilde{T}_1 \oplus \widetilde{T}_2, \widetilde{I}_1 \otimes \widetilde{I}_2, \widetilde{F}_1 \otimes \widetilde{F}_2\right)$

$$= \bigcup_{\substack{\gamma_1 \in \widetilde{T}_1, \mu_1 \in \widetilde{I}_1, \varphi_1 \in \widetilde{F}_1 \\ \gamma_2 \in \widetilde{T}_2, \mu_2 \in \widetilde{I}_2, \varphi_2 \in \widetilde{F}_2}} (\gamma_1 + \gamma_2 - \gamma_1\gamma_2, \mu_1\mu_2, \varphi_1\varphi_2)$$

(2) $\widetilde{n}_1 \otimes \widetilde{n}_2 = \left(\widetilde{T}_1 \otimes \widetilde{T}_2, \widetilde{I}_1 \oplus \widetilde{I}_2, \widetilde{F}_1 \oplus \widetilde{F}_2\right)$

$$= \bigcup_{\substack{\gamma_1 \in \widetilde{T}_1, \mu_1 \in \widetilde{I}_1, \varphi_1 \in \widetilde{F}_1 \\ \gamma_2 \in \widetilde{T}_2, \mu_2 \in \widetilde{I}_2, \varphi_2 \in \widetilde{F}_2}} (\gamma_1\gamma_2, \mu_1 + \mu_2 - \mu_1\mu_2, , \varphi_1 + \varphi_2 - \varphi_1\varphi_2)$$

(3) $k\, \widetilde{n}_1 = \bigcup_{\gamma_1 \in \widetilde{T}_1, \mu_1 \in \widetilde{I}_1, \varphi_1 \in \widetilde{F}_1} \left(1 - (1-\gamma_1)^k, \mu_1^k, \varphi_1^k\right), k > 0$

(4) $\widetilde{n}_1^k = \bigcup_{\gamma_1 \in \widetilde{T}_1, \mu_1 \in \widetilde{I}_1, \varphi_1 \in \widetilde{F}_1} \left(\gamma_1^k, 1 - (1-\mu_1)^k, 1 - (1-\varphi_1)^k\right), k > 0$

For many real world problems (e.g. ordering results of a journal, ordering results of an institute and inclusion degrees), multipolar information exists. The notion of m-polar fuzzy set was put forward to deal with those problems where m is an arbitrary ordinal number [34].

2.2. *Multi-Polar Neutrosophic Set*

Definition 2.4. [9] An m-polar neutrosophic set (mNS) on a universal set Z is a mapping

$X = \{(s_1 \circ T_X(z), s_2 \circ T_X(z), ..., s_m \circ T_X(z)), (s_1 \circ I_X(z), s_2 \circ I_X(z), ..., s_m \circ I_X(z)), (s_1 \circ F_X(z), s_2 \circ F_X(z), ..., s_m \circ F_X(z))\} : Z \longrightarrow ([0,1]^m, [0,1]^m, [0,1]^m)$

where ith mapping is defined as
$s_i \circ T_X : Z \longrightarrow [0,1]$
$s_i \circ I_X : Z \longrightarrow [0,1]$
$s_i \circ F_X : Z \longrightarrow [0,1]$
and $0 \leq s_i \circ T_X(z) + s_i \circ I_X(z) + s_i \circ F_X(z) \leq 3$
for all $i = 1, 2, ..., m$ and $z \in Z$.

Example 2.5. Let $Z = \{z_1, z_2, z_3\}$ be a universal set. Then

$$X = \left\{ \begin{array}{l} ((0.4, 0.6, 0.7), (0.1, 0.2, 0.3), (0.3, 0.5, 0.6))/z_1 \\ ((0.2, 0.4, 0.5), (0.6, 0.7, 0.8), (0.7, 0.8, 0.9))/z_2 \\ ((0.2, 0.5, 0.6), (0.3, 0.4, 0.6), (0.4, 0.6, 0.8))/z_3 \end{array} \right\}$$

represents an 3-polar neutrosophic set (3NS).

2.3. *Neutrosophic Soft Set*

Let Z be a universal set and E be the set of attributes of elements in Z. Take X to be a subset of E.

Definition 2.6. [4] An neutrosophic soft set (NSS) (ω, X) over Z is a mapping from X to $P(Z)$ and is defined as

$\Omega_X = (\omega, X) = \{(e, \omega_X(e)) : e \in E, \omega_X(e) \in P(Z)\}$

where $P(Z)$ denotes the collection of all neutrosophic subsets of Z,

$\omega_X(e) = \{z, T_X(e)(z), I_X(e)(z), F_X(e)(z) : z \in Z\}$

and each of $T_X(e)(z)$, $I_X(e)(z)$ and $F_X(e)(z)$ is a mapping from Z to interval $[0, 1]$ with

$0 \leq T_X(e)(z) + I_X(e)(z) + F_X(e)(z) \leq 3$

for all $e \in E$ and $z \in Z$.

Definition 2.7. [5] An m-polar neutrosophic soft set (mNSS) (ω, X) over Z is a mapping from X to $P(Z)$ and is defined as

$\Omega_X = (\omega, X) = \{(e, \omega_X(e)) : e \in E, \omega_X(e) \in P(Z)\}$

where $P(Z)$ denotes collection of all neutrosophic subsets of Z,

$\omega_X(e) = \{z, s_i \circ T_X(e)(z), s_i \circ I_X(e)(z), s_i \circ F_X(e)(z) : z \in Z\}$

and

$0 \leq s_i \circ T_X(e)(z) + s_i \circ I_X(e)(z) + s_i \circ F_X(e)(z) \leq 3$

for all $i = 1, 2, ..., m$; $e \in E, z \in Z$.

2.4. *Bonferroni mean operator*

Definition 2.8. [37] Let l, m be two natural numbers and $x_i \geq 0$ where $i \in \{1, 2, ..., n\}$ then Bonferroni mean $B^{l,m}$is defined as follow:

$$B^{l,m} = \left(\frac{1}{n(n-1)} \sum_{i,j=1, i \neq j}^{n} x_i^l x_j^m \right)^{\frac{1}{l+m}}.$$

Definition 2.9. [37] Let l, m be two natural numbers and $x_i \geq 0$ $(i = 1, 2, ..., n)$ and $W = [w_i \geq 0]^T$ be the weight vector of $[x_i]$ with the condition $\sum_{i=1}^{n} w_i = 1$, then weighted Bonferroni mean (WBM) is defined as follow:

$$WBM^{l,m} = \left(\frac{1}{n(n-1)} \sum_{i,j=1, i\neq j}^{n} (w_i x_i)^l (w_j x_j)^m \right)^{\frac{1}{l+m}}.$$

2.5. *Fuzzy Differential Equations*

2.5.1. *Fuzzy Numbers and Fuzzy Functions*

Definition 2.10. [6] A fuzzy number x is defined by a pair $x = (\underline{x}, \overline{x})$ of functions $\underline{x}, \overline{x} : [0,1] \longrightarrow R$, satisfying the three conditions:

(1) $\underline{x}(\alpha)$ is a bounded, monotonically increasing left-continuous function for all $\alpha \in (0,1]$ and right-continous for $\alpha = 0$,
(2) $\overline{x}(\alpha)$ is a bounded, monotonically decreasing left-continuous function for all $\alpha \in (0,1]$ and right-continous for $\alpha = 0$,
(3) For all $\alpha \in (0,1]$ we have: $\underline{x} \leq \overline{x}$.

For every $x = (\underline{x}, \overline{x})$, $y = (\underline{y}, \overline{y})$ and $k > 0$, $\alpha \in (0,1]$, we define addition and multiplication as follows:

- $\underline{(x+y)}(\alpha) = \underline{x}(\alpha) + \underline{y}(\alpha)$,
- $\overline{(x+y)}(\alpha) = \overline{x}(\alpha) + \overline{y}(\alpha)$,
- $(\underline{kx})(\alpha) = k\underline{x}(\alpha)$,
- $(\overline{kx})(\alpha) = k\overline{x}(\alpha)$.

With this definition of addition and multiplication, the collection of all fuzzy numbers is denoted by E^1. For $0 < \alpha \leq 1$, we define $\alpha-$cuts of fuzzy number u with $[x]^\alpha = \{u \in R \mid x(u) \geq \alpha\}$ and for $\alpha = 0$, the support of x is defined as $[x]^0 = \{u \in R \mid x(u) > 0\}$.

Definition 2.11. [6] Let $x = (\underline{x}, \overline{x})$ and $y = (\underline{y}, \overline{y})$ be two arbitrary numbers, then distance between x and y is defined as follows:

$$d(x,y) = \sup_{\alpha \in [0,1]} \{\max[\mid \underline{x(\alpha)} - \underline{y(\alpha)} \mid, \mid \overline{x(\alpha)} - \overline{y(\alpha)} \mid]\}.$$

Definition 2.12. [6] A fuzzy function $g : R^1 \to E^1$ is said to be continuous if for an arbitrary fixed $t_\circ \in R^1$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that:

$$|t - t_\circ| < \delta \Rightarrow d(g(t), g(\hat{t})) < \varepsilon,$$

then g is said to be continuous.

Definition 2.13. [6] Let $x, y \in E^1$. If there exists $z \in E^1$ such that $x = y + z$ then z is called the H-difference of x, y and it is denoted by $x - y$.

Definition 2.14. [6] A function $g : (c, d) \to E^1$ is said to be H-differentiable at $t_\circ \in (c, d)$ if for a small $h > 0$, there exist the H-differences $g(t_\circ) - g(t_\circ - h)$, $g(t_\circ + h) - g(t_\circ)$ and the element $g'(t_\circ) \in E^1$ such that:

$$0 = \lim_{h\to 0^+} d\left(\frac{g(t_\circ) - g(t_\circ - h)}{h}, g'(t_\circ)\right) = \lim_{h\to 0^+} d\left(\frac{g(t_\circ + h) - g(t_\circ)}{h}, g'(t_\circ)\right),$$

then $g'(t_\circ)$ is called the fuzzy derivative of g at $t_\circ$.

Definition 2.15. [7] The triangular fuzzy numbers are common and are denoted by $x = (\alpha, c, \beta)$ and defined by:

$$x = \begin{cases} \frac{u-\alpha}{c-\alpha}, & \text{if } \alpha \leq u \leq c, \\ \frac{\beta-u}{\beta-c}, & \text{if } c \leq u \leq \beta, \\ 0, & \text{otherwise.} \end{cases}$$

2.5.2. *First Order Fuzzy Differential Equations*

A first order fuzzy differential equation is written in the following form:

$$x'(t) = g(t, x(t))$$

where $g(t, x)$ is a fuzzy function of the crisp variable t and the fuzzy variable x and x is a fuzzy function of t. Here x' is the fuzzy derivative of x. Consider the initial value problem

$$x'(t) = g(t, x(t)), \qquad x(t_\circ) = x_\circ, \tag{1}$$

a mapping $x : R^1 \to E^1$ is a solution to the problem (1) if and only if it is continuous and satisfies the integral equation

$$x(t) = x_\circ + \int_{t_\circ}^{t} g(s, x(s))ds$$

for all $t \in R^1$ [8]. Moreover, sufficient conditions for the existence of a unique solution to Eq (1) are:

- f is continuous,
- A Lipschitz condition $d(g(t, x), g(t, y)) \leq Ld(x, y)$ satisfied for some $L > 0$.

To obtain the solution of Eq (1), it can be replaced by the following equivalent system:

$$\begin{aligned} \underline{x}'(t) &= \underline{g}(t, x(t)), \qquad \underline{x}(t_\circ) = \underline{x_\circ}, \\ \overline{x}'(t) &= \overline{g}(t, x(t)), \qquad \overline{x}(t_\circ) = \overline{x_\circ}. \end{aligned}$$

For example, to solve

$$\tfrac{dx}{dt} = t^2 x, \quad x(0) = (0, \tfrac{1}{2}, 1),$$

it is replaced by

$$\begin{aligned} \tfrac{d\underline{x}}{dt} &= t^2 \underline{x}, \quad \underline{x}(0) = (0, \tfrac{1}{2}), \\ \tfrac{d\overline{x}}{dt} &= t^2 \overline{x}, \quad \overline{x}(0) = (\tfrac{1}{2}, 1), \end{aligned}$$

$\underline{x}(0) = (0, \frac{1}{2})$ and $\overline{x}(0) = (\frac{1}{2}, 1)$ are replaced by parametric forms $\underline{x}(0) = 2\alpha$ and $\overline{x}(0) = 2(1-\alpha)$, respectively where $\alpha \in [0, 1]$. And the solution is:

$$x = (2\alpha e^{\frac{t^3}{3}}, 2(1-\alpha)e^{\frac{t^3}{3}}), \quad \alpha \in [0, 1].$$

2.6. *Human attitude analysis after a decision*

The constants which provide the base to rank the alternatives, can also provide a support for further process of rethinking after a decision. These constants were utilized in fuzzy soft differential equations [41] and in developing the influence matrix which can play a vital role in influence model and doubly extended TOPSIS [42, 48].

2.6.1. *Human attitude analysis based on fuzzy soft differential equations*

A system of linear fuzzy soft differential equations is developed by Beg et al. [41].

$$\begin{aligned} \frac{dP_1}{dt} &= a^1_{P_1}P_1 + a^1_{P_2}P_2, \\ \frac{dP_2}{dt} &= a^2_{P_1}P_1 + a^2_{P_2}P_2 \end{aligned} \tag{2}$$

where P_1 and P_2 are the variables representing the attitude of two persons after taking a decision at time t, $\frac{dP_1}{dt}$ and $\frac{dP_2}{dt}$ represent the change in persons' attitudes after some time due to that decision and $a^i_{P_j}$ $(i, j = 1, 2)$ are optimum fuzzy soft constants (OFSCs) taken as signed fuzzy numbers denoting the influence on ith person of his internal feelings and jth person's feelings. Positive sign is assigned to $a^i_{P_j}$ when the attitude of jth person for ith person is supportive, otherwise a negative sign is assigned to it. Stability of system (2) depends upon eigen values of the matrix $\begin{bmatrix} a^1_{P_1} & a^1_{P_2} \\ a^2_{P_1} & a^2_{P_2} \end{bmatrix}$

3. **Multi valued Multi-Polar Neutrosophic Set**

Multi-valued multi-polar neutrosophic set (MVmNS) is a generalization and composition of MVNS and mNS.

Definition 3.1. Let Z be a non empty set. An MVmNS X is a mapping defined as

$$X : Z \to \begin{pmatrix} m \text{ sets of discrete values in } [0,1], \\ m \text{ sets of discrete values in } [0,1], \\ m \text{ sets of discrete values in } [0,1] \end{pmatrix}$$

$$X(z) = \begin{pmatrix} \left(s_1 \circ \widetilde{T}_X(z), s_2 \circ \widetilde{T}_X(z), ..., s_m \circ \widetilde{T}_X(z)\right), \\ \left(s_1 \circ \widetilde{I}_X(z), s_2 \circ \widetilde{I}_X(z), ..., s_m \circ \widetilde{I}_X(z)\right), \\ \left(s_1 \circ \widetilde{F}_X(z), s_2 \circ \widetilde{F}_X(z), ..., s_m \circ \widetilde{F}_X(z)\right) \end{pmatrix}$$

where $s_i \circ \widetilde{T}_X(z)$, $s_i \circ \widetilde{I}_X(z)$ and $s_i \circ \widetilde{F}_X(z)$ $(i = 1, 2, ..., m)$ are the collections of discrete real values γ_i, μ_i and φ_i denoting the truth-membership degree, the indeterminacy-membership degree and the falsity-membership degree of the element $z \in Z$ to the set X respectively with $0 \leq \gamma_i,\ \mu_i,\ \varphi_i \leq 1$, $0 \leq \gamma_i^+ + \mu_i^+ + \varphi_i^+ \leq 3$, $\gamma_i^+ \in \sup\left(s_i \circ \widetilde{T}_X(z)\right)$, $\mu_i^+ \in \sup\left(s_i \circ \widetilde{I}_X(z)\right)$, $\varphi_i^+ \in \sup\left(s_i \circ \widetilde{F}_X(z)\right)$.

3.1. *Multi valued Multi-Polar Neutrosophic Soft Set*

Definition 3.2. Let Z be a universal set and E be a set of parameters with $X \subseteq E$. Define $\omega : X \to P(Z)$, where $P(Z)$ is the collection of all MVmN subsets of the set Z. Then (ω, X) is said to be an multi-valued m-polar neutrosophic soft set (MVmNSS) over Z which is represented as $\Omega_X = (\omega, X) = \{e, \omega_X(e) : e \in E, \omega_X(e) \in P(Z)\}$ and $\omega_X(e)$ is an MVmNS over Z.

3.2. *Operations on MVmNS*

$$\text{Let } X(z_1) = \begin{pmatrix} \left(s_1 \circ \widetilde{T}_X(z_1), s_2 \circ \widetilde{T}_X(z_1), ..., s_m \circ \widetilde{T}_X(z_1)\right), \\ \left(s_1 \circ \widetilde{I}_X(z_1), s_2 \circ \widetilde{I}_X(z_1), ..., s_m \circ \widetilde{I}_X(z_1)\right), \\ \left(s_1 \circ \widetilde{F}_X(z_1), s_2 \circ \widetilde{F}_X(z_1), ..., s_m \circ \widetilde{F}_X(z_1)\right) \end{pmatrix}$$

$$= \left(\left(\cup\{\gamma_1^{(z_1)}\}, ..., \cup\{\gamma_m^{(z_1)}\}\right), \left(\cup\{\mu_1^{(z_1)}\}, ..., \cup\{\mu_m^{(z_1)}\}\right), \left(\cup\{\varphi_1^{(z_1)}\}, ..., \cup\{\varphi_m^{(z_1)}\}\right)\right)$$

$$\text{and } X(z_2) = \begin{pmatrix} \left(s_1 \circ \widetilde{T}_X(z_2), s_2 \circ \widetilde{T}_X(z_2), ..., s_m \circ \widetilde{T}_X(z_2)\right), \\ \left(s_1 \circ \widetilde{I}_X(z_2), s_2 \circ \widetilde{I}_X(z_2), ..., s_m \circ \widetilde{I}_X(z_2)\right), \\ \left(s_1 \circ \widetilde{F}_X(z_2), s_2 \circ \widetilde{F}_X(z_2), ..., s_m \circ \widetilde{F}_X(z_2)\right) \end{pmatrix}$$

$$= \left(\left(\cup\{\gamma_1^{(z_2)}\}, ..., \cup\{\gamma_m^{(z_2)}\}\right), \left(\cup\{\mu_1^{(z_2)}\}, ..., \cup\{\mu_m^{(z_2)}\}\right), \left(\cup\{\varphi_1^{(z_2)}\}, ..., \cup\{\varphi_m^{(z_2)}\}\right)\right)$$

be two elements of an MVmNS. Then their operational laws are defined as

$$(1)\ \ (X(z_1))^c = \begin{pmatrix} \left(s_1 \circ \widetilde{F}_X(z_1), s_2 \circ \widetilde{F}_X(z_1), ..., s_m \circ \widetilde{F}_X(z_1)\right), \\ \left(s_1 \circ \widetilde{I}'_X(z_1), s_2 \circ \widetilde{I}'_X(z_1), ..., s_m \circ \widetilde{I}'_X(z_1)\right), \\ \left(s_1 \circ \widetilde{T}_X(z_1), s_2 \circ \widetilde{T}_X(z_1), ..., s_m \circ \widetilde{T}_X(z_1)\right) \end{pmatrix}$$

$$= \left(\left(\cup\{\varphi_1^{(z_1)}\}, ..., \cup\{\varphi_m^{(z_1)}\}\right), \left(\cup\{1-\mu_1^{(z_1)}\}, ..., \cup\{1-\mu_m^{(z_1)}\}\right), \left(\cup\{\gamma_1^{(z_1)}\}, ..., \cup\{\gamma_m^{(z_1)}\}\right)\right).$$

(2) $X(z_1) \oplus X(z_2)$

$$= \begin{pmatrix} \left(s_1 \circ \widetilde{T}_X(z_1), s_2 \circ \widetilde{T}_X(z_1), ..., s_m \circ \widetilde{T}_X(z_1)\right), \\ \left(s_1 \circ \widetilde{I}_X(z_1), s_2 \circ \widetilde{I}_X(z_1), ..., s_m \circ \widetilde{I}_X(z_1)\right), \\ \left(s_1 \circ \widetilde{F}_X(z_1), s_2 \circ \widetilde{F}_X(z_1), ..., s_m \circ \widetilde{F}_X(z_1)\right) \end{pmatrix} \oplus \begin{pmatrix} \left(s_1 \circ \widetilde{T}_X(z_2), s_2 \circ \widetilde{T}_X(z_2), ..., s_m \circ \widetilde{T}_X(z_2)\right), \\ \left(s_1 \circ \widetilde{I}_X(z_2), s_2 \circ \widetilde{I}_X(z_2), ..., s_m \circ \widetilde{I}_X(z_2)\right), \\ \left(s_1 \circ \widetilde{F}_X(z_2), s_2 \circ \widetilde{F}_X(z_2), ..., s_m \circ \widetilde{F}_X(z_2)\right) \end{pmatrix}$$

$$= \begin{pmatrix} \left(s_1 \circ \widetilde{T}_X(z_1), s_2 \circ \widetilde{T}_X(z_1), ..., s_m \circ \widetilde{T}_X(z_1)\right) \oplus \left(s_1 \circ \widetilde{T}_X(z_2), s_2 \circ \widetilde{T}_X(z_2), ..., s_m \circ \widetilde{T}_X(z_2)\right), \\ \left(s_1 \circ \widetilde{I}_X(z_1), s_2 \circ \widetilde{I}_X(z_1), ..., s_m \circ \widetilde{I}_X(z_1)\right) \otimes \left(s_1 \circ \widetilde{I}_X(z_2), s_2 \circ \widetilde{I}_X(z_2), ..., s_m \circ \widetilde{I}_X(z_2)\right), \\ \left(s_1 \circ \widetilde{F}_X(z_1), s_2 \circ \widetilde{F}_X(z_1), ..., s_m \circ \widetilde{F}_X(z_1)\right) \otimes \left(s_1 \circ \widetilde{F}_X(z_2), s_2 \circ \widetilde{F}_X(z_2), ..., s_m \circ \widetilde{F}_X(z_2)\right) \end{pmatrix}$$

$$= \begin{pmatrix} \left(\left(s_1 \circ \widetilde{T}_X(z_1)\right) \oplus \left(s_1 \circ \widetilde{T}_X(z_2)\right), ..., \left(s_m \circ \widetilde{T}_X(z_1)\right) \oplus \left(s_m \circ \widetilde{T}_X(z_2)\right)\right), \\ \left(\left(s_1 \circ \widetilde{I}_X(z_1)\right) \otimes \left(s_1 \circ \widetilde{I}_X(z_2)\right), ..., \left(s_m \circ \widetilde{I}_X(z_1)\right) \otimes \left(s_m \circ \widetilde{I}_X(z_2)\right)\right), \\ \left(\left(s_1 \circ \widetilde{F}_X(z_1)\right) \otimes \left(s_1 \circ \widetilde{F}_X(z_2)\right), ..., \left(s_m \circ \widetilde{F}_X(z_1)\right) \otimes \left(s_m \circ \widetilde{F}_X(z_2)\right)\right) \end{pmatrix}$$

$$= \begin{pmatrix} \left(\cup\{\gamma_1^{(z_1)} + \gamma_1^{(z_2)} - \gamma_1^{(z_1)}\gamma_1^{(z_2)}\}, ..., \cup\{\gamma_m^{(z_1)} + \gamma_m^{(z_2)} - \gamma_m^{(z_1)}\gamma_m^{(z_2)}\}\right), \\ \left(\cup\{\mu_1^{(z_1)}\mu_1^{(z_2)}\}, ..., \cup\{\mu_m^{(z_1)}\mu_m^{(z_2)}\}\right), \\ \left(\cup\{\varphi_1^{(z_1)}\varphi_1^{(z_2)}\}, ..., \cup\{\varphi_m^{(z_1)}\varphi_m^{(z_2)}\}\right) \end{pmatrix}.$$

(3) $X(z_1) \otimes X(z_2)$

$$= \begin{pmatrix} \left(s_1 \circ \widetilde{T}_X(z_1), s_2 \circ \widetilde{T}_X(z_1), ..., s_m \circ \widetilde{T}_X(z_1)\right), \\ \left(s_1 \circ \widetilde{I}_X(z_1), s_2 \circ \widetilde{I}_X(z_1), ..., s_m \circ \widetilde{I}_X(z_1)\right), \\ \left(s_1 \circ \widetilde{F}_X(z_1), s_2 \circ \widetilde{F}_X(z_1), ..., s_m \circ \widetilde{F}_X(z_1)\right) \end{pmatrix} \otimes \begin{pmatrix} \left(s_1 \circ \widetilde{T}_X(z_2), s_2 \circ \widetilde{T}_X(z_2), ..., s_m \circ \widetilde{T}_X(z_2)\right), \\ \left(s_1 \circ \widetilde{I}_X(z_2), s_2 \circ \widetilde{I}_X(z_2), ..., s_m \circ \widetilde{I}_X(z_2)\right), \\ \left(s_1 \circ \widetilde{F}_X(z_2), s_2 \circ \widetilde{F}_X(z_2), ..., s_m \circ \widetilde{F}_X(z_2)\right) \end{pmatrix}$$

$$= \begin{pmatrix} \left(s_1 \circ \widetilde{T}_X(z_1), s_2 \circ \widetilde{T}_X(z_1), ..., s_m \circ \widetilde{T}_X(z_1)\right) \otimes \left(s_1 \circ \widetilde{T}_X(z_2), s_2 \circ \widetilde{T}_X(z_2), ..., s_m \circ \widetilde{T}_X(z_2)\right), \\ \left(s_1 \circ \widetilde{I}_X(z_1), s_2 \circ \widetilde{I}_X(z_1), ..., s_m \circ \widetilde{I}_X(z_1)\right) \oplus \left(s_1 \circ \widetilde{I}_X(z_2), s_2 \circ \widetilde{I}_X(z_2), ..., s_m \circ \widetilde{I}_X(z_2)\right), \\ \left(s_1 \circ \widetilde{F}_X(z_1), s_2 \circ \widetilde{F}_X(z_1), ..., s_m \circ \widetilde{F}_X(z_1)\right) \oplus \left(s_1 \circ \widetilde{F}_X(z_2), s_2 \circ \widetilde{F}_X(z_2), ..., s_m \circ \widetilde{F}_X(z_2)\right) \end{pmatrix}$$

$$= \begin{pmatrix} \left(\left(s_1 \circ \widetilde{T}_X(z_1)\right) \otimes \left(s_1 \circ \widetilde{T}_X(z_2)\right), ..., \left(s_m \circ \widetilde{T}_X(z_1)\right) \otimes \left(s_m \circ \widetilde{T}_X(z_2)\right)\right), \\ \left(\left(s_1 \circ \widetilde{I}_X(z_1)\right) \oplus \left(s_1 \circ \widetilde{I}_X(z_2)\right), ..., \left(s_m \circ \widetilde{I}_X(z_1)\right) \oplus \left(s_m \circ \widetilde{I}_X(z_2)\right)\right), \\ \left(\left(s_1 \circ \widetilde{F}_X(z_1)\right) \oplus \left(s_1 \circ \widetilde{F}_X(z_2)\right), ..., \left(s_m \circ \widetilde{F}_X(z_1)\right) \oplus \left(s_m \circ \widetilde{F}_X(z_2)\right)\right) \end{pmatrix}$$

$$= \begin{pmatrix} \left(\cup\{\gamma_1^{(z_1)}\gamma_1^{(z_2)}\}, ..., \cup\{\gamma_m^{(z_1)}\gamma_m^{(z_2)}\}\right), \\ \left(\cup\{\mu_1^{(z_1)} + \mu_1^{(z_2)} - \mu_1^{(z_1)}\mu_1^{(z_2)}\}, ..., \cup\{\mu_m^{(z_1)} + \mu_m^{(z_2)} - \mu_m^{(z_1)}\mu_m^{(z_2)}\}\right), \\ \left(\cup\{\varphi_1^{(z_1)} + \varphi_1^{(z_2)} - \varphi_1^{(z_1)}\varphi_1^{(z_2)}\}, ..., \cup\{\varphi_m^{(z_1)} + \varphi_m^{(z_2)} - \varphi_m^{(z_1)}\varphi_m^{(z_2)}\}\right) \end{pmatrix}.$$

(4) $$kX(z_1) = \begin{pmatrix} \left(\cup\{1 - \left(1 - \gamma_1^{(z_1)}\right)^k\}, ..., \cup\{1 - \left(1 - \gamma_m^{(z_1)}\right)^k\}\right), \\ \left(\cup\{\left(\mu_1^{(z_1)}\right)^k\}, ..., \cup\{\left(\mu_m^{(z_1)}\right)^k\}\right), \\ \left(\cup\{\left(\varphi_1^{(z_1)}\right)^k\}, ..., \cup\{\left(\varphi_m^{(z_1)}\right)^k\}\right) \end{pmatrix}.$$

$$(5)\ \ (X(z_1))^k = \begin{pmatrix} \left(\cup\{\left(\gamma_1^{(z_1)}\right)^k\}, ..., \cup\{\left(\gamma_m^{(z_1)}\right)^k\}\right), \\ \left(\cup\{1-\left(1-\mu_1^{(z_1)}\right)^k\}, ..., \cup\{1-\left(1-\mu_m^{(z_1)}\right)^k\}\right), \\ \left(\cup\{1-\left(1-\varphi_1^{(z_1)}\right)^k\}, ..., \cup\{1-\left(1-\varphi_m^{(z_1)}\right)^k\}\right) \end{pmatrix}.$$

Example 3.3. Let

$x_1 = ((\{0.3\}, \{0.4, 0.5\}, \{0.5, 0.6\}), (\{0.4, 0.5\}, \{0.2\}, \{0.7\}), (\{0.3\}, \{0.5\}, \{0.8\}))$ and

$x_2 = ((\{0.1\}, \{0.3\}, \{0.6\}), (\{0.2, 0.4\}, \{0.6\}, \{0.7, 0.8\}), (\{0.5\}, \{0.6\}, \{0.7, 0.8\}))$

be two elements of an MV3NS. Then

$(x_1)^c = ((\{0.3\}, \{0.5\}, \{0.8\}), (\{0.6, 0.5\}, \{0.8\}, \{0.3\}), (\{0.3\}, \{0.4, 0.5\}, \{0.5, 0.6\}))$,

$$x_1 \oplus x_2 = \begin{pmatrix} (\{0.37\}, \{0.58, 0.65\}, \{0.8, 0.84\}), \\ (\{0.08, 0.2\}, \{0.12\}, \{0.49, 0.56\}), \\ (\{0.15\}, \{0.30\}, \{0.56, 0.64\}) \end{pmatrix},$$

$$x_1 \otimes x_2 = \begin{pmatrix} (\{0.03\}, \{0.12, 0.15\}, \{0.3, 0.36\}), \\ (\{0.52, 0.7\}, \{0.68\}, \{0.91, 0.94\}), \\ (\{0.65\}, \{0.8\}, \{0.94, 0.96\}) \end{pmatrix},$$

$$2x_1 = \begin{pmatrix} (\{0.51\}, \{0.64, 0.75\}, \{0.75, 0.84\}), \\ (\{0.16, 0.25\}, \{0.04\}, \{0.49\}), (\{0.09\}, \{0.25\}, \{0.64\}) \end{pmatrix},$$

$$(x_1)^2 = \begin{pmatrix} (\{0.09\}, \{0.16, 0.25\}, \{0.25, 0.36\}), \\ (\{0.64, 0.75\}, \{0.36\}, \{0.91\}), (\{0.51\}, \{0.75\}, \{0.96\}) \end{pmatrix}.$$

Definition 3.4. Score function $s(X(z))$ and accuracy function $a(X(z))$ of an element $X(z)$ of an MVmNS is defined as follows:

$$s(X(z)) = \frac{1}{m}\sum_{i=1}^{m}\left(\frac{1}{l_{iT}l_{iI}l_{iF}}\sum\left(\frac{\gamma_i - \mu_i - \varphi_i}{3}\right)\right),$$
$$a(X(z)) = \frac{1}{m}\sum_{i=1}^{m}\left(\frac{1}{l_{iT}l_{iI}l_{iF}}\sum\left(\frac{\gamma_i + \mu_i + \varphi_i}{3}\right)\right),$$

where $\gamma_i \in s_i \circ \widetilde{T}_X(z)$, $\mu_i \in s_i \circ \widetilde{I}_X(z)$, $\varphi_i \in s_i \circ \widetilde{F}_X(z)$ and l_{iT}, l_{iI}, l_{iF} are the number of elements in $s_i \circ \widetilde{T}_X(z)$, $s_i \circ \widetilde{I}_X(z)$ and $s_i \circ \widetilde{F}_X(z)$ respectively.

It can be observed that, the score function and accuracy function satisfy the following properties:

(1) For an element $X(z)$ of an MVmNS,
$$-\frac{2}{3} \le s(X(z)) \le \frac{1}{3}.$$

(2) For an element $X(z)$ of an MVmNS,
$$0 \le a(X(z)) \le 1.$$

Definition 3.5. Two elements $X(z_1)$ and $X(z_2)$ of an MVmNS are compared as:

- if $s\left(X\left(z_1\right)\right) > s\left(X\left(z_2\right)\right)$, then $X\left(z_1\right) > X\left(z_2\right)$,
- if $s\left(X\left(z_1\right)\right) = s\left(X\left(z_2\right)\right)$ and
  - if $a\left(X\left(z_1\right)\right) > a\left(X\left(z_2\right)\right)$, then $X(z_1) > X\left(z_2\right)$,
  - if $a\left(X\left(z_1\right)\right) = a\left(X\left(z_2\right)\right)$, then $X(z_1) = X\left(z_2\right)$.

Definition 3.6. Let $X(z_i)$, $(i = 1, 2, ..., n)$ be the elements of an MVmNS. Then for two natural numbers p, q, MVmNBM operator is defined as

$$\mathrm{MV}m\mathrm{NBM}^{p,q}\left(X\left(z_1\right), X\left(z_2\right), ..., X\left(z_n\right)\right) = \left(\frac{1}{n(n-1)}\left(\underset{i,j=1, j\neq i}{\overset{n}{\oplus}}\left(\left(X(z_i)\right)^p \otimes \left(X\left(z_j\right)\right)^q\right)\right)\right)^{\frac{1}{p+q}}$$

Theorem 3.7. *Let $X(z_i)$, $(i = 1, 2, ..., n)$ be n elements of an MVmNS, then MVmNBM operator can be expressed as:*

$$MVmNBM^{p,q}\left(X\left(z_1\right), X\left(z_2\right), ..., X\left(z_n\right)\right) =$$

$$\left(\begin{array}{c}
\left(\cup\left\{\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(\gamma_1^{(z_i)}\right)^p\left(\gamma_1^{(z_j)}\right)^q\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right\}, ..., \cup\left\{\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(\gamma_m^{(z_i)}\right)^p\left(\gamma_m^{(z_j)}\right)^q\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right\}\right), \\
\left(\cup\left\{1-\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\mu_1^{(z_i)}\right)^p\left(1-\mu_1^{(z_j)}\right)^q\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right\}, ..., \cup\left\{1-\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\mu_m^{(z_i)}\right)^p\left(1-\mu_m^{(z_j)}\right)^q\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right\}\right), \\
\left(\cup\left\{1-\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\varphi_1^{(z_i)}\right)^p\left(1-\varphi_1^{(z_j)}\right)^q\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right\}, ..., \cup\left\{1-\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\varphi_m^{(z_i)}\right)^p\left(1-\varphi_m^{(z_j)}\right)^q\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right\}\right)
\end{array}\right)$$

Proof. Let

$$X(z_i) = \left(\left(\cup\{\gamma_1^{(z_i)}\}, ..., \cup\{\gamma_m^{(z_i)}\}\right), \left(\cup\{\mu_1^{(z_i)}\}, ..., \cup\{\mu_m^{(z_i)}\}\right), \left(\cup\{\varphi_1^{(z_i)}\}, ..., \cup\{\varphi_m^{(z_i)}\}\right)\right)$$

and $X\left(z_j\right) = \left(\left(\cup\{\gamma_1^{(z_j)}\}, ..., \cup\{\gamma_m^{(z_j)}\}\right), \left(\cup\{\mu_1^{(z_j)}\}, ..., \cup\{\mu_m^{(z_j)}\}\right), \left(\cup\{\varphi_1^{(z_j)}\}, ..., \cup\{\varphi_m^{(z_j)}\}\right)\right)$

$$\left(X\left(z_i\right)\right)^p = \left(\begin{array}{c}
\left(\cup\{\left(\gamma_1^{(z_i)}\right)^p\}, ..., \cup\{\left(\gamma_m^{(z_i)}\right)^p\}\right), \\
\left(\cup\{1-\left(1-\mu_1^{(z_i)}\right)^p\}, ..., \cup\{1-\left(1-\mu_m^{(z_i)}\right)^p\}\right), \\
\left(\cup\{1-\left(1-\varphi_1^{(z_i)}\right)^p\}, ..., \cup\{1-\left(1-\varphi_m^{(z_i)}\right)^p\}\right)
\end{array}\right)$$

$$\left(X\left(z_j\right)\right)^q = \left(\begin{array}{c}
\left(\cup\{\left(\gamma_1^{(z_j)}\right)^q\}, ..., \cup\{\left(\gamma_m^{(z_j)}\right)^q\}\right), \\
\left(\cup\{1-\left(1-\mu_1^{(z_j)}\right)^q\}, ..., \cup\{1-\left(1-\mu_m^{(z_j)}\right)^q\}\right), \\
\left(\cup\{1-\left(1-\varphi_1^{(z_j)}\right)^q\}, ..., \cup\{1-\left(1-\varphi_m^{(z_j)}\right)^q\}\right)
\end{array}\right)$$

$$\left(X\left(z_i\right)\right)^p \otimes \left(X\left(z_j\right)\right)^q =$$

$$\left(\begin{array}{c}
\left(\cup\left\{\left(\gamma_1^{(z_i)}\right)^p\left(\gamma_1^{(z_j)}\right)^q\right\}, ..., \cup\left\{\left(\gamma_m^{(z_i)}\right)^p\left(\gamma_m^{(z_j)}\right)^q\right\}\right), \\
\left(\cup\left\{1-\left(1-\mu_1^{(z_i)}\right)^p\left(1-\mu_1^{(z_j)}\right)^q\right\}, ..., \cup\left\{1-\left(1-\mu_m^{(z_i)}\right)^p\left(1-\mu_m^{(z_j)}\right)^q\right\}\right), \\
\left(\cup\left\{1-\left(1-\varphi_1^{(z_i)}\right)^p\left(1-\varphi_1^{(z_j)}\right)^q\right\}, ..., \cup\left\{1-\left(1-\varphi_m^{(z_i)}\right)^p\left(1-\varphi_m^{(z_j)}\right)^q\right\}\right)
\end{array}\right)$$

$$\underset{i,j=1,j\neq i}{\overset{n}{\oplus}}((X(z_i))^p \otimes (X(z_j))^q) =$$

$$\left(\begin{array}{c} \left(\cup\left\{ 1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(\gamma_1^{(z_i)}\right)^p\left(\gamma_1^{(z_j)}\right)^q\right)\right\},...,\cup\left\{1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(\gamma_m^{(z_i)}\right)^p\left(\gamma_m^{(z_j)}\right)^q\right)\right\}\right), \\ \left(\cup\left\{ \prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\mu_1^{(z_i)}\right)^p\left(1-\mu_1^{(z_j)}\right)^q\right)\right\},...,\cup\left\{\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\mu_m^{(z_i)}\right)^p\left(1-\mu_m^{(z_j)}\right)^q\right)\right\}\right), \\ \left(\cup\left\{ \prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\varphi_1^{(z_i)}\right)^p\left(1-\varphi_1^{(z_j)}\right)^q\right)\right\},...,\cup\left\{\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\varphi_m^{(z_i)}\right)^p\left(1-\varphi_m^{(z_j)}\right)^q\right)\right\}\right) \end{array} \right)$$

Finally, the required result is obtained by using operations 4 and 5, presented in section 3.2.

□

Definition 3.8. Let $X(z_i)$, $(i = 1, 2, ..., n)$ be the elements of an MVmNS with weight vector $W = (w_1, w_2, ..., w_n)^T$ satisfying $w_i \geq 0$ and $\sum_{i=1}^{n} w_i = 1$. Then for two natural numbers p, q, MVmNWBM operator is defined as

$$MVmNWBM^{p,q}\left(X\left(z_1\right), X\left(z_2\right), ..., X\left(z_n\right)\right) = \left(\frac{1}{n(n-1)}\left(\underset{i,j=1,j\neq i}{\overset{n}{\oplus}}((w_iX(z_i))^p \otimes (w_jX\left(z_j\right))^q)\right)\right)^{\frac{1}{p+q}}$$

Theorem 3.9. *Let $X(z_i)$, $(i = 1, 2, ..., n)$ be n elements of an MVmNS with weight vector $W = (w_1, w_2, ..., w_n)^T$ satisfying $w_i \geq 0$ and $\sum_{i=1}^{n} w_i = 1$, then MVmNWBM operator can be expressed as:*

$$MVmNWBM^{p,q}\left(X\left(z_1\right), X\left(z_2\right), ..., X\left(z_n\right)\right) =$$

$$
\left(\begin{array}{l}
\left(\begin{array}{l}
\cup\left\{\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(1-\gamma_1^{(z_i)}\right)^{w_i}\right)^{p}\left(1-\left(1-\gamma_1^{(z_j)}\right)^{w_j}\right)^{q}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right\},\ldots,\\
\cup\left\{\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(1-\gamma_m^{(z_i)}\right)^{w_i}\right)^{p}\left(1-\left(1-\gamma_m^{(z_j)}\right)^{w_j}\right)^{q}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right\}
\end{array}\right),\\
\left(\begin{array}{l}
\cup\left\{1-\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(\mu_1^{(z_i)}\right)^{w_i}\right)^{p}\left(1-\left(\mu_1^{(z_j)}\right)^{w_j}\right)^{q}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right\},\ldots,\\
\cup\left\{1-\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(\mu_m^{(z_i)}\right)^{w_i}\right)^{p}\left(1-\left(\mu_m^{(z_j)}\right)^{w_j}\right)^{q}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right\}
\end{array}\right),\\
\left(\begin{array}{l}
\cup\left\{1-\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(\varphi_1^{(z_i)}\right)^{w_i}\right)^{p}\left(1-\left(\varphi_1^{(z_j)}\right)^{w_j}\right)^{q}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right\},\ldots,\\
\cup\left\{1-\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(\varphi_m^{(z_i)}\right)^{w_i}\right)^{p}\left(1-\left(\varphi_m^{(z_j)}\right)^{w_j}\right)^{q}\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right\}
\end{array}\right)
\end{array}\right)
$$

Proof. This result can be obtained similarly as the previous one. □

MVmNBM and MVmNWBM operators satisfy the following properties:

- Permutation
- Monotonicity
- Boundedness

Theorem 3.10. ***(Permutation)*** *Let $X(z_i)$, $(i = 1, 2, ..., n)$ be a set of n elements of an MVmNS. If $\acute{X}(z_i)$, $(i = 1, 2, ..., n)$ is a permutation of $X(z_i)$, $(i = 1, 2, ..., n)$, then*

$$MVmNWBM(X(z_1), X(z_2), ..., X(z_n)) = MVmNWBM(\acute{X}(z_1), \acute{X}(z_2), ..., \acute{X}(z_n))$$

Proof. Following result can be obtained by definition

$$\left(\frac{1}{n(n-1)}\left(\underset{i,j=1,j\neq i}{\overset{n}{\oplus}}((w_iX(z_1))^p \otimes (w_jX(z_2))^q)\right)\right)^{\frac{1}{p+q}} = \left(\frac{1}{n(n-1)}\left(\underset{i,j=1,j\neq i}{\overset{n}{\oplus}}\left(\left(w_i\acute{X}(z_1)\right)^p \otimes \left(w_j\acute{X}(z_2)\right)^q\right)\right)\right)^{\frac{1}{p+q}}.$$

Hence the result. □

Theorem 3.11. ***(Monotonicity)*** *Let $X(z_i)$, $(i = 1, 2, ..., n)$ and $\acute{X}(z_i)$, $(i = 1, 2, ..., n)$ be two sets of elements of an MVmNS. If $X(z_i) \geq \acute{X}(z_i)$ for all $i = 1, 2, ..., n$ then*

$$MVmNWBM^{p,q}(X(z_1), X(z_2), ..., X(z_n)) \geq MVmNWBM^{p,q}\left(\acute{X}(z_1), \acute{X}(z_2), ..., \acute{X}(z_n)\right).$$

Proof. Let

$X(z_i) = \left(\left(\cup\{\gamma_1^{(z_i)}\}, ..., \cup\{\gamma_m^{(z_i)}\}\right), \left(\cup\{\mu_1^{(z_i)}\}, ..., \cup\{\mu_m^{(z_i)}\}\right), \left(\cup\{\varphi_1^{(z_i)}\}, ..., \cup\{\varphi_m^{(z_i)}\}\right)\right)$

and $\acute{X}(z_i) = \left(\left(\cup\{\acute{\gamma}_1^{(z_i)}\}, ..., \cup\{\acute{\gamma}_m^{(z_i)}\}\right), \left(\cup\{\acute{\mu}_1^{(z_i)}\}, ..., \cup\{\acute{\mu}_m^{(z_i)}\}\right), \left(\cup\{\acute{\varphi}_1^{(z_i)}\}, ..., \cup\{\acute{\varphi}_m^{(z_i)}\}\right)\right)$.

(1) $X(z_i) \geq \acute{X}(z_i) \Longrightarrow \gamma_1^{(z_i)} \geq \acute{\gamma}_1^{(z_i)} \Longrightarrow 1 - \left(1 - \gamma_1^{(z_i)}\right)^{w_i} \geq 1 - \left(1 - \acute{\gamma}_1^{(z_i)}\right)^{w_i}$

$$\Longrightarrow \left(1 - \left(1 - \gamma_1^{(z_i)}\right)^{w_i}\right)^p \left(1 - \left(1 - \gamma_1^{(z_j)}\right)^{wj}\right)^q \geq \left(1 - \left(1 - \acute{\gamma}_1^{(z_i)}\right)^{w_i}\right)^p \left(1 - \left(1 - \acute{\gamma}_1^{(z_j)}\right)^{wj}\right)^q$$

for all $j = 1, 2, \ldots, m$,

$$\Longrightarrow 1 - \left(1 - \left(1 - \acute{\gamma}_1^{(z_i)}\right)^{w_i}\right)^p \left(1 - \left(1 - \acute{\gamma}_1^{(z_j)}\right)^{wj}\right)^q \leq 1 - \left(1 - \left(1 - \gamma_1^{(z_i)}\right)^{w_i}\right)^p \left(1 - \left(1 - \gamma_1^{(z_j)}\right)^{wj}\right)^q$$

$$\Longrightarrow \prod_{\substack{i,j=1 \\ i\neq j}}^{n} \left(1 - \left(1 - \left(1 - \acute{\gamma}_1^{(z_i)}\right)^{w_i}\right)^p \left(1 - \left(1 - \acute{\gamma}_1^{(z_j)}\right)^{wj}\right)^q\right) \leq \prod_{\substack{i,j=1 \\ i\neq j}}^{n} \left(1 - \left(1 - \left(1 - \gamma_1^{(z_i)}\right)^{w_i}\right)^p \left(1 - \left(1 - \gamma_1^{(z_j)}\right)^{wj}\right)^q\right)$$

$$\Longrightarrow 1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(1-\gamma_1^{(z_i)}\right)^{w_i}\right)^p\left(1-\left(1-\gamma_1^{(z_j)}\right)^{wj}\right)^q\right)\geq$$

$$1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(1-\acute{\gamma}_1^{(z_i)}\right)^{w_i}\right)^p\left(1-\left(1-\acute{\gamma}_1^{(z_j)}\right)^{wj}\right)^q\right)$$

$$\Longrightarrow\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(1-\gamma_1^{(z_i)}\right)^{w_i}\right)^p\left(1-\left(1-\gamma_1^{(z_j)}\right)^{w_j}\right)^q\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\geq$$

$$\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(1-\acute{\gamma}_1^{(z_i)}\right)^{w_i}\right)^p\left(1-\left(1-\acute{\gamma}_1^{(z_j)}\right)^{w_j}\right)^q\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}$$

Similarly

$$\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(1-\gamma_k^{(z_i)}\right)^{w_i}\right)^p\left(1-\left(1-\gamma_k^{(z_j)}\right)^{w_j}\right)^q\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\geq$$

$$\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(1-\acute{\gamma}_k^{(z_i)}\right)^{w_i}\right)^p\left(1-\left(1-\acute{\gamma}_k^{(z_j)}\right)^{w_j}\right)^q\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}$$

for $k=1,2,...,m$.

(2) $X(z_i)\geq \acute{X}(z_i)\Longrightarrow \mu_1^{(z_i)}\leq \acute{\mu}_1^{(z_i)}\Longrightarrow\left(\mu_1^{(z_i)}\right)^{w_i}\leq\left(\acute{\mu}_1^{(z_i)}\right)^{w_i}$

$\Longrightarrow\left(1-\left(\mu_1^{(z_i)}\right)^{w_i}\right)^p\left(1-\left(\mu_1^{(z_j)}\right)^{w_j}\right)^q\geq\left(1-\left(\acute{\mu}_1^{(z_i)}\right)^{w_i}\right)^p\left(1-\left(\acute{\mu}_1^{(z_j)}\right)^{w_j}\right)^q$

$$\Longrightarrow 1-\left(1-\left(\mu_1^{(z_i)}\right)^{w_i}\right)^p\left(1-\left(\mu_1^{(z_j)}\right)^{w_j}\right)^q \leq 1-\left(1-\left(\mu_1^{\prime(z_i)}\right)^{w_i}\right)^p\left(1-\left(\mu_1^{\prime(z_j)}\right)^{w_j}\right)^q$$

$$\Longrightarrow \prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(\mu_1^{(z_i)}\right)^{w_i}\right)^p\left(1-\left(\mu_1^{(z_j)}\right)^{w_j}\right)^q\right) \leq \prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(\mu_1^{\prime(z_i)}\right)^{w_i}\right)^p\left(1-\left(\mu_1^{\prime(z_j)}\right)^{w_j}\right)^q\right)$$

$$\Longrightarrow 1-\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(\mu_1^{(z_i)}\right)^{w_i}\right)^p\left(1-\left(\mu_1^{(z_j)}\right)^{w_j}\right)^q\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}} \leq$$

$$1-\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(\mu_1^{\prime(z_i)}\right)^{w_i}\right)^p\left(1-\left(\mu_1^{\prime(z_j)}\right)^{w_j}\right)^q\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}$$

Similarly

$$1-\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(\mu_k^{(z_i)}\right)^{w_i}\right)^p\left(1-\left(\mu_k^{(z_j)}\right)^{w_j}\right)^q\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}} \leq$$

$$1-\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-\left(\mu_k^{\prime(z_i)}\right)^{w_i}\right)^p\left(1-\left(\mu_k^{\prime(z_j)}\right)^{w_j}\right)^q\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}$$

for $k = 1, 2, ..., m$.

$$(3)1-\left(1-\left(\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(1-\left(\varphi_k^{(z_i)}\right)^{w_i}\right)^p\left(1-\left(\varphi_k^{(z_j)}\right)^{w_j}\right)^q\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\leq$$

$$1-\left(1-\left(\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(1-\left(\varphi_k^{(z_i)}\right)^{w_i}\right)^p\left(1-\left(\varphi_k^{(z_j)}\right)^{w_j}\right)^q\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}$$

for $k = 1, 2, ..., m$.

From (1)-(3), the required result is obtained. □

Theorem 3.12. ***(Boundedness)*** *Let $X(z_i)$, $(i = 1, 2, ..., n)$ be a set of n elements of an MVmNS, then*

$$\min\left(X\left(z_1\right), X\left(z_2\right), ..., X\left(z_n\right)\right) \leq MVmNWBM\left(X\left(z_1\right), X\left(z_2\right), ..., X\left(z_n\right)\right)$$
$$\leq \max\left(X\left(z_1\right), X\left(z_2\right), ..., X\left(z_n\right)\right)$$

Proof. Let $m = \min\left(X\left(z_1\right), X\left(z_2\right), ..., X\left(z_n\right)\right)$ and $M = \max\left(X\left(z_1\right), X\left(z_2\right), ..., X\left(z_n\right)\right)$

Since $m \leq X(z_i) \leq M$ so by using previous theorem

$$m \leq MVmNWBM\left(X\left(z_1\right), X\left(z_2\right), ..., X\left(z_n\right)\right),$$
$$MVmNWBM\left(X\left(z_1\right), X\left(z_2\right), ..., X\left(z_n\right)\right) \leq M.$$

Hence the result.

3.3. *Multi-Valued Multi-Polar Neutrosophic Soft Set*

Definition 3.13. Let Z be a universal set and E be a set of parameters with $X \subseteq E$. Define $\omega : X \to P(Z)$, where $P(Z)$ is the collection of all MVmN subsets of the set Z. Then (ω, X) is said to be an multi-valued m-polar neutrosophic soft set (MVmNSS) over Z which is represented as $\Omega_X = (\omega, X) = \{e, \omega_X(e) : e \in E, \omega_X(e) \in P(Z)\}$ and $\omega_X(e)$ is an MVmNS over Z.

TABLE 1. Representation of a MV3NSS.

	C_1	C_2	C_3
A_1	$\begin{pmatrix} (\{0.6\},\{0.4,0.5\},\{0.4,0.5\}), \\ (\{0.8,\},\{0.1,0.4\},\{0.5\}), \\ (\{0.2\},\{0.8\},\{0.3,0.6\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.6\},\{0.7\},\{0.4,0.5\}), \\ (\{0.3,0.7\},\{0.6\},\{0.8,0.9\}), \\ (\{0.4\},\{0.8,0.9\},\{0.5\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.3,0.5\},\{0.7\},\{0.9\}), \\ (\{0.6,0.9\},\{0.8\},\{0.6\}), \\ (\{0.4,0.5\},\{0.4\},\{0.7\}) \end{pmatrix}$
A_2	$\begin{pmatrix} (\{0.4,0.6\},\{0.5\},\{0.2\}), \\ (\{0.6,0.7\},\{0.7\},\{0.2,0.4\}), \\ (\{0.4\},\{0.6\},\{0.6,0.8\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.4,0.6\},\{0.6\},\{0.6,0.9\}), \\ (\{0.6\},\{0.6,0.8\},\{0.7\}), \\ (\{0.4\},\{0.6\},\{0.5,0.9\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.6,0.8\},\{0.6\},\{0.5,0.7\}), \\ (\{0.2,0.6\},\{0.3,0.4\},\{0.4\}), \\ (\{0.4\},\{0.5\},\{0.5,0.7\}) \end{pmatrix}$
A_3	$\begin{pmatrix} (\{0.4\},\{0.3\},\{0.8,0.9\}), \\ (\{0.4\},\{0.3\},\{0.9\}), \\ (\{0.7,0.9\},\{0.8\},\{0.5\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.6\},\{0.4,0.6\},\{0.8\}), \\ (\{0.7,0.8\},\{0.9\},\{0.5\}), \\ (\{0.3,0.5\},\{0.8\},\{0.7,0.8\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.9\},\{0.7\},\{0.8,0.9\}), \\ (\{0.4\},\{0.7,0.9\},\{0.8\}), \\ (\{0.4\},\{0.8\},\{0.7,0.9\}) \end{pmatrix}$
A_4	$\begin{pmatrix} (\{0.6\},\{0.6\},\{0.5,0.7\}), \\ (\{0.5,0.7\},\{0.7\},\{0.9\}), \\ (\{0.4\},\{0.8,0.9\},\{0.5,0.6\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.6\},\{0.4\},\{0.1,0.3\}), \\ (\{0.5\},\{0.8\},\{0.6,0.7\}), \\ (\{0.3\},\{0.6,0.7\},\{0.6,0.9\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.7\},\{0.3,0.6\},\{0.7\}), \\ (\{0.7\},\{0.9\},\{0.4,0.5\}), \\ (\{0.4\},\{0.3\},\{0.6\}) \end{pmatrix}$

Following example shows an MV3NS where three poles represent three different opinion leaders and decision makers are considered as opinion followers. Opinion leaders have an influence power for updating process of opinion followers' opinions [40].

Example 3.14. Let $\{A_1, A_2, A_3, A_4\}$ be a set of four companies where an investor wants to invest a suitable amount and $\{C_1, C_2, C_3\}$ be a set of criteria, then an MV3NSS is represented in Table 1.

3.3.1. *Operations on Multi-Valued m-Neutrosophic Soft Set*

Some operations in MVmNSS are defined in this section.

Definition 3.15. Let Z be a universal set and E be a set of parameters with $U, V \subseteq E$. For two MVmNSSs Ω_U and Ψ_V, $\Omega_U \breve{\subseteq} \Psi_V$ if

(1) $U \subseteq V$,
(2) $\Omega_U(e) \subseteq \Psi_V(e)$ for all $e \in U$ *i.e.* $s\left(\Omega_U(e)(z)\right) \leq s\left(\Psi_V(e)(z)\right)$ for all $e \in U$, $z \in Z$.

Example 3.16. Let $Z = \{z_1, z_2\}$ and $E = \{e_1, e_2, e_3\}$. $U = \{e_1, e_2\}$ and $V = \{e_1, e_2\}$ be subsets of E. Let Ω_U and Ψ_V be two MV3NSSs defined as:

$$\begin{aligned}\Omega_U = \ & \{(e_1,(z_1,(\{0.3,0.4\},\{0.4,0.6,0.7\},\{0.2,0.5\}),(\{0.5,0.7\},\{0.6,0.8,0.9\},\{0.7,0.8\}),\\ & (\{0.4,0.6\},\{0.5,0.7\},\{0.5,0.7,0.8\})),(z_2,(\{0.3,0.4\},\{0.6,0.9\},\{0.8,0.9\}),\\ & (\{0.4,0.5\},\{0.6,0.8\},\{0.4,0.6,0.7\}),(\{0.1,0.3,0.5\},\{0.6,0.7\},\{0.7,0.8\}))),\\ & (e_2,(z_1,(\{0.4,0.5\},\{0.4,0.6\},\{0.6,0.9\}),(\{0.2,0.4,0.5\},\{0.6,0.7\},\{0.7,0.8\}),\\ & (\{0.3,0.5\},\{0.6,0.7,0.8\},\{0.7,0.9\})),(z_2,(\{0.4,0.6\},\{0.4,0.6,0.7\},\{0.6,0.7\}),\\ & (\{0.1,0.2,0.4\},\{0.5,0.6,0.7\},\{0.6,0.8\}),(\{0.4,0.5\},\{0.5,0.6\},\{0.5,0.7\}))))\}\\ \Psi_V = \ & \{(e_1,(z_1,(\{0.8,0.9\},\{0.3,0.5\},\{0.5,0.6\}),(\{0.5,0.6\},\{0.2,0.3\},\{0.4,0.5\}),\\ & (\{0.8,0.9\},\{0.3,0.4,0.5\},\{0.3,0.4,0.5\})),(z_2,(\{0.7,0.8,0.9\},\{0.4,0.5\},\{0.5,0.6\}),\\ & (\{0.6,0.7\},\{0.6,0.7\},\{0.3,0.5\}),(\{0.7,0.8\},\{0.4,0.6\},\{0.2,0.5\}))),\\ & (e_2,(z_1,(\{0.6,0.8\},\{0.4,0.5\},\{0.4,0.5,0.6\}),(\{0.6,0.8,0.9\},\{0.4,0.6\},\{0.6,0.7\}),\\ & (\{0.7,0.8\},\{0.6,0.7\},\{0.5,0.7\})),(z_2,(\{0.6,0.8\},\{0.4,0.5\},\{0.4,0.5\}),\\ & (\{0.5,0.8\},\{0.4,0.5\},\{0.1,0.4\}),(\{0.9\},\{0.4,0.5\},\{0.5,0.7\}))))\}.\end{aligned}$$

Since $s(\Omega_U(e)(z)) \leq s(\Psi_V(e)(z))$ for all $e \in U$, $z \in Z \Rightarrow \Omega_U \breve{\subseteq} \Psi_V$ (one of the different choices of e and z is explained as: $s(\Omega_U(e_1)(z_1)) = -0.3288 \leq -0.083 = s(\Psi_U(e_1)(z_1))$).

Definition 3.17. Let Z be a universal set and Ω_U, Ψ_V be two MVmNS sets, where U and V are subsets of E. Ω_U and Ψ_V are said to be equal if $\Omega_U \breve{\subseteq} \Psi_V$ and $\Psi_V \breve{\subseteq} \Omega_U$.

4. Distance Measures

Let $Z = \{z_1, z_2, ..., z_n\}$ be a universal set, $E = \{e_1, e_2, ..., e_p\}$ be a set of attributes and $U, V \subseteq E$. Let Ω_U and Ψ_V be two MVmNS sets over Z with their respective MVmN mappings:

$$\omega_U(e_j) = \left\{\left(z_k, s_i \circ \widetilde{T}_U(e_j)(z_k), s_i \circ \widetilde{I}_U(e_j)(z_k), s_i \circ \widetilde{F}_U(e_j)(z_k)\right)\right\},$$
$$\psi_V(e_j) = \left\{\left(z_k, s_i \circ \widetilde{T}_V(e_j)(z_k), s_i \circ \widetilde{I}_V(e_j)(z_k), s_i \circ \widetilde{F}_V(e_j)(z_k)\right)\right\},$$

for all $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, p$ and $k = 1, 2, \ldots, n$, then the distance measures between Ω_U and Ψ_V are defined as:

4.1. *Hamming Distance*

$$d_H(\Omega_U, \Psi_V) = \frac{1}{3mp}\left\{\sum_{i=1}^{m}\sum_{j=1}^{p}\sum_{k=1}^{n}\left(\begin{array}{c}\left|s_i \circ \widetilde{T}_U(e_j)(z_k) - s_i \circ \widetilde{T}_V(e_j)(z_k)\right| + \\ \left|s_i \circ \widetilde{I}_U(e_j)(z_k) - s_i \circ \widetilde{I}_V(e_j)(z_k)\right| + \\ \left|s_i \circ \widetilde{F}_U(e_j)(z_k) - s_i \circ \widetilde{F}_V(e_j)(z_k)\right|\end{array}\right)\right\}.$$

4.2. *Normalized Hamming Distance*

$$d_{NH}\left(\Omega_U,\Psi_V\right)=\frac{1}{3mpn}\left\{\sum_{i=1}^{m}\sum_{j=1}^{p}\sum_{k=1}^{n}\left(\begin{array}{c}\left|s_i\circ\widetilde{T}_U\left(e_j\right)\left(z_k\right)-s_i\circ\widetilde{T}_V\left(e_j\right)\left(z_k\right)\right|+\\\left|s_i\circ\widetilde{I}_U\left(e_j\right)\left(z_k\right)-s_i\circ\widetilde{I}_V\left(e_j\right)\left(z_k\right)\right|+\\\left|s_i\circ\widetilde{F}_U\left(e_j\right)\left(z_k\right)-s_i\circ\widetilde{F}_V\left(e_j\right)\left(z_k\right)\right|\end{array}\right)\right\}.$$

4.3. *Euclidean Distance*

$$d_{E}\left(\Omega_U,\Psi_V\right)=\frac{1}{3mp}\left\{\sum_{i=1}^{m}\sum_{j=1}^{p}\sum_{k=1}^{n}\left(\begin{array}{c}\left(s_i\circ\widetilde{T}_U\left(e_j\right)\left(z_k\right)-s_i\circ\widetilde{T}_V\left(e_j\right)\left(z_k\right)\right)^2+\\\left(s_i\circ\widetilde{I}_U\left(e_j\right)\left(z_k\right)-s_i\circ\widetilde{I}_V\left(e_j\right)\left(z_k\right)\right)^2+\\\left(s_i\circ\widetilde{F}_U\left(e_j\right)\left(z_k\right)-s_i\circ\widetilde{F}_V\left(e_j\right)\left(z_k\right)\right)^2\end{array}\right)\right\}^{\frac{1}{2}}.$$

4.4. *Normalized Euclidean Distance*

$$d_{NE}\left(\Omega_U,\Psi_V\right)=\frac{1}{3mpn}\left\{\sum_{i=1}^{m}\sum_{j=1}^{p}\sum_{k=1}^{n}\left(\begin{array}{c}\left(s_i\circ\widetilde{T}_U\left(e_j\right)\left(z_k\right)-s_i\circ\widetilde{T}_V\left(e_j\right)\left(z_k\right)\right)^2+\\\left(s_i\circ\widetilde{I}_U\left(e_j\right)\left(z_k\right)-s_i\circ\widetilde{I}_V\left(e_j\right)\left(z_k\right)\right)^2+\\\left(s_i\circ\widetilde{F}_U\left(e_j\right)\left(z_k\right)-s_i\circ\widetilde{F}_V\left(e_j\right)\left(z_k\right)\right)^2\end{array}\right)\right\}^{\frac{1}{2}}.$$

Some distance measures are defined with weight vector $W=(w_1,w_2,...,w_p)^T$ satisfying $w_j\geq 0$ and $\sum\limits_{j=1}^{p}w_j=1$.

4.5. *Weighted Hamming Distance*

$$d_{WH}\left(\Omega_U,\Psi_V\right)=\frac{1}{3mp}\left\{\sum_{i=1}^{m}\sum_{j=1}^{p}\sum_{k=1}^{n}w_j\left(\begin{array}{c}\left|s_i\circ\widetilde{T}_U\left(e_j\right)\left(z_k\right)-s_i\circ\widetilde{T}_V\left(e_j\right)\left(z_k\right)\right|+\\\left|s_i\circ\widetilde{I}_U\left(e_j\right)\left(z_k\right)-s_i\circ\widetilde{I}_V\left(e_j\right)\left(z_k\right)\right|+\\\left|s_i\circ\widetilde{F}_U\left(e_j\right)\left(z_k\right)-s_i\circ\widetilde{F}_V\left(e_j\right)\left(z_k\right)\right|\end{array}\right)\right\}.$$

4.6. *Weighted Euclidean Distance*

$$d_{E}\left(\Omega_U,\Psi_V\right)=\frac{1}{3mp}\left\{\sum_{i=1}^{m}\sum_{j=1}^{p}\sum_{k=1}^{n}w_j\left(\begin{array}{c}\left(s_i\circ\widetilde{T}_U\left(e_j\right)\left(z_k\right)-s_i\circ\widetilde{T}_V\left(e_j\right)\left(z_k\right)\right)^2+\\\left(s_i\circ\widetilde{I}_U\left(e_j\right)\left(z_k\right)-s_i\circ\widetilde{I}_V\left(e_j\right)\left(z_k\right)\right)^2+\\\left(s_i\circ\widetilde{F}_U\left(e_j\right)\left(z_k\right)-s_i\circ\widetilde{F}_V\left(e_j\right)\left(z_k\right)\right)^2\end{array}\right)\right\}^{\frac{1}{2}}.$$

5. MCDM Based on MVmNSS by using TOPSIS

Ordering of the elements of MVmNS and formulation of distance measures between them leads us to develop a stepwise algorithm of TOPSIS.

Step 1: Construct a set of alternatives $X = \{x_1, x_2, \ldots, x_n\}$ and a set of attributes $E = \{e_1, e_2, \ldots, e_p\}$.

Step 2: A decision matrix is constructed by a decision maker which is the representation of an MVmNSS. In case of group decision, decision matrices are obtained from the experts and then an aggregated matrix D is obtained by using MVmNBM (Definition 3.6), and is represented as:

$$D(x_k) = \left\{ \left(e_j, s_i \circ \widetilde{T}_D(e_j), s_i \circ \widetilde{I}_D(e_j), s_i \circ \widetilde{F}_D(e_j) \right) \right\},$$

for an alternative x_k, $k = 1, 2, \ldots, n$.

Step 3: Choose the positive and negative ideal solutions by calculating the score values of the entries of decision matrices,

$$\begin{aligned} PIS &= \left\{ \left(e_j, s_i \circ \widetilde{T}_P(e_j), s_i \circ \widetilde{I}_P(e_j), s_i \circ \widetilde{F}_P(e_j) \right) \right\}, \\ NIS &= \left\{ \left(e_j, s_i \circ \widetilde{T}_N(e_j), s_i \circ \widetilde{I}_N(e_j), s_i \circ \widetilde{F}_N(e_j) \right) \right\}. \end{aligned}$$

Step 4: Find the distances of the elements of the aggregated matrix from PIS and NIS for each alternative x_k, $k = 1, 2, \ldots, n$ by using one of the following group of distance measures:

- $$d_{WH}(D_{x_k}, PIS) = \frac{1}{3mp} \left\{ \sum_{i=1}^{m} \sum_{j=1}^{p} w_j \left(\begin{array}{c} \left| s_i \circ \widetilde{T}_D(e_j) - s_i \circ \widetilde{T}_P(e_j) \right| + \\ \left| s_i \circ \widetilde{I}_D(e_j) - s_i \circ \widetilde{I}_P(e_j) \right| + \\ \left| s_i \circ \widetilde{F}_D(e_j) - s_i \circ \widetilde{F}_P(e_j) \right| \end{array} \right) \right\},$$

  $$d_{WH}(D_{x_k}, NIS) = \frac{1}{3mp} \left\{ \sum_{i=1}^{m} \sum_{j=1}^{p} w_j \left(\begin{array}{c} \left| s_i \circ \widetilde{T}_D(e_j) - s_i \circ \widetilde{T}_N(e_j) \right| + \\ \left| s_i \circ \widetilde{I}_D(e_j) - s_i \circ \widetilde{I}_N(e_j) \right| + \\ \left| s_i \circ \widetilde{F}_D(e_j) - s_i \circ \widetilde{F}_N(e_j) \right| \end{array} \right) \right\},$$

- $$d_{WE}(D_{x_k}, PIS) = \frac{1}{3mp} \left\{ \sum_{i=1}^{m} \sum_{j=1}^{p} w_j \left(\begin{array}{c} \left(s_i \circ \widetilde{T}_D(e_j) - s_i \circ \widetilde{T}_P(e_j) \right)^2 + \\ \left(s_i \circ \widetilde{I}_D(e_j) - s_i \circ \widetilde{I}_P(e_j) \right)^2 + \\ \left(s_i \circ \widetilde{F}_D(e_j) - s_i \circ \widetilde{F}_P(e_j) \right)^2 \end{array} \right) \right\}^{\frac{1}{2}},$$

$$d_{WE}\left(D_{x_k}, NIS\right) = \frac{1}{3mp}\left\{\sum_{i=1}^{m}\sum_{j=1}^{p} w_j \left(\begin{array}{c} \left(s_i \circ \widetilde{T}_D\left(e_j\right) - s_i \circ \widetilde{T}_N\left(e_j\right)\right)^2 + \\ \left(s_i \circ \widetilde{I}_D\left(e_j\right) - s_i \circ \widetilde{I}_N\left(e_j\right)\right)^2 + \\ \left(s_i \circ \widetilde{F}_D\left(e_j\right) - s_i \circ \widetilde{F}_N\left(e_j\right)\right)^2 \end{array}\right)\right\}^{\frac{1}{2}}.$$

Step 5: Calculate the co-efficients of relative closeness (RC) for the alternatives by using one of the following formulae:

$$RC(x_k) = \frac{d_{WH}\left(D_{x_k}, NIS\right)}{d_{WH}\left(D_{x_k}, NIS\right) + d_{WH}\left(D_{x_k}, PIS\right)},$$

or

$$RC(x_k) = \frac{d_{WE}\left(D_{x_k}, NIS\right)}{d_{WE}\left(D_{x_k}, NIS\right) + d_{WE}\left(D_{x_k}, PIS\right)},$$

$k = 1, 2, ..., n$, according to the distance measure used in step 4.

Step 6: Rank the alternatives.

5.1. *An Application Example*

Let $X = \{x_1, x_2, x_3, x_4\}$ be a set of alternatives, $E = \{e_1, e_2, e_3\}$ be a set of attributes and $D = \{d_1, d_2, d_3\}$ be the set of decision makers. Ranking of alternatives by the experts and observation of their attitudes is done here by two techniques:

(1) MVmNBM
(2) TOPSIS

By using first technique, stepwise procedure is as under:

Step 1: Obtain the MV2NSSs from the decision makers d_1, d_2 and d_3 which can be represented in Table 2, Table 3 and Table 4 respectively.

Step 2: Obtain an MVmNSS d^{agg} by calculating MV2NBM (Definition 3.6) for the respective values of Table 2, Table 3 and Table 4.

Step 3: Let $W_1 = \begin{pmatrix} 0.3 & 0.5 & 0.2 \end{pmatrix}$, $W_2 = \begin{pmatrix} 0.2 & 0.4 & 0.4 \end{pmatrix}$ and $W_3 = \begin{pmatrix} 0.7 & 0.1 & 0.2 \end{pmatrix}$ be three weight vectors for the attributes provided by three decision makers d_1, d_2 and d_3 respectively. Their weighted aggregated values are obtained from Definition 3.7 and are shown in Table 6, Table 7 and Table 8.

Step 4: Now by using the score function (Definition 3.4), find the single values for each alternative.

Score values for d_1:
$S(x_1) = -0.3128$
$S(x_2) = -0.3341$
$S(x_3) = -0.3009$
$S(x_4) = -0.3147$

TABLE 2. Decision matrix from the decision maker d_1.

1	e_1	e_2	e_3
x_1	$\begin{pmatrix} (\{0.3, 0.4\}, \{0.4\}), \\ (\{0.5\}, \{0.4\}), \\ (\{0.6\}, \{0.7\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.6, 0.7\}, \{0.8\}), \\ (\{0.5, 0.7\}, \{0.4, 0.6\}), \\ (\{0.9\}, \{0.9\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.9\}, \{0.6, 0.8\}), \\ (\{0.4, 0.5\}, \{0.7\}), \\ (\{0.3\}, \{0.6\}) \end{pmatrix}$
x_2	$\begin{pmatrix} (\{0.1\}, \{0.5\}), \\ (\{0.5\}, \{0.9\}), \\ (\{0.6, 0.7\}, \{0.9\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.7, 0.8\}, \{0.5\}), \\ (\{0.6, 0.7\}, \{0.6\}), \\ (\{0.6, 0.8\}, \{0.9\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.3\}, \{0.7\}), \\ (\{0.4\}, \{0.9\}), \\ (\{0.2\}, \{0.3, 0.5\}) \end{pmatrix}$
x_3	$\begin{pmatrix} (\{0.6\}, \{0.8\}), \\ (\{0.7, 0.8\}, \{0.1\}), \\ (\{0.6\}, \{0.1\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.9\}, \{0.8\}), \\ (\{0.7\}, \{0.6\}), \\ (\{0.5, 0.6\}, \{0.5\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.5\}, \{0.6\}), \\ (\{0.4\}, \{0.9\}), \\ (\{0.1, 0.4\}, \{0.5\}) \end{pmatrix}$
x_4	$\begin{pmatrix} (\{0.5\}, \{0.9\}), \\ (\{0.2, 0.6\}, \{0.4\}), \\ (\{0.7\}, \{0.4\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.1, 0.4\}, \{0.5\}), \\ (\{0.7\}, \{0.2\}), \\ (\{0.5\}, \{0.3, 0.7\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.3\}, \{0.8\}), \\ (\{0.4\}, \{0.5, 0.6\}), \\ (\{0.4, 0.6\}, \{0.7\}) \end{pmatrix}$

TABLE 3. Decision matrix from the decision maker d_2.

2	e_1	e_2	e_3
x_1	$\begin{pmatrix} (\{0.1\}, \{0.3\}), \\ (\{0.2\}, \{0.1, 0.5\}), \\ (\{0.4\}, \{0.6\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.7\}, \{0.5\}), \\ (\{0.4\}, \{0.5\}), \\ (\{0.3, 0.5\}, \{0.4\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.3, 0.5\}, \{0.5, 0.6\}), \\ (\{0.4\}, \{0.8\}), \\ (\{0.3\}, \{0.7\}) \end{pmatrix}$
x_2	$\begin{pmatrix} (\{0.4, 0.6\}, \{0.9\}), \\ (\{0.8\}, \{0.5\}), \\ (\{0.6\}, \{0.6\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.1\}, \{0.5, 0.7\}), \\ (\{0.6\}, \{0.7\}), \\ (\{0.1\}, \{0.3\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.4\}, \{0.9\}), \\ (\{0.5\}, \{0.6\}), \\ (\{0.7\}, \{0.8\}) \end{pmatrix}$
x_3	$\begin{pmatrix} (\{0.2\}, \{0.4\}), \\ (\{0.5\}, \{0.6, 0.7\}), \\ (\{0.2\}, \{0.7\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.4\}, \{0.7, 0.8\}), \\ (\{0.2\}, \{0.1\}), \\ (\{0.2\}, \{0.7\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.2\}, \{0.8, 0.9\}), \\ (\{0.5\}, \{0.8\}), \\ (\{0.1\}, \{0.5, 0.9\}) \end{pmatrix}$
x_4	$\begin{pmatrix} (\{0.5, 0.7\}, \{0.4\}), \\ (\{0.4, 0.8\}, \{0.9\}), \\ (\{0.1\}, \{0.5\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.9\}, \{0.8\}), \\ (\{0.6\}, \{0.7, 0.9\}), \\ (\{0.3\}, \{0.6\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.3\}, \{0.4, 0.5\}), \\ (\{0.3, 0.4\}, \{0.6\}), \\ (\{0.7\}, \{0.2, 0.3\}) \end{pmatrix}$

Score values for d_2:

$S(x_1) = -0.3084$

$S(x_2) = -0.3326$

$S(x_3) = -0.2952$

$S(x_4) = -0.3167$

Score values for d_3:

TABLE 4. Decision matrix from the decision maker d_3.

3	e_1	e_2	e_3
x_1	$\begin{pmatrix} (\{0.6\},\{0.7\}), \\ (\{0.8,0.9\},\{0.4\}), \\ (\{0.3\},\{0.6\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.4,0.6\},\{0.7\}), \\ (\{0.4\},\{0.6\}), \\ (\{0.5\},\{0.7,0.9\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.1,0.3\},\{0.4\}), \\ (\{0.6\},\{0.5,0.8\}), \\ (\{0.3,0.5\},\{0.4\}) \end{pmatrix}$
x_2	$\begin{pmatrix} (\{0.4\},\{0.5\}), \\ (\{0.6\},\{0.7\}), \\ (\{0.8\},\{0.6\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.3\},\{0.4\}), \\ (\{0.5,0.6\},\{0.6\}), \\ (\{0.4\},\{0.7\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.5,0.8\},\{0.5\}), \\ (\{0.2\},\{0.3\}), \\ (\{0.4\},\{0.9\}) \end{pmatrix}$
x_3	$\begin{pmatrix} (\{0.2\},\{0.3,0.6\}), \\ (\{0.4\},\{0.5\}), \\ (\{0.7\},\{0.8\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.6\},\{0.7,0.8\}), \\ (\{0.7\},\{0.9\}), \\ (\{0.4,0.5\},\{0.8\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.9\},\{0.7\}), \\ (\{0.4\},\{0.6,0.7\}), \\ (\{0.6\},\{0.8\}) \end{pmatrix}$
x_4	$\begin{pmatrix} (\{0.2\},\{0.6\}), \\ (\{0.4,0.6\},\{0.7\}), \\ (\{0.6\},\{0.8,0.9\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.6\},\{0.4\}), \\ (\{0.4\},\{0.9\}), \\ (\{0.3\},\{0.6,0.7\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.2\},\{0.3,0.5\}), \\ (\{0.7,0.8\},\{0.9\}), \\ (\{0.4\},\{0.1\}) \end{pmatrix}$

TABLE 5. Aggregated matrix d^{agg}.

Agg1	e_1	e_2	e_3
x_1	$\begin{pmatrix} (\{0.3039,0.3437\},\{0.4539\}), \\ (\{0.5187,0.56\},\{0.3025,0.4338\}), \\ (\{0.4362\},\{0.6346\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.5638,0.6666\},\{0.6666\}), \\ (\{0.4338,0.5050\},\{0.5022,0.5677\}), \\ (\{0.5891,0.6507\},\{0.6961,0.7911\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.3691,0.5488\},\{0.4978,0.5955\}), \\ (\{0.4687,0.5022\},\{0.6774,0.7690\}), \\ (\{0.3,0.3672\},\{0.5740\}) \end{pmatrix}$
x_2	$\begin{pmatrix} (\{0.2860,0.3437\},\{0.6246\}), \\ (\{0.6418\},\{0.7204\}), \\ (\{0.6722,0.7050\},\{0.7140\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.3268,0.3484\},\{0.4659,0.5285\}), \\ (\{0.5677,0.6346\},\{0.6346\}), \\ (\{0.3732,0.4454\},\{0.6732\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.3966,0.4806\},\{0.7003\}), \\ (\{0.3693\},\{0.6299\}), \\ (\{0.4404\},\{0.7207,0.7607\}) \end{pmatrix}$
x_3	$\begin{pmatrix} (\{0.3068\},\{0.4806,0.5955\}), \\ (\{0.5387,0.5775\},\{0.4130,0.4512\}), \\ (\{0.5194\},\{0.5824\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.6268\},\{0.7334,0.8\}), \\ (\{0.5607\},\{0.5803\}), \\ (\{0.3693,0.4419\},\{0.6774\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.5095\},\{0.7,0.7334\}), \\ (\{0.4338\},\{0.7832,0.8081\}), \\ (\{0.2430,0.3732\},\{0.6088,0.7607\}) \end{pmatrix}$
x_4	$\begin{pmatrix} (\{0.3912,0.4524\},\{0.6268\}), \\ (\{0.3346,0.6722\},\{0.6961\}), \\ (\{0.4936\},\{0.6193\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.5169,0.6268\},\{0.5581\}), \\ (\{0.5740\},\{0.6516,0.7608\}), \\ (\{0.3672\},\{0.5080,0.6682\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.2648\},\{0.4806,0.5947\}), \\ (\{0.4724,0.5432\},\{0.6842,0.7140\}), \\ (\{0.5050,0.5740\},\{0.32,0.3656\}) \end{pmatrix}$

$S(x_1) = -0.3444$
$S(x_2) = -0.3663$
$S(x_3) = -0.3365$
$S(x_4) = -0.3503$

$S(x_2) < S(x_4) < S(x_1) < S(x_3)$ is the ranking of alternatives which is similar for all three decision makers. Alternative x_3 is the best one to select.

Step 5: To analyze the future attitude of the decision makers, system of differential equations (2) is developed by selecting $a^j_{p_i}$, $i, j = 1, 2, 3$ from the score values.$a^1_{p_1} = 0.6991, a^1_{p_2} = 0.7048, a^2_{p_1} = 0.6991, a^2_{p_2} = 0.7048$

$$\begin{aligned} \tfrac{dP_1}{dt} &= 0.6991P_1 + 0.7048P_2 \\ \tfrac{dP_2}{dt} &= 0.6991P_1 + 0.7048P_2 \end{aligned} \quad (3)$$

TABLE 6. Weighted aggregated values for d_1.

x_1	$\begin{pmatrix} (\{0.6455, 0.7069\}, \{0.7202, 0.7360\}), \\ (\{0.7883, 0.8137\}, \{0.7889, 0.8375\}), \\ (\{0.7778, 0.8019\}, \{0.8684, 0.8858\}) \end{pmatrix}$
x_2	$\begin{pmatrix} (\{0.5922, 0.6256\}, \{0.7444, 0.7571\}), \\ (\{0.8186, 0.8313\}, \{0.8771\}), \\ (\{0.8, 0.8185\}, \{0.8924, 0.8968\}) \end{pmatrix}$
x_3	$\begin{pmatrix} (\{0.6843\}, \{0.7723, 0.8112\}), \\ (\{0.8109, 0.8172\}, \{0.8381, 0.8473\}), \\ (\{0.7346, 0.7749\}, \{0.8617, 0.8789\}) \end{pmatrix}$
x_4	$\begin{pmatrix} (\{0.6379, 0.6756\}, \{0.7298, 0.7465\}), \\ (\{0.7832, 0.8494\}, \{0.8824, 0.9030\}), \\ (\{0.7760, 0.7859\}, \{0.7963, 0.8345\}) \end{pmatrix}$

TABLE 7. Weighted aggregated values for d_2.

x_1	$\begin{pmatrix} (\{0.6527, 0.7225\}, \{0.7263, 0.7470\}), \\ (\{0.7825, 0.8081\}, \{0.8027, 0.8508\}), \\ (\{0.7683, 0.7942\}, \{0.8634, 0.8808\}) \end{pmatrix}$
x_2	$\begin{pmatrix} (\{0.6019, 0.6362\}, \{0.7520, 0.7644), \\ (\{0.8064, 0.8201\}, \{0.8716\}), \\ (\{0.7848, 0.8033\}, \{0.8912, 0.8976\}) \end{pmatrix}$
x_3	$\begin{pmatrix} (\{0.7001\}, \{0.7867, 0.8221\}), \\ (\{0.8033, 0.8084\}, \{0.8527, 0.8623\}), \\ (\{0.7179, 0.7647\}, \{0.8595, 0.8840\}) \end{pmatrix}$
x_4	$\begin{pmatrix} (\{0.6304, 0.6637\}, \{0.7270, 0.7496\}), \\ (\{0.7826, 0.8418\}, \{0.8802, 0.9015\}), \\ (\{0.7730, 0.7872\}, \{0.7815, 0.8234\}) \end{pmatrix}$

Line graph for the system (3) (Figure 2) shows the same future behaviour of the decision makers d_1 and d_2, since lines are overlapping and phase portrait (Figure 1) shows that the system is unstable. It means that the experts may change their attitudes in future. A similar conclusion can be observed between d_3 and d_2 or d_1 and d_3. Future attitudes of d_1 and d_2 can also be analyzed (Figure 3) with the following fuzzy initial conditions (FICs):

$P_1(0) = (-1, 0, 1)$,

$P_2(0) = (-1, 0, 1)$,

or (α-cut representation)

$P_1(0) = (-1 + \alpha, 1 - \alpha) \qquad \alpha \in [0, 1]$,

$P_2(0) = (-1 + \alpha, 1 - \alpha) \qquad \alpha \in [0, 1]$.

TABLE 8. Weighted aggregated values for d_3.

x_1	$\begin{pmatrix} (\{0.5941, 0.6529\}, \{0.6708, 0.6887\}), \\ (\{0.8239, 0.8432\}, \{0.8310, 0.8709\}), \\ (\{0.7917, 0.8105\}, \{0.8813, 0.8881\}) \end{pmatrix}$
x_2	$\begin{pmatrix} (\{0.5629, 0.6\}, \{0.7245, 0.7304\}), \\ (\{0.8437, 0.8488\}, \{0.8980\}), \\ (\{0.8431, 0.8567\}, \{0.9106, 0.9155\}) \end{pmatrix}$
x_3	$\begin{pmatrix} (\{0.6299\}, \{0.7240, 0.7646\}), \\ (\{0.8336, 0.8407\}, \{0.8720, 0.8806\}), \\ (\{0.7762, 0.8090\}, \{0.8767, 0.8985\}) \end{pmatrix}$
x_4	$\begin{pmatrix} (\{0.5891, 0.6212\}, \{0.6886, 0.7115\}), \\ (\{0.8066, 0.8744\}, \{0.9019, 0.9135\}), \\ (\{0.8179, 0.8283\}, \{0.8261, 0.8461\}) \end{pmatrix}$

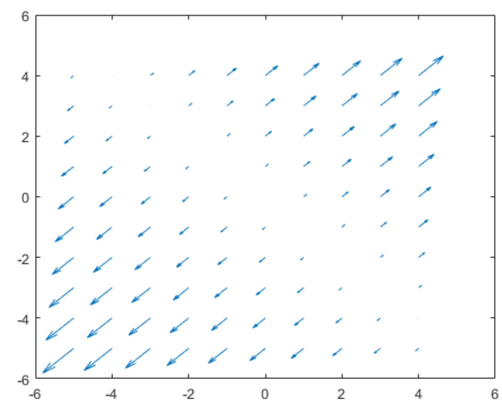

FIGURE 1. Phase portrait for the system (3).

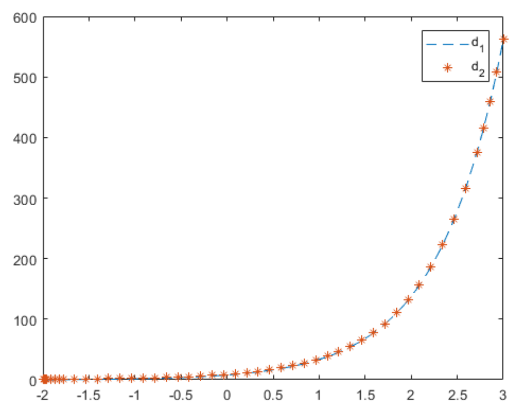

FIGURE 2. Line graph for the system (3).

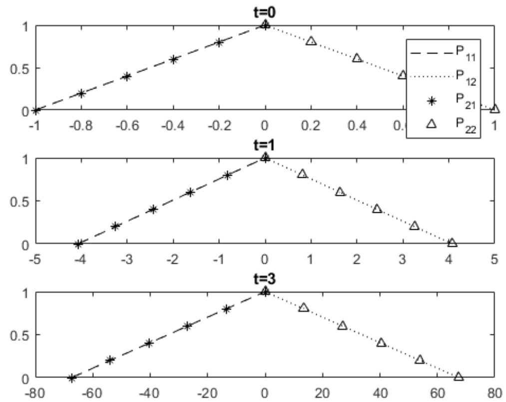

FIGURE 3. Line graph for the system (3) with FICs.

TABLE 9. Score values from the decision maker d_1.

d_1	e_1	e_2	e_3
x_1	-0.2416	-0.2416	-0.075
x_2	-0.3916	-0.2666	-0.15
x_3	-0.025	-0.1083	-0.1583
x_4	-0.0833	-0.1916	-0.175

TABLE 10. Score values from the decision maker d_2.

d_2	e_1	e_2	e_3
x_1	-0.1833	-0.0833	-0.2083
x_2	-0.1833	-0.1666	-0.2166
x_3	-0.2416	-0.0083	-0.175
x_4	-0.1833	-0.1	-0.1916

Now the stepwise procedure of the second technique is as under:

Step 1 Same as in first technique

Step 2 Same as in first technique.

Step 3 Find the score values of the entries of Table 2, Table 3 and Table 4 by Definition 3.4. Respective score values are represented in Table 9, Table 10 and Table 11.

Step 4 By comparing the score values of the alternatives in Table 9, Table 10 and Table 11, select the PIS and NIS from Table 2, Table 3 and Table 4.

Step 5 Find the weighted distances between the entries of Table 5 and Table 12 as described in section 4.5 with $W_1 = \begin{pmatrix} 0.3 & 0.5 & 0.2 \end{pmatrix}$ and $W_2 = \begin{pmatrix} 0.2 & 0.4 & 0.4 \end{pmatrix}$. Here weighted Hamming distance is utilized.

Table 11. Score values from the decision maker d_3.

d_3	e_1	e_2	e_3
x_1	-0.1416	-0.1833	-0.2416
x_2	-0.3	-0.2583	-0.1
x_3	-0.2916	-0.25	-0.1416
x_4	-0.3083	-0.2083	-0.2583

Table 12. Positive and negative ideal solution.

	e_1	e_2	e_3
PIS	$\begin{pmatrix} (\{0.6\},\{0.8\}), \\ (\{0.7,0.8\},\{0.1\}), \\ (\{0.6\},\{0.1\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.4\},\{0.7,0.8\}), \\ (\{0.2\},\{0.1\}), \\ (\{0.2\},\{0.7\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.9\},\{0.6,0.8\}), \\ (\{0.4,0.5\},\{0.7\}), \\ (\{0.3\},\{0.6\}) \end{pmatrix}$
NIS	$\begin{pmatrix} (\{0.1\},\{0.5\}), \\ (\{0.5\},\{0.9\}), \\ (\{0.6,0.7\},\{0.9\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.7,0.8\},\{0.5\}), \\ (\{0.6,0.7\},\{0.6\}), \\ (\{0.6,0.8\},\{0.9\}) \end{pmatrix}$	$\begin{pmatrix} (\{0.2\},\{0.3,0.5\}), \\ (\{0.7,0.8\},\{0.9\}), \\ (\{0.4\},\{0.1\}) \end{pmatrix}$

$$
\begin{array}{r|l}
d_{W_1H}(D_{x_1},PIS)=0.4059 & d_{W_2H}(D_{x_1},PIS)=0.3875 \\
d_{W_1H}(D_{x_2},PIS)=0.4284 & d_{W_2H}(D_{x_2},PIS)=0.4131 \\
d_{W_1H}(D_{x_3},PIS)=0.4048 & d_{W_2H}(D_{x_3},PIS)=0.3886 \\
d_{W_1H}(D_{x_4},PIS)=0.4523 & d_{W_2H}(D_{x_4},PIS)=0.4485 \\
d_{W_1H}(D_{x_1},NIS)=0.3838 & d_{W_2H}(D_{x_1},NIS)=0.4064 \\
d_{W_1H}(D_{x_2},NIS)=0.3928 & d_{W_2H}(D_{x_2},NIS)=0.3824 \\
d_{W_1H}(D_{x_3},NIS)=0.4060 & d_{W_2H}(D_{x_3},NIS)=0.4109 \\
d_{W_1H}(D_{x_4},NIS)=0.4122 & d_{W_2H}(D_{x_4},NIS)=0.4597
\end{array}
$$

Step 6 Find the Coefficients of relative closeness for each alternative and rank the alternatives.

$$
\begin{array}{c|c}
RC_{W_1}(x_1)=0.4860 & RC_{W_2}(x_1)=0.5119 \\
RC_{W_1}(x_2)=0.4783 & RC_{W_2}(x_2)=0.4807 \\
RC_{W_1}(x_3)=0.5007 & RC_{W_2}(x_3)=0.5139 \\
RC_{W_1}(x_4)=0.4768 & RC_{W_2}(x_4)=0.5061 \\
S(x_4)<S(x_2)<S(x_1)<S(x_3) & S(x_2)<S(x_4)<S(x_1)<S(x_3)
\end{array}
$$

Both experts select the same alternative and their future attitude is same as discussed in previous technique.

6. Conclusion

MVmNSS can model the problems of MCDM with undetermined information better than MVNSS and mNSS. It engages not only the multi-polar information but also multi-valued data. The multi-valued neutrosophic set has the membership, non-membership and indeterminacy values which can be treated as in hesitant fuzzy set or dual hesitant fuzzy set when operational laws (Definition 2.3) are defined. An analysis of experts' attitudes after their decisions can also be done by utilizing the MVmNBM. This study has also been carried out by Beg et al. [41] with a fuzzy soft matrix as the initial data which does not captivate the degrees of falsity-membership and indeterminacy-membership. MVmNSS handles these complicated uncertainties and can be aggregated by MVmNBM. In the future, other MCDM methods (TOPSIS, VIKOR, etc.) can be applied in group decision problems by defining the distance and similarity measures in MVmNSs. Another aspect of this research is the utilization of differential equations with FICs which does not produce different results.

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Pythagorean m-polar Fuzzy Neutrosophic Metric Spaces

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Abstract. Neutrosophy deals with the study of neutrosophic logic, set and probability. A Pythagorean m-polar neutrosophic set is indeed an expansion of crisp, fuzzy, intuitionistic fuzzy, neutrosophic and Pythagorean m-polar fuzzy sets. In this paper, we develop the perception of Pythagorean m-polar fuzzy neutrosophic metric space defined over Pythagorean m-polar fuzzy neutrosophic set relying on the classical definition of metric spaces defined on a crisp set. We present some related results and illustrations to perceive the conceptions. We present many examples of metrics which hold true for classical sets but fail to make sense in Pythagorean m-polar fuzzy environment. We also render a practical utility of the proposed metrics in pattern recognition.

Keywords: Pythagorean m-polar fuzzy neutrosophic set; Pythagorean m-polar fuzzy neutrosophic subset; Pythagorean m-polar fuzzy neutrosophic metric spaces; Pattern recognition

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1. Introduction

In the wake of advancement of classical sets to fuzzy sets by Zadeh [33], the scientists around the globe started working on diverse aspects of fuzzy sets and its expansions. Contrary to classical sets, an element is allowed to partially belong to the set, as specified in fuzzy set. In [2,3] Atanassov unveiled the notion of intuitionistic fuzzy sets (IFSs) by including the so called non-membership grade to already included membership grade in a fuzzy set. Yager [32] comforted the decision makers by enhancing the space for the the choice of association and dissociation grades prevailing in the IFSs and called the resulting model as Pythagorean fuzzy set. Naeem *et al.* [21] expanded the conception given by Yager to Pythagorean m-polar fuzzy sets and rendered a fascinating practical implementation to advertisement mode selection problem. Later, Riaz *et al.* [24] further generalized the thought to Pythagorean fuzzy soft sets. Naeem *et al.* further explored the chief characteristics of Pythagorean m-polar fuzzy sets in [22].

Maurice René Fréchet, a French mathematician, floated the idea of metric spaces in 1906. Deng [12], Diamond and Kloden [13], Atefi and Jehadi [4], Chaudhuri and Rosenfeld [5], George and Veeramani [14], and Gregori and Romaguera [15] are among the mathematicians who studied and explored different aspects of fuzzy metric spaces. The scientists who explored metric spaces in the framework of IFSs mainly include Gregori and Romaguera [16], Li *et al.* [19], and Park *et al.* [23].

Smarandache [28] presented yet another expansion of fuzzy sets called Neutrosophic sets. He made further explorations in [29] and [30]. The series of fascinating explorations by Smarandache is continued. Wang *et al.* [31] presented single valued neutrosophic sets. Arockiarani *et al.* [1] studied fuzzy neutrosophic soft topological spaces. Şimşek and Kirşci [27] explored fixed points in the context of neutrosophic metric spaces. Ishtiaq *et al.* [17] presented fixed points results in orthogonal neutrosophic metric spaces. Jansi and Mohana [18] studied, in recent times, pairwise Pythagorean neutrosophic P-spaces (with dependent neutrosophic components between T and F). In recent times, Siraj *et al.* [25] unveiled the apprehension of Pythagorean m-polar fuzzy neutrosophic topology with applications towards handling economic crises caused due to COVID-19 and the root cause behind scarcity of water in Thar desert of Pakistan.

Das *et al.* [6] presented the notion of neutrosophic fuzzy matrices with their algebraic operations. Das and Tripathy [7] studied neutrosophic multiset topological space. Mukherjee and Das [20] explored neutrosophic bipolar vague soft set and its application. Das *et al.* [8] unveiled the notions of neutro algebra and neutro group. Das and Das [9] presented neutrosophic separation axioms. Recently, Das *et al.* [10] rendered the idea of pentapartitioned neutrosophic probability distributions. Das *et al.* [11] studied topology on ultra neutrosophic set.

There arise many situations in real life where we have to think time and again before reaching at some decision–a decision that may be thought as flawless. It is in fact the process of multipolarity. The ever-expanding applications of neutrosophic sets are not concealed from the world. Pythagorean neutrosophic environment provides the enhanced facility of choosing values for the three membership functions (truth, indeterminacy and falsity) from a broader space.

In this article, we explore some notions of Pythagorean m-polar fuzzy neutrosophic metric spaces. Section 2 presents some basic notions necessary to conceive the main topic of this study. The third section presents main study of this article. In this section, the notion of

Pythagorean m-polar fuzzy neutrosophic metric spaces has been put forward. A large number of examples and illustrations are presented to conceive the perception. Section 4 presents a practical implementation of the proposed metrics in pattern recognition. Section 5 concludes the paper.

2. Preliminaries

Definition 2.1. [28, 29] A *neutrosophic set* $\mathbb{N}$ on the underlying set X is specified as

$$\mathbb{N} = \{< \Upsilon, T_{\mathbb{N}}(\Upsilon), I_{\mathbb{N}}(\Upsilon), F_{\mathbb{N}}(\Upsilon) >: \Upsilon \in X\}$$

where $T, I, F : X \mapsto]^-0, 1^+[$ accompanied by the constraint $^-0 \leq T_{\mathbb{N}}(\Upsilon)+I_{\mathbb{N}}(\Upsilon)+F_{\mathbb{N}}(\Upsilon) \leq 3^+$. Here $T_{\mathbb{N}}(\Upsilon)$, $I_{\mathbb{N}}(\Upsilon)$ and $F_{\mathbb{N}}(\Upsilon)$ are the degrees of membership, indeterminacy and falsity (non-membership) of members of the given set, respectively. T, I and F are acknowledged as the neutrosophic components.

Definition 2.2. [1] A *fuzzy neutrosophic set* ($\mathfrak{fn}$-set) over X is delineated as

$$A = \{< \Upsilon, T_A(\Upsilon), I_A(\Upsilon), F_A(\Upsilon) >: \Upsilon \in X\}$$

where $T, I, F : X \mapsto [0, 1]$ in such a way that $0 \leq T_A(\Upsilon) + I_A(\Upsilon) + F_A(\Upsilon) \leq 3$.

Definition 2.3. [25] A *Pythagorean m-polar fuzzy neutrosophic set* (PmFNS) $\Im$ over a basic set X is marked by three mappings $T_{\Im}^{(i)} : X \to [0, 1]^m$, $I_{\Im}^{(i)} : X \to [0, 1]^m$ and $F_{\Im}^{(i)} : X \to [0, 1]^m$, where m is a natural number, $\forall i = 1, 2, \cdots, m$, with the limitation that

$$0 \leq \left(T_{\Im}^{(i)}(\Upsilon)\right)^2 + \left(I_{\Im}^{(i)}(\Upsilon)\right)^2 + \left(F_{\Im}^{(i)}(\Upsilon)\right)^2 \leq 2$$

for all $\Upsilon \in X$.

A PmFNS may be expressed as

$$\begin{aligned}
\Im &= \left\{(\Upsilon, \left((T_{\Im}^{(1)}(\Upsilon), I_{\Im}^{(1)}(\Upsilon), F_{\Im}^{(1)}(\Upsilon)), \cdots, (T_{\Im}^{(m)}(\Upsilon), I_{\Im}^{(m)}(\Upsilon), F_{\Im}^{(m)}(\Upsilon))\right) : \Upsilon \in X\right\} \\
&= \left\{\frac{\Upsilon}{(T_{\Im}^{(1)}(\Upsilon), I_{\Im}^{(1)}(\Upsilon), F_{\Im}^{(1)}(\Upsilon)), \cdots, (T_{\Im}^{(m)}(\Upsilon), I_{\Im}^{(m)}(\Upsilon), F_{\Im}^{(m)}(\Upsilon))} : \Upsilon \in X\right\} \\
&= \left\{\frac{\Upsilon}{(T_{\Im}^{(i)}(\Upsilon), I_{\Im}^{(i)}(\Upsilon), F_{\Im}^{(i)}(\Upsilon))} : \Upsilon \in X, i = 1, 2, \cdots, m\right\}
\end{aligned}$$

If cardinality of X is l, then tabular structure of $\Im$ is as in Table 1:

TABLE 1. Tabular representation of PmFNS $\Im$

$\Im$				
$\curlyvee_1$	$(T^{(1)}_{\Im}(\curlyvee_1), I^{(1)}_{\Im}(\curlyvee_1), F^{(1)}_{\Im}(\curlyvee_1))$	$(T^{(2)}_{\Im}(\curlyvee_1), I^{(2)}_{\Im}(\curlyvee_1), F^{(2)}_{\Im}(\curlyvee_1))$	$\cdots$	$(T^{(m)}_{\Im}(\curlyvee_1), I^{(m)}_{\Im}(\curlyvee_1), F^{(m)}_{\Im}(\curlyvee_1))$
$\curlyvee_2$	$(T^{(1)}_{\Im}(\curlyvee_2), I^{(1)}_{\Im}(\curlyvee_2), F^{(1)}_{\Im}(\curlyvee_2))$	$(T^{(2)}_{\Im}(\curlyvee_2), I^{(2)}_{\Im}(\curlyvee_2), F^{(2)}_{\Im}(\curlyvee_2))$	$\cdots$	$(T^{(m)}_{\Im}(\curlyvee_2), I^{(m)}_{\Im}(\curlyvee_2), F^{(m)}_{\Im}(\curlyvee_2))$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\curlyvee_l$	$(T^{(1)}_{\Im}(\curlyvee_l), I^{(1)}_{\Im}(\curlyvee_l), F^{(1)}_{\Im}(\curlyvee_l))$	$(T^{(2)}_{\Im}(\curlyvee_l), I^{(2)}_{\Im}(\curlyvee_l), F^{(2)}_{\Im}(\curlyvee_l))$	$\cdots$	$(T^{(m)}_{\Im}(\curlyvee_l), I^{(m)}_{\Im}(\curlyvee_l), F^{(m)}_{\Im}(\curlyvee_l))$

The corresponding matrix format is

$$\Im = \begin{pmatrix} (T^{(1)}_{\Im}(\curlyvee_1), I^{(1)}_{\Im}(\curlyvee_1), F^{(1)}_{\Im}(\curlyvee_1)) & (T^{(2)}_{\Im}(\curlyvee_1), I^{(2)}_{\Im}(\curlyvee_1), F^{(2)}_{\Im}(\curlyvee_1)) & \cdots & (T^{(m)}_{\Im}(\curlyvee_1), I^{(m)}_{\Im}(\curlyvee_1), F^{(m)}_{\Im}(\curlyvee_1)) \\ (T^{(1)}_{\Im}(\curlyvee_2), I^{(1)}_{\Im}(\curlyvee_2), F^{(1)}_{\Im}(\curlyvee_2)) & (T^{(2)}_{\Im}(\curlyvee_2), I^{(2)}_{\Im}(\curlyvee_2), F^{(2)}_{\Im}(\curlyvee_2)) & \cdots & (T^{(m)}_{\Im}(\curlyvee_2), I^{(m)}_{\Im}(\curlyvee_2), F^{(m)}_{\Im}(\curlyvee_2)) \\ \vdots & \vdots & \ddots & \vdots \\ (T^{(1)}_{\Im}(\curlyvee_l), I^{(1)}_{\Im}(\curlyvee_l), F^{(1)}_{\Im}(\curlyvee_l)) & (T^{(2)}_{\Im}(\curlyvee_l), I^{(2)}_{\Im}(\curlyvee_l), F^{(2)}_{\Im}(\curlyvee_l)) & \cdots & (T^{(m)}_{\Im}(\curlyvee_l), I^{(m)}_{\Im}(\curlyvee_l), F^{(m)}_{\Im}(\curlyvee_l)) \end{pmatrix}$$

This $l \times m$ matrix is known as *PmFN matrix*. The assortment of each PmFNS characterized over universe would be designated by PmFNS(X).

Definition 2.4. [25] Let $\Im_1$ and $\Im_2$ be PmFNSs over X. $\Im_1$ is acknowledged as a *subset* of $\Im_2$, written as $\Im_1 \subseteq \Im_2$, $\forall \Im \in X$ and each values of i ranging from 1 to m, if

1) $T^{(i)}_{\Im_1}(\curlyvee) \leq T^{(i)}_{\Im_2}(\curlyvee)$,
2) $I^{(i)}_{\Im_1}(\curlyvee) \geq I^{(i)}_{\Im_2}(\curlyvee)$,
3) $F^{(i)}_{\Im_1}(\curlyvee) \geq F^{(i)}_{\Im_2}(\curlyvee)$.

$\Im_1$ and $\Im_2$ are said to be *equal* if $\Im_1 \subseteq \Im_2 \subseteq \Im_1$ and is written as $\Im_1 = \Im_2$.

Definition 2.5. [25] A PmFNS $\Im$ over X is known as *null PmFNS* if $T^{(i)}_{\Im}(\curlyvee) = 0$, $I^{(i)}_{\Im}(\curlyvee) = 1$ and $F^{(i)}_{\Im}(\curlyvee) = 1$, $\forall \curlyvee \in X$ and all acceptable values of i. It is designated by Φ.
Thus,

$$\Phi = \begin{pmatrix} (0,1,1) & (0,1,1) & \cdots & (0,1,1) \\ (0,1,1) & (0,1,1) & \cdots & (0,1,1) \\ \vdots & \vdots & \ddots & \vdots \\ (0,1,1) & (0,1,1) & \cdots & (0,1,1) \end{pmatrix}.$$

Definition 2.6. [25] A PmFNS $\Im$ over X is called an *absolute PmFNS* if $T^{(i)}_{\Im}(\curlyvee) = 1$, $I^{(i)}_{\Im}(\curlyvee) = 0$, and $F^{(i)}_{\Im}(\curlyvee) = 0$, $\forall \curlyvee \in X$. It is denoted by $\breve{\chi}$.
Thus,

$$\breve{\chi} = \begin{pmatrix} (1,0,0) & (1,0,0) & \cdots & (1,0,0) \\ (1,0,0) & (1,0,0) & \cdots & (1,0,0) \\ \vdots & \vdots & \ddots & \vdots \\ (1,0,0) & (1,0,0) & \cdots & (1,0,0) \end{pmatrix}.$$

Definition 2.7. [25] The *complement* of a PmFNS

$$\Im = \left\{ \frac{\Upsilon}{(T_{\Im}^{(i)}(\Upsilon), I_{\Im}^{(i)}(\Upsilon), F_{\Im}^{(i)}(\Upsilon))} : \Upsilon \in X, i = 1, \cdots, m \right\}$$

over X is defined as

$$\Im^c = \left\{ \frac{\Upsilon}{(F_{\Im}^{(i)}(\Upsilon), 1 - I_{\Im}^{(i)}(\Upsilon), T_{\Im}^{(i)}(\Upsilon))} : \Upsilon \in X, i = 1, \cdots, m \right\}.$$

Definition 2.8. [25] The *union* of any PmFNSs $\Im_1$ and $\Im_2$ expressed over the same universe X is represented as

$$\Im_1 \cup_{\mathfrak{M}} \Im_2 = \left\{ \frac{\Upsilon}{(\max(T_{\Im1}^{(i)}(\Upsilon), T_{\Im2}^{(i)}(\Upsilon)), \min(I_{\Im1}^{(i)}(\Upsilon), I_{\Im2}^{(i)}(\Upsilon)), \min(F_{\Im1}^{(i)}(\Upsilon), F_{\Im2}^{(i)}(\Upsilon))} : \Upsilon \in X, i = 1, \cdots, m \right\}$$

Definition 2.9. [25] The *intersection* of any PmFNSs $\Im_1$ and $\Im_2$ expressed over the same universe X is represented as

$$\Im_1 \cap_{\mathfrak{M}} \Im_2 = \left\{ \frac{\Upsilon}{(\min(T_{\Im1}^{(i)}(\Upsilon), T_{\Im2}^{(i)}(\Upsilon)), \max(I_{\Im1}^{(i)}(\Upsilon), I_{\Im2}^{(i)}(\Upsilon)), \max(F_{\Im1}^{(i)}(\Upsilon), F_{\Im2}^{(i)}(\Upsilon))} : \Upsilon \in X, i = 1, \cdots, m \right\}$$

3. Pythagorean m-Polar Fuzzy Neutrosophic Metric Spaces

In this section, we introduce the notion of Pythagorean m-polar fuzzy neutrosophic metric space along with its prime characteristics and illustrations. The superscript i, wherever used, will run from 1 to m, unless stated otherwise.

Definition 3.1. Let $\eth_1$, $\eth_2$ and $\eth_3$ be three PmFNSs on $\underline{\mathrm{X}}$. The mapping $\underline{\mathbb{M}}^s : PmFN(\underline{\mathrm{X}}) \times PmFN(\underline{\mathrm{X}}) \mapsto [0, 2]$ is said to be a *Pythagorean m-polar fuzzy neutrosophic metric* on $PmFN(\underline{\mathrm{X}})$ if it ensures the following postulates:

- $\underline{\mathbb{M}}^s_1$: $0 \leq \underline{\mathbb{M}}^s(\eth_1, \eth_2) \leq 2$
- $\underline{\mathbb{M}}^s_2$: $\underline{\mathbb{M}}^s(\eth_1, \eth_2) = \underline{\mathbb{M}}^s(\eth_2, \eth_1)$
- $\underline{\mathbb{M}}^s_3$: $\underline{\mathbb{M}}^s(\eth_1, \eth_2) = 0 \Leftrightarrow \eth_1 = \eth_2$
- $\underline{\mathbb{M}}^s_4$: $\underline{\mathbb{M}}^s(\eth_1, \eth_3) \leq \underline{\mathbb{M}}^s(\eth_1, \eth_2) + \underline{\mathbb{M}}^s(\eth_2, \eth_3)$
- $\underline{\mathbb{M}}^s_5$: If $\eth_1 \subseteq \eth_2 \subseteq \eth_3$, then $\underline{\mathbb{M}}^s(\eth_1, \eth_2) \leq \underline{\mathbb{M}}^s(\eth_1, \eth_3)$ and $\underline{\mathbb{M}}^s(\eth_2, \eth_3) \leq \underline{\mathbb{M}}^s(\eth_1, \eth_3)$

for all $\eth_1$, $\eth_2$ and $\eth_3 \Subset PmFN(\underline{\mathrm{X}})$.

The pair $(PmFN(\underline{\mathrm{X}}), \underline{\mathbb{M}}^s)$ is said to be the *Pythagorean m-polar fuzzy neutrosophic metric space* (PmFNMS). $PmFN(\underline{\mathrm{X}})$ is known as the *Pythagorean m-polar fuzzy neutrosophic underlying set* (PmFN-underlying set) or the *Pythagorean m-polar fuzzy neutrosophic ground set* (PmFN-ground set). The elements of $PmFN(\underline{\mathrm{X}})$ are called the *Pythagorean m-polar fuzzy neutrosophic points* (PmFN-points) of the PmFNMS $(PmFN(\underline{\mathrm{X}}), \underline{\mathbb{M}}^s)$.

Remark 3.2. If $\eth_1,\ \eth_2,\ \eth_3,\cdots,\eth_{n-1},\ \eth_n$ are n distinct PmFN points of the PmFNMS $(PmFN(\underline{X}),\ \underline{\mathbb{M}}^s)$, then the fourth postulate may be generalized as

$$\underline{\mathbb{M}}^s(\eth_1,\ \eth_n)\ \leq\ \underline{\mathbb{M}}^s(\eth_1,\ \eth_2)\ +\ \underline{\mathbb{M}}^s(\eth_2,\ \eth_3)\ +\ \underline{\mathbb{M}}^s(\eth_3,\ \eth_4)\ +\ \cdots\ +\underline{\mathbb{M}}^s(\eth_{n-1},\eth_n)$$

Example 3.3. Let

$$\eth_1\ =\ \{\langle(\underline{\daleth}_1^{(i)},\ \underline{I}_1^{(i)},\ \underline{E}_1^{(i)})\rangle\},$$

and

$$\eth_2\ =\ \{\langle(\underline{\daleth}_2^{(i)},\ \underline{I}_2^{(i)},\ \underline{E}_2^{(i)})\rangle\}$$

be members of $PmFN(\underline{X})$. We establish that

$$\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)=\sqrt[2m]{\sum_{i=1}^{m}\left\{(\underline{\daleth}_1^{(i)}-\underline{\daleth}_2^{(i)})^{2m}+(\underline{I}_1^{(i)}-\underline{I}_2^{(i)})^{2m}+(\underline{E}_1^{(i)}-\underline{E}_2^{(i)})^{2m}\right\}}$$

is a PmFNMS on $PmFN(\underline{X})$.

$\underline{\mathbb{M}}_2^s$ and $\underline{\mathbb{M}}_3^s$ of Definition 3.1 are obvious. We establish the remaining requirements.

$\underline{\mathbb{M}}_1^s$: Since $0\leq(\underline{\daleth}_1^{(i)}-\underline{\daleth}_2^{(i)})^{2m}\leq 1,\ 0\leq(\underline{I}_1^{(i)}-\underline{I}_2^{(i)})^{2m}\leq 1,\ 0\leq(\underline{E}_1^{(i)}-\underline{E}_2^{(i)})^{2m}\leq 1$
$\Rightarrow 0\leq\ \sqrt[2m]{\sum_{i=1}^{m}\left\{(\underline{\daleth}_1^{(i)}-\underline{\daleth}_2^{(i)})^{2m}+(\underline{I}_1^{(i)}-\underline{I}_2^{(i)})^{2m}+(\underline{E}_1^{(i)}-\underline{E}_2^{(i)})^{2m}\right\}}\leq 2$
Thus,

$$0\leq\ \underline{\mathbb{M}}^s(\eth_1,\ \eth_2)\ \leq\ 2$$

$\forall\ \eth_1,\ \eth_2\in PmFN(\underline{X})$.

$\underline{\mathbb{M}}_4^s$: By virtue of Minkowski's inequality, we have
$[(\underline{\daleth}_1^{(i)}-\underline{\daleth}_3^{(i)})^{2m}+(\underline{I}_1^{(i)}-\underline{I}_3^{(i)})^{2m}+(\underline{E}_1^{(i)}-\underline{E}_3^{(i)})^{2m}]^{\frac{1}{2m}}\ \leq\ [(\underline{\daleth}_1^{(i)}-\underline{\daleth}_2^{(i)})^{2m}+(\underline{I}_1^{(i)}-\underline{I}_2^{(i)})^{2m}+(\underline{E}_1^{(i)}-\underline{E}_2^{(i)})^{2m}]^{\frac{1}{2m}}\ +\ [(\underline{\daleth}_2^{(i)}-\underline{\daleth}_3^{(i)})^{2m}+(\underline{I}_2^{(i)}-\underline{I}_3^{(i)})^{2m}+(\underline{E}_2^{(i)}-\underline{E}_3^{(i)})^{2m}]^{\frac{1}{2m}}$
$\Rightarrow\ \underline{\mathbb{M}}^s(\eth_1,\ \eth_3)\ \leq\ \underline{\mathbb{M}}^s(\eth_1,\ \eth_2)\ +\ \underline{\mathbb{M}}^s(\eth_2,\ \eth_3)$
$\forall\ \eth_1,\ \eth_2,\ \eth_3\in PmFN(\underline{X})$.

$\underline{\mathbb{M}}_5^s$: If $\eth_1\ \subseteq\ \eth_2\ \subseteq\ \eth_3$, then

$$\begin{aligned}\underline{\daleth}_1^{(i)}\ &\leq\ \underline{\daleth}_2^{(i)}\ \leq\ \underline{\daleth}_3^{(i)},\\ \underline{I}_1^{(i)}\ &\geq\ \underline{I}_2^{(i)}\ \geq\ \underline{I}_3^{(i)},\\ \underline{E}_1^{(i)}\ &\geq\ \underline{E}_2^{(i)}\ \geq\ \underline{E}_3^{(i)}\end{aligned}$$

so that
$\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)\ =\ [(\underline{\daleth}_1^{(i)}-\underline{\daleth}_2^{(i)})^{2m}+(\underline{I}_1^{(i)}-\underline{I}_2^{(i)})^{2m}+(\underline{E}_1^{(i)}-\underline{E}_2^{(i)})^{2m}]^{\frac{1}{2m}}$
$\because\ \underline{\mathbb{M}}^s(\eth_1,\ \eth_2)\ =\ \underline{\mathbb{M}}^s(\eth_2,\ \eth_1)$, from $\underline{\mathbb{M}}_2^s$
So,
$\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)\ =\ [(\underline{\daleth}_2^{(i)}-\underline{\daleth}_1^{(i)})^{2m}+(\underline{I}_2^{(i)}-\underline{I}_1^{(i)})^{2m}+(\underline{E}_2^{(i)}-\underline{E}_1^{(i)})^{2m}]^{\frac{1}{2m}}$
and
$\underline{\mathbb{M}}^s(\eth_1,\ \eth_3)\ =\ [(\underline{\daleth}_1^{(i)}-\underline{\daleth}_3^{(i)})^{2m}+(\underline{I}_1^{(i)}-\underline{I}_3^{(i)})^{2m}+(\underline{E}_1^{(i)}-\underline{E}_3^{(i)})^{2m}]^{\frac{1}{2m}}$

again from $\underline{\mathbb{M}}_2^s$, we have
$\underline{\mathbb{M}}^s(\eth_1,\ \eth_3) \ = \ [(\underline{\daleth}_3^{(i)} - \underline{\daleth}_1^{(i)})^{2m} + (\underline{I}_3^{(i)} - \underline{I}_1^{(i)})^{2m} + (\underline{F}_3^{(i)} - \underline{F}_1^{(i)})^{2m}]^{\frac{1}{2m}}$
Now, if $\eth_2 \ \subseteq \ \eth_3$, then

$$\begin{aligned} \underline{\daleth}_2^{(i)} &\leq \underline{\daleth}_3^{(i)} \\ \Rightarrow \underline{\daleth}_2^{(i)} - \underline{\daleth}_1^{(i)} &\leq \underline{\daleth}_3^{(i)} - \underline{\daleth}_1^{(i)} \\ \Rightarrow (\underline{\daleth}_2^{(i)} - \underline{\daleth}_1^{(i)})^{2m} &\leq (\underline{\daleth}_3^{(i)} - \underline{\daleth}_1^{(i)})^{2m} \end{aligned}$$

Also,

$$\begin{aligned} \underline{I}_2^{(i)} &\geq \underline{I}_3^{(i)} \\ \Rightarrow -\underline{I}_2^{(i)} &\leq -\underline{I}_3^{(i)} \\ \Rightarrow \underline{I}_1^{(i)} - \underline{I}_2^{(i)} &\leq \underline{I}_1^{(i)} - \underline{I}_3^{(i)} \\ \Rightarrow (\underline{I}_1^{(i)} - \underline{I}_2^{(i)})^{2m} &\leq (\underline{I}_1^{(i)} - \underline{I}_3^{(i)})^{2m} \end{aligned}$$

and

$$\begin{aligned} \underline{I}_2^{(i)} &\geq \underline{I}_3^{(i)} \\ \Rightarrow -\underline{F}_2^{(i)} &\leq -\underline{F}_3^{(i)} \\ \Rightarrow \underline{F}_1^{(i)} - \underline{F}_2^{(i)} &\leq \underline{F}_1^{(i)} - \underline{F}_3^{(i)} \\ \Rightarrow (\underline{F}_1^{(i)} - \underline{F}_2^{(i)})^{2m} &\leq (\underline{F}_1^{(i)} - \underline{F}_3^{(i)})^{2m} \end{aligned}$$

It follows from above inequalities that $\underline{\mathbb{M}}^s(\eth_1,\ \eth_2) \leq \underline{\mathbb{M}}^s(\eth_1,\ \eth_3)$.

The other inclusion may be established on the parallel track.

Thus, $\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)$ is a PmFNMS on $PmFN(\underline{X})$.

Example 3.4. Consider the PmFNSs $\eth_1,\ \eth_2$ and $\eth_3$ given in Example 3.3. Then, none of the following is a PmFNMS on $PmFN(\underline{X})$:

(1) $\underline{\mathbb{M}}_r^s(\eth_1,\ \eth_2) = \max_i\{\underline{\daleth}_1^{(i)}, \underline{\daleth}_2^{(i)}\} + \max_i\{\underline{I}_1^{(i)}, \underline{I}_2^{(i)}\} + \max_i\{\underline{F}_1^{(i)}, \underline{F}_2^{(i)}\}$.
(2) $\underline{\mathbb{M}}_t^s(\eth_1,\ \eth_2) = \min_i\{\underline{\daleth}_1^{(i)}, \underline{\daleth}_2^{(i)}\} + \min_i\{\underline{I}_1^{(i)}, \underline{I}_2^{(i)}\} + \min_i\{\underline{F}_1^{(i)}, \underline{F}_2^{(i)}\}$.
(3) $\underline{\mathbb{M}}_b^s(\eth_1, \eth_2) = \sum_{i=1}^m \left(\underline{\daleth}_1^{(i)} + \underline{\daleth}_2^{(i)} + \underline{I}_1^{(i)} + \underline{I}_2^{(i)} + \underline{F}_1^{(i)} + \underline{F}_2^{(i)}\right)$.
(4) $\underline{\mathbb{M}}_c^s(\eth_1, \eth_2) = \sum_{i=1}^m \left\{(\underline{\daleth}_1^{(i)} + \underline{\daleth}_2^{(i)})^2 + (\underline{I}_1^{(i)} + \underline{I}_2^{(i)})^2 + (\underline{F}_1^{(i)} + \underline{F}_2^{(i)})^2\right\}$.
(5) $\underline{\mathbb{M}}_d^s(\eth_1, \eth_2) = \sqrt{\sum_{i=1}^m \left\{(\underline{\daleth}_1^{(i)} + \underline{\daleth}_2^{(i)})^2 + (\underline{I}_1^{(i)} + \underline{I}_2^{(i)})^2 + (\underline{F}_1^{(i)} + \underline{F}_2^{(i)})^2\right\}}$.
(6) $\underline{\mathbb{M}}_e^s(\eth_1, \eth_2) = \sum_{i=1}^m \left\{\left((\underline{\daleth}_1^{(i)} + \underline{\daleth}_2^{(i)})^2 + (\underline{I}_1^{(i)} + \underline{I}_2^{(i)})^2 + (\underline{F}_1^{(i)} + \underline{F}_2^{(i)})^2\right)^n\right\}$, where $n \in \mathbb{R}$.
(7) $\underline{\mathbb{M}}_u^s(\eth_1,\ \eth_2) = \max_i\{\underline{\daleth}_1^{(i)} + \underline{\daleth}_2^{(i)}, \underline{I}_1^{(i)} + \underline{I}_2^{(i)}, \underline{F}_1^{(i)} + \underline{F}_2^{(i)}\}$.

In case of (1), if $\eth_1 = \eth_2$ i.e. if $\underline{\daleth}_1^{(i)} = \underline{\daleth}_2^{(i)},\ \underline{I}_1^{(i)} = \underline{I}_2^{(i)}$ and $\underline{F}_1^{(i)} = \underline{F}_2^{(i)}$, then it is not necessary that $\underline{\daleth}_1^{(i)} = \underline{\daleth}_2^{(i)} = \ \underline{I}_1^{(i)} = \underline{I}_2^{(i)} = \underline{F}_1^{(i)} = \underline{F}_2^{(i)} = 0$.
Therefore, $\eth_1 = \eth_2 \not\Rightarrow \underline{\mathbb{M}}_s^r(\eth_1,\ \eth_2) \ = \ 0$. Hence, $\underline{\mathbb{M}}_r^s(\eth_1,\ \eth_2)$ is not a PmFNMS on $PmFN(\underline{X})$.
Same reasoning holds good for (2).

For (3), it is not guaranteed that the sum on the RHS will not exceed 2. So, $\underline{\mathbb{M}}_b^s$ fails to be a PmFNMS on $PmFN(\underline{\mathbb{X}})$. The same argument is valid for (4), (5) and (6).
The analogous issue arises in case of (7).

Example 3.5. Let $PmFN(\underline{\mathbb{X}}) = \{x_1, x_2\}$ be the universal sets with three P3FNSs as given in Tables 2 - 4.

TABLE 2. P3FNS $\mathbb{M}_1$

$\mathbb{M}_1$			
x_1	(0.417, 0.312, 0.356)	(0.012, 0.374, 0.436)	(0.811, 0.363, 0.272)
x_2	(0.712, 0.117, 0.562)	(0.333, 0.672, 0.891)	(0.068, 0.772, 0.921)

TABLE 3. P3FNS $\mathbb{N}_1$

$\mathbb{N}_1$			
x_1	(0.811, 0.062, 0.211)	(0.312, 0.270, 0.137)	(0.921, 0.266, 0.152)
x_2	(0.932, 0.101, 0.431)	(0.466, 0.352, 0.721)	(0.368, 0.572, 0.900)

and

TABLE 4. P3FNS $\mathbb{O}_1$

$\mathbb{O}_1$			
x_1	(0.932, 0.001, 0.200)	(0.527, 0.170, 0.007)	(1.000, 0.062, 0.008)
x_2	(0.982, 0.001, 0.231)	(0.667, 0.252, 0.421)	(0.766, 0.423, 0.262)

where $\mathbb{M}_1,\ \mathbb{N}_1,\ \mathbb{O}_1 \Subset P3FN(\underline{\mathbb{X}})$ and $\mathbb{M}_1 \subset \mathbb{N}_1 \subset \mathbb{O}_1$. We show that $\underline{\mathbb{M}}^s(\mathbb{M}_1,\ \mathbb{N}_1)$ is a P3FNMS on $PmFN(\underline{\mathbb{X}})$ if

$$\underline{\mathbb{M}}_\alpha^s(\mathbb{M}_1,\ \mathbb{N}_1) \ = \ \sqrt{\sum_i \left\{(\daleth_1^{(i)} - \daleth_2^{(i)})^2 + (\underline{I}_1^{(i)} - \underline{I}_2^{(i)})^2 + (\underline{F}_1^{(i)} - \underline{F}_2^{(i)})^2\right\}}$$

$\underline{\mathbb{M}}_1^s$: $\underline{\mathbb{M}}_\alpha^s(\mathbb{M}_1,\ \mathbb{N}_1) = \sqrt{0.239 + 0.190 + 0.036 + 0.066 + 0.149 + 0.130} = 0.900$
$\Rightarrow\ 0\ \leq \underline{\mathbb{M}}_\alpha^s(\mathbb{M}_1,\ \mathbb{N}_1) \leq 2.$

$\underline{\mathbb{M}}_2^s$: Obvious.

$\underline{\mathbb{M}}_3^s$: Obvious.

$\underline{\mathbb{M}}_4^s$: Since $\underline{\mathbb{M}}_\alpha^s(\mathbb{M}_1,\ \mathbb{O}_1) = 1.680$, $\underline{\mathbb{M}}_\alpha^s(\mathbb{N}_1,\ \mathbb{O}_1) = 0.970$, and $\underline{\mathbb{M}}_\alpha^s(\mathbb{M}_1,\ \mathbb{N}_1) = 0.900$, so $\underline{\mathbb{M}}_\alpha^s(\mathbb{M}_1,\ \mathbb{O}_1) \ \leq\ \underline{\mathbb{M}}_\alpha^s(\mathbb{M}_1,\ \mathbb{N}_1) + \underline{\mathbb{M}}_\alpha^s(\mathbb{N}_1,\ \mathbb{O}_1)$

$\underline{\mathbb{M}}_5^s$: $\mathbb{M}_1 \subset \mathbb{N}_1 \subset \mathbb{O}_1 \Rightarrow \underline{\mathbb{M}}_\alpha^s(\mathbb{M}_1,\ \mathbb{N}_1)\ <\ \underline{\mathbb{M}}_\alpha^s(\mathbb{M}_1,\ \mathbb{O}_1)$ and $\underline{\mathbb{M}}_\alpha^s(\mathbb{N}_1,\ \mathbb{O}_1)\ <\ \underline{\mathbb{M}}_\alpha^s(\mathbb{M}_1,\ \mathbb{O}_1)$ follows from above computations.

Thus, $\underline{\mathbb{M}}^s_\alpha(\mathbb{M}_1,\ \mathbb{N}_1)$ is a P3FNMS on $PmFN(\underline{\mathrm{X}})$, for $\mathbb{M}_1,\ \mathbb{N}_1,\ \mathbb{O}_1 \Subset P3FN(\underline{\mathrm{X}})$.

Proposition 3.6. *Let* $\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)$ *and* $\underline{\mathbb{M}}^s(\eth_3,\ \eth_4)$ *be two PmFNMSs on a PmFNS* $PmFN(\underline{X})$*, then* $\underline{\mathbb{M}}^s_f[(\eth_1,\ \eth_3),(\eth_2,\ \eth_4)] = \underline{\mathbb{M}}^s(\eth_1,\ \eth_2) + \underline{\mathbb{M}}^s(\eth_3,\ \eth_4)$ *is not a PmFNMS on* $PmFN(\underline{X}) \times PmFN(\underline{X})$.

Proof. Since $\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)$ and $\underline{\mathbb{M}}^s(\eth_3,\ \eth_4)$ are two PmFNMSs. Therefore, by definition

$$0 \ \leq\ \underline{\mathbb{M}}^s(\eth_1,\ \eth_2)\ \leq\ 2 \text{ and } 0\ \leq\ \underline{\mathbb{M}}^s(\eth_3,\ \eth_4)\ \leq\ 2$$
$$\Rightarrow\ 0\ \leq\ \underline{\mathbb{M}}^s(\eth_1,\ \eth_2) + \underline{\mathbb{M}}^s(\eth_3,\ \eth_4)\ \leq\ 4$$
$$\Rightarrow\ 0\ \leq\ \underline{\mathbb{M}}^s_f[(\eth_1,\ \eth_3),(\eth_2,\ \eth_4)]\ \leq\ 4$$

So, $\underline{\mathbb{M}}^s_f[(\eth_1,\ \eth_3),(\eth_2,\ \eth_4)]$ is not a PmFNMS on $PmFN(\underline{\mathrm{X}}) \times PmFN(\underline{\mathrm{X}})$.
0.1cm□

Remark 3.7. It is interesting to note that the distance defined in the way as in Proposition 3.6 yields metric space in crisp sets but fails to hold in PmFNSs.

Example 3.8. Consider the PmFNSs $PmFN(\underline{\mathrm{X}})$, $\mathbb{M}_1,\ \mathbb{N}_1$ and $\mathbb{O}_1$ given in Example 3.5 and $\mathbb{M}_2,\ \mathbb{N}_2$ and $\mathbb{O}_2$ given in Tables 5, 6 and 7, respectively:

Table 5. P3FNS $\mathbb{M}_2$

$\mathbb{M}_2$			
x_1	(0.444, 0.123, 0.256)	(0.114, 0.274, 0.336)	(0.901, 0.269, 0.117)
x_2	(0.882, 0.107, 0.432)	(0.441, 0.521, 0.742)	(0.172, 0.710, 0.916)

Table 6. P3FNS $\mathbb{N}_2$

$\mathbb{N}_2$			
x_1	(0.844, 0.002, 0.201)	(0.332, 0.260, 0.037)	(0.922, 0.261, 0.109)
x_2	(0.936, 0.006, 0.331)	(0.470, 0.262, 0.621)	(0.468, 0.472, 0.889)

Table 7. P3FNS $\mathbb{O}_2$

$\mathbb{O}_2$			
x_1	(0.992, 0.001, 0.169)	(0.627, 0.070, 0.006)	(1.000, 0.032, 0.006)
x_2	(0.988, 0.001, 0.201)	(0.676, 0.152, 0.411)	(0.862, 0.413, 0.216)

$\forall\ \mathbb{M}_1,\ \mathbb{M}_2,\ \mathbb{N}_1,\ \mathbb{N}_2,\ \mathbb{O}_1,\ \mathbb{O}_2 \Subset P3FN(\underline{X})$. We check whether $\underline{\mathbb{M}}^s[(\mathbb{M}_1,\mathbb{M}_2),(\mathbb{N}_1,\mathbb{N}_2)] = \sum_{r=1}^{2}\underline{\mathbb{M}}^s(\mathbb{M}_r,\mathbb{N}_r)$ is a PmFNMS on $PmFN(\underline{X}) \times PmFN(\underline{X})$ or not?

Since $\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1) = 0.900$ and $\underline{\mathbb{M}}^s(\mathbb{M}_2,\mathbb{N}_2) = 0.756$, so that $\underline{\mathbb{M}}^s[(\mathbb{M}_1,\mathbb{M}_2),(\mathbb{N}_1,\mathbb{N}_2)] = 1.656$

$$\Rightarrow\ 0 \leq \underline{\mathbb{M}}^s[(\mathbb{M}_1,\mathbb{M}_2),(\mathbb{N}_1,\mathbb{N}_2)] \leq 2$$

But,

$$\begin{aligned}
\underline{\mathbb{M}}^s[(\mathbb{M}_1,\mathbb{M}_2),(\mathbb{O}_1,\mathbb{O}_2)] &= \underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{O}_1) + \underline{\mathbb{M}}^s(\mathbb{M}_2,\mathbb{O}_2) \\
&= \sqrt{0.386 + 0.491 + 0.196 + 0.196 + 0.509 + 1.043} \\
&+ \sqrt{0.323 + 0.414 + 0.095 + 0.076 + 0.301 + 1.054} \\
&= \sqrt{2.821} + \sqrt{2.263} \\
&= 3.182 \nleq 2
\end{aligned}$$

So, $\underline{\mathbb{M}}^s[(\mathbb{M}_1,\mathbb{M}_2),(\mathbb{N}_1,\mathbb{N}_2)]$ is not a P3FNMS on $PmFN(\underline{X}) \times PmFN(\underline{X})$.

Proposition 3.9. *Let* $\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)$ *and* $\underline{\mathbb{M}}^s(\eth_3,\ \eth_4)$ *be two PmFNMSs on PmFNS* $PmFN(\underline{X})$*, then*

(i) $\underline{\mathbb{M}}^s_g[(\eth_1,\ \eth_3),(\eth_2,\ \eth_4)] = \max\{\underline{\mathbb{M}}^s(\eth_1,\ \eth_2),\underline{\mathbb{M}}^s(\eth_3,\ \eth_4)\}$

(ii) $\underline{\mathbb{M}}^s_g[(\eth_1,\ \eth_3),(\eth_2,\ \eth_4)] = \min\{\underline{\mathbb{M}}^s(\eth_1,\ \eth_2),\underline{\mathbb{M}}^s(\eth_3,\ \eth_4)\}$

are PmFNMSs on $PmFN(\underline{X}) \times PmFN(\underline{X})$*.*

Proof. We prove (i) here. The proof of (ii) may be furnished on the parallel track.

$\underline{\mathbb{M}}^s_1$: Since $\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)$ and $\underline{\mathbb{M}}^s(\eth_3,\ \eth_4)$ are PmFNMSs on $PmFN(\underline{X})$.

$\Rightarrow\ 0 \leq \underline{\mathbb{M}}^s(\eth_1,\ \eth_2) \leq 2$ and $0 \leq \underline{\mathbb{M}}^s(\eth_3,\ \eth_4) \leq 2$

But then, $\max\{\underline{\mathbb{M}}^s(\eth_1,\ \eth_2),\underline{\mathbb{M}}^s(\eth_3,\ \eth_4)\}$, being the maximum of two non-negative and less than or equal to 2 quantities, is also non-negative and less than or equal to 2.

$\underline{\mathbb{M}}^s_2$: Obvious.

$\underline{\mathbb{M}}^s_3$:

$$\begin{aligned}
\underline{\mathbb{M}}^s_g[(\eth_1,\ \eth_3),(\eth_2,\ \eth_4)] = 0 \ &\Leftrightarrow\ \max\{\underline{\mathbb{M}}^s(\eth_1,\ \eth_2),\underline{\mathbb{M}}^s(\eth_3,\ \eth_4)\} = 0 \\
&\Leftrightarrow\ \underline{\mathbb{M}}^s(\eth_1,\ \eth_2) = 0,\ \underline{\mathbb{M}}^s(\eth_3,\ \eth_4) = 0 \\
&\Leftrightarrow\ \eth_1 = \eth_2,\ \eth_3 = \eth_4 \\
&\Leftrightarrow\ (\eth_1,\eth_3) = (\eth_2,\eth_4)
\end{aligned}$$

$\underline{\mathbb{M}}^s_4$: $\underline{\mathbb{M}}^s_g[(\eth_1,\ \eth_3),(\eth_5,\ \eth_6)] = \max\{\underline{\mathbb{M}}^s(\eth_1,\ \eth_5),\underline{\mathbb{M}}^s(\eth_3,\ \eth_6)\}$

Let $\underline{\mathbb{M}}^s_g[(\eth_1,\ \eth_3),(\eth_5,\ \eth_6)] = \underline{\mathbb{M}}^s(\eth_1,\ \eth_5)$. Then,

$\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)\ \leq\ \max\{\underline{\mathbb{M}}^s(\eth_1,\ \eth_2),\underline{\mathbb{M}}^s(\eth_3,\ \eth_4)\}$

and

$\underline{\mathbb{M}}^s(\eth_2,\ \eth_5)\ \leq\ \max\{\underline{\mathbb{M}}^s(\eth_2,\ \eth_5), \underline{\mathbb{M}}^s(\eth_4,\ \eth_6)\}$

Since $\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)$ is a PmFNMS. Therefore,

$\underline{\mathbb{M}}^s(\eth_1,\ \eth_5)\ \leq\ \underline{\mathbb{M}}^s(\eth_1,\ \eth_2) + \underline{\mathbb{M}}^s(\eth_2,\ \eth_5)$

$\Rightarrow \max\ [\underline{\mathbb{M}}^s(\eth_1,\ \eth_5), \underline{\mathbb{M}}^s(\eth_3,\ \eth_6)]\ \leq\ \underline{\mathbb{M}}^s(\eth_1,\ \eth_2) + \underline{\mathbb{M}}^s(\eth_2,\ \eth_5)\ \leq \max\{\underline{\mathbb{M}}^s(\eth_1,\ \eth_2), \underline{\mathbb{M}}^s(\eth_3,\ \eth_4)\} + [\underline{\mathbb{M}}^s(\eth_2,\ \eth_5), \underline{\mathbb{M}}^s(\eth_4,\ \eth_6)]$

$\Rightarrow\ \underline{\mathbb{M}}^s_g[(\eth_1,\ \eth_3),(\eth_5,\ \eth_6)] \leq \underline{\mathbb{M}}^s_g[(\eth_1,\ \eth_3),(\eth_2,\ \eth_4)] + \underline{\mathbb{M}}^s_g[(\eth_2,\ \eth_4),(\eth_5,\ \eth_6)]$

$\underline{\mathbb{M}}^s_5$: Straight forward.

Thus, $\underline{\mathbb{M}}^s_g[(\eth_1,\ \eth_3),(\eth_2,\ \eth_4)],\ \forall\ \eth_1,\ \eth_2,\ \eth_3,\ \eth_4,\ \eth_5,\ \eth_6 \Subset PmFN(\underline{\mathbb{X}})$ is a PmFNMS on $PmFN(\underline{\mathbb{X}}) \times PmFN(\underline{\mathbb{X}})$.

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Remark 3.10. Proposition 3.9 shows that more than one PmFNMSs can be defined on a single PmFNS.

Example 3.11. Consider the P3FNSs given in Example 3.5 and 3.8. We check whether

(i) $\underline{\mathbb{M}}^s_1[(\mathbb{M}_1, \mathbb{M}_2),(\mathbb{N}_1, \mathbb{N}_2)] = \max\{\underline{\mathbb{M}}^s(\mathbb{M}_1, \mathbb{N}_1), \underline{\mathbb{M}}^s(\mathbb{M}_2, \mathbb{N}_2)\}$

(ii) $\underline{\mathbb{M}}^s_2[(\mathbb{M}_1, \mathbb{M}_2),(\mathbb{N}_1, \mathbb{N}_2)] = \min\{\underline{\mathbb{M}}^s(\mathbb{M}_1, \mathbb{N}_1), \underline{\mathbb{M}}^s(\mathbb{M}_2, \mathbb{N}_2)\}$

where $\underline{\mathbb{M}}^s(\mathbb{M}_1, \mathbb{M}_2)$ and $\underline{\mathbb{M}}^s[(\mathbb{M}_3, \mathbb{M}_4)$ are P3FNMSs on $PmFN(\underline{\mathbb{X}}) \times PmFN(\underline{\mathbb{X}})$ or not?

For (i):

$\underline{\mathbb{M}}^s_1$:

$$\begin{aligned} \underline{\mathbb{M}}^s_1[(\mathbb{M}_1, \mathbb{M}_2),(\mathbb{N}_1, \mathbb{N}_2)] &= \max\{\underline{\mathbb{M}}^s(\mathbb{M}_1, \mathbb{N}_1), \underline{\mathbb{M}}^s(\mathbb{M}_2, \mathbb{N}_2)\} \\ &= \max\{0.900, 0.756\} \\ &= 0.900 \end{aligned}$$

$$\Rightarrow\ 0 \leq \underline{\mathbb{M}}^s_1[(\mathbb{M}_1, \mathbb{M}_2),(\mathbb{N}_1, \mathbb{N}_2)] \leq 2$$

$\underline{\mathbb{M}}^s_2$: Obvious.

$\underline{\mathbb{M}}^s_3$: Obvious.

$\underline{\mathbb{M}}^s_4$: $\underline{\mathbb{M}}^s_1[(\mathbb{M}_1, \mathbb{M}_2),(\mathbb{O}_1, \mathbb{O}_2)] = \max\{\underline{\mathbb{M}}^s(\mathbb{M}_1, \mathbb{O}_1), \underline{\mathbb{M}}^s(\mathbb{M}_2, \mathbb{O}_2)\} = \max\{1.680, 1.504\} = 1.680$.

Also,

$\underline{\mathbb{M}}^s_1[(\mathbb{M}_1, \mathbb{M}_2),(\mathbb{N}_1, \mathbb{N}_2)] = \max\{\underline{\mathbb{M}}^s(\mathbb{M}_1, \mathbb{N}_1), \underline{\mathbb{M}}^s(\mathbb{M}_2, \mathbb{N}_2)] = \max\{0.900, 0.756\} = 0.900$,

and

$\underline{\mathbb{M}}^s_1[(\mathbb{N}_1, \mathbb{N}_2),(\mathbb{O}_1, \mathbb{O}_2)] = \max\{\underline{\mathbb{M}}^s(\mathbb{N}_1, \mathbb{O}_1), \underline{\mathbb{M}}^s(\mathbb{N}_2, \mathbb{O}_2)\} = \max\{0.970, 0.973\} = 0.973$.

$\Rightarrow\ \underline{\mathbb{M}}^s_1[(\mathbb{M}_1, \mathbb{M}_2),(\mathbb{N}_1, \mathbb{N}_2)] + \underline{\mathbb{M}}^s_1[(\mathbb{N}_1, \mathbb{N}_2),(\mathbb{O}_1, \mathbb{O}_2)] = 0.900 + 0.973 = 1.873$

Thus,

$$\underline{\mathbb{M}}_1^s[(\mathbb{M}_1,\mathbb{M}_2),(\mathbb{O}_1,\mathbb{O}_2)] < \underline{\mathbb{M}}_1^s[(\mathbb{M}_1,\mathbb{M}_2),(\mathbb{N}_1,\mathbb{N}_2)] + \underline{\mathbb{M}}_1^s[(\mathbb{N}_1,\mathbb{N}_2),(\mathbb{O}_1,\mathbb{O}_2)]$$

$\underline{\mathbb{M}}_5^s$: As given, $\mathbb{M}_1 \subseteq \mathbb{N}_1 \subseteq \mathbb{O}_1$ and $\mathbb{M}_2 \subseteq \mathbb{N}_2 \subseteq \mathbb{O}_2$

$$\underline{\mathbb{M}}_1^s[(\mathbb{M}_1,\mathbb{M}_2),(\mathbb{N}_1,\mathbb{N}_2)] = 0.900,$$
$$\underline{\mathbb{M}}_1^s[(\mathbb{M}_1,\mathbb{M}_2),(\mathbb{O}_1,\mathbb{O}_2)] = 1.680,$$

and

$$\underline{\mathbb{M}}_1^s[(\mathbb{N}_1,\mathbb{N}_2),(\mathbb{O}_1,\mathbb{O}_2)] = 0.973.$$

It may be observed that

$$\underline{\mathbb{M}}_1^s[(\mathbb{M}_1,\mathbb{M}_2),(\mathbb{N}_1,\mathbb{N}_2)] < \underline{\mathbb{M}}_1^s[(\mathbb{M}_1,\mathbb{M}_2),(\mathbb{O}_1,\mathbb{O}_2)]$$

and

$$\underline{\mathbb{M}}_1^s[(\mathbb{N}_1,\mathbb{N}_2),(\mathbb{O}_1,\mathbb{O}_2)] < \underline{\mathbb{M}}_1^s[(\mathbb{M}_1,\mathbb{M}_2),(\mathbb{O}_1,\mathbb{O}_2)]$$

Hence, $\underline{\mathbb{M}}_1^s[(\mathbb{M}_1,\mathbb{M}_2),(\mathbb{N}_1,\mathbb{N}_2)]$ is a P3FNMS on $PmFN(\underline{\mathbb{X}}) \times PmFN(\underline{\mathbb{X}})$.

(ii) may be proved likewise.

Example 3.12. Consider the P3FNMS given in Example 3.5, then following are not P3FNMSs on $PmFN(\underline{\mathbb{X}})$.

(i) $\underline{\mathbb{M}}_4^s(\mathbb{M}_1,\mathbb{N}_1) = \frac{\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1)}{1-\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1)}$

(ii) $\underline{\mathbb{M}}_5^s(\mathbb{M}_1,\mathbb{N}_1) = \frac{1-\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1)}{\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1)}$

(iii) $\underline{\mathbb{M}}_6^s(\mathbb{M}_1,\mathbb{N}_1) = \frac{3-\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1)}{1+\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1)}$

(iv) $\underline{\mathbb{M}}_7^s(\mathbb{M}_1,\mathbb{N}_1) = \frac{\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1)}{3-\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1)}$

(v) $\underline{\mathbb{M}}_8^s(\mathbb{M}_1,\mathbb{N}_1) = \frac{1-\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1)}{1+\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1)}$

We prove them one by one as follows:

(i) Since $\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1) = 0.900$, so

$$\underline{\mathbb{M}}_4^s(\mathbb{M}_1,\mathbb{N}_1) = \frac{\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1)}{1-\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1)} = \frac{0.900}{1-0.900} = 9 \nleq 2$$

and hence $\underline{\mathbb{M}}_4^s(\mathbb{M}_1,\mathbb{N}_1) = \frac{\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1)}{1-\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1)}$ is not a P3FNMS on $PmFN(\underline{\mathbb{X}})$.

(ii) Since

$$\underline{\mathbb{M}}_5^s(\mathbb{M}_1,\mathbb{M}_1) = \frac{1-\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{M}_1)}{\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{M}_1)} = \frac{1-0}{0} = \frac{1}{0}$$

which is undefined. So, $\underline{\mathbb{M}}_5^s(\mathbb{M}_1,\mathbb{N}_1)$ is not a P3FNMS on $PmFN(\underline{\mathbb{X}})$.

(iii) Since

$$\underline{\mathbb{M}}_6^s(\mathbb{M}_1,\mathbb{M}_1) = \frac{3-\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{M}_1)}{1+\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{M}_1)} = \frac{3-0}{1+0} = 3 \neq 0$$

Hence, $\underline{\mathbb{M}}_6^s(\mathbb{M}_1,\mathbb{N}_1)$ is not a P3FNMS on $PmFN(\underline{\mathbb{X}})$.

(iv) Since

$$\underline{\mathbb{M}}_7^s(\mathbb{M}_1,\mathbb{O}_1) = \frac{\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{O}_1)}{3-\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{O}_1)} = \frac{1.680}{3-1.680} = 1.273$$
$$\underline{\mathbb{M}}_7^s(\mathbb{M}_1,\mathbb{N}_1) = \frac{\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1)}{3-\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1)} = \frac{0.900}{3-0.900} = 0.429$$

and

$$\underline{\mathbb{M}}_7^s(\mathbb{N}_1, \mathbb{O}_1) = \frac{\underline{\mathbb{M}}^s(\mathbb{N}_1,\mathbb{O}_1)}{3-\underline{\mathbb{M}}^s(\mathbb{N}_1,\mathbb{O}_1)} = \frac{0.970}{3-0.970} = 0.478$$

so,

$$\underline{\mathbb{M}}_7^s(\mathbb{M}_1, \mathbb{O}_1) \not\leq \underline{\mathbb{M}}_7^s(\mathbb{M}_1, \mathbb{N}_1) + \underline{\mathbb{M}}_7^s(\mathbb{N}_1, \mathbb{O}_1)$$

and hence $\underline{\mathbb{M}}_7^s(\mathbb{M}_1, \mathbb{N}_1)$ is not a P3FNMS on $PmFN(\underline{\mathrm{X}})$.

(v) Since

$$\underline{\mathbb{M}}_8^s(\mathbb{M}_1, \mathbb{M}_1) = \frac{1-\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{M}_1)}{1+\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{M}_1)} = \frac{1-0}{1+0} = 1 \neq 0$$

Thus, $\underline{\mathbb{M}}_8^s(\mathbb{M}_1, \mathbb{N}_1)$ is not a P3FNMS on $PmFN(\underline{\mathrm{X}})$.

Remark 3.13. Let $\underline{\mathbb{M}}^s(\mathbb{M}_1, \mathbb{N}_1)$ be a PmFNMS on non-empty universal PmFNS $PmFN(\underline{\mathrm{X}})$, then

(1) $\underline{\mathbb{M}}^{s*}(\mathbb{M}_1, \mathbb{N}_1) = \frac{\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1)}{q-\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1)}$, where q is any integer, is not a PmFNMS on $PmFN(\underline{\mathrm{X}})$.

(2) $\underline{\mathbb{M}}^{s**}(\mathbb{M}_1, \mathbb{N}_1) = \frac{\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1)}{n+\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1)}$, where n is any natural number, is a PmFNMS on $PmFN(\underline{\mathrm{X}})$.

(3) Distance defined in this way yields metric spaces in crisp set but fails to hold in PmFNMSs.

Proposition 3.14. *Let* $\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)$ *be a PmFNMS on a non-empty universal PmFNS* $PmFN(\underline{X})$ *then* $\underline{\mathbb{M}}_f^s(\eth_1,\ \eth_2) = \frac{\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)}{n+\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)}$, *where* n *is any natural number, is also a PmFNMS on* $PmFN(\underline{X})$.

Proof. $\underline{\mathbb{M}}_1^s$: Since $\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)$ is a PmFNMS on $PmFN(\underline{\mathrm{X}})$. So,

$$0 \leq \underline{\mathbb{M}}^s(\eth_1,\ \eth_2) \leq 2 \Rightarrow 0 \leq \frac{\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)}{n+\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)} \leq 2 \Rightarrow 0 \leq \underline{\mathbb{M}}_f^s(\eth_1,\ \eth_2) \leq 2$$

$\underline{\mathbb{M}}_2^s$:

$$\begin{aligned} \underline{\mathbb{M}}_f^s(\eth_1,\ \eth_2) &= \frac{\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)}{n+\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)} \\ &= \frac{\underline{\mathbb{M}}^s(\eth_2,\ \eth_1)}{n+\underline{\mathbb{M}}^s(\eth_2,\ \eth_1)} \ (\because \underline{\mathbb{M}}^s(\eth_1,\ \eth_2) \text{ is a P}m\text{FNMS on } PmFN(\underline{\mathrm{X}})) \\ &= \underline{\mathbb{M}}_f^s(\eth_2,\ \eth_1) \end{aligned}$$

$\underline{\mathbb{M}}_3^s$:

$$\begin{aligned} \underline{\mathbb{M}}_f^s(\eth_1,\ \eth_2) = 0 \ &\Leftrightarrow \ \frac{\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)}{n+\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)} = 0 \\ &\Leftrightarrow \ \underline{\mathbb{M}}^s(\eth_1,\ \eth_2) = 0 \\ &\Leftrightarrow \ \eth_1 = \eth_2 \end{aligned}$$

Since $\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)$ is a PmFNMS on $PmFN(\underline{\mathrm{X}})$.

$\underline{\mathbb{M}}^s_4$: Since $\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)$ is a PmFNMS on $PmFN(\underline{\mathrm{X}})$. So,

$$\begin{aligned}
\underline{\mathbb{M}}^s(\eth_1,\ \eth_5) &\leq \underline{\mathbb{M}}^s(\eth_1,\ \eth_2)+\underline{\mathbb{M}}^s(\eth_2,\ \eth_5)\\
n+\underline{\mathbb{M}}^s(\eth_1,\ \eth_5) &\leq n+\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)+\underline{\mathbb{M}}^s(\eth_2,\ \eth_5)\\
\frac{1}{n+\underline{\mathbb{M}}^s(\eth_1,\ \eth_5)} &\geq \frac{1}{n+\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)+\underline{\mathbb{M}}^s(\eth_2,\ \eth_5)}\\
\frac{-n}{n+\underline{\mathbb{M}}^s(\eth_1,\ \eth_5)} &\leq \frac{-n}{n+\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)+\underline{\mathbb{M}}^s(\eth_2,\ \eth_5)}\\
1-\frac{n}{n+\underline{\mathbb{M}}^s(\eth_1,\ \eth_5)} &\leq 1-\frac{n}{n+\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)+\underline{\mathbb{M}}^s(\eth_2,\ \eth_5)}\\
\frac{n+\underline{\mathbb{M}}^s(\eth_1,\ \eth_5)-n}{n+\underline{\mathbb{M}}^s(\eth_1,\ \eth_5)} &\leq \frac{n+\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)+\underline{\mathbb{M}}^s(\eth_2,\ \eth_5)-n}{n+\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)+\underline{\mathbb{M}}^s(\eth_2,\ \eth_5)}\\
\frac{\underline{\mathbb{M}}^s(\eth_1,\ \eth_5)}{n+\underline{\mathbb{M}}^s(\eth_1,\ \eth_5)} &\leq \frac{\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)+\underline{\mathbb{M}}^s(\eth_2,\ \eth_5)}{n+\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)+\underline{\mathbb{M}}^s(\eth_2,\ \eth_5)}\\
&= \frac{\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)}{n+\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)+\underline{\mathbb{M}}^s(\eth_2,\ \eth_5)}+\frac{\underline{\mathbb{M}}^s(\eth_2,\ \eth_5)}{n+\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)+\underline{\mathbb{M}}^s(\eth_2,\ \eth_5)}\\
&\leq \frac{\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)}{n+\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)}+\frac{\underline{\mathbb{M}}^s(\eth_2,\ \eth_5)}{n+\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)}\\
\Rightarrow \underline{\mathbb{M}}^s_f(\eth_1,\ \eth_5) &\leq \underline{\mathbb{M}}^s_f(\eth_1,\ \eth_2)+\underline{\mathbb{M}}^s_f(\eth_2,\ \eth_5)
\end{aligned}$$

$\underline{\mathbb{M}}^s_5$: Since $\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)$ is a PmFNMS on $PmFN(\underline{\mathrm{X}})$. So, if $\eth_1 \subseteq \eth_2 \subseteq \eth_5$ then $\underline{\mathbb{M}}^s(\eth_1,\ \eth_2) \leq \underline{\mathbb{M}}^s(\eth_1,\ \eth_5)$ and $\underline{\mathbb{M}}^s(\eth_2,\ \eth_5) \leq \underline{\mathbb{M}}^s(\eth_1,\ \eth_5)$ then follows directly from definition.

Hence, $\underline{\mathbb{M}}^s_f(\eth_1,\ \eth_2)$ is also a PmFNMS on $PmFN(\underline{\mathrm{X}})$. 0.1cm□

Example 3.15. Consider the PmFNMS given in Example 3.5. We show that $\underline{\mathbb{M}}^s_3(\mathbb{M}_1,\mathbb{N}_1) = \frac{\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1)}{1+\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1)}$ is a PmFNMS on $PmFN(\underline{\mathrm{X}})$.

$\underline{\mathbb{M}}^s_1$: Since $\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{M}_2) = 0.900$ from Example 3.5, so
$\underline{\mathbb{M}}^s_3(\mathbb{M}_1,\mathbb{N}_1) = \frac{\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1)}{1+\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1)} = \frac{0.900}{1+0.900} = \frac{0.900}{1.900} = 0.474 \ \Rightarrow\ 0 \leq \underline{\mathbb{M}}^s_3(\mathbb{M}_1,\mathbb{N}_1) \leq 2.$

$\underline{\mathbb{M}}^s_2$: Obvious.

$\underline{\mathbb{M}}^s_3$: Obvious.

$\underline{\mathbb{M}}^s_4$: $\underline{\mathbb{M}}^s_3(\mathbb{M}_1,\mathbb{O}_1) = \frac{\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{O}_1)}{1+\underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{O}_1)} = \frac{1.680}{1+1.680} = 0.627$
$\underline{\mathbb{M}}^s_3(\mathbb{M}_1,\mathbb{N}_1) = 0.474$ and
$\underline{\mathbb{M}}^s_3(\mathbb{N}_1,\mathbb{O}_1) = \frac{\underline{\mathbb{M}}^s(\mathbb{N}_1,\mathbb{O}_1)}{1+\underline{\mathbb{M}}^s(\mathbb{N}_1,\mathbb{O}_1)} = \frac{0.970}{1+0.970} = 0.492$
$\therefore\ \underline{\mathbb{M}}^s_3(\mathbb{M}_1,\mathbb{O}_1) < \underline{\mathbb{M}}^s(\mathbb{M}_1,\mathbb{N}_1) + \underline{\mathbb{M}}^s_3(\mathbb{N}_1,\mathbb{O}_1)$

$\underline{\mathbb{M}}^s_5$: $\underline{\mathbb{M}}^s_3(\mathbb{M}_1,\mathbb{N}_1) = 0.474 < 0.627 = \underline{\mathbb{M}}^s_3(\mathbb{M}_1,\mathbb{O}_1)$
and $\underline{\mathbb{M}}^s_3(\mathbb{N}_1,\mathbb{O}_1) = 0.492 < 0.627 = \underline{\mathbb{M}}^s_3(\mathbb{M}_1,\mathbb{O}_1)$

Thus, $\underline{\mathbb{M}}^s_3(\mathbb{M}_1,\mathbb{N}_1)$ is also a PmFNMS on $PmFN(\underline{\mathrm{X}})$.

4. Application of proposed metrics in pattern recognition

In this section, we present an application of suggested metrics in pattern recognition. Pattern recognition is the science endued with diverse utilizations, mainly including speech and fingerprint recognition, medical imaging and diagnosis, aerial photo interpretation, image processing, and optical character recognition in scanned documents such as contracts and photographs.

Example 4.1. Let PmFN($\underline{Z}$) $= \{z_1, z_2, z_3\}$ be the universal set with model P3FNS $\mathbb{M}$ and three P3FNSs $\mathbb{M}_1$, $\mathbb{M}_2$ and $\mathbb{M}_3$ as given in Tables 8, 9, 10 and 11, respectively.

TABLE 8. Model P3FNS $\mathbb{M}$

$\mathbb{M}$			
z_1	(0.206, 0.101, 0.135)	(0.010, 0.153, 0.215)	(0.600, 0.142, 0.051)
z_2	(0.114, 0.100, 0.215)	(0.080, 0.093, 0.435)	(0.090, 0.002, 0.981)
z_3	(0.087, 0.132, 0.156)	(0.090, 0.123, 0.204)	(0.340, 0.642, 0.131)

TABLE 9. P3FNS $\mathbb{M}_1$

$\mathbb{M}_1$			
z_1	(0.307, 0.202, 0.246)	(0.002, 0.264, 0.326)	(0.701, 0.253, 0.162)
z_2	(0.542, 0.002, 0.254)	(0.143, 0.876, 0.796)	(0.214, 0.005, 0.214)
z_3	(0.053, 0.007, 0.760)	(0.320, 0.432, 0.324)	(0.530, 0.241, 0.964)

TABLE 10. P3FNS $\mathbb{M}_2$

$\mathbb{M}_2$			
z_1	(0.701, 0.052, 0.101)	(0.202, 0.160, 0.027)	(0.811, 0.156, 0.042)
z_2	(0.262, 0.001, 0.003)	(0.290, 0.980, 0.017)	(0.041, 0.126, 0.022)
z_3	(0.754, 0.023, 0.100)	(0.192, 0.360, 0.023)	(0.408, 0.134, 0.702)

and

TABLE 11. P3FNS $\mathbb{M}_3$

$\mathbb{M}_3$			
z_1	(0.822, 0.001, 0.100)	(0.417, 0.060, 0.007)	(1.000, 0.052, 0.008)
z_2	(0.143, 0.084, 0.098)	(0.009, 0.170, 0.037)	(0.000, 0.402, 0.064)
z_3	(0.632, 0.340, 0.132)	(0.128, 0.604, 0.215)	(0.800, 0.322, 0.609)

where $\mathbb{M}, \mathbb{M}_3, \mathbb{M}_4, \mathbb{M}_5 \in P3FN(\underline{Z})$. We use the metrics defined in Example 3.3, which is

$$\underline{\mathbb{M}}^s(\eth_1,\ \eth_2) = \sqrt[2m]{\sum_{i=1}^{m}\{(\daleth_1^{(i)} - \daleth_2^{(i)})^{2m} + (\underline{\mathrm{I}}_1^{(i)} - \underline{\mathrm{I}}_2^{(i)})^{2m} + (\underline{F}_1^{(i)} - \underline{F}_2^{(i)})^{2m}\}},$$

in Example 3.5, which is

$$\underline{\mathbb{M}}_\alpha^s(\mathbb{M}_1,\ \mathbb{N}_1) \ = \ \sqrt{\sum_i \{(\daleth_1^{(i)} - \daleth_2^{(i)})^2 + (\underline{\mathrm{I}}_1^{(i)} - \underline{\mathrm{I}}_2^{(i)})^2 + (\underline{F}_1^{(i)} - \underline{F}_2^{(i)})^2\}}$$

and that in Example 3.14, which is

$$\underline{\mathbb{M}}_f^s(\eth_1,\ \eth_2) = \frac{\underline{\mathbb{M}}^s(\eth_1,\ \eth_2)}{n + \underline{\mathbb{M}}^s(\eth_1,\ \eth_2)}$$

taking $n = 2$, to determine pattern similarity between $\mathbb{M}$ and $\mathbb{M}_i$'s. The results so computed are tabulated in Table 12.

TABLE 12. Metrics between $\mathbb{M}$ and $\mathbb{M}_i$'s

Metric	$(\mathbb{M}, \mathbb{M}_1)$	$(\mathbb{M}, \mathbb{M}_2)$	$(\mathbb{M}, \mathbb{M}_3)$
$\underline{\mathbb{M}}^s$	0.969	1.068	**0.948**
$\underline{\mathbb{M}}_\alpha^s$	1.753	1.880	**1.650**
$\underline{\mathbb{M}}_f^s$	0.326	0.348	**0.322**

Above results show that pattern of $\mathbb{M}_3$ is most recognizable with $\mathbb{M}$. These results are depicted in Figure 1.

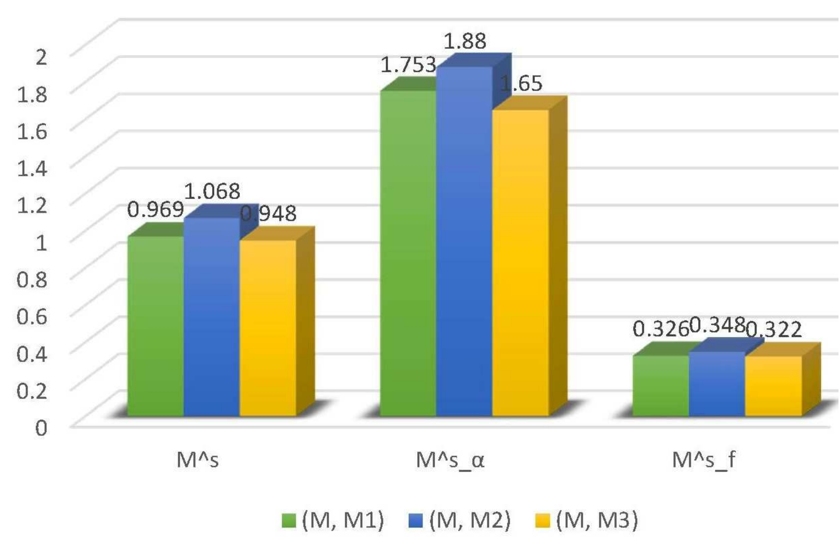

FIGURE 1. Chart of metrics between $\mathbb{M}$ and $\mathbb{M}_i$'s

5. Conclusion

We have inculcated the axiomatic definition of Pythagorean m-polar fuzzy neutrosophic metric space with the help of Pythagorean m-polar neutrosophic sets and classical metric space in this study. We provided a large number of examples to perceive the notion clearly. The cases which are metrics in classical sets but fail to be so in the environment of Pythagorean m-polar neutrosophic setting have also been made part of the study. The results presented also hold good in the case of Pythagorean fuzzy neutrosophic sets. We presented an application of the proposed metrics in pattern recognition. We computed three metrics there and exhibited that these metrics yield the same optimal choice. The results computed are displayed with the assistance of a statistical chart. We hope that this article will give new ideas to the researchers to promote research in various fields.

Conflicts of Interest: The authors declare that there is no conflict of interests.
Authors' Contributions: The authors contributed to each part of this paper equally.

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On Superhyper BCK-Algebras

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Abstract. BCK-algebras are algebraic structures in universal algebra such that are based on logical axioms and have some applications. This paper introduces the concept of super hyper BCK-algebras as a generalization of BCK-algebras and investigates some properties of this novel concept.

Keywords: BCK-algebra, hyper BCK-algebra, super hyper BCK-algebra, generalized operation.

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1. **Introduction**

Smarandache introduced a new concept in neutrosophy branches as neutro-algebra as a generalization of partial algebra. A neutro algebra is an algebra which has at least one neutro-operation (an operation that is partially well-defined, partially indeterminate, and partially outer-defined) or one neutro-axiom (axiom that is true for some elements, indeterminate for other elements, and false for the other elements). A partial algebra is an algebra that has at least one partial operation, and all its axioms are classical (i.e. axioms true for all elements). Through a theorem he proved that Neutro-algebra is a generalization of partial algebra, and he gave examples of neutro-algebras that are not partial algebras. He also introduced the neutro-function (and neutro-operation). Recently, Smarandache, introduced a new concept as a generalization of hypergraphs to n-super hypergraph, plithogenic n-super hypergraph {with super-vertices (that are groups of vertices) and hyper-edges {defined on power-set of power-set...} that is the most general form of graph as today}, and n-ary hyperalgebra, n-ary neutro hyperalgebra, n-ary anti hyperalgebra respectively, which have several properties and are connected with the real world [2,8]. Recently in the scope of neutro logical (hyper) algebra, Hamidi, et al. introduced the concept of neutro BCK-subalgebras [4], neutro d-subalgebras [3] and single-valued neutro hyper BCK-subalgebras [5] as a generalization of BCK-algebras and hyper BCK-subalgebras, respectively and presented the main results in this regard. Also

Smarandache a novel concept as super hyperalgebra with its super hyperoperations and super hyperaxioms, then is introduced some concepts such as super hypertopology and especially the super hyperfunction and neutrosophic super hyperfunction [10, 11].

Regarding these points, we try to develop the notation of BCK-algebras to the concept of super hyper BCK-algebras and so we want to seek the connection between BCK-algebras and super hyper BCK-algebras.

2. Preliminaries

In this section, we recall some concepts that need to our work.

Definition 2.1. [6] Let $X \neq \emptyset$. Then a universal algebra $(X, \vartheta, 0)$ of type $(2, 0)$ is called a *BCK-algebra*, if $\forall\ x, y, z \in X$:
(BCI-1) $((x\vartheta\ y)\vartheta\ (x\vartheta\ z))\vartheta\ (z\vartheta\ y) = 0$,
(BCI-2) $(x\vartheta\ (x\vartheta\ y))\vartheta\ y = 0$,
(BCI-3) $x\vartheta\ x = 0$,
(BCI-4) $x\vartheta\ y = 0$ and $y\vartheta\ x = 0$ imply $x = y$,
(BCK-5) $0\vartheta\ x = 0$,
where $\vartheta(x, y)$ is denoted by $x\vartheta\ y$.

Definition 2.2. [1,7] Let $X \neq \emptyset$ and $P^*(X) = \{Y \mid \emptyset \neq Y \subseteq X\}$. Then for a map $\varrho : X^2 \to P^*(X)$ a hyperalgebraic system $(X, \varrho, 0)$ is called a *hyper BCK-algebra*, if $\forall\ x, y, z \in X$:
$(H1)$ $(x\ \varrho\ z)\ \varrho\ (y\ \varrho\ z) \ll x\ \varrho\ y$,
$(H2)$ $(x\ \varrho\ y)\ \varrho\ z = (x\ \varrho\ z)\ \varrho\ y$,
$(H3)$ $x\ \varrho\ X \ll x$,
$(H4)$ $x \ll y$ and $y \ll x$ imply $x = y$,
where $x \ll y$ is defined by $0 \in x\ \varrho\ y$, $\forall\ W, Z \subseteq X$, $W \ll Z \Leftrightarrow\ \forall\ a \in W\ \exists\ b \in Z$ s.t $a \ll b$, $(W\ \varrho\ Z) = \bigcup\limits_{a \in W, b \in Z} (a\ \varrho\ b)$ and $\varrho(x, y)$ is denoted by $x\varrho\ y$.

We will call X is a *weak commutative* hyper BCK-algebra if, $\forall\ x, y \in X, (x\ \varrho\ (x\ \varrho\ y)) \cap (y\ \varrho\ (y\ \varrho\ x)) \neq \emptyset$.

Theorem 2.3. [7] *Let $(X, \varrho, 0)$ be a hyper BCK-algebra. Then $\forall\ x, y, z \in X$ and $W, Z \subseteq X$,*

(i) $(0\ \varrho\ 0) = 0, 0 \ll x, (0\ \varrho\ x) = 0$, $x \in (x\ \varrho\ 0)$ *and* $(W \ll 0 \Rightarrow W = 0)$,
(ii) $x \ll x$, $x\ \varrho\ y \ll x$ *and* $(y \ll z \Rightarrow x\ \varrho\ z \ll x\ \varrho\ y)$,
(iii) $W\ \varrho\ Z \ll W$, $W \ll W$ *and* $(W \subseteq Z \Rightarrow W \ll Z)$.

Definition 2.4. [10, 11] Let X be a nonempty set and $0 \in X$. Then $(X, \circ^*_{(m,n)})$ is called an (m,n)-super hyperalgebra, where $\circ^*_{(m,n)} : X^m \to P^n_*(X)$ is called an (m,n)-super hyperoperation, $P^n_*(X)$ is the n^{th} powerset of the set $X, \emptyset \notin P^n_*(X)$, for any $A \in P^n_*(X)$, we identify $\{A\}$ with $A, m, \geq 2, n \geq 0, X^m = \underbrace{X \times X \times \ldots X}_{m-times}$ and $P^0_*(X) = X$.

3. **Superhyper BCK-subalgebra**

In this section, we make the concept of superhyper BCK-subalgebras as an extension of BCK-subalgebras and seek some of their properties.

Proposition 3.1. *Let $(X, \vartheta, 0)$ be a BCK-algebra. Then for all $x, y, z \in X$,*

(i) $\vartheta(\vartheta(x,y), \vartheta(x,z)) = \vartheta(\vartheta(\vartheta(x,y), \vartheta(x,z)), 0)$.

(ii) $\vartheta(\vartheta(x), \vartheta(x,y)) = \vartheta(\vartheta(\vartheta(x), \vartheta(x,y)), 0)$.

Proof. Since for all $x \in X, \vartheta(x,0) = x$, results are clear. □

By Proposition 3.1, we define the concept of (m,n)-super hyper BCK-subalgebras.

Definition 3.2. Let X be a nonempty set and $0 \in X$ and $\alpha = \underbrace{0, 0, \ldots 0}_{(m-1)-times}$. Then $(X, \circ^*_{(m,n)})$ is called an (m,n)-super hyper BCK-subalgebra, if

(i) $0 \in \circ^*_{(m,n)}\big(\circ^*_{(m,n)} (\circ^*_{(m,n)}(x^1_1, x^1_2, \ldots, x^1_m), \ldots, \circ^*_{(m,n)}(x^1_1, x^m_2, \ldots, x^m_m)), \alpha, \circ^*_{(m,n)}(x^m_m, x^{m-1}_m, \ldots, x^1_m)\big)$,

(ii) $0 \in \circ^*_{(m,n)}\big(\circ^*_{(m,n)} (x^1_1, \underbrace{0, 0, \ldots 0}_{(m-2)-times}, \circ^*_{(m,n)}(x^1_1, x^1_2, \ldots, x^1_m)), \underbrace{0, 0, \ldots 0}_{(m-1)-times}, x^1_m)\big)$,

(iii) $0 \in \circ^*_{(m,n)}\big(x, x, \ldots, x\big)$,

(iv) if $0 \in \circ^*_{(m,n)}\big(x_1, x_2, \ldots, x_m\big)$ and $0 \in \circ^*_{(m,n)}\big(x_m, x_{m-1}, \ldots, x_1\big)$, then $x_i = x_j$, where $i + j = m + 1$,

(v) $0 \in \circ^*_{(m,n)}\big(0, 0, \ldots, x\big)$,

Example 3.3. *(i)* Let $(X, \circ^*_{(m,n)})$ be a (m,n)-super hyper BCK-subalgebra. Then $(X, \circ^*_{(2,0)})$ is a BCK-subalgebra.

(ii) Let $(X, \circ^*_{(m,n)})$ be a (m,n)-super hyper BCK-subalgebra. Then $(X, \circ^*_{(2,1)})$ is a hyper BCK-subalgebra.

Example 3.4. Let $X = \{0, a\}$.

(i) Then $(X, \circ^*)$ is a $(3,3)$-super hyper BCK-subalgebra as follows:

$$\circ^*_{(3,3)}(x,y,z) = \begin{cases} P^3_*(\{0,x,z\}) & \text{if } x = z \\ P^3_*(\{0,z\}) & \text{if } x = y = 0 \\ P^3_*(\{a\}) & o.w \end{cases},$$

where

$$P_*(\{a\}) = P_*^2(\{a\}) = P_*^3(\{a\}) = \{a\}, P_*(\{0,a\}) = \{0, a, \{0,a\}\},$$

$$P_*^2(\{0,a\}) = \{0, a, \{0,a\}, \{0, \{0,a\}\}, \{a, \{0,a\}\}\},$$

$$P_*^3(\{0,a\}) = \big\{0, a, \{0,a\}, \{0, \{0,a\}\}, \{a, \{0,a\}\}, \{0, \{0, \{0,a\}\}, \{0, \{a, \{0,a\}\}, \{a, \{0, \{0,a\}\},$$
$$\{a, \{a, \{0,a\}\}, \{\{0,a\}, \{0, \{0,a\}\}\}, \{\{0,a\}, \{a, \{0,a\}\}, \{\{0, \{0,a\}\}, \{a, \{0,a\}\}\}\big\}.$$

(i) By definition, $\circ^*_{(3,3)}\big(\circ^*_{(3,3)}(\circ^*_{(3,3)}(x,y,z), \circ^*_{(3,3)}(x',y',z'), \circ^*_{(3,3)}(x'',y'',z'')), 0, \circ^*_{(3,3)}(z'',z',z)\big) \subseteq \{0,a\}$. (ii) It is similar to item (i).

(iii) By definition, $\circ^*_{(3,3)}(a,a,a) = \{0,a\}$.

(iv) By definition, if $0 \in \circ^*_{(3,3)}(x,y,z)$ and $0 \in \circ^*_{(3,3)}(z,y,x)$, then $x = z$ and so $(x,y,z) = (z,y,x)$.

(v) By definition, $\circ^*_{(3,3)}(0,0,a) = \{0,a\}$.

(ii) Then $(X, \circ^*)$ is a $(3,0)$-super hyper BCK-subalgebra as follows:

$$\circ^*_{(3,1)}(x,y,z) = \begin{cases} 0 & \text{if } x = y = z \\ x & o.w \end{cases},$$

Theorem 3.5. *Let $(X, \circ^*_{(m,n)})$ be an (m,n)-super hyper BCK-subalgebra. Then for any $k \geq n$, $(X, \circ^*_{(m,n)})$ is an (m,k)-super hyper BCK-subalgebra.*

Proof. Let $(X, \circ^*_{(m,n)})$ be an (m,n)-super hyper BCK-subalgebra and $k \geq n$. Since $P_*^n(X) \subseteq P_*^k(X)$, for any $x_1, x_2, \ldots, x_m \in X, \circ^*_{(m,n)}(x_1, x_2, \ldots, x_m) \subseteq \circ^*_{(m,k)}(x_1, x_2, \ldots, x_m)$. Thus $0 \in \circ^*_{(m,n)}(x_1, x_2, \ldots, x_m)$ implies that $0 \in \circ^*_{(m,k)}(x_1, x_2, \ldots, x_m)$ and all axioms are valid. □

Example 3.6. Let $X = \{0,a\}$. Then for any $n \geq 3$, by Theorem 3.5, $(X, \circ^*)$ is a $(3,n)$-super hyper BCK-subalgebra as follows:

$$\circ^*_{(3,3)}(x,y,z) = \begin{cases} P_*^n(\{0,x,z\}) & \text{if } x = z \\ P_*^n(\{0,z\}) & \text{if } x = y = 0 \\ P_*^n(\{a\}) & o.w \end{cases}.$$

Let m be an even and $(X, \circ^*_{(m,n)})$ be an (m,n)-super hyper BCK-subalgebra. For any given $x_1, x_2, \ldots, x_m \in X$, define $(x_1, x_2, \ldots, x_{\frac{m}{2}}) \leq (x_{\frac{m}{2}+1}, x_{\frac{m}{2}+2}, \ldots, x_m)$ if and only if $0 \in \circ^*_{(m,n)}(x_1, x_2, \ldots, x_m)$.

Theorem 3.7. *Let m be an even and $x_1, x_2, \ldots, x_m \in X$. Then $(X, \circ^*_{(m,n)})$ is an (m,n)-super hyper BCK-subalgebra if and only if*

(i) $\circ^*_{(m,n)}(\circ^*_{(m,n)}(x_1^1, x_2^1, \ldots, x_m^1), \ldots, \circ^*_{(m,n)}(x_1^1, x_2^m, \ldots, x_m^m))) \leq \circ^*_{(m,n)}(x_m^m, x_m^{m-1}, \ldots, x_m^1)$,

(ii) $\circ^*_{(m,n)}(x_1^1, \underbrace{0,0,\ldots 0}_{(m-2)-times}, \circ^*_{(m,n)}(x_1^1,x_2^1,\ldots,x_m^1)) \leq \circ^*_{(m,n)}(\underbrace{0,0,\ldots 0}_{(m-1)-times}, x_m^1)),$

(iii) $(\underbrace{x,x,\ldots,x}_{(\frac{m}{2})-times}) \leq (\underbrace{x,x,\ldots,x}_{(\frac{m}{2})-times}),$

(iv) *if* $(x_1,x_2,\ldots,x_{\frac{m}{2}}) \leq (x_{\frac{m}{2}+1},x_{\frac{m}{2}+2},\ldots,x_m)$ *and* $(x_{\frac{m}{2}+1},x_{\frac{m}{2}+2},\ldots,x_m) \leq (x_1,x_2,\ldots,x_{\frac{m}{2}})$, *then* $x_i = x_j$, *where* $|i-j| = 2$,

(v) $(\underbrace{0,0,\ldots,0}_{(\frac{m}{2})-times}) \leq (x_{\frac{m}{2}+1},x_{\frac{m}{2}+2},\ldots,x_m),$

(vi) $(x_1,x_2,\ldots,x_{\frac{m}{2}}) \leq (x_{\frac{m}{2}+1},x_{\frac{m}{2}+2},\ldots,x_m)$ *if and only if* $0 \in \circ^*_{(m,n)}(x_1,x_2,\ldots,x_m)$.

Proof. Immediate by definition. □

Theorem 3.8. *Let m be an even and $(X,\circ^*_{(m,n)})$ be an (m,n)-super hyper BCK-subalgebra and $x_1,x_2,\ldots,x_{\frac{m}{2}},y_1,y_2,\ldots,y_{\frac{m}{2}},z_1,z_2,\ldots,z_{\frac{m}{2}} \in X$. If $0 \in \circ^*_{(m,n)}(x_1,x_2,\ldots,x_{\frac{m}{2}},y_1,y_2,\ldots,y_{\frac{m}{2}})$, then $0 \in \circ^*_{(m,n)}\big(\circ^*_{(m,n)}(z_1,z_2,\ldots,z_{\frac{m}{2}},y_1,y_2,\ldots,y_{\frac{m}{2}}), \underbrace{0,0,\ldots 0}_{(m-2)-times}, \circ^*_{(m,n)}(z_1,z_2,\ldots,z_{\frac{m}{2}},x_1,x_2,\ldots,x_{\frac{m}{2}})\big)$.*

Proof. Let $x_1,x_2,\ldots,x_{\frac{m}{2}},y_1,y_2,\ldots,y_{\frac{m}{2}},z_1,z_2,\ldots,z_{\frac{m}{2}} \in X$. Clearly,

$$\circ^*_{(m,n)}(\circ^*_{(m,n)}(z_1,z_2,\ldots,z_{\frac{m}{2}},y_1,y_2,\ldots,y_{\frac{m}{2}}), \underbrace{0,0,\ldots 0}_{(m-2)-times}, \circ^*_{(m,n)}(z_1,z_2,\ldots,z_{\frac{m}{2}},x_1,x_2,\ldots,x_{\frac{m}{2}})))$$
$$\leq \circ^*_{(m,n)}(x_1,x_2,\ldots,x_{\frac{m}{2}},y_1,y_2,\ldots,y_{\frac{m}{2}}).$$

Since $0 \in \circ^*_{(m,n)}(x_1,x_2,\ldots,x_{\frac{m}{2}},y_1,y_2,\ldots,y_{\frac{m}{2}})$, we get that

$$0 \in \circ^*_{(m,n)}\big(\circ^*_{(m,n)}(z_1,z_2,\ldots,z_{\frac{m}{2}},y_1,y_2,\ldots,y_{\frac{m}{2}}), \underbrace{0,0,\ldots 0}_{(m-2)-times}, \circ^*_{(m,n)}(z_1,z_2,\ldots,z_{\frac{m}{2}},y_1,y_2,\ldots,y_{\frac{m}{2}})\big).$$

□

Theorem 3.9. *Let m be an even and $(X,\circ^*_{(m,n)})$ be an (m,n)-super hyper BCK-subalgebra and $x_1,x_2,\ldots,x_{\frac{m}{2}},y_1,y_2,\ldots,y_{\frac{m}{2}},z_1,z_2,\ldots,z_{\frac{m}{2}} \in X$. If*

$$0 \in \circ^*_{(m,n)}(x_1,x_2,\ldots,x_{\frac{m}{2}},y_1,y_2,\ldots,y_{\frac{m}{2}}) \cap \circ^*_{(m,n)}(y_1,y_2,\ldots,y_{\frac{m}{2}},z_1,z_2,\ldots,z_{\frac{m}{2}}),$$

then $0 \in \circ^*_{(m,n)}(x_1,x_2,\ldots,x_{\frac{m}{2}},z_1,z_2,\ldots,z_{\frac{m}{2}})$.

Proof. Let $x_1,x_2,\ldots,x_{\frac{m}{2}},y_1,y_2,\ldots,y_{\frac{m}{2}},z_1,z_2,\ldots,z_{\frac{m}{2}} \in X$. Since

$$0 \in \circ^*_{(m,n)}(x_1,x_2,\ldots,x_{\frac{m}{2}},y_1,y_2,\ldots,y_{\frac{m}{2}}) \cap \circ^*_{(m,n)}(y_1,y_2,\ldots,y_{\frac{m}{2}},z_1,z_2,\ldots,z_{\frac{m}{2}}),$$

by Theorem 3.8, we get that

$$0 \in \circ^*_{(m,n)}\big(\circ^*_{(m,n)}(z_1,z_2,\ldots,z_{\frac{m}{2}},y_1,y_2,\ldots,y_{\frac{m}{2}}), \underbrace{0,0,\ldots 0}_{(m-2)-times}, \circ^*_{(m,n)}(z_1,z_2,\ldots,z_{\frac{m}{2}},x_1,x_2,\ldots,x_{\frac{m}{2}})\big)$$

and

$$0 \in \circ^*_{(m,n)}\big(\circ^*_{(m,n)}(x_1, x_2, \ldots, x_{\frac{m}{2}}, z_1, z_2, \ldots, z_{\frac{m}{2}}), \underbrace{0, 0, \ldots 0}_{(m-2)-times}, \circ^*_{(m,n)}(x_1, x_2, \ldots, x_{\frac{m}{2}}, y_1, y_2, \ldots, y_{\frac{m}{2}})\big).$$

It follows that $0 \in \circ^*_{(m,n)}(x_1, x_2, \ldots, x_{\frac{m}{2}}, z_1, z_2, \ldots, z_{\frac{m}{2}})$. □

Let $(X, \circ^*_{(m,n)})$ be an (m, n)-super hyper BCK-subalgebra and $A, B \subseteq X$. If $\circ^*_{(m,n)}(A) \cap \circ^*_{(m,n)}(B) \neq \emptyset$, will denote it by $A \approx B$.

Theorem 3.10. *Let m be an even and $(X, \circ^*_{(m,n)})$ be an (m, n)-super hyper BCK-subalgebra and $x_1, x_2, \ldots, x_{\frac{m}{2}}, y_1, y_2, \ldots, y_{\frac{m}{2}}, z_1, z_2, \ldots, z_{\frac{m}{2}} \in X$. If $\alpha = \underbrace{0, \ldots, 0}_{(\frac{m}{2}-1)-times}$, then*

$$\circ^*_{(m,n)}(\circ^*_{(m,n)}(x_1, \ldots, x_{\frac{m}{2}}, y_1, \ldots, y_{\frac{m}{2}}), \alpha, z_1, \ldots, z_{\frac{m}{2}}) \approx \circ^*_{(m,n)}(\circ^*_{(m,n)}(x_1, \ldots, x_{\frac{m}{2}}, z_1, \ldots, z_{\frac{m}{2}}), \alpha, y_1, \ldots, y_{\frac{m}{2}}).$$

Proof. Let $x_1, x_2, \ldots, x_{\frac{m}{2}}, y_1, y_2, \ldots, y_{\frac{m}{2}}, z_1, z_2, \ldots, z_{\frac{m}{2}} \in X$. Since $0 \in \circ^*_{(m,n)}\big(\circ^*_{(m,n)}(x_1, x_2, \ldots, x_{\frac{m}{2}}, (\circ^*_{(m,n)}(x_1, x_2, \ldots, x_{\frac{m}{2}}, z_1, z_2, \ldots, z_{\frac{m}{2}})), z_1, z_2, \ldots, z_{\frac{m}{2}})\big)$, we get that

$$\begin{aligned}&\circ^*_{(m,n)}\big(\circ^*_{(m,n)}(x_1, x_2, \ldots, x_{\frac{m}{2}}, y_1, y_2, \ldots, y_{\frac{m}{2}}), \underbrace{0, 0, \ldots 0}_{(m-2)-times}, z_1, z_2, \ldots, z_{\frac{m}{2}})\big)\\ &\leq \circ^*_{(m,n)}\big(\circ^*_{(m,n)}(x_1, x_2, \ldots, x_{\frac{m}{2}}, z_1, z_2, \ldots, z_{\frac{m}{2}}), \underbrace{0, 0, \ldots 0}_{(m-2)-times}, y_1, y_2, \ldots, y_{\frac{m}{2}})\big)\end{aligned}$$

and in similar to

$$\begin{aligned}&\circ^*_{(m,n)}\big(\circ^*_{(m,n)}(x_1, x_2, \ldots, x_{\frac{m}{2}}, z_1, z_2, \ldots, z_{\frac{m}{2}}), \underbrace{0, 0, \ldots 0}_{(m-2)-times}, y_1, y_2, \ldots, y_{\frac{m}{2}})\big)\\ &\leq \circ^*_{(m,n)}\big(\circ^*_{(m,n)}(x_1, x_2, \ldots, x_{\frac{m}{2}}, y_1, y_2, \ldots, y_{\frac{m}{2}}), \underbrace{0, 0, \ldots 0}_{(m-2)-times}, z_1, z_2, \ldots, z_{\frac{m}{2}})\big).\end{aligned}$$

It follows that

$$\circ^*_{(m,n)}(\circ^*_{(m,n)}(x_1, \ldots, x_{\frac{m}{2}}, y_1, \ldots, y_{\frac{m}{2}}), \alpha, z_1, \ldots, z_{\frac{m}{2}}) \approx \circ^*_{(m,n)}(\circ^*_{(m,n)}(x_1, \ldots, x_{\frac{m}{2}}, z_1, \ldots, z_{\frac{m}{2}}), \alpha, y_1, \ldots, y_{\frac{m}{2}}).$$

□

Corollary 3.11. *Let m be an even and $(X, \circ^*_{(m,n)})$ be an (m, n)-super hyper BCK-subalgebra and $x_1, x_2, \ldots, x_{\frac{m}{2}}, y_1, y_2, \ldots, y_{\frac{m}{2}}, z_1, z_2, \ldots, z_{\frac{m}{2}} \in X$. If*

$$0 \approx \circ^*_{(m,n)}(\circ^*_{(m,n)}(x_1, \ldots, x_{\frac{m}{2}}, y_1, \ldots, y_{\frac{m}{2}}), \underbrace{0, \ldots 0}_{(\frac{m}{2}-1)-times}, z_1, \ldots, z_{\frac{m}{2}})$$

then

$$0 \approx \circ^*_{(m,n)}(\circ^*_{(m,n)}(x_1, \ldots, x_{\frac{m}{2}}, z_1, \ldots, z_{\frac{m}{2}}), \underbrace{0, \ldots 0}_{(\frac{m}{2}-1)-times}, y_1, \ldots, y_{\frac{m}{2}}).$$

Example 3.12. Consider the $(3,3)$-super hyper BCK-subalgebra in Example 3.4. Clearly

$$\begin{aligned}\circ^*_{(3,3)}(\circ^*_{(3,3)}(0,a,0),0,a) &= \circ^*_{(3,3)}(P^3_*(\{0\}),0,a) = \circ^*_{(3,3)}(0,0,a) = P^3_*(\{0,a\})\\ &= \circ^*_{(3,3)}(a,0,a) = \circ^*_{(3,3)}(P^3_*(\{a\}),0,a) = \circ^*_{(3,3)}(\circ^*_{(3,3)}(0,a,a),0,a).\end{aligned}$$

Thus $\circ^*_{(3,3)}(\circ^*_{(3,3)}(0,a,0),0,a) = \circ^*_{(3,3)}(\circ^*_{(3,3)}(0,a,a),0,a)$, while m is an odd. It follows that the converse of Theorem 3.10, is not necessarily true.

Theorem 3.13. *Let m be an even and $(X,\circ^*_{(m,n)})$ be an (m,n)-super hyper BCK-subalgebra and $x_1,x_2,\ldots,x_m,y_1,y_2,\ldots,y_m \in X$. If $\alpha = \underbrace{0,\ldots,0}_{(\frac{m}{2}-1)-times}$, then*

(*i*)

$$\circ^*_{(m,n)}(\circ^*_{(m,n)}(x_1,\ldots,x_{\frac{m}{2}},x_1,\ldots,x_{\frac{m}{2}}),\alpha,y_1,\ldots,y_{\frac{m}{2}}) \approx \circ^*_{(m,n)}(\circ^*_{(m,n)}(x_1,\ldots,x_{\frac{m}{2}},y_1,\ldots,y_{\frac{m}{2}}),\alpha,x_1,\ldots,x_{\frac{m}{2}}).$$

(*ii*)

$$\circ^*_{(m,n)}(\underbrace{0,\ldots0}_{(\frac{m}{2})-times},y_1,\ldots,y_{\frac{m}{2}}) \approx \circ^*_{(m,n)}(\circ^*_{(m,n)}(x_1,\ldots,x_{\frac{m}{2}},y_1,\ldots,y_{\frac{m}{2}}),\underbrace{0,\ldots,0}_{(\frac{m}{2}-1)-times},x_1,\ldots,x_{\frac{m}{2}}).$$

(*iii*)

$$\circ^*_{(m,n)}(\circ^*_{(m,n)}(x_1,\ldots,x_{\frac{m}{2}},x_1,\ldots,x_{\frac{m}{2}}),\underbrace{0,\ldots0}_{(\frac{m}{2}-1)-times},y_1,\ldots,y_{\frac{m}{2}}) \approx \circ^*_{(m,n)}(\underbrace{0,\ldots,0}_{(\frac{m}{2})-times},y_1,\ldots,y_{\frac{m}{2}}).$$

Proof. $(i),(ii),(iii)$ Let $x_1,x_2,\ldots,x_{\frac{m}{2}},y_1,y_2,\ldots,y_{\frac{m}{2}},z_1,z_2,\ldots,z_{\frac{m}{2}} \in X$. Using Corollary 3.11, we get that

$$0 \approx \circ^*_{(m,n)}(\circ^*_{(m,n)}(x_1,\ldots,x_{\frac{m}{2}},y_1,\ldots,y_{\frac{m}{2}}),\underbrace{0,\ldots0}_{(\frac{m}{2}-1)-times},x_1,\ldots,x_{\frac{m}{2}})$$

and

$$0 \approx \circ^*_{(m,n)}(\circ^*_{(m,n)}(x_1,\ldots,x_{\frac{m}{2}},x_1,\ldots,x_{\frac{m}{2}}),\underbrace{0,\ldots0}_{(\frac{m}{2}-1)-times},y_1,\ldots,y_{\frac{m}{2}}).$$

In addition, by definition we get that $0 \approx \circ^*_{(m,n)}(\underbrace{0,\ldots0}_{(\frac{m}{2})-times},y_1,\ldots,y_{\frac{m}{2}})$, hence the proof is completed. □

Corollary 3.14. *Let m be an even and $(X,\circ^*_{(m,n)})$ be an (m,n)-super hyper BCK-subalgebra and $x_1,x_2,\ldots,x_{m-1},y_1,y_2,\ldots,y_{m-1} \in X$. Then*

(*i*)

$$\circ^*_{(m,n)}(\circ^*_{(m,n)}(x_1,\ldots,x_{\frac{m}{2}},x_1,\ldots,x_{\frac{m}{2}}),y_1,\ldots,y_{m-1}) \approx \circ^*_{(m,n)}(\circ^*_{(m,n)}(x_1,\ldots,x_{\frac{m}{2}},y_1,\ldots,y_{\frac{m}{2}}),x_1,\ldots,x_{m-1}).$$

(ii)

$$\circ^*_{(m,n)}(0, y_1, \ldots, y_{m-1}) \approx \circ^*_{(m,n)}(\circ^*_{(m,n)}(x_1, \ldots, x_{\frac{m}{2}}, y_1, \ldots, y_{\frac{m}{2}}), x_1, \ldots, x_{m-1}).$$

(iii)

$$\circ^*_{(m,n)}(\circ^*_{(m,n)}(x_1, \ldots, x_{\frac{m}{2}}, x_1, \ldots, x_{\frac{m}{2}}), y_1, \ldots, y_{m-1}) \approx \circ^*_{(m,n)}(0, y_1, \ldots, y_{m-1}).$$

Theorem 3.15. *Let m be an even and $(X, \circ^*_{(m,n)})$ be an (m,n)-super hyper BCK-subalgebra and $x_1, x_2, \ldots, x_{m-1} \in X$. Then $(x_1, \ldots, x_{m-1}) \approx \circ^*_{(m,n)}(x_1, \ldots, x_{m-1}, 0)$.*

Proof. Let $x_1, x_2, \ldots, x_m \in X$. Then $0 \approx \circ^*_{(m,n)}(x_1, x_2, \ldots, x_{m-1}, \circ^*_{(m,n)}(x_1, x_2, \ldots, x_{m-1}, 0))$. Moreover by Theorem 3.13, we have $0 \approx \circ^*_{(m,n)}(\circ^*_{(m,n)}(x_1, \ldots, x_{m-1}, 0), x_1, \ldots, x_{m-1})$. Thus we conclude that $(x_1, \ldots, x_{m-1}) \approx \circ^*_{(m,n)}(x_1, \ldots, x_{m-1}, 0)$. □

4. **Conclusion**

The concept of super hyper BCK-algebras as a generalization of BCK-algebras is introduced in this paper such that for special cases, we can obtain the concepts of BCK-algebras and hyper BCK-algebras. We wish this research is important for the next studies in logical super hyperalgebras. In our future studies, we hope to obtain more results regarding single-valued neutrosophic super(hyper)BCK-subalgebras and their applications in handing information regarding various aspects of uncertainty, non-classical mathematics (fuzzy mathematics or great extension and development of classical mathematics) that are considered to be a more powerful technique than classical mathematics.

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Application of Neutrosophic Interval valued Goal Programming to a Supply Chain Inventory Model for Deteriorating Items with Time Dependent Demand

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Abstract. A single deteriorating product's EOQ model has been examined in the literature, where it is considered that the product deteriorate continuously but has a maximum lifespan. It has also been assumed that market demand is linearly related to time. Additionally, the credit-risk is required for the retailer to pay the purchase price is offered by the supplier. The total annual relevant cost has been demonstrated to be convex, suggesting that not only does the ideal replenishment cycle time exist, but that it is also singular. We identify the system's ideal replenishment strategy, which reduces the overall cost per unit of time. To generalize the model we used Neutrosophic triangular numbers for the parameters. Finally, an numerical example is given to illustrate the theoretical results of this model.

Keywords:EOQ model; deterioration; time dependent demand; Neutrosophic interval valued goal programming

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1. Introduction

In real life, certain type of products either deteriorate or become obsolete and can not serve the need of the customer for an extended period of time. For example, in the food industry items deteriorate continuously. Also, how items are stored also has an impact on the lifespan of the products. For example in the durg industry items become obsolete after a fixed period. So, deterioration impacts how well can a customer be served. A lot of literature explored inventory model with deterioration. Some recent papers including including [8–12] explored deterioration inventory models. A supply chain environment to determine retailer's optimal credit period and cycle time was consided by Mahata [29]. A two-warehouse inventory model for decaying goods having imperfect quality was considered by [28] . Mahata [30] considered supply chain inventory model for deteriorating items with maximum lifetime and partial trade credit to credit-risk customers. In the article, they showed that the total annual relevant cost is convex. An EOQ inventory model for non-instantaneous deteriorating products with advertisement and price sensitive demand under order quantity dependent trade credit was discussed in [31]. Some modified mathematical derivations of the annual total relevant cost of the inventory model with two levels of trade credit in the supply chain system was analyzed in [32]. Liao et. al. [27] studied Lot-sizing decisions for deteriorating items with two warehouses under an order-size-dependent trade credit. Abdel et. al. [2] discussed a hybrid approach of neutrosophic sets and DEMATEL method for the development in supplier selection criteria. [3] has done a case study using the integrated neutrosophic ANP and VIKOR technique to achieve sustainable supplier selection. [4] provided a Comprehensive Framework to evaluate sustainable green building indicators under an uncertain environment. [5] provided a bipolar neutrosophic multi criteria decision making technique for making a professional selection. [6] developed a hybrid multi-criteria decision making technique to evaluate thesustainable photovoltaic farms locations. [7] introduced an approach of TOPSIS technique for developing supplier selection with group decision making under type-2 neutrosophic number.

In most of the articles above, the market demand is taken to be constant. Furthermore all the articles are formulated with the assumption all the data available in hand are precise. But, in real life problems the data available may not be exact. Using the arguments presented above, this paper investigate two important elements. Firstly, the market demand is taken to be a linear function of time. And secondly, the parameters are taken to be neutrosopic triangular numbers to consider the fuzziness in the data. Zadeh [26] developed the idea of fuzzy set. Bellman [21] explained the decision making in fuzzy systems. Zimmermann [22] implemented this concept for solving linear programming problem with several objective functions. Atanassov [19] introduced the concept of intuitionistic fuzzy set, where the sum of the membership degree and non-membership degreeis less than equal to one. Smarandache [25] developed the concept of neutrosophic by adding another independent membership function called as indeterminacy membership along with truth and falsity membership functions. Smarandache [14,18] introduced the idea of Neutrosophic interval valued number. Some basic properties as have been established in that paper. Ye [13] explained some basic properties and developed a linear programming method. Banerjee [17] dealt with a single objective linear goal programming model with neutrosophic numbers. [1] developed a EOQ model with trade credit model with deterioration with constant demand.

In this paper, we have developed a EOQ model where the said item deteriorates continuously with time and the demand is linearly dependent on time. Also to generalize the model we have taken demand as the triangular neutrosophic number. The rest of the manuscript is organized as follows; 2 provides some basic definitions. 3 presents the model. 4 forms the fuzzy problem, where demand is taken to be Neutrosophic triangular number. 5 provides an illustrative example. 6 gives the conclusion.

2. Preliminaries

2.1. *Some Definitions*

Definition 2.1 (Fuzzy Sets)**.** According to [26], a fuzzy set $\widetilde{A}$ in a universe of discourse X is defined as the ordered pairs $\widetilde{A} = \{(x, M_{\widetilde{A}}(x)) : x \in X\}$ where $M_{\widetilde{A}} : X \to [0, 1]$ is a function known as the membership function of the set $\widetilde{A}$. $M_{\widetilde{A}}(x)$ is the degree of membership of x $\in X$ in the fuzzy set $\widetilde{A}$. Higher value of $M_{\widetilde{A}}(x)$ indicates a higher degree of membership in $\widetilde{A}$.

Definition 2.2 (Neutrosophic sets)**.** According to [25], let X be a universe of discourse and let $x \in X$. A neutrosophic set A in X is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity- membership function $F_A(x)$, where $T_A(x), I_A(x), F_A(x) \in (0, 1), \forall x \in X$ and $0^+ \leq supT_A(x) + supI_A(x) + supF_A(x) \leq 3^-$.

Definition 2.3 (Single valued neutrosophic sets)**.** According to [24], if X is a universe of discourse and if $x \in X$, a single valued neutrosophic set A is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity- membership function $F_A(x)$, where $T_A(x), I_A(x), F_A(x) \in [0, 1], \forall x \in X$ and $0 \leq supT_A(x) + supI_A(x) + supF_A(x) \leq 3$.

Definition 2.4 (Interval valued Neutrosophic number)**.** As in [18], A neutrosophic number $\alpha = a + bI$ where a is the determinate part, b is the indeterminate part and I is the indeterminacy. Here $a, b \in \Re$ and I is an real interval.

$\alpha = a + bI$, where $I = [I^l, I^u] \implies \alpha = [a + bI^l, a + bI^u] = \{x \in \Re | a + bI^l < x < a + bI^u\} = [\alpha^l, \alpha^u]$(say).

Example: Let $\alpha = 1 + 2[0.1, 0.2]$ where 1 is the determinate part and 2 is the indeterminate part. Assume that $I \in [0.1, 0.2]$, then α becomes an interval $\alpha = [1.2, 1.4]$.

2.2. *Neutrosophic interval valued linear programming*

In this section we briefly discuss neutrosophic interval valued linear programming as in [13, 17].

$$\text{Minimize } Z_n = \sum_{i=1}^{n} [c_{ni}^l, c_{ni}^u] x_i \tag{1}$$

Subject to,

$$\sum_{k=1}^{K}[a_{mk}^{l}, a_{mk}^{u}]x_k \leq [b_m^l, b_m^u] \qquad m = 1, 2, ..., M \tag{2}$$

$$x_k \geq 0 \qquad k = 1, 2, ..., K \tag{3}$$

where Z_n for n=1,2,3,...,N are objective functions, $[c_{ni}^l, c_{ni}^u]$ are the interval coefficients for the p^{th} objective function, $[a_{mk}^l, a_{mk}^u]$, $[b_m^l, b_m^u]$ are the interval coefficients of the constraints.

Accordingly in [15, 16], the constraints in 2 can be transformed into two following inequalities,

$$\sum_{k=1}^{K} a_{mk}^{l} x_k \leq b_m^u \qquad m = 1, 2, ..., M \tag{4}$$

$$\sum_{k=1}^{K} a_{mk}^{u} x_k \leq b_m^l \qquad m = 1, 2, ..., M \tag{5}$$

Therefore then minimization problem stated above can be written as,

$$\text{Minimize } Z_n = \sum_{i=1}^{n}[c_{ni}^l, c_{ni}^u]x_i \tag{6}$$

Subject to,

$$\sum_{k=1}^{K} a_{mk}^{l} x_k \leq b_m^u \qquad m = 1, 2, ..., M \tag{7}$$

$$\sum_{k=1}^{K} a_{mk}^{u} x_k \leq b_m^l \qquad m = 1, 2, ..., M \tag{8}$$

$$x_k \geq 0 \qquad k = 1, 2, ..., K \tag{9}$$

For the best possible solution, we solve the problem

$$\text{Minimize } Z_n = \sum_{i=1}^{n} c_{ni}^l x_i = Z^l(\text{say}) \tag{10}$$

Subject to,

$$\sum_{k=1}^{K} a_{mk}^{l} x_k \leq b_m^u \qquad m = 1, 2, ..., M \tag{11}$$

$$x_k \geq 0 \qquad k = 1, 2, ..., K \tag{12}$$

And for the worst possible solution, we solve the problem

$$\text{Minimize } Z_n = \sum_{i=1}^{n} c_{ni}^u x_i = Z^u(\text{say}) \tag{13}$$

Subject to,

$$\sum_{k=1}^{K} a_{mk}^{u} x_k \leq b_m^{l} \qquad m = 1, 2, ..., M \tag{14}$$

$$x_k \geq 0 \qquad k = 1, 2, ..., K \tag{15}$$

Let the best and worst possible solution respectively be $Z_n^b(x_n^b)$ and $Z_n^w(x_n^w)$. So the optimal solution lies in the interval $[Z_n^b(x_n^b), Z_n^w(x_n^w)]$. So, for the decision maker the objective function Z lies in $[Z_n^b(x_n^b), Z_n^w(x_n^w)]$. If $d^l, d^u \geq 0$ be devotional variables, then the goal achievement functions can be written as,

$$-Z^u + d^u = -Z_n^b(x_n^b) \text{ and } Z^l + d^l = Z_n^w(x_n^w) \tag{16}$$

So, the goal programming problem according to [17],

$$\text{Minimize}(d^u + d^l) \tag{17}$$

subject to,

$$-Z^u + d^u = -Z_n^b(x_n^b) \tag{18}$$

$$Z^l + d^l = Z_n^w(x_n^w) \tag{19}$$

$$\sum_{k=1}^{K} a_{mk}^{l} x_k \leq b_m^{u} \qquad m = 1, 2, ..., M \tag{20}$$

$$\sum_{k=1}^{K} a_{mk}^{u} x_k \leq b_m^{l} \qquad m = 1, 2, ..., M \tag{21}$$

$$d^l \geq 0 \tag{22}$$

$$d^u \geq 0 \tag{23}$$

$$x_k \geq 0 \qquad k = 1, 2, ..., K \tag{24}$$

(25)

3. Mathematical Model

In this manuscript, we study a supply chain system where a supplier supplies retailers with deteriorating products. Also, the give credit to pay the credit-risk of the retailer's accounts. For this we have the following notations and assumptions.

3.1. *Notations*

o per unit order cost

c per unit cost of purchasing

p per unit selling price

h per unit annual holding costs excluding interest costs

I_e interest that the retailer earns each year
I_p interest accrued annually
I(t) inventory level at any time t
$\theta(t)$ non-decreasing deterioration rate at any time t
m maximum lifetime of the products in years
M trade credit term in years set by the supplier
D time dependent demand rate per year
Q order amount in units per replenishment cycle
T the number of years in the replenishing cycle
Z(t) the total relevant yearly cost
T^* the optimal duration between replenishment cycles

3.2. *Assumptions*

(1) The item deteriorate continuously. Also, the item expires after the maximum lifetime. The deterioration rate is assumed to be closed to 1 when time approaches to the expiration date m. The deterioration rate is assumed to be same as that in [23] as follows:

$$\theta(t) = \frac{1}{1+m-t} \qquad (0 \leq t \leq T \leq m) \tag{26}$$

Clearly, $0 \leq \theta(t) \leq 1$, $\theta(m) = 1$ and $\theta^{'}(t) \geq 1$
(2) The market demand is a linear function of time as follows: $D(t) = a + bt$
(3) Shortages are not allowed.
(4) There is no delay in replenishment and also the lead time is zero.
(5) Time horizon is assumed to be infinite.
(6) The trade credit agreement is assumed to be as follow:
- The retailer initially borrows money to pay the supplier's procurement costs, after which interest charges are incurred during the time interval (0,M].
- In the event that the retailer does not settle the balance by time M, the supplier asks the retailer to pay the unpaid balance plus interest with interest rate I_p. The retailer then uses the earnings from the sale to settle the supplier's outstanding debt. Once all accounts have been settled, the retailer keeps the profit and uses sales income to earn interest for the course of the replenishment cycle (T).

3.3. *Model formulation*

At any time $t \in [0,T]$, the inventory level is depleting from the demand and deterioration. The inventory level is described by the following differential equation:

$$\begin{aligned}\frac{\mathrm{d}I(t)}{\mathrm{d}t} &= -D-\theta(t)I(t)\\ &= -(a+bt)-\frac{1}{1+m-t}I(t) \qquad (0\le t\le T\le m)\end{aligned} \tag{27}$$

With the boundary condition I(T)=0.

Solving the differential eq. 27, we get,

$$I(t) = (1+m-t)[(a+(1+m)b)\ln\left(\frac{1+m-t}{1+m-T}\right)+b(t-T)] \tag{28}$$

Furthermore, by assumption 6, the retailer takes loan to pay off the supplier. So the amount of loan the retailer has to take $\int_0^T cDdt = c[aT+b\frac{T^2}{2}]$. Then the amount of interest charged $cMI_p[aT+\frac{bT^2}{2}]$ during the time interval (0, M]. Additionally, the retailer keeps the payments and receives interest during the same time period.,i.e. ,

$$p\int_0^M (a+bt)dt + pI_e\int_0^M (a+bt)tdt = p(aM+\frac{bM^2}{2})+pI_e(\frac{aM^2}{2}+\frac{bM^3}{3})$$

Now, if $p(aM+\frac{bM^2}{2})+pI_e(\frac{aM^2}{2}+\frac{bM^3}{3}) \ge c[aT+b\frac{T^2}{2}]$ then the retailer succeeds in paying off the loan and keeps earning interest on the remaining balance given by,

$$p(aM+\frac{bM^2}{2})+pI_e(\frac{aM^2}{2}+\frac{bM^3}{3}) > -c[aT+b\frac{T^2}{2}]$$

If, $p(aM+\frac{bM^2}{2})+pI_e(\frac{aM^2}{2}+\frac{bM^3}{3}) < c[aT+b\frac{T^2}{2}]$ then the retailer fails to pay off the loan and he/she has to reduce the loan amount from sales revenue.

Additionally, the retailer encounters the following costs,

(1)

$$\text{Annual ordering cost} = \frac{o}{T} \tag{29}$$

(2)

$$\text{Annual procurement cost} = \frac{cQ}{T} = \frac{cI(0)}{T} = \frac{c(1+m)[(a+(1+m)b)\ln\left(\frac{1+m}{1+m-T}\right)+b(-T)]}{T} \tag{30}$$

(3)

Annual holding cost excluding interest charge,

$$\begin{aligned}&= \frac{h}{T}\int_0^T I(t)dt\\ &= \frac{h(a+b(1+m))}{T}[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right)+\frac{T^2}{4}-\frac{(1+m)T}{2}-\frac{bTh}{2}]\end{aligned} \tag{31}$$

In this connection two cases arise. The first case is for when the replenishment cycle is less than trade credit period and the second one is for when replenishment cycle is greater than trade credit period.

Case 1: In this case we examine the case where the retailer pays off the loan in time t=M and $T \leq M$ Here, the retailer keeps the profit and sells revenue and earns profit on it untill the the replenishment cycle time T.

The annual interest payable is given by,

$$I_p c(aM + \frac{cM^2}{2}) \tag{32}$$

Interest earned by the retailer from t=0 to t=T with an interest rate I_e is given by,

$$I_e p \int_0^T (a+bt)tdt = I_e p[\frac{aT^2}{2} + \frac{bT^3}{3}]. \tag{33}$$

Additionally, the interest earned starting from the time t=T to t=M is given by,

$$I_e[p(aT + \frac{bT^2}{2}) + pI_e(\frac{aT^2}{2} + \frac{bT^3}{3})](M-T) \tag{34}$$

Hence, the total annual interest earned is given by,

$$\frac{1}{T}[I_e p(\frac{aT^2}{2} + \frac{bT^3}{3}) + I_e(M-T)\{p(aT + \frac{bT^2}{2}) + pI_e(\frac{aT^2}{2} + \frac{bT^3}{3})\}] \tag{35}$$

So, the annual opportunity cost of capital is given by,

$$Z_1 = \frac{o}{T} + \frac{c(1+m)[(a+(1+m)b)\ln\left(\frac{1+m}{1+m-T}\right) + b(-T)]}{T} + \frac{h(a+b(1+m))}{T}[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right) + \frac{T^2}{4} - \frac{(1+m)T}{2} - \frac{bTh}{2}] + I_p c(aM + \frac{bM^2}{2}) - \frac{1}{T}[I_e p(\frac{aT^2}{2} + \frac{bT^3}{3}) + I_e(M-T)\{p(aT + \frac{bT^2}{2}) + pI_e(\frac{aT^2}{2} + \frac{bT^3}{3})\}] \tag{36}$$

With the inventory constant, $I(0) \leq I$

Case 2: In this case, we study the case when the replenishment cycle is greater than trade credit i.e. $M \leq T$. Similar to the previous case, the retailer has to pay the annual interest,

$$I_p c(aM + \frac{bM^2}{2}) \tag{37}$$

The retailer earns interest on sales revenue from t=0 to t=M and it is given by,

$$I_e p \int_0^M (a+bt)tdt = I_e p[\frac{aM^2}{2} + \frac{bM^3}{3}]. \tag{38}$$

After paying the loan interest, the retailer uses the remaining revenue to earn more interest. Since the retailer pays off in time t=M, he earns interest on the net revenue from time t=M to t=T on every replenishment cycle. So, the annual interest earned is given by,

$$I_e[(p(aM + \frac{bM^2}{2})) + pI_e(\frac{aM^2}{2} + \frac{bM^3}{3}) - c(aT + \frac{bT^2}{2})](T-M) + pI_e(\frac{a(T-m)^2}{2} + \frac{b(T-M)^3}{3}) \tag{39}$$

So, the annual opportunity cost of capital is given by,

$$Z_2 = \frac{o}{T} + \frac{c(1+m)[(a+(1+m)b)\ln\left(\frac{1+m}{1+m-T}\right) + b(-T)]}{T} + \frac{h(a+b(1+m))}{T}[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right) + \frac{T^2}{4} - \frac{(1+m)T}{2} - \frac{bTh}{2}] + I_p c(aM + \frac{bM^2}{2}) - \frac{1}{T}[I_e p\left(\frac{aM^2}{2} + \frac{bM^3}{3}\right) + I_e[(p(aM + \frac{bM^2}{2})) + pI_e(\frac{aM^2}{2} + \frac{bM^3}{3}) - c(aT + \frac{bT^2}{2})](T-M) + pI_e(\frac{a(T-m)^2}{2} + \frac{b(T-M)^3}{3})] \tag{40}$$

So we have,

$$Z(T) = \begin{cases} Z_1(T) & \text{if} \quad T \leq M \\ Z_2(T) & \text{if} \quad T \geq M \end{cases} \tag{41}$$

4. **Fuzzy Model formulation**

Sometimes it is hard to predict the market demand precisely. The approximate demand within a range may be predicted. So for generalization we form the same problem with the help of neutrosophic triangular number. We take the market demand as $D = [a^l, a^u] + [b^l, b^u]t$, where $[a^l, a^u], [b^l, b^u]$ are interval coefficients of fuzzy demand function. Here again the inventory level is described by the following differential equation:

$$\begin{aligned} \frac{\mathrm{d}I(t)}{\mathrm{d}t} &= -D - \theta(t)I(t) \\ &= -([a^l, a^u] + [b^l, b^u]t) - \frac{1}{1+m-t}I(t) \qquad (0 \leq t \leq T \leq m) \end{aligned} \tag{42}$$

Solving the differential eq. we get,

$$I(t) = (1+m-t)[([a^l, a^u] + (1+m)[b^l, b^u])\ln\left(\frac{1+m-t}{1+m-T}\right) + [b^l, b^u](t-T) \tag{43}$$

Proceeding similar way, the loan amount will be $c[[a^l, a^u]T + [b^l, b^u]\frac{T^2}{2}]$ and the interest charged will be $cMI_p[[a^l, a^u]T + [b^l, b^u]\frac{T^2}{2}]$. Here the costs the retailer encounters are,

(1)

$$\text{Annual ordering cost} = \frac{o}{T} \tag{44}$$

(2)

$$\text{Annual procurement cost} = \frac{cQ}{T} = \frac{cI(0)}{T} = \frac{c(1+m)[([a^l, a^u] + (1+m)[b^l, b^u])\ln\left(\frac{1+m}{1+m-T}\right) + [b^l, b^u](-T)]}{T} \tag{45}$$

(3)

Annual holding cost excluding interest charge,

$$= \frac{h}{T}\int_0^T I(t)dt$$
$$= \frac{h([a^l,a^u]+[b^l,b^u](1+m))}{T}[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right)+\frac{T^2}{4}-\frac{(1+m)T}{2}-\frac{[b^l,b^u]Th}{2}] \tag{46}$$

Again two cases arise.

Case 1: Similarly as the crisp cases we get the following annual opportunity cost of capital,

$$\begin{aligned}\tilde{Z}_1 =& \frac{o}{T}+\frac{c(1+m)[([a^l,a^u]+(1+m)[b^l,b^u])\ln\left(\frac{1+m}{1+m-T}\right)+[b^l,b^u](-T)]}{T}\\ &+\frac{h([a^l,a^u]+[b^l,b^u](1+m))}{T}[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right)+\frac{T^2}{4}\\ &-\frac{(1+m)T}{2}-\frac{[b^l,b^u]Th}{2}]+I_pc([a^l,a^u]M+\frac{[b^l,b^u]M^2}{2})-\frac{1}{T}[I_ep(\frac{[a^l,a^u]T^2}{2}+\frac{[b^l,b^u]T^3}{3})+I_e(M-T)\{p([a^l,a^u]\\ &+\frac{[b^l,b^u]T^2}{2})+pI_e(\frac{[a^l,a^u]T^2}{2}+\frac{[b^l,b^u]T^3}{3})\}]\end{aligned} \tag{47}$$

With the inventory constant, $I(0) \leq I$

We solve the problem, using neutrosophic goal programming method.

Case 2: Here again, as before we get the following annual opportunity cost of capital,

$$\begin{aligned}\tilde{Z}_2 =& \frac{o}{T}+\frac{c(1+m)[([a^l,a^u]+(1+m)[b^l,b^u])\ln\left(\frac{1+m}{1+m-T}\right)+[b^l,b^u](-T)]}{T}\\ &+\frac{h([a^l,a^u]+[b^l,b^u](1+m))}{T}[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right)+\frac{T^2}{4}-\frac{(1+m)T}{2}-\frac{[b^l,b^u]Th}{2}]\\ &+I_pc([a^l,a^u]M+\frac{[b^l,b^u]M^2}{2})-\frac{1}{T}[I_ep\left(\frac{[a^l,a^u]M^2}{2}+\frac{[b^l,b^u]M^3}{3}\right)+I_e[(p([a^l,a^u]M+\frac{[b^l,b^u]M^2}{2}))\\ &+pI_e(\frac{[a^l,a^u]M^2}{2}+\frac{[b^l,b^u]M^3}{3})-c([a^l,a^u]T+\frac{[b^l,b^u]T^2}{2})](T-M)\\ &+pI_e(\frac{[a^l,a^u](T-m)^2}{2}+\frac{[b^l,b^u](T-M)^3}{3})]\end{aligned} \tag{48}$$

Similarly we have,

$$\tilde{Z}(T) = \begin{cases}\tilde{Z}_1(T) & \text{if} \quad T \leq M\\ \tilde{Z}_2(T) & \text{if} \quad T \geq M\end{cases} \tag{49}$$

5. **Numerical example**

In this section we discuss the numerical results in two cases. In the first case the crisp model is discussed. And in the second case the fuzzy model is discussed.

Case 1: For the following example, we have taken the ordering cost 10$ per order. Per unit cost of purchasing and selling are taken respectively 10$,15$. We have taken holding cost per unit per year excluding interest charge to be 1$. We assumed the retailer earns 0.12$ per year and the retailer pays 0.15$ per year interest. We have the following problem,

$$\text{Minimize } \tilde{Z}(T) \tag{50}$$

$$Z(T) = \begin{cases} Z_1(T) & \text{if} \quad T \leq M \\ Z_2(T) & \text{if} \quad T \geq M \end{cases} \tag{51}$$

We have used Lingo software for solving this optimization problem.

We have the following results,

Table 1

a	b	o	c	p	h	I_e	I_p	M	m	T^*	$Z^*(T^*)$
100	.1	10	10	15	10	.12	.15	1	1	0.1071494	653.5241
105	.11	10	10	15	10	.12	.15	1	1	0.1046234	628.8297
110	.12	10	10	15	10	.12	.15	1	1	0.1022692	599.0090
95	.1	10	10	15	10	.12	.15	1	1	0.1098596	625.4176

5.1. *Numerical example(Fuzzy)*

In this section we solve the problem using the neutrosophic interval valued linear programming to solve the problem as discussed in 2.2. Again for the following example, we have taken the ordering cost 10$ per order. Per unit cost of purchasing and selling are taken respectively 10$,15$. We have taken holding cost per unit per year excluding interest charge to be 1$. We assumed the retailer earns 0.12$ per year and the retailer pays 0.15$ per year interest.

The problem is,

$$\text{Minimize } \tilde{Z}(T) \tag{52}$$

where,

$$\tilde{Z}(T) = \begin{cases} \tilde{Z}_1(T) & \text{if} \quad T \leq M \\ \tilde{Z}_2(T) & \text{if} \quad T \geq M \end{cases} \tag{53}$$

Similarly using Lingo program we get the following results for different values of the market demand.

By similar arguments, for different values of the score functions we get the following results,

Table 2

$[a^l, a^u]$	$[b^l, b^u]$	o	c	p	h	I_e	I_p	M	m	T^*	$Z^*(T^*)$
[103,115]	[.11,.15]	10	10	15	10	.12	.25	1	1	0.3269101	[448.55,618.64]
[101,110]	[.10,.13]	10	10	15	10	.12	.25	1	1	0.3656319	[543.78,659.13]
[99,108]	[.12,.16]	10	10	15	10	.12	.25	1	1	0.2551095	[372.84,548.5184]
[105,125]	[.09,.12]	10	10	15	10	.12	.25	1	1	0.4338459	[667.10,734.18]

So, when we use the neutrosophic interval valued number we get an range of value for the objective function rather than getting a fixed value. So, the decision maker has more freedom in choosing the approximate demand.

6. Conclusion

In this paper we have developed a EOQ model for a single deteriorating product which deteriorate continuously. Also, the market demand is considered to be linearly dependent of time. The retailer is given trade credit with a fixed interest. The retailer assumed to be earning interest on the profit. Since the market demand cannot be predicted precisely, the model is further generalized using neutrosophic triangular numbers for the parameters. The final model is solved using neutrosophic interval valued goal programming method. Through an example we have shown that the retailer has more freedom in choosing the approximate demand for the later case.

The model has been formed assuming the demand function is a linear function of time. For future work, the demand function can be assumed to be more complex functions of time or other parameters. Furthermore, the model can be more generalized by considering the other parameters as neutrosophic triangular number.

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Mappings on Bipolar Hypersoft Classes

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Abstract. Mappings are significant mathematical tools with many applications in our daily lives. The bipolar hypersoft set is one of the effective tools for dealing with ambiguity and vagueness. The purpose of this article is to define mappings between the classes of bipolar hypersoft sets. The notions of bipolar hypersoft image and bipolar hypersoft inverse image of bipolar hypersoft sets are then defined, and some of their properties are studied. Moreover, we discuss the relations between the bipolar hypersoft image and the bipolar hypersoft inverse image of the bipolar hypersoft sets. This proposed work can be extended to *IndetermSoft Set*, *IndetermHyperSoft Set* and *TreeSoft Set* and their corresponding Fuzzy, Intuitionistic Fuzzy, Neutrosophic forms and other Fuzzy-extension.

Keywords: bipolar hypersoft mapping; bipolar hypersoft image; bipolar hypersoft inverse image; bipolar hypersoft set; hypersoft set; soft set

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1. Introduction

In all real-life disciplines, such as environmental science, social science, engineering and economics, there is ambiguity, inaccuracy, and inadequate information. Many researchers have attempted to process such data in the past and present. In 1999, Molodtsov [14] proposed the theory of soft set as a completely flexible mathematical approach to modeling uncertainties. In 2003, Maji et al. [12] developed the theory of soft sets by defining several essential operations like subset, the equal set and the complement of a soft set. Shabir and Naz [25] proposed and studied the concept of bipolar soft sets (a combination of the soft set and the bipolarity structure) and its use in decision-making (2013).

The traditional soft set is built on a determinate function, however there are numerous sources in our world that, due to ignorance or a lack of knowledge, present indeterminate information. Due to the uncertainty in our world, they can be modeled by operators with some degree of uncertainty. As a result, Smarandache [27,28] extended the soft set to hypersoft set in 2018, then both of them to IndetermSoft Set and IndetermHyperSoft Set [29, 30] respectively in 2022, and introduced TreeSoft Set [31] as extension of the MultiSoft Set [26]. Several applications are presented for each type of soft set. Musa and Asaad [4,15,16] applied hypersoft set to present some topological concepts such as connectedness and separation axioms.

Defining relations and mappings on soft sets, bipolar soft sets and hypersoft sets was one of the most important steps in the development of these theories. Babitha and Sunil [6] initiated the notion of soft relations and soft functions. Qin et al. [21] introduced the concept of soft relation which is a generalization of soft set relation presented in [6]. They supported their work with an application to information systems. Majumdar and Samanta [13] examined the concept of crisp (soft) set images using soft mappings. Kharal and Ahmad [11] defined the idea of soft class mappings and discussed the characteristics of soft images and soft pre-images. Furthermore, they provided an application of soft mapping in medical diagnosis. Addis et al. [2] has developed a new method to define soft mappings and studying their properties. They used this concept in a new way to study soft homomorphisms and soft homomorphism theorems on groups. They also built a soft mapping to model a symptom–disease relationship in medical diagnosis. The notion of mappings between two collections of bipolar soft sets was introduced by Al-shami [1] and exhaustively studied by Fadel and Dzul-Kifli [8]. Saeed et al. [23, 24] introduced mappings to the hypersoft set environment. They defined hypersoft image and hypersoft pre-image and studied some of their properties. Moreover, the validity and dominance of their suggested technique is demonstrated through practical application and comparative analysis. Other searches for mappings can be seen [3, 5, 7, 9, 10, 22, 32–34].

Musa and Asaad [17], came up with the concept of bipolar hypersoft set as a mixture of hypersoft set and bipolarity structure and is created by looking at not only a collection of carefully chosen parameters, but also a set associated with parameters with opposing meanings known as "not set of parameters". They also presented an application of bipolar hypersoft sets in a decision-making problem [18]. In addition, the authors [19,20] studied the topological structures of bipolar hypersoft sets. Motivated by the interest of researchers for mappings and their applications. We continue to study bipolar hypersoft sets by defining bipolar hypersoft mapping and discuss some of its characteristics.

The rest of the article is organized in the following order: Section 2 provides an overview of several fundamental concepts that are necessary to understand our research. In section 3, we define the concept of bipolar hypersoft mapping and study its properties. In section 4, we introduce bipolar hypersoft inverse image and related results. We conclude this section by presenting the relationship between the bipolar hypersoft image and the bipolar hypersoft inverse image. Section 5 provide a summary of ongoing work as well as a suggestion for future study.

2. **Preliminaries**

Throughout this work, $\Re$ and $\aleph$ denote the universal sets; $2^{\Re}$ and $2^{\aleph}$ denote the power sets of $\Re$ and $\aleph$, respectively; $\Sigma = \sigma_1 \times \sigma_2 \times \ldots \times \sigma_n$ and $\acute{\Sigma} = \acute{\sigma}_1 \times \acute{\sigma}_2 \times \ldots \times \acute{\sigma}_n$ denote the parameter sets with $\sigma_i \cap \sigma_j = \phi$, $\acute{\sigma}_i \cap \acute{\sigma}_j = \phi$ where $i \neq j$; and $\Lambda, \Delta, \acute{\Lambda}, \acute{\Delta}$ are non-empty sets of parameters where $\Lambda, \Delta \subseteq \Sigma$ and $\acute{\Lambda}, \acute{\Delta} \subseteq \acute{\Sigma}$.

The basic definitions and results introduced in [17] will be collected in this section.

Definition 2.1. A triple $(g, \widehat{g}, \Lambda)$ is called a bipolar hypersoft set over $\Re$, where g and $\widehat{g}$ are mappings given by $g : \Lambda \to 2^{\Re}$ and $\widehat{g} : \neg\Lambda \to 2^{\Re}$ such that $g(\ell) \cap \widehat{g}(\neg\ell) = \phi$ for all $\ell \in \Lambda$.

We represent a bipolar hypersoft set $(g, \widehat{g}, \Lambda)$ as:

$$(g, \widehat{g}, \Lambda) = \{(\ell, g(\ell), \widehat{g}(\neg\ell)) : \ell \in \Lambda \text{ and } g(\ell) \cap \widehat{g}(\neg\ell) = \phi\}.$$

The collection of all bipolar hypersoft sets on $\Re$ (resp., $\aleph$) with the set of parameters Σ (resp., $\acute{\Sigma}$) is denoted by $\Omega_{(\Re,\Sigma)}$ (resp., $\Omega_{(\aleph,\acute{\Sigma})}$).

Definition 2.2. Let $(g, \widehat{g}, \Lambda)$, $(f, \widehat{f}, \Delta) \in \Omega_{(\Re,\Sigma)}$. Then

i. $(g, \widehat{g}, \Lambda)$ is a bipolar hypersoft subset of $(f, \widehat{f}, \Delta)$, denoted by $(g, \widehat{g}, \Lambda) \,\widetilde{\widetilde{\sqsubseteq}}\, (f, \widehat{f}, \Delta)$, if $\Lambda \subseteq \Delta$ and $g(\ell) \subseteq f(\ell)$, $\widehat{f}(\neg\ell) \subseteq \widehat{g}(\neg\ell)$ for all $\ell \in \Lambda$.

ii. $(g, \widehat{g}, \Lambda)$ and $(f, \widehat{f}, \Delta)$ are bipolar hypersoft equal, if $(g, \widehat{g}, \Lambda) \,\widetilde{\widetilde{\sqsubseteq}}\, (f, \widehat{f}, \Delta)$ and $(f, \widehat{f}, \Delta) \,\widetilde{\widetilde{\sqsubseteq}}\, (g, \widehat{g}, \Lambda)$.

iii. If $g(\ell) = \phi$ and $\widehat{g}(\neg\ell) = \Re$ for all $\ell \in \Lambda$, then $(g, \widehat{g}, \Lambda)$ is called a relative null bipolar hypersoft set and denoted by $(\widetilde{\phi}, \widetilde{\Re}, \Lambda)$.

iv. If $g(\ell) = \Re$ and $\widehat{g}(\neg\ell) = \phi$ for all $\ell \in \Lambda$, then $(g, \widehat{g}, \Lambda)$ is called a relative whole bipolar hypersoft set and denoted by $(\widetilde{\Re}, \widetilde{\phi}, \Lambda)$.

v. The complement of $(g, \widehat{g}, \Lambda)$ is a bipolar hypersoft set $(g, \widehat{g}, \Lambda)^c = (g^c, \widehat{g}^c, \Lambda)$ where $g^c(\ell) = \widehat{g}(\neg\ell)$ and $\widehat{g}^c(\neg\ell) = g(\ell)$ for all $\ell \in \Lambda$.

vi. The union of $(g, \widehat{g}, \Lambda)$ and $(f, \widehat{f}, \Delta)$, denoted by $(g, \widehat{g}, \Lambda) \,\widetilde{\widetilde{\sqcup}}\, (f, \widehat{f}, \Delta)$, is a bipolar hypersoft set $(h, \widehat{h}, C)$, where $C = \Lambda \cup \Delta$ and for all $\ell \in C$:

$$h(\ell) = \begin{cases} g(\ell) & \text{if } \ell \in \Lambda \setminus \Delta \\ f(\ell) & \text{if } \ell \in \Delta \setminus \Lambda \\ g(\ell) \cup f(\ell) & \text{if } \ell \in \Lambda \cap \Delta \end{cases}$$

$$\widehat{h}(\neg\ell) = \begin{cases} \widehat{g}(\neg\ell) & \text{if } \neg\ell \in \neg\Lambda \setminus \neg\Delta \\ \widehat{f}(\neg\ell) & \text{if } \neg\ell \in \neg\Delta \setminus \neg\Lambda \\ \widehat{g}(\neg\ell) \cap \widehat{f}(\neg\ell) & \text{if } \neg\ell \in \neg\Lambda \cap \neg\Delta \end{cases}$$

vii. The extended intersection of $(g, \widehat{g}, \Lambda)$ and $(f, \widehat{f}, \Delta)$, denoted by $(g, \widehat{g}, \Lambda) \;\tilde{\tilde{\sqcap}}_{\varepsilon}\; (f, \widehat{f}, \Delta)$, is a bipolar hypersoft set $(h, \widehat{h}, C)$, where $C = \Lambda \cup \Delta$ and for all $\ell \in C$:

$$h(\ell) = \begin{cases} g(\ell) & \text{if } \ell \in \Lambda \setminus \Delta \\ f(\ell) & \text{if } \ell \in \Delta \setminus \Lambda \\ g(\ell) \cap f(\ell) & \text{if } \ell \in \Lambda \cap \Delta \end{cases}$$

$$\widehat{h}(\neg\ell) = \begin{cases} \widehat{g}(\neg\ell) & \text{if } \neg\ell \in \neg\Lambda \setminus \neg\Delta \\ \widehat{f}(\neg\ell) & \text{if } \neg\ell \in \neg\Delta \setminus \neg\Lambda \\ \widehat{g}(\neg\ell) \cup \widehat{f}(\neg\ell) & \text{if } \neg\ell \in \neg\Lambda \cap \neg\Delta \end{cases}$$

viii. The restricted union of $(g, \widehat{g}, \Lambda)$ and $(f, \widehat{f}, \Delta)$, denoted by $(g, \widehat{g}, \Lambda) \;\tilde{\tilde{\sqcup}}_{\mathcal{R}}\; (f, \widehat{f}, \Delta)$, is a bipolar hypersoft set $(h, \widehat{h}, C)$, where $C = \Lambda \cap \Delta$ and for all $\ell \in C$: $h(\ell) = g(\ell) \cup f(\ell)$ and $\widehat{h}(\neg\ell) = \widehat{g}(\neg\ell) \cap \widehat{f}(\neg\ell)$.

ix. The intersection of $(g, \widehat{g}, \Lambda)$ and $(f, \widehat{f}, \Delta)$, denoted by $(g, \widehat{g}, \Lambda) \;\tilde{\tilde{\sqcap}}\; (f, \widehat{f}, \Delta)$, is a bipolar hypersoft set $(h, \widehat{h}, C)$, where $C = \Lambda \cap \Delta$ and for all $\ell \in C$: $h(\ell) = g(\ell) \cap f(\ell)$ and $\widehat{h}(\neg\ell) = \widehat{g}(\neg\ell) \cup \widehat{f}(\neg\ell)$.

Proposition 2.3. *Let* $(g, \widehat{g}, \Lambda)$, $(f, \widehat{f}, \Lambda) \in \Omega_{(\Re,\Sigma)}$. *Then*

i. $((g, \widehat{g}, \Lambda)^c)^c = (g, \widehat{g}, \Lambda)$.

ii. *If* $(g, \widehat{g}, \Lambda) \;\tilde{\tilde{\sqsubseteq}}\; (f, \widehat{f}, \Lambda)$, *then* $(f, \widehat{f}, \Lambda)^c \;\tilde{\tilde{\sqsubseteq}}\; (g, \widehat{g}, \Lambda)^c$.

iii. $(\widetilde{\phi}, \widetilde{\Re}, \Lambda) \;\tilde{\tilde{\sqsubseteq}}\; (g, \widehat{g}, \Lambda) \;\tilde{\tilde{\sqcap}}\; (g, \widehat{g}, \Lambda)^c \;\tilde{\tilde{\sqsubseteq}}\; (g, \widehat{g}, \Lambda) \;\tilde{\tilde{\sqcup}}\; (g, \widehat{g}, \Lambda)^c \;\tilde{\tilde{\sqsubseteq}}\; (\widetilde{\Re}, \widetilde{\phi}, \Lambda)$.

iv. $(g, \widehat{g}, \Lambda) \;\tilde{\tilde{\sqcup}}\; (f, \widehat{f}, \Lambda) = (g, \widehat{g}, \Lambda) \;\tilde{\tilde{\sqcup}}_{\mathcal{R}}\; (f, \widehat{f}, \Lambda)$.

v. $(g, \widehat{g}, \Lambda) \;\tilde{\tilde{\sqcap}}\; (f, \widehat{f}, \Lambda) = (g, \widehat{g}, \Lambda) \;\tilde{\tilde{\sqcap}}_{\varepsilon}\; (f, \widehat{f}, \Lambda)$.

Proposition 2.4. *Let* $(g, \widehat{g}, \Lambda)$, $(f, \widehat{f}, \Delta) \in \Omega_{(\Re,\Sigma)}$. *Then*

i. $((g, \widehat{g}, \Lambda) \;\tilde{\tilde{\sqcup}}\; (f, \widehat{f}, \Delta))^c = (g, \widehat{g}, \Lambda)^c \;\tilde{\tilde{\sqcap}}\; (f, \widehat{f}, \Delta)^c$.

ii. $((g, \widehat{g}, \Lambda) \;\tilde{\tilde{\sqcap}}\; (f, \widehat{f}, \Delta))^c = (g, \widehat{g}, \Lambda)^c \;\tilde{\tilde{\sqcup}}\; (f, \widehat{f}, B)^c$.

3. **Bipolar Hypersoft Mappings**

In this section, we study mappings between families of bipolar hypersoft sets with different universes and sets of parameters. In addition, illustrative examples are offered to help understand the main results.

Definition 3.1. Let $\gamma : \Re \to \aleph$ be an injective mapping. Let $\delta : \Sigma \to \acute{\Sigma}$ and $\lambda : \neg\Sigma \to \neg\acute{\Sigma}$ be two mappings such that $\lambda(\neg\ell) = \neg\delta(\ell)$ for all $\neg\ell \in \neg\Sigma$. Then a bipolar hypersoft mapping $\Psi_{\gamma\delta\lambda} : \Omega_{(\Re,\Sigma)} \to \Omega_{(\aleph,\acute{\Sigma})}$ is defined as: for any bipolar hypersoft set $(\mathcal{g}, \widehat{\mathcal{g}}, \Lambda) \in \Omega_{(\Re,\Sigma)}$, the image of $(\mathcal{g}, \widehat{\mathcal{g}}, \Lambda)$ under $\Psi_{\gamma\delta\lambda}$, $\Psi_{\gamma\delta\lambda}((\mathcal{g}, \widehat{\mathcal{g}}, \Lambda)) = (\Psi_{\gamma\delta\lambda}(\mathcal{g}), \Psi_{\gamma\delta\lambda}(\widehat{\mathcal{g}}), \acute{\Sigma})$ is a bipolar hypersoft set in $\Omega_{(\aleph,\acute{\Sigma})}$ given as, for all $\acute{\ell} \in \acute{\Sigma}$:

$$\Psi_{\gamma\delta\lambda}(\mathcal{g})(\acute{\ell}) = \begin{cases} \gamma\left(\bigcup_{\ell\in\delta^{-1}(\acute{\ell})\cap\Lambda} \mathcal{g}(\ell)\right), & \text{if } \delta^{-1}(\acute{\ell}) \cap \Lambda \neq \phi \\ \phi, & \text{otherwise} \end{cases}$$

$$\Psi_{\gamma\delta\lambda}(\widehat{\mathcal{g}})(\neg\acute{\ell}) = \begin{cases} \gamma\left(\bigcap_{\neg\ell\in\lambda^{-1}(\neg\acute{\ell})\cap\neg\Lambda} \widehat{\mathcal{g}}(\neg\ell)\right), & \text{if } \lambda^{-1}(\neg\acute{\ell}) \cap \neg\Lambda \neq \phi \\ \aleph, & \text{otherwise} \end{cases}$$

Example 3.2. Let $\Re = \{r_1, r_2, r_3\}$ and $\aleph = \{\eta_1, \eta_2, \eta_3, \eta_4\}$ be two sets, $\sigma_1 = \{\ell_1, \ell_2, \ell_3, \ell_4\}, \sigma_2 = \{\ell_5\}, \sigma_3 = \{\ell_6\}$, and $\acute{\sigma_1} = \{\acute{\ell_1}, \acute{\ell_2}, \acute{\ell_3}, \acute{\ell_4}\}, \acute{\sigma_2} = \{\acute{\ell_5}\}, \acute{\sigma_3} = \{\acute{\ell_6}\}$ be sets of parameters, $\gamma : \Re \to \aleph$ be a mapping defined as $\gamma(r_i) = \eta_i$ for $i = 1, 2, 3$, the mapping $\delta : \Sigma \to \acute{\Sigma}$ be defined as $\delta((\ell_1, \ell_5, \ell_6)) = \delta((\ell_2, \ell_5, \ell_6)) = (\acute{\ell_1}, \acute{\ell_5}, \acute{\ell_6})$, $\delta((\ell_3, \ell_5, \ell_6)) = (\acute{\ell_3}, \acute{\ell_5}, \acute{\ell_6})$, $\delta((\ell_4, \ell_5, \ell_6)) = (\acute{\ell_4}, \acute{\ell_5}, \acute{\ell_6})$, the mapping $\lambda : \neg\Sigma \to \neg\acute{\Sigma}$ be defined as $\lambda(\neg\ell_i) = \neg\delta(\ell_i)$ for $i = 1, 2, 3$, and $\Psi_{\gamma\delta\lambda} : \Omega_{(\Re,\Sigma)} \to \Omega_{(\aleph,\acute{\Sigma})}$ be a bipolar hypersoft mapping. Let $\Lambda_1 = \{\ell_1, \ell_2, \ell_3\}, \Lambda_2 = \{\ell_5\}, \Lambda_3 = \{\ell_6\}$ and $(\mathcal{g}, \widehat{\mathcal{g}}, \Lambda) = \{((\ell_1, \ell_5, \ell_6), \{r_1\}, \{r_2\}), ((\ell_2, \ell_5, \ell_6), \{r_3\}, \{r_1, r_2\}), ((\ell_3, \ell_5, \ell_6), \{r_3\}, \{r_1\})\}$. Then, the bipolar hypersoft image of $(\mathcal{g}, \widehat{\mathcal{g}}, \Lambda)$:

Since $\delta(\Lambda) = \delta(\{(\ell_1, \ell_5, \ell_6), (\ell_2, \ell_5, \ell_6), (\ell_3, \ell_5, \ell_6)\}) = \{(\acute{\ell_1}, \acute{\ell_5}, \acute{\ell_6}), (\acute{\ell_3}, \acute{\ell_5}, \acute{\ell_6})\}$, then for $(\acute{\ell_1}, \acute{\ell_5}, \acute{\ell_6}) : \delta^{-1}((\acute{\ell_1}, \acute{\ell_5}, \acute{\ell_6})) \cap \Lambda = \{(\ell_1, \ell_5, \ell_6), (\ell_2, \ell_5, \ell_6)\} \cap \{(\ell_1, \ell_5, \ell_6), (\ell_2, \ell_5, \ell_6), (\ell_3, \ell_5, \ell_6)\} = \{(\ell_1, \ell_5, \ell_6), (\ell_2, \ell_5, \ell_6)\}$. We have

$\Psi_{\gamma\delta\lambda}(\mathcal{g})(\acute{\ell_1}, \acute{\ell_5}, \acute{\ell_6}) = \gamma\left(\bigcup_{\ell\in\delta^{-1}(\acute{\ell_1},\acute{\ell_5},\acute{\ell_6})\cap\Lambda} \mathcal{g}(\ell)\right) = \gamma\left(\mathcal{g}(\ell_1, \ell_5, \ell_6) \cup \mathcal{g}(\ell_2, \ell_5, \ell_6)\right) = \gamma\left(\{r_1\} \cup \{r_3\}\right) = \gamma\left(\{r_1, r_3\}\right) = \{\eta_1, \eta_3\}$.

Also, $\lambda(\neg\Lambda) = \{\neg(\acute{\ell_1}, \acute{\ell_5}, \acute{\ell_6}), \neg(\acute{\ell_3}, \acute{\ell_5}, \acute{\ell_6})\}$, then for $\neg(\acute{\ell_1}, \acute{\ell_5}, \acute{\ell_6}) : \lambda^{-1}(\neg(\acute{\ell_1}, \acute{\ell_5}, \acute{\ell_6})) \cap \neg\Lambda = \{\neg(\ell_1, \ell_5, \ell_6), \neg(\ell_2, \ell_5, \ell_6)\}$. We have

$\Psi_{\gamma\delta\lambda}(\widehat{\mathcal{g}})(\neg(\acute{\ell_1}, \acute{\ell_5}, \acute{\ell_6})) = \gamma\left(\bigcap_{\neg\ell\in\lambda^{-1}(\neg(\acute{\ell_1},\acute{\ell_5},\acute{\ell_6}))\cap\neg\Lambda} \widehat{\mathcal{g}}(\neg\ell)\right) = \gamma(\widehat{\mathcal{g}}(\neg(\ell_1, \ell_5, \ell_6)) \cap \widehat{\mathcal{g}}(\neg(\ell_2, \ell_5, \ell_6))) = \gamma\left(\{r_2\} \cap \{r_1, r_2\}\right) = \gamma\left(\{r_2\}\right) = \{\eta_2\}$

Then, $\Psi_{\gamma\delta\lambda}((\mathcal{g}, \widehat{\mathcal{g}}, \Lambda))((\acute{\ell_1}, \acute{\ell_5}, \acute{\ell_6})) = ((\acute{\ell_1}, \acute{\ell_5}, \acute{\ell_6}), \{\eta_1, \eta_3\}, \{\eta_2\})$.

Now, for $(\acute{\ell_3}, \acute{\ell_5}, \acute{\ell_6}) : \delta^{-1}((\acute{\ell_3}, \acute{\ell_5}, \acute{\ell_6})) \cap \Lambda = \{(\ell_3, \ell_5, \ell_6)\} \cap \{(\ell_1, \ell_5, \ell_6), (\ell_2, \ell_5, \ell_6), (\ell_3, \ell_5, \ell_6)\}$ $= \{(\ell_3, \ell_5, \ell_6)\}$. We have

$$\Psi_{\gamma\delta\lambda}(g)((\acute{\ell_3}, \acute{\ell_5}, \acute{\ell_6})) = \gamma\left(\bigcup_{\ell \in \delta^{-1}((\acute{\ell_3}, \acute{\ell_5}, \acute{\ell_6}))\cap\Lambda} g(\ell)\right) = \gamma\left(g(\ell_3, \ell_5, \ell_6)\right) = \gamma\left(\{r_3\}\right) = \{\eta_3\}.$$

Also, for $\neg(\acute{\ell_3}, \acute{\ell_5}, \acute{\ell_6}) : \lambda^{-1}(\neg(\acute{\ell_3}, \acute{\ell_5}, \acute{\ell_6})) \cap \neg\Lambda = \{\neg(\ell_3, \ell_5, \ell_6)\}$. We have

$$\Psi_{\gamma\delta\lambda}(\widehat{g})(\neg(\acute{\ell_3}, \acute{\ell_5}, \acute{\ell_6})) = \gamma\left(\bigcap_{\neg\ell \in \lambda^{-1}(\neg(\acute{\ell_3}, \acute{\ell_5}, \acute{\ell_6}))\cap\neg\Lambda} \widehat{g}(\neg\ell)\right) = \gamma\left(\widehat{g}(\neg(\ell_3, \ell_5, \ell_6))\right) = \gamma\left(\{r_1\}\right) = \{\eta_1\}.$$

Then, $\Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Lambda))((\acute{\ell_3}, \acute{\ell_5}, \acute{\ell_6})) = ((\acute{\ell_3}, \acute{\ell_5}, \acute{\ell_6}), \{\eta_3\}, \{\eta_1\})$.
Hence, $\Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Lambda)) = \{((\acute{\ell_1}, \acute{\ell_5}, \acute{\ell_6}), \{\eta_1, \eta_3\}, \{\eta_2\}), ((\acute{\ell_2}, \acute{\ell_5}, \acute{\ell_6}), \phi, \aleph), ((\acute{\ell_3}, \acute{\ell_5}, \acute{\ell_6}), \{\eta_3\}, \{\eta_1\}),$ $((\acute{\ell_4}, \acute{\ell_5}, \acute{\ell_6}), \phi, \aleph)\}$.

Remark 3.3. In the next example, we illustrate the reason for choosing the mapping $\gamma : \Re \to \aleph$ in Definition 3.1 to be injective .

Example 3.4. Suppose $\Psi_{\gamma\delta\lambda}$ and $(g, \widehat{g}, \Lambda)$ be the same as in Example 3.2 but $\gamma(r_2) = \eta_1$ instead of $\gamma(r_2) = \eta_2$, then $\Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Lambda))((\acute{\ell_1}, \acute{\ell_5}, \acute{\ell_6})) = ((\acute{\ell_1}, \acute{\ell_5}, \acute{\ell_6}), \{\eta_1, \eta_3\}, \{\eta_1\})$ which contradicts the definition of bipolar hypersoft set since $\Psi_{\gamma\delta\lambda}(g)((\acute{\ell_1}, \acute{\ell_5}, \acute{\ell_6})) \cap \Psi_{\gamma\delta\lambda}(\widehat{g})(\neg(\acute{\ell_1}, \acute{\ell_5}, \acute{\ell_6}))$ $\neq \phi$.

Definition 3.5. Suppose that $\Psi_{\gamma\delta\lambda} : \Omega_{(\Re,\Sigma)} \to \Omega_{(\aleph,\acute{\Sigma})}$ is a bipolar hypersoft mapping and $(g, \widehat{g}, \Lambda)$, $(f, \widehat{f}, \Delta) \in \Omega_{(\Re,\Sigma)}$. Then:

(1) The union of bipolar hypersoft image of $(g, \widehat{g}, \Lambda)$, $(f, \widehat{f}, \Delta) \in \Omega_{(\Re,\Sigma)}$ is defined as, for all $\acute{\ell} \in \acute{\Sigma}$,

$$\left(\Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Lambda)) \widetilde{\widetilde{\sqcup}} \Psi_{\gamma\delta\lambda}((f, \widehat{f}, \Delta))\right)(\acute{\ell}) =$$

$$\left(\acute{\ell}, \Psi_{\gamma\delta\lambda}(g)(\acute{\ell}) \cup \Psi_{\gamma\delta\lambda}(f)(\acute{\ell}), \Psi_{\gamma\delta\lambda}(\widehat{g})(\neg\acute{\ell}) \cap \Psi_{\gamma\delta\lambda}(\widehat{f})(\neg\acute{\ell})\right).$$

(2) The intersection of bipolar hypersoft image of $(g, \widehat{g}, \Lambda)$, $(f, \widehat{f}, \Delta) \in \Omega_{(\Re,\Sigma)}$ is defined as, for all $\acute{\ell} \in \acute{\Sigma}$,

$$\left(\Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Lambda)) \widetilde{\widetilde{\sqcap}} \Psi_{\gamma\delta\lambda}((f, \widehat{f}, \Delta))\right)(\acute{\ell}) =$$

$$\left(\acute{\ell}, \Psi_{\gamma\delta\lambda}(g)(\acute{\ell}) \cap \Psi_{\gamma\delta\lambda}(f)(\acute{\ell}), \Psi_{\gamma\delta\lambda}(\widehat{g})(\neg\acute{\ell}) \cup \Psi_{\gamma\delta\lambda}(\widehat{f})(\neg\acute{\ell})\right).$$

Definition 3.6. Suppose that $\Psi_{\gamma\delta\lambda} : \Omega_{(\Re,\Sigma)} \to \Omega_{(\aleph,\acute{\Sigma})}$ is a bipolar hypersoft mapping, where $\gamma : \Re \to \aleph$ is an injective mapping, $\delta : \Sigma \to \acute{\Sigma}$ and $\lambda : \neg\Sigma \to \neg\acute{\Sigma}$ are two mappings such that $\lambda(\neg\ell) = \neg\delta(\ell)$ for all $\neg\ell \in \neg\Sigma$. Then a bipolar hypersoft mapping $\Psi_{\gamma\delta\lambda}$ is called:

(1) A bipolar hypersoft surjective mapping if γ and δ are surjective mappings.

(2) A bipolar hypersoft injective mapping if γ and δ are injective mappings. (Provided that any bipolar hypersoft sets in $\Omega_{(\Re,\Sigma)}$ must have the same sets of parameters.)

(3) A bipolar hypersoft bijective mapping if γ and δ are bijective mappings.

Proposition 3.7. *Suppose that $\Psi_{\gamma\delta\lambda} : \Omega_{(\Re,\Sigma)} \to \Omega_{(\aleph,\acute{\Sigma})}$ is a bipolar hypersoft mapping, where $\gamma : \Re \to \aleph$ is an injective mapping, $\delta : \Sigma \to \acute{\Sigma}$ and $\lambda : \neg\Sigma \to \neg\acute{\Sigma}$ are two mappings such that $\lambda(\neg\ell) = \neg\delta(\ell)$ for all $\neg\ell \in \neg\Sigma$. If $(g, \widehat{g}, \Lambda)$, $(f, \widehat{f}, \Delta) \in \Omega_{(\Re,\Sigma)}$ then:*

(1) $\Psi_{\gamma\delta\lambda}((\Phi, \widehat{\Re}, \Sigma)) \;\tilde{\tilde{\sqsupseteq}}\; ((\Phi, \widehat{\aleph}, \acute{\Sigma}))$. The equality holds if γ is a surjective mapping.

(2) $\Psi_{\gamma\delta\lambda}((\widehat{\Re}, \Phi, \Sigma)) \;\tilde{\tilde{\sqsubseteq}}\; ((\widehat{\aleph}, \Phi, \acute{\Sigma}))$.

(3) If $(g, \widehat{g}, \Lambda) \;\tilde{\tilde{\sqsubseteq}}\; (f, \widehat{f}, \Delta)$, then $\Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Lambda)) \;\tilde{\tilde{\sqsubseteq}}\; \Psi_{\gamma\delta\lambda}((f, \widehat{f}, \Delta))$.

(4) $\Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Lambda)\tilde{\tilde{\sqcup}}(f, \widehat{f}, \Delta)) = \Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Lambda)) \;\tilde{\tilde{\sqcup}}\; \Psi_{\gamma\delta\lambda}((f, \widehat{f}, \Delta))$.

(5) $\Psi_{\gamma\delta\lambda}\left((g, \widehat{g}, \Lambda)\tilde{\tilde{\sqcap}}(f, \widehat{f}, \Delta) = (h, \widehat{h}, \Lambda \cap \Delta)\right) \;\tilde{\tilde{\sqsubseteq}}\; \Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Lambda)) \;\tilde{\tilde{\sqcap}}\; \Psi_{\gamma\delta\lambda}((f, \widehat{f}, \Delta))$. The equality holds if $\Psi_{\gamma\delta\lambda}$ is a bipolar hypersoft injective mapping.

Proof. 1. and 2. are straightforward.

3. Let $(g, \widehat{g}, \Lambda) \;\tilde{\tilde{\sqsubseteq}}\; (f, \widehat{f}, \Delta)$, then we want to show that, for all $\acute{\ell} \in \acute{\Sigma}$, $\Psi_{\gamma\delta\lambda}(g)(\acute{\ell}) \subseteq \Psi_{\gamma\delta\lambda}(f)(\acute{\ell})$ and, for all $\neg\acute{\ell} \in \neg\acute{\Sigma}$, $\Psi_{\gamma\delta\lambda}(\widehat{f})(\neg\acute{\ell}) \subseteq \Psi_{\gamma\delta\lambda}(\widehat{g})(\neg\acute{\ell})$. Let $\acute{\ell} \in \delta(\Lambda) \subseteq \delta(\Delta) \subseteq \acute{\Sigma}$ (if $\acute{\ell} \notin \delta(\Lambda)$, then $\Psi_{\gamma\delta\lambda}(g)(\acute{\ell}) = \phi \subseteq \Psi_{\gamma\delta\lambda}(f)(\acute{\ell})$), then

$$\begin{aligned}
\Psi_{\gamma\delta\lambda}(g)(\acute{\ell}) &= \gamma\left(\bigcup_{\ell\in\delta^{-1}(\acute{\ell})\cap\Lambda} g(\ell)\right)\\
&\subseteq \gamma\left(\bigcup_{\ell\in\delta^{-1}(\acute{\ell})\cap\Delta} f(\ell)\right), since\, g(\ell) \subseteq f(\ell)\, for\, all\, \ell \in \Lambda\\
&= \Psi_{\gamma\delta\lambda}(f)(\acute{\ell}).
\end{aligned}$$

Now, for $\neg\acute{\ell} \in \lambda(\neg\Lambda) \subseteq \lambda(\neg\Delta) \subseteq \neg\acute{\Sigma}$ (if $\neg\acute{\ell} \notin \lambda(\neg\Lambda)$, then $\Psi_{\gamma\delta\lambda}(\widehat{g})(\neg\acute{\ell}) = \aleph \supseteq \Psi_{\gamma\delta\lambda}(\widehat{f})(\neg\acute{\ell})$), we have

$$\begin{aligned}
\Psi_{\gamma\delta\lambda}(\widehat{f})(\neg\acute{\ell}) &= \gamma\left(\bigcap_{\neg\ell\in\lambda^{-1}(\neg\acute{\ell})\cap\neg\Delta} \widehat{f}(\neg\ell)\right)\\
&\subseteq \gamma\left(\bigcap_{\neg\ell\in\lambda^{-1}(\neg\acute{\ell})\cap\neg\Lambda} \widehat{g}(\neg\ell)\right), since\, \widehat{f}(\neg\ell) \subseteq \widehat{g}(\ell)\, for\, all\, \neg\ell \in \neg\Lambda\\
&= \Psi_{\gamma\delta\lambda}(\widehat{g})(\acute{\neg\ell}).
\end{aligned}$$

Hence, $\Psi_{\gamma\delta\lambda}((g,\widehat{g},\Lambda)) \,\widetilde{\widetilde{\sqsubseteq}}\, \Psi_{\gamma\delta\lambda}((f,\widehat{f},\Delta))$.

4. To keep things simple, let

$$\Psi_{\gamma\delta\lambda}((g,\widehat{g},\Lambda))\widetilde{\widetilde{\sqcup}}\Psi_{\gamma\delta\lambda}((f,\widehat{f},\Delta)) = (h,\widehat{h},\acute{\Sigma})$$

$$\Psi_{\gamma\delta\lambda}((g,\widehat{g},\Lambda)\widetilde{\widetilde{\sqcup}}(f,\widehat{f},\Delta)) = \Psi_{\gamma\delta\lambda}((I,\widehat{I},\Lambda\cup\Delta)) = (\mathcal{J},\widehat{\mathcal{J}},\acute{\Sigma}).$$

We want to prove that , for all $\acute{\ell}\in\acute{\Sigma}$, $\mathcal{J}(\acute{\ell}) = h(\acute{\ell})$ and, for all $\neg\acute{\ell}\in\acute{\Sigma}$, $\widehat{\mathcal{J}}(\neg\acute{\ell}) = \widehat{h}(\neg\acute{\ell})$. For non-trivial case, let $\acute{\ell}\in\delta(\Lambda\cup\Delta) = \delta(\Lambda)\cup\delta(\Lambda) = \acute{\Lambda}\cup\acute{\Delta}$, then

$$\mathcal{J}(\acute{\ell}) = \Psi_{\gamma\delta\lambda}(I)(\acute{\ell}) = \gamma\left(\bigcup\nolimits_{\ell\in\delta^{-1}(\acute{\ell})\cap(\Lambda\cup\Delta)} I(\ell)\right)$$

$$= \begin{cases} \gamma\left(\bigcup_{\ell\in\delta^{-1}(\acute{\ell})\cap(\Lambda\setminus\Delta)} g(\ell)\right), & \text{if } \acute{\ell}\in\acute{\Lambda}\setminus\acute{\Delta} \\ \gamma\left(\bigcup_{\ell\in\delta^{-1}(\acute{\ell})\cap(\Delta\setminus\Lambda)} f(\ell)\right), & \text{if } \acute{\ell}\in\acute{\Delta}\setminus\acute{\Lambda} \\ \gamma\left(\bigcup_{\ell\in\delta^{-1}(\acute{\ell})\cap\Lambda} g(\ell)\right)\cup\gamma\left(\bigcup_{\ell\in\delta^{-1}(\acute{\ell})\cap\Delta} f(\ell)\right), & \text{if } \acute{\ell}\in\acute{\Lambda}\cap\acute{\Delta} \end{cases}$$

$$= \begin{cases} \Psi_{\gamma\delta\lambda}(g)(\acute{\ell}), & \text{if } \acute{\ell}\in\acute{\Lambda}\setminus\acute{\Delta} \\ \Psi_{\gamma\delta\lambda}(f)(\acute{\ell}), & \text{if } \acute{\ell}\in\acute{\Delta}\setminus\acute{\Lambda} \\ \Psi_{\gamma\delta\lambda}(g)(\acute{\ell})\cup\Psi_{\gamma\delta\lambda}(f)(\acute{\ell}), & \text{if } \acute{\ell}\in\acute{\Lambda}\cap\acute{\Delta} \end{cases}$$

Since $\Psi_{\gamma\delta\lambda}(f)(\acute{\ell}) = \phi$ for $\acute{\ell}\in\acute{\Lambda}\setminus\acute{\Delta}$ and $\Psi_{\gamma\delta\lambda}(g)(\acute{\ell}) = \phi$ for $\acute{\ell}\in\acute{\Delta}\setminus\acute{\Lambda}$, then for all $\acute{\ell}\in\acute{\Sigma}$, we have

$$\mathcal{J}(\acute{\ell})) = \Psi_{\gamma\delta\lambda}(g)(\acute{\ell})\cup\Psi_{\gamma\delta\lambda}(f)(\acute{\ell})$$

$$= h(\acute{\ell})), \text{ by Definition 3.5 (1.).}$$

Also, for non-trivial case, let $\neg\acute{\ell}\in\neg(\acute{\Lambda}\cup\acute{\Delta}) = \neg\acute{\Lambda}\cup\neg\acute{\Delta}$, then

$$\widehat{\mathcal{J}}(\neg\acute{\ell}) = \Psi_{\gamma\delta\lambda}(\widehat{I})(\neg\acute{\ell}) = \gamma\left(\bigcap\nolimits_{\neg\ell\in\lambda^{-1}(\neg\acute{\ell})\cap(\neg\Lambda\cup\neg\Delta)} \widehat{I}(\neg\ell)\right)$$

$$= \begin{cases} \gamma\left(\bigcap_{\neg\ell\in\lambda^{-1}(\neg\acute{\ell})\cap(\neg\Lambda\setminus\neg\Delta)} \widehat{g}(\neg\ell)\right), & \text{if } \neg\acute{\ell}\in\neg\acute{\Lambda}\setminus\neg\acute{\Delta} \\ \gamma\left(\bigcap_{\neg\ell\in\lambda^{-1}(\neg\acute{\ell})\cap(\neg\Delta\setminus\neg\Lambda)} \widehat{f}(\neg\ell)\right), & \text{if } \neg\acute{\ell}\in\neg\acute{\Delta}\setminus\neg\acute{\Lambda} \\ \gamma\left(\bigcap_{\neg\ell\in\lambda^{-1}(\neg\acute{\ell})\cap\neg\Lambda} \widehat{g}(\neg\ell)\right)\cap\gamma\left(\bigcap_{\neg\ell\in\lambda^{-1}(\neg\acute{\ell})\cap\neg\Delta} \widehat{f}(\neg\ell)\right), & \text{if } \neg\acute{\ell}\in\neg\acute{\Lambda}\cap\neg\acute{\Delta} \end{cases}$$

$$= \begin{cases} \Psi_{\gamma\delta\lambda}(\widehat{g})(\neg\acute{\ell}), & \text{if } \neg\acute{\ell}\in\neg\acute{\Lambda}\setminus\neg\acute{\Delta} \\ \Psi_{\gamma\delta\lambda}(\widehat{f})(\neg\acute{\ell}), & \text{if } \neg\acute{\ell}\in\neg\acute{\Delta}\setminus\neg\acute{\Lambda} \\ \Psi_{\gamma\delta\lambda}(\widehat{g})(\neg\acute{\ell})\cap\Psi_{\gamma\delta\lambda}(\widehat{f})(\neg\acute{\ell}), & \text{if } \neg\acute{\ell}\in\neg\acute{\Lambda}\cap\neg\acute{\Delta} \end{cases}$$

Since $\Psi_{\gamma\delta\lambda}(\widehat{f})(\neg\acute{\ell}) = \aleph$ for $\neg\acute{\ell}\in\neg\acute{\Lambda}\setminus\neg\acute{\Delta}$ and $\Psi_{\gamma\delta\lambda}(\widehat{g})(\neg\acute{\ell}) = \aleph$ for $\neg\acute{\ell}\in\neg\acute{\Delta}\setminus\neg\acute{\Lambda}$, then for all $\neg\acute{\ell}\in\neg\acute{\Sigma}$, we have

$$\widehat{\mathcal{J}}(\neg\acute{\ell})) = \Psi_{\gamma\delta\lambda}(\widehat{g})(\neg\acute{\ell})\cap\Psi_{\gamma\delta\lambda}(\widehat{f})(\neg\acute{\ell})$$

$$= \widehat{h}(\neg\acute{\ell})), \text{ by Definition 3.5 (2.).}$$

Hence, $\Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Lambda)\widetilde{\widetilde{\sqcup}}(f, \widehat{f}, \Delta)) = \Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Lambda)) \ \widetilde{\widetilde{\sqcup}} \ \Psi_{\gamma\delta\lambda}((f, \widehat{f}, \Delta))$.

5. Simply, let

$$\Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Lambda)\widetilde{\widetilde{\sqcap}}(f, \widehat{f}, \Delta)) = \Psi_{\gamma\delta\lambda}((h, \widehat{h}, \Lambda \cap \Delta))$$

$$\Psi_{\gamma\delta\lambda}((g, \widehat{g}, \Lambda))\widetilde{\widetilde{\sqcap}}\Psi_{\gamma\delta\lambda}((f, \widehat{f}, \Delta)) = (I, \widehat{I}, \acute{\Sigma}).$$

We want to show that, for all $\acute{\ell} \in \acute{\Sigma}$, $\Psi_{\gamma\delta\lambda}(h)(\acute{\ell}) \subseteq (I)(\acute{\ell})$ and, for all $\neg\acute{\ell} \in \neg\acute{\Sigma}$, $(\widehat{I})(\neg\acute{\ell}) \subseteq \Psi_{\gamma\delta\lambda}(\widehat{h})(\neg\acute{\ell})$. For a non-trivial case, let $\acute{\ell} \in \delta(\Lambda \cap \Delta) \subseteq \acute{\Sigma}$, then

$$\begin{aligned}
\Psi_{\gamma\delta\lambda}(h)(\acute{\ell}) &= \gamma\left(\bigcup_{\ell\in\delta^{-1}(\acute{\ell})\cap(\Lambda\cap\Delta)} h(\ell)\right) \\
&= \gamma\left(\bigcup_{\ell\in\delta^{-1}(\acute{\ell})\cap(\Lambda\cap\Delta)} g(\ell)\cap f(\ell)\right) \\
&= \gamma\left(\bigcup_{\ell\in\delta^{-1}(\acute{\ell})\cap(\Lambda\cap\Delta)} g(\ell)\right) \cap \gamma\left(\bigcup_{\ell\in\delta^{-1}(\acute{\ell})\cap(\Lambda\cap\Delta)} f(\ell)\right) \\
&\subseteq \gamma\left(\bigcup_{\ell\in\delta^{-1}(\acute{\ell})\cap\Lambda} g(\ell)\right) \cap \gamma\left(\bigcup_{\ell\in\delta^{-1}(\acute{\ell})\cap\Delta} f(\ell)\right) \\
&= \Psi_{\gamma\delta\lambda}(g)(\acute{\ell}) \cap \Psi_{\gamma\delta\lambda}(f)(\acute{\ell}) \\
&= I(\acute{\ell}).
\end{aligned}$$

Now, for a non-trivial case, let $\neg\acute{\ell} \in \lambda(\neg\Lambda \cap \neg\Delta) \subseteq \neg\acute{\Sigma}$, then

$$\begin{aligned}
\Psi_{\gamma\delta\lambda}(\widehat{h})(\neg\acute{\ell}) &= \gamma\left(\bigcap_{\neg\ell\in\lambda^{-1}(\neg\acute{\ell})\cap(\neg\Lambda\cap\neg\Delta)} \widehat{h}(\neg\ell)\right) \\
&= \gamma\left(\bigcap_{\neg\ell\in\lambda^{-1}(\neg\acute{\ell})\cap(\neg\Lambda\cap\neg\Delta)} \widehat{g}(\neg\ell)\cap\widehat{f}(\neg\ell)\right) \\
&\supseteq \gamma\left(\bigcap_{\neg\ell\in\lambda^{-1}(\neg\acute{\ell})\cap(\neg\Lambda\cap\neg\Delta)} \widehat{g}(\ell)\right) \cup \gamma\left(\bigcap_{\neg\ell\in\lambda^{-1}(\neg\acute{\ell})\cap(\neg\Lambda\cap\neg\Delta)} \widehat{f}(\neg\ell)\right) \\
&\supseteq \gamma\left(\bigcap_{\neg\ell\in\lambda^{-1}(\neg\acute{\ell})\cap\neg\Lambda} \widehat{g}(\neg\ell)\right) \cup \gamma\left(\bigcap_{\neg\ell\in\lambda^{-1}(\neg\acute{\ell})\cap\neg\Delta} \widehat{f}(\neg\ell)\right) \\
&= \Psi_{\gamma\delta\lambda}(\widehat{g})(\neg\acute{\ell}) \cup \Psi_{\gamma\delta\lambda}(\widehat{f})(\neg\acute{\ell}) \\
&= \widehat{I}(\neg\acute{\ell}).
\end{aligned}$$

Therefore, $\Psi_{\gamma\delta\lambda}((g,\widehat{g},\Lambda)\tilde{\tilde{\sqcap}}(f,\widehat{f},\Delta)) = \Psi_{\gamma\delta\lambda}((h,\widehat{h},\Lambda\cap\Delta)) \tilde{\tilde{\sqsubseteq}} \Psi_{\gamma\delta\lambda}((g,\widehat{g},\Lambda)) \tilde{\tilde{\sqcap}} \Psi_{\gamma\delta\lambda}((f,\widehat{f},\Delta))$.

□

Remark 3.8. The reverse of Proposition 3.7 (5) is incorrect.

Example 3.9. Let $\Psi_{\gamma\delta\lambda}$ and $(g,\widehat{g},\Lambda)$ be the same as in Example 3.2. Let $(f,\widehat{f},\Delta) = \{((\ell_1,\ell_5,\ell_6),\{r_2\},\{r_1,r_3\}),((\ell_2,\ell_5,\ell_6),\{r_1\},\{r_2\}),((\ell_3,\ell_5,\ell_6),\Re,\phi)),((\ell_4,\ell_5,\ell_6),\{r_3\},\{r_1\}))\}$, then $\Psi_{\gamma\delta\lambda}((f,\widehat{f},\Delta)) = \{((\acute{\ell_1},\acute{\ell_5},\acute{\ell_6}),\{\eta_1,\eta_2\},\phi),\ ((\acute{\ell_2},\acute{\ell_5},\acute{\ell_6}),\phi,\aleph),\ ((\acute{\ell_3},\acute{\ell_5},\acute{\ell_6}),\ \{\eta_1,\eta_2,\eta_3\},\phi),\ ((\acute{\ell_4},\acute{\ell_5},\acute{\ell_6}),\{\eta_3\},\{\eta_1\})\}$. Now,

$\Psi_{\gamma\delta\lambda}((g,\widehat{g},\Lambda))\tilde{\tilde{\sqcap}}\Psi_{\gamma\delta\lambda}((f,\widehat{f},\Delta)) = \{((\acute{\ell_1},\acute{\ell_5},\acute{\ell_6}),\{\eta_1\},\{\eta_2\}),\ ((\acute{\ell_2},\acute{\ell_5},\acute{\ell_6}),\phi,\aleph),\ ((\acute{\ell_3},\acute{\ell_5},\acute{\ell_6}),\{\eta_3\},\{\eta_1\}),\ ((\acute{\ell_4},\acute{\ell_5},\acute{\ell_6}),\phi,\aleph)\}$.

On the other hand, $(g,\widehat{g},\Lambda)\tilde{\tilde{\sqcap}}(f,\widehat{f},\Delta) = \{((\ell_1,\ell_5,\ell_6),\phi,\{r_1,r_2,r_3\}),((\ell_2,\ell_5,\ell_6),\phi,\{r_1,r_2\}),((\ell_3,\ell_5,\ell_6),\{r_3\},\{r_1\})\}$, then

$\Psi_{\gamma\delta\lambda}((g,\widehat{g},\Lambda)\tilde{\tilde{\sqcap}}(f,\widehat{f},\Delta)) = \{((\acute{\ell_1},\acute{\ell_5},\acute{\ell_6}),\phi,\{\eta_1,\eta_2\}),\ ((\acute{\ell_2},\acute{\ell_5},\acute{\ell_6}),\phi,\aleph),\ ((\acute{\ell_3},\acute{\ell_5},\acute{\ell_6}),\{\eta_3\},\{\eta_1\}),\ ((\acute{\ell_4},\acute{\ell_5},\acute{\ell_6}),\phi,\aleph)\}$.

Therefore, $\Psi_{\gamma\delta\lambda}((g,\widehat{g},\Lambda)) \tilde{\tilde{\sqcap}} \Psi_{\gamma\delta\lambda}((f,\widehat{f},\Delta)) \tilde{\tilde{\not\sqsubseteq}} \Psi_{\gamma\delta\lambda}((g,\widehat{g},\Lambda)\tilde{\tilde{\sqcap}}(f,\widehat{f},\Delta))$.

Remark 3.10. In Proposition 3.7 (5.), $\Psi_{\gamma\delta\lambda}\left((g,\widehat{g},\Lambda)\tilde{\tilde{\sqcap}}(f,\widehat{f},\Delta) = (h,\widehat{h},\Lambda\cup\Delta)\right) \tilde{\tilde{\not\sqsubseteq}} \Psi_{\gamma\delta\lambda}((g,\widehat{g},\Lambda)) \tilde{\tilde{\sqcap}} \Psi_{\gamma\delta\lambda}((f,\widehat{f},\Delta))$.

Example 3.11. Let $\Psi_{\gamma\delta\lambda}$ and $(g,\widehat{g},\Lambda)$ be the same as in Example 3.2. Let $(f,\widehat{f},\Delta) = \{((\ell_1,\ell_5,\ell_6),\{r_1,r_2\},\phi),((\ell_3,\ell_5,\ell_6),\{r_3\},\{r_1\}),((\ell_4,\ell_5,\ell_6),\Re,\phi)\}$, then $\Psi_{\gamma\delta\lambda}((f,\widehat{f},\Delta)) = \{((\acute{\ell_1},\acute{\ell_5},\acute{\ell_6}),\{\eta_1,\eta_2\},\phi),\ ((\acute{\ell_2},\acute{\ell_5},\acute{\ell_6}),\phi,\aleph),\ ((\acute{\ell_3},\acute{\ell_5},\acute{\ell_6}),\ \{\eta_3\},\{\eta_1\}),\ ((\acute{\ell_4},\acute{\ell_5},\acute{\ell_6}),\{\eta_1,\eta_2,\eta_3\},\phi)\}$. Now,

$\Psi_{\gamma\delta\lambda}((g,\widehat{g},\Lambda))\tilde{\tilde{\sqcap}}\Psi_{\gamma\delta\lambda}((f,\widehat{f},\Delta)) = \{((\acute{\ell_1},\acute{\ell_5},\acute{\ell_6}),\{\eta_1\},\{\eta_2\}),\ ((\acute{\ell_2},\acute{\ell_5},\acute{\ell_6}),\phi,\aleph),\ ((\acute{\ell_3},\acute{\ell_5},\acute{\ell_6}),\{\eta_3\},\{\eta_1\}),\ ((\acute{\ell_4},\acute{\ell_5},\acute{\ell_6}),\phi,\aleph)\}$.

On the other hand, $(g,\widehat{g},\Lambda)\tilde{\tilde{\sqcap}}(f,\widehat{f},\Delta) = \{((\ell_1,\ell_5,\ell_6),\{r_1\},\{r_2\}),((\ell_2,\ell_5,\ell_6),\{r_3\},\{r_1,r_2\}),((\ell_3,\ell_5,\ell_6),\{r_3\},\{r_1\}),((\ell_4,\ell_5,\ell_6),\Re,\phi\})\}$, then

$\Psi_{\gamma\delta\lambda}((g,\widehat{g},\Lambda)\tilde{\tilde{\sqcap}}(f,\widehat{f},\Delta)) = \{((\acute{\ell_1},\acute{\ell_5},\acute{\ell_6}),\{\eta_1,\eta_3\},\{\eta_2\}),\ ((\acute{\ell_2},\acute{\ell_5},\acute{\ell_6}),\phi,\aleph),\ ((\acute{\ell_3},\acute{\ell_5},\acute{\ell_6}),\{\eta_3\},\{\eta_1\}),\ ((\acute{\ell_4},\acute{\ell_5},\acute{\ell_6}),\{\eta_1,\eta_2,\eta_3\},\phi)\}$.

Therefore, $\Psi_{\gamma\delta\lambda}((g,\widehat{g},\Lambda)\widetilde{\widetilde{\sqcap}}(f,\widehat{f},\Delta)) \widetilde{\widetilde{\nsupseteq}} \Psi_{\gamma\delta\lambda}((g,\widehat{g},\Lambda)) \widetilde{\widetilde{\sqcap}} \Psi_{\gamma\delta\lambda}((f,\widehat{f},\Delta))$.

4. **Bipolar Hypersoft Inverse Image**

This section focuses on bipolar hypersoft inverse image and its relation to the bipolar hypersoft image on bipolar hypersoft sets.

Definition 4.1. Let $\gamma : \Re \to \aleph$ be an injective mapping. Let $\delta : \Sigma \to \acute{\Sigma}$ and $\lambda : \neg\Sigma \to \neg\acute{\Sigma}$ be two mappings such that $\lambda(\neg\ell) = \neg\delta(\ell)$ for all $\neg\ell \in \neg\Sigma$, and $\Psi_{\gamma\delta\lambda} : \Omega_{(\Re,\Sigma)} \to \Omega_{(\aleph,\acute{\Sigma})}$ be a bipolar hypersoft mapping. The inverse image of a bipolar hypersoft set $(g,\widehat{g},\acute{\Lambda})$ under $\Psi_{\gamma\delta\lambda}$, $\Psi^{-1}_{\gamma\delta\lambda}((g,\widehat{g},\Lambda)) = (\Psi^{-1}_{\gamma\delta\lambda}(g),\Psi^{-1}_{\gamma\delta\lambda}(\widehat{g}),\Sigma)$ is a bipolar hypersoft set in $\Omega_{(\Re,\Sigma)}$ given as, for all $\ell \in \Sigma$:

$$\Psi^{-1}_{\gamma\delta\lambda}(g)(\ell) = \begin{cases} \gamma^{-1}\left(g(\delta(\ell))\right), & \text{if } \delta(\ell) \in \acute{\Lambda} \\ \phi, & \text{if } \delta(\ell) \notin \acute{\Lambda} \end{cases}$$

$$\Psi^{-1}_{\gamma\delta\lambda}(\widehat{g})(\neg\ell) = \begin{cases} \gamma^{-1}(\widehat{g}(\lambda(\neg\ell))), & \text{if } \lambda(\neg\ell) \in \neg\acute{\Lambda} \\ \Re, & \text{if } \lambda(\neg\ell) \notin \neg\acute{\Lambda} \end{cases}$$

Example 4.2. Let $\Psi_{\gamma\delta\lambda}$ be the same as in Example 3.2. Let $(g,\widehat{g},\acute{\Lambda}) = \{((\acute{\ell_3},\acute{\ell_5},\acute{\ell_6}),\aleph,\phi),((\acute{\ell_4},\acute{\ell_5},\acute{\ell_6}),\{\eta_1,\eta_3\},\{\eta_2\})\}$.

Since $\delta^{-1}(\acute{\Lambda}) = \delta^{-1}(\{(\acute{\ell_3},\acute{\ell_5},\acute{\ell_6}),(\acute{\ell_4},\acute{\ell_5},\acute{\ell_6})\}) = \{(\ell_3,\ell_5,\ell_6),(\ell_4,\ell_5,\ell_6)\}$ and $\delta((\ell_3,\ell_5,\ell_6)) = (\acute{\ell_3},\acute{\ell_5},\acute{\ell_6}) \in \acute{\Lambda}$, then

$$\Psi^{-1}_{\gamma\delta\lambda}(g)(\ell_3,\ell_5,\ell_6) = \gamma^{-1}(g(\delta((\ell_3,\ell_5,\ell_6)))) = \gamma^{-1}(g((\acute{\ell_3},\acute{\ell_5},\acute{\ell_6}))) = \gamma^{-1}(\aleph) = \Re.$$

Also, $\lambda^{-1}(\neg\acute{\Lambda}) = \lambda^{-1}(\{(\neg\acute{\ell_3},\neg\acute{\ell_5},\neg\acute{\ell_6}),(\neg\acute{\ell_4},\neg\acute{\ell_5},\neg\acute{\ell_6})\}) = \{(\neg\ell_3,\neg\ell_5,\neg\ell_6),(\neg\ell_4,\neg\ell_5,\neg\ell_6)\}$ and $\lambda((\neg\ell_3,\neg\ell_5,\neg\ell_6)) = (\neg\acute{\ell_3},\neg\acute{\ell_5},\neg\acute{\ell_6}) \in \neg\acute{\Lambda}$, then

$$\Psi^{-1}_{\gamma\delta\lambda}(g)(\ell_3,\ell_5,\ell_6) = \gamma^{-1}(g(\delta((\ell_3,\ell_5,\ell_6)))) = \gamma^{-1}(\widehat{g}((\neg\acute{\ell_3},\neg\acute{\ell_5},\neg\acute{\ell_6}))) = \gamma^{-1}(\phi) = \phi.$$

Then, $\Psi^{-1}_{\gamma\delta\lambda}((g,\widehat{g},\acute{\Lambda}))((\ell_3,\ell_5,\ell_6)) = ((\ell_3,\ell_5,\ell_6),\Re,\phi)$.

Now, for (ℓ_4,ℓ_5,ℓ_6): $\delta((\ell_4,\ell_5,\ell_6)) = (\acute{\ell_4},\acute{\ell_5},\acute{\ell_6}) \in \acute{\Lambda}$, then

$$\Psi^{-1}_{\gamma\delta\lambda}(g)(\ell_4,\ell_5,\ell_6) = \gamma^{-1}(g(\delta((\ell_4,\ell_5,\ell_6)))) = \gamma^{-1}(g((\acute{\ell_4},\acute{\ell_5},\acute{\ell_6}))) = \gamma^{-1}(\{\eta_1,\eta_3\}) = \{r_1,r_3\}.$$

Also, $\lambda((\neg\ell_4, \neg\ell_5, \neg\ell_6)) = (\neg\acute{\ell_4}, \neg\acute{\ell_5}, \neg\acute{\ell_6}) \in \neg\acute{\Lambda}$, then

$\Psi^{-1}_{\gamma\delta\lambda}(g)(\ell_4, \ell_5, \ell_6) = \gamma^{-1}(g(\delta((\ell_4, \ell_5, \ell_6)))) = \gamma^{-1}(\widehat{g}((\neg\acute{\ell_4}, \neg\acute{\ell_5}, \neg\acute{\ell_6}))) = \gamma^{-1}(\{\eta_2\}) = \{r_2\}$.
Then, $\Psi^{-1}_{\gamma\delta\lambda}((g, \widehat{g}, \acute{\Lambda}))((\ell_4, \ell_5, \ell_6)) = ((\ell_4, \ell_5, \ell_6), \{r_1, r_3\}, \{r_2\})$.

Hence, $\Psi^{-1}_{\gamma\delta\lambda}((g, \widehat{g}, \acute{\Lambda})) = \{((\ell_1, \ell_5, \ell_6), \phi, \Re), ((\ell_2, \ell_5, \ell_6), \phi, \Re), ((\ell_3, \ell_5, \ell_6), \Re, \phi), ((\ell_4, \ell_5, \ell_6), \{r_1, r_3\}, \{r_2\})\}$.

Definition 4.3. Suppose that $\Psi_{\gamma\delta\lambda} : \Omega_{(\Re,\Sigma)} \to \Omega_{(\aleph,\acute{\Sigma})}$ is a bipolar hypersoft mapping and $(g, \widehat{g}, \acute{\Lambda}), (f, \widehat{f}, \acute{\Delta}) \in \Omega_{(\aleph,\acute{\Sigma})}$. Then:

(1) The union of bipolar hypersoft inverse image of $(g, \widehat{g}, \acute{\Lambda}), (f, \widehat{f}, \acute{\Delta}) \in \Omega_{(\aleph,\acute{\Sigma})}$ is defined as, for all $\ell \in \Sigma$,

$$\left(\Psi^{-1}_{\gamma\delta\lambda}((g, \widehat{g}, \acute{\Lambda}))\tilde{\tilde{\sqcup}}\Psi^{-1}_{\gamma\delta\lambda}((f, \widehat{f}, \acute{\Delta}))\right)(\ell) =$$

$$\left(\ell, \Psi^{-1}_{\gamma\delta\lambda}(g)(\ell) \cup \Psi^{-1}_{\gamma\delta\lambda}(f)(\ell), \Psi^{-1}_{\gamma\delta\lambda}(\widehat{g})(\neg\ell) \cap \Psi^{-1}_{\gamma\delta\lambda}(\widehat{f})(\neg\ell)\right).$$

(2) The intersection of bipolar hypersoft inverse image of $(g, \widehat{g}, \acute{\Lambda}), (f, \widehat{f}, \acute{\Delta}) \in \Omega_{(\aleph,\acute{\Sigma})}$ is defined as, for all $\ell \in \Sigma$,

$$\left(\Psi^{-1}_{\gamma\delta\lambda}((g, \widehat{g}, \acute{\Lambda}))\tilde{\tilde{\sqcap}}\Psi^{-1}_{\gamma\delta\lambda}((f, \widehat{f}, \acute{\Delta}))\right)(\ell) =$$

$$\left(\ell, \Psi^{-1}_{\gamma\delta\lambda}(g)(\ell) \cap \Psi^{-1}_{\gamma\delta\lambda}(f)(\ell), \Psi^{-1}_{\gamma\delta\lambda}(\widehat{g})(\neg\ell) \cup \Psi^{-1}_{\gamma\delta\lambda}(\widehat{f})(\neg\ell)\right).$$

Proposition 4.4. *Suppose that $\Psi_{\gamma\delta\lambda} : \Omega_{(\Re,\Sigma)} \to \Omega_{(\aleph,\acute{\Sigma})}$ is a bipolar hypersoft mapping, where $\gamma : \Re \to \aleph$ is an injective mapping, $\delta : \Sigma \to \acute{\Sigma}$ and $\lambda : \neg\Sigma \to \neg\acute{\Sigma}$ are two mappings such that $\lambda(\neg\ell) = \neg\delta(\ell)$ for all $\neg\ell \in \neg\Sigma$. If $(g, \widehat{g}, \acute{\Lambda}), (f, \widehat{f}, \acute{\Delta}) \in \Omega_{(\aleph,\acute{\Sigma})}$ then:*

(1) $\Psi^{-1}_{\gamma\delta\lambda}((\Phi, \widehat{\aleph}, \acute{\Sigma})) = (\Phi, \widehat{\Re}, \Sigma)$.
(2) $\Psi^{-1}_{\gamma\delta\lambda}((\widehat{\aleph}, \Phi, \acute{\Sigma})) = (\widehat{\Re}, \Phi, \Sigma)$.
(3) If $(g, \widehat{g}, \acute{\Lambda}) \tilde{\tilde{\sqsubseteq}} (f, \widehat{f}, \acute{\Delta})$, *then* $\Psi^{-1}_{\gamma\delta\lambda}((g, \widehat{g}, \acute{\Lambda})) \tilde{\tilde{\sqsubseteq}} \Psi^{-1}_{\gamma\delta\lambda}((f, \widehat{f}, \acute{\Delta}))$.
(4) $\Psi^{-1}_{\gamma\delta\lambda}((g, \widehat{g}, \acute{\Lambda})\tilde{\tilde{\sqcup}}(f, \widehat{f}, \acute{\Delta})) = \Psi^{-1}_{\gamma\delta\lambda}((g, \widehat{g}, \acute{\Lambda})) \tilde{\tilde{\sqcup}} \Psi^{-1}_{\gamma\delta\lambda}((f, \widehat{f}, \acute{\Delta}))$.
(5) $\Psi^{-1}_{\gamma\delta\lambda}\left((g, \widehat{g}, \acute{\Lambda})\tilde{\tilde{\sqcap}}(f, \widehat{f}, \acute{\Delta}) = (h, \widehat{h}, \acute{\Lambda} \cap \acute{\Delta})\right) = \Psi^{-1}_{\gamma\delta\lambda}((g, \widehat{g}, \acute{\Lambda})) \tilde{\tilde{\sqcap}} \Psi^{-1}_{\gamma\delta\lambda}((f, \widehat{f}, \acute{\Delta}))$.
(6) $\Psi^{-1}_{\gamma\delta\lambda}((g, \widehat{g}, \acute{\Sigma})^c) = (\Psi^{-1}_{\gamma\delta\lambda}((g, \widehat{g}, \acute{\Sigma})))^c$.

Proof. 1. and 2. are straightforward.
3. Let $(g, \widehat{g}, \acute{\Lambda}) \tilde{\tilde{\sqsubseteq}} (f, \widehat{f}, \acute{\Delta})$, then we want to show that, for all $\ell \in \Sigma$, $\Psi^{-1}_{\gamma\delta\lambda}(g)(\ell) \subseteq \Psi^{-1}_{\gamma\delta\lambda}(f)(\ell)$ and, for all $\neg\ell \in \neg\Sigma$, $\Psi^{-1}_{\gamma\delta\lambda}(\widehat{f})(\neg\ell) \subseteq \Psi^{-1}_{\gamma\delta\lambda}(\widehat{g})(\neg\ell)$. Let $\ell \in \Sigma$ where $\delta(\ell) \in \acute{\Lambda} \subseteq \acute{\Delta}$ (if $\delta(\ell) \notin \acute{\Lambda}$,

then $\Psi^{-1}_{\gamma\delta\lambda}(g)(\ell) = \phi \subseteq \Psi^{-1}_{\gamma\delta\lambda}(f)(\ell))$, then

$$\begin{aligned}\Psi^{-1}_{\gamma\delta\lambda}(g)(\ell) &= \gamma^{-1}\left(g(\delta(\ell))\right)\\ &\subseteq \gamma^{-1}(f(\delta(\ell))), since\, g(\acute{\ell}) \subseteq f(\acute{\ell})\, for\, all\, \acute{\ell} \in \acute{\Lambda}\\ &= \Psi^{-1}_{\gamma\delta\lambda}(f)(\ell).\end{aligned}$$

Now, for $\neg\ell \in \neg\Sigma$ where $\lambda(\neg\ell) \in \neg\acute{\Lambda} \subseteq \neg\acute{\Delta}$ (if $\lambda(\neg\ell) \notin \neg\acute{\Lambda}$), then $\Psi^{-1}_{\gamma\delta\lambda}(\widehat{f})(\neg\ell) \subseteq \Psi^{-1}_{\gamma\delta\lambda}(\widehat{g})(\neg\ell) = \Re$, we have

$$\begin{aligned}\Psi^{-1}_{\gamma\delta\lambda}(\widehat{f}(\neg\ell)) &= \gamma^{-1}(\widehat{f}(\lambda(\neg\ell)))\\ &\subseteq \gamma^{-1}(\widehat{g}(\lambda(\neg\ell))), since\, \widehat{f}(\neg\acute{\ell}) \subseteq \widehat{g}(\acute{\ell})\, for\, all\, \neg\acute{\ell} \in \neg\acute{\Lambda}\\ &= \Psi^{-1}_{\gamma\delta\lambda}(\widehat{g})(\neg\ell).\end{aligned}$$

Hence, $\Psi^{-1}_{\gamma\delta\lambda}((g,\widehat{g},\acute{\Lambda})) \mathrel{\tilde{\tilde{\sqsubseteq}}} \Psi^{-1}_{\gamma\delta\lambda}((f,\widehat{f},\acute{\Delta}))$.

4. To keep things simple, let

$$\Psi^{-1}_{\gamma\delta\lambda}((g,\widehat{g},\acute{\Lambda}) \mathrel{\tilde{\tilde{\sqcup}}} (f,\widehat{f},\acute{\Delta})) = \Psi^{-1}_{\gamma\delta\lambda}((I,\widehat{I},\acute{\Lambda}\cup\acute{\Delta})) = (\mathcal{J},\widehat{\mathcal{J}},\Sigma)$$
$$\Psi^{-1}_{\gamma\delta\lambda}((g,\widehat{g},\acute{\Lambda})) \mathrel{\tilde{\tilde{\sqcup}}} \Psi^{-1}_{\gamma\delta\lambda}((f,\widehat{f},\acute{\Delta})) = (\hbar,\widehat{\hbar},\Sigma).$$

We want to prove that, for all $\ell \in \Sigma$, $\mathcal{J}(\ell) = \hbar(\ell)$ and, for all $\neg\ell \in \neg\Sigma$, $\widehat{\mathcal{J}}(\neg\ell) = \widehat{\hbar}(\neg\ell)$. For a non-trivial case, let $\ell \in \Sigma$ where $\delta(\ell) \in \acute{\Lambda}\cup\acute{\Delta}$, then

$$\mathcal{J}(\ell) = \Psi^{-1}_{\gamma\delta\lambda}(I)(\ell) = \gamma^{-1}(I(\delta(\ell)))$$

$$= \begin{cases} \gamma^{-1}(g(\delta(\ell))), & \text{if } \delta(\ell) \in \acute{\Lambda}\setminus\acute{\Delta}\\ \gamma^{-1}(f(\delta(\ell))), & \text{if } \delta(\ell) \in \acute{\Delta}\setminus\acute{\Lambda}\\ \gamma^{-1}(g(\delta(\ell))\cup f(\delta(\ell))), & \text{if } \delta(\ell) \in \acute{\Lambda}\cap\acute{\Delta}\end{cases}$$

$$= \begin{cases} \gamma^{-1}(g(\delta(\ell))), & \text{if } \delta(\ell) \in \acute{\Lambda}\setminus\acute{\Delta}\\ \gamma^{-1}(f(\delta(\ell))), & \text{if } \delta(\ell) \in \acute{\Delta}\setminus\acute{\Lambda}\\ \gamma^{-1}(g(\delta(\ell)))\cup\gamma^{-1}(f(\delta(\ell))), & \text{if } \delta(\ell) \in \acute{\Lambda}\cap\acute{\Delta}\end{cases}$$

$$= \begin{cases} \Psi^{-1}_{\gamma\delta\lambda}(g(\ell)), & \text{if } \delta(\ell) \in \acute{\Lambda}\setminus\acute{\Delta}\\ \Psi^{-1}_{\gamma\delta\lambda}(f(\ell)), & \text{if } \delta(\ell) \in \acute{\Delta}\setminus\acute{\Lambda}\\ \Psi^{-1}(g(\ell))\cup\Psi^{-1}(f(\ell)), & \text{if } \delta(\ell) \in \acute{\Lambda}\cap\acute{\Delta}\end{cases}$$

Since $\Psi^{-1}_{\gamma\delta\lambda}(f)(\ell) = \phi$ for $\delta(\ell) \in \acute{\Lambda}\setminus\acute{\Delta}$ and $\Psi^{-1}_{\gamma\delta\lambda}(g)(\acute{\ell}) = \phi$ for $\delta(\ell) \in \acute{\Delta}\setminus\acute{\Lambda}$, then for all $\ell \in \Sigma$, we have

$$\begin{aligned}\mathcal{J}(\ell) &= \Psi^{-1}_{\gamma\delta\lambda}(g)(\ell)\cup\Psi^{-1}_{\gamma\delta\lambda}(f)(\ell)\\ &= \hbar(\ell), \text{ by Definition 4.3 (1.).}\end{aligned}$$

Also, for a non-trivial case, let $\neg\ell \in \neg\Sigma$ where $\lambda(\neg\ell) \in \neg\acute{\Lambda} \cup \neg\acute{\Delta}$, then

$$\mathcal{J}(\neg\ell) = \Psi^{-1}_{\gamma\delta\lambda}(\widehat{I})(\neg\ell) = \gamma^{-1}(\widehat{I}(\lambda(\neg\ell)))$$

$$= \begin{cases} \gamma^{-1}(\widehat{g}(\lambda(\neg\ell))), & \text{if } \lambda(\neg\ell) \in \neg\acute{\Lambda} \setminus \neg\acute{\Delta} \\ \gamma^{-1}(\widehat{f}(\lambda(\neg\ell))), & \text{if } \lambda(\neg\ell) \in \neg\acute{\Delta} \setminus \neg\acute{\Lambda} \\ \gamma^{-1}(\widehat{g}(\lambda(\neg\ell)) \cap \widehat{f}(\lambda(\neg\ell))), & \text{if } \lambda(\neg\ell) \in \neg\acute{\Lambda} \cap \neg\acute{\Delta} \end{cases}$$

$$= \begin{cases} \gamma^{-1}(\widehat{g}(\lambda(\neg\ell))), & \text{if } \lambda(\neg\ell) \in \neg\acute{\Lambda} \setminus \neg\acute{\Delta} \\ \gamma^{-1}(\widehat{f}(\lambda(\neg\ell))), & \text{if } \lambda(\neg\ell) \in \neg\acute{\Delta} \setminus \neg\acute{\Lambda} \\ \gamma^{-1}(\widehat{g}(\lambda(\neg\ell))) \cap \gamma^{-1}(\widehat{f}(\lambda(\neg\ell))), & \text{if } \lambda(\neg\ell) \in \neg\acute{\Lambda} \cap \neg\acute{\Delta} \end{cases}$$

$$= \begin{cases} \Psi^{-1}_{\gamma\delta\lambda}(\widehat{g}(\neg\ell)), & \text{if } \lambda(\neg\ell) \in \neg\acute{\Lambda} \setminus \neg\acute{\Delta} \\ \Psi^{-1}_{\gamma\delta\lambda}(\widehat{f}(\neg\ell)), & \text{if } \lambda(\neg\ell) \in \neg\acute{\Delta} \setminus \neg\acute{\Lambda} \\ \Psi^{-1}(\widehat{g}(\neg\ell)) \cap \Psi^{-1}(\widehat{f}(\neg\ell)), & \text{if } \lambda(\neg\ell) \in \neg\acute{\Lambda} \cap \neg\acute{\Delta} \end{cases}$$

Since $\Psi^{-1}_{\gamma\delta\lambda}(\widehat{f})(\neg\ell) = \Re$ for $\lambda(\neg\ell) \in \neg\acute{\Lambda} \setminus \neg\acute{\Delta}$ and $\Psi^{-1}_{\gamma\delta\lambda}(\widehat{g})(\neg\acute{\ell}) = \Re$ for $\lambda(\neg\ell) \in \neg\acute{\Delta} \setminus \neg\acute{\Lambda}$, then for all $\neg\ell \in \neg\Sigma$, we have

$$\widehat{\mathcal{J}}(\neg\ell) = \Psi^{-1}_{\gamma\delta\lambda}(\widehat{g})(\neg\ell) \cap \Psi^{-1}_{\gamma\delta\lambda}(\widehat{f})(\neg\ell)$$

$$= \widehat{h}(\neg\ell), \text{ by Definition 4.3 (2.).}$$

Hence, $\Psi^{-1}_{\gamma\delta\lambda}((g,\widehat{g},\acute{\Lambda})\widetilde{\widetilde{\sqcup}}(f,\widehat{f},\acute{\Delta})) = \Psi^{-1}_{\gamma\delta\lambda}((g,\widehat{g},\acute{\Lambda}))\ \widetilde{\widetilde{\sqcup}}\ \Psi^{-1}_{\gamma\delta\lambda}((f,\widehat{f},\acute{\Delta}))$.

5. Simply, let

$$\Psi^{-1}_{\gamma\delta\lambda}((g,\widehat{g},\acute{\Lambda})\widetilde{\widetilde{\sqcap}}(f,\widehat{f},\acute{\Delta})) = \Psi^{-1}_{\gamma\delta\lambda}((I,\widehat{I},\acute{\Lambda}\cap\acute{\Delta})) = (\mathcal{J},\widehat{\mathcal{J}},\Sigma)$$

$$\Psi^{-1}_{\gamma\delta\lambda}((g,\widehat{g},\acute{\Lambda}))\widetilde{\widetilde{\sqcap}}\Psi^{-1}_{\gamma\delta\lambda}((f,\widehat{f},\acute{\Delta})) = (h,\widehat{h},\Sigma).$$

We want to prove that, for all $\ell \in \Sigma$, $\mathcal{J}(\ell) = h(\ell)$ and, for all $\neg\ell \in \neg\Sigma$, $\widehat{\mathcal{J}}(\neg\ell) = \widehat{h}(\neg\ell)$. For a non-trivial case, let $\ell \in \delta^{-1}(\acute{\Lambda} \cap \acute{\Delta}) = \delta^{-1}(\acute{\Lambda}) \cap \delta^{-1}(\acute{\Delta})$, then

$$\begin{aligned} \mathcal{J}(\ell) &= \Psi^{-1}_{\gamma\delta\lambda}(I)(\ell) \\ &= \gamma^{-1}(g(\delta(\ell)) \cap f(\delta(\ell))) \\ &= \gamma^{-1}(g(\delta(\ell))) \cap \gamma^{-1}(f(\delta(\ell))) \\ &= \Psi^{-1}_{\gamma\delta\lambda}(g)(\ell) \cap \Psi^{-1}_{\gamma\delta\lambda}(f(\delta(\ell))) \\ &= h(\ell). \end{aligned}$$

Also, let $\neg\ell \in \lambda^{-1}(\neg\acute{\Lambda} \cap \neg\acute{\Delta}) = \lambda^{-1}(\neg\acute{\Lambda}) \cap \lambda^{-1}(\neg\acute{\Delta})$, then

$$\begin{aligned}\widehat{\mathcal{J}}(\neg\ell) &= \Psi^{-1}_{\gamma\delta\lambda}(\widehat{I})(\neg\ell) \\ &= \gamma^{-1}(\widehat{g}(\lambda(\neg\ell)) \cup \widehat{f}(\lambda(\neg\ell))) \\ &= \gamma^{-1}(\widehat{g}(\lambda(\neg\ell))) \cup \gamma^{-1}(\widehat{f}(\lambda(\neg\ell))) \\ &= \Psi^{-1}_{\gamma\delta\lambda}(\widehat{g})(\neg\ell) \cup \Psi^{-1}_{\gamma\delta\lambda}(\widehat{f}(\lambda(\neg\ell))) \\ &= \widehat{h}(\neg\ell).\end{aligned}$$

Hence, $\Psi^{-1}_{\gamma\delta\lambda}((g,\widehat{g},\acute{\Lambda})\tilde{\tilde{\sqcap}}(f,\widehat{f},\acute{\Delta})) = \Psi^{-1}_{\gamma\delta\lambda}((g,\widehat{g},\acute{\Lambda})) \tilde{\tilde{\sqcap}} \Psi^{-1}_{\gamma\delta\lambda}((f,\widehat{f},\acute{\Delta}))$.

6. Simply, let $\Psi^{-1}_{\gamma\delta\lambda}((g,\widehat{g},\acute{\Sigma})) = (h,\widehat{h},\Sigma)$. Let $\ell \in \Sigma$, then

$$\begin{aligned}\left(\Psi^{-1}_{\gamma\delta\lambda}(g)(\ell)\right)^c &= \left(\gamma^{-1}(g(\delta(\ell)))\right)^c \\ &= \left(h(\ell)\right)^c \\ &= \widehat{h}(\neg\ell).\end{aligned}$$

Again, $(g,\widehat{g},\acute{\Sigma})^c = (g^c,\widehat{g}^c,\acute{\Sigma})$, then

$$\begin{aligned}\Psi^{-1}_{\gamma\delta\lambda}(g^c)(\ell) &= \gamma^{-1}(g^c(\delta(\ell))) \\ &= \gamma^{-1}(\widehat{g}(\neg\delta(\ell))) \\ &= \gamma^{-1}(\widehat{g}(\lambda(\neg\ell))) \\ &= \widehat{h}(\neg\ell).\end{aligned}$$

Hence, $\left(\Psi^{-1}_{\gamma\delta\lambda}(g)(\ell)\right)^c = \Psi^{-1}_{\gamma\delta\lambda}(g)^c(\ell)$. Using the same technique, we can show that $\left(\Psi^{-1}_{\gamma\delta\lambda}(\widehat{g})(\neg\ell)\right)^c = \Psi^{-1}_{\gamma\delta\lambda}(\widehat{g})^c(\neg\ell)$ for all $\neg\ell \in \neg\Sigma$.

Therefore, $\Psi^{-1}_{\gamma\delta\lambda}((g,\widehat{g},\acute{\Sigma})^c) = (\Psi^{-1}_{\gamma\delta\lambda}((g,\widehat{g},\acute{\Sigma})))^c$. □

Remark 4.5. In Proposition 4.4 (5.), $\Psi^{-1}_{\gamma\delta\lambda}\left((g,\widehat{g},\acute{\Lambda})\tilde{\tilde{\sqcap}}(f,\widehat{f},\acute{\Delta}) = (h,\widehat{h},\acute{\Lambda}\cup\acute{\Delta})\right) \neq \Psi^{-1}_{\gamma\delta\lambda}((g,\widehat{g},\acute{\Lambda})) \tilde{\tilde{\sqcap}} \Psi^{-1}_{\gamma\delta\lambda}((f,\widehat{f},\acute{\Delta}))$.

Example 4.6. Consider $\Psi_{\gamma\delta\lambda}$ in Example 3.2 and $(g,\widehat{g},\acute{\Lambda})$ in Example 4.2. Let $(f,\widehat{f},\acute{\Delta}) = \{((\acute{\ell}_1,\acute{\ell}_5,\acute{\ell}_6),\{\eta_4\},\{\eta_3\}),((\acute{\ell}_2,\acute{\ell}_5,\acute{\ell}_6),\{\eta_1\},\{\eta_2\}),((\acute{\ell}_3,\acute{\ell}_5,\acute{\ell}_6),\{\eta_1\},\{\eta_3,\eta_4\}),((\acute{\ell}_4,\acute{\ell}_5,\acute{\ell}_6),\phi,\aleph)\}$, then $\Psi^{-1}_{\gamma\delta\lambda}((f,\widehat{f},\acute{\Delta})) = \{((\ell_1,\ell_5,\ell_6),\phi,\{r_3\}), ((\ell_2,\ell_5,\ell_6),\phi,\{r_3\}), ((\ell_3,\ell_5,\ell_6), \{r_1\},\{r_3\}), ((\ell_4,\ell_5,\ell_6),\phi,\Re)\}$. Now,

$\Psi^{-1}_{\gamma\delta\lambda}((g,\widehat{g},\acute{\Lambda}))\tilde{\tilde{\sqcap}}\Psi^{-1}_{\gamma\delta\lambda}((f,\widehat{f},\acute{\Delta})) = \{((\ell_1,\ell_5,\ell_6),\phi,\Re), ((\ell_2,\ell_5,\ell_6),\phi,\Re), ((\ell_3,\ell_5,\ell_6), \{r_1\},\{r_3\}), ((\ell_4,\ell_5,\ell_6),\phi,\Re)\}$.

On the other hand, $(g,\widehat{g},\acute{\Lambda}) \tilde{\tilde{\sqcap}} (f,\widehat{f},\acute{\Delta}) = (f,\widehat{f},\acute{\Delta})$, then $\Psi^{-1}_{\gamma\delta\lambda}((g,\widehat{g},\Lambda)\tilde{\tilde{\sqcap}}(f,\widehat{f},\Delta)) = \Psi^{-1}_{\gamma\delta\lambda}((f,\widehat{f},\acute{\Delta}))$.

Therefore, $\Psi^{-1}_{\gamma\delta\lambda}\left((g,\widehat{g},\acute{\Lambda})\widetilde{\widetilde{\sqcap}}(f,\widehat{f},\acute{\Delta})\right) \neq \Psi^{-1}_{\gamma\delta\lambda}((g,\widehat{g},\acute{\Lambda}))\ \widetilde{\widetilde{\sqcap}}\ \Psi^{-1}_{\gamma\delta\lambda}((f,\widehat{f},\acute{\Delta}))$.

In what follows, the bipolar hypersoft image and the bipolar hypersoft inverse image of bipolar hypersoft sets are discussed.

Proposition 4.7. *Suppose that $\Psi_{\gamma\delta\lambda} : \Omega_{(\Re,\Sigma)} \to \Omega_{(\aleph,\acute{\Sigma})}$ is a bipolar hypersoft mapping, where $\gamma : \Re \to \aleph$ is an injective mapping, $\delta : \Sigma \to \acute{\Sigma}$ and $\lambda : \neg\Sigma \to \neg\acute{\Sigma}$ are two mappings such that $\lambda(\neg\ell) = \neg\delta(\ell)$ for all $\neg\ell \in \neg\Sigma$. If $(g,\widehat{g},\Lambda) \in \Omega_{(\Re,\Sigma)}$, then $(g,\widehat{g},\Lambda)\ \widetilde{\widetilde{\sqsubseteq}}\ \Psi^{-1}_{\gamma\delta\lambda}(\Psi_{\gamma\delta\lambda}((g,\widehat{g},\Lambda)))$. The equality holds if $\Lambda = \Sigma$ and $\Psi_{\gamma\delta\lambda}$ is a bipolar hypersoft injective mapping.*

Proof. Let $\Psi^{-1}_{\gamma\delta\lambda}(\Psi_{\gamma\delta\lambda}((g,\widehat{g},\Lambda))) = \Psi^{-1}_{\gamma\delta\lambda}((f,\widehat{f},\acute{\Sigma})) = (h,\widehat{h},\Sigma)$. We want to show that, for all $\ell \in \Lambda$, $\Psi_{\gamma\delta\lambda}(g)(\ell) \subseteq \Psi_{\gamma\delta\lambda}(h)(\ell)$ and, for all $\neg\ell \in \neg\Lambda$, $\Psi_{\gamma\delta\lambda}(\widehat{h})(\neg\ell) \subseteq \Psi_{\gamma\delta\lambda}(\widehat{g})(\neg\ell)$. Let $\ell \in \Lambda$, then

$$\begin{aligned}
h(\ell) &= \Psi^{-1}_{\gamma\delta\lambda}(f)(\ell)\\
&= \gamma^{-1}(f(\delta(\ell)))\\
&= \gamma^{-1}\left(\gamma(\bigcup_{\ell\in\delta^{-1}(\delta(\ell))\cap\Lambda} g(\ell))\right)\\
&= \bigcup_{\ell\in\delta^{-1}(\delta(\ell))\cap\Lambda} \gamma^{-1}(\gamma(g(\ell)))\\
&= \bigcup_{\ell\in\delta^{-1}(\delta(\ell))\cap\Lambda} g(\ell),\ since\,\gamma\,is\,injective\,mapping\\
&\supseteq g(\ell).
\end{aligned}$$

Also, for $\neg\ell \in \neg\Lambda$, then

$$\begin{aligned}
\widehat{h}(\neg\ell) &= \Psi^{-1}_{\gamma\delta\lambda}(\widehat{f})(\neg\ell)\\
&= \gamma^{-1}(\widehat{f}(\lambda(\neg\ell)))\\
&= \gamma^{-1}\left(\gamma(\bigcap_{\neg\ell\in\lambda^{-1}(\lambda(\neg\ell))\cap\neg\Lambda} \widehat{g}(\neg\ell))\right)\\
&= \bigcap_{\neg\ell\in\lambda^{-1}(\lambda(\neg\ell))\cap\neg\Lambda} \gamma^{-1}(\gamma(\widehat{g}(\neg\ell))),\ since\,\gamma\,is\,injective\,mapping\\
&= \bigcap_{\neg\ell\in\lambda^{-1}(\lambda(\neg\ell))\cap\neg\Lambda} \widehat{g}(\neg\ell),\ since\,\gamma\,is\,injective\,mapping\\
&\subseteq \widehat{g}(\neg\ell).
\end{aligned}$$

Hence, the proof is completed. □

Remark 4.8. The equality is false in Proposition 4.7.

Example 4.9. Consider $\Psi_{\gamma\delta\lambda}$, $(g,\widehat{g},\Lambda)$, and $\Psi_{\gamma\delta\lambda}((g,\widehat{g},\Lambda))$ in Example 3.2. Then $\Psi^{-1}_{\gamma\delta\lambda}(\Psi_{\gamma\delta\lambda}((g,\widehat{g},\Lambda))) = \{((\ell_1,\ell_5,\ell_6),\{r_1,r_3\},\{r_2\}),((\ell_2,\ell_5,\ell_6),\{r_1,r_3\},\{r_2\}),$ $((\ell_3,\ell_5,\ell_6),\{r_3\},\{r_1\}),((\ell_4,\ell_5,\ell_6),\phi,\Re)\}$. Hence, $(g,\widehat{g},\Lambda) \neq \Psi^{-1}_{\gamma\delta\lambda}(\Psi_{\gamma\delta\lambda}((g,\widehat{g},\Lambda)))$.

Proposition 4.10. *Suppose that $\Psi_{\gamma\delta\lambda} : \Omega_{(\Re,\Sigma)} \to \Omega_{(\aleph,\acute{\Sigma})}$ is a bipolar hypersoft mapping, where $\gamma : \Re \to \aleph$ is a bijective mapping, $\delta : \Sigma \to \acute{\Sigma}$ and $\lambda : \neg\Sigma \to \neg\acute{\Sigma}$ are two mappings such that $\lambda(\neg\ell) = \neg\delta(\ell)$ for all $\neg\ell \in \neg\Sigma$. If $(f,\widehat{f},\acute{\Sigma}) \in \Omega_{(\aleph,\acute{\Sigma})}$, then $\Psi_{\gamma\delta\lambda}(\Psi^{-1}_{\gamma\delta\lambda}((f,\widehat{f},\acute{\Sigma}))) \,\widetilde{\widetilde{\sqsubseteq}}\, (f,\widehat{f},\acute{\Sigma})$. The equality holds if $\Psi_{\gamma\delta\lambda}$ is a bipolar hypersoft surjective mapping.*

Proof. Let $\Psi_{\gamma\delta\lambda}(\Psi^{-1}_{\gamma\delta\lambda}((f,\widehat{f},\acute{\Sigma}))) = \Psi_{\gamma\delta\lambda}((g,\widehat{g},\Sigma)) = (h,\widehat{h},\acute{\Sigma})$. We want to show that, for all $\acute{\ell} \in \acute{\Sigma}$, $\Psi_{\gamma\delta\lambda}(h)(\acute{\ell}) \subseteq \Psi_{\gamma\delta\lambda}(f)(\acute{\ell})$ and, for all $\neg\acute{\ell} \in \neg\acute{\Sigma}$, $\Psi_{\gamma\delta\lambda}(\widehat{f})(\neg\acute{\ell}) \subseteq \Psi_{\gamma\delta\lambda}(\widehat{h})(\neg\acute{\ell})$. Let $\acute{\ell} \in \delta(\delta^{-1}(\acute{\Sigma})) \subseteq \acute{\Sigma}$ (if $\acute{\ell} \in \acute{\Sigma} \setminus \delta(\delta^{-1}(\acute{\Sigma}))$, then $h(\ell) = \phi \subseteq g(\ell)$), then

$$\begin{aligned}
h(\acute{\ell}) &= \Psi_{\gamma\delta\lambda}(g)(\acute{\ell})\\
&= \gamma\left(\bigcup_{\ell\in\delta^{-1}(\acute{\ell})\cap\Sigma} g(\ell)\right)\\
&= \gamma\left(\bigcup_{\ell\in\delta^{-1}(\acute{\ell})} \gamma^{-1}(f(\delta(\ell)))\right)\\
&= \gamma\left(\gamma^{-1}(\bigcup_{\ell\in\delta^{-1}(\acute{\ell})} f(\delta(\ell)))\right)\\
&= \gamma\left(\gamma^{-1}(f(\acute{\ell}))\right), \mathit{since}\, f(\delta(\ell)) = f(\acute{\ell})\, \mathit{for\, all}\, \ell \in \delta^{-1}(\acute{\ell})\\
&= f(\acute{\ell}),\ \mathit{since}\, \gamma\, \mathit{is\, surjective\, mapping}.
\end{aligned}$$

Also, for $\neg\acute{\ell} \in \lambda(\lambda^{-1}(\neg\acute{\Sigma})) \subseteq \neg\acute{\Sigma}$ (if $\neg\acute{\ell} \in \neg\acute{\Sigma} \setminus \lambda(\lambda^{-1}(\neg\acute{\Sigma}))$, then $\widehat{h}(\neg\ell) = \aleph \supseteq \widehat{f}(\neg\ell)$), then

$$\begin{aligned}
\widehat{h}(\neg\acute{\ell}) &= \Psi_{\gamma\delta\lambda}(\widehat{g})(\neg\acute{\ell})\\
&= \gamma\left(\bigcap_{\neg\ell\in\lambda^{-1}(\neg\acute{\ell})\cap\neg\Sigma} \widehat{g}(\neg\ell)\right)\\
&= \gamma\left(\bigcap_{\neg\ell\in\lambda^{-1}(\neg\acute{\ell})} \gamma^{-1}(\widehat{f}(\lambda(\neg\ell)))\right)\\
&= \gamma\left(\gamma^{-1}(\bigcap_{\neg\ell\in\lambda^{-1}(\neg\acute{\ell})} \widehat{f}(\lambda(\neg\ell)))\right)\\
&= \gamma\left(\gamma^{-1}(\widehat{f}(\neg\acute{\ell}))\right), \mathit{since}\, \widehat{f}(\lambda(\neg\ell)) = \widehat{f}(\neg\acute{\ell})\, \mathit{for\, all}\, \neg\ell \in \lambda^{-1}(\neg\acute{\ell})\\
&= \widehat{f}(\neg\acute{\ell}),\ \mathit{since}\, \gamma\, \mathit{is\, surjective\, mapping}.
\end{aligned}$$

Hence, the proof is completed. □

Remark 4.11. If γ is not surjective in Proposition 4.10, then the subset relation is not true in general.

Example 4.12. Consider $\Psi_{\gamma\delta\lambda}$ in Example 3.2 and $(f, \widehat{f}, \acute{\Lambda} = \acute{\Sigma})$, $\Psi^{-1}_{\gamma\delta\lambda}((f, \widehat{f}, \acute{\Sigma}))$ in Example 4.6. Then $\Psi_{\gamma\delta\lambda}(\Psi^{-1}_{\gamma\delta\lambda}((f, \widehat{f}, \acute{\Sigma}))) = \{((\acute{\ell}_1, \acute{\ell}_5, \acute{\ell}_6), \phi, \{\eta_3\}), ((\acute{\ell}_2, \acute{\ell}_5, \acute{\ell}_6), \phi, \aleph), ((\acute{\ell}_3, \acute{\ell}_5, \acute{\ell}_6), \{\eta_1\}, \{\eta_3\}), ((\acute{\ell}_4, \acute{\ell}_5, \acute{\ell}_6), \phi, \{\eta_1, \eta_2, \eta_3\})\}$. Hence, $\Psi_{\gamma\delta\lambda}(\Psi^{-1}_{\gamma\delta\lambda}((f, \widehat{f}, \acute{\Sigma}))) \tilde{\tilde{\nsubseteq}} (f, \widehat{f}, \acute{\Sigma})$.

Remark 4.13. The equality does not hold in Proposition 4.10.

Example 4.14. Consider $\Psi_{\gamma\delta\lambda}$ in Example 3.2 but if we take $\aleph = \{\eta_1, \eta_2, \eta_3\}$ instead of $\aleph = \{\eta_1, \eta_2, \eta_3, \eta_4\}$, then γ will be a bijective mapping. Let $(f, \widehat{f}, \acute{\Sigma}) = \{((\acute{\ell}_1, \acute{\ell}_5, \acute{\ell}_6), \{\eta_1, \eta_2\}, \phi), ((\acute{\ell}_2, \acute{\ell}_5, \acute{\ell}_6), \aleph, \phi), ((\acute{\ell}_3, \acute{\ell}_5, \acute{\ell}_6), \aleph, \phi), ((\acute{\ell}_4, \acute{\ell}_5, \acute{\ell}_6), \{\eta_3\}, \{\eta_1\})\}$, then $\Psi^{-1}_{\gamma\delta\lambda}((f, \widehat{f}, \acute{\Sigma})) = \{((\ell_1, \ell_5, \ell_6), \{r_1, r_2\}, \phi), ((\ell_2, \ell_5, \ell_6), \{r_1, r_2\}, \phi), ((\ell_3, \ell_5, \ell_6), \Re, \phi), ((\ell_4, \ell_5, \ell_6), \{r_3\}, \{r_1\})\}$. Therefore, $\Psi_{\gamma\delta\lambda}(\Psi^{-1}_{\gamma\delta\lambda}((f, \widehat{f}, \acute{\Sigma}))) = \{((\acute{\ell}_1, \acute{\ell}_5, \acute{\ell}_6), \{\eta_1, \eta_2\}, \phi), ((\acute{\ell}_2, \acute{\ell}_5, \acute{\ell}_6), \phi, \aleph), ((\acute{\ell}_3, \acute{\ell}_5, \acute{\ell}_6), \aleph, \phi), ((\acute{\ell}_4, \acute{\ell}_5, \acute{\ell}_6), \{\eta_3\}, \{\eta_1\})\}$. Hence, $\Psi_{\gamma\delta\lambda}(\Psi^{-1}_{\gamma\delta\lambda}((f, \widehat{f}, \acute{\Sigma}))) \neq (f, \widehat{f}, \acute{\Sigma})$.

5. Conclusions

Throughout this study, we have introduced bipolar hypersoft mapping as well as various associated concepts and properties. Also, the definition of the bipolar hypersoft inverse image along with some of the related results are then presented. We examined, on a bipolar hypersoft set, the relationship between bipolar hypersoft image and the bipolar hypersoft inverse image. In the future, we strongly recommend applying these results and suggestions to real-life problems in decision-making and medical diagnosis, as well as examining the behavior of specific topological and algebraic concepts.

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Linear Diophantine Neutrosophic Sets and Their Properties

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Abstract: In 2019, Riaz et al. introduced the notion of linear Diophantine fuzzy set(LDFS) where there is an addition of reference parameters that help to address the issues that cannot be managed by the existing theories such as fuzzy sets(FSs), intuitionistic fuzzy sets(IFSs), Pythagorean fuzzy sets(PFSs), and q-rung orthopair fuzzy sets(q-ROFSs). But all these theories are not capable to describe indeterminacy that exists in numerous real-world problems. For this purpose, neutrosophic sets(NSs), single-valued neutrosophic sets(SVNSs), Pythagorean neutrosophic sets(PNSs) are introduced. In PNS, each object x in the universe is characterized by a dependent truth $\left(\mu_T(x)\right)$ and falsity $\left(\gamma_F(x)\right)$ membership values and indeterminacy $\left(\nu_I(x)\right)$ membership value with the restriction $0 \le \left(\mu_T(x)\right)^2 + \left(\gamma_F(x)\right)^2 + \left(\nu_I(x)\right)^2 \le 2$. If we consider a neutrosophic triplet as $\langle 0.9, 0.9, 0.9 \rangle$ then $0.9^2 + 0.9^2 + 0.9^2$ will give 2.43, which is > 2. Such a problem cannot be handled by the decision-makers under the Pythagorean neutrosophic environment. To take care of such an issue there is an urgency to develop another mathematical model. This lead to an introduction of linear Diophantine neutrosophic set(LDNS) as an extension of PNS. Thus, the main purpose of this paper is to introduce the LDNS model with an aid of reference parameters to ensure that through this new model the decision-makers can freely choose the neutrosophic membership values with an extended domain. Therefore, in a broad sense, the LDNSs are a new idea that removes the restrictions present in the existing concepts such as FSs, IFSs, PFSs, q-ROFSs, PNSs, LDFSs, etc. From example 3.1.1, it is quite visible that this new structure helps to classify the problem by changing the physical nature of reference parameters. Moreover, some basic properties and operations on LDNSs are investigated. We also define the score and accuracy function based on linear Diophantine neutrosophic number(LDNN). With the help of a novel linear Diophantine single-valued neutrosophic weighted arithmetic-geometric aggregation (LDSVNWAGA) operator, an algorithm has been developed for decision-making. Finally, the proposed algorithm has been successfully executed with the help of a numerical application.

Keywords: Neutrosophic set; Linear Diophantine neutrosophic set; Reference parameter; Decision-Making.

1. Introduction

Presently, in the real-world we are facing complicated problems that cannot be solved by the traditional mathematical tools. It is due to the involvement of uncertainty or vagueness in real-life situations. The crisp concept is no more valid to define ambiguity. A crisp set A can be characterized by a characteristic function χ_A and the values of χ_A corresponding to all the objects in A are either 0 or 1. Boolean algebra also useful to address the same situation. In mathematics, we find some linguistic terms such as "excellent", "beautiful", "intelligent" etc, which are subjective. To eradicate such a problem to some extent, Zadeh introduced the fuzzy set [1] in 1965 and fuzzy logic [2] in 1996. A fuzzy set is a significant mathematical tool to model vagueness or uncertainty in the data or information, that has been attracted the attention of many researchers across the globe in the last decades. A fuzzy set X be characterized by its membership function $\mu : X \to [0,1]$, which assigns a real value in the unit closed interval $[0,1]$ to each object of the universe. Thus, a fuzzy set is an extension of a crisp set whose boundary is blurred. The researchers have been studied fuzzy sets as problem-solving techniques in various fields including, engineering, computer science, medical science, social science, economics, environments, robotics, etc., having various uncertainties. Some significant works associated with fuzzy sets are studied in [3-7]. Later on, in 2010, Bustince [8] introduced an interval-valued fuzzy set (IVFS), where the membership function defined as $\mu : X \to \operatorname{int}([0,1])$, $\operatorname{int}([0,1])$ denotes the collection of all subsets of $[0,1]$. To define the incomplete information, Atanassov [9] introduced intuitionistic fuzzy set(IFS) as a direct extension of the fuzzy set by using the notion of membership degree (μ) and the non-membership degree (γ), where both the membership values belong to the interval $[0,1]$ with a restriction that their sum cannot exceed the unity and the hesitancy degree is calculated as $\pi = 1 - \mu - \gamma$. Bustince [10] defined vague sets are intuitionistic fuzzy sets, in [11], Garg et al. presented an improved possibility degree method to find the rank of intuitionistic fuzzy numbers(IFNs), Gou et al. [12] defined exponential operations for IFNs, Heilpern [13] proposed an application of fuzzy numbers(FNs), Nayagam et al. [14] defined ranking of IFNs, Szmidt et al. [15] gives an application of IFS, Wang et al. [16] proposed IFS and L-FS, Zeng et al. [17] presented multiattribute decision-making based on novel score function of intuitionistic fuzzy values and modified VIKOR method. If a decision-maker assigns an ordered pair $(0.65.0.55)$ to an alternative, then it is not an IFN, as $0.65+0.55>1$. To tackle such a case, Yager [18] introduced a Pythagorean fuzzy set(PFS) where the sum of the squares of Pythagorean fuzzy membership grades should not exceed unity. So, we have an enlarged space for PFSs as compared to IFSs. In

◎19◎ Wan et al. introduced Pythagorean fuzzy number(PFN). PFSs have been further extended by introducing q-ROFSs◎20-24◎ Some novel works associated with PFSs and PFNs are proposed in ◎25-36◎ In 2019, Jansi et al.◎37◎introduced correlation measure for Pythagorean neutrosophic sets where truthfulness and falseness are dependent components. Ajay et al.◎38◎introduced the Pythagorean neutrosophic fuzzy graphs.

In some real-life problems, the sum of the membership grade and non-membership grade to which an alternative satisfying an attribute provided by the decision-maker (DM) may be larger than 1 (e.g 0.8◎0.7◎1) and their sum of the squares is also larger than 1 (e.g 0.8^2◎0.7^2◎1). Thus, IFS and PFS fail to hold in such situations. To overcome these deficiencies, the restrictions on membership and non-membership grades are altered to $0 \le \mu^q + \gamma^q \le 1$ in the case of q-rung orthopair fuzzy set(q-ROFS). Even for very large values of "q", we can deal with membership and non-membership grades independently to some extent. In some practical problems, when $\mu = \gamma = 1$, we obtain $1^q + 1^q \ge 1$, which contradicts the constraint of q-ROFS. It makes the MADM limited and affects the optimum decision. Linear Diophantine fuzzy set (LDFS) ◎39◎ can deal with such situations to some extent. LDFS provides a large number of applications to the MADM for such real-world problems. So, through the model of LDFS, we can deal with the intuitionistic, Pythagorean, and q-rung orthopair nature of attributes under the effect of reference parameters (α, β). For example, let (0.7◎0.6◎1), we can introduce reference parameters (α, β) such that $(\alpha)(0.7)+(\beta)(0.6)<1$, where (α, β) denotes the reference parameters concerning for to membership and non-membership grade respectively. Some recent works related to LDFS are given in ◎40-42◎

The term neutrosophy denotes the study of neutralities and it is proposed by Smarandache◎43◎ Neutrosophy can be treated as a branch of philosophy. If we consider $\langle A \rangle$ be an idea or proposition or an axiom or theorem then its opposite notion is denoted by $\langle \text{anti}A \rangle$ and for completeness property we consider another concept known as $\langle \text{nor}\, A \rangle$. But, some concepts are there which lie in between $\langle A \rangle$ and $\langle \text{anti}A \rangle$, they are denoted by $\langle \text{neut}\, A \rangle$.

So, realizing the importance of the study of neutrality, Smarandache◎44◎introduced a neutrosophic set(NS), as an extension of IFS. For technical use, Wang et al. ◎45◎introduced a single-valued neutrosophic set(SVNS). Some recent research works associated with NSs are in the following: data development analysis for simplified NS is studied in ◎46◎ Another data envelopment analysis under a triangular neutrosophic number environment has been done in ◎47◎ In ◎48◎ Edalatpanah introduced the neutrosophic structured element. A triangular neutrosophic linear programming model is presented in ◎49◎ Martin et al. ◎50◎introduced the COVID-19 diagnostic model by using a new pilthogenic cognitive maps approach. Debnath◎51◎presented the neutrosophic statistical data to assess the knowledge, attitude, and symptoms of reproductive tract infection(RTI) among women in selected villages in India.

By using IFS, PFS, and q-ROFS we only define the incomplete information present in the data. But, in real life some information is there whichis partially true and partially false i.e., they are indeterminate or inconsistent. To overcome such problems, the concept of the neutrosophic theory is very helpful. For the sake of computation, throughout the paper, we use SVNS instead of NS.

The main motivation behind presenting this paper is to extend the notion of LDFS to LDNS. Some MADM problems exist in real life which involves indeterminate attributes. To handle such problems we need a powerful tool to tackle. This leads to the introduction of LDNSs. Also, we have investigated some operations and properties based on LDNSs. Further, we have introduced an algorithm that can be applied successfully in solving real MADM problems with the help of a suitable example.

1.1 Novelty

There exists some real-world-based complex phenomenon that cannot be solved by using the existing fuzzy theories and their extensions. Such phenomenon can be tackled with addition of reference parameters that build a bridge between the existing theories and the physical world. For this purpose, we have introduced a novel concept known as linear Diophantine neutrosophic set(LDNS) to apply it in different MADM problems by categorizing the data using reference parameters. Therefore, the LDNS model surely provides a powerful mathematical tool for the further development of the neutrosophic theory. The objectives of the proposed study are discussed in the following manner:

- The PNS ◎37, 38◎ is developed to generalize the PFS◎18◎ and the SVNS with dependent neutrosophic components. But, in some real-life situation, the sum of squares of a membership grade, non-membership grade, and indeterminacy grade to an attribute provided by a decision-maker may be > 2. Such problems cannot be described by FS, IFS, PFS, SVNS, PNS,q-ROFS, LDFS. To remove such inadequacy, the LDNS is introduced to deal with a large number of MADM problems by enlarging the domain with an aid of reference parameters.

  For better understanding, suppose the neutrosophic triplet of an attribute provided by the decision-maker is $\langle 0.8, 0.9, 0.9 \rangle$. The sum of their squares gives 2.26 > 2. Corresponding to the neutrosophic triplet if we assign the grades of the reference parameters triplet as $\langle 0.5, 0.6, 0.7 \rangle$. Then, $0.8\times 0.5 + 0.9\times 0.6 + 0.9\times 0.7$ ◎1.57 < 2. It looks similar to the linear Diophantine equation $ax + by + cz = d$ which is a popular topic in number theory. So, the name of the proposed model is logical in this sense. Thus, by introducing LDNS, we fill the research gap.

- FS, IFS, NS, SVNS, PFS, q-ROFS, PNS cannot deal with parameters. So, by introducing reference parameters in LDNS, there is a huge scope for a decision-maker to address various types of MADM problems by changing the physical nature of the reference parameters.

- Define the linear Diophantine neutrosophic numbers (LDNNs) and study their properties.
- Define a new aggregate operator called LDSVNWAGA operator that helps to obtain the rank of the alternatives.
- Construction of a new algorithm for solving MADM problems by using the new aggregate operator.
- Justify the algorithm with the help of a numerical application based on real life.

1.2 Structure of the paper

The manuscript is organized in the following manner: Section 2 includes the basic definitions of FS, IFS, PFS, q-ROFS, PNS, LDFS which are useful to build the proposed study. Section 3 contains the definition of LDNNs and their properties. Section 4 contains the definitions of score function, accuracy function, and aggregate operator based on LDNNs. In Section 5, an algorithm is constructed for MADM problems. In Section 6, a numerical example is presented to justify the proposed algorithm. Section 7 contains a comparative study between the proposed and the existing theories. Conclusion and the future scope have been studied in Section 8.

2. Preliminaries

In this section, we review some basic definitions with examples that are very useful for the subsequent sections of this paper.

Definition 2.1 [1, 2, 6] Let X be an initial universe and $\mu_A : X \to [0,1]$ be the membership function. Then a fuzzy set A is defined by

$$A = \{(x, \mu_A(x)) : x \in X\}$$

$$= \sum \mu_A(x)/x \text{ , when X is discrete}$$

$$\circledcirc \int \mu_A(x)/x \text{ , when X is continuous}$$

Here $\mu_A(x)$ denotes the degree of membership of x to the fuzzy set A. The value of the membership function $\mu_A(x)$ can be chosen by different experts may be different depending upon their experiences, perceptions, perspectives, etc. The collection of all fuzzy sets in X is denoted by I^X.

Example 2.1.1 Let $X = \{x_1, x_2, x_3, x_4, x_5\}$ be a collection of beautiful students, and then the fuzzy set A associated with X is defined by a decision-maker (DM) as

$$A = \{(x_1, 0.5), (x_2, 0.6), (x_3, 1.0), (x_4, 0.0), (x_5, 0.3)\}$$

If all the membership values in A are either 0 or 1, then A reduces to a crisp set. So, a crisp set is a particular class of a fuzzy set.

Definition 2.2 [9] An intuitionistic fuzzy set(IFS) A over the universe X is defined as

$A=\{(x,\mu_A(x),\gamma_A(x)):x\in X\}$ such that $0\le\mu_A(x)+\gamma_A(x)\le1$, $\forall x\in X$ where $\mu_A:X\to[0,1]$ and $\gamma_A:X\to[0,1]$ denote the membership function and the non-membership function, respectively. However, the hesitancy degree is given by $\pi_A(x)=1-\mu_A(x)-\gamma_A(x), \forall x\in X$.

Definition 2.3 [18, 26, 36] A Pythagorean fuzzy set(PFS) P over the universe X is defined by $P=\{(x,\mu_P(x),\gamma_P(x)):x\in X\}$ where $\mu_P,\gamma_P:X\to[0,1]$ with the restriction $0\le(\mu_P(x))^2+(\gamma_P(x))^2\le1.$

Hence PFSs have a wide range of space of application as compared to IFSs.

The degree of hesitancy may be computed as $I_P(x)=\sqrt{1-(\mu_P(x))^2-(\gamma_P(x))^2}$

Definition 2.4 [20, 21] Let $\omega=\{\xi_1,\xi_2,....,\xi_n\}$ be a finite universal set, then a q-ROFS, Q in ω can be defined as follows:

$Q=\{\langle\xi,\mu_Q(\xi),\gamma_Q(\xi)\rangle:\xi\in\omega\}$ where $\mu_Q,\gamma_Q:\omega\to[0,1]$ with the condition $0\le(\mu_Q(\xi))^q+(\gamma_Q(\xi))^q\le1,\ q\ge1,\ \forall\xi\in\omega.$

The value $\pi_Q(\xi)=\sqrt{1-(\mu_Q(\xi))^q-(\gamma_Q(\xi))^q}$ is called the degree of indeterminacy of Q in ω.

Also, $0\le\pi_Q(\xi)\le1, \forall\xi\in\omega.$

Definition 2.5 [39] Let Q be the non-empty reference set. An LDFS £D on Q is an object of the form:

£D= $\{(\varsigma,\langle\mu_D(\varsigma),\gamma_D(\varsigma)\rangle,\langle\alpha,\beta\rangle):\varsigma\in Q\}$ where, $\mu_D(\varsigma),\gamma_D(\varsigma),\alpha,\beta\in[0,1]$ are membership, non-membership and reference parameters with the following conditions:

$0\le\alpha\mu_D(\varsigma)+\beta\gamma_D(\varsigma)\le1, \forall\varsigma\in Q$ and $0\le\alpha+\beta\le1$. These reference parameters can help in defining or classifying a particular system. The hesitation part can be evaluated as:

$\xi\pi_D=1-\alpha\mu_D(\varsigma)-\beta\gamma_D(\varsigma)$ where ξ is the reference parameters related to the degree of hesitancy.

Definition 2.6 [37, 38] Let X be a non-empty universal set. A Pythagorean neutrosophic set with T and F are dependent neutrosophic components A over X is an object of the form $\{\langle x,T_A(x),I_A(x),F_A(x)\rangle:x\in X\}$, where $T_A(x),I_A(x),F_A(x)\in[0,1]$,

$0\le T_A^2(x)+I_A^2(x)+F_A^2(x)\le2$, for all $x\in X$. Here,

$T_A(x), I_A(x)$ and $F_A(x)$ respectively denote the degree of truth membership, degree of indeterminacy membership, and the degree of falsity membership.

3. Linear Diophantine Neutrosophic Set(LDNS)

Definition 3.1

Let Q be the non-empty reference set. A LDNS ⊚ND on Q is an object of the form:

⊚ND⊚ $\left\{\left(\varsigma, \left\langle \mu_{ND}(\varsigma), \nu_{ND}(\varsigma), \gamma_{ND}(\varsigma)\right\rangle, \left\langle \alpha, \delta, \beta \right\rangle\right) : \varsigma \in Q\right\}$ where, $\mu_{ND}(\varsigma), \nu_{ND}(\varsigma), \gamma_{ND}(\varsigma)$, $\alpha, \delta, \beta \in$ ⊚0,1⊚are truth-membership, indeterminacy-membership, falsity-membership, and their reference parameters respectively with the following conditions:

$0 \le \alpha\mu_{ND}(\varsigma) + \delta\nu_{ND}(\varsigma) + \beta\gamma_{ND}(\varsigma) \le 2$, $\forall \varsigma \in Q$ and $0 \le \alpha + \delta + \beta \le 2$. These reference parameters can help in defining or classifying a particular system. The hesitation part can be evaluated as:

$\xi\pi_{ND} = 2 - \left(\alpha\mu_{ND}(\varsigma) + \delta\nu_{ND}(\varsigma) + \beta\gamma_{ND}(\varsigma)\right)$ where ξ is the reference parameter related to the degree of indeterminacy. Simply $\Lambda = \left(\left\langle \mu_{ND}, \nu_{ND}, \gamma_{ND} \right\rangle, \left\langle \alpha, \delta, \beta \right\rangle\right)$ is called linear Diophantine neutrosophic number(LDNN) with $0 \le \alpha\mu_{ND} + \delta\nu_{ND} + \beta\gamma_{ND} \le 2$ and $0 \le \alpha + \delta + \beta \le 2$.

Since the proposed model looks similar to the well-known linear Diophantine equation $ax + by + cz = d$ in the number theory, so LDNS is the most suitable name for the proposed model. The proposed model enhances the existing methodologies and the decision-maker (DM) can choose the grades with more liberty as compared to the other existing theories. This structure also categorizes the problem by changing the physical sense of reference alternatives in MADM.

Example 3.1.1 Chemical bonding can be described as a force that binds two or more atoms together to form molecules or ionic compounds. Chemical bonds form because the overall energy of the bonded atoms is less than the atoms have separately. Atoms form bonds to attain a noble gas configuration. There are two main types of bonds such as ionic bonds and covalent bonds. Covalent bonds are dived into polar and non-polar covalent bonds. Some atoms have high electro negativity (e.g. fluorine), some have low electro negativity (e.g. cesium) and some are neutral (e.g. carbon) in nature.

Let Q⊚ $\{\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6\}$ be a collection of atoms having different electro negativity and by combining two or more of them, molecule is formed. If we consider the reference or control parameters as:

α ⊚polar covalent bond, δ ⊚ionic bond and β ⊚non-polar covalent bond

Then its LDNS is given in Table 1

Alternatives	LDNSs
η_1	$(\langle 0.871, 0.563, 0.643\rangle, \langle 0.321, 0.564, 0.456\rangle)$
η_2	$(\langle 0.862, 0.573, 0.776\rangle, \langle 0.354, 0.567, 0.786\rangle)$
η_3	$(\langle 0.578, 0.654, 0.456\rangle, \langle 0.567, 0.865, 0.546\rangle)$
η_4	$(\langle 0.525, 0.943, 0.654\rangle, \langle 0.324, 0.456, 0.567\rangle)$
η_5	$(\langle 0.675, 0.765, 0.845\rangle, \langle 0.865, 0.467, 0.656\rangle)$
η_6	$(\langle 0.456, 0.678, 0.897\rangle, \langle 0.564, 0.867, 0.567\rangle)$

Table 1. LDNS for Molecule

Definition 3.2

A LDNS on Q of the form ^{1}◎$_{ND}$ = $\left\{\left(\varsigma, \langle 1,0,0\rangle, \langle 1,0,0\rangle\right) : \varsigma \in Q\right\}$ is called absolute LDNS and

^{0}◎$_{ND}$◎ $\left\{\left(\varsigma, \langle 0,1,1\rangle, \langle 0,1,1\rangle\right) : \varsigma \in Q\right\}$ is called empty or void LDNS.

Now, we define some operations on LDNNs associated with LDNSs

Definition 3.3

Let $\Lambda_Q = \left(\left\langle {}^Q\mu_{ND}, {}^Q\nu_{ND}, {}^Q\gamma_{ND}\right\rangle, \left\langle {}^Q\alpha, {}^Q\delta, {}^Q\beta\right\rangle\right)$, where $Q \in \omega$ be an assembling of LDNNs , then

(i) $\Lambda^c{}_Q = \left(\left\langle {}^Q\gamma_{ND}, {}^Q\nu_{ND}, {}^Q\mu_{ND}\right\rangle, \left\langle {}^Q\beta, {}^Q\delta, {}^Q\alpha\right\rangle\right)$

(ii) $\Lambda_1 = \Lambda_2 \Leftrightarrow {}^1\mu_{ND} = {}^2\mu_{ND}, {}^1\nu_{ND} = {}^2\nu_{ND}, {}^1\gamma_{ND} = {}^2\gamma_{ND}, {}^1\alpha = {}^2\alpha, {}^1\delta = {}^2\delta, {}^1\beta = {}^2\beta$

(iii) $\Lambda_1 \subseteq \Lambda_2 \Leftrightarrow {}^1\mu_{ND} \leq {}^2\mu_{ND}, {}^1\nu_{ND} \geq {}^2\nu_{ND}, {}^1\gamma_{ND} \geq {}^2\gamma_{ND}, {}^1\alpha \leq {}^2\alpha, {}^1\delta \geq {}^2\delta, {}^1\beta \geq {}^2\beta$

(iv) $\bigcup_{Q\in\omega} \Lambda_Q = \left(\left\langle \sup_{Q\in\omega}{}^Q\mu_{ND}, \inf_{Q\in\omega}{}^Q\nu_{ND}, \inf_{Q\in\omega}{}^Q\gamma_{ND}\right\rangle, \left\langle \sup_{Q\in\omega}{}^Q\alpha_{ND}, \inf_{Q\in\omega}{}^Q\delta_{ND}, \inf_{Q\in\omega}{}^Q\beta_{ND}\right\rangle\right)$

(V) $\bigcap_{Q\in\omega} \Lambda_Q = \left(\left\langle \inf_{Q\in\omega}{}^Q\mu_{ND}, \sup_{Q\in\omega}{}^Q\nu_{ND}, \sup_{Q\in\omega}{}^Q\gamma_{ND}\right\rangle, \left\langle \inf_{Q\in\omega}{}^Q\alpha_{ND}, \sup_{Q\in\omega}{}^Q\delta_{ND}, \sup_{Q\in\omega}{}^Q\beta_{ND}\right\rangle\right)$

(vi)

$$\Lambda_1 \oplus \Lambda_2 = \left(\left\langle {}^1\mu_{ND} + {}^2\mu_{ND} - {}^1\mu_{ND}{}^2\mu_{ND}, {}^1\nu_{ND}{}^2\nu_{ND}, {}^1\gamma_{ND}{}^2\gamma_{ND} \right\rangle, \left\langle {}^1\alpha + {}^2\alpha - {}^1\alpha^2\alpha, {}^1\delta^2\delta, {}^1\beta^2\beta \right\rangle \right)$$

(vii) $$\Lambda_1 \otimes \Lambda_2 = \begin{pmatrix} \left\langle {}^1\mu_{ND}{}^2\mu_{ND}, {}^1\nu_{ND} + {}^2\nu_{ND} - {}^1\nu_{ND}{}^2\nu_{ND}, {}^1\gamma_{ND} + {}^2\gamma_{ND} - {}^1\gamma_{ND}{}^2\gamma_{ND} \right\rangle, \\ \left\langle {}^1\alpha^2\alpha, {}^1\delta + {}^2\delta - {}^1\delta^2\delta, {}^1\beta + {}^2\beta - {}^1\beta^2\beta \right\rangle \end{pmatrix}$$

(viii) $$\lambda\Lambda_1 = \left(\left\langle 1 - \left(1 - {}^1\mu_{ND}\right)^{\lambda}, {}^1\nu_{ND}^{\lambda}, {}^1\gamma_{ND}^{\lambda} \right\rangle, \left\langle 1 - \left(1 - {}^1\alpha_{ND}\right)^{\lambda}, {}^1\delta_{ND}^{\lambda}, {}^1\beta_{ND}^{\lambda} \right\rangle \right), \lambda > 0$$

(ix)

$$\Lambda^{\lambda}{}_1 = \left(\left\langle {}^1\mu_{ND}^{\lambda}, 1 - \left(1 - {}^1\nu_{ND}\right)^{\lambda}, 1 - \left(1 - {}^1\gamma_{ND}\right)^{\lambda} \right\rangle, \left\langle {}^1\alpha_{ND}^{\lambda}, 1 - \left(1 - {}^1\delta_{ND}\right)^{\lambda}, 1 - \left(1 - {}^1\beta_{ND}\right)^{\lambda} \right\rangle \right), \lambda > 0$$

It is to be noted that LDNNs don't obey De Morgan's laws. It is one of the drawbacks of using LDNNs.

Example 3.3.1 Let

$$\Lambda_1 = \left(\langle 0.55, 0.65, 0.84 \rangle, \langle 0.56, 0.64, 0.46 \rangle \right) \text{ and } \Lambda_2 = \left(\langle 0.65, 0.45, 0.54 \rangle, \langle 0.66, 0.34, 0.36 \rangle \right)$$

be two LDNNs. Then, we obtain the following results:

$$(\Lambda_1)^c = \left(\langle 0.84, 0.65, 0.55 \rangle, \langle 0.46, 0.64, 0.56 \rangle \right) \text{ and } (\Lambda_2)^c = \left(\langle 0.54, 0.45, 0.65 \rangle, \langle 0.36, 0.34, 0.66 \rangle \right)$$

Here, $\Lambda_1 \subseteq \Lambda_2$ (By definition 3.3)

Now,

$$\Lambda_1 \cup \Lambda_2 \circledcirc \left(\langle 0.65, 0.45, 0.54 \rangle, \langle 0.66, 0.34, 0.36 \rangle \right) \circledcirc \Lambda_2 \text{ and } \Lambda_1 \cap \Lambda_2 \circledcirc \Lambda_1$$

$$\Lambda_1 \oplus \Lambda_2 \circledcirc \left(\langle 0.8425, 0.2925, 0.4536 \rangle, \langle 0.8504, 0.2176, 0.1656 \rangle \right)$$

$$\Lambda_1 \otimes \Lambda_2 \circledcirc \left(\langle 0.3575, 0.8075, 0.9264 \rangle, \langle 0.3696, 0.7624, 0.6544 \rangle \right)$$

For $\lambda \circledcirc 0.4$, $\lambda\Lambda_1 = \left(\langle 0.273, 0.841, 0.932 \rangle, \langle 0.279, 0.836, 0.732 \rangle \right)$

For $\lambda \circledcirc 0.2$, $\Lambda^{\lambda}{}_1 = \left(\langle 0.887, 0.189, 0.315 \rangle, \langle 0.89, 0.184, 0.115 \rangle \right)$

Proposition 3.4 Let Λ_1, Λ_2 and Λ_3 be three LDNNs then we have the following results:

(i) If $\Lambda_1 \subseteq \Lambda_2$ and $\Lambda_2 \subseteq \Lambda_3 \Rightarrow \Lambda_1 \subseteq \Lambda_3$ (Transitivity)

(ii) $\Lambda_1 \cup \Lambda_2 = \Lambda_2 \cup \Lambda_1$ and $\Lambda_1 \cap \Lambda_2 = \Lambda_2 \cap \Lambda_1$ (commutativity)

(iii) $\Lambda_1 \cup (\Lambda_2 \cup \Lambda_3) = (\Lambda_1 \cup \Lambda_2) \cup \Lambda_3$ and $\Lambda_1 \cap (\Lambda_2 \cap \Lambda_3) = (\Lambda_1 \cap \Lambda_2) \cap \Lambda_3$ (Associativity)

(iv) $\Lambda_1 \cap (\Lambda_2 \cup \Lambda_3) = (\Lambda_1 \cap \Lambda_2) \cup (\Lambda_1 \cap \Lambda_3)$ and $\Lambda_1 \cup (\Lambda_2 \cap \Lambda_3) = (\Lambda_1 \cup \Lambda_2) \cap (\Lambda_1 \cup \Lambda_3)$

(Distributivity)

Proof. All proofs are straightforward.

4. Linear Diophantine single-valued neutrosophic weighted arithmetic and geometric aggregation(LDSVNWAGA) operator

In this section, we describe the score and accuracy function for the comparative analysis in MADM of LDNNs. The notion of score and accuracy function of neutrosophic numbers proposed by Smarandache in [52]. However, hybrid arithmetic and geometric aggregation operators of single-valued neutrosophic numbers are proposed in [53].

Definition 4.1

Let $\Lambda_Q = \left(\left\langle {}^Q\mu_{ND}, {}^Q\nu_{ND}, {}^Q\gamma_{ND} \right\rangle, \left\langle {}^Q\alpha, {}^Q\delta, {}^Q\beta \right\rangle \right)$ be a LDNN, then the score function(SF) on Λ_Q can be defined by the mapping $\varphi : LDNN(Q) \rightarrow [0,1]$ and given by

$$\varphi(\Lambda_Q) = \varphi_{\Lambda_Q} = \frac{1}{3}\left[\left(2 + {}^Q\mu_{ND} - {}^Q\nu_{ND} - {}^Q\gamma_{ND}\right) + \left(2 + {}^Q\alpha - {}^Q\delta - {}^Q\beta\right) \right]$$

where $LDNN(Q)$ is an assembling of $LDNNs$ on Q.

Definition 4.2

Let $\Lambda_Q = \left(\left\langle {}^Q\mu_{ND}, {}^Q\nu_{ND}, {}^Q\gamma_{ND} \right\rangle, \left\langle {}^Q\alpha, {}^Q\delta, {}^Q\beta \right\rangle \right)$ be a LDNN, then the accuracy function(AF) on Λ_Q can be defined by the mapping $\psi : LDNN(Q) \rightarrow [-1,1]$ and given by

$$\psi(\Lambda_Q) = \psi_{\Lambda_Q} = \frac{1}{3}\left[\left({}^Q\mu_{ND} - {}^Q\gamma_{ND}\right) + \left({}^Q\alpha - {}^Q\beta\right) \right]$$

Definition 4.3

Let Λ_{Q_1} and Λ_{Q_2} be two LDNNs, then on the context of SF and AF we can compare the two LDNNs as follows:

(i) If $\varphi_{\Lambda_{Q_1}} < \varphi_{\Lambda_{Q_2}}$, then $\Lambda_{Q_1} < \Lambda_{Q_2}$

(ii) If $\varphi_{\Lambda_{Q_1}} > \varphi_{\Lambda_{Q_2}}$, then $\Lambda_{Q_1} > \Lambda_{Q_2}$

(iii) If $\varphi_{\Lambda_{Q_1}} = \varphi_{\Lambda_{Q_2}}$ then,

(a) If $\psi_{\Lambda_{Q_1}} < \psi_{\Lambda_{Q_2}}$ then $\Lambda_{Q_1} < \Lambda_{Q_2}$

(b) If $\psi_{\Lambda_{Q_1}} > \psi_{\Lambda_{Q_2}}$ then $\Lambda_{Q_1} > \Lambda_{Q_2}$

(c) If $\psi_{\Lambda_{Q_1}} = \psi_{\Lambda_{Q_2}}$ then $\Lambda_{Q_1} \approx \Lambda_{Q_2}$

Definition 4.4

Let $\Lambda_{ND_\tau} = \left\{ \left(\left\langle {}^{\tau}\mu_{ND}, {}^{\tau}\nu_{ND}, {}^{\tau}\gamma_{ND} \right\rangle, \left\langle {}^{\tau}\alpha, {}^{\tau}\delta, {}^{\tau}\beta \right\rangle \right) : \tau = 1, 2, ..., n \right\}$ be an assembling of LDNNs on the reference set $\circledcirc$ and $\omega = \left(\omega_1, \omega_2, ..., \omega_n\right)^T$ be the weight vector with $\sum_{\tau=1}^{n} \omega_\tau = 1$, then the linear Diophantine single-valued neutrosophic weighted Arithmetic geometric aggregation $(LDSVNWAGA)$ operator defined as

$$LDSVNWAGA\left(\Lambda_{ND_1}, \Lambda_{ND_2}, ..., \Lambda_{ND_n}\right) =$$

$$\left\langle \Sigma\left(\left(1 - \prod_{j=1}^{n} \left(1 - {}^{j}\mu_{ND}\right)^{\omega_j}\right)^{j_\alpha} \left(\prod_{j=1}^{n} {}^{j}\mu^{\omega_j}{}_{ND}\right)^{(1-j_\alpha)} \right), \Sigma\left(1 - \left(1 - \prod_{j=1}^{n} \left({}^{j}\nu_{ND}\right)^{\omega_j}\right)^{j_\delta} \left(\prod_{j=1}^{n} \left(1 - {}^{j}\nu_{ND}\right)^{\omega_j}\right)^{(1-j_\delta)} \right), \Sigma\left(1 - \left(1 - \prod_{j=1}^{n} \left({}^{j}\gamma_{ND}\right)^{\omega_j}\right)^{j_\beta} \left(\prod_{j=1}^{n} \left(1 - {}^{j}\gamma_{ND}\right)^{\omega_j}\right)^{(1-j_\beta)} \right) \right\rangle$$

5. An algorithmic approach

For mathematical modeling, we construct an algorithm that is based on LDNNs. The steps of the algorithm are given in the following:

Algorithm:

Step1: Input the opinion of the expert's $\wp_l\ (l = 1, 2, ..., n)$ in the form of LDNNs for each attribute.

Step2: Input the weight vector of the experts.

Step3: Calculate the aggregate value of each attribute by using *LDSVNWAGA* operator proposed in definition 4.4

Step4: Find the total weight of the aggregate value of each alternative.

Step5: Rank the weight in ascending order and choose the attribute having the highest weight. If more than one attributes having the same weight then we repeat all the previous steps by reassessing the expert's opinion.

6. Numerical Example

In this section, we cite an example of the real world that helps to realize the importance of LDNNs in real decision-making problems. We consider the following example:

Suppose that Mr. Advik, together want to invest their money in any one of investment plans belong to the set given by

$\zeta = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5\}$

Where

ζ_1 =Monthly Income Plan(MIP)

ζ_2 ◎Mutual Fund(MF)

ζ_3 ◎Public Provident Fund(PPF)

ζ_4 ◎Life Insurance Plan(LIP)

ζ_5 ◎Unit Linked Insurance Plan(ULIP)

According to the performance of the above investment plans, there associated three risk factors, they are denoted by the set of three reference parameters given by

$\xi = \{\alpha, \beta, \gamma\}$, where α ◎low-risk investment, β ◎medium-risk investment, and γ ◎high-risk investment.

To choose the best investment scheme influenced by three risk factors, Mr. Advik takes the advice of three experts(decision-makers) denoted by the set

$\omega = \{\omega_1, \omega_2, \omega_3\}$.

The set of LDNNs of the set of attributes of the three experts are shown below in the form of the following tables:

Alternatives	LDNSs
ζ_1	$(\langle 0.8, 0.9, 0.7\rangle, \langle 0.7, 0.8, 0.6\rangle)$
ζ_2	$(\langle 0.5, 0.6, 0.8\rangle, \langle 0.9, 0.7, 0.8\rangle)$
ζ_3	$(\langle 0.7, 0.6, 0.9\rangle, \langle 0.5, 0.8, 0.6\rangle)$
ζ_4	$(\langle 0.7, 0.6, 0.4\rangle, \langle 0.3, 0.9, 0.6\rangle)$
ζ_5	$(\langle 0.9, 0.8, 0.6\rangle, \langle 0.8, 0.6, 0.7\rangle)$

Table2. LDNS for ω_1

Alternatives	LDNSs
ζ_1	$(\langle 0.6, 0.8, 0.8\rangle, \langle 0.9, 0.5, 0.8\rangle)$
ζ_2	$(\langle 0.4, 0.3, 0.7\rangle, \langle 0.8, 0.9, 0.6\rangle)$

ζ_3	$(\langle 0.9, 0.7, 0.8\rangle, \langle 0.7, 0.9, 0.7\rangle)$
ζ_4	$(\langle 0.7, 0.8, 0.7\rangle, \langle 0.8, 0.6, 0.8\rangle)$
ζ_5	$(\langle 0.9, 0.7, 0.4\rangle, \langle 0.5, 0.8, 0.6\rangle)$

Table3. LDNS for ω_2

Alternatives	LDNSs
ζ_1	$(\langle 0.7, 0.6, 0.7\rangle, \langle 0.7, 0.5, 0.8\rangle)$
ζ_2	$(\langle 0.8, 0.7, 0.6\rangle, \langle 0.7, 0.6, 0.5\rangle)$
ζ_3	$(\langle 0.8, 0.7, 0.9\rangle, \langle 0.6, 0.7, 0.8\rangle)$
ζ_4	$(\langle 0.8, 0.7, 0.6\rangle, \langle 0.6, 0.8, 0.7\rangle)$
ζ_5	$(\langle 0.8, 0.6, 0.6\rangle, \langle 0.7, 0.5, 0.8\rangle)$

Table4. LDNS for ω_3

According to the experience of the experts, we consider the weight vector as

$$\omega = (0.3,\ 0.4,\ 0.3)$$

Now, the aggregate value of each alternative, by using *LDSVNWAGA* operator is given by:

$$LDSVNWAGA(\zeta_1) = (2.0938,\ 0.9499,\ 1.3212)$$

$$LDSVNWAGA(\zeta_2) = (1.7335,\ 0.9318,\ 1.031)$$

$$LDSVNWAGA(\zeta_3) = (2.4582,\ 1.3853,\ 1.2407)$$

$$LDSVNWAGA(\zeta_4) = (2.0399,\ 1.3804,\ 1.0146)$$

$$LDSVNWAGA(\zeta_5) = (2.6224,\ 1.031,\ 0.9207)$$

Next, we calculate the total weight of the aggregate values of all the alternatives given by,

$$\varpi(\zeta_1) \circledcirc 4.3649$$

$$\varpi(\zeta_2) \circledcirc 3.6963$$

$$\varpi(\zeta_3) \circledcirc 5.0842$$

$\varpi(\zeta_4)$◎4.4349

$\varpi(\zeta_5)$◎4.5741

The rank of the total weight in ascending order is given by

$$\varpi(\zeta_2) \prec \varpi(\zeta_1) \prec \varpi(\zeta_4) \prec \varpi(\zeta_5) \prec \varpi(\zeta_3)$$

From the ascending order of the rank, we observe that ζ_3 has the highest value. Thus, we conclude that Mr. Advik will select Public Provident Fund to invest his money and earn the maximum return in the future.

Thus, by using the reference parameters in LDNS, we can handle another particular class of neutrosophic data.

7. Comparison Analysis of LDNS model with the existing models in the literature

Types of set	**Uncertainty**	**Falsity**	**Indeterminacy**	**Hesitancy**	**Parametrization**
FS◎1◎	✓	×	×	×	×
IFS◎9◎	✓	✓	×	✓	×
PFS◎18◎	✓	✓	×	✓	×
q-ROFS◎20◎	✓	✓	×	✓	×
SVNS◎45◎	✓	✓	✓	×	×
PNS◎37,38◎	✓	✓	✓	✓	×
LDFS◎39◎	✓	✓	×	✓	✓
LDNS(Proposed)	✓	✓	✓	✓	✓

Table5. Comparison analysis of LDNS model with the existing models in the literature

8. Conclusion and Future Scope

In this work, we have introduced the notion of LDNS which can be viewed as an extension of FS, IFS, PFS, q-ROFS, PNS, etc. LDNS is a new structure that deals with uncertainty and indeterminacy with the support of reference parameters. The LDNS model can transform the problem related to the physical world into numerical form due to its parametric nature. Therefore, it provides more flexibility to handle uncertainty as compared to the existing theories. We have discussed some properties of LDNSs. For comparison of LDNNs, we have defined score and accuracy functions. Moreover, we have introduced *LDSVNWAGA* operator for solving MADM problems with the help of an algorithm. We have presented an illustrative example to give a practical application of the proposed method. Finally, we have presented the comparative analysis of the proposed model and the existing models which gives a clear picture to the researchers of the importance of this study and it will surely motivate them to enrich the present study by introducing many other important theories and results associated to LDNNs and apply them in various practical applications.

In the future, we hope that there is a huge scope for the researchers and the policymakers(decision-makers) to explore several practical real-world applications related to topics based on linear Diophantine interval neutrosophic set(LDINS), linear Diophantine neutrosophic rough set(LDNRS), linear Diophantine neutrosophic graph(LDNG), linguistic linear Diophantine neutrosophic set(LLDNS), linear Diophantine hesitant neutrosophic set(LDHNS). The proposed study may be further extended by introducing TOPSIS, VIKOR, AHP, aggregate operators, several distance-based similarity measures.

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On Some Estimation Methods of Neutrosophic Continuous Probability Distributions Using One-Dimensional AH-Isometry

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Abstract: In this research, we introduce an algebraic approach to define the concept of neutrosophic maximum likelihood estimation method based on neutrosophic continuous probability distributions based on classical neutrosophic numbers of the form $N = a + bI; I^2 = I$ i.e., I is a letter not a numerical set. We prove that the neutrosophic loglikelihood function gives the same estimators given by neutrosophic likelihood function. Also, we present the concept of neutrosophic moments estimation method which produces system of neutrosophic equations to derive the neutrosophic estimators using an algebraic isomorphism. Estimators based on two mentioned methods were derived successfully for some neutrosophic continuous probability distributions. Concept of neutrosophic Fisher information is also presented. Theorems were proved using an algebraic approach depending on the one-dimensional AH-Isometry. A simulation study is also presented to show the efficiency of the presented estimators.

Keywords: AH Isometry; Neutrosophic Field of Reals; Maximum Likelihood; Moments; Probability Density Functions; Neutrosophic Fisher Information.

1. Introduction

Neutrosophic field of reals is an extension to field of reals adding new algebraic structure I satisfies $I^2 = I$ so we get $R(I) = R \cup \{I\} = \{a + bI\ ; a, b \in R\ , I^2 = I\}$ which is neutrosophic field of reals. [1]

Many mathematical studies were done based on the neutrosophic set of reals in many fields of abstract mathematics including abstract algebra, probability theory, topology, number theory, etc.[2-7].

In [8] Abobala and Hatip presented an isometry called AH-Isometry which transfers mathematical problems from $R(I)$ to $R \times R$ and an inverse isometry transfers the mathematical problem from $R \times R$ to $R(I)$. This isometry is very applicable to solve and study many types of mathematical problems including real analysis, complex analysis, algebraic structures, probability theory, operations research, etc.

Many previous studies about neutrosophic probability theory were done assuming that parameters of probability distribution functions are indeterminant, i.e. parameter θ is an interval neutrosophic number, so it can be noted by $\theta_N \in [\theta^L, \theta^U]$.[9-15]

In [16], [17] Zahid Khan, Sultan Salem et al. presented neutrosophic lognormal model and studied its critical properties then applied this model to environmental data and in lifetime data where they treated problems with interval neutrosophic numbers, in [18] Zahid Khan et al. presented neutrosophic gamma distribution and applied it to a real dataset for the purpose of dealing with inaccurate statistical data which is also described by interval neutrosophic numbers. Many other extensions were done to other types of distributions like neutrosophic exponential distribution, neutrosophic maxwell distribution, etc. and these extensions were applied successfully in may real datasets. [19-21]. Notice that all the mentioned studies are done by using neutrosophic interval numbers $N = d + I$ where $I \in [a, b]$ and not neutrosophic classical numbers of the form $N = a + bI, I^2 = I$ and this is the main difference between our study and the previous studies, so, we are going to study neutrosophic probability distribution assuming that there is uncertainty in its parameters and the random variable itself, i.e. $f(x;\theta)$ is $f(x_N;\theta_N)$ based on its algebraic structure, i.e. $x_N = x + yI, \theta_N = \theta_1 + \theta_2 I ; x, y, \theta_1, \theta_2 \in R, I^2 = I, 0 \cdot I = 0$ by using AH-Isometry which will transfer each neutrosophic probability density function into two crisp probability distribution functions. Based on this transformation we have successfully defined the neutrosophic log-likelihood function and studied its properties then found estimators of neutrosophic probability distributions based on maximum likelihood estimation method, also on the same algorithm we have succeed to define moments estimation method and finally the neutrosophic fisher information about the estimated parameters. One can also define many other estimation methods based on the same algorithm presented in this paper. Many examples were solved successfully and estimators of many neutrosophic probability distributions were successfully derived.

2. Preliminaries

Definition 2.1 [5] [8]

Let $R(I) = \{a + bI ; a, b \in R, I^2 = I, 0 \cdot I = 0\}$ be the neutrosophic field of reals. The one-dimensional AH-isometry is defined as follows:

$$T: R(I) \to R^2 : T(a + bI) = (a, a + b) \qquad (1)$$

And its inverse is defined as follows:

$$T^{-1}: R^2 \to R(I) : T^{-1}(a, b) = a + (b - a)I \qquad (2)$$

Remark:

We will call the form $a + bI$ the formal of a neutrosophic number.

Definition 2.2 [6]

Let $f: R(I) \to R(I); f = f(x_N)$ where $x_N = x + yI \in R(I)$ then f is called a neutrosophic real function with one neutrosophic variable.

Definition 2.3 [1]

A neutrosophic random variable can be defined as follows: [6] [22]

$$X_N = X + YI ; I^2 = I, 0 \cdot I = 0 \qquad (3)$$

Where X, Y are crisp random variables taking values on R.

Definition 2.4 [4]

Let $R(I)$ be the neutrosophic field of reals, and let $a_N = a_1 + a_2 I, b_N = b_1 + b_2 I \in R(I)$. We say that $a_N \geq_N b_N$ iff:

$$a_1 \geq b_1 \text{ and } a_1 + a_2 \geq b_1 + b_2$$

Definition 2.5 [22]

Let $R(I)$ be the neutrosophic field of reals, the neutrosophic logarithmic function can be defined as:

$$ln(x + yI) = ln\, x + [ln(x + y) - ln(x)]I, \text{ where } x + yI >_N 0.$$

3. Results and Discussion

Definition 3.1

Suppose that $\mathbb{X}_N = X_{1N}, X_{2N}, \dots, X_{nN}$ is a sequence of neutrosophic random variables, we say that $\mathbb{X}_N$ is a neutrosophic random sample drawn from neutrosophic random variable X_N if $X_{1N}, X_{2N}, \dots, X_{nN}$ are dependent and have the same probability distribution as X_N.

Definition 3.2

Let $\mathbb{X}_N$ be a random sample drawn from X_N, we call the function:

$$L_N = L(\mathbb{X}_N; \Theta_N) = f(\mathbb{X}_N; \Theta_N) = \prod_{i=1}^{n} f(X_{iN}; \Theta_N) \quad (4)$$

The neutrosophic likelihood function where $\Theta_N = \Theta_1 + \Theta_2 I = (\theta_{1N}, \theta_{2N}, \dots, \theta_{rN})$ is a vector of unknown parameters.

Theorem 1

The formal form of neutrosophic likelihood function L_N is:

$$L_N = L(\mathbb{X}_N; \Theta_N) = L(\mathbb{X}; \Theta_1) + [L(\mathbb{X} + \mathbb{Y}; \Theta_1 + \Theta_2) - L(\mathbb{X}; \Theta_1)]I \quad (5)$$

Proof:

Using the one-dimensional AH-Isometry:

$$T\big(L(\mathbb{X}_N; \Theta_N)\big) = T\left(\prod_{i=1}^{n} f(x_i + y_i I; \Theta_1 + \Theta_2 I)\right)$$

$$= \prod_{i=1}^{n} f\big((x_i, x_i + y_i); (\Theta_1, \Theta_1 + \Theta_2)\big)$$

$$= \left(\prod_{i=1}^{n} f(x_i; \Theta_1), \prod_{i=1}^{n} f(x_i + y_i; \Theta_1 + \Theta_2)\right) \quad (6)$$

Now taking the inverse isometry T^{-1}:

$$L(\mathbb{X}_N; \Theta_N) = T^{-1}\left(\left(\prod_{i=1}^{n} f(x_i; \Theta_1), \prod_{i=1}^{n} f(x_i + y_i; \Theta_1 + \Theta_2)\right)\right)$$

$$= \prod_{i=1}^{n} f(x_i, \Theta_1) + \left[\prod_{i=1}^{n} f(x_i + y_i; \Theta_1 + \Theta_2) - \prod_{i=1}^{n} f(x_i, \Theta_1)\right] I$$

$$= L(\mathbb{X}; \Theta_1) + [L(\mathbb{X} + \mathbb{Y}; \Theta_1 + \Theta_2) - L(\mathbb{X}; \Theta_1)]I \quad (7)$$

Definition 3.3

We call $\mathcal{L}_N = ln\, L(\mathbb{X}_N; \Theta_N)$ the neutrosophic loglikelihood function.

Theorem 2

The formal form of neutrosophic loglikelihood function is:

$$\mathcal{L}_N = \mathcal{L}(\mathbb{X}; \Theta_1) + [\mathcal{L}(\mathbb{X} + \mathbb{Y}; \Theta_1 + \Theta_2) - \mathcal{L}(\mathbb{X}; \Theta_1)]I \quad (8)$$

Proof:

Similar to theorem 1.

Definition 3.4

The neutrosophic statistic $\widehat{\Theta}_N$ based on random sample that maximize the neutrosophic likelihood function is called the neutrosophic likelihood estimator.

Theorem 3

The neutrosophic statistic based on random sample that maximize the neutrosophic likelihood function is the same statistic that maximize the neutrosophic loglikelihood function.

Proof:

The neutrosophic statistic $\widehat{\Theta}_N$ that maximize the likelihood function fulfills the following conditions:

$$\frac{\partial}{\partial \Theta_N} L(\mathbb{X}_N;\Theta_N)|_{\Theta_N=\widehat{\Theta}_N} = 0 \,, \frac{\partial^2}{\partial \Theta_N^2} L(\mathbb{X}_N;\Theta_N)|_{\Theta_N=\widehat{\Theta}_N} <_N 0 \qquad (9)$$

Using theorem 1 the conditions become:

$$\frac{\partial L(\mathbb{X};\Theta_1)|_{\Theta_1=\widehat{\Theta}_1}}{\partial \Theta_1} + \left[\frac{\partial L(\mathbb{X}+\mathbb{Y};\Theta_1+\Theta_2)|_{\Theta_1+\Theta_2=\widehat{\Theta}_1+\widehat{\Theta}_2}}{\partial(\Theta_1+\Theta_2)} - \frac{\partial L(\mathbb{X};\Theta_1)|_{\Theta_1=\widehat{\Theta}_1}}{\partial \Theta_1}\right] I = 0 \qquad (10)$$

Which means that:

$$\frac{\partial}{\partial \Theta_1} L(\mathbb{X};\Theta_1)|_{\Theta_1=\widehat{\Theta}_1} = 0 \qquad (11)$$

$$\frac{\partial}{\partial(\Theta_1+\Theta_2)} L(\mathbb{X}+\mathbb{Y};\Theta_1+\Theta_2)|_{\Theta_1+\Theta_2=\widehat{\Theta}_1+\widehat{\Theta}_2} = 0 \qquad (12)$$

The same to the second condition which yields to:

$$\frac{\partial^2}{\partial \Theta_1^2} L(\mathbb{X};\Theta_1)|_{\Theta_1=\widehat{\Theta}_1} < 0 \qquad (13)$$

$$\frac{\partial^2}{\partial(\Theta_1+\Theta_2)^2} L(\mathbb{X}+\mathbb{Y};\Theta_1+\Theta_2)|_{\Theta_1+\Theta_2=\widehat{\Theta}_1+\widehat{\Theta}_2} < 0 \qquad (14)$$

If we apply the same conditions to the neutrosophic loglikelihood function we get:

$$\frac{\partial}{\partial \Theta_N}\mathcal{L}_N = \frac{\partial \mathcal{L}(\mathbb{X};\Theta_1)|_{\Theta_1=\widehat{\Theta}_1}}{\partial \Theta_1} + \left[\frac{\partial \mathcal{L}(\mathbb{X}+\mathbb{Y};\Theta_1+\Theta_2)|_{\Theta_1+\Theta_2=\widehat{\Theta}_1+\widehat{\Theta}_2}}{\partial(\Theta_1+\Theta_2)} - \frac{\partial \mathcal{L}(\mathbb{X};\Theta_1)|_{\Theta_1=\widehat{\Theta}_1}}{\partial \Theta_1}\right] I = 0 \qquad (15)$$

Since $\mathcal{L}_N = \ln L_N$, we know that $\frac{\partial}{\partial \Theta_N}\mathcal{L}_N = \frac{\partial}{\partial \Theta_N} \ln L_N = \frac{\frac{\partial}{\partial \Theta_N} L_N}{L_N}$

So, the first condition become:

$$\frac{\frac{\partial}{\partial \Theta_1} L(\mathbb{X};\Theta_1)}{L(\mathbb{X};\Theta_1)}|_{\Theta_1=\widehat{\Theta}_1} + \left[\frac{\frac{\partial}{\partial(\Theta_1+\Theta_2)} L(\mathbb{X}+\mathbb{Y};\Theta_1+\Theta_2)}{L(\mathbb{X}+\mathbb{Y};\Theta_1+\Theta_2)}|_{\Theta_1+\Theta_2=\widehat{\Theta}_1+\widehat{\Theta}_2} - \frac{\frac{\partial}{\partial \Theta_1} L(\mathbb{X};\Theta_1)}{L(\mathbb{X};\Theta_1)}|_{\Theta_1=\widehat{\Theta}_1}\right] I = 0 \quad (16)$$

Which means that both following equations hold:

$$\frac{\frac{\partial}{\partial \Theta_1} L(\mathbb{X};\Theta_1)}{L(\mathbb{X};\Theta_1)}|_{\Theta_1=\widehat{\Theta}_1} = 0 \qquad (17)$$

$$\frac{\frac{\partial}{\partial(\Theta_1+\Theta_2)} L(\mathbb{X}+\mathbb{Y};\Theta_1+\Theta_2)}{L(\mathbb{X}+\mathbb{Y};\Theta_1+\Theta_2)}|_{\Theta_1+\Theta_2=\widehat{\Theta}_1+\widehat{\Theta}_2} = 0 \qquad (18)$$

And this yields to:

$$\frac{\partial}{\partial \Theta_1} L(\mathbb{X};\Theta_1)|_{\Theta_1=\widehat{\Theta}_1} = 0 \qquad (19)$$

$$\frac{\partial}{\partial(\Theta_1+\Theta_2)} L(\mathbb{X}+\mathbb{Y};\Theta_1+\Theta_2)|_{\Theta_1+\Theta_2=\widehat{\Theta}_1+\widehat{\Theta}_2} = 0 \qquad (20)$$

And these are the same equations as (11), (12).

Same proof can be applied to the second condition.

Example 1:

Let $X_{1N}, X_{2N}, \dots, X_{nN}$ be a neutrosophic random sample drawn from the density of neutrosophic power distribution:

$$f(x_N;\theta_N) = \theta_N x_N^{\theta_N-1}; 0 \le_N x_N \le_N 1$$

Let's take AH-Isometry to $f(x_N;\theta_N)$:

$$\begin{aligned} T[f(x_N;\theta_N)] &= T\left[\theta_N x_N^{\theta_N-1}\right] = T\left[(\theta_1+\theta_2 I)(x+yI)^{(\theta_1+\theta_2 I)-1}\right] \\ &= T[(\theta_1+\theta_2 I)]T\left[(x+yI)^{(\theta_1+\theta_2 I)-1}\right] \\ &= T[(\theta_1+\theta_2 I)]T[(x+yI)]^{T[(\theta_1+\theta_2 I)-1]} \\ &= (\theta_1,\theta_1+\theta_2)(x,x+y)^{(\theta_1,\theta_1+\theta_2)-(1,1)} \\ &= \left(\theta_1 x^{\theta_1-1}, (\theta_1+\theta_2)(x+y)^{(\theta_1+\theta_2)-1}\right) \\ &= \left(f(x;\theta_1), f(x+y;\theta_1+\theta_2)\right) \end{aligned}$$

So, by applying equation (15) considering properties of probability density functions we get:

$$T\left[\frac{\partial}{\partial\theta_N} ln\, L(\mathbb{X}_N;\theta_N)\right] = \left(\sum_{i=1}^{n}\frac{\partial}{\partial\theta_1} ln\, f(x_i;\theta_1), \sum_{i=1}^{n}\frac{\partial}{\partial(\theta_1+\theta_2)} ln\, f(x_i+y_i;\theta_1+\theta_2)\right)$$

$$= \left(\sum_{i=1}^{n}\frac{\partial}{\partial\theta_1} ln\left(\theta_1 x_i^{\theta_1-1}\right), \sum_{i=1}^{n}\frac{\partial}{\partial(\theta_1+\theta_2)} ln\left((\theta_1+\theta_2)(x_i+y_i)^{(\theta_1+\theta_2)-1}\right)\right)$$

$$= \left(\sum_{i=1}^{n}\frac{\partial\, ln\,\theta_1}{\partial\theta_1} + \frac{\partial(\theta_1-1)\, ln\, x_i}{\partial\theta_1}, \sum_{i=1}^{n}\frac{\partial\, ln(\theta_1+\theta_2)}{\partial(\theta_1+\theta_2)} + \frac{\partial(\theta_1+\theta_2-1)\, ln(x_i+y_i)}{\partial(\theta_1+\theta_2)}\right)$$

$$= \left(\sum_{i=1}^{n}\frac{1}{\theta_1} + ln\, x_i, \sum_{i=1}^{n}\frac{1}{\theta_1+\theta_2} + ln(x_i+y_i)\right) = \left(\frac{n}{\theta_1} + \sum_{i=1}^{n} ln\, x_i, \frac{n}{\theta_1+\theta_2} + \sum_{i=1}^{n} ln(x_i+y_i)\right)$$

$$T\left[\frac{\partial}{\partial\theta_N} ln\, L(\mathbb{X}_N;\theta_N)\right] = T[0]$$

$$\left(\frac{n}{\hat\theta_1} + \sum_{i=1}^{n} ln\, x_i, \frac{n}{\hat\theta_1+\hat\theta_2} + \sum_{i=1}^{n} ln(x_i+y_i)\right) = (0,0)$$

$$\left(\hat\theta_1, \hat\theta_1+\hat\theta_2\right) = \left(-\frac{n}{\sum_{i=1}^{n} ln\, x_i}, -\frac{n}{\sum_{i=1}^{n} ln(x_i+y_i)}\right)$$

$$T^{-1}\left(\hat\theta_1, \hat\theta_1+\hat\theta_2\right) = T^{-1}\left(-\frac{n}{\sum_{i=1}^{n} ln\, x_i}, -\frac{n}{\sum_{i=1}^{n} ln(x_i+y_i)}\right)$$

$$\Rightarrow \hat\theta_N = -\frac{n}{\sum_{i=1}^{n} ln\, x_i} + \left[-\frac{n}{\sum_{i=1}^{n} ln(x_i+y_i)} + \frac{n}{\sum_{i=1}^{n} ln\, x_i}\right] I$$

Example 2:

Let $X_{1N}, X_{2N}, \dots, X_{nN}$ be a neutrosophic random sample drawn from the density of neutrosophic Maxwell distribution:

$$f(x_N;\theta_N) = \sqrt{\frac{2}{\pi}}\,\theta_N^{\frac{3}{2}} x_N^2 e^{-\frac{1}{2}\theta_N x_N^2};\, x_N >_N 0$$

So:

$$\left(\sum_{i=1}^{n}\frac{\partial}{\partial\theta_1} ln\, f(x_i;\theta_1), \sum_{i=1}^{n}\frac{\partial}{\partial(\theta_1+\theta_2)} ln\, f(x_i+y_i;\theta_1+\theta_2)\right) = (0,0)$$

$$\left(\sum_{i=1}^{n}\frac{\partial}{\partial\theta_1} ln\left(\sqrt{\frac{2}{\pi}}\,\theta_1^{\frac{3}{2}} x^2 e^{-\frac{1}{2}\theta_1 x^2}\right), \sum_{i=1}^{n}\frac{\partial}{\partial(\theta_1+\theta_2)} ln\left(\sqrt{\frac{2}{\pi}}(\theta_1+\theta_2)^{\frac{3}{2}}(x+y)^2 e^{-\frac{1}{2}(\theta_1+\theta_2)(x+y)^2}\right)\right) = (0,0)$$

$$\left(\frac{3n}{2\hat\theta_1} - \frac{1}{2}\sum_{i=1}^{n} x_i^2, \frac{3n}{2\left(\hat\theta_1+\hat\theta_2\right)} - \sum_{i=1}^{n}(x_i+y_i)^2\right) = (0,0)$$

$$\left(\hat\theta_1, \hat\theta_1+\hat\theta_2\right) = \left(\frac{3n}{\sum_{i=1}^{n} x_i^2}, \frac{3n}{\sum_{i=1}^{n}(x_i+y_i)^2}\right)$$

Taking T^{-1}:

$$T^{-1}\left(\hat\theta_1, \hat\theta_1+\hat\theta_2\right) = T^{-1}\left(\frac{3n}{\sum_{i=1}^{n} x_i^2}, \frac{3n}{\sum_{i=1}^{n}(x_i+y_i)^2}\right)$$

$$\hat\theta_N = \frac{3n}{\sum_{i=1}^{n} x_i^2} + \left[\frac{3n}{\sum_{i=1}^{n}(x_i+y_i)^2} - \frac{3n}{\sum_{i=1}^{n} x_i^2}\right] I$$

Example 3:

Let $X_{1N}, X_{2N}, \dots, X_{nN}$ be a neutrosophic random sample drawn from the density of neutrosophic exponential distribution:

$$f(x_N;\theta_N) = \frac{1}{\theta_N} e^{-\frac{x_N}{\theta_N}} \quad ; \quad x_N >_N 0$$

So:

$$\left(\sum_{i=1}^{n}\frac{\partial}{\partial\theta_1} ln\, f(x_i;\theta_1), \sum_{i=1}^{n}\frac{\partial}{\partial(\theta_1+\theta_2)} ln\, f(x_i+y_i;\theta_1+\theta_2)\right) = (0,0)$$

$$\left(\sum_{i=1}^{n}\frac{\partial}{\partial\theta_1}ln\frac{1}{\theta_1}e^{-\frac{x}{\theta_1}},\sum_{i=1}^{n}\frac{\partial}{\partial(\theta_1+\theta_2)}ln\frac{1}{\theta_1+\theta_2}e^{-\frac{(x+y)}{(\theta_1+\theta_2)}}\right)=(0,0)$$
$$\left(-\frac{n}{\hat{\theta}_1}+\frac{n\bar{X}}{\theta_1^2},-\frac{n}{\hat{\theta}_1+\hat{\theta}_2}+\frac{n\bar{X}+n\bar{Y}}{\left(\hat{\theta}_1+\hat{\theta}_2\right)^2}\right)=(0,0)$$
$$\left(\hat{\theta}_1,\hat{\theta}_1+\hat{\theta}_2\right)=(\bar{X},\bar{X}+\bar{Y})$$
$$T^{-1}\left(\hat{\theta}_1,\hat{\theta}_1+\hat{\theta}_2\right)=T^{-1}(\bar{X},\bar{X}+\bar{Y})$$
$$\Rightarrow\hat{\theta}_N=\bar{X}+\bar{Y}I$$

Definition 3.5

Let X_N be a neutrosophic random variable, we call $\alpha_{kN}=E(X_N^k)$ the k^th^ moment of the neutrosophic random variable X_N.

Definition 3.6

Let $X_{1N},X_{2N},\dots,X_{nN}$ be a neutrosophic random sample drawn from the neutrosophic random variable X_N, we call $A_{kN}(X)=\frac{1}{n}\sum_{i=1}^{n}X_{iN}^k$ the sample moment of order k.

Definition 3.7

The parameter that satisfies the following system of equations:

$$\alpha_{kN}=A_{kN}(X) \qquad (21)$$

Is called the moments estimator where k is the number of unknown parameters.

Theorem 4

Equations (21) can be written in R^2 in the following form:

$$\left(\int_{-\infty}^{+\infty}x^kf(x;\theta_1)\,dx,\int_{-\infty}^{+\infty}(x+y)^kf(x+y;\theta_1+\theta_2)\,d(x+y)\right)=\left(\frac{1}{n}\sum_{i=1}^{n}x_i^k,\frac{1}{n}\sum_{i=1}^{n}(x_i+y_i)^k\right) \qquad (22)$$

Proof:

$$\alpha_{kN}=\alpha_{kN}(\theta_N)=E(X_N^k)=\int_{-\infty}^{+\infty}x_N^kf(x_N;\theta_N)\,dx_N \qquad (23)$$

Taking AH-Isometry:

$$T[\alpha_{kN}]=T[E(X_N^k)]=T\left[\int_{-\infty}^{+\infty}x_N^kf(x_N;\theta_N)\,dx_N\right]$$
$$=\left(\int_{-\infty}^{+\infty}x^kf(x;\theta_1)\,dx,\int_{-\infty}^{+\infty}(x+y)^kf(x+y;\theta_1+\theta_2)\,d(x+y)\right) \qquad (24)$$

Also:

$$A_{kN}(X)=\frac{1}{n}\sum_{i=1}^{n}X_{iN}^k \qquad (25)$$

And taking the AH-Isometry:

$$T[A_{kN}(X)]=T\left[\frac{1}{n}\sum_{i=1}^{n}X_{iN}^k\right]=\left(\frac{1}{n}\sum_{i=1}^{n}x_i^k,\frac{1}{n}\sum_{i=1}^{n}(x_i+y_i)^k\right) \qquad (26)$$

Equations (24) and (26) proves the theorem.

e.g., for one parameter, we substitute $k=1$:

$$\left(\int_{-\infty}^{+\infty}xf(x;\theta_1)\,dx,\int_{-\infty}^{+\infty}(x+y)f(x+y;\theta_1+\theta_2)\,d(x+y)\right)=(\bar{X},\bar{X}+\bar{Y}) \qquad (27)$$

for two parameters, we substitute $k=2$:

$$\left(\int_{-\infty}^{+\infty}x^2f(x;\theta_1)\,dx,\int_{-\infty}^{+\infty}(x+y)^2f(x+y;\theta_1+\theta_2)\,d(x+y)\right)=\left(\frac{1}{n}\sum_{i=1}^{n}x_i^2,\frac{1}{n}\sum_{i=1}^{n}(x_i+y_i)^2\right) \qquad (28)$$

And so on.

Example 4:

Let $X_{1N},X_{2N},\dots,X_{nN}$ be a neutrosophic random sample drawn from the density given in example 1, then to find the moments estimator we have to solve the equation:

$$\alpha_{1N}=A_{1N}$$
$$\left(\int_{-\infty}^{+\infty}xf(x;\theta_1)\,dx,\int_{-\infty}^{+\infty}(x+y)f(x+y;\theta_1+\theta_2)\,d(x+y)\right)=(\bar{X},\bar{X}+\bar{Y})$$

$$\left(\int_0^1 x\theta_1 x^{\theta_1-1}\,dx, \int_0^1 (x+y)(\theta_1+\theta_2)(x+y)^{\theta_1+\theta_2-1}\,d(x+y)\right) = (\bar{X}, \bar{X}+\bar{Y})$$

$$\left(\frac{\hat{\theta}_1}{\hat{\theta}_1+1}, \frac{\hat{\theta}_1+\hat{\theta}_2}{\hat{\theta}_1+\hat{\theta}_2+1}\right) = (\bar{X}, \bar{X}+\bar{Y})$$

$$(\hat{\theta}_1, \hat{\theta}_1+\hat{\theta}_2) = \left(\frac{\bar{X}}{1-\bar{X}}, \frac{\bar{X}+\bar{Y}}{1-(\bar{X}+\bar{Y})}\right)$$

$$T^{-1}(\hat{\theta}_1, \hat{\theta}_1+\hat{\theta}_2) = T^{-1}\left(\frac{\bar{X}}{1-\bar{X}}, \frac{\bar{X}+\bar{Y}}{1-(\bar{X}+\bar{Y})}\right)$$

$$\hat{\theta}_N = \hat{\theta}_1 + \hat{\theta}_2 I = \frac{\bar{X}}{1-\bar{X}} + \left[\frac{\bar{X}+\bar{Y}}{1-(\bar{X}+\bar{Y})} - \frac{\bar{X}}{1-\bar{X}}\right] I$$

Example 5:

Let $X_{1N}, X_{2N}, \ldots, X_{nN}$ be a neutrosophic random sample drawn from the density given in example 2, then to find the moments estimator we have to solve the equation:
Using equation (27):

$$\left(\int_0^{+\infty} x\sqrt{\frac{2}{\pi}}\theta_1^{\frac{3}{2}} x^2 e^{-\frac{1}{2}\theta_1 x^2}\,dx, \int_0^{+\infty} (x+y)\sqrt{\frac{2}{\pi}}(\theta_1+\theta_2)^{\frac{3}{2}}(x+y)^2 e^{-\frac{1}{2}(\theta_1+\theta_2)(x+y)^2}\,d(x+y)\right) = (\bar{X}, \bar{X}+\bar{Y})$$

$$\left(\sqrt{\frac{8}{\pi\hat{\theta}_1}}, \sqrt{\frac{8}{\pi(\hat{\theta}_1+\hat{\theta}_2)}}\right) = (\bar{X}, \bar{X}+\bar{Y})$$

$$(\hat{\theta}_1, \hat{\theta}_1+\hat{\theta}_2) = \left(\frac{8}{\pi\bar{X}^2}, \frac{8}{\pi(\bar{X}+\bar{Y})^2}\right)$$

$$\hat{\theta}_N = T^{-1}\left(\frac{8}{\pi\bar{X}^2}, \frac{8}{\pi(\bar{X}+\bar{Y})^2}\right) = \frac{8}{\pi\bar{X}^2} + \left[\frac{8}{\pi(\bar{X}+\bar{Y})^2} - \frac{8}{\pi\bar{X}^2}\right] I$$

Example 6:

Let $X_{1N}, X_{2N}, \ldots, X_{nN}$ be a neutrosophic random sample drawn from the density given in example 3, then to find the moments estimator we have to solve the equation:
Using equation (27):

$$\left(\int_0^{+\infty} x\frac{1}{\theta_1} e^{-\frac{x}{\theta_1}}\,dx, \int_0^{+\infty} (x+y)\frac{1}{\theta_1+\theta_2} e^{-\frac{(x+y)}{(\theta_1+\theta_2)}}\,d(x+y)\right) = (\bar{X}, \bar{X}+\bar{Y})$$

$$(\hat{\theta}_1, \hat{\theta}_1+\hat{\theta}_2) = (\bar{X}, \bar{X}+\bar{Y})$$

$$T^{-1}(\hat{\theta}_1, \hat{\theta}_1+\hat{\theta}_2) = T^{-1}(\bar{X}, \bar{X}+\bar{Y})$$

$$\hat{\theta}_N = \bar{X} + \bar{Y} I$$

Definition 3.8

We call the partial derivative of neutrosophic log-likelihood function the neutrosophic score function and we denote it by:

$$U(\mathbb{X}_N; \theta_N) = \frac{\partial}{\partial\Theta_N}\mathcal{L}_N \qquad (29)$$

Remark:

Notice that equation (29) is a neutrosophic random sample since it is a function of $\mathbb{X}_N$.

Theorem 5

Expected value of neutrosophic score function is equal to zero.

Proof:

$$T\left[\int_{-\infty}^{+\infty} L(\mathbb{x}_N; \Theta_N)\,d\mathbb{x}_N\right] = T[1] \qquad (30)$$

Where $\int_{-\infty}^{+\infty} L(\mathbb{x}_N; \Theta_N)\,d\mathbb{x}_N = 1$ because $L(\mathbb{x}_N; \Theta_N)$ is a neutrosophic probability density function.

$$T\left[\frac{\partial}{\partial\Theta_N}\int_{-\infty}^{+\infty} L(\mathbb{x}_N; \Theta_N)\,d\mathbb{x}_N\right] = T\left[\frac{\partial}{\partial\Theta_N}1\right] \qquad (31)$$

$$\left(\frac{\partial}{\partial\Theta_1}\int_{-\infty}^{+\infty} L(\mathbb{x};\Theta_1)\,d\mathbb{x}, \frac{\partial}{\partial(\Theta_1+\Theta_2)}\int_{-\infty}^{+\infty} L(\mathbb{x}+\mathbb{y};\Theta_1+\Theta_2)\,d(\mathbb{x}+\mathbb{y})\right) = (0,0) \quad (32)$$

$$\left(\int_{-\infty}^{+\infty} \frac{\partial}{\partial\Theta_1} ln\, L(\mathbb{x};\Theta_1)\, L(\mathbb{x};\Theta_1)\, dx_1, \int_{-\infty}^{+\infty} \frac{\partial}{\partial(\Theta_1+\Theta_2)} ln\, L(\mathbb{x}+\mathbb{y};\Theta_1+\Theta_2)\, L(\mathbb{x}+\mathbb{y};\Theta_1+\Theta_2)\, d(\mathbb{x}+\mathbb{y})\right) = (0,0) \quad (33)$$

$$\left(E\left[\frac{\partial}{\partial\Theta_1} ln\, L(\mathbb{X};\Theta_1)\right], E\left[\frac{\partial}{\partial(\Theta_1+\Theta_2)} ln\, L(\mathbb{X}+\mathbb{Y};\Theta_1+\Theta_2)\right]\right) = (0,0) \quad (34)$$

$$(E[U(\mathbb{X};\Theta_1)], E[U(\mathbb{X}+\mathbb{Y};\Theta_1+\Theta_2)]) = (0,0) \quad (35)$$

Taking T^{-1} get:

$$E\big(U(\mathbb{X}_N;\theta_N)\big) = 0 \quad (36)$$

Definition 3.9

We will call variance of neutrosophic score function the neutrosophic Fisher information about the neutrosophic parameter Θ_N $(NFI_n(\Theta_N))$ i.e.:

$$NFI_n(\Theta_N) = Var\big(U(\mathbb{X}_N;\theta_N)\big) = E\big(U^2(\mathbb{X}_N;\theta_N)\big) \quad (37)$$

Theorem 6

$$Var\big(U(\mathbb{X}_N;\theta_N)\big) = E([U(\mathbb{X}_N;\theta_N)]^2) = -nE\left(\frac{\partial^2 \ln f(x_N;\theta_N)}{\partial\theta_N^2}\right) \quad (38)$$

Proof:

$$\frac{\partial}{\partial\Theta_N} E\big(U(\mathbb{X}_N;\theta_N)\big) = 0 \quad (39)$$

$$T\left[\frac{\partial}{\partial\Theta_N} E\big(U(\mathbb{X}_N;\theta_N)\big)\right] = T[0] \quad (40)$$

$$\left(\frac{\partial}{\partial\Theta_1}\int_{-\infty}^{+\infty} \frac{\partial}{\partial\Theta_1} ln\, L(\mathbb{x};\Theta_1)\, L(\mathbb{x};\Theta_1)\, d\mathbb{x}, \frac{\partial}{\partial(\Theta_1+\Theta_2)}\int_{-\infty}^{+\infty} \frac{\partial}{\partial(\Theta_1+\Theta_2)} ln\, L(\mathbb{x}+\mathbb{y};\Theta_1+\Theta_2)\, L(\mathbb{x}+\mathbb{y};\Theta_1+\Theta_2)\, d(\mathbb{x}+\mathbb{y})\right) \quad (41)$$

$$\left(\int_{-\infty}^{+\infty}\left(\frac{\partial^2 ln\, L(\mathbb{x};\Theta_1) L(\mathbb{x};\Theta_1)}{\partial\Theta_1^2} + \frac{\partial L(\mathbb{x};\Theta_1)}{\partial\Theta_1}\frac{\partial\, ln\, L(\mathbb{x};\Theta_1)}{\partial\Theta_1}\right) d\mathbb{x}, \left(\int_{-\infty}^{+\infty} \frac{\partial^2 ln\, L(\mathbb{x}+\mathbb{y};\Theta_1+\Theta_2) L(\mathbb{x}+\mathbb{y};\Theta_1+\Theta_2)}{\partial(\Theta_1+\Theta_2)^2} + \frac{\partial L(\mathbb{x}+\mathbb{y};\Theta_1+\Theta_2)}{\partial(\Theta_1+\Theta_2)}\frac{\partial\, ln\, L(\mathbb{x}+\mathbb{y};\Theta_1+\Theta_2)}{\partial(\Theta_1+\Theta_2)}\right) d(\mathbb{x}+\mathbb{y})\right) = (0,0) \quad (42)$$

$$\left(\int_{-\infty}^{+\infty}\left(\frac{\partial^2 ln\, L(\mathbb{x};\Theta_1) L(\mathbb{x};\Theta_1)}{\partial\Theta_1^2} + \left(\frac{\partial\, ln\, L(\mathbb{x};\Theta_1)}{\partial\Theta_1}\right)^2 L(\mathbb{x};\Theta_1)\right) d\mathbb{x}, \int_{-\infty}^{+\infty}\left(\frac{\partial^2 ln\, L(\mathbb{x}+\mathbb{y};\Theta_1+\Theta_2) L(\mathbb{x}+\mathbb{y};\Theta_1+\Theta_2)}{\partial(\Theta_1+\Theta_2)^2} + \left(\frac{\partial\, ln\, L(\mathbb{x}+\mathbb{y};\Theta_1+\Theta_2)}{\partial(\Theta_1+\Theta_2)}\right)^2 L(\mathbb{x}+\mathbb{y};\Theta_1+\Theta_2)\right) d(\mathbb{x}+\mathbb{y})\right) = (0,0) \quad (43)$$

$$\left(E\left(\frac{\partial^2 ln\, L(\mathbb{X};\Theta_1)}{\partial\Theta_1^2}\right) + E\left(\frac{\partial\, ln\, L(\mathbb{X};\Theta_1)}{\partial\Theta_1}\right)^2, E\left(\frac{\partial^2 ln\, L(\mathbb{X}+\mathbb{Y};\Theta_1+\Theta_2)}{\partial(\Theta_1+\Theta_2)^2}\right) + E\left(\frac{\partial\, ln\, L(\mathbb{X}+\mathbb{Y};\Theta_1+\Theta_2)}{\partial(\Theta_1+\Theta_2)}\right)^2\right) = (0,0) \quad (44)$$

$$E\left[\frac{\partial^2 ln\, L(\mathbb{X};\Theta_1)}{\partial\Theta_1^2}\right] + E\left[\left(\frac{\partial\, ln\, L(\mathbb{X};\Theta_1)}{\partial\Theta_1}\right)^2\right] + \left[E\left[\left(\frac{\partial^2 ln\, L(\mathbb{X}+\mathbb{Y};\Theta_1+\Theta_2)}{\partial(\Theta_1+\Theta_2)^2}\right)^2\right] + E\left[\left(\frac{\partial\, ln\, L(\mathbb{X}+\mathbb{Y};\Theta_1+\Theta_2)}{\partial(\Theta_1+\Theta_2)}\right)^2\right] - \left(E\left[\frac{\partial^2 ln\, L(\mathbb{X};\Theta_1)}{\partial\Theta_1^2}\right] + E\left[\left(\frac{\partial\, ln\, L(\mathbb{X};\Theta_1)}{\partial\Theta_1}\right)^2\right]\right)\right] I = 0 \quad (45)$$

$$E\left[\frac{\partial^2}{\partial\Theta_N^2} ln\, L(\mathbb{X}_N;\Theta_N)\right] + E\left[\left(\frac{\partial}{\partial\Theta_N} ln\, L(\mathbb{X}_N;\Theta_N)\right)^2\right] = 0 \quad (46)$$

$$E([U(\mathbb{X}_N;\theta_N)]^2) = -nE\left(\frac{\partial^2 \ln f(x_N;\theta_N)}{\partial\theta_N^2}\right) \quad (47)$$

Theorem 7

Neutrosophic Fisher information can be written in the following form:

$$NFI_n(\theta_N) = FI_n(\theta_1) + [FI_n(\theta_1+\theta_2) - FI_n(\theta_1)]I \quad (48)$$

Where:

$$FI_n(\theta) = nE\frac{\partial^2}{\partial\theta^2} ln\, f(x;\theta) \quad (49)$$

Proof:

Using equations (47) and properties of AH-Isometry we get:

$$NFI_n(\theta_N) = E([U(\mathbb{X}_N;\theta_N)]^2) = -nE\left(\frac{\partial^2 \ln f(x_N;\theta_N)}{\partial\theta_N^2}\right) \quad (50)$$

$$T\big(NFI_n(\theta_N)\big) = T\left(-nE\left(\frac{\partial^2}{\partial(\theta_1+\theta_2 I)^2}\ln f(x+yI;\theta_1+\theta_2 I)\right)\right)$$

$$= \left(-nE\frac{\partial^2}{\partial\theta_1^2}\ln f(x;\theta_1), -nE\frac{\partial^2}{\partial(\theta_1+\theta_2)^2}\ln f(x+y;\theta_1+\theta_1)\right) \quad (51)$$

Taking T^{-1}get:

$$NFI_n(\theta_N) = T^{-1}\left(-nE\frac{\partial^2}{\partial\theta_1^2}\ln f(x;\theta_1), -nE\frac{\partial^2}{\partial(\theta_1+\theta_2)^2}\ln f(x+y;\theta_1+\theta_1)\right)$$

$$= -nE\frac{\partial^2}{\partial\theta_1^2}\ln f(x;\theta_1) + \left[-nE\frac{\partial^2}{\partial(\theta_1+\theta_2)^2}\ln f(x+y;\theta_1+\theta_2) + nE\frac{\partial^2}{\partial\theta_1^2}\ln f(x;\theta_1)\right]I$$

$$= FI_n(\theta_1) + [FI_n(\theta_1+\theta_2) - FI_n(\theta_1)]I \quad (52)$$

Example 7:

Let $\mathbb{X}_N$ be a neutrosophic random sample of distribution given in example 3, then:

$$T\big(f(x_N;\theta_N)\big) = \left(\frac{1}{\theta_1}e^{-\frac{x}{\theta_1}}, \frac{1}{(\theta_1+\theta_2)}e^{-\frac{(x+y)}{(\theta_1+\theta_2)}}\right)$$

$$T(\ln f(x_N;\theta_N)) = \left(-\ln\theta_1 - \frac{x}{\theta_1}, -\ln(\theta_1+\theta_2) - \frac{(x+y)}{(\theta_1+\theta_2)}\right)$$

$$T\left(\frac{\partial}{\partial\theta_N}\ln f(x_N;\theta_N)\right) = \left(-\frac{1}{\theta_1}+\frac{x}{\theta_1^2}, -\frac{1}{(\theta_1+\theta_2)}+\frac{(x+y)}{(\theta_1+\theta_2)^2}\right)$$

$$T\left(\frac{\partial^2}{\partial\theta_N^2}\ln f(x_N;\theta_N)\right) = \left(\frac{1}{\theta_1^2}-\frac{2x}{\theta_1^3}, \frac{1}{(\theta_1+\theta_2)^2}-\frac{2(x+y)}{(\theta_1+\theta_2)^3}\right)$$

$$T\left(-nE\left(\frac{\partial^2}{\partial\theta_N^2}\ln f(x_N;\theta_N)\right)\right) = \left(-nE\left(\frac{1}{\theta_1^2}-\frac{2x}{\theta_1^3}\right), -nE\left(\frac{1}{(\theta_1+\theta_2)^2}-\frac{2(x+y)}{(\theta_1+\theta_2)^3}\right)\right)$$

$$T\left(-nE\left(\frac{\partial^2}{\partial\theta_N^2}\ln f(x_N;\theta_N)\right)\right) = \left(\frac{n}{\theta_1^2}, \frac{n}{(\theta_1+\theta_2)^2}\right)$$

$$-nE\left(\frac{\partial^2}{\partial\theta_N^2}\ln f(x_N;\theta_N)\right) = T^{-1}\left(\left(\frac{n}{\theta_1^2}, \frac{n}{(\theta_1+\theta_2)^2}\right)\right) = \frac{n}{\theta_1^2} + \left[\frac{n}{(\theta_1+\theta_2)^2} - \frac{n}{\theta_1^2}\right]I = NIF_n(\theta_N)$$

Simulation Analysis:

In this part, performance of two estimation methods was evaluated based on Monte Carlo simulation to the three studied neutrosophic probability distributions using R software with various sample sizes and with total replication of $N = 10000$ times with sample sizes of 5,15,30,50 and 100 and with fixed parameter $\theta_N = 2 + I$. Goodness of estimation was assessed depending on average bias and root mean square error defined below: [18]

$$AB = \frac{\sum_{i=1}^{N}(\hat{\theta}_{Ni} - \theta_N)}{N}$$

$$RMSE = \sqrt{\frac{\sum_{i=1}^{N}(\hat{\theta}_{Ni} - \theta_N)^2}{N}}$$

Table (1) shows results of simulation analysis for neutrosophic power distribution and compares the two proposed estimation methods, notice that average bias of moments estimator is decreasing faster than maximum likelihood's average bias, which proves by simulation that moments estimator is asymptotically unbiased.

Table 1: Simulation performance of Neutrosophic Power Distribution.

n	Maximum Likelihood			Moments		
	$RMSE$	AB	Average $\hat{\theta}_N$	$RMSE$	AB	Average $\hat{\theta}_N$
5	$1.48 + 0.76I$	$0.4992 + 0.2356I$	$2.50 + 1.24I$	$1.45 + 0.75I$	$0.3877 + 0.2213I$	$2.39 + 1.22I$

15	$0.61 + 0.29I$	$0.1443 + 0.0638I$	$2.14 + 1.06I$	$0.63 + 0.28I$	$0.1101 + 0.0559I$	$2.11 + 1.06I$
30	$0.39 + 0.22I$	$0.0678 + 0.0422I$	$2.07 + 1.04I$	$0.41 + 0.21I$	$0.0505 + 0.0406I$	$2.05 + 1.04I$
50	$0.29 + 0.16I$	$0.0386 + 0.0204I$	$2.04 + 1.02I$	$0.31 + 0.15I$	$0.0282 + 0.0193I$	$2.03 + 1.02I$
100	$0.21 + 0.10I$	$0.0192 + 0.0126I$	$2.02 + 1.01I$	$0.22 + 0.10I$	$0.0147 + 0.0104I$	$2.01 + 1.01I$

Table (2) shows results of simulation analysis for neutrosophic Exponential distribution and compares the two proposed estimation methods and we see that both methods give the same estimators.

Table 2: Simulation performance of Neutrosophic Exponential Distribution.

n	Maximum Likelihood			Moments		
	$RMSE$	AB	Average $\hat{\theta}_N$	$RMSE$	AB	Average $\hat{\theta}_N$
5	$0.89 + 0.46I$	$-0.0046 + 0.0097I$	$2.00 + 1.01I$	$0.89 + 0.46I$	$-0.0046 + 0.0097I$	$2.00 + 1.01I$
15	$0.61 + 0.29I$	$0.0067 - 0.0178I$	$2.01 + 0.98I$	$0.61 + 0.29I$	$0.0067 - 0.0178I$	$2.01 + 0.98I$
30	$0.39 + 0.22I$	$-0.0024 + 0.0121I$	$2.00 + 1.01I$	$0.39 + 0.22I$	$-0.0024 + 0.0121I$	$2.00 + 1.01I$
50	$0.29 + 0.16I$	$-0.0024 - 0.0013I$	$2.00 + 1.00I$	$0.29 + 0.16I$	$-0.0024 - 0.0013I$	$2.00 + 1.00I$
100	$0.21 + 0.10I$	$-0.0008 + 0.0033I$	$2.00 + 1.00I$	$0.21 + 0.10I$	$-0.0008 + 0.0033I$	$2.00 + 1.00I$

Table (3) shows results of simulation analysis for neutrosophic Maxwell distribution and compares the two proposed estimation methods, notice that average bias of moments estimator is decreasing faster than maximum likelihood's average bias, which proves by simulation that moments estimator is asymptotically unbiased.

Table 3: Simulation performance of Neutrosophic Maxwell Distribution.

n	Maximum Likelihood			Moments		
	$RMSE$	AB	Average $\hat{\theta}_N$	$RMSE$	AB	Average $\hat{\theta}_N$
5	$1.04 + 0.49I$	$0.3119 + 0.1552I$	$2.31 + 1.16I$	$1.02 + 0.48I$	$0.2470 + 0.1260I$	$2.25 + 1.13I$
15	$0.47 + 0.23I$	$0.0882 + 0.0568I$	$2.09 + 1.06I$	$0.47 + 0.24I$	$0.0687 + 0.0488I$	$2.07 + 1.05I$
30	$0.31 + 0.16I$	$0.0429 + 0.0185I$	$2.04 + 1.02I$	$0.32 + 0.16I$	$0.0351 + 0.0113I$	$2.04 + 1.01I$
50	$0.24 + 0.12I$	$0.0274 + 0.0105I$	$2.03 + 1.01I$	$0.24 + 0.12I$	$0.0231 + 0.0073I$	$2.02 + 1.01I$
100	$0.17 + 0.08I$	$0.0160 + 0.0046I$	$2.02 + 1.00I$	$0.17 + 0.08I$	$0.0135 + 0.0032I$	$2.01 + 1.00I$

6. Conclusions and future research directions

In this paper we have introduced the concept of neutrosophic likelihood estimation method and neutrosophic moments estimation method and studied its properties based on AH-Isometry. We also presented theorems on these two estimation methods. We see that two estimation methods yields to different estimators. We also presented the concept of neutrosophic fisher information and presented some theorems related to it. In future work we are looking forward to study the properties of estimators like biasness, consistency and sufficiency. This paper opens the way to study the theory of neutrosophic statistical inference which is using neutrosophic classical numbers $N = a + bI; I^2 = I$ (not interval neutrosophic numbers).

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Introduction to the Symbolic Plithogenic Algebraic Structures (revisited)

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Abstract: In this paper, we recall and study the new type of algebraic structures called Symbolic Plithogenic Algebraic Structures. Their operations are given under the Absorbance Law and the Prevalence Order.

Keywords: Absorbance Law; Prevalence Order; Neutrosophic Quadruple Numbers; Plithogenic Set; Type-k Neutrosophic Set; Type-k Plithogenic Set; Hybridization of Classical, Fuzzy and Fuzzy Extension Sets; Symbolic Plithogenic Components; Symbolic Plithogenic Operations; Plithogenic Numbers; Symbolic Plithogenic Algebraic Structures; Symbolic Plithogenic Group; Symbolic Plithogenic Ring.

1. Introduction

The plithogeny, plithogenic set, plithogenic logic, plithogenic probability, plithogenic statistics, and the symbolic plithogenic algebraic structures were introduced in 2018-2019 by Smarandache [1, 2, 3, 4, 5].

Plithogeny is the genesis or origination, creation, formation, development, and evolution of new entities from dynamics and organic fusions of contradictory and/or neutrals and/or non-contradictory multiple old entities.

And **plithogenic** means what is pertaining to plithogeny.

Plithogeny is an extension of neutrosophy, which is an extension of paradoxism.

While **paradoxism** [6] is based on using opposite ideas, contradictions, paradoxes in arts, letters, and science creations,

neutrosophy is based on the dynamics of a pair of opposites (<A> ,<antiA>) and their neutral (indeterminacy) <neutA>,

but **plithogeny** on the dynamics of many pairs of opposites (<A_1>, <antiA_1>), …, (<A_k>, <antiA_k>) and their neutralities <neutA_1>, …, <neutA_k>, for $k \geq 2$ ["plitho" means "many" in Greek language].

Plithogenic Set was extended to Type-n Plithogenic Set, for integer $n \geq 1$.

Symbolic Plithogenic Numbers are generalizations of Neutrosophic Quadruple Numbers, Refined Neutrosophic Quadruple Numbers, and Symbolic Turiyam Numbers.

Consequently, the Symbolic Plithogenic Algebraic Structures (semigroup, group, ring, etc.) are generalization of the corresponding algebraic structures built on these particular cases described above.

2. Informal Definition of Plithogenic Set

A plithogenic set (PS) is a set whose elements are characterized, as in our real world, by many attributes (parameters): $P_1, P_2, \ldots, P_n$.

$PS = \{x(P_1, P_2, \ldots, P_n), x \in U\}$, where U is a universe of discourse.

A generic element x belongs to the plithogenic set PS in a certain degree $d(P_i)$ with respect to each attribute (parameter) P_i. The degree of appurtenance of an element to the plithogenic set may be: classical, fuzzy, intuitionistic fuzzy, neutrosophic, refined neutrosophic, and any other type of extended fuzzy.

In a better descriptive way, emphasizing the degrees, we may re-write it as:

$$PS = \{x(d(P_1), d(P_2), ..., d(P_n)), x \in U\}$$

The attributes (parameters) $P_1, P_2, \ldots, P_n$ may be independent, dependent, or partially independent and partially dependent of each other - according to each application.

This is also called Type-1 Plithogenic Set.

3. Type-k Plithogenic Set

The Type-k Neutrosophic Set [13] has been extended to Type-k Plithogenic Set.

If the parameters $P_i, 1 \le i \le n,$ depend on sub-parameters $P_{i1}, P_{i2}, ..., P_{im_i}$ for $m_i \ge 1$ then one gets a Type-2 Plithogenic Set.

Afterward, if the sub-parameters $P_{ij}, 1 \le i \le n, 1 \le j \le m_i$ are formed by sub-sub-parameters $P_{ij1}, P_{ij2}, ..., P_{ijm_j}$ for $m_j \ge 1$ then one gets a Type-3 Plithogenic Set.

And so on, up to Type-k Plithogenic Set.

Passing to degrees, one may write:

$$PS_1 = \{x(d_1(P_1), d_1(P_2), ..., d_1(P_n)), x \in U\}$$

Type-2 Plithogenic Set

$$PS_2 = \{x(d_2(d_1(P_1)), d_2(d_1(P_2)), ..., d_2(d_1(P_n))), x \in U\}$$

In general, Type-n Plithogenic Set

$$PS_k = \{x(d_k(...d_2(d_1(P_1))...), d_k(...d_2(d_1(P_2))...), ..., d_k(...d_2(d_1(P_n))...), x \in U\}.$$

4. Hybridization of Classical, Fuzzy, and Fuzzy Extension Sets

The real applications require many times to deal with multiple types of classical, fuzzy, and fuzzy extension sets.

Assume that, starting from a neutrosopohic element of the form x(T, I, F), with $0 \le T + I + F \le 3$, one has be combined it with a Picture_Fuzzy form (T, N, F), with $0 \le T + N + F \le 1$, then one gets: the neutrosophic-picture_fuzzy hybrid form: $((TT, TN, TF), (IT, IN, IF), (FT, FN, FF))$,

with $0 \le TT + TN + TF \le 1, 0 \le IT + IN + IF \le 1, 0 \le FT + FN + FF \le 1,$

where T was split into TT, TN, TF representing the confidence in T, neutral-confidence in T, and nonconfidence in T respectively; similarly for I and F.

Further on, let's combine the result with the Spherical_Fuzzy Set, where the sum of squares of components is between 0 and 1, then one obtains a neutrosophic-picture_fuzzy-spherical_fuzzy hybrid form: $((TT, TN, TF), (IT, IN, IF), (FT, FN, FF))$, with

$$0 \le TT^2 + TN^2 + TF^2 \le 1, 0 \le IT^2 + IN^2 + IF^2 \le 1, 0 \le FT^2 + FN^2 + FF^2 \le 1.$$

The hybridization chain may be as long as needed, and may deal with various types of classical, fuzzy, and fuzzy extension sets – including repeated types.

5. Definitions of Symbolic Plithogenic Set & Symbolic Plithogenic Algebraic Structures

Let SPS be a non-empty set, included in a universe of discourse U, defined as follows:

$SPS = \{x \mid x = a_0 + a_1P_1 + a_2P_2 + ... + a_nP_n, n \ge 1, a_i \in R$ or $a_i \in C$ or a_i belong to some given algebraic structure, for $0 \le i \le n$, where R = the set of real numbers, C = the set of complex numbers, and all P_i are letters (or variables) and are called *Symbolic (Literal) Plithogenic Components (Variables)*}, where *1, P_1, P_2, …, P_n* act like a base for the elements of the above set *SPS*.

$a_0, a_1, a_2, \ldots, a_n$ are called coefficients.

SPS is called a Symbolic Plithogenic Set. And the algebraic structures defined on this set are called *Symbolic Plithogenic Algebraic Structures.*

In general, Symbolic (or Literal) Plithogenic Theory is referring to the use of abstract symbols {i.e. the letters/parameters) *P_1, P_2, …, P_n*, representing the plithogenic components (variables) as above} in some theory.

6. Definition of Plithogenic Numbers *(PN)*

The numbers of the form $PN = a_0 + a_1P_1 + a_2P_2 + ... + a_nP_n$ defined as above are called Plithogenic Numbers, where a_nP_n is called the *leading (strongest) term*.

7. Prevalence Order *(PO)*

The experts establish a *prevalence order* [1], or total order, according to the importance of each attribute/parameter (P_i) into the application. To obtain a total order among the symbolic plithogenic components $\{P_1, P_2, ..., P_n\}$, one defines some relationships (laws) between them.

The most used one is the absorbance law.

8. Absorbance Law

We recall and use now our 2015 Absorbance Law [1], simply defined as:

the greater absorbs the smaller [the bigger fish eats the smaller fish].

9. Multiplication and Power of Symbolic Plithogenic Components under the Absorbance Law

We assume that in the above definition of the plithogenic numbers, the symbolic plithogenic components are ranked increasingly, or

$$P_1 < P_2 < ... < P_n \quad (\textit{prevalence order})$$

where "<" may signify: smaller, less important, under, inferior, etc.

Whence, the multiplication and power of symbolic plithogenic components are:

$P_i \cdot P_j = P_{\max\{i,j\}}$, whence $(P_i)^2 = P_i$.

In general, $P_{i_1} \cdot P_{i_2} \cdot ... \cdot P_{i_k} = P_{\max\{i_1, i_2, ..., i_k\}}$ and $(P_i)^m = P_i$ for integer $m \geq 1$.

Negative powers of Symbolic Plithogenic Components do not exist, $\left(P_i\right)^{-m} = \dfrac{1}{\left(P_i\right)^m}$ does not exist. For example, $\left(P_i\right)^{-1} = \dfrac{1}{P_i}$ does not exist.

And P_i to the power zero is equal to 1 by definition: $\left(P_i\right)^0 \triangleq 1$.

10. m-th Root of the Symbolic Plithogenic Components

$\sqrt[m]{P_i} = P_i, 1 \leq i \leq n$, for integer $m \geq 2$, because $\left(\sqrt[m]{P_i}\right)^m = (P_i)^m$, or $P_i = P_i$.

$\left(\sqrt[m]{P_i}\right)^m$ cannot be equal to P_{i-1} or lower, nor P_{i+1} or upper, because the last two raised to the power m would not give P_i.

Examples: $\sqrt{P_1} = P_1$, $\sqrt[3]{P_7} = P_7$, $\sqrt[4]{16P_9} = \sqrt[4]{16} \cdot \sqrt[4]{P_9} = 2P_9$.

11. Example of Plithogenic Set

Let's have a classical set

S = {John, George, Mary},

and each element is characterized with respect to the attributes: *Weight (W), Tallness (T), Oldness (()), Beauty (B), Health (H).*

Each person/element has some (classical, fuzzy, or any fuzzy extension) degree (*d*) with respect to each attribute (parameter): d(Weight), d(Tallness), d(Oldness), d(Beauty), d(Health).

And thus one transforms the classical set into a plithogenic set:

PS = {John[d(Weight), d(Tallness), d(Oldness), d(Beauty), d(Health)],
George[d(Weight), d(Tallness), d(Oldness), d(Beauty), d(Health)],
Mary[d(Weight), d(Tallness), d(Oldness), d(Beauty), d(Health)]}.

As a numerical example, see below, as evaluated by Expert 1:

PS_1 *= {John(0.5, 0.6, 0.3, 0.1, 0.7), George(0.1, 0.8, 0.3, 0.1, 0.4), Mary(0.9, 0.4, 0.6, 0.1, 0.2)}.*

PS_1 is a *Type-1 Plithogenic Set.*

Which means that on some corresponding scales, John's fuzzy degree of Weight is 0.5, fuzzy degree of Tallness 0.6, fuzzy degree of Oldness 0.3, fuzzy degree of Beauty 0.1, and fuzzy degree of Health 0.7. {Of course, one may consider all kind of degrees: not only fuzzy, but also: classical, intuitionistic fuzzy, neutrosophic, refined neutrosophic, and other fuzzy extensions.}

Similarly for George's and Mary's degrees.

12. Example of Type-2 Plithogenic Set

Assume that Expert 2 is not totally confident on the evaluation of the Expert 1 in the above example. Thus, he decides to evaluate the first evaluation. Expert 2 may, as well, use any types of degrees – according to the expert desire and tools, not necessarily the same as in the previous evaluation.

For the sake of simplicity, let's consider that Expert 2 also uses fuzzy degrees. Now one gets a Type-2 Plithogenic Set:

$$PS_2 = \{John[0.5(0.9), 0.6(0.4), 0.3(1.0), 0.1(0.0), 0.7(0.8)],$$
$$George[0.1(0.3),\ 0.8(0.4),\ 0.3(0.5),\ 0.1(0.7),\ 0.4(0.9)],$$
$$Mary[0.9(0.1),\ 0.4(0.5),\ 0.6(0.6),\ 0.1(0.8),\ 0.2(0.9)]\}$$

Which are interpreted as follows:

0.5(0.9) means that Expert 2 is 0.9 (90%) confident in John's fuzzy degree of Weight of 0.5 assigned by Expert 1;

0.6(0.4) means that Expert 2 is 0.4 (40%) confident in John's fuzzy degree of Tallness of 0.6 assigned by Expert 1;

0.3(1.0) means that Expert 2 is 1.0 (100%) confident in John's fuzzy degree of Oldness of 0.3 assigned by Expert 1;

0.1(0.0) means that Expert 2 is 0.0 (0%) confident in John's fuzzy degree of Beauty of 0.1 assigned by Expert 1;

0.7(0.8) means that Expert 2 is 0.8 (80%) confident in John's fuzzy degree of Health of 0.7 assigned by Expert 1.

And similarly for George's and Mary's second round of degrees.

13. Example of Type-3 Plithogenic Set

The process may go on and have an Expert 3 evaluate the Expert 2. Assume Expert 3 uses neutrosophic degrees.

$$PS_3 = \{John\{0.5[0.9(0.6, 0.7, 0.3)], 0.6[0.4(0.6, 0.7, 0.3)],$$
$$0.3[1.0(0.6, 0.7, 0.3)], 0.1[0.0(0.6, 0.7, 0.3)], 0.7[0.8(0.6, 0.7, 0.3)]\},$$
$$George[0.1[0.3(0.4,0.4,0.06)], 0.8[0.4(0.9,0.1,0.03)], 0.3[0.5(0.9,1.0,0.2)],$$
$$0.1[0.7(0.7,0.3,0.6)], 0.4[0.9(0.1,0.0,0.4)],$$
$$Mary\{0.9[0.1(0.2,0.3,0.4)], 0.4[0.5(0.7,0.8,0.7)],$$
$$0.6[0.6(1.0,0.0,0.0)], 0.1[0.8(0.1,0.4,0.6)], 0.2[0.9(0.0,0.0,0.0)]\}$$

Therefore,
$0.5[0.9(0.6, 0.7, 0.3)]$
means that Expert 3 assigns the neutrosophic degrees of truth = 0.6, indeterminacy = 0.7, and falsehood = 0.3, to the Expert 2's evaluation of 0.9 (90%) confidence on Expert 1's evaluation of 0.5 degree of John's Weight.
And so on for all others.

One may generalize to **Type-k Plithogenic Set**, recurrently going on from a type to the next type, but it becomes more sophisticated and not usable in practice.

14. Example of Symbolic Plithogenic Numbers

The corresponding Symbolic Plithogenic Algebraic Structure is based on the symbolic (or literal) plithogenic components *W, T, O, B, H*, and we get the **plithogenic numbers** (*PN*) of the form:

$$PN = a + bW + cT + dO + eB + fH,$$

where *a, b, c, d, e, f* are real, or complex numbers, or they may belong to a set of a given classical algebraic structure. As a particular example, let $PN_1 = 2 - 3W + 5T - O + 6B - 4H$.

In this example, let's assume that the *prevalence order* is:

$W < T < O < B < H$, where "<" means "less important",

or *W* is less important than *T*, which is less important than *O*, which is less important that B, which is less important than *H*.

The *absorbance law* is defined as follows: the most important absorbs the less important in the multiplication operation, for example $W \cdot T = T$, since *T* absorbs *W* because *T* is more important (bigger) than *W*. Similarly for the other multiplications.

15. Operations with Plithogenic Numbers

Let's consider two plithogenic numbers:

$$PN_1 = a_0 + a_1P_1 + a_2P_2 + ... + a_nP_n$$
$$PN_2 = b_0 + b_1P_1 + b_2P_2 + ... + b_nP_n.$$

15.1. Addition of Plithogenic Numbers

$$PN_1 + PN_2 = (a_0 + b_0) + \sum_{i=1}^{n}(a_i + b_i)P_i$$

15.2. Subtraction of Plithogenic Numbers

$$PN_1 - PN_2 = (a_0 - b_0) + \sum_{i=1}^{n}(a_i - b_i)P_i$$

(SPS, +) is a Symbolic Plithogenic Commutative Group

15.3. Scalar Multiplication of Plithogenic Numbers

$$c \cdot PN_1 = c \cdot (a_0 + a_1P_1 + a_2P_2 + ... + a_nP_n) = c \cdot a_0 + c \cdot a_1P_1 + c \cdot a_2P_2 + ... + c \cdot a_nP_n$$

15.4. Multiplication of and Ppower of Plithogenic Numbers

$PN_1 \cdot PN_2 = (a_0 + a_1P_1 + a_2P_2 + ... + a_nP_n) \cdot (b_0 + b_1P_1 + b_2P_2 + ... + b_nP_n)$

and then one multiplies them, term by term $(a_iP_i) \cdot (a_jP_j) = a_i \cdot a_j \cdot P_{\max\{i,j\}}$, where $\cdot$ is the classical multiplication, as in classical algebra, using the above multiplication of symbolic plithogenic components.

As particular case: $0 \cdot P_i = 0$.

$(SPS, +, \cdot)$ is a Symbolic Plithogenic Commutative Ring, with the plithogenic unitary element: $1_{\cdot} \equiv 1 + 0 \cdot P_1 + 0 \cdot P_2 + ... + 0 \cdot P_n$.

The symbolic plithogenic components P_i's are not inversible, therefore the elements of SPS are non-inversible (except the plithogenic unitary element 1_*).

$(PN_1)^m = \underbrace{PN_1 \cdot PN_1 \cdot ... \cdot PN_1}_{m-times}$ for integer $m \geq 1$;

The negative power of a plithogenic number $(PN_1)^{-m}$ does not exist.

15.5. Alternative Multiplication of Plithogenic Numbers

$PN_1 \otimes PN_2 = a_0 \cdot b_0 + a_1 \cdot b_1 \cdot P_1 + a_2 \cdot b_2 \cdot P_2 + ... + a_n \cdot b_n \cdot P_n$.

$(SPS, +, \otimes)$, is a Symbolic Plithogenic Commutative Ring, with the unitary element: $1_{\otimes} \equiv 1 + 1 \cdot P_1 + 1 \cdot P_2 + ... + 1 \cdot P_n$.

The plithogenic numbers that have coefficients equal to zero do not have an inverse, for example: $2 + 3P_1 - 5P_3 = 2 + 3P_1 + 0P_2 - 5P_3$ is not inversible.

15.6. Division of Symbolic Plithogenic Components

$$\frac{P_i}{P_j} = \begin{cases} x_0 + x_1P_1 + x_2P_2 + ... + x_jP_j + P_i & x_0 + x_1 + ... + x_j = 0 \quad i > j \\ x_0 + x_1P_1 + x_2P_2 + ... + x_iP_i & x_0 + x_1 + ... + x_i = 1 \quad i = j \\ \phi & i < j \end{cases}$$

where all coefficients $x_0, x_1, x_2, ...x_i, ..., x_j, ... \in SPS$.

There are j-tuple infinities of quotients when i > j,
also i-tuple infinities of quotients when i = j,
and no quotient (indeterminate division) when i < j.
Therefore, the operation of division d(. , .) of symbolic plithogenic components

$d(P_i, P_j) : \{P_1, P_2, ..., P_n\}^2 \rightarrow SPS$

is a NeutroOperation, because:

it is well-defined (inner-defined) for no elements, since one never gets a single quotient, or $d(P_i, P_j) \notin SPS$;

it is indeterminate (cannot be calculated) for some elements (when $P_i < P_j$) with $d(P_i, P_j)$ being indeterminate;

and outer-defined (when $P_i = P_j$ and $P_i > P_j$) with $d(P_i, P_j) \notin SPS$

but $d(P_i, P_j) \in P(SPS)$ the powerset of *SPS*.

15.6.1. Example 1 of Division of Symbolic Plithogenic Components

$i > j$ [j-tuple infinities of quotients]

Let's divide P_5 by P_2.

$\frac{P_5}{P_2} = x$

where $x = x_0 + x_1P_1 + x_2P_2 + ... + x_nP_n \in SPS$.

$P_5 = x \cdot P_2$

Since the multiplication $x \cdot P_2$ should not exceed P_5 we take $n = 5$ into the formula of x, or

$$x \cdot P_2 = (x_0 + x_1P_1 + x_2P_2 + x_3P_3 + x_4P_4 + x_5P_5) \cdot P_2$$
$$= x_0P_2 + x_1P_1P_2 + x_2P_2P_2 + x_3P_3P_2 + x_4P_4P_2 + x_5P_5P_2$$
$$= (x_0 + x_1 + x_2)P_2 + x_3P_3 + x_4P_4 + x_5P_5$$
$$\equiv P_5 \equiv 0 + 0P_1 + 0P_2 + 0P_3 + 0P_4 + 1P_5$$

Therefore, $x_5 = 1$, $x_4 = 0$, $x_3 = 0$, $x_0 + x_1 + x_2 = 0$.

Whence, $\frac{P_5}{P_2} = x_0 + x_1P_1 + x_2P_2 + P_5$, with $x_0 + x_1 + x_2 = 0$, and the coefficients $x_0, x_1, x_2 \in SPS$ [2-tuple infinities of quotients].

15.6.2. Example 2 of Division of Symbolic Plithogenic Components

$i = j$ [i-tuple infinities of quotients]

$\frac{P_3}{P_3} = x$ or

$$P_3 = P_3 \cdot x = P_3 \cdot (x_0 + x_1P_1 + x_2P_2 + x_3P_3) = x_0P_3 + x_1P_1P_3 + x_2P_2P_3 + x_3P_3P_3$$
$$= x_0P_3 + x_1P_3 + x_2P_3 + x_3P_3 = (x_0 + x_1 + x_2 + x_3)P_3 \equiv 1 \cdot P_3$$

whence $x_0 + x_1 + x_2 + x_3 = 1$.

Thus, $\frac{P_3}{P_3} = x_0 + x_1P_1 + x_2P_2 + x_3P_3$,

where $x_0 + x_1 + x_2 + x_3 = 1$, and the coefficients $x_0, x_1, x_2, x_3 \in SPS$

15.6.3. Example 3 of Division of Symbolic Plithogenic Components

$i < j$ [indeterminate, no quotient]

$\frac{P_2}{P_4} = x$ or $P_2 = P_4 \cdot x \geq P_4 > P_2$ or $P_2 > P_2$, which is impossible.

This multiplication, P_4 times any of $1, P_1, P_2, \ldots, P_n$, will give a result that is greater than or equal to P_4 according to the absorbing law.

This division is undefined (indeterminate).

15.7. Division of Symbolic Plithogenic Numbers

Let consider two symbolic plithogenic numbers as below:

$PN_3 = a_0 + a_1P_1 + a_2P_2 + ... + a_rP_r$ and $PN_4 = b_0 + b_1P_1 + b_2P_2 + ... + b_sP_s$

where $r, s \leq n$ are positive integers, and the leading coefficients (the coefficients of the highest/largest symbolic plithogenic components P_r and respectively P_s) are nonnull, $a_r \neq 0, b_s \neq 0$. The division is also based on the absorbance law.

$$\frac{PN_r}{PN_s} = \frac{a_0 + a_1P_1 + a_2P_2 + ... + a_rP_r}{b_0 + b_1P_1 + b_2P_2 + ... + b_sP_s} = x$$

We need to find $x \in SPS$.

$$a_0 + a_1P_1 + a_2P_2 + ... + a_rP_r \equiv x \cdot (b_0 + b_1P_1 + b_2P_2 + ... + b_sP_s)$$

We are focusing first on the division of their leading symbolic plithogenic components: $\frac{P_r}{P_s}$ as we did on the previous section. Of course the leading coefficients $a_r \neq 0, b_s \neq 0$.

$\frac{PN_r}{PN_s} = \begin{Bmatrix} none, one, many & r \geq s \\ \phi & r < s \end{Bmatrix}$ This is also a NeutroOperation since it has indeterminate cases.

For $r \geq s$ there may be: none, one, or many (including infinitely many) quotients.

For r < s no quotient. Indeterminacy.

We prove these through several examples:

15.7.1. Example 1 (no quotient)

$$\frac{P_1+1}{P_1} = ?$$

$\frac{P_1+1}{P_1} = x = (x_0 + x_1P_1)$, we need to solve for x (actually for x_0 and x_1).

$$P_1 + 1 = (x_0 + x_1P_1) \cdot P_1 = x_0P_1 + x_1P_1P_1 = x_0P_1 + x_1P_1 = (x_0 + x_1)P_1$$

We may set $x_0 + x_1 = 1$, but we are not able to catch the free coefficient 1 from the left-hand side, i.e.

$$P_1 + 1 \neq P_1$$

15.7.2. Example 3 (one quotient only)

$$\frac{P_1+2}{P_1+1} = ?$$

$$\frac{P_1+2}{P_1+1} = x = x_0 + x_1P_1$$

whence

$$P_1 + 2 = (x_0 + x_1P_1) \cdot (P_1 + 1) = x_0P_1 + x_1P_1P_1 + x_0 + x_1P_1$$

$$= x_0P_1 + x_1P_1 + x_0P_1 + x_1P_1 = x_0 + (x_0 + 2x_1)P_1$$

we get

$x_0 = 2, x_0 + 2x_1 = 1,$ then $x_1 = -0.5$.

There is only one quotient (solution): $x_0 + x_1P_1 = 2 - 0.5P_1 = -0.5P_1 + 2$.

Let's check the result:

$$(P_1+1) \cdot (-0.5P_1 + 2) = -0.5P_1P_1 + 2P_1 - 0.5P_1 + 2 = -0.5P_1 + 2P_1 - 0.5P_1 + 2$$

$$= P_1 + 2.$$

15.7.3. Example 3 (double infinities of quotients (solutions))

$$\frac{5P_3}{P_3 - 2P_2} = ?$$

$$\frac{5P_3}{P_3 - 2P_2} = x = x_0 + x_1P_1 + x_2P_2 + x_3P_3.$$

We need to find the coefficients x_0, x_1, x_2, x_3.

$$5P_3 = (x_0 + x_1P_1 + x_2P_2 + x_3P_3) \cdot (P_3 - 2P_2)$$

$$= (x_0 + x_1P_1 + x_2P_2 + x_3P_3) \cdot P_3 + (x_0 + x_1P_1 + x_2P_2 + x_3P_3) \cdot (-2P_2)$$

$$= (x_0 + x_1 + x_2 + x_3)P_3 - (2x_0 + 2x_1 + 2x_2)P_2 - 2x_3P_3$$

$$= (x_0 + x_1 + x_2 - x_3)P_3 - (2x_0 + 2x_1 + 2x_2)P_2 \equiv 5P_3 + 0P_2$$

Whence we get two equations:

$$x_0 + x_1 + x_2 - x_3 = 5$$

$$2x_0 + 2x_1 + 2x_2 = 0$$

Hence $2(x_0 + x_1 + x_2) = 0$, or $x_0 + x_1 + x_2 = 0$.

Replace it into the first equation:

0 – x_3 = 5, then x_3 = -5.

$\frac{5P_3}{P_3 - 2P_2} = x_0 + x_1P_1 + x_2P_2 + x_3P_3 = x_0 + x_1P_1 + x_2P_2 - 5P_3,$

where $x_0 + x_1 + x_2 = 0$.

15.8. m-th Root of the Plithogenic Number

$$\sqrt[m]{PN_1} = \sqrt[m]{a_0 + a_1P_1 + a_2P_2 + ... + a_nP_n} = x_0 + x_1P_1 + x_2P_2 + ... + x_nP_n .$$

We need to find the coefficients $x_0, x_1, x_2, ..., x_n$.

Raising to the power *m* both sides, one gets:

$a_0 + a_1P_1 + a_2P_2 + ... + a_nP_n = (x_0 + x_1P_1 + x_2P_2 + ... + x_nP_n)^m$, where $x_0, x_1, x_2, ..., x_n$ are coefficients that we need to find out. After raising to the power *k* the right-hand side, we identify the coefficients two by two.

The *m-root* of a plithogenic number may have: no solution, several solutions, or infinitely many solutions.

Example 1 of *m-th* Root of the Plithogenic Number with real coefficients (several solutions)

$\sqrt{4-3P_1} = ?$

$\sqrt{4-3P_1} = x_0 + x_1P_1$, where we need to find x_0 and x_1.

Raise both sides to the second power:

$(\sqrt{4-3P_1})^2 = (x_0 + x_1P_1)^2$ or

$4-3P = (x_0)^2 + 2x_0x_1P_1 + (x_1)^2(P_1)^2 = (x_0)^2 + 2x_0x_1P_1 + (x_1)^2P_1$

$= (x_0)^2 + [2x_0x_1 + (x_1)^2]P_1 \equiv 4-3P_1$

Identify the coefficients:

$$\begin{Bmatrix} (x_0)^2 = 4 \\ 2x_0x_1 + (x_1)^2 = -3 \end{Bmatrix}$$

Whence $x_0 = 2, -2$ from the first equation. Replaced into the second equation one gets:

$\pm 4x_1 + (x_1)^2 = -3$, or two quadratic equations $(x_1)^2 \pm 4x_1 + 3 = 0$ that we need to solve.

For $x_0 = 2$, $(x_1)^2 + 4x_1 + 3 = 0$, has the solutions $x_1 = -1, -3$,

thus $(x_0, x_1) = (2, -1)$ or $(2, -3)$.

For $x_0 = -2$, $(x_1)^2 - 4x_1 + 3 = 0$, has the solutions $x_1 = 1, 3$,

thus $(x_0, x_1) = (-2, 1)$ or $(-2, 3)$.

Final answer:

$\sqrt{4-3P_1} = x_0 + x_1P_1 = 2 - P_1, 2 - 3P_1, -2 + P_1, -2 + 3P_1$ (four solutions).

Example 2 of m-th Root of the Plithogenic Number with real coefficients (no solution)

$\sqrt{-4-3P_1}$ has no solution since one gets, in the above calculation $(x_0)^2 = -4$, which does not work in the set of real numbers.

15.9. Remark 1

Other operations may be constructed on the Symbolic Plithogenic Set (SPS), giving birth to various symbolic plithogenic algebraic structures.

15.10. Remark 2

All previous operations are valid for the absorbance law and prevalence order defined above. If different law and order are defined by the experts, then different operations and results one gets.

16. Particular Cases of Symbolic Plithogenic Algebraic Structures

16.1. Neutrosophic Quadruple Numbers

Let's consider an entity (i.e. a number, an idea, an object, etc.) which is represented by a known part (a) and an unknown part ($bT + cI + dF$).
Numbers of the form

$NQ = a + bT + cI + dF$,

where a, b, c, d are real (or complex) numbers (or intervals, or in general subsets), and
T = truth / membership / probability,
I = indeterminacy / neutrality,
F = false / membership / improbability,
are called Neutrosophic Quadruple (Real respectively Complex) Numbers (or Intervals, or in general Subsets) [1].

"a" is called the known part of NQ,

while "$bT + cI + dF$" is called the unknown part of NQ.

Neutrosophic Quadruple Numbers [1] are particular case of the Plithogenic Numbers, since one takes $n = 3$, and P_1, P_2, P_3 are more general than $T, I,$ and F respectively.

16.2. Refined Neutrosophic Quadruple Numbers

The Refined Neutrosophic Quadruple Numbers [1, 7] have the form:

$$RQN = a + \sum_{j=1}^{p} b_j T_j + \sum_{k=1}^{r} c_k I_k + \sum_{l=1}^{s} d_l F_l$$

where a, all b_i, all c_j, and all d_k are real (or complex) numbers, intervals, or, in general, subsets, while $T_1, T_2, \dots, T_p$ are refinements of T;
$I_1, I_2, \dots, I_r$ are refinements of I;
and $F_1, F_2, \dots, F_s$ are refinements of F,

for integers $p, r, s \geq 0$ and at least one of them be ≥ 2, with $p + r + s = n$.

Refined Neutrosophic Quadruple Numbers are also particular case of the Plithogenic Numbers, since instead of symbolic sub-truths / sub-indeterminacies / sub-falsehoods T_j, I_k, F_l one may use all kinds of symbolic plithogenic components $P_1, P_2, \dots, P_n$.

All, Neutrosophic Quadruple Numbers and Refined Neutrosophic Quadruple Numbers, together with the Prevalence Order and Absorbance Law, were introduced by Smarandache [1] in 2015.

16.3. (Symbolic) Turiyam Set

Turiyam Set (TS) was introduced by P. K Singh [9] in 2021, who added to the neutrosophic components *T (Truth), I (Indeterminacy), F(Falsehood),* another component Y (called state of *awareness*).

According to him, Turiyam component (Y) means: "Rejection of both acceptation and rejection of attribute at the given time i.e. unknown region (l). It needs Turiyam consciousness to explore it" [9].

Turiyam Set is very similar to Belnap's Logic, based on: True (T), False (F), Unknown (U), and Contradiction (C), where T, F, U, C are taken as symbols, not numbers. Belnap's Logic is a particular case of Refined Neutrosophic Logic [10].

Turiyam Set was defined as:

$TS = \{(a_0, a_1T, a_2F, a_3I, a_4Y), a_i \in A\}$, where A is a given set, or it is the set of a given classical algebraic structure.

The Symbolic Turiyam Numbers have the form:

$$STN = a_0 + a_1T + a_2F + a_3I + a_4Y$$

where $a_i \in A$.

It is clear that Turiyam Set (2021) is a particular case of the Plithogenic Set, because one replaces $n = 4$, and P_1, P_2, P_3, P_4 by T, F, I, Y respectively, since the symbolic plithogenic components may be either independent, or dependent, or partially independent/dependent as we desire.

The operations on *TS* were defined as particular cases to Smarandache's 2015 neutrosophic quadruple numbers and absorbance law [1] and 2019 symbolic plithogenic numbers [5].

Let

$$x = (a_0, a_1T, a_2F, a_3I, a_4Y) = a_0 + a_1T + a_2F + a_3I + a_4Y$$

$$y = (b_0, b_1T, b_2F, b_3I, b_4Y) = b_0 + b_1T + b_2F + b_3I + b_4Y$$

be two STNs, and *c* be a scalar.

Then the addition

$$x + y = (a_0 + b_0, (a_1 + b_1)T, (a_2 + b_2)F, (a_3 + b_3)I, (a_4 + b_4)Y)$$

$$= (a_0 + b_0) + (a_1 + b_1)T + (a_2 + b_2)F + (a_3 + b_3)I + (a_4 + b_4)Y$$

The multiplication of the symbolic components T, F, I, Y were more complicated listed in [12], as:

$$T \cdot T = T^2 = T, F \cdot F = F^2 = F, I \cdot I = I^2 = I, Y \cdot Y = Y^2 = Y, T \cdot Y = Y \cdot T = Y,$$

$$T \cdot F = F \cdot T = F, T \cdot I = I \cdot T = I, I \cdot Y = Y \cdot I = I, F \cdot Y = Y \cdot F = Y, F \cdot I = I \cdot F = I.$$

While using the absorbance law (the stronger absorbs the weaker) and the prevalence order T < F < I < Y (as chosen by author Singh [12]) it would have been much simpler.

Multiplication of STNs:

$$x \cdot y = (a_0 + a_1T + a_2F + a_3I + a_4Y) \cdot (b_0 + b_1T + b_2F + b_3I + b_4Y)$$

Then similarly multiply them term by term, taking into consideration the multiplication of symbolic components T, F, I, Y as explained above.

Scalar Multiplication in the similar way:

$$c \cdot x = c \cdot (a_0 + a_1T + a_2F + a_3I + a_4Y) = c \cdot a_0 + c \cdot a_1T + c \cdot a_2F + c \cdot a_3I + c \cdot a_4Y.$$

Consequently, the Symbolic Turiyam Group [11] and Symbolic Turiyan Ring [12], as algebraic structures, are particular cases of the **Symbolic Plithogenic Commutative Group** (defined above in sections 15.1 & 15.2), and respectively **Symbolic Plithogenic Commutative Ring** (defined above in sections 15.4 or 15.5).

17. Practical Application

Since the cases $n = 3$ and 4 of Symbolic Plithogenic Algebraic Structures have been investigated, the reader may try to develop it for the case when $n = 5$, using Hexagonal Plithogenic Numbers (*HPN*), hexa since the dimension of *HPN* is $5 + 1 = 6$ because one has 6 vectors into the base: $1, P_1, P_2, P_3, P_4, P_5$.

$HPN = a_0 + a_1P_1 + a_2P_2 + a_3P_3 + a_4P_4 + a_5P_5$, where all coefficients a_i belong to a given set.

As practical application, for example, assume that the parameters represent various colors, C_1, C_2, C_3, C_4, C_5, then we denote it as:

$HPN = a_0 + a_1C_1 + a_2C_2 + a_3C_3 + a_4C_4 + a_5C_5$.

As multiplication law of the symbolic plithogenic components C_i with C_j one adopts a law from the real world. For example, if C_1 = *yellow*, and C_2 = *red*, then it makes sense to consider $C_1 \cdot C_2$ = *pink* (because *yellow* mixed with *red* give *pink*), and so on.

In this practical application, the absorbance law does not work, that's why one designs a new law in order to be able to multiply the components.

18. Open Question

Future possible study for researchers would be to investigate the *infinite-case*, we mean when each element in the plithogenic set (section 2 above) is characterized by infinitely many attributes (parameters), and similarly the symbolic plithogenic numbers (section 3 above) have infinitely many symbolic plithogenic components $P_1, P_2, ..., P_\infty$ and, eventually, their applications.

19. Conclusion

In this paper, the new types of algebraic structures from 2018-2019, called Symbolic Plithogenic Algebraic Structures, were revisited, and afterwards compared to other related structures.

We proved that the Symbolic Plithogenic Numbers are generalizations of Neutrosophic Quadruple Numbers, Refined Neutrosophic Quadruple Numbers, and Symbolic Turiyam Numbers.

Consequently, the Symbolic Plithogenic algebraic structures (semigroup, group, ring, etc.) are generalization of the corresponding algebraic structures built on these particular cases described above. We recalled the Symbolic Plithogenic Group and Ring.

Many examples and practical applications were also revealed.

Any future application may require a special multiplication law of the components and of plithogenic numbers that the experts should design themselves.

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