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# Neutrosophic Sets and Systems

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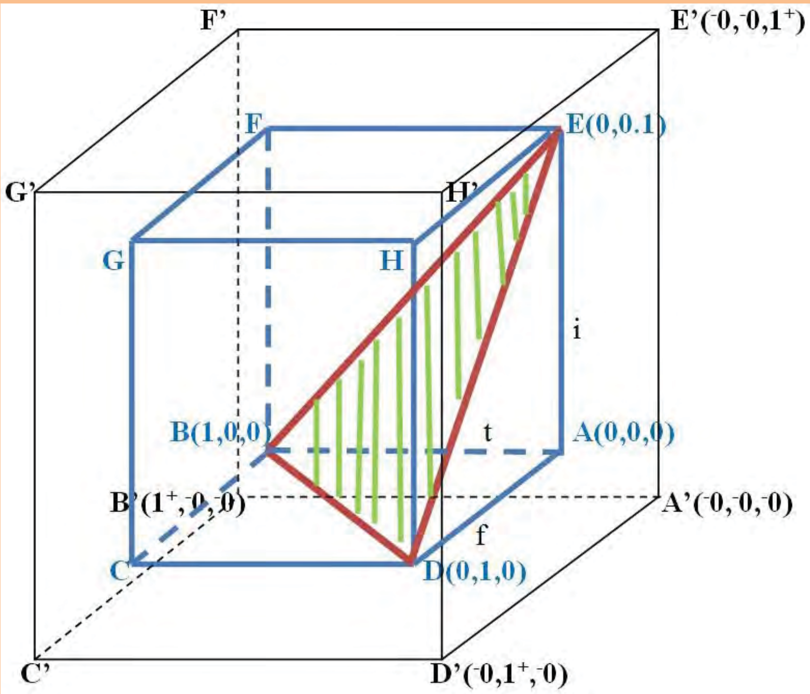


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University of New Mexico

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"Neutrosophic Sets and Systems" has been created for publications on advanced studies in neutrosophy, neutrosophic set, neutrosophic logic, neutrosophic probability, neutrosophic statistics that started in 1995 and their applications in any field, such as the neutrosophic structures developed in algebra, geometry, topology, etc.

The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

*Neutrosophy* is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

*Neutrosophic Set* and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth ($T$), a degree of indeterminacy ($I$), and a degree of falsity ($F$), where $T, I, F$ are standard or non-standard subsets of $]^-0, 1^+[$.

*Neutrosophic Probability* is a generalization of the classical probability and imprecise probability.

*Neutrosophic Statistics* is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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# Contents

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# Optimization of EOQ Model with Limited Storage Capacity by Neutrosophic Geometric Programming

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**Abstract:** In this article, we present deterministic single objective economic order quantity model with limited storage capacity in neutrosophic environment. We consider variable limit production cost and time dependent holding cost into account. Here we minimize total average cost of proposed model by applying neutrosophic geometric programming, which is obtained by extending existing fuzzy and intuitionistic fuzzy geometric programming for solving resultant non-linear optimization model. Next we consider numerical application to show that optimal solution obtained by neutrosophic geometric programming is more desirable than that of crisp, fuzzy and intuitionistic fuzzy geometric programming. Also we perform sensitivity analysis of parameters and present key managerial insights. Finally we draw the conclusions.

**Keywords:** Economic Order Quantity, Neutrosophic geometric programming, Non-linear optimization, Limited storage capacity, Shape parameter.

# 1 Introduction

We define inventory as an idle resource of any enterprise. Although idle, a certain amount of inventory is essential for smooth conduction of organisational activities. We find control of inventory as one of the key areas for operational management. We observe that an adequate control of inventory significantly brings down operating cost and increases efficiency [1, 2]. So we determine Economic Order Quantity (EOQ) to minimize total cost of inventory e.g., holding cost, order cost, and shortage cost. In most cases, optimization of corresponding mathematical model requires Non-Linear Programming(NLP). And one of the most popular and constructive method for solving NLP problem is Geometric Programming (GP). It is convenient in applications of variety of optimization models and is under general class of signomial problems. We employ it to solve large scale, real life based models by quantifying them into an equivalent optimization problem. Also GP allows sensitivity analysis to be performed efficiently.

Historically, F. Harries [3, 4] first presented concept of EOQ and subsequently Wilson applied it. Hoon Jung et al. [5,6] discussed optimal inventory policies for maximizing profit of EOQ models under various cost functions. N. K. Mondal et al. [7] considered a model with deteriorating items. S. Islam [8] formulated multi-objective inventory model with capacity constraint and shortage cost. S. Sadjadi et al. [2] considered a model with cubic demand function. L. Janseen et al. [9] presented one extensive literature review of deteriorating inventory models. In recent era, development of EOQ model is primarily based on several constraints, among which budgetary and limited storage capacitys generate considerable attention of researchers.

Again through investigation of cost minimization techniques for engineering and designing problem [10, 11] Zenner introduced notion of GP. Later, Duffin et al. [12,13] presented the mathematical formulations of GP. Kochenberger [14] was the first scientist to solve non-linear EOQ problem by GP. Beightler and Phillips [15] studied advantages of applying GP to real life based problems. Afterwards, Cheng [16, 17] formulated an EOQ model with unit production cost. Lee [18] proposed GP formulations for optimal order quantities and prices with storage capacity limitations. Nezami et al. [19] determined optimal demand rate and production quantity by GP. Sadjadi et al. [1,2] investigated the integrated pricing, lot sizing and marketing planning model and reviewed literature of last two decades. Tabatabaei et al. [20] discussed optimal pricing and marketing planning for deteriorating items. Again total inventory cost of an EOQ model is controlled by constraints in real life based imprecise environment. Among numerous constraints that affect optimal inventory cost, e.g. ceiling on storage capacity, number of orders and production cost, In this article, we consider upper limit on storage capacity to be imprecise in nature. The much needed paradigm shift to bring impreciseness in math-ematics was formaly acted by Zadeh [21]. Next Bellman and Zadeh [22] used fuzzy set in decision making problems. Tanaka et al. [23] proposed objectives as fuzzy goals. Zimmerman [24] presented solution method for multi-objective linear programming problem in fuzzy environment. In subsequent years, mathematicians developed various optimization methods and employed them in different directions. Sommer [25] employed fuzzy concept to inventory and production-scheduling problem. Park [26] examined fuzzy EOQ model. Roy and Maiti [27] solved single objective EOQ model using GP technique in fuzzy environment. Islam and Mon-dal [28] formulated one fuzzy Eeconomic Production Quantity (EPQ) model having flexibility and reliability considerations. Mahapatra et al. [29] considered fuzzy EPQ model and solved by applying parametric GP technique.

On the other hand, fuzzy set theory has been widely developed and recently several modifications have appeared. Atanassov presented Intuitionistic Fuzzy (IF) set theory, where we consider non-membership function along with membership function of imprecise information. Whereas Atanassov and Gargov [30] listed optimization in IF environment as an open problem, Angelov [31] developed optimization technique in IF environment. Pramanik and Roy [32] analyzed vector operational problem using IF goal programming. A transportation model was elucidated by Jana and Roy [33] by using multi-objective IF linear programming. Chakraborty et al. [34] applied IF optimization technique for Pareto optimal solution of manufacturing inventory model with shortages. Garai et al. [35, 36] worked on T-Sets based on optimization technique in air quality strategies and supply chain management respectively. Pramanik and Roy [37–39] applied IF goal programming approach to solve quality control problem and multi objective transportation problem also they investigated bilevel programming in said environment.

Again F. Smarandache [40, 41] introduced Neutrosophic (NS) Set, by combining nature with philosophy. It is the study of neutralities as an extension of dialectics. Interestingly, whereas IF sets can only handle incomplete information but failed in case of indeterminacy, NS set can manipulate both incomplete and imprecise information [40]. We characterize NS set by membership function (or, truth membership degree), hesitancy function (or, indeterminacy membership degree) and non-membership function (or, falsity membership degree). In NS environment, decision maker maximizes degree of membership function, minimizes both degree of indeterminacy and degree of non-membership function. Whereas we find application of NS in different directions of research, in this article, we concentrate on optimization in NS environment. Roy and Das [42] solved multi-criteria production planning problem by NS linear programming approach. Baset et al. [43] presented NS Goal Programming (NSGP) problem. Pramanik et al. [44] presented TOPSIS method for multi-attribute group decision-making under single-valued NS environment Basset et al. [45] used Analytic Hierarchy Process (AHP) in multi-criteria group decision making problems in NS environment. Also they extended AHP-SWOT analysis in NS environment [46]. Sarkar et al. [47] used NS optimization technique in truss design and multi-objective cylindrical skin plate design problem. S. Pramanik [48, 49] discussed multi-objective linear goal programming problem in neutrosophic number environment.

Recently Several researcher has worked on Multi-Criteria Decision Making (MCDM) or Multi-Attribute Decision Making (MADM) problem using neutrosophic environment. Biswas et al. [50] discussed neutrosophic MADM with unknown weight information. Mondal and Pramanik [51] extended Multi-Criteria Group Decision Making (MCGDM) approach for teacher recruitment in higher education in neutrosophic environment. Also, Biswas et al. [52] discussed MADM using entropy based grey relational analysis method under SVNSs environment. Afterwards, Mondal and Pramanik [53] explained neutrosophic decision making model for school choice. Pramanik et al. [54] investigated the contribution of some indian researchers to MADM in neutrosophic environment. Later on, Mondal and Pramanik [55] applied tangent similarity measure to neutrosophic MADM process. Mondal et al. [56] developed MADM process for SVNSs using similarity measures based on hyperbolic sine functions. Mondal et al. [57, 58] used hybrid binary logarithm similarity measure and refined similarity measure based on cotangent function to solve Multi-Attribute Group Decision Making (MAGDM) problem under SVNSs environment. Recently mondal et al. [59] analyzed interval neutrosophic tangent similarity measure based MADM strategy and its application to MADM problems. In recent era, Pramanik et al. [60–62] solved MAGDM problem using NS and IN cross entropy, also they investigate MAGDM problem for logistic center location selection. Recently Biswas et al. [63–69] discussed distance measure based MADM and TOPSIS strategies with interval trapeziodal neutrosophic numbers, also they worked on aggrega-tion of triangular fuzzy neutrosophic set, value and ambiguity index based ranking method of SVTNs, hybrid vector similarity measures and their application to MADM problem respectively.

Although we have performed extensive literature reviews and have found case studies of EOQ models in NS environment, we observe that in most cases, models are optimized through various existing software packages only. In this article, we consider one EOQ model with limited storage capacity. Next we solve it by using NSGP method.

We organize the rest of the article as follows. In Section 2, we present elementary definitions. In Section 3, we construct single objective EOQ model with limited storage capacity. In Section 4, we solve the model in crisp environment by applying classical GP. In Section 5, we present optimal solution of proposed model in fuzzy GP. In Section 6, we present optimal solution of proposed model in IFGP. In Section 7, we consider the model in NS environment and solve it by applying NSGP. Next numerical application in Section 8.1 shows that optimal solution in NS environment is more preferable than crisp, fuzzy and IF environment. Also we perform sensitivity analysis and present key managerial insights. Finally in Section 9, we draw conclusions and discuss future scopes of research.

# 2 Definitions

## 2.1 Intuitionistic fuzzy set

Let $X$ be an universal set. An intuitionistic fuzzy set $A$ in $X$ is an object of the form:

$$A = \left\{ (x, \mu_A(x), \nu_A(x)) : x \in X \right\}.$$

Here $\mu_A(x) : X \to [0,1]$ and $\nu_A(x) : X \to [0,1]$ are membership function and non-membership function of $A$ in $X$ respectively and satisfy the condition $0 \leq \mu_A(x) + \nu_A(x) \leq 1, \forall x \in X$.

## 2.2 Neutrosophic set

Let $X$ be an universal set. A neutrosophic (NS) set $A \in X$ is defined by:

$$A = \left\{ (x, \mu_A(x), \sigma_A(x), \nu_A(x)) : x \in X \right\}.$$

Here $\mu_A(x), \sigma_A(x)$ and $\nu_A(x)$ are called membership function, hesitancy function and non-membership function respectively. They are respectively defined by:

$$\mu_A(x) : X \to \left]0^-, 1^+\right[, \ \sigma_A(x) : X \to \left]0^-, 1^+\right[, \ \nu_A(x) : X \to \left]0^-, 1^+\right[$$

subject to $0^- \leq sup\, \mu_A(x) + sup\, \sigma_A(x) + sup\, \nu_A(x) \leq 3^+$.

## 2.3 Single valued NS set

Let $X$ be an universal set. A single valued NS set $A \in X$ is defined by:

$$\mu_A(x) : X \to [0,1], \sigma_A(x) : X \to [0,1], \nu_A(x) : X \to [0,1]$$

subject to $0 \leq \mu_A(x) + \sigma_A(x) + \nu_A(x) \leq 3$. here $\mu_A(x), \sigma_A(x)$ and $\nu_A(x)$ are called membership function, hesitancy function and non-membership function respectively.

## 2.4 Union of two NS sets

Let $X$ be an universal set and $A$ and $B$ are any two subsets of $X$. Here $\mu_A(x) : X \to [0,1]$, $\sigma_A(x) : X \to [0,1]$ and $\nu_A(x) : X \to [0,1]$ are membership function, hesitancy function and non-membership function of $A$ respectively. Then union of $A$ and $B$ is denoted by $A \cup B$ and is definded as:

$$A \cup B = \left\{ (x, max(\mu_A(x), \mu_B(x)), max(\sigma_A(x), \sigma_B(x)), min(\nu_A(x), \nu_B(x))) : x \in X \right\}.$$

## 2.5 Intersection of two NS sets

Let $X$ be an universal set and $A$ and $B$ are any two subsets of $X$. Here $\mu_A(x) : X \to [0,1]$, $\sigma_A(x) : X \to [0,1]$ and $\nu_A(x) : X \to [0,1]$ are membership function, hesitancy function and non-membership function of $A$

respectively. Then intersection of $A$ and $B$ is denoted by $A \cap B$ and is definded as:

$$A \cap B = \{(x, min(\mu_A(x), \mu_B(x)), min(\sigma_A(x), \sigma_B(x)), max(\nu_A(x), \nu_B(x))) : x \in X\}.$$

# 3 Formulation of single objective EOQ model with limited storage capacity

In this article, we take a single objective EOQ model, along with limited storage capacity. Here we take following unit production cost:

$$P(D, S) = \theta D^{-x} S^{-1}$$

We note that shape parameter ($x$) should lie within pre-determined values so as to satisfy positivity conditions of Dual Geometric Programming Problem (DGPP). We present the notations and assumptions of proposed model, for which explanations are given in Table 9, as follows:

## 3.1 Assumptions

To specify scopes of study and to further simplify the proposed EOQ model, we consider following assumptions
(i) proposed EOQ model shall involve exactly one item;
(ii) we consider infinite rate for instantaneously replenishment;
(iii) lead time is negligible;
(iv) we take demand rate as constant;
(v) the holding cost of proposed model is a funtion of time, i.e. we take $H(t) = at$;
(vi) upgradation to modern machineries involves higher costs, which is a part of set up cost. Since these machineries have higher production rates and other advantages, large scale production can bring down the unit production cost and it is generally adopted when demand is high. Therefore we find that unit production cost is inversely releated to set-up cost and rate of demand. Hence we get as follows:

$$P(D, S) = \theta D^{-x} S^{-1}; \ \ \theta, x \in R^{+}$$

(vii) We do not allow any shortage in inventory.

## 3.2 Formulation of model

In this article, we take initial inventory level at $t = 0$ as $Q$. Also inventory level gradually decreases in [0, T] and it is zero at time T. Since we do not allow shortage, the cycle is repeated over time period T. We illustrate the proposed inventory model graphically in Fig.1. Here inventory level at any time t in [0, T] is denoted by Q(t). Hence differential equation for instantaneous inventory level Q(t) at time t in [0, T] is as follows:

$$\frac{dI(t)}{dt} = -D \qquad for \ \ 0 \leqslant t \leqslant T$$

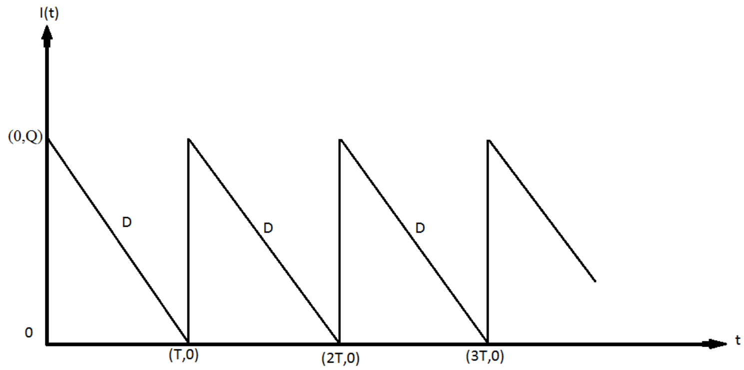


Figure 1: Inventory model

with boundary conditions as $I(0) = Q$, $I(T) = 0$.
By applying those conditions, we obtain as follows:

$$I(t) = D(T - t)$$

Therefore inventory holding cost becomes as follows:

$$\int_0^T H(t)I(t)d(t) = \frac{aQ^3}{6D^2}$$

Hence total average inventory cost per cycle [0, T] is as follows:

$$TAC(D, S, Q) = \frac{SD}{Q} + \frac{aQ^2}{6D} + \theta D^{1-x} S^{-1}$$

Here maximum floor capacity for storing items in warehouse is $W$. So storage area $w_0Q$ for production quantity $Q$ can never go beyond maximum floor capacity in warehouse for storing items at any time $t$. Therefore limited storage capacity is as follows:

$$w_0Q \leqslant W$$

Finally we have inventory model in crisp environment as follows:

$$\min TAC(D, S, Q) = \frac{SD}{Q} + \frac{aQ^2}{6D} + \theta D^{1-x} S^{-1} \tag{3.1}$$

subject to

$$S(Q) \equiv w_0Q \leqslant W$$

$$D, S, Q > 0.$$

# 4 Solution of EOQ model by crisp GP

We apply classical or crisp GP to solve proposed EOQ model. Here DD is 0. We apply Duffin and Peterson theorem [13] of GP on equation (3.1) and obtain DGPP as follows:

$$\begin{aligned}
&\max \mathrm{d(w)} = \left(\frac{1}{w_{01}}\right)^{w_{01}} \left(\frac{a}{6w_{02}}\right)^{w_{02}} \left(\frac{\theta}{w_{03}}\right)^{w_{03}} \left(\frac{w_0}{Ww_{11}}\right)^{w_{11}} {w_{11}}^{w_{11}} \\
&\text{subject to} \\
&w_{01} + w_{02} + w_{03} = 1, \\
&w_{01} - w_{02} + (1-x)w_{03} = 0, \\
&w_{01} - w_{03} = 0, \\
&-w_{01} + 2w_{02} + w_{11} = 0, \\
&w_{01}, w_{02}, w_{03}, w_{11} \geq 0.
\end{aligned}$$

The optimal solution in crisp environment is as follows:

$$w_{01}^* = w_{03}^* = \frac{1}{4-x}, \;\; w_{02}^* = \frac{2-x}{4-x}, \;\; w_{11}^* = \frac{2x-3}{4-x}.$$

Since value of shape paremeter $x$ has to lie in interval $[1.5, 2]$, all dual variables remain positive. Thus optimal values of primal variables are as follows:

$$\begin{aligned}
D^* &= \left\{ \frac{1}{\theta} \left(\frac{a}{6(2-x)}\right)^2 \left(\frac{W}{w_0}\right)^5 \right\}^{\frac{1}{4-x}}, \\
S^* &= \left\{ \left(\frac{\theta W}{w_0}\right)^2 \left(\frac{6w_0^3(2-x)}{aW^3}\right)^x \right\}^{\frac{1}{4-x}}, \\
Q^* &= \frac{W}{w_0}\,.
\end{aligned}$$

with optimal TAC as follows:

$$TAC^*(D^*, S^*, Q^*) = (4-x) \left\{ \theta \, \left(\frac{w_0}{W}\right)^{(2x-3)} \left(\frac{a}{6(2-x)}\right)^{(2-x)} \right\}^{\frac{1}{4-x}} = T_1 \;\; (say)$$

# 5 Solution of EOQ model by fuzzy GP

We apply max-additive operator to solve proposed EOQ model in fuzzy environment. Here we compute individual optimum values of objective function: TAC and constraint: limited storage capacity of model (3.1), as given in Table 1. Also DM supplies goal and goal plus tolerance values for membership functions of objective function and constraint. For sake of simplicity, we consider linear membership function for TAC and limited storage capacity as follows:

Table 1: Individual maximum and minimum values of decision variables and TAC

| | Maximum value | Minimum value |
|---|---|---|
| Demand per unit time $(D)$ | $\left\{\frac{1}{\theta}\left(\frac{a}{6(2-x)}\right)^2\left(\frac{W}{w_0}\right)^5\right\}^{\frac{1}{4-x}}$ | $\left\{\frac{1}{\theta}\left(\frac{a}{6(2-x)}\right)^2\left(\frac{W+w_p}{w_0}\right)^5,\right\}^{\frac{1}{4-x}}$ |
| Set up cost $(S)$ | $\left\{\left(\frac{\theta W}{w_0}\right)^2\left(\frac{6w_0^3(2-x)}{aW^3}\right)^x\right\}^{\frac{1}{4-x}}$ | $\left\{\left(\frac{\theta(W+w_p)}{w_0}\right)^2\left(\frac{6w_0^3(2-x)}{a(W+w_p)^3}\right)^x\right\}^{\frac{1}{4-x}}$ |
| Production quantity per batch $(Q)$ | $\frac{W}{w_0}$ | $\frac{W+w_p}{w_0}$ |
| Total Average Cost TAC(D, S, Q) | $(4-x)\left\{\theta\ \left(\frac{w_0}{W}\right)^{(2x-3)}\left(\frac{a}{6(2-x)}\right)^{(2-x)}\right\}^{\frac{1}{4-x}}$ | $(4-x)\left\{\theta\ \left(\frac{w_0}{W+w_p}\right)^{(2x-3)}\left(\frac{a}{6(2-x)}\right)^{(2-x)}\right\}^{\frac{1}{4-x}}$ |

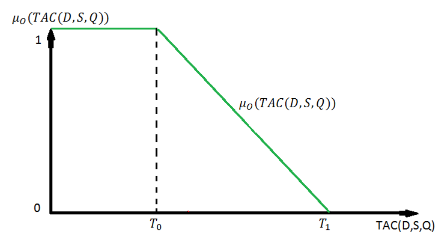


Figure 2: Membership function of fuzzy objective function

$$\mu_{\tilde{O}}(TAC(D,S,Q)) = \begin{cases} 1 & \text{if} \quad TAC(D,S,Q) \leqslant T_0 \\ \frac{T_1 - TAC(D,S,Q)}{T_1 - T_0} & \text{if} \quad T_0 \leqslant TAC(D,S,Q) \leqslant T_1 \\ 0 & \text{if} \quad \text{otherwise.} \end{cases}$$

$$\mu_{\tilde{C}}(S(Q)) = \begin{cases} 1 & \text{if} \quad w_0 Q \leqslant W \\ \frac{W + w_p - w_0 Q}{w_p} & \text{if} \quad W \leqslant w_0 Q \leqslant W + w_p \\ 0 & \text{if} \quad \text{otherwise.} \end{cases}$$

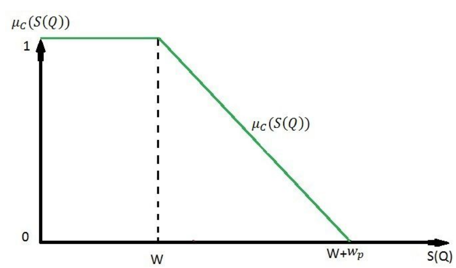


Figure 3: Membership function of fuzzy constraint

Next we formulate the mathematical model as follows:

$$\begin{aligned} &\max \ \ \{\mu_{\tilde{O}}(TAC(D,S,Q))\mu_{\tilde{C}}(S(Q))\} \\ &\text{subject to} \\ &0 < \mu_{\tilde{O}}(TAC(D,S,Q)) + \mu_{\tilde{C}}(S(Q)) < 1, \\ &D, S, Q > 0. \end{aligned}$$

By applying max-additive operator, we get crisp Primal Geometric Programming Problem (PGPP) and use convex combination operator to obtain as follows:

$$\max VF_{FA}(D,S,Q) = F_K - VF_{FA1}(D,S,Q)$$

Here $F_K = \frac{T_0}{T_1-T_0} + \frac{W+w_p}{w_p}$ and $VF_{FA1}(D,S,Q) = \frac{TAC(D,S,Q)}{T_1-T_0} + \frac{w_0 Q}{w_p}$.
Therefore the problem reduces to the following model:

$$\begin{aligned}
&\min VF_{FA1}(D,S,Q) = \frac{SD}{Q(T_1-T_0)} + \frac{aQ^2}{6D(T_1-T_0)} + \frac{\theta D^{1-x}}{(T_1-T_0)S} + \frac{w_0 Q}{w_p}\\
&\text{subject to}\\
&D,S>0, Q \in \left[\frac{W}{w_0}, \frac{W+w_p}{w_0}\right], TAC(D,S,Q) \in [T_0,T_1].
\end{aligned} \tag{5.1}$$

It is unconstrained PGPP with DD = 0. Hence optimal values for primal variables of model (5.1) are as follows:

$$\begin{aligned}
D^* &= \frac{3}{2}\left\{\theta\, 6^{(x-2)}\left(\frac{a}{2-x}\right)^3\left(\frac{w_0}{w_p}\right)^{(2x-3)}\left(\frac{2x-3}{T_1-T_0}\right)^5\right\}^{\frac{1}{x+1}},\\
S^* &= 2\left\{\theta\, 6^{(x-2)}\left(\frac{T_1-T_0}{2x-3}\right)^{(3x-2)}\left(\frac{w_0}{w_p}\right)^{(2x-3)}\left(\frac{a}{2-x}\right)^{(1-2x)}\right\}^{\frac{1}{x+1}},\\
Q^* &= 3(2x-3)\left\{\theta\left(\frac{1}{T_1-T_0}\right)^{(4-x)}\left(\frac{a}{6(2-x)}\right)^{(2-x)}\left(\frac{w_0}{w_p(2x-3)}\right)^{(2x-3)}\right\}^{\frac{1}{x+1}},
\end{aligned}$$

with optimal TAC as follows:

$$TAC^*(D^*,S^*,Q^*) = \left[\left\{\theta\left(\frac{a}{6(2-x)}\right)^{(2-x)}\left(\frac{w_0(T_1-T_0)}{w_p(2x-3)}\right)^{(2x-3)}\right\}^{\frac{1}{x+1}}\left\{1+\left(\frac{2-x}{6^x}\right)^{\frac{1}{x+1}}+\left(\frac{w_p}{3w_0}\right)^{(2x-3)}\right\}\right]$$

provided $Q^* \in \left[\frac{W}{w_0}, \frac{W+w_p}{w_0}\right]$, $\text{TAC}^*(D^*,S^*,Q^*) \in [T_0,T_1]$.

# 6 Solution of EOQ model by IFGP

We employ IF optimization method and solve proposed EOQ model (3.1). Goal and goal plus tolerance values of non-membership functions of TAC and limited storage capacity, as obtained from DM, are given in Table 1. Based on these values, we construct following linear non-membership functions of TAC and limited storage capacity:

$$\nu_{\tilde{O}}(TAC(D,S,Q)) = \begin{cases} 0 & \text{if } TAC(D,S,Q) \leqslant T_0+\epsilon_o\\ \frac{TAC(D,S,Q)-T_0-\epsilon_o}{T_1-T_0-\epsilon_o} & \text{if } T_0+\epsilon_o \leqslant TAC(D,S,Q) \leqslant T_1\\ 1 & \text{otherwise.}\end{cases}$$

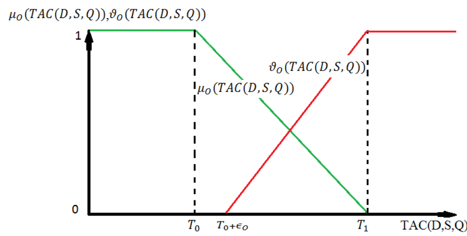


Figure 4: Membership and non-membership function of IF objective function

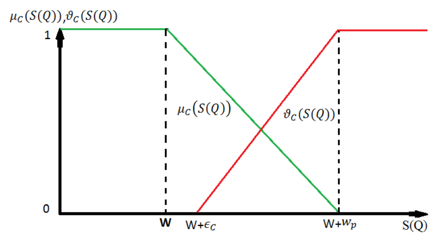


Figure 5: Membership and non-membership function for IF constraint

$$\nu_{\tilde{C}}(S(Q)) = \begin{cases} 0 & \text{if} \quad w_0 Q \leqslant W + \epsilon_C \\ \frac{w_0 Q - W - \epsilon_C}{w_p - \epsilon_C} & \text{if} \quad W + \epsilon_C \leqslant w_0 Q \leqslant W + w_p \\ 1 & \text{if} \quad w_0 Q \geq W + w_p \end{cases}$$

Next we formulate EOQ model as follows:

$$\begin{aligned} & \max \quad \{\mu_{\tilde{O}}(TAC(D,S,Q))\mu_{\tilde{C}}(S(Q))\} \\ & \min \quad \{\nu_{\tilde{O}}(T(D,S,Q)), \nu_{\tilde{C}}(S(Q))\} \\ & \text{subject to} \\ & 0 < \mu_{\tilde{O}}(TAC(D,S,Q)) + \nu_{\tilde{O}}(TAC(D,S,Q)) < 1; \\ & 0 < \mu_{\tilde{C}}(S(Q)) + \nu_{\tilde{C}}(S(Q)) < 1; \\ & D, S, Q > 0. \end{aligned}$$

By applying max-additive operator and then GP in IF environment, we obtain optimal decision variables as follows:

$$D^* = \left\{ \theta \left( \frac{a}{6(2-x)} \right)^3 \left( \frac{I_{K1}\,(2x-3)}{I_{K2}\, w_0} \right)^5 \right\}^{\frac{1}{x+1}},$$

$$S^* = \left\{ \theta \, \left( \frac{I_{K2}\, w_0}{I_{K1}\,(2x-3)} \right)^{(3x-2)} \left( \frac{a}{6(2-x)} \right)^{(1-2x)} \right\}^{\frac{1}{x+1}},$$

$$Q^* = \left\{ \theta \, \left( \frac{I_{K1}\,(2x-3)}{I_{K2}\, w_0} \right)^{(4-x)} \left( \frac{a}{6(2-x)} \right)^{(2-x)} \right\}^{\frac{1}{x+1}}$$

with optimal TAC as follows:

$$TAC^*(D^*, S^*, Q^*) = \left[ \left\{ \theta \left(\frac{a}{6}\right)^{(2-x)} \left(\frac{IK_1(2x-3)}{Ik_2\, w_0}\right)^{(3-2x)} \right\}^{\frac{1}{x+1}} \left\{ 2\left(\frac{1}{2-x}\right)^{\left(\frac{2-x}{x+1}\right)} + \left(\frac{1}{2-x}\right)^{\left(\frac{1-2x}{x+1}\right)} \right\} \right]$$

provided$Q^* \in \left[\frac{W+\epsilon_C}{w_0}, \frac{W+w_p}{w_0}\right]$, $\mathrm{TAC^*(D^*, S^*, Q^*)} \in [\mathrm{T_0} + \epsilon_{\mathrm{O}}, \mathrm{T_1}]$.

# 7 Solution of EOQ model by NSGP

The world is full of indeterminacy and hence we require more precise imprecision. Thus the concept of NS set comes into picture. We consider membership function, hesitancy function, non-membership function for each objective function and constraint of proposed model. we consider same memebership function, as given in Section 5 and same non-membership function, as given in Section 6. We take hesitancy functions for objective function and constraint as follows:

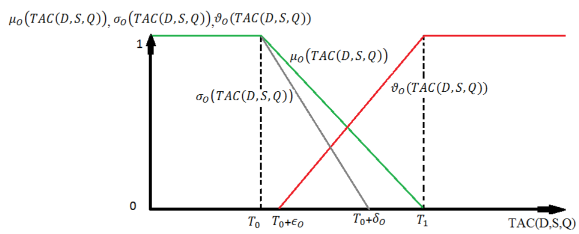


Figure 6: Membership, hesitancy and non-membership function of objective function in NS environment.

$$\sigma_{\tilde{O}}(TAC(D,S,Q)) = \begin{cases} 1 & \text{if } \ TAC(D,S,Q) \leqslant T_0 \\ \frac{T_0+\delta_o-TAC(D,S,Q)}{\delta_o} & \text{if } \ T_0 \leqslant TAC(D,S,Q) \leqslant T_0+\delta_o \\ 0 & \text{if } \ TAC(D,S,Q) \geq T_0+\delta_o \end{cases}$$

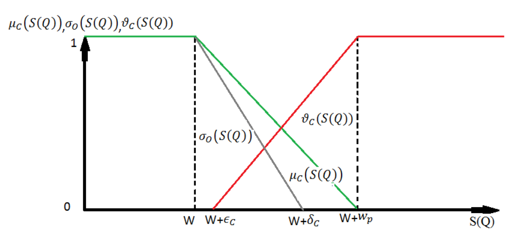


Figure 7: Membership, hesitancy and non-membership function of constraint in NS environment.

$$\sigma_{\tilde{C}}(S(Q)) = \begin{cases} 1 & \text{if } \ w_0 Q \leqslant W \\ \frac{W+\delta_c-w_0 Q}{\delta_c} & \text{if } \ W \leqslant w_0 Q \leqslant W+\delta_c \\ 0 & \text{if } \ w_0 Q \geq W+\delta_C \end{cases}$$

We note that $0 < \epsilon_C$ , $\delta_c < w_p$. Here we consider the case when hesitancy function behaves like non-membership function. We present several more cases in Table 2. Then linear hesitancy functions of objective function and constraint are as follows:

Table 2: On different natures of hesitancy function

| Nature of hesitancy function in | | Value of parameter | |
|---|---|---|---|
| Objective function | Constraint | $N_{k1}$ | $N_{k2}$ |
| non-increasing | non-increasing | $\left(\frac{1}{T_1-T_0}+\frac{1}{\delta_o}+\frac{1}{T_1-T_0-\epsilon_O}\right)$ | $\left(\frac{1}{w_p}+\frac{1}{\delta_c}+\frac{1}{w_p-\epsilon_C}\right)$ |
| non-decreasing | non-decreasing | $\left(\frac{1}{T_1-T_0}+\frac{1}{T_1-T_0-\delta_o}+\frac{1}{T_1-T_0-\epsilon_O}\right)$ | $\left(\frac{1}{w_p}+\frac{1}{w_p-\delta_c}+\frac{1}{w_p-\epsilon_C}\right)$ |
| non-increasing | non-decreasing | $\left(\frac{1}{T_1-T_0}+\frac{1}{\delta_o}+\frac{1}{T_1-T_0-\epsilon_O}\right)$ | $\left(\frac{1}{w_p}+\frac{1}{w_p-\delta_c}+\frac{1}{w_p-\epsilon_C}\right)$ |
| non-decreasing | non-increasing | $\left(\frac{1}{T_1-T_0}+\frac{1}{T_1-T_0-\delta_o}+\frac{1}{T_1-T_0-\epsilon_O}\right)$ | $\left(\frac{1}{w_p}+\frac{1}{\delta_c}+\frac{1}{w_p-\epsilon_C}\right)$ |

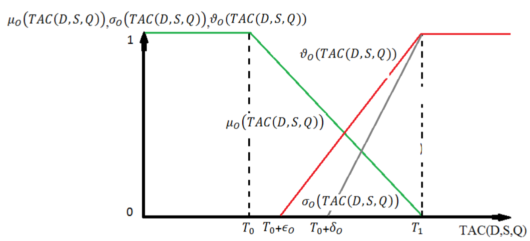


Figure 8: Membership, hesitancy and non-membership function of objective function in NS environment.

$$\sigma_{\tilde{O}}(TAC(D,S,Q)) = \begin{cases} 0 & \text{if} \quad TAC(D,S,Q) \leqslant T_0+\delta_o \\ \frac{TAC(D,S,Q)-(T_0+\delta_o)}{T_1-T_0-\delta_o} & \text{if} \quad T_0+\delta_o \leqslant TAC(D,S,Q) \leqslant T_1 \\ 1 & \text{if} \quad TAC(D,S,Q) \geq T_1 \end{cases}$$

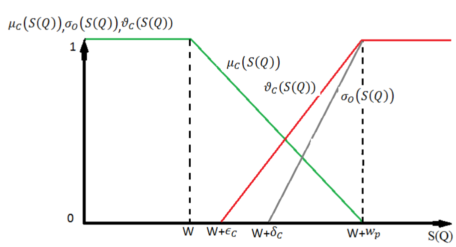


Figure 9: Membership, hesitancy and non-membership function of constraint in NS environment.

$$\sigma_{\tilde{C}}(S(Q)) = \begin{cases} 0 & \text{if} \quad w_0Q \leqslant W+\delta_c \\ \frac{w_0Q-(W+\delta_c)}{w_p-\delta_c} & \text{if} \quad W+\delta_c \leqslant w_0Q \leqslant W+w_p \\ 1 & \text{if} \;\; w_0Q \geq W+w_p \end{cases}$$

Then we obtain following optimization model in NS environment:

$$
\begin{aligned}
&\max \ \{\mu_{\tilde{O}}(TAC(D,S,Q))\mu_{\tilde{C}}(S(Q))\}\\
&\max \ \{\sigma_{\tilde{O}}(T(D,S,Q)),\sigma_{\tilde{C}}(S(Q))\}\\
&\min \ \{\nu_{\tilde{O}}(T(D,S,Q)),\nu_{\tilde{C}}(S(Q))\}\\
&\text{subject to}\\
&\mu_{\tilde{O}}(TAC(D,S,Q)) \geq \sigma_{\tilde{O}}(TAC(D,S,Q)), \mu_{\tilde{C}}(S(Q)) \geq \sigma_{\tilde{C}}(S(Q))\\
&\mu_{\tilde{O}}(TAC(D,S,Q)) \geq \nu_{\tilde{O}}(TAC(D,S,Q)), \mu_{\tilde{C}}(S(Q)) \geq \nu_{\tilde{C}}(S(Q))\\
&0 \leqslant \mu_{\tilde{O}}(TAC(D,S,Q)), \sigma_{\tilde{O}}(TAC(D,S,Q)), \nu_{\tilde{O}}(TAC(D,S,Q)) \leq 1\\
&0 \leq \mu_{\tilde{C}}(S(Q)), \sigma_{\tilde{C}}(S(Q)), \nu_{\tilde{C}}(S(Q)) \leq 1\\
&D,S,Q > 0.
\end{aligned}
$$

The corrosponding single objective optimization model is as follows:

$$
\begin{aligned}
Max\, VF_{NFA}(D,S,Q) = {} & \mu_{\tilde{O}}(TAC(D,S,Q)) + \mu_{\tilde{C}}(S(Q)) + \sigma_{\tilde{O}}(TAC(D,S,Q))\\
& + \sigma_{\tilde{C}}(S(Q)) \quad - \nu_{\tilde{O}}(TAC(D,S,Q)) - \nu_{\tilde{C}}(S(Q))\\
&\text{subject to}\\
&\mu_{\tilde{O}}(TAC(D,S,Q)) \geq \sigma_{\tilde{O}}(TAC(D,S,Q)), \mu_{\tilde{C}}(S(Q)) \geq \sigma_{\tilde{C}}(S(Q))\\
&\mu_{\tilde{O}}(TAC(D,S,Q)) \geq \nu_{\tilde{O}}(TAC(D,S,Q)), \mu_{\tilde{C}}(S(Q)) \geq \nu_{\tilde{C}}(S(Q))\\
&0 \leqslant \mu_{\tilde{O}}(TAC(D,S,Q)), \sigma_{\tilde{O}}(TAC(D,S,Q)), \nu_{\tilde{O}}(TAC(D,S,Q)) \leq 1\\
&0 \leq \mu_{\tilde{C}}(S(Q)), \sigma_{\tilde{C}}(S(Q)), \nu_{\tilde{C}}(S(Q)) \leq 1;\\
&D,S,Q > 0.
\end{aligned}
$$

We rewrite the above model as follows:

$$
\begin{aligned}
&\max \mathrm{VF_{NFA}(D,S,Q) = N_K - VF_{NFA1}(D,S,Q)}\\
&\text{subject to}\\
&D,S > 0, Q \in \left[\frac{W+\epsilon_C}{w_0}, \frac{W+w_p}{w_0}\right], \mathrm{TAC(D,S,Q)} \in [\mathrm{T_0} + \epsilon_{\mathrm{O}}, \mathrm{T_1}].
\end{aligned}
$$

Here $N_K = \left(\frac{T_1}{T_1-T_0} + \frac{T_0+\delta_o}{\delta_o} + \frac{T_0+\epsilon_O}{T_1-T_0-\epsilon_O}\right) + \left(\frac{W+w_p}{w_p} + \frac{W+\delta_c}{\delta_c} + \frac{W+\epsilon_C}{w_p-\epsilon_C}\right)$,
$VF_{NFA1}(D,S,Q) = \frac{N_{K1}\, SD}{Q} + \frac{N_{K1}\, aQ^2}{6D} + N_{K1}\, \theta D^{1-x} S^{-1} + N_{K2}\, w_0 Q$,
with $N_{K1} = \left(\frac{1}{T_1-T_0} + \frac{1}{\delta_o} + \frac{1}{T_1-T_0-\epsilon_O}\right)$ *and* $N_{K2} = \left(\frac{1}{w_p} + \frac{1}{\delta_c} + \frac{1}{w_p-\epsilon_C}\right)$.
Hence unconstrainted PGPP is as follows:

$$
\begin{aligned}
&\min \mathrm{VF_{NFA1}(D,S,Q)} = \frac{\mathrm{N_{K1}\ SD}}{\mathrm{Q}} + \frac{\mathrm{N_{K1}\ aQ^2}}{\mathrm{6D}} + \frac{\mathrm{N_{K1}}\ \theta \mathrm{D^{1-x}}}{\mathrm{S}} + \mathrm{N_{K2}\ w_0 Q}\\
&\text{subject to}\\
&D,S > 0, Q \in \left[\frac{W+\epsilon_C}{w_0}, \frac{W+w_p}{w_0}\right], \mathrm{TAC(D,S,Q)} \in [\mathrm{T_0} + \epsilon_{\mathrm{O}}, \mathrm{T_1}]. \qquad (7.1)
\end{aligned}
$$

Here DD=0. we solve above model by NSGP [8, 14] and obtain as follows:

$$\max d(w) = \left(\frac{N_{K1}}{w_{01}}\right)^{w_{01}} \left(\frac{aN_{K1}}{6w_{02}}\right)^{w_{02}} \left(\frac{\theta N_{K1}}{w_{03}}\right)^{w_{03}} \left(\frac{w_0 N_{K2}}{w_{04}}\right)^{w_{04}}$$

subject to

$$w_{01} + w_{02} + w_{03} + w_{04} = 1,$$
$$w_{01} - w_{02} + (1-x)w_{03} = 0,$$
$$w_{01} - w_{03} = 0,$$
$$-w_{01} + 2w_{02} + w_{04} = 0,$$
$$w_{01}, w_{02}, w_{03}, w_{04} \geq 0.$$

Therefore optimal dual variables are as follows:

$$w^*_{01} = \frac{1}{4-x}, w^*_{02} = \frac{2-x}{4-x}, w^*_{03} = \frac{1}{4-x}, w^*_{04} = \frac{2x-3}{4-x}.$$

Hence optimal decision variables are as follows:

$$D^* = \left\{\theta \left(\frac{a}{6(2-x)}\right)^3 \left(\frac{N_{K1}\,(2x-3)}{N_{K2}\,w_0}\right)^5\right\}^{\frac{1}{x+1}}$$

$$S^* = \left\{\theta\, \left(\frac{N_{K2}\,w_0}{N_{K1}\,(2x-3)}\right)^{(3x-2)} \left(\frac{a}{6(2-x)}\right)^{(1-2x)}\right\}^{\frac{1}{x+1}}$$

$$Q^* = \left\{\theta\, \left(\frac{N_{K1}\,(2x-3)}{N_{K2}\,w_0}\right)^{(4-x)} \left(\frac{a}{6(2-x)}\right)^{(2-x)}\right\}^{\frac{1}{x+1}}$$

with optimal TAC as follows:

$$TAC^*(D^*, S^*, Q^*) = \left[\left\{\theta \left(\frac{a}{6}\right)^{(2-x)} \left(\frac{N_{KI}(2x-3)}{N_{K2}\,w_0}\right)^{(3-2x)}\right\}^{\frac{1}{x+1}} \left\{2\left(\frac{1}{2-x}\right)^{\left(\frac{2-x}{x+1}\right)} + \left(\frac{1}{2-x}\right)^{\left(\frac{1-2x}{x+1}\right)}\right\}\right]$$

provided$Q^* \in \left[\frac{W+\epsilon_C}{w_0}, \frac{W+w_p}{w_0}\right]$, $\mathrm{TAC}^*(\mathrm{D}^*, \mathrm{S}^*, \mathrm{Q}^*) \in [\mathrm{T}_0 + \epsilon_\mathrm{O}, \mathrm{T}_1]$.

# 8 Numerical application

We consider a simple numerical application to solve proposed model in NS environment as follows:

A manufacturing company produces machines $PBA_{597}$. The inventory carrying cost for the machines is $Rs.105$ per unit per year. The production cost of this machine varies inversely with the demand and set-up cost. From the past experiences, we can consider the production cost of the machine $PBA_{597}$ at about $120D^{-0.75}S^{-1}$, where $D$ is the demand rate and $S$ is the set-up cost. The company has storage capacity area per unit time $(w_0)$ and total storage capacity area $(W)$ as $100$ sq. ft. and $2000$ sq. ft. respectively. The task is

to determine the optimal demand rate $(D)$, set-up cost $(S)$, production quantity $(Q)$ and hence optimal TAC of the production system.
Here mathematical model is of the following form:

$$\min TAC(D,S,Q) = \frac{SD}{Q} + \frac{105Q^2}{6D} + 120D^{-0.75}S^{-1} \tag{8.1}$$
subject to
$$S(Q) \equiv 100Q \leqslant 2000,$$
$$D,S,Q > 0.$$

We consider goal and goal plus tolerance values for TAC and limited storage capacity as given in Table 3. Based on these values, we construct following linear membership, hesitancy and non-membership functions of TAC and limited storage capacity:

Table 3: Goal and goal plus tolerance values of TAC and variables

| | Demand $(D)$ | Set-up cost $(S)$ | Production quantity $(Q)$ | Total Average Cost (TAC$(D,S,Q)$) |
|---|---|---|---|---|
| Goal | 4047.477 | 0.034 | 20.000 | 15.565 |
| Goal plus tolerance | 5521.645 | 0.028 | 23.000 | 15.089 |

$$\mu_{\tilde{O}}(TAC(D,S,Q)) = \begin{cases} 1 & \text{if } TAC(D,S,Q) \leqslant 15.089 \\ \frac{15.565-TAC(D,S,Q)}{0.476} & \text{if } 15.089 \leqslant TAC(D,S,Q) \leqslant 15.565 \\ 0 & \text{if otherwise.} \end{cases}$$

$$\mu_{\tilde{C}}(S(Q)) = \begin{cases} 1 & \text{if } 100Q \leqslant 2000 \\ \frac{2300-100Q}{300} & \text{if } 2000 \leqslant 100Q \leqslant 2300 \\ 0 & \text{if otherwise.} \end{cases}$$

$$\sigma_{\tilde{O}}(TAC(D,S,Q)) = \begin{cases} 1 & \text{if } TAC(D,S,Q) \leqslant 15.089 \\ \frac{15.389-TAC(D,S,Q)}{0.3} & \text{if } 15.089 \leqslant TAC(D,S,Q) \leqslant 15.389 \\ 0 & \text{if } TAC(D,S,Q) \geq 15.389 \end{cases}$$

$$\sigma_{\tilde{C}}(S(Q)) = \begin{cases} 1 & \text{if } 100Q \leqslant 2000 \\ \frac{2170-100Q}{170} & \text{if } 2000 \leqslant 100Q \leqslant 2170 \\ 0 & \text{if } 100Q \geq 2170 \end{cases}$$

$$\nu_{\tilde{O}}(TAC(D,S,Q)) = \begin{cases} 0 & \text{if } TAC(D,S,Q) \leqslant 15.306 \\ \frac{TAC(D,S,Q)-15.306}{0.259} & \text{if } 15.306 \leqslant TAC(D,S,Q) \leqslant 15.565 \\ 1 & \text{if otherwise.} \end{cases}$$

$$\nu_{\tilde{C}}(S(Q)) = \begin{cases} 0 & \text{if } 100Q \leqslant 2070 \\ \frac{100Q-2070}{230} & \text{if } 2070 \leqslant 100Q \leqslant 2300 \\ 1 & \text{if } 100Q \geq 2300 \end{cases}$$

Therefore single objective EOQ model with limited storage capacity is as follows:

$$\begin{aligned}
&\min TAC(D,S,Q) = \frac{9.295SD}{Q} + \frac{162.663Q^2}{D} + 1115.4D^{-0.75}S^{-1} + 1.356Q \\
&\text{subject to} \\
&D,S>0, Q\in[20.5,23], \mathrm{TAC}(D,S,Q)=[15.089,15.565]
\end{aligned} \tag{8.2}$$

We solve the model (8.2) by GP. Here DD = 0. Hence DGPP of (8.2) is as follows:

$$\begin{aligned}
&\max d(w) = \left(\frac{9.295}{w_{01}}\right)^{w_{01}} \left(\frac{162.663}{w_{02}}\right)^{w_{02}} \left(\frac{1115.4}{w_{03}}\right)^{w_{03}} \left(\frac{1.356}{w_{04}}\right)^{w_{04}} \\
&\text{subject to} \\
&w_{01}+w_{02}+w_{03}+w_{04}=1, \\
&w_{01}-w_{02}+(1-x)w_{03}=0, \\
&w_{01}-w_{03}=0, \\
&-w_{01}+2w_{02}+w_{04}=0, \\
&w_{01},w_{02},w_{03},w_{04}\geq 0.
\end{aligned}$$

Therefore optimal values of dual variables are as follows:

$$w_{01}^*=0.444, w_{02}^*=0.111, w_{03}^*=0.444, w_{04}^*=0.222.$$

Hence optimal values of decision variables are as follows:

$$D^*=5575.110, S^*=0.028, Q^*=22.998, TAC^*(D^*,S^*,Q^*)=15.094.$$

We note that optimal TAC is $15.094$ units with demand as $5575.110$ units, set-up cost as $0.030$ units and production quantity as $22.998$ units. Also the optimal order quantity and TAC satify the necessary conditions. Next we compare the relative performance of proposed model by comparing its result with that obtained by employing crisp GP, fuzzy GP and IFGP and present it in Table 4. We find that optimal TAC is more preferable in NS environment than that of crisp, fuzzy and IF environments. Also NS environment yields higher demand for the machine $PBA_{597}$ with lower set-up cost. Moreover production quantity increases in NS environment.

Table 4: Optimal solutions of model (3.1) in different environments

| Environment | Demand ($D$) | Set-up cost ($S$) | Production quantity ($Q$) | Total Average Cost (TAC($D,S,Q$)) |
|---|---|---|---|---|
| Crisp | 4047.477 | 0.034 | 20.000 | 15.565 |
| Fuzzy | 4742.869 | 0.031 | 21.479 | 15.320 |
| IF | 4998.630 | 0.030 | 21.993 | 15.240 |
| NS | *5575.110* | *0.028* | *22.998* | *15.094* |

## 8.1 Sensitivity analysis

In this article, we investigate optimal policy of DM of proposed model in real life based NS environment. We perform sensitivity analysis of following key parameters
$(i)$ storage capacity per machine '$w_0$' (Table 5)
$(ii)$ shape parameter '$x$' (Table 6)
$(iii)$ variational parameter '$a$' (Table 7)
$(iv)$ shape parameter '$\theta$' (Table 8)
and present corresponding optimal solution in NS environment.

### 8.1.1 Managerial insights

We present phenomenon of change of storage capacity per machine '$w_0$' in Table 5 . We observe that optimal TAC is most preferable to DM in NS environment, which is well explained in Fig.10 . Also we find that each reduction in storage capacity per machine reduces TAC not only in NS environment but also in other environ-ments. Hence the management should trim down the size of packet of finished goods to reduce TAC.

Table 5: Sensitivity analysis in different environments of storage capacity per machine $'w_0'$

| TAC in | Storage capacity $'w_0'$ | | | | |
|---|---|---|---|---|---|
| | 80 | 90 | 100 | 110 | 120 |
| Crisp environment | 14.812 | 15.205 | 15.565 | 15.898 | 16.209 |
| Fuzzy environment | 14.718 | 15.032 | 15.320 | 15.595 | 15.840 |
| IF environment | 14.513 | 14.826 | 15.240 | 15.379 | 15.623 |
| NS environment | *14.494* | *14.805* | *15.094* | *15.355* | *15.600* |

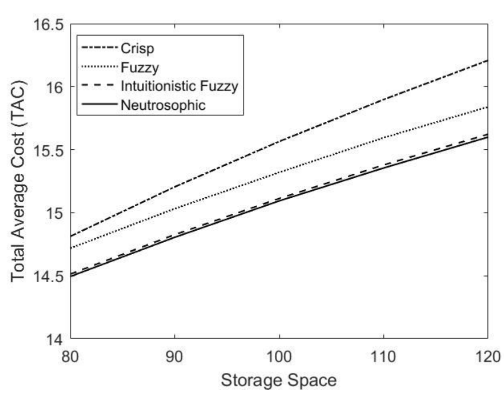


Figure 10: Effect on TAC in different environment due to change in storage space per machine '$w_0$'.

Next we consider change of shape parameter '$x$' in Table 6. Here we find that optimal TAC rapidly reduces for every increment in value of shape parameter and hence for every rise in demand in each of the said environments. It is consistent with common knowledge. Also in nearly all cases, we get most preferable optimal TAC in NS environment. This can be observed in Fig. 11. Again we perform sensitivity analysis of variational parameter '$a$' and present in Table 7. Here in all cases, we obtain most desirable TAC in NS environment among said environments. It can be visualized in Fig. 12. Also optimal TAC reduces as holding cost decreases in all said environments.

Table 6: Sensitivity analysis in different environments of shape parameter $'x'$

| TAC in | Shape parameter $'x'$ | | | | |
|---|---|---|---|---|---|
| | 1.6 | 1.7 | 1.75 | 1.8 | 1.9 |
| Crisp environment | *25.799* | 18.612 | 15.565 | 12.860 | 8.385 |
| Fuzzy environment | 26.582 | 18.554 | 15.320 | 12.585 | 8.408 |
| IF environment | 26.429 | 18.349 | 15.240 | 12.385 | 8.198 |
| NS environment | 26.413 | *18.328* | *15.094* | *12.366* | *8.179* |

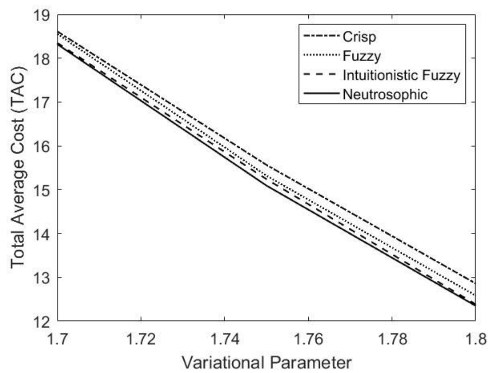


Figure 11: Effect on TAC in different environments due to change in shape parameter'$x$'.

Table 7: Sensitivity analysis in different environments of variational parameter $'a'$

| TAC in | Variational parameter $'a'$ | | | | |
|---|---|---|---|---|---|
| | 95 | 100 | 105 | 110 | 115 |
| Crisp environment | 15.393 | 15.481 | 15.565 | 15.646 | 15.723 |
| Fuzzy environment | 15.182 | 15.253 | 15.320 | 15.387 | 15.448 |
| IF environment | 15.103 | 15.174 | 15.240 | 15.307 | 15.368 |
| NS environment | *14.956* | *15.024* | *15.094* | *15.155* | *15.217* |

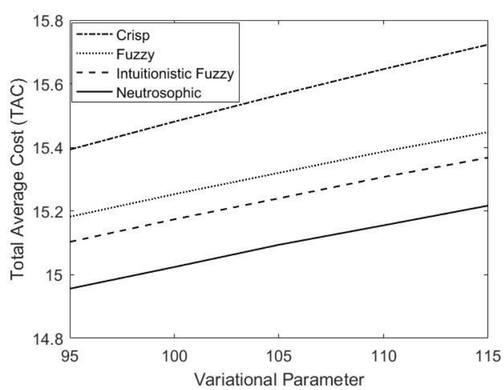


Figure 12: Effect on TAC in different environments due to change in variational parameter '$a$'.

Also we consider change of shape parameter '$\theta$' and present result in Table 8. As before, we find that optimal TAC is most favourable to DM in NS environment among said environments. Fig.13 brings clarity to this phenomenon. Additionally, we observe that optimal TAC can be further reduced by decreasing the value of shape parameter.

Table 8: Sensitivity analysis in different environments of shape parameter $'\theta'$

| TAC in | Shape parameter $'x'$ | | | | |
|---|---|---|---|---|---|
| | 100 | 110 | 120 | 130 | 140 |
| Crisp environment | 14.354 | 14.975 | 15.565 | 16.129 | 16.669 |
| Fuzzy environment | 14.338 | 14.843 | 15.320 | 15.773 | 16.037 |
| IF environment | 14.146 | 14.643 | 15.240 | 15.560 | 15.985 |
| NS environment | *14.123* | *14.622* | *15.094* | *15.536* | *15.962* |

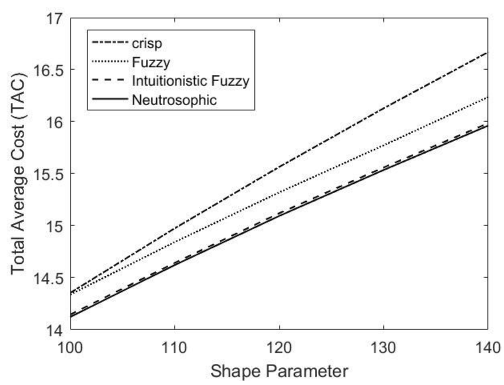


Figure 13: Effect on TAC in different environments due to change in shape parameter '$\theta$'.

# 9 Conclusions

In this article, we consider deterministic single objective EOQ model with limited storage capacity and solve it by applying GP in NS environment. We know it well that fuzzy set can better represent real life cases than crisp set. Again Ranjit Biswas [70] has shown how IF set can better represent real life cases than fuzzy set in many cases. Next Smarandache introduced NS set by generalizing IF set and at which we consider hesitancy function along with membership and non-membership function with appropriate constraints. Again advantages of GP among non-linear optimization methods are manifold. As per Cao [71], GP provides us with a systematic approach for solving a class of non-linear optimization problems by determining optimal values of decision variables and objective functions.

Whereas existing literature survey finds that GP is extended and thereby employed to solve mathematical models in fuzzy and IF environment, we can find very few articles, where EOQ models with limited storage capacity are solved by GP in NS environment. In this article, we employ max-additive operator to convert EOQ model with limited storage capacity to single objective PGPP and thereby solve it by applying NSGP. In numerical application, we find that optimal solution, obtained by NSGP is more preferable to DM than those obtained in crisp GP, fuzzy GP and IFGP. Next we perform sensitivity analysis of key parameters of proposed model and list several key managerial insights. Also we explain them graphically.

*Future scopes of research*
We locate lot of scopes for further research and enlist few of them as follows:
(i) We can consider multiple products scenario. In this case, we can employ modified GP in NS environment.
(ii) Shape parameters can be neutrsophic in nature.
(iii) We can allow shortage of items in inventory and update the mathematical model accordingly.
(iv) We can use other optimization methods to solve non-linear models in NS environment.
(v) And last but not the least, we can discuss present model in other imprecise environments.

## Acknowledgments

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Table 9: Notations and their explanations

| | |
|---|---|
| D | Demand per unit time, which is constant |
| H(t) | Holding cost per unit item, which is time (t) depended |
| I(t) | Inventory level at any time, $t \geq 0$ |
| P$(D, S)$ | Unit demand (D) and set-up cost (S) dependent production cost |
| Q | Production quantity per batch |
| S | Set-up cost per unit time |
| T | Period of cycle |
| TAC$(D, S, Q)$ | Total average cost per unit time |
| W | Total storage capacity area |
| $w_0$ | capacity area per unit quantity |

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# Some Neutrosophic Probability Distributions

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**Abstract.** In this paper, we introduce and study some neutrosophic probability distributions, The study is done through generalization of some classical probability distributions as Poisson distribution, Exponential distribution and Uniform distribution, this study opens the way for dealing with issues that follow the classical distributions and at the same time contain data not specified accurately.

**Keywords**: Poisson, Exponential & Uniform distributions, Classical Logic, Neutrosophic Logic, Neutrosophic crisp sets.

**1 Introduction**: Neutrosophy theory introduced by Smarandache in 1995. It is a new branch of philosophy, presented as a generalization for the fuzzy logic [5] and as a generalization for the intuitionistic fuzzy logic [6]. The fundamental concepts of neutrosophic set, introduced by Smarandache in [7, 8, 9, 10], and Salama et al. in [11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23], provides a new foundation for dealing with issues that have indeterminate data. The indeterminate data may be numbers, and the neutrosophic numbers have been defined in [24, 25, 26, 27]. In this paper, we highlight the use of neutrosophic crisp sets theory [3,4] with the classical probability distributions, particularly Poisson distribution, Exponential distribution and Uniform distribution, which opens the way for dealing with issues that follow the classical distributions and at the same time contain data not specified accurately. The extension of classical distributions according to the neutrosophic logic , means that parameters of classical distribution take undetermined values, which allows dealing with all the situations that one may encounter while working with statistical data and especially when working with vague and inaccurate statistical data, Florentin Smarandache presented the neutrosophic binomial distribution and the neutrosophic natural distribution [1,2] in 2014 , In this paper, we will discuss continuous random distributions such as the Exponential distribution and Uniform distribution , and discontinuous random distribution such as Poisson distribution by using neutrosophic logic.

**2 TERMINOLOGIES:** We recollect some relevant basic preliminaries, and in particular, the work of Smarandache in [1, 9, 10], and Salama et al. [22, 23]. We consider the following classic statistical distributions Poisson distribution, Exponential distribution and Uniform distribution.

## 3 Neutrosophic probability Distributions:

### 3.1 Neutrosophic Poisson Distribution:

**3.1.a Definition**: Neutrosophic Poisson distribution of a discrete variable X is a classical Poisson distribution of X, but its parameter is imprecise. For example, λ can be set with two or more elements. The most common such distribution is when λ is interval.

$$NP(\mathrm{x}) = \mathrm{e}^{-\lambda_N} \cdot \frac{(\lambda_N)^{\mathrm{x}}}{\mathrm{x}!} \quad ; \quad x = 0,1, \dots ..$$

$\lambda_N$ : Is the distribution parameater .

- $\lambda_N$ : is equal to the expected value and the variance .

$NE(x) = NV(x) = \lambda_N$

Where, N = d + I is a neutrosophic statistical number in [2].

**3.1.b Example for Case study**:

In a company, Phone employee receives phone calls, the calls arrive with rate of [1, 3] calls per minute, we will calculate the probability that:

- The employee will not receive any call within a minute:

Assuming x: the number of calls in a minute.

Then:

$$NP(\mathrm{x}=0) = \mathrm{e}^{-\lambda_N} \cdot \frac{(\lambda_N)^0}{0!} = \mathrm{e}^{-\lambda_N} = \mathrm{e}^{-[1,3]}$$

For $\lambda = 1$:

$$NP(0) = \mathrm{e}^{-1} = 0.3679$$

For $\lambda = 3$ :

$$NP(0) = \mathrm{e}^{-3} = 0.0498$$

Thus, the probability that employee won't receive any call, within a minute

, ranges between [0.0498, 0.3697].

- the probability that employee won't receive any call ,within 5 minutes:

  Then:

$$\lambda_N = 5.[1,3] = [5\,,15]$$

$$NP(\mathrm{x}) = \mathrm{e}^{-[5,15]} \cdot \frac{([5,15])^{\mathrm{x}}}{\mathrm{x}!} \quad ; \quad x = 0,1,\ldots..$$

$$NP(\mathrm{x}=0) = \mathrm{e}^{-\lambda_N} \cdot \frac{(\lambda_N)^0}{0!} = \mathrm{e}^{-\lambda_N} = \mathrm{e}^{-[5,15]}$$

For $\lambda = 5$:

$$NP(0) = \mathrm{e}^{-5} = 0.0067$$

For $\lambda = 15$:

$$NP(0) = \mathrm{e}^{-15} = 0.000000306$$

Thus, the probability that the employee will not receive any call within 5 minutes ranges between [0.000000306, 0.0067].

### 3.2 The Neutrosophic Exponential Distribution:

**3.2.a Definition**: Neutrosophic exponential distribution [21] is defined as a generalization of classical exponential distribution , Neutrosophic exponential distribution can deals with all the data even non-specific, we express the density function as:

$X_N \sim \exp(\lambda_N) = f_N(x) = \lambda_N\, e^{-x\,.\lambda_N} \quad ; \qquad 0 < x < \infty \quad ,$

$\exp(\lambda_N)$ : Neutrosophic Exponential Distribution.

$X_N$ : X neutrosophic random variable [22].

$\lambda_N$ : distribution parameater.

**3.2.b the distribution properties**:

1- Expected value:

$$E(x) = \frac{1}{\lambda_N}$$

Variance:

$$var(x) = \frac{1}{(\lambda_N)^2}$$

2- Distribution function:

Probability to terminate the client's service in less than a minute:

$NF(x) = NP(X \leq x) = \left(1 - e^{-x\,.\lambda_N}\right)$

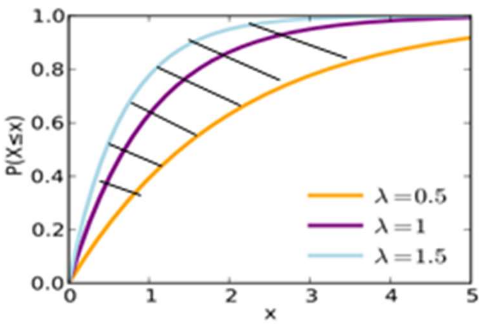


**Figure 1**

**3.2.c. Example for Case study**:

The time required to terminate client's service in the bank follows an exponential distribution, with an average of one minute, let us write a density function that represents the time required for terminating client's service, and then calculate the probability of terminating client's service in less than a minute.

**Solution:**

-Assuming x: represents the time required for termination of the client's service per minute.

-The average $1/\lambda = 1 \Rightarrow \lambda = 1$

Therefore, the Probability density function:

$$f(x) = e^{-x} \quad ; \quad 0 < x < \infty$$

-The possibility of client's service terminated in less than a minute:

$$p(X \leq 1) = (1 - e^{-x}) = \left(1 - e^{-(1)}\right) = 0.63$$

-The above example is a simple example practically, but if it is changed to the following:

The time required to terminate client's service in the bank follow an exponential distribution, with an average of [0.67, 2] minute. We know that classical exponential distribution only deals with data defined accurately, note that the average here is an interval, how we will deal with this situation.
So, we will turn to the neutrosophic exponential distribution to solve this issue:

For exponential distribution, its average [0.67, 2] minutes, we write:

$$\frac{1}{\lambda_N} = [0.67\,, 2\,] \Rightarrow \lambda_N = \frac{1}{[0.67\,, 2\,]} = [0.5\,, 1.5]$$

The probability density function:

$$f_N(x) = \lambda_N\, e^{-x\,.\lambda_N} \quad ; \quad 0 < x < \infty \quad ,$$

$$f_N(x) = [0.5\,, 1.5]\, e^{-[0.5\,, 1.5]\,x} \quad ; \quad 0 < x < \infty$$

Probability to terminate the client's service in less than a minute:

$$NF(x) = NP(X \leq x) = \left(1 - e^{-x\,.\lambda_N}\right)$$

$$NP(X \leq 1) = \left(1 - e^{-[0.5\,, 1.5]\,x}\right) = \left(1 - e^{-[0.5\,, 1.5](1)}\right) = 1 - e^{-[0.5\,, 1.5]}$$

We note:

For $\lambda = 0.5$ :

$$NP(X \leq 1) = 1 - e^{-0.5} = 0.39$$

For $\lambda = 1.5$ :

$$NP(X \leq 1) = 1 - e^{-1.5} = 0.78$$

That is, the probability of terminating client's service in less than a minute ranges between [0.39, 0.78].

- Note that, the value of the classic probability to terminate client's service in less than a minute is one of the domain values for the neutrosophic probability:

$$p(X \leq 1) = 0.63 \in [0.39 , 0.78] = NP(X \leq 1)$$

And the solutions are the shaded area in Figure 1.

**3.2.d Note:** We also mention the relationship of exponential distribution with Poisson distribution, If the occurrence of events follows the Poisson distribution, the duration between the occurrence of two events follow exponential distribution. For example, arrival of customers to a service centre follows the Poisson distribution, the time between the arrival of a customer and the next customer follow the exponential distribution. Thus, when the parameter λ is inaccurately defined, we are dealing with the neutrosophic exponential distribution and the neutrosophic Poisson distribution
, and we write:
If an event is repeated in time according to the neutrosophic Poisson distribution:

$$NP(\mathrm{x}) = \mathrm{e}^{-\lambda_\mathrm{N}} . \frac{(\lambda_\mathrm{N})^\mathrm{x}}{\mathrm{x}!} \quad ; \quad x = 0,1, \dots ..$$

Then, the time between two events follows the neutrosophic exponential distribution:

$$f_\mathrm{N}(\mathrm{t}) = \lambda_\mathrm{N} \ . \ \mathrm{e}^{-\lambda_\mathrm{N} \ t} \quad ; \quad t > 0$$

**3.2.d.i. Example**:
Assuming that we have a machine in a factory. The rate of machine breakdowns is [1, 2] per week, let's calculate the possibility of no breakdowns per week, and calculate the possibility that at least two weeks pass before the appearance of the following breakdowns.

**Solution**:

- The possibility of no breakdowns in the week:

Assume x: variable represents occurrence of breakdowns in the week.

We note that, x is a variable that is subject to the neutrosophic Poisson distribution, the distribution parameter is

$\lambda_N =$ [1,2] , thus:

$$NP(\mathrm{x} = 0) = \mathrm{e}^{-\lambda_\mathrm{N}} . \frac{(\lambda_\mathrm{N})^\mathrm{x}}{\mathrm{x}!} = \mathrm{e}^{-\lambda_\mathrm{N}} . \frac{(\lambda_\mathrm{N})^0}{0!} = \mathrm{e}^{-\lambda_\mathrm{N}} = \mathrm{e}^{-[1,2]}$$

Then, the possibility of no breakdowns in the week ranges between [0.135, 0.368].

- Assuming y: is represent the time before the appearance of the following breakdowns, we note that y is a variable following the neutrosophic exponential distribution, then:

$$NF(x) = NP(X \leq x) = \left(1 - e^{-\mathrm{x} . \lambda_N}\right)$$

$$NP(y > 2) = 1 - NP(y \leq 2) = 1 - NF(2) = 1 - \left(1 - \mathrm{e}^{-2.\lambda_\mathrm{N}}\right)$$

$$= \mathrm{e}^{-2.\lambda_\mathrm{N}} = \mathrm{e}^{-2[1,2]} = \mathrm{e}^{[-4,-2]}$$

Thus, the possibility that at least two weeks pass before the appearance of the following breakdowns, ranges between [0.018, 0.135].

**3.3 Neutrosophic Uniform Distribution**:

**3.3.a Definition**: Neutrosophic Uniform distribution of a continuous variable X, is a classical Uniform distribution , but distribution parameters a or b or both are imprecise. For example, a or b or both are sets with two or more elements (may a or b or both are intervals) with $a < b$ .

**3.3.b Example for Case study**:

Assuming x is a variable represents a person's waiting time to passengers' bus (in minutes), bus's arrival time is not specified, the station official said:

1- the bus arrival time is: either from now to 5 minutes [0,5] or will arrive after 15 to 20 minutes[15,20], then:

a= [0, 5] , b= [15, 20]

Then, the density function:

$$f_N(x) = \frac{1}{b-a} = \frac{1}{[15,20]-[0,5]} = \frac{1}{[10,20]} = [\,0.05\,,0.1]$$

The solution in the Graph is the shaded area, with the probability to moving (a) between [0, 5] and (b) between [15, 20].

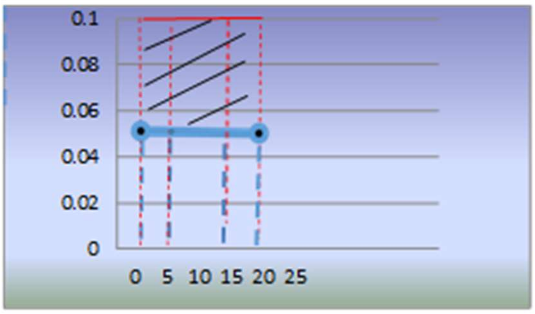


**Figure 2**

2- The bus arrives after five minutes or will arrive after 15 to 20 minutes [15 , 20] , then:
a=5 b= [15, 20]

Then, the density function:

$$f_N(x) = \frac{1}{b-a} = \frac{1}{[15{,}20]-5} = \frac{1}{[10\,,15]} = [0.067\,,0.1]$$

The solution is the shaded area, with the probability to moving (b) between [15, 20].

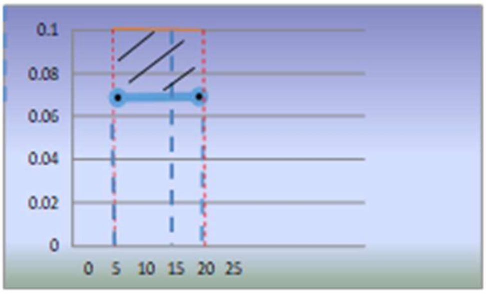


**Figure 3**

- There are many non-specific situations that we encounter about the values a, b such as a, b or both are intervals (a, b or both are sets with two or more elements), we deal with these situations as the cases studied above.

## 4 The research gap

The classical probability distributions only deal with the specified data. The classical distribution parameters are always given with a specified value. This paper contributes to the study of classical distributions with undetermined values, and distribution parameters such as periods. We call these distributions neutrosophic probability distributions.

## Conclusion:

We conclude from this paper that the neutrosophic probability distributions gives us a more general and clarity study of the studied issue, So that the classical probability is one solution among the solutions resulting from the study, of course, this is produced by giving the distribution parameters several options possible and does not remain linked to a single value. This paper is to present some the neutrosophic probability distributions, and we present various solved for the problems that classic logic is not deal with it. We look forward in the future to study other types of probability distributions according to the neutrosophic logic, especially the gamma distribution and student distribution and other distributions that have not yet been studied.

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# A framework for selecting cloud computing services based on consensus under single valued neutrosophic numbers

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**Abstract**.Many organizations are seeking to contract services from cloud computing with many cloud services available and numerous criteria that should be counted in the selection process. Therefore, the selection process of cloud services can be considered as a type of multi-criteria decision analysis problems with multiples stakeholders. In this paper a framework for selecting cloud services taking into account consensus and using single valued neutrosophic numbers for indeterminacy representation is presented.The proposed framework is composed of five activities: framework, gathering parameters, eliciting preferences, computing consensus degree, advice generation, rating alternatives and cloud service selection. The model includes automatic search mechanisms for conflict areas and recommendations to the experts to bring closer their preferences.An illustrative example that corroborates the applicability of the model is presented.The paper ends with conclusion and further areas or research recommendations.

**Keywords:**Decision Analysis, SVN Numbers, Cloud Service, Consensus.

## 1 Introduction

Cloud Computing is experiencing a strong adoption in the market and this trend is expected to continue [1]. Due to the diversity of cloud service providers, it is a very significant defy for organizations to select the appropriate cloud services which can fulfil requirements, as numerous criteria should be counted in the selection process of cloud services and diverse stakeholders are involved. Therefore, the selection process of cloud services can be considered as a type of multi-criteria multi-expert decision analysis problems [2, 3]. In this research paper, we present how to aid a decision maker to evaluate different cloud services by providing a neutrosophic multi-criteria decision analysis including a consensus process. To demonstrate the pertinence of the proposed model and illustrative example is presented.

Neutrosophy is mathematical theory developed by Florentín Smarandache for dealing with indeterminacy[4-6]. It has been the base for developing of new methods to handle indeterminate and inconsistent information like neutrosophic sets and neutrosophic logic, especiallyused on decision making problems [6-8] . Because of the imprecise nature of the linguistic assessments new techniques have been developed. Single valued neutrosophic sets (SVNS) [9] for handling indeterminate and inconsistent information is a relatively new approach.In this paper a new model for cloud service selection is developed based on single valued neutrosophic numbers (SVN-number) allowing the use of linguistic variables [10, 11]. Group decision support and complex systems modelling makes recommendable to develop a consensus process [12-14]. Consensus is defined as a state of agreement among members of a group. A consensus reaching process is iterative process comprising several rounds where the experts adapt their preferences [13].

This paper is structured as follows: Section 2 reviews some preliminaries concepts about neutrosophic decision analysis and consensus process. In Section 3, a framework for selecting cloud computing services based on single valued neutrosophic numbers and consensus process is presented. Section 4 shows an illustrative example of the proposed model. The paper ends with conclusions and further work recommendations.

## 2 Preliminaries

In this section, we first provide a brief revision of neutrosophic multicriteria decision analysis, consensus process and cloud computing.

### 2.1Neutrosophic multicriteria decision analysis

Fuzzy logic was initiallyproposedby Zadeh [15], for helping in modeling knowledge in a more natural way. The basic idea is the notion of the membership relation which takes truth values in the interval [0, 1] [16].
The intuitionistic fuzzy set(IFS) on a universe was introduced by K. Atanassov as a generalization of fuzzy sets [17, 18]. In IFS besides the degree of membership ($\boldsymbol{\mu_A(x)} \in [\mathbf{0}, \mathbf{1}]$ ) of each element $x \in X$ to a set A there was considered a degree of non-membership $\boldsymbol{\nu_A(x)} \in [\mathbf{0}, \mathbf{1}]$, such that:

$$\forall \boldsymbol{x} \in \boldsymbol{X} \boldsymbol{\mu_A}(\boldsymbol{x}) + \boldsymbol{\nu_A}(\boldsymbol{x}) \leq \mathbf{1} \qquad (1)$$

Neutrosophic set (NS) introduced the degree of indeterminacy (i) as independent component [6, 19].
The truth value in neutrosophic set is as follows [20, 21]:
Let $\boldsymbol{N}$ be a set defined as: $\boldsymbol{N} = \{(\boldsymbol{T}, \boldsymbol{I}, \boldsymbol{F}) : \boldsymbol{T}, \boldsymbol{I}, \boldsymbol{F} \subseteq [\mathbf{0}, \mathbf{1}]\}$, a neutrosophic valuation n is a mapping from the set of propositional formulas to $\boldsymbol{N}$ , that is for each sentence p we have $\boldsymbol{v}\ (\mathbf{p}) = (\boldsymbol{T}, \boldsymbol{I}, \boldsymbol{F})$.
Single valued neutrosophic set (SVNS ) [9] was developed to facilitate real world applications ofneutrosophic set and set-theoretic operators. A single valued neutrosophic set is a special case of neutrosophic set proposed as a generalization of intuitionistic fuzzy sets in order to deal with incomplete information [10].
Single valued neutrosophic numbers (SVN number) are denoted by $A$= ($a$,b,$c$), where $a,b,c \in [0,1]$ and $a+b+c \leq 3$[22] . In real world problems, sometimes we can use linguistic terms such as 'good ', 'bad ' to obtain preferences aboutan alternativeand cannot use some numbers to express some qualitative information [23, 24].Some classical multicriteria decision models[25, 26] have been adapted to neutrosophic for exampleAHP[27], TOPSIS[28] and DEMATEL [29].

### 2.2 Consensus reaching process

Consensus is an active area of research in fields such as group decision making and learning [30, 31]. A consensus reaching process is defined as a dynamic and iterative process composed by several rounds where the experts express, discuss, and modify their opinions or preferences[13, 32]. The process is generally supervised by a moderator (Fig. 1), who helps the experts to make their preferences closer to each other's.

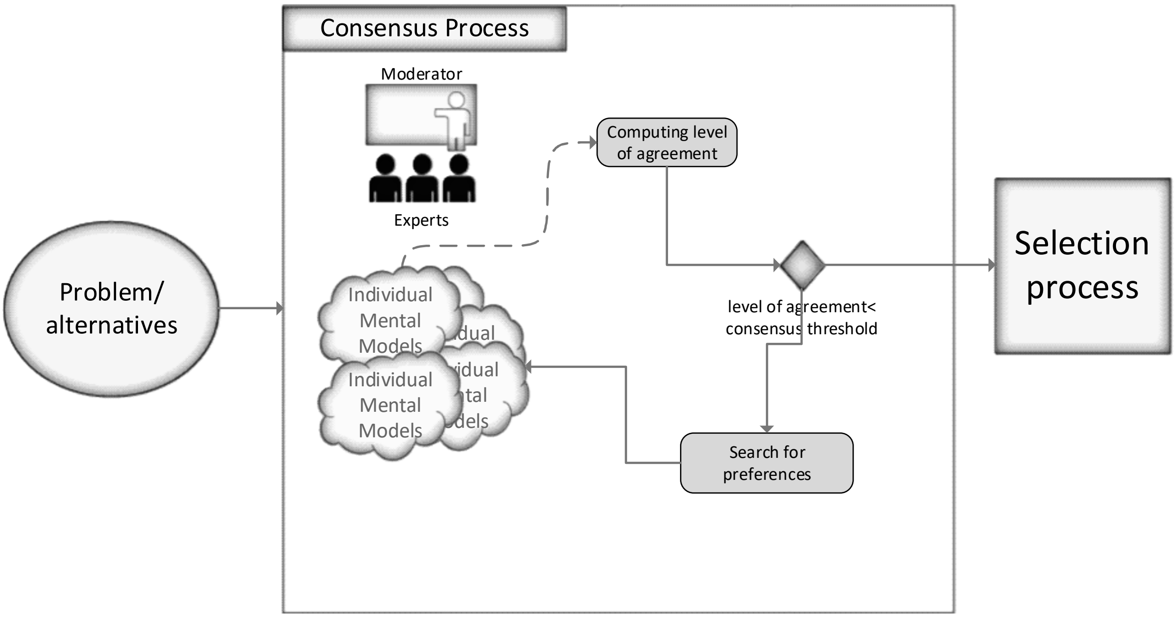


**Figure1.** Phases of the consensus process supervised by the moderator [32].

A frequent approach to consensus modeling involves the aggregation of preferences and the computing of individual differences with that value[33]. In each round the moderator helps to make closer the opinions with discussions and advices to experts to change preferences in case [12]. A consensus previous to group decision making allows the discussion and change of preferences helping to reach a state of agreement satisfying experts. Consensual points of view obtained from this process provide a stable base for decisions making[31].

### 2.3 Cloud computing services

Cloud computing has emerged as a paradigm to deliver on demand resources like infrastructure, platform, software, among others,to customers similar to other utilities. Traditionally, small and medium enterprises (SMEs) had to make high capital investment for procuring IT/software infrastructure, skilled developers and system administrators, which results in a high cost of ownership. Cloud computing aims to deliver virtual services so that users can access them from anywhere in the world on subscription at competitive costs for SMEs [34]..
Due to the fast expansion of cloud computing, many cloud services have been developed [35]. Therefore, given the diversity of Cloud service offerings, an important challenge for customers is to discover who are the ‘‘right’’ Cloud providers that can satisfy their requirements. Numerous criteria should be counted in the selection process of cloud services and various stakeholders are involved. Consequently, the selection process of cloud services can be considered as a type of multi-criteria multi-expert decision analysis problems [2, 3].

## 3 Proposed framework.

Our aim is to develop a framework for cloud service provider selection based on a consensus process. The model consists of the following phases (fig. 2).

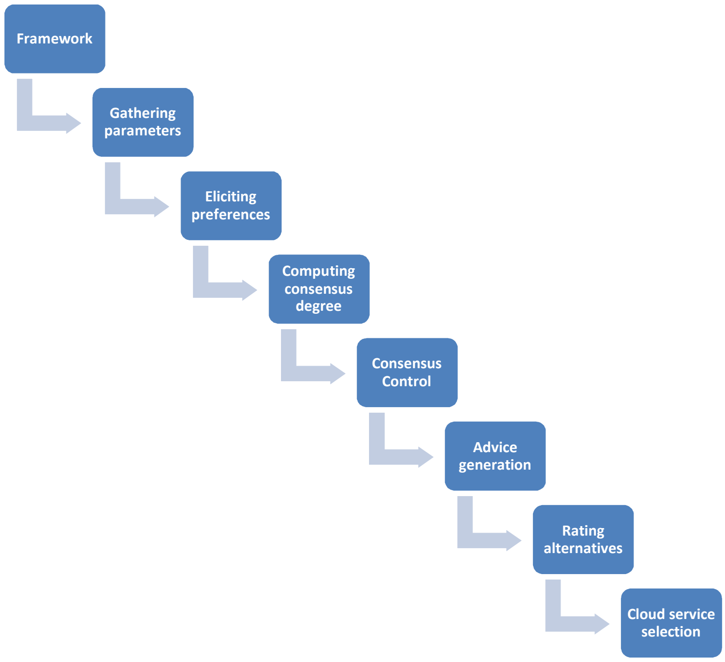


**Figure 2:** A framework for cloud serivece selection.

The proposed framework is composed of five activities:

- Framework
- Gathering parameters
- Eliciting preferences
- Computing consensus degree
- Advice generation
- Rating alternatives
- Cloud service selection.

Following, the proposed decision method is described in further detail, showing the operation of each phase

1. Framework: In this phase, the evaluation framework, for the decision problem of cloud serviceselection is defined. The framework is established as follows:

- C=$\{c_1, c_2, \ldots, c_n\}$ with $n \geq 2$ , a set of criteria.
- E=$\{e_1, e_2, \ldots, e_k\}$ with $k \geq 2$, a set of experts.
- $X = \{x_1, x_2, \ldots, x_m\}$ with $m \geq 2$, a finite set of information technologies cloud services alternatives.

Criteria and experts might be grouped. The set of experts will provide the assessments of the decision problem. Main criteria for cloud service selection are visually summarizes as follows.

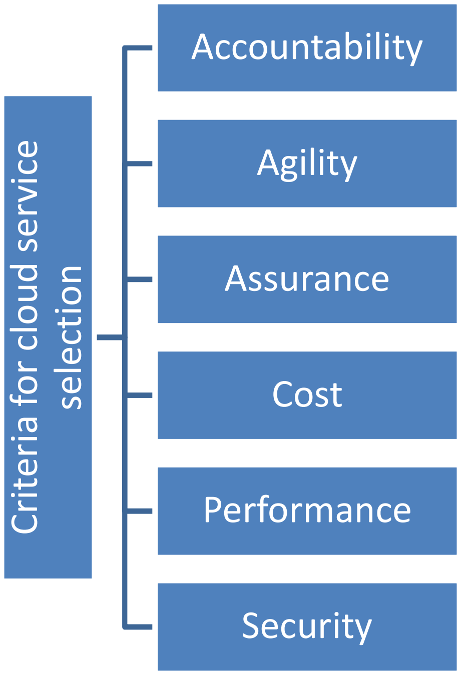


**Figure 3.** Cloud service selection criteria.

2. Gathering parameters: The granularity of the linguistic term is selected. Parameters are gathered for controlling the consensus process: consensus threshold $\mu \in [0,1]$ and $MAXROUND \in \mathbb{N}$ to limit the maximum number of discussion rounds. Acceptability threshold $\varepsilon \geq 0$, to allow a margin of acceptability for prevents generating unnecessary recommendations is also gathered.

3. Eliciting preferences: for each expert his /her preference is gathered using the linguistic term set chosen. In this phase, each expert, $e_k$provides the assessments by means of assessment vectors:
$U^K = (v_i^k, i = 1,..,n, j = 1,..,m)$ (2)
The assessment$v_i^k$, provided by each expert$e_k$, for each criterion $c_i$ of each cloud service alternative$x_j$, is expressed using SVN numbers.

4. Computing consensus degree: The degree of collective agreement is computed in [0,1].
For each pair of experts $e_k$, $e_t$, $(k < t)$, a similarity[36, 37] vector $SM_{kt} = (sm_i^{kt})$ , $sm_i^{kt} \in [0,1]$, is computed:

$$\mathrm{sm}_i^{kt} = 1 - \left(\frac{1}{3}\sum_{j=1}^{n}\left\{\left(\left|t_i^k - t_i^t\right|\right)^2 + \left(\left|i_i^k - i_i^t\right|\right)^2 + \left(\left|f_i^k - f_i^t\right|\right)^2\right\}\right)^{\frac{1}{2}} \quad (3)$$

$(i = 1,2, \dots, m)$

A consensus vector $CM = (cm_i)$ is obtained by aggregating similarity values:

$$cm_i = OAG_1(SIM_i) \quad (4)$$

where $OAG_1$is an aggregationoperator, $SIM_i = \{sm_i^{12}, \dots, sm_i^{1m}, \dots, sm_i^{(m-1)m}\}$ represents all pairs of experts' similarities in their opinion on preference between $(v_i, v_j)$ and $cm_i$ is the degree of consensus achieved by the group in their opinion.

Finally, an overall consensus degree is computed:

$$cg = \frac{\sum_{i=1}^{n} cv_i}{n} \quad (5)$$

5. Consensus Control: Consensus degree $cg$ is compared with the consensus threshold ($\mu$). If $cg \geq \mu$, the consensus process ends; otherwise, the process requires additional discussion. The number of rounds is compared with parameter $MAXROUND$ to limit the maximum number of discussion rounds.

6. Advice generation: When $cg < \mu$, experts must modify the preferences relations to make their preferencescloser to each other and increase the consensus degree in the following round. Advice generation begin computing a collective preferences $W^c$. This collective preference model is computed aggregating each experts'preference vector:

$$w_i^c = OAG_2(v_1^1, \ldots, v_i^m) \quad (6)$$

where $v \in U$ and $OAG_2$is anaggregation operator.

After that, a proximity vector ($PP^k$) between each one of the $e_k$experts and $W^c$ is obtained.Proximity values, $pp_{ij}^k \in [0,1]$ are computed as follows:

$$pp_i^k = 1 - \left(\frac{1}{3}\sum_{j=1}^{n}\left\{\left(\left|t_i{}^k - t_i^c\right|\right)^2 + \left(\left|i_i{}^k - i_i^c\right|\right)^2 + \left(\left|f_i{}^k - f_i{}^c\right|\right)^2\right\}\right)^{\frac{1}{2}} \quad (7)$$

Afterwards, preferences relations to change (CC) are identified. preference relation between criteria$c_i$ and$c_j$ with consensus degree under the defined (μ) are identified:

$$CC = \{w_i^c | cm_i < \mu\} \quad (8)$$

Later, based on CC, those experts who should change preference are identified. To compute an average proximity $pp_i^A$ , proximity measures are aggregate

$$pp_i^A = OAG_2(pp_i^1, \ldots, pp_i^m) \quad (9)$$

where $OAG_2$ is a SVN aggregation operator.

Experts $e_k$ whose $pp_i^k < pp_i^A$ are advised to modify their preference relation $w_i{}^k$.

Finally direction rules are checked to suggest the direction of changes proposed. Threshold$\varepsilon \geq 0$is established to prevents generating an excessive number of unnecessary advice .

DR 1: If $v_i^k - w_i^c < -\varepsilon$then $e_k$ should increase his/her the value of preference relation $v_i$.

DR 2: If $v_i^k - w_i^c > \varepsilon$ then $e_k$ should decrease his/her the value of preference relation $v_i$.

DR 3: If$-\varepsilon \leq v_i^k - w_i^c \leq \varepsilon$ then $e_k$ should not modify his/her the value of preference relation $v_i$.

Step from3-6 are repeated until consensus reachedor maximum number of rounds.

7. Rating alternatives: The aim of this phase is to obtain a global assessment for each alternative. Taking into account the previous phase, an assessment for each alternative is computed, using the selected solving process that allows managing the information expressed in the decision framework.

In this case alternatives are rated according to single valued neutrosophic weighted averaging (SVNWA) aggregation operator as proposed by Ye [38] for SVNSs as follows[10]:

$$F_w(A_1, A_2, \ldots, A_n) = \langle 1 - \prod_{j=1}^{n}\left(1 - T_{A_j}(x)\right)^{w_j}, \prod_{j=1}^{n}\left(I_{A_j}(x)\right)^{w_j}, \prod_{j=1}^{n}\left(F_{A_j}(x)\right)^{w_j}\rangle \qquad )$$

where $W = (w_1, w_1, \ldots, w_n)$ is the weighting vector of$A_j (j = 1, 2, \ldots, n)$ , $w_n \in [0, 1]$and $\sum_j^n w_j = 1$.

8. Cloudserviceselection: In this phase of the alternatives are rankedand the most desirable one is chosen-by the score function [39, 40].According to the scoring and accuracy functions for SVN-sets, a ranking order of the set of the alternatives can be generated [41]. Selecting the option(s) with higher scores.

For ordering alternatives a scoring function is used [42]:

$$s(V_j) = 2 + T_j - F_j - I_j \qquad (11)$$

Additionally an accuracy function is defined:

$$a(V_j) = T_j - F_j \qquad (12)$$

And then

1. If $s(Vj) < s(Vi)$, then $Vj$ is smaller than$Vi$, denoted by $Vj < Vi$
2. If$s(Vj) = s(Vi)$
    a. If $a(Vj) < a(Vi)$, then $Vj$ is smaller than $Vi$, denoted by $Vj < Vi$
    b. If $a(Vj) = a(Vi)$, then $Vj$ and $Vi$ are the same, denoted by $Vj = Vi$

Another option is to use the scoring function proposed in[28]:

$$s(V_j) = (1 + T_j - 2F_j - I_j)/2 \qquad (13)$$

where $s(V_j) \in [-1,1]$.

If $s(Vj) < s(Vi)$, then $Vj$ is smaller than$Vi$, denotedby $Vj < Vi$

According to the scoring function ranking method of SVN-sets, the ranking order of the set of cloud service alternatives can be generated and the best alternative can be determined.

## 4 Illustrative example

In this case study three experts $\mathrm{E} = \{\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3\}(\mathrm{n} = 3)$ are inquired about their preferences. A linguistic term sets with cardinality nine (Table 1) is used.

| **Linguistic terms** | **SVNSs** |
|---|---|
| Extremely good (EG) | (1,0,0) |
| Very very good (VVG) | (0.9, 0.1, 0.1) |
| Very good (VG) | (0.8,0,15,0.20) |
| Good (G) | (0.70,0.25,0.30) |
| Medium good (MG) | (0.60,0.35,0.40) |
| Medium (M) | (0.50,0.50,0.50) |
| Medium bad (MB) | (0.40,0.65,0.60) |
| Bad (B) | (0.30,0.75,0.70) |
| Very bad (VB) | (0.20,0.85,0.80) |
| Very very bad (VVB) | (0.10,0.90,0.90) |
| Extremely bad (EB) | (0,1,1) |

**Table 1.** Linguistic terms used to provide the assessments [28]

The scope of the consensus processis defined by five criteria $C = (c_1, \ldots, c_5)$ shown in Table 2.

| Node | Description |
|---|---|
| A | Accountability |
| B | Agility |
| C | Assurance |
| D | Cost |
| E | Performance |
| F | Security |

**Table 2.** Criteria for Cloud service selection

Parameters used in this case study are shown in Table 3.

| | |
|---|---|
| **Consensus threshold** | $\boldsymbol{\mu = 0.9}$ |
| **Maximum number of discussion rounds** | $MAXROND = 10$ |
| **Acceptability threshold** | $\varepsilon = 0.15$ |

**Table 3.**Parameters defined

Initially, the experts provide the following preferences.

| | A | B | C | D | E |
|---|---|---|---|---|---|
| E1 | G | M | B | G | B |
| E2 | VG | VG | M | G | VB |
| E3 | G | G | G | G | VG |

**Table 4**. Preferences Round 1.

**First round**

Similarity vector are obtained

$S^{12}$=[0.9, 0.682, 0.782, 1,0.9]

$S^{13}$=[1, 0.782, 0.564, 1,0.465]

$S^{23}$= [0.9, 0.9, 0.782, 1, and 0.365]

The consensus vector CV=[0.933, 0.676, 0.79, 1, 0,577]

Finally, an overall consensus degree is computed:$cg = 0.795$

Because cg $= 0.795 < \mu = 0.9$ the advice generation is activated.

The collective preferences is calculated using the SVNWA operator giving in this case equal importance to each expert$W^c$ =[(0.64, 0.246, 0.377),(0.591,0.303, 0.427), (0.437,0.492, 0.578), (0.62, 0.287,0.416), (0.428, 0.495, 0.587)]

Proximity vectors are calculated $PP^k$:

$PP^1$= [0.944, 0.68, 0.817, 0.916, 0.823]

$PP^2$= [0.852, 0.801, 0.942, 0.916, 0.632]

$PP^3$=[0.944,0.899, 0.739, 0.916, 0.632]

After that preference to change (CC) are identified (11).

$$CC = \{W_i|cv_i < 0.9\} = \{w_2, w_3, w_5\}$$

Average proximity for this value is computed as follows:

$(pp_2^A = 0.793, pp_3^A = 0.833, pp_5^A = 0.696)$

Proximity values for each expert in preferences $\{w_2, w_3, w_5\}$ is as follows:

$(pp_2^1 = 0.68, pp_3^1 = 0.817\ pp_5^1 = 0.823)$

$(pp_2^2 = 0.81, pp_3^2 = 0.942, pp_5^2 = 0.632))$

$(pp_2^3 = 0.899, pp_3^3 = 0.739, pp_5^3 = 0.632)$

The sets of preferences to change ($pp_i^k < pp_i^A$) are:

$$\{v_2^1, v_3^1, v_5^2, v_3^3, v_5^3\}$$

According to rule DR1, the experts are required to increase the following relations:

$$\{v_3^1, v_5^2\}$$

According to rule DR2, the experts are required to decrease the following relations:

$$\{v_3^3, v_5^3\}$$

And According to rule DR3 this relations should not be changed:

$$\{v_2^1\}$$

**Second Round**

According to the previous advices, the experts implemented changes, and the new elicited preferences

| | A | B | C | D | E |
|---|---|---|---|---|---|
| E1 | G | M | M | G | B |
| E2 | VG | VG | M | G | B |
| E3 | G | G | M | G | B |

**Table 4.** Preferences Round 2.

Similarity vector are obtained again:

$S^{12}$=[0.9, 0.682, 1, 1,1]

$S^{13}$=[1, 0.782, 1, 1,1]

$S^{23}$=[0.9, 0.9, 1, 1,1]

The consensus vector CV=[0.933, 0.676, 1, 1, 1]

Finally, an overall consensus degree is computed:

$$cg = 0.922$$

Because cg=0.93>$\mu = 0.9$ the desired level of consensus is achieved.

## 5 Conclusions.

The fast expansion of cloud computinghas caused the development of many cloud services. Given the diversity of cloud service offerings, an important challenge for customers is to discover who are the ‘‘right’’ cloud providers that can satisfy their requirements with numerous criteria that should be counted in the selection process and diverse stakeholders involved. Therefore, the selection process of cloud services can be considered as a type of multi-criteria multi-expert decision analysis problems.A consensus process allows developing a better group decision process.

Recently, neutrosophic sets and its application to multiple attribute decision making have become a topic of great importance for researchers and practitioners alike. In this paper a new framework for selecting cloud services taking into account consensus and using single valued neutrosophic numbers for indeterminacy representation is presented.

The proposed framework is composed of five activities: framework, gathering parameters, eliciting preferences, computing consensus degree, advice generation, rating alternatives and cloud service selection. The model includes automatic search mechanisms for conflict areas and recommendations to the experts to bring closer their preferences. To demonstrate the applicability of the proposed model and illustrative example ispresented.

Further works will concentrate in extending the model for dealing with heterogeneous information and the development of a software tool. New measures of consensus based on neutrosophic theory will be additionally developed.

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# The Basic Notions for (over, off, under) Neutrosophic Geometric Programming Problems

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**Abstract.** Neutrosophic (over, off, under) set and logic were defined for the first time in 1995 by Florentin Smarandache, and presented during 1995-2018 to various national and international conferences and seminars. The (over, off, under) neutrosophic geometric programming was put forward by Huda et al. in (2016) [8], in an attempt to define a new type of geometric programming using (over, off, under) neutrosophic less than or equal to. This paper completes the basic notions of (over, off, under) neutrosophic geometric programming illustrating its convexity condition, and its decomposition theorems. The definitions of $(\alpha, \beta, \gamma) - cut$ and strong $(\alpha, \beta, \gamma) - cut$ are introduced, and some of their important properties are proved.

**Keyword:** Neutrosophic Set (NS), Neutrosophic Geometric Programming (NGP), (Over, Off, Under) Neutrosophic Convex Set, (sleeves, neut-sleeves, anti-sleeves) of Neutrosophic Sets, Ideal Sleeves, $(\alpha, \beta, \gamma) - \mathrm{cut}$, Strong $(\alpha, \beta, \gamma) - \mathrm{cut}$ , Excluded Middle Law, Decomposition Theorems;

## Introduction

B. Y. Cao set up the mathematical fundamentals of fuzzy geometric programming (FGP) [1], and introduced it at the second IFSA conference, in 1987, in (Tokyo). The formulation and uniqueness of the maximum solution of fuzzy neutrosophic geometric programming in the type of relational equations were firstly introduced by H.E. Khalid [14], later there was a novel method for finding the minimum solution in the same fuzzy neutrosophic relational equations on geometric programming presented on 2016 [15]. The most important paper which related with the basic role of this paper which regarded as the first attempt to present the notion of (over, off, under) neutrosophic less than or equal in geometric programming was established by Florentin S. and Huda E. [8]. The NGP method has been admitted by specialists and created a new branch of neutrosophic mathematics. Inspired by Smarandache's neutrosophic sets theory and (over, off, under) neutrosophic set theory [2, 5, 6], neutrosophic geometric programming emerges from the combination of neutrosophic sets with geometric programming. The present paper intends to discuss the (over, off, under) convexity in neutrosophic sets, introducing a new definition for convexity, and graphing the geometrical representations for (over, off, under) convexity property. Neutrosophic sleeves, neutrosophic neut-sleeves and neutrosophic anti-sleeves are also introduced in this research. Because each neutrosophic set can uniquely be represented by the family of all its $(\alpha, \beta, \gamma) - cut$, it is useful to enunciate the definition of $(\alpha, \beta, \gamma) - cut$ and prove some of its properties, similarly talking for strong $(\alpha, \beta, \gamma) - cut$. Any neutrosophic mathematical programming cannot be generated from the womb of fuzzy mathematical programming without the passage through intuitionistic fuzzy mathematical programming [17, 18], so we should be familiar with all aspects of intuitionistic mathematical programming fundamentals from the point of view of K. T. Atanassov [13, 16].

### 1 (Over, Off, Under) Convexity Property in Neutrosophic Sets

In this section, a new convexity behavior of the neutrosophic set will be given. Let X be an ordinary set whose generic elements are denoted by x. N(X) is the set of all neutrosophic sets included in X.

### 1.1 Definition [19]

A neutrosophic set $A \in N(x)$ is defined as $A = \{< \mu_A(x), \sigma_A(x), v_A(x) >: x \in X\}$ where $\mu_A(x), \sigma_A(x), v_A(x)$ represent the membership function, the indeterminacy function, the non-membership function respectively.

### 1.2 Definition [4]

A mapping $A: X \to [0,1]$, $x \to \mu_A(x)$, $x \to \sigma_A(x)$, $x \to v_A(x)$ is called a collection of neutrosophic elements, where $\mu_A$ a membership x corresponding to a neutrosophic set A, $\sigma_A(x)$ an indeterminacy membership x corresponding to a neutrosophic set A, $v_A(x)$ a non-membership x corresponding to a neutrosophic set A.

### 1.3 Definition

Suppose $A \in N(x)$. If $\forall x_1, x_2 \in X$, we call that A is an (over, off, under) convex neutrosophic set, iff the following conditions hold together:

1- $\mu_A(\lambda\, x_1 + (1-\lambda)x_2) > \min(\mu_A(x_1), \mu_A(x_2)$ $\forall x_1, x_2 \in X$.
2- Let $\sigma_A(x) = \mu_A(x) \cap v_A(x)$ and
   a- $\sigma_A(x)$ satisfies the convex condition,
      i.e. $\sigma_A(\lambda\, x_1 + (1-\lambda)x_2) > \min(\sigma_A(x_1), \sigma_A(x_2)$ for some $x_1, x_2 \in X$.
   b- $\sigma_A(x)$ satisfies the concave condition,
      i.e. $\sigma_A(\lambda\, x_1 + (1-\lambda)x_2) < \max(\sigma_A(x_1), \sigma_A(x_2))$ for some $x_1, x_2 \in X$.
   c- $\sigma_A(x)$ is neither convex nor concave at $t_1 \in X$, where $t_1 = \lambda\, x_2 + (\mathbf{1}-\lambda)x_1$, and $\lambda = 0.5$, (i.e. $\sigma_A(x_1) = \sigma_A(x_2) = \sigma_A(t_1) = 0$).
3- $v_A(\lambda\, x_1 + (1-\lambda)x_2) < \max(v_A(x_1), v_A(x_2))$ $\forall x_1, x_2 \in X$.

For more details, see Figures 1, 2, and 3.

## 2 Geometrical Representation

This section illustrates the geometrical representation of the (over, off, under) convexity behavior in neutrosophic sets. Figures 1,2 and 3 illustrate the given notion as follow:

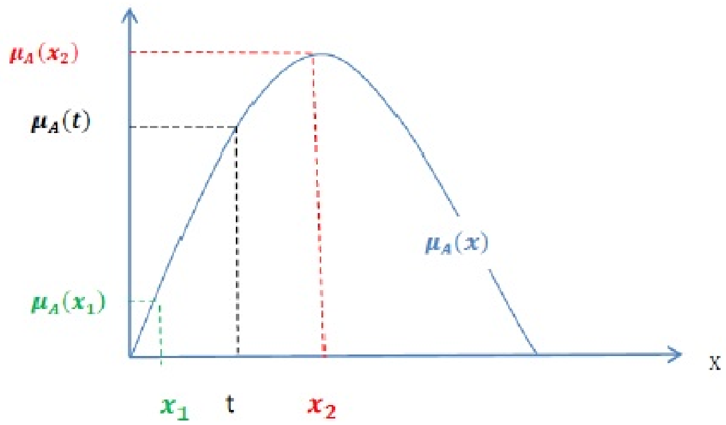


Figure 1: The convex condition of the truth membership function $\mu_A(x)$.

Here, $\mu_A(t) > \min(\mu_A(x_1), \mu_A(x_2)$, where $t = (\lambda\, x_1 + (1-\lambda)x_2)$ ; $\lambda \in [0,1]$, satisfying the condition for all $x_1, x_2 \in X$.

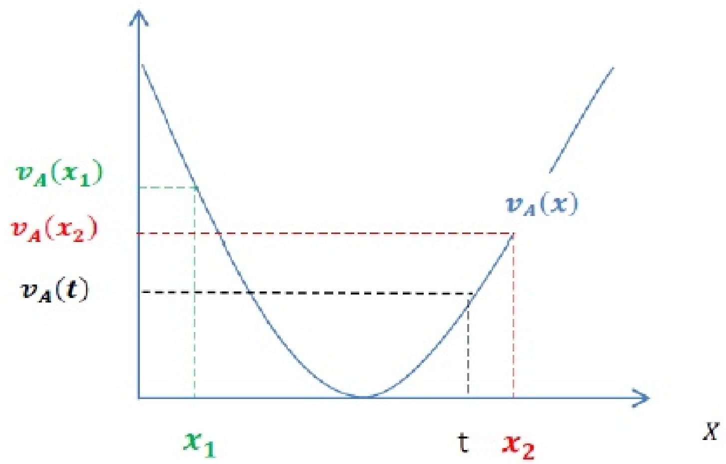


Figure 2: The concave condition of the falsehood membership function $v_A(x)$

Here, $v_A(t) < \max\ (v_A(x_1), v_A(x_2))$ , where $t = (\lambda x_1 + (1-\lambda)x_2)$ ; $\lambda \in [0,1]$ the condition is happening for all $x_1, x_2 \in X$.

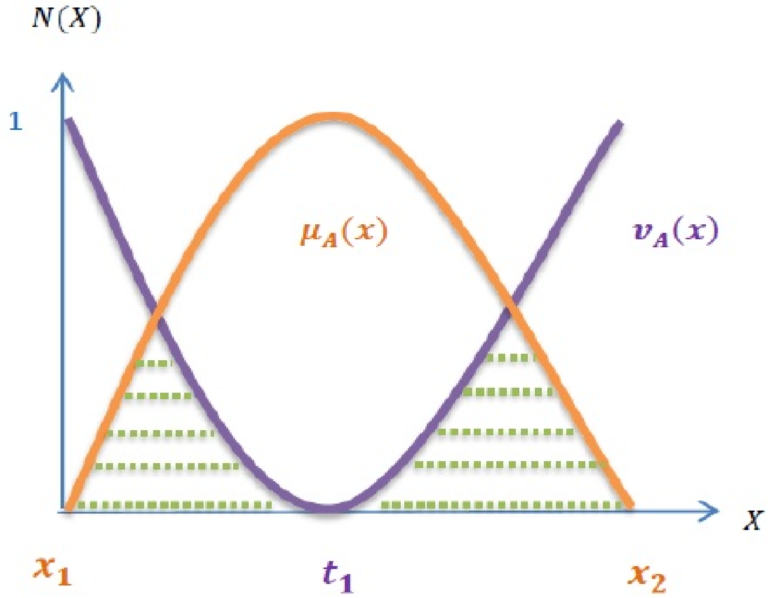


Figure 3: Here the indeterminate function is constructed from the intersection between the truth and falsehood membership functions; i.e. $\sigma_A(x) = \mu_A(x) \cap v_A(x)$. In this figure, the dashed point lines (i.e. shaded with green points represent the indeterminate region, here $\sigma_A(x)$ is neither convex nor concave at $\sigma_A(x_1) = \sigma_A(x_2) = \sigma_A(t_1) = 0$, where $t_1 = \lambda x_2 + (1-\lambda)x_1$, and $\lambda = 0.5$.

## 3 Neutrosophic Sleeves, Neutrosophic Anti-sleeves, Neutrosophic Neut-sleeves

This section introduces for the first time the notion of neutrosophic sleeves, its contradiction and its neutrality. Together with the definitions of the neutrosophic sleeve, neutrosophic anti-sleeve, neutrosophic unit-sleeve, we provided graphs; however, the graphs are imprecise, offering an illustration of the meaning of composite sleeves.

### 3.1 Definition

If a set-valued mapping $\mathrm{H}: [0,1] \rightarrow \mathrm{N}(\mathrm{x})$ satisfies $\forall\, \alpha_1, \alpha_2 \in [0,1]$, $\alpha_1 < \alpha_2 \Rightarrow \mathrm{H}(\alpha_1) \supseteq \mathrm{H}(\alpha_2)$, then $H$ is called a collection neutrosophic sleeve on $X$ . A set composed of all the collections of neutrosophic sleeves on $X$ is written as $(X)$ . The ideal sleeve occurs when $H(\alpha_1) = H(\alpha_2)$.

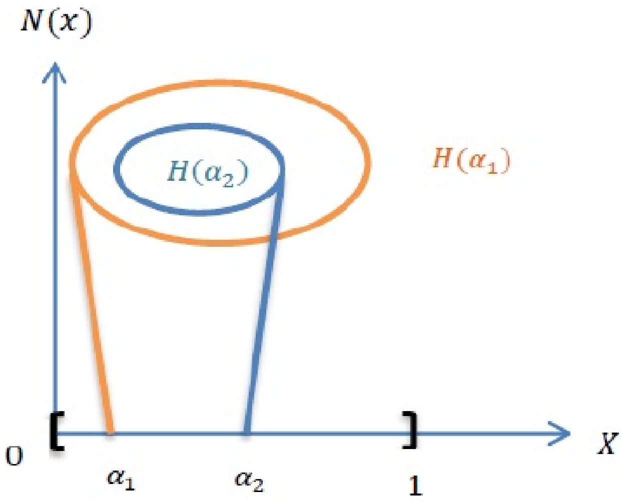


Figure 4: Neutrosophic sleeve

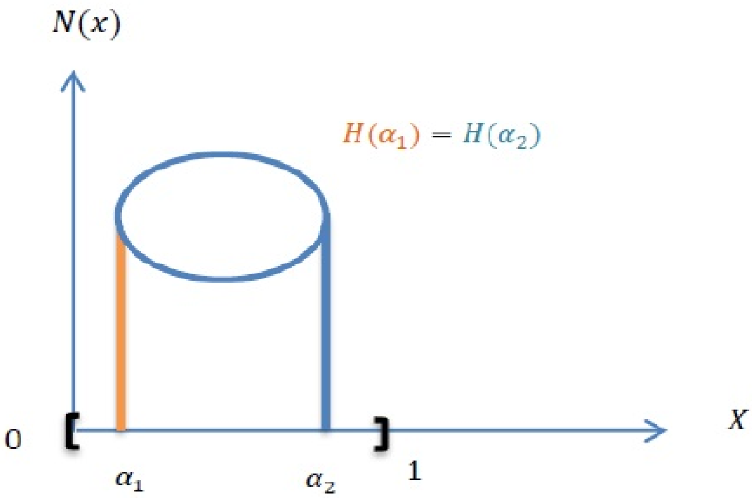


Figure 5: Neutrosophic ideal sleeve

### 3.2 Definition

If a set-valued mapping $H: [0,1] \rightarrow N(x)$ satisfies $\forall\, \alpha_1, \alpha_2, 2\alpha_2 - \alpha_1 \in [0,1]\,, \alpha_2 < 2\alpha_2 - \alpha_1 \Rightarrow H(\alpha_2) \supseteq H(2\alpha_2 - \alpha_1)$, then $H$ is called a collection of neutrosophic anti-sleeves on $X$. A set composed of all the collection of neutrosophic anti-sleeves on $X$ is written as $anti\ U(X)$. The ideal neutrosophic anti-sleeve on $X$ occures when $H(\alpha_2) = H(2\alpha_2 - \alpha_1)$.

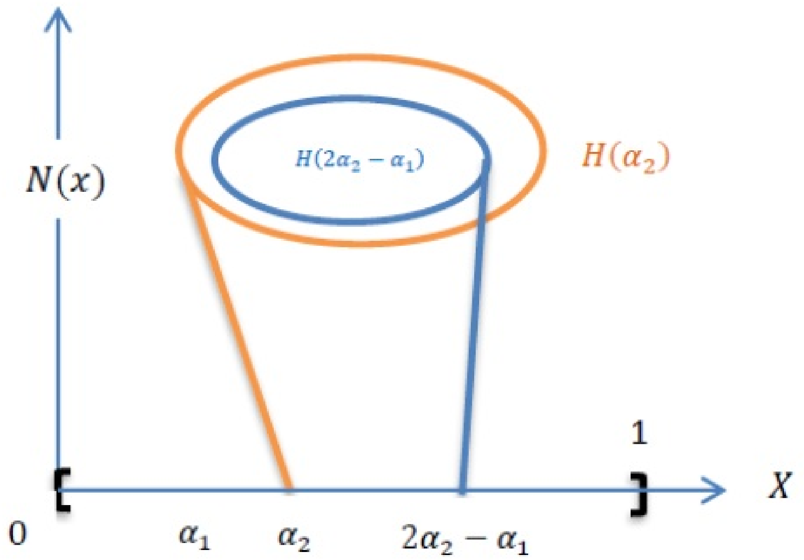


Figure 6: Neutrosophic anti-sleeve

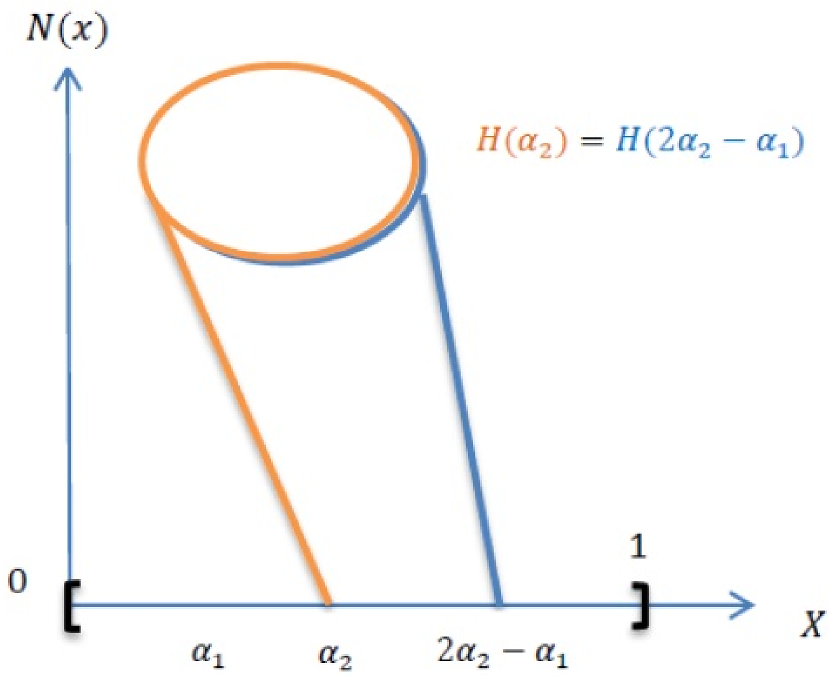


Figure 7: Neutrosophic ideal anti-sleeve

**3.3 Definition**

If a set-valued mapping $H: [0,1] \to N(x)$ satisfies $\forall\, \alpha_1, \alpha_2, 2\alpha_2 - \alpha_1 \in [0,1]\,, \alpha_1 < \alpha_2 < 2\alpha_2 - \alpha_1 \Rightarrow H(\alpha_1) \wedge H(2\alpha_2 - \alpha_1) = \min\{\, H(\alpha_1), H(2\alpha_2 - \alpha_1)\} = H(\alpha_2)$, then $H$ is called a collection of neutrosophic neut-sleeves on $X$ . A set composed of all the collection of neutrosophic neut-sleeves on $X$ is written as $neut\ U(X)$. The ideal neutrosophic neut-sleeve on $X$ occurs in the case of $0 < \alpha_1 < \alpha_2 < 2\alpha_2 - \alpha_1 < 1 \Rightarrow H(\alpha_1) = H(\alpha_2) = H(2\alpha_2 - \alpha_1)$ .

**Note that:**

The ideal case of neutrosophic neut-sleeve is composed from the ideal case of the neutrosophic sleeve combined with the ideal case of the neutrosophic anti-sleeve.

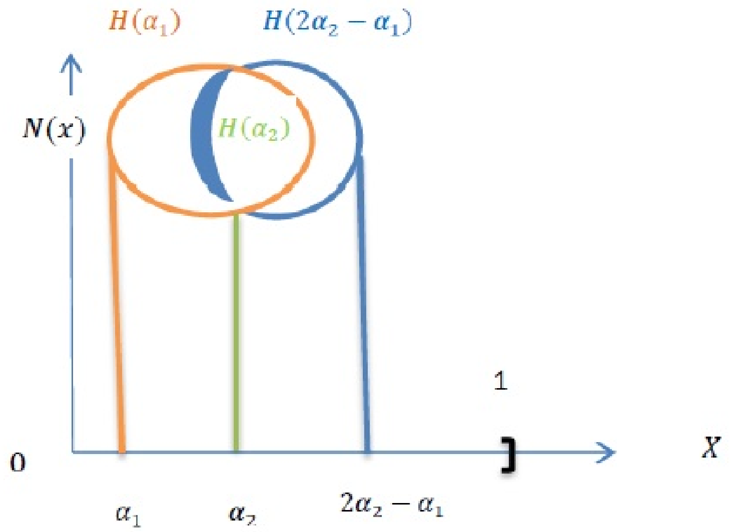


Figure 8: Neutrosophic neut-sleeve

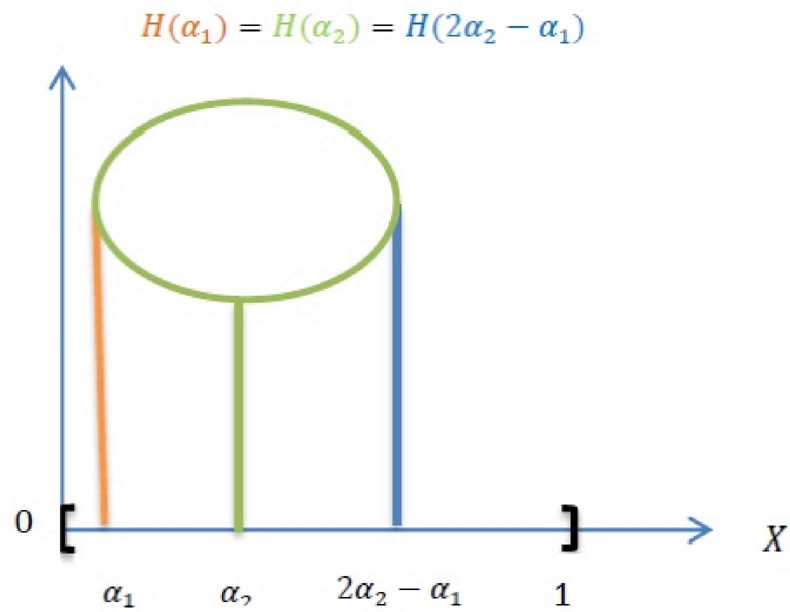


Figure 9: Neutrosophic ideal neut-sleeve

**Note that:**

All figures from 4 to 9 are just indicative graphs employed to understand the meaning of neutrosophic sleeves, neutrosophic anti-sleeves and neutrosophic neut-sleeves, but are not necessary accurate.

## 4 A new insight of the excluded-middle law in neutrosophic theory

### 4.1 The excluded-middle law in classical and fuzzy logics

In classical dialectics, the excluded middle law is the third of the three classic laws of thought. It states that for any proposal, either that proposal is true, or its contradictory is true. The earliest known formulation was in Aristotle's discussion of the principle of non-contradiction, where he said that of two contradictory suggestions, one of them must be true, and the other is false. Aristotle 384 BC, said that it is necessary for every claim there are two opposite parts, either confirm or deny and it is unattainable that there should be anything between the two parts of an opposition. We point out that fuzzy logic, intuitionistic fuzzy logic and neutrosophic logic no longer satisfy the excluded-middle law [3].

Let $X$ be an ordinary fuzzy set, whose generic elements are denoted by $x$, a mapping $A: X \to [0,1]$, $x \to \mu_A(x)$ called a fuzzy set $A$, and let the complement of $A$ be $A^c$ with its membership function meaning $\mu_{A^c}(x) = 1 - \mu_A(x)$, then it is obvious that the excluded-middle law is not satisfied.

$$\mu_A(x) \cup \mu_{A^c}(x) \neq X \tag{1}$$

$$\mu_A(x) \cap \mu_{A^c}(x) \neq \emptyset \tag{2}$$

Example:

Let $X = [0,1], \mu_A(x) = x$ , then $\mu_{A^c}(x) = 1 - x$, while

$$(\mu_A \cup \mu_{A^c})(x) = \begin{cases} 1 - x & x \leq \frac{1}{2} \\ x & x > \frac{1}{2} \end{cases} \tag{3}$$

$$(\mu_A \cap \mu_{A^c})(x) = \begin{cases} x & x \leq \frac{1}{2} \\ 1 - x & x > \frac{1}{2} \end{cases} \tag{4}$$

Hence

$$\mu_A \cup \mu_{A^c} \neq X \qquad \& \qquad \mu_A \cap \mu_{A^c} \neq \emptyset$$

Especially,

$$(\mu_A \cup \mu_{A^c})\left(\frac{1}{2}\right) = (\mu_A \cap \mu_{A^c})\left(\frac{1}{2}\right) = \frac{1}{2} \tag{5}$$

A fuzzy set operation does not satisfy the excluded-middle law, which complicates the study of fuzzy sets. The fuzzy sets can provide more objective properties than the classical sets [1].

### 4.2 The excluded middle law with the perspective of (over, off, under) neutrosophic geometric programming

In the two-valued logic, all the designated values as types of truth and all the anti-designated values as types of untruth with gaps between truth-value (or falsehood-value). In the neutrosophic theory, one specifies the non-designated values as types of indeterminacy and thus, each neutrosophic consequences have degrees of designated, non-designated, and anti-designated values. However, the excluded middle law in the neutrosophic system does no longer work [7].

Even more, Smarandache (2014) [3] generalized the Law of Included Middle to the *Law of Included Multiple-Middles*, showing that in refined neutrosophic logic (2013), between truth ($T$) and falsehood ($F$) there are multiple types of sub-indeterminacies ($I_1$, $I_2$, ...) [10,11,12].

In upcoming definitions, the authors affirm that $\mu_A(x) \cap \mu_{A^c}(x) \neq \emptyset$ in neutrosophic environment; for example but not limited to, the nonlinear neutrosophic programming ( i.e. example of neutrosophic geometric programming (NGP)).

### 4.2.1 Definition

Let $N(\mathrm{X})$ be the set of all neutrosophic variable vectors $x_i$, $i = 1,2,\dots,m$, i.e. $N(X) = \{(x_1, x_2, \dots, x_m)^T \mid x_i \in X\}$. The function g(x): N(X) $\rightarrow$ R $\cup$ $I$ is said to be neutrosophic GP function of $x$, where $g(x) = \sum_{k=1}^{J} c_k \prod_{l=1}^{m} x_l^{\gamma_{kl}}$, $c_k \geq 0$ is a constant, $\gamma_{kl}$ being an arbitrary real number.

### 4.2.2 Definition

Let $g(x)$ be a neutrosophic geometric function in any neutrosophic geometric programming, and let $A_0$ be the neutrosophic set for all functions $g(x)$ that are neutrosophically less than or equal to one.

$$A_0 = \{x_i \in \mathrm{X}:\ g(x) <\!\!\!\!\!\!\mathrm{N}\ 1\} = \{x_i \in \mathrm{X}:\ g(x) < 1, anti(\,g(x)) > 1, neut(\,g(x)) = 1\} \qquad (6)$$

### 4.2.3 Definition

Let $g(x)$ be any neutrosophic geometric function written as a constraint in any neutrosophic geometric programming (NGP), where $x_i \in X = [0,1] \cup [0, nI]$ and $x = (x_1, x_2, \dots, x_m)^T$ is an m-dimensional neutrosophic variable vector.

Call the inequality

$$g(x) <\!\!\!\!\!\!\mathrm{N}\ 1 \qquad (7)$$

where " $<\!\!\!\!\!\!\mathrm{N}$" denotes the neutrosophied version of " $\leq$ " with the linguistic interpretation being "less than (the original claimed), greater than (the anti-claim of the original less than), or equal (neither the original claim nor the anti-claim)"

The constraint (7) can be redefined into three constraints as follow:-

$$\left.\begin{array}{r} g(x) < 1 \\ anti\ (g(x)) > 1 \\ neut(\,g(x)) = 1 \end{array}\right\} \qquad (8)$$

### 4.2.4 Definition

Let $A_0$ be the set of all neutrosophic geometric functions that neutrosophically less than or equal to one,

i.e. $A_0 = \{x_i \in \mathrm{X}, g(x) <\!\!\!\!\!\!\mathrm{N}\ 1\} \Leftrightarrow A_0 = \{x_i \in \mathrm{X}: g(x) < 1, anti(\,g(x)) > 1, neut(\,g(x)) = 1\}$

It is significant to define the following membership functions:

$$\mu_{A_o}(\,g(x)) = \begin{cases} 1 & , \quad 0 \leq g(x) \leq 1 \\ \left(\left(e^{\frac{-1}{d_o}(g(x)-1)} + e^{\frac{-1}{d_o}(anti(\,g(x))-1)} - 1\right)\right), & \quad 1 < g(x) \leq 1 - d_o \ln 0.5 \end{cases} \qquad (9)$$

$$\mu_{A_o}(anti(\,g(x))) = \begin{cases} 0 & , \quad 0 \leq g(x) \leq 1 \\ \left(\left(1 - e^{\frac{-1}{d_o}(anti(\,g(x))-1)} - e^{\frac{-1}{d_o}(g(x)-1)}\right)\right), & \quad 1 - d_o \ln 0.5 \leq g(x) \leq 1 + d_o \end{cases} \qquad (10)$$

It is clear that $\mu_{A_o}(neut(\,g(x)))$ consists from the intersection of the following functions:

$$e^{\frac{-1}{d_o}(g(x)-1)}\ , \quad 1 - e^{\frac{-1}{d_o}(anti(\,g(x))-1)}$$

i.e.

$$\mu_{A_o}(neut(\,g(x))) = \begin{cases} 1 - e^{\frac{-1}{d_o}(anti(\,g(x))-1)}\ , & 1 \leq g(x) \leq 1 - d_o \ln 0.5 \\ e^{\frac{-1}{d_o}(g(x)-1)} & , 1 - d_o \ln 0.5 < g(x) \leq 1 + d_o \end{cases} \qquad (11)$$

Note that $d_o > 0$ is a constant expressing a limit of the admissible violation of the neutrosophic geometric function $g(x)$.

Consequently,

$$\mu_{A_o}(\,g(x)) \cap \mu_{A_o}\left(anti(\,g(x))\right) \neq \emptyset\,,$$

Here $\mu_{A_o}(\,g(x)) \cap \mu_{A_o}\left(anti(\,g(x))\right) = \mu_{A_o}(neut(\,g(x)))$.

## 5 $(\alpha, \beta, \gamma) - \mathbf{cut}$ and strong $(\alpha, \beta, \gamma) - \mathbf{cut}$ of Neutrosophic sets

We put the following definitions as an initial step to prepare to prove the properties of $(\alpha, \beta, \gamma) - cut$ and strong $(\alpha, \beta, \gamma) - cut$ of neutrosophic sets.

### 5.1 Definition

Let $A \in N(x), \forall\ (\alpha, \beta, \gamma) \in [0,1]$ , written $A_{(\alpha,\beta,\gamma)} = \{x: \mu_A(x) \geq \alpha, \sigma_A(x) \geq \beta, v_A(x) \leq \gamma\}$, $A_{(\alpha,\beta,\gamma)}$ is said to be an $(\alpha, \beta, \gamma) - cut$ set of a neutrosophic set $A$ . Again, we write $A_{(\alpha,\beta,\gamma)^+} = \{x: \mu_A(x) > \alpha, \sigma_A(x) > \beta, v_A(x) < \gamma\}$, $A_{(\alpha,\beta,\gamma)^+}$ is said to be a strong $(\alpha, \beta, \gamma) - cut$ set of a neutrosophic set $A$, $(\alpha, \beta, \gamma)$ are confidence levels and $\alpha + \beta + \gamma \leq 3$.

### 5.2 Definition

Let $A \in N(x)$, written $A_{(0,0,1)^+} = \{x: \mu_A(x) > 0, \sigma_A(x) > 0, v_A(x) < 1\} = supp\ A$, $A_{(0,0,1)^+}$ is called a support of a neutrosophic set $A$. Again, $ker\ A = \{x: \mu_A(x) = 1, \sigma_A(x) = 0, v_A(x) = 0\}$ is called a kernel of neutrosophic set $A$, and $A$ is a normal neutrosophic set for $\ker A \neq \emptyset$.

### 5.3 Definition

Let $A \in N(x)$, written $A \cup B = \{\langle x, \max\left(\mu_A(x), \mu_B(x)\right),$

$\max(\sigma_A(x), \sigma_B(x)),\ \min(v_A(x), v_B(x))\rangle : x \in X\}$, the union of $A\&B$

$A \cap B = \{\langle x, \min\left(\mu_A(x), \mu_B(x)\right), \min(\sigma_A(x), \sigma_B(x)),\ \max(v_A(x), v_B(x))\rangle : x \in X\}$, the intersection of $A\&B$.

### 5.4 Theorem

We have the following properties for $(\alpha, \beta, \gamma) - \text{cut}$ and strong $(\alpha, \beta, \gamma) - \text{cut}$ neutrosophic sets:

1- $A \subseteq B \Rightarrow A_{(\alpha,\beta,\gamma)} \subseteq B_{(\alpha,\beta,\gamma)}$

2- $(A \cup B)_{(\alpha,\beta,\gamma)} \supseteq A_{(\alpha,\beta,\gamma)} \cup B_{(\alpha,\beta,\gamma)}$ equality holds if $\alpha + \beta + \gamma = 3$.
   $(A \cap B)_{(\alpha,\beta,\gamma)} = A_{(\alpha,\beta,\gamma)} \cap B_{(\alpha,\beta,\gamma)}$.

3- $(A \cup B)_{(\alpha,\beta,\gamma)^+} \supseteq A_{(\alpha,\beta,\gamma)^+} \cup B_{(\alpha,\beta,\gamma)^+}$, equality holds if $\alpha + \beta + \gamma = 3$.
   $(A \cap B)_{(\alpha,\beta,\gamma)^+} = A_{(\alpha,\beta,\gamma)^+} \cap B_{(\alpha,\beta,\gamma)^+}$.

**Proof**

1- Let $x \in A_{(\alpha,\beta,\gamma)} \Rightarrow \mu_A(x) \geq \alpha, \sigma_A(x) \geq \beta, v_A(x) \leq \gamma$

But $B \supseteq A \Rightarrow \mu_B(x) \geq \mu_A(x) \geq \alpha, \sigma_B(x) \geq \sigma_A(x) \geq \beta, v_B(x) \leq v_A(x) \leq \gamma$

$\Rightarrow \mu_B(x) \geq \alpha, \sigma_B(x) \geq \beta, v_B(x) \leq \gamma$

$\Rightarrow x \in B_{(\alpha,\beta,\gamma)}$, therefore $A_{(\alpha,\beta,\gamma)} \subseteq B_{(\alpha,\beta,\gamma)}$

2- $(A \cup B)_{(\alpha,\beta,\gamma)} \supseteq A_{(\alpha,\beta,\gamma)} \cup B_{(\alpha,\beta,\gamma)}$

Since $A \subseteq (A \cup B)$, $B \subseteq (A \cup B)$ and from 1 above, we have:

$A_{(\alpha,\beta,\gamma)} \subseteq (A \cup B)_{(\alpha,\beta,\gamma)}$ (12)

$B_{(\alpha,\beta,\gamma)} \subseteq (A \cup B)_{(\alpha,\beta,\gamma)}$ (13)

Combine (12) with (13). The proof of property 2 is complete, i.e.

$(A \cup B)_{(\alpha,\beta,\gamma)} \supseteq A_{(\alpha,\beta,\gamma)} \cup B_{(\alpha,\beta,\gamma)}$ (14)

If $\alpha + \beta + \gamma = 3$, we show that $(A \cup B)_{(\alpha,\beta,\gamma)} = A_{(\alpha,\beta,\gamma)} \cup B_{(\alpha,\beta,\gamma)}$

Let $x \in (A \cup B)_{(\alpha,\beta,\gamma)} \Rightarrow \mu_A(x) \cup \mu_B(x) \geq \alpha, \sigma_A(x) \cup \sigma_B(x) \geq \beta, v_A(x) \cup v_B(x) \leq \gamma$

if $\mu_A(x) \geq \alpha$ *and* $\sigma_A(x) \geq \beta$ then $v_A(x) \leq 3 - \alpha - \beta = \gamma \Rightarrow x \in A_{(\alpha,\beta,\gamma)} \subseteq A_{(\alpha,\beta,\gamma)} \cup B_{(\alpha,\beta,\gamma)}$

also if $\mu_B(x) \geq \alpha$ *and* $\sigma_B(x) \geq \beta$ then $v_B(x) \leq 3 - \alpha - \beta = \gamma \Rightarrow x \in B_{(\alpha,\beta,\gamma)} \subseteq A_{(\alpha,\beta,\gamma)} \cup B_{(\alpha,\beta,\gamma)}$

$\Rightarrow x \in A_{(\alpha,\beta,\gamma)} \cup B_{(\alpha,\beta,\gamma)}$

and so $(A \cup B)_{(\alpha,\beta,\gamma)} \subseteq A_{(\alpha,\beta,\gamma)} \cup B_{(\alpha,\beta,\gamma)}$ (15)

From (14) and (15), we get

$(A \cup B)_{(\alpha,\beta,\gamma)} = A_{(\alpha,\beta,\gamma)} \cup B_{(\alpha,\beta,\gamma)}$

We still need to prove that $(A \cap B)_{(\alpha,\beta,\gamma)} = A_{(\alpha,\beta,\gamma)} \cap B_{(\alpha,\beta,\gamma)}$

Proof

Since $A \cap B \subseteq A$ *and* $A \cap B \subseteq B$

$\Rightarrow (A \cap B)_{(\alpha,\beta,\gamma)} \subseteq A_{(\alpha,\beta,\gamma)}$ & $(A \cap B)_{(\alpha,\beta,\gamma)} \subseteq B_{(\alpha,\beta,\gamma)}$ (16)

Let $x \in A_{(\alpha,\beta,\gamma)} \cap B_{(\alpha,\beta,\gamma)}$

$\Rightarrow x \in A_{(\alpha,\beta,\gamma)}$ & $x \in B_{(\alpha,\beta,\gamma)}$

$\Rightarrow \mu_A(x) \geq \alpha, \sigma_A(x) \geq \beta, v_A(x) \leq \gamma$ and $\mu_B(x) \geq \alpha, \sigma_B(x) \geq \beta, v_B(x) \leq \gamma$

$\Rightarrow \mu_A(x) \cap \mu_B(x) \geq \alpha, \sigma_A(x) \cap \sigma_B(x) \geq \beta, v_A(x) \cup v_B(x) \leq \gamma$

$\Rightarrow x \in (A \cap B)_{(\alpha,\beta,\gamma)}$

$\Rightarrow A_{(\alpha,\beta,\gamma)} \cap B_{(\alpha,\beta,\gamma)} \subseteq (A \cap B)_{(\alpha,\beta,\gamma)}$ (17)

From (16) and (17), we have

$(A \cap B)_{(\alpha,\beta,\gamma)} = A_{(\alpha,\beta,\gamma)} \cap B_{(\alpha,\beta,\gamma)}$

Note that:

The same technique that used for proving 2 will be employed to prove the properties of strong (α,β,γ)-cut in 3.

## 6 Representations of neutrosophic sets

The decomposition theorems of neutrosophic sets is a bridge between neutrosophic sets and ordinary ones. The principal feature of $(\alpha, \beta, \gamma) -$ cut and strong $(\alpha, \beta, \gamma) -$ cut sets in neutrosophic set theory is the capability to represent neutrosophic sets. We show in this section that each neutrosophic set can uniquely be represented by either the family of all its $(\alpha, \beta, \gamma) -$ cuts or the family of all its strong $(\alpha, \beta, \gamma) -$ cuts.

We can convert each of $(\alpha, \beta, \gamma) -$ cut and strong $(\alpha, \beta, \gamma) -$ cut to special neutrosophic sets denoted by ${}_{(\alpha,\beta,\gamma)}A$ *and* ${}_{(\alpha,\beta,\gamma)}^{\;+}A$, as follows:

for $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma \leq 3$, we have:

$$_{(\alpha,\beta,\gamma)}A = \begin{cases} (\alpha,\beta,\gamma) & if\ x \in A_{(\alpha,\beta,\gamma)} \\ (0,0,1) & if\ x \notin A_{(\alpha,\beta,\gamma)} \end{cases} \tag{18}$$

$$_{(\alpha,\beta,\gamma)}^{+}A = \begin{cases} (\alpha,\beta,\gamma) & if\ x \in A_{(\alpha,\beta,\gamma)^+} \\ (0,0,1) & if\ x \notin A_{(\alpha,\beta,\gamma)^+} \end{cases} \tag{19}$$

The representation of an arbitrary neutrosophic set $A$ in terms of the special neutrosophic sets $_{(\alpha,\beta,\gamma)}A$, which are defined in terms of the $(\alpha,\beta,\gamma)-$cuts of $A$ by (18), is usually referred to as decomposition of $A$. In the following, we formulate and prove two basic decomposition theorems of neutrosophic sets.

### 6.1 First Decomposition theorem of neutrosophic set (NS)

For every $A \in N(x)$, $A = \underset{\alpha,\beta,\gamma \in [0,1]}{\cup} {}_{(\alpha,\beta,\gamma)}A$, where $_{(\alpha,\beta,\gamma)}A$ is defined by (18) and $\cup$ denotes the standard neutrosophic union.

**Proof**

For each particular $x \in X$, let

$\mu_A(x) = \mathrm{a}, \sigma_A(x) = \mathrm{b}, v_A(x) = c$. Then,

$$\left(\underset{\alpha,\beta,\gamma \in [0,1]}{\cup} {}_{(\alpha,\beta,\gamma)}A\right)(x) = \left(\underset{\alpha \in [0,1]}{\sup} \mu_A(x), \quad \underset{\beta \in [0,1]}{\sup} \sigma_A(x), \quad \underset{\gamma \in [0,1]}{\inf} v_A(x)\right)$$

$$= max\left[\left(\underset{\alpha \in [0,a]}{\sup} \mu_A(x), \quad \underset{\beta \in [0,1]}{\sup} \sigma_A(x), \quad \underset{\gamma \in [0,1]}{\inf} v_A(x)\right), \quad \left(\underset{\alpha \in (a,1]}{\sup} \mu_A(x), \quad \underset{\beta \in [0,1]}{\sup} \sigma_A(x), \quad \underset{\gamma \in [0,1]}{\inf} v_A(x)\right)\right]$$

For each $\alpha \in (a, 1]$, we have $\mu_A(x) = a < \alpha$ and, therefore, $_{(\alpha,\beta,\gamma)}A = (0,0,1)$. On the other hand, for each $\alpha \in [0, a]$, we have $\mu_A(x) = a \geq \alpha$, therefore, $_{(\alpha,\beta,\gamma)}A = (\alpha,\beta,\gamma)$.

The second step of the prove is to complete the maximum value for the second component $\left(i.e.\ \underset{\beta \in [0,1]}{\sup} \sigma_A(x)\right)$ as follow:

$$\left(\underset{\alpha,\beta,\gamma \in [0,1]}{\cup} {}_{(\alpha,\beta,\gamma)}A\right)(x)$$

$$= max\left[\left(\underset{\alpha \in [0,a]}{\sup} \mu_A(x), \ \underset{\beta \in [0,b]}{\sup} \sigma_A(x), \ \underset{\gamma \in [0,1]}{\inf} v_A(x)\right), \left(\underset{\alpha \in [0,a]}{\sup} \mu_A(x), \ \underset{\beta \in (b,1]}{\sup} \sigma_A(x), \ \underset{\gamma \in [0,1]}{\inf} v_A(x)\right)\right]$$

For each $\beta \in (b, 1]$, we have $\sigma_A(x) = b < \beta$ and, therefore, $_{(\alpha,\beta,\gamma)}A = (0,0,1)$. On the other hand, for each $\beta \in [0, b]$, we have $\sigma_A(x) = b \geq \beta$, therefore, $_{(\alpha,\beta,\gamma)}A = (\alpha,\beta,\gamma)$.

$$= \left(\underset{\alpha \in [0,a]}{\sup} \mu_A(x), \ \underset{\beta \in [0,b]}{\sup} \sigma_A(x), \ \underset{\gamma \in [0,1]}{\inf} v_A(x)\right)$$

The final step of the proof is to complete the maximum value for the third component $\left(i.e.\ \underset{\gamma \in [0,1]}{\inf} v_A(x)\right)$,

$$\left(\underset{\alpha,\beta,\gamma \in [0,1]}{\cup} {}_{(\alpha,\beta,\gamma)}A\right)(x)$$

$$= max\left[\left(\underset{\alpha \in [0,a]}{\sup} \mu_A(x), \underset{\beta \in [0,b]}{\sup} \sigma_A(x), \ \underset{\gamma \in [0,c)}{\inf} v_A(x)\right), \quad \left(\underset{\alpha \in [0,a]}{\sup} \mu_A(x), \underset{\beta \in [0,b]}{\sup} \sigma_A(x), \underset{\gamma \in [c,1]}{\inf} v_A(x)\right)\right]$$

For each $\gamma \in [c, 1]$, we have $v_A(x) = c \leq \gamma$, therefore, $_{(\alpha,\beta,\gamma)}A = (\alpha,\beta,\gamma)$. On the other hand, for each $\gamma \in [0, c)$, we have $v_A(x) = c > \gamma$, therefore, $_{(\alpha,\beta,\gamma)}A = (0,0,1)$.

Consequently,

$$\left(\underset{\alpha,\beta,\gamma \in [0,1]}{\cup} {}_{(\alpha,\beta,\gamma)}A\right)(x) = \left(\underset{\alpha \in [0,a]}{\sup} \mu_A(x), \underset{\beta \in [0,b]}{\sup} \sigma_A(x), \ \underset{\gamma \in [c,1]}{\inf} v_A(x)\right) = (\alpha,\beta,\gamma) = A(x)$$

Since the same argument is valid for each $x \in X$, the theorem is proved.

■

### 6.2 Second Decomposition Theorem of Neutrosophic Set (NS)

Let $X$ be any non-empty set. For a neutrosophic subset $A \in N(X)$,
$A = \underset{\alpha,\beta,\gamma \in [0,1]}{\cup} {}_{(\alpha,\beta,\gamma)}\overset{+}{A}$ , where ${}_{(\alpha,\beta,\gamma)}\overset{+}{A}$ is defined by (19) ,and $\cup$ denotes the standard neutrosophic union.

**Proof:**
For each particular $x \in X$, let $\mu_A(x) = \mathrm{a}, \sigma_A(x) = \mathrm{b}, v_A(x) = c$. Then

$$\left(\underset{\alpha,\beta,\gamma \in [0,1]}{\cup} {}_{(\alpha,\beta,\gamma)}\overset{+}{A}\right)(x) = \left(\sup_{\alpha \in [0,1]} \mu_A(x),\ \sup_{\beta \in [0,1]} \sigma_A(x),\ \inf_{\gamma \in [0,1]} v_A(x)\right)$$

$$= max\left[\left(\sup_{\alpha \in [0,a)} \mu_A(x), \sup_{\beta \in [0,1]} \sigma_A(x), \inf_{\gamma \in [0,1]} v_A(x)\right),\quad \left(\sup_{\alpha \in [a,1]} \mu_A(x), \sup_{\beta \in [0,1]} \sigma_A(x),\ \inf_{\gamma \in [0,1]} v_A(x)\right)\right]$$

For each $\alpha \in [a, 1]$, we have $\mu_A(x) = a \le \alpha$ and, therefore, ${}_{(\alpha,\beta,\gamma)}\overset{+}{A} = (0,0,1)$. On the other hand, for each $\alpha \in [0, a)$, we have $\mu_A(x) = a > \alpha$, therefore, ${}_{(\alpha,\beta,\gamma)}\overset{+}{A} = (\alpha, \beta, \gamma)$ the second step of the proof is to complete the maximum value for the second component $\left(i.e.\ \sup_{\beta \in [0,1]} \sigma_A(x)\right)$.
Again,

$$\left(\underset{\alpha,\beta,\gamma \in [0,1]}{\cup} {}_{(\alpha,\beta,\gamma)}\overset{+}{A}\right)(x)$$
$$= max\left[\left(\sup_{\alpha \in [0,a)} \mu_A(x),\ \sup_{\beta \in [0,b)} \sigma_A(x), \inf_{\gamma \in [0,1]} v_A(x)\right),\quad \left(\sup_{\alpha \in [a,1]} \mu_A(x),\ \sup_{\beta \in [b,1]} \sigma_A(x),\ \inf_{\gamma \in [0,1]} v_A(x)\right)\right]$$

For each $\beta \in [b, 1]$, we have $\sigma_A(x) = b \le \beta$ and, therefore, ${}_{(\alpha,\beta,\gamma)}\overset{+}{A} = (0,0,1)$. On the other hand, for each $\beta \in [0, b)$, we have $\sigma_A(x) = b > \beta$, therefore, ${}_{(\alpha,\beta,\gamma)}\overset{+}{A} = (\alpha, \beta, \gamma)$.

$$\left(\underset{\alpha,\beta,\gamma \in [0,1]}{\cup} {}_{(\alpha,\beta,\gamma)}\overset{+}{A}\right)(x) = \left(\sup_{\alpha \in [0,a)} \mu_A(x),\quad \sup_{\beta \in [0,b)} \sigma_A(x),\ \inf_{\gamma \in [0,1]} v_A(x)\right)$$

The final step of the proof is to complete the maximum value for the third component $\left(i.e.\ \sup_{\beta \in [0,1]} \sigma_A(x)\right)$.
Finally,

$$\left(\underset{\alpha,\beta,\gamma \in [0,1]}{\cup} {}_{(\alpha,\beta,\gamma)}\overset{+}{A}\right)(x)$$
$$= max\left[\left(\sup_{\alpha \in [0,a)} \mu_A(x),\ \sup_{\beta \in [0,b)} \sigma_A(x),\ \inf_{\gamma \in [0,c]} v_A(x)\right),\quad \left(\sup_{\alpha \in [0,a)} \mu_A(x),\ \sup_{\beta \in [0,b)} \sigma_A(x),\ \inf_{\gamma \in (c,1]} v_A(x)\right)\right]$$

For each $\gamma \in (c, 1]$, we have $v_A(x) = c < \gamma$, therefore, ${}_{(\alpha,\beta,\gamma)}\overset{+}{A} = (\alpha, \beta, \gamma)$. On the other hand, for each $\gamma \in [0, c]$, we have $v_A(x) = c \ge \beta$, therefore, ${}_{(\alpha,\beta,\gamma)}\overset{+}{A} = (0,0,1)$.
Consequently,

$$\left(\underset{\alpha,\beta,\gamma \in [0,1]}{\cup} {}_{(\alpha,\beta,\gamma)}\overset{+}{A}\right)(x) = \left(\sup_{\alpha \in [0,a)} \mu_A(x),\ \sup_{\beta \in [0,b)} \sigma_A(x), \inf_{\gamma \in (c,1]} v_A(x)\right) = (\alpha, \beta, \gamma) = A(x)$$

Since the same argument is valid for each $x \in X$, therefore the theorem is proved.

■

### Conclusion

Neutrosophic geometric programming (NGP) can find many application areas, such as power engineering, postal services, look for exemplars for eliminating waste-water in a power plant, or determining the power equipping radius in the electrical transformers. All the above-mentioned applications require building a strong neutrosophic theory for neutrosophic geometric programming (NGP), these aims lead the authors to present the (over, off, under) convexity condition in neutrosophic geometric functions. The needed of establishing the aspects of sleeves, neut sleeves and anti-sleeves were necessary. Furthermore, the basic concept of (α,β,γ)-cut and strong (α,β,γ)-cut of neutrosophic sets have been given. By strong definitions and given example, the authors proved that the excluded middle law has no longer satisfied in neutrosophic theory, this proof has been made by neutrosophic geometrical programming.

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# A New Neutrosophic Cognitive Map with Neutrosophic Sets on Connections, Application in Project Management

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**Abstract**. Neutrosophic sets and their application to decision support have become a very important topic. In real situations, there are different sources of indeterminacy. This paper suggests a new decision-making model based on Neutrosophic Cognitive Maps (NCMs) for making comprehensive decisions from a multi-objective approach (diagnosis, decisions, and prediction) during the execution of many projects simultaneously. A Soft Computing technique like Fuzzy Cognitive Maps (FCMs) has been widely used for decision-making process in project management, but this technique has the limitation of not considering the indeterminacy between concepts. This limitation is overcome by the proposed model since NCMs can represent the indeterminacy or neutrality. The new model includes neutrosophic sets in the map's connections. Finally, the suggested model has been compared with traditional FCM-based model considering efficiency and efficacy

**Keywords:** Neutrosophic cognitive maps, neutrosophic sets, project management, single valued neutrosophic numbers.

## 1 Introduction

Project management is characterized by being a complex and dynamic system with high degrees of uncertainty [1]. Consequently, there is a low percentage of success in projects as shown in the reports of the Standish Group International Incorporated. Standish Group is continuously studying the behavior of different companies since 2004; these studies address around 5000 projects annually. Reports of this group show that the numbers of satisfactorily delivered, closed or failed, and renegotiated projects have moved by around 35%, 18%, and 43% respectively [2].

To mitigate this situation, many international project management schools like Project Management Institute (PMI) with its PMBOK standard [3], ISO 21500 standard developed by the International Standards Organization (ISO) [4], and CMMI proposed by Software Engineering Institute [5] have developed guidelines and recommendations for project managers. However, these guides are very generic and frequently need to be personalized to be applied in different contexts. Besides, the techniques they propose do not define clearly how to deal with uncertainty, impression, and incomplete information [6]. In other words, these guidelines are not enough to solve the problems and limitations still presented in project management. Nevertheless, many of these problems are associated with the decision-making process in project management.

In this sense, different authors have referred to decision making as an essential process in project management [7], and others like Trumper et al. [8] defined the project management as the art of making right decisions. Cunha et al. [9] stated that project success depends on how software project managers deal with the problems and make decisions.

In project management, the main decision-making process occurs in project's cuts, see Figure 1. In this respect, decisions should be made out of a multi-objective approach, which includes a diagnosis to know the real state of the ongoing project, making corrective decisions in order to mitigate delays and deviations, and finally predict the project evolution according to the decisions made. It is also important in project management to consider an adequate balance between time, cost, and quality [10].

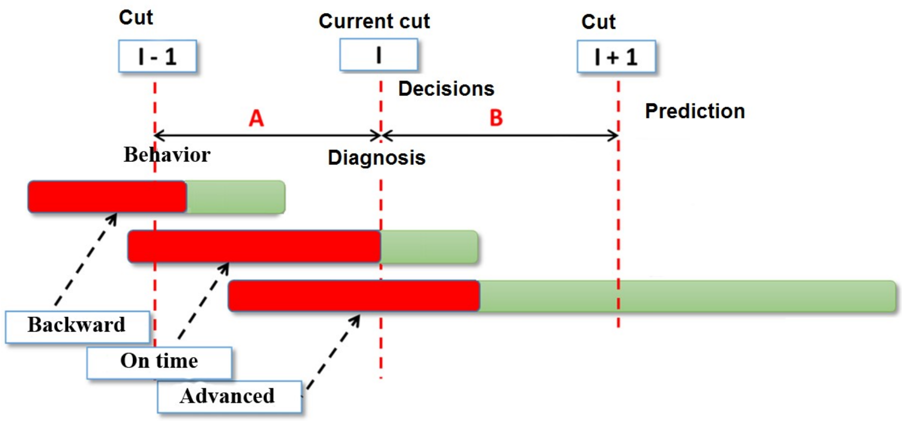


Figure 1. Decision making by cuts in projects.

Generally, the decision-making process in project management involves multiple stakeholders with different degrees of expertise; hence, a consensus process should be carried out in order to reach a generally accepted opinion. Besides, experts on many occasions need to express indeterminacy relationships existing between concepts.

From the previous analysis, it is perceived that there are opportunities to improve the guidelines and recommendations provided by international Standards and schools for project management through the use of Soft Computing techniques. Fuzzy Cognitive Maps (FCMs) is a suitable tool for the representation and simulation of dynamic and complex systems with the presence of uncertainty and incomplete information [11].

FCMs were introduced by Kosko in 1986 [12] as an extension of the Cognitive Maps Theory developed by Axelrod in 1976 [13]. In FCMs, there are three possible types of relations between concepts: positive relation, negative relation, or non-existence of relations, see Figure 2. The widespread use of Fuzzy Cognitive Maps is due to its features of simplicity, adaptability, and capability of dealing with uncertainty, vagueness and incomplete information, besides their capacity to represent feedback relationship [14]. For this reason, FCMs have been widely employed for modeling complex and dynamic systems such as project management [15].

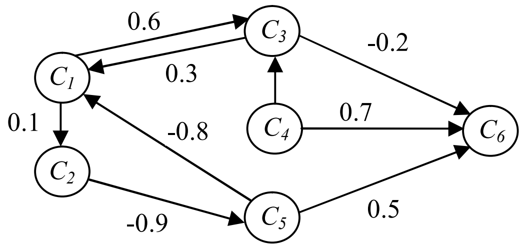


Figure 2. Representation of a basic Fuzzy Cognitive Map.

Some authors have focused on IT projects, such is the case of Rodriguez-Repiso, Setchi, and Salmeron who used FCMs in [16] to model critical success factors and the relationships between them. Following this line, Salmeron et al. [17] presented a model for predicting the impact of risks in ERP maintenance projects. Leyva et al. [18] presented a model to select IT projects using the business modeling and FCMs. Zare Ravasan et al. [19] proposed a dynamic model based on FCMs to identify the most important ERP projects failure factors.

In construction projects, Ahn et al. [20] used FCMs for the prediction of labor productivity. Bağdatlı, Akbıyıklı, and Papageorgiou developed in [21] a decision-making model based on FCMs for the cost-benefit analysis, taking into account the risk analysis. Khanzadi et al. [22] presented an FCM model for dynamic analysis of changes in construction projects.

Regarding risk analysis, Jamshidia et. al [23] had developed an FCM model for risk analysis in maintenance outsourcing projects, and in another work [24] he proposed the use of FCM to support decision making for the dynamic risk assessment in project management, taking into account the probability of occurrence and the impact of each risk factor. Other authors following different approaches developed models based on FCMs for stakeholders evaluation [25], and project schedule overrun prediction [26].

From the previous revision, it was perceived that the mentioned models based on traditional FCMs in project management do not consider, despite its importance, the indeterminacy between concepts. Indeterminacy is frequently presented in the decision-making process [27], mainly, when experts are not sure if one factor may or may not impact another.

However, traditional FCMs have the limitation of not considering the indeterminacy relations between concepts [28]. In this respect, Smarandache in 1995 introduced the Neutrosophic Theory, making possible the representation of indeterminacy [29]. Such characteristic is helpful for modeling decision-making problems [30] since it considers all aspects of decision such as agree, not sure, and disagree [31]. For this reason, Neutrosophic Logic has been widely used in decision-making environments [32], [33], [34].

An application of this theory in FCMs is the Neutrosophic Cognitive Maps (NCMs) developed by Vasantha & Smarandache in 2003 [35]. NCMs overcomes the drawback presented in traditional FCMs of not representing the indeterminacy relations between concepts.

However, NCMs have been little applied in project management. Bhutani et al. [36] used NCMs for the identification and evaluation of success factors in IT projects. Betancourt, Leyva, and Pérez proposed in [37] a new method for modeling risk interdependencies ing projects. In another context, Pramanik and Chackrabarti carried out in [38] a study to assess the impact of problems faced by construction workers in West Bengal, India, based on NCMs to find its solutions. Following this line, Monda and Pramanik modeled in [39] the problems of Hijras in West Bengal, India, using NCM.

It was noticed that in the previous papers, the linguistic evaluations are not represented by neutrosophic sets, but by a single number which represents the degree of causality between two concepts or by the letter *I* to indicate the indeterminacy, without sufficiently exploiting all the potentialities of neutrosophic sets.

In general, many of the aforementioned articles about decision making in project management, only make a diagnosis without making decisions or predictions. In some of them, the map is constructed with the help of multiple experts without proposing any method for the consensus process. On the other hand, in the majority of these papers, the concepts of time, cost, and quality were not properly considered.

This article aims at proposing a model based on neutrosophic cognitive maps for making comprehensive decisions out of a multi-objective approach (diagnosis, decision, and prediction) during projects execution, considering the indeterminacy relations between concepts. In the proposed model, experts' evaluations are expressed by neutrosophic sets, taking into account an adequate balance between cost, time, and quality.

The remaining of the paper is structured as follows: Section 2 describes preliminary concepts and notation of Neutrosophic theory. In section 3, the neutrosophic cognitive map for decision making in project management is introduced. In section 4, authors compare the results of project management decisions by using a traditional fuzzy cognitive map with the neutrosophic cognitive map. The paper ends with conclusions in Section 5.

## 2 Preliminary concepts and notation

The traditional fuzzy set introduced by Zadeh in 1965 [40] uses one real value $\mu A(x) \in [0,1]$ to represent the grade of membership of fuzzy set A defined on universe X.

**Definition 1.** A fuzzy set consists of two elements, a linguistic label, and a membership function $\mu$. Function $\mu$ of $X$ is a mapping from the set $X$ to the unit interval $\mu : X \rightarrow [0, 1]$, where $\mu$ (x) is called a degree of membership.

An example of a fuzzy set is represented by the triangular functions as shown in Figure 3. Let A be a fuzzy set represented by the following function of membership:

$$\mu_A(x) = \begin{cases} 0, x < a \\ \dfrac{x-a}{b-a}, a \le x \le b \\ \dfrac{c-x}{c-b}, b \le x \le c \\ 0, x > c \end{cases}$$

High

$\mu_A$ (x)

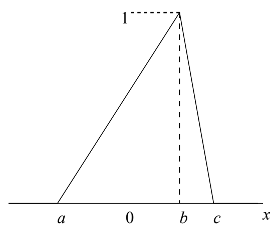


Figure 3. Fuzzy set "High" based on triangular membership $\mu_A$ (x) with values (a, b, c) and its graphical representation.

In this case, a triangular number (a, b, c) represents the membership function. However, fuzzy set only considers the membership degree of an element x of a fuzzy set *A* and fails to consider falsity-membership [41].

In 1986, Atanassov introduced the intuitionistic fuzzy sets (IFS) [42] which is a generalization of fuzzy sets. The intuitionistic fuzzy sets consider both truth-membership $\mu_A(x)$ and falsity-membership $f_A(x)$, with $\mu_A(x)$, $f_A(x)$

$\in [0,1]$ and $0 \leq \mu_A(x)+f_A(x) \leq 1$. Intuitionistic fuzzy sets can only handle incomplete information but not the indeterminate and inconsistent information [43].

An alternative that extends the theory of fuzzy logic and helps to improve the treatment of uncertainty is the introduction of concepts of neutrality dealing with neutrosophic numbers. Neutrosophy logic was introduced by Smarandache in 1995 [44].

**Definition 2.** Let $M$ be a neutrosophic set in universe $X$ characterized by a quintuple $(Label, X, \mu_M(x), \tau_M(x), \sigma_M(x))$ where: *Label* is a linguistic term which represents the name of set, X represents the universe of discourse, $\mu_M(x) \in [0,1]$ represents a membership function, $\tau_M(x) \in [0,1]$ represents a indeterminacy-membership function, and $\sigma_M(x) \in [0,1]$ represents a falsity-membership function, where $0 \leq \mu_M(x) + \tau_M(x) + \sigma_M(x) \leq 3$.

This definition implies that each value of the domain $x \in X$ when evaluated in neutrosophic set $M$, such that $M(x)$ returns the value $(\mu_M(x), \tau_M(x), \sigma_M(x))$ where the first component represents the membership degree of the value x to the set M, the second component represents the indetermination degree of the value x to the set $M$, and the third component means the non-membership degree of the value $x$ to the set $M$.

Single Valued Neutrosophic Set (SVNS) is an instance of a neutrosophic set which can be used in real scientific and engineering applications, see definition 3.

**Definition 3.** Let $X$ be a space of points (objects), with a generic element in $X$ denoted by $x$ represents a single valued neutrosophic number (SVN) and is characterized by a vector $(V, I, F)$ where $V$ indicates truth-value, $I$ indeterminacy-value, and $F$ falsity-value.

In order to extend fuzzy logic definitions with neutrosophic theory, authors include a definition 4 of neutrosophic linguistic variables.

**Definition 4.** A neutrosophic linguistic variable consists of quintuple $(Var, T(x), X, G, M)$ in which $Var$ is the name of the variable, $T(X)$ is the set of linguistic terms associated with the variable, $X$ is the universe of discourse, $M$ is a semantic rule which associates to each linguistic value $z \in T(x)$ its meaning $M(z)$, where $M(z)$ denotes a neutrosophic set in $X$, see definition 2, and $G$ is the set of syntactic rules for the generation of compound terms, based on the atomic terms that make up the sentences that give place to each linguistic value.

$$\mu_A(x) = \begin{cases} (x-a)u_A/(b-a) & (a \leq x < b) \\ u_A & (x = b) \\ (c-x)u_A/(c-b) & (b < x \leq c) \\ 0 & otherwise \end{cases}$$

$$\tau_A(x) = \begin{cases} \big(b - x + v_A(x-a)\big)/(b-a) & (a \leq x < b) \\ r_A & (x = b) \\ \big(x - b + v_A(c-x)\big)/(c-b) & (b < x \leq c) \\ 1 & otherwise \end{cases}$$

$$\sigma_A(x) = \begin{cases} \big(b - x + f_A(x-a)\big)/(b-a) & (a \leq x < b) \\ f_A & (x = b) \\ \big(x - b + f_A(c-x)\big)/(c-b) & (b < x \leq c) \\ 1 & otherwise \end{cases}$$

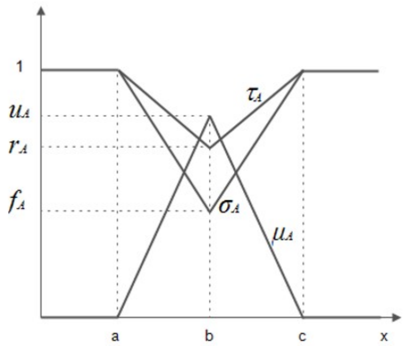


Figure 4. Neutrosophic set, $\mu_A(x)$ membership function, $\tau_A(x)$ indeterminacy-membership function and $\sigma_A(x)$ falsity-membership function.

Other important concept is a T-norm and S-Conorms functions. Let $T$ be a T-norm function and $S$ a co-norm function:

TNorma funtion $T: [0,1] \times [0,1] \rightarrow [0,1]$ for example (min), with following properties:

- $T(a, b) = T(b, a)$ *Commutativity*
- $T(T(a, b), c) = T(a, T(b, c))$ *Associativity*
- *Si* $a \geq b$ *y* $c \geq d$ *then* $T(a, c) \geq T(b, d)$ *Monotony*
- $T(a, 1) = a$ *Neutro element*

Conorma funtion $S: [0,1] \times [0,1] \rightarrow [0,1]$ for example (max), with following properties:

- $S(a, b) = S(b, a)$ *Commutativity*
- $S(S(a, b), c) = S(a, S(b, c))$ *Associativity*
- *Si* $a \geq b$ *y* $c \geq d$ *then* $S(a, c) \geq S(b, d)$ *Monotony*
- $S(a, 0) = a$ *Neutro element*

In order to operate with single valued triangular neutrosophic numbers, Şahin, Kargın, and Smarandache in [45] describe operations as follows:

Let $A_1$ be represented by number $((a_1, b_1, c_1); u_A, r_A, f_A)$ and $B_2$ is represented by number $((a_2, b_2, c_2); u_B, r_B, f_B)$ with T as T-norm and $S_1$, $S_2$ two co-norms then:

Sum: $A_1 (+) B_2 = ((a_1+a_1, b_1+b_2, c_1+c_2); T(u_A, u_B), S_1(r_A, r_B), S_2(f_A, f_B))$ (1)

Difference: $A_1$ (-) $B_2$= (($a_1$-$c_2$, $b_1$-$b_2$, $c_1$-$a_2$); $T(u_A, u_B)$, $S_1(r_A, r_B)$, $S_2(f_A, f_B)$) (2)
Product: $A_1$ (*) $B_2$= (($a_1a_2$, $b_1b_2$, $c_1c_2$); $T(u_A, u_B)$, $S_1(r_A, r_B)$, $S_2(f_A, f_B)$) if $c_1$, $c_2 > 0$ (3)
$A_1$ (*) $B_2$= (($a_1c_2$, $b_1b_2$, $c_1a_2$); $T(u_A, u_B)$, $S_1(r_A, r_B)$, $S_2(f_A, f_B)$) if $c_1 < 0$, $c_2 > 0$ (4)
$A_1$ (*) $B_2$= (($c_1c_2$, $b_1b_2$, $a_1a_2$); $T(u_A, u_B)$, $S_1(r_A, r_B)$, $S_2(f_A, f_B)$) if $c_1$, $c_2 < 0$ (5)
Division: $A_1$ (/) $B_2$= (($a_1/c_2$, $b_1/b_2$, $c_1/a_2$); $T(u_A, u_B)$, $S_1(r_A, r_B)$, $S_2(f_A, f_B)$) if $c_1$, $c_2 > 0$ (6)
$A_1$ (/) $B_2$= (($a_1/a_2$, $b_1/b_2$, $c_1/c_2$); $T(u_A, u_B)$, $S_1(r_A, r_B)$, $S_2(f_A, f_B)$) if $c_1 < 0$, $c_2 > 0$ (7)
$A_1$ (/) $B_2$= (($c_1/a_2$, $b_1/b_2$, $a_1/c_2$); $T(u_A, u_B)$, $S_1(r_A, r_B)$, $S_2(f_A, f_B)$) if $c_1$, $c_2 < 0$ (8)
Product by scalar: $kA_1 = ((ka_1, kb_1, kc_1); u_A, r_A, f_A)$ if $k > 0$ (9)
$kA_1 = ((kc_1, kb_1, ka_1); u_A, r_A, f_A)$ if $k < 0$ (10)
Inverse: $(A_1)^{-1} = ((1/c_1, 1/b_1, 1/a_1); u_A, r_A, f_A)$ with $A_1 \neq (0,0,0)$ (11)

## 3 A Neutrosophic Cognitive Map for decision making in Project Management

In this section, the model based on a NCMs for making comprehensive decisions out of a multi-objective approach (diagnosis, decision, and prediction) during projects execution, considering the indeterminacy relations between concepts is proposed. The main characteristics of the model are:

- It is based on expert triangulation methods to avoid high dependence of one expert and mitigate the experts slant.
- Introduces a new representation of neutrosophic cognitive maps by including neutrosophic sets into maps connections.
- Manages two types of indeterminacy, the first one is when the value of some indicators is unknown and the second one is when experts declare indeterminacy between two concepts, see Figure .
- Takes into account an adequate balance between cost, time, and quality.
- Provides solutions for diagnosis, decision, and prediction simultaneously.
- Uses computing with word techniques to aggregate the individual cognitive maps.

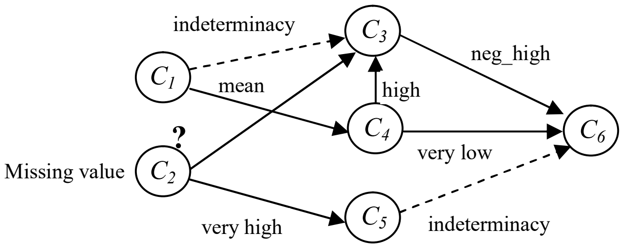


Figure 5. Neutrosophic cognitive map with indeterminacy relationships between concepts

The proposed model consists of two algorithms:

***Algorithm PM_NCM 1: the construction of the neutrosophic cognitive map***

1. *Defining the project management's problems, context, and particularities.*
2. *Selecting k experts in project management.*
3. *Evaluating the expertise degree for each expert, by using co-evaluation methods and computing with word techniques (t-tuples technique* [46]*).*
   *for each Expert$_i$*
   *for each Expert$_j$ : $i \neq j$*
   *Expert$_i$ = Aggregate(Expert$_i$ , Evaluation (Expert$_{ji}$ ) ) // Expert$_{ji}$ represents evaluation of expert j over expert i.*
   *end for*
   *end for*
4. *Establishing indicator, diagnosis, decision, and prediction concepts associated with project management problems by using brain storming techniques (carrousel style).*
5. *Building the individual maps for each expert*
   *For each Expert$_i$*
   *QueueMaps ← Expert$_i$ builds a map by considering identified concepts in the previous step. Each map edge is represented by a neutrosophic set represented in Table 1*

*End for*

6. *Ncm = AggregateMaps(QueueMaps) //Finally getting a map, where each map's node represents one of the following elements: a cause of project delay, a decision, a prognosis.*

In the proposed model, five experts were selected, who identified the following concepts for the construction of the individual maps:

- Indicators: these reflect the current state of the project under evaluating, they can be expressed in different domains, numerical, linguistic or interval. the Indicators are calculated by project management systems and they are associated with each project management knowledge area such as performance, quality, and logistic. For the construction of the individual maps, experts have identified the following indicators concepts:
  - SPI: scheduling performance indicator.
  - CPI: cost performance indicator.
  - EFPI: efficacy performance indicator, represents the quality of the project.
  - LPI: logistic performance indicator.
  - DQPI: data quality performance indicator, representing the quality of data in project management information system.
  - HRCPI: human resource correlation performance indicator, which represents the correlation between the plan and real-time in human resource scheduling.
  - HREPI: human resource efficacy performance indicator, which represents the efficacy of human resources.
  - HREFI: human resource efficiency performance indicator.
- Diagnosis concepts reflect the causes of project difficulties. In order to improve the project performance, these elements should be identified carefully, since they have a crucial impact on projects decisions. The following factors were selected by experts as diagnosis concepts:
  - F1. Defects quality control.
  - F2. Defects tasks control.
  - F3. Defects HR efficiency: defects on human resource efficiency.
  - F4. Defects HR efficacy: defects on human resource efficacy.
  - F5. Defects on scheduling.
  - F6. Defects on logistic.
  - F7. Defects on cost management.
  - F8. Defects on cost scheduling.
- Decision concepts represent the possible decisions to be made in order to correct project deviation and they are mainly related to the causes of the problems. Decision concepts were identified as follows:
  - D1. Increase quality control.
  - D2. Leave the project manager.
  - D3. Increase control milestones.
  - D4. Rewards HR: rewards to human resources.
  - D5. Penalize HR: penalize to human resources.
  - D6. New HR contracts: contracts more human resources.
  - D7. Rescheduling.
  - D8. Extra hours scheduling.
  - D9. Improve logistics management.
  - D10. Decrease cost.
  - D11. Rescheduling scope.
- Prediction concepts are identified to know what will happen to the project if a certain decision is made. Experts defined the following prediction concepts:
  - P1. Improve quality.
  - P2. Recover delays.
  - P3. Improve cost balance.
  - P4. Increase perceived quality.
  - P5. Increase HR motivation: increase human resource motivation.
  - P6. Decrease quality.
  - P7. Increase delays.
  - P8. Increase costs defects.
  - P9. Increase scope defects.
  - P10. Decrease HR motivation: decrease human resource motivation.

In the algorithm 1, each expert builds his own map establishing his preferences by using the neutrosophic sets defined in Table 1. The relations are represented with positive influence, negative influence or without influence (indeterminacy). Experts describe their preferences by using the following linguistic terms *LBTL = {negative highest, negative very high, negative high, negative mean, negative low, negative very low, none, very low, low, mean, high, very high, highest, indeterminacy}*

| **Linguistic terms** | **Neutrosophic sets based on triangular functions** |
|---|---|
| neg_highest | (-0.83, -1.0, -1.0, 0.95, 0.45, 0.15) |
| neg_very high | (0.67, -0.83, -1.0, 0.95, 0.45, 0.15) |
| neg_high | (-0.5, -0.67, -0.83, 0.95, 0.45, 0.15) |
| neg_mean | (-0.33, -0.5, -0.67, 0.95, 0.45, 0.15) |
| neg_low | (-0.17, -0.33, -0.5, 0.95, 0.45, 0.15) |
| neg_very low | (0, -0.17, -0.33, 0.95, 0.45, 0.15) |
| none | (0, 0.07, 0.17, 0.95, 0.45, 0.15) |
| very low | (0, 0.17, 0.33, 0.95, 0.45, 0.15) |
| low | (0.17, 0.33, 0.5, 0.95, 0.45, 0.15) |
| mean | (0.33, 0.5, 0.67, 0.95, 0.45, 0.15) |
| high | (0.5, 0.67, 0.83, 0.95, 0.45, 0.15) |
| very high | (0.67, 0.83, 1.0, 0.95, 0.45, 0.15) |
| highest | (0.83, 1.0, 1.0, 0.95, 0.45, 0.15) |
| indeterminacy | (0, 0, 0, 0.1, 0.9, 0.1) |

Table 1. Neutrosophic sets to represent map relationships.

***Algorithm PM_NCM 2: the simulation process of the neutrosophic cognitive map***

```
Inputs
   ncm: neutrosophic cognitive map
   maxepoch: max number of epoch
   indicators: means project indicators to evaluate project during cut
Initialize(prediction_memory)
continue_criteriom = true
diagnosis = do_initial_diagnosis(indicator, ncm)
epoch = 1
while continue_criteriom && epoch <= maxepoch do
   decisions = do_aggregate_svns(diagnostic, decisions, ncm)
   prediction = do_aggregate_svns(decisions, prediction, , ncm)
   continue_criteriom = do_compare_distance (prediction_memory, prediction, epsilon)
      if continue_criteriom
         prediction_memory  = prediction
         diagnosis = do_aggregate_svns(prediction, diagnosis, ncm)
      end if
   epoch += 1
end while
Sort diagnosis, decisions, prediction
Return diagnosis, decisions, prediction
```

In the simulation process, users can exploit the map, using it to make comprehensive decisions. The process is started with the activation of some of the map's indicator nods during the execution of a project, triggering off the activation of the rest of the map's concepts (diagnosis, decisions, and predictions). The simulation process is carried out by using equations (12) and (13), where $W_{ij}$ represents a neutrosophic set, not a single value, such is the case of traditional FCMs or other neutrosophic cognitive maps' approaches. In order to operate with neutrosophic sets between neutrosophic sets and numbers, the equations (2), (3), (4), (5), (6), (7), (8), (9), (10) and (11) were used.

$$A_i(epoch+1) = f\left( A_i(\text{epoch}) + \sum_{i=1}^{n} \mathrm{W}_{\mathrm{ji}} \cdot \mathrm{A}_{\mathrm{j}}(\text{epoch}) \right) \quad (12)$$

The hyperbolic tangent function is used in order to force the concept value to be monotonically mapped into the range [-1,1] [47].

$$S_i(W_{ij}) = \tanh(\lambda W_{ij}) = \frac{e^{\lambda W_{ji}} - e^{-\lambda W_{ji}}}{e^{\lambda W_{ji}} + e^{-\lambda W_{ji}}} \qquad (13)$$

## 4 Analysis and discussion.

To validate the proposed model, the authors selected the database DPME5 from the Research Database Repository for Project Management provided by UCI [48]. This database contains 6115 records with 8 attributes. All attributes are represented by real values in [0, 1] interval. This database contains 3175 projects evaluated as "bad performance", 607 project evaluated as "regular performance", and 2333 projects evaluated as "correct performance".

Authors implemented two cognitive maps. The first map "PM_FCM" is based on a traditional fuzzy cognitive map FCM for decision making in project management. The Map was constructed by means of the concepts and relations illustrated in tables 2.3.4.6.

| | **Defects quality control F1** | **Defects tasks control F2** | **Defects HR efficiency F3** | **Defects HR efficacy F4** | **Defects on scheduling F5** | **Defects on logistic F6** | **Defects on cost management F7** | **Defects cost scheduling F8** |
|---|---|---|---|---|---|---|---|---|
| **SPI** | very low | very high | very high | low | very high | mean | very low | mean |
| **CPI** | very low | very high | very high | none | mean | high | very high | very high |
| **EFPI** | highest | mean | none | very high | mean | none | none | none |
| **LPI** | low | low | very low | very low | low | very high | high | mean |
| **DQPI** | high | mean | very low | high | none | none | low | none |
| **HRCPI** | very low | very low | high | low | very high | very low | very low | very low |
| **HREPI** | very high | mean | very low | very high | none | none | none | none |
| **HRFPI** | very low | high | very high | very low | high | very low | low | very low |

Table 2. Initial diagnosis: the relations between indicators and diagnosis concepts.

| | **D1** | **D2** | **D3** | **D4** | **D5** | **D6** | **D7** | **D8** | **D9** | **D10** | **D11** |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **F1** | highest | low | mean | very low | low | very low | low | low | very low | very low | mean |
| **F2** | mean | highest | highest | low | low | very low | mean | low | very low | very low | mean |
| **F3** | very low | very high | mean | very high | very high | high | mean | high | low | low | mean |
| **F4** | very high | high | mean | very high | very high | low | mean | mean | low | very low | high |
| **F5** | mean | high | highest | low | low | low | very high | low | mean | low | high |
| **F6** | low | very high | low | low | very low | very low | mean | very low | very high | low | mean |
| **F7** | low | low | highest | very low | very low | very low | mean | very low | high | very high | mean |
| **F8** | very low | highest | low | very low | very low | very low | very high | very low | mean | mean | very high |

Table 3. Represents the decision process through relations between diagnosis and decisions concepts.

| | **P1** | **P2** | **P3** | **P4** | **P5** | **P6** | **P7** | **P8** | **P9** | **P10** |
|---|---|---|---|---|---|---|---|---|---|---|
| **D1** | very high | low | low | very high | very low | neg_very high | neg_mean | mean | neg_highest | low |
| **D2** | mean | mean | none | mean | mean | low | neg_high | very low | very low | low |
| **D3** | high | very high | high | mean | low | neg_very high | very low | none | low | low |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| **D4** | high | high | mean | high | very high | neg_high | neg_very high | very high | low | very low |
| **D5** | mean | mean | mean | mean | low | mean | high | neg_very high | mean | mean |
| **D6** | low | high | low | none | low | very high | mean | neg_highest | very high | low |
| **D7** | high | very high | high | low | mean | none | neg_mean | low | low | low |
| **D8** | mean | very high | high | mean | low | none | neg_very high | neg_low | very low | mean |
| **D9** | mean | high | very high | mean | low | very low | low | neg_very low | low | low |
| **D10** | none | low | very high | low | low | mean | very low | neg_highest | low | low |
| **D11** | high | very high | high | low | mean | low | neg_very high | low | mean | low |

Table 4. Represents the prediction process through relations between decisions and prediction concepts, used by PM_FCM model.

| | **P1** | **P2** | **P3** | **P4** | **P5** | **P6** | **P7** | **P8** | **P9** | **P10** |
|---|---|---|---|---|---|---|---|---|---|---|
| **D1** | very high | neg_low | low | very high | very low | neg_very high | neg_mean | mean | neg_highest | low |
| **D2** | mean | mean | indeterminacy | mean | mean | low | neg_high | very low | very low | low |
| **D3** | high | very high | high | indeterminacy | low | neg_very high | very low | none | low | low |
| **D4** | high | high | indeterminacy | high | very high | neg_high | neg_very high | very high | low | very low |
| **D5** | mean | mean | mean | mean | low | mean | high | neg_very high | mean | mean |
| **D6** | low | high | low | indeterminacy | low | very high | mean | neg_highest | very high | low |
| **D7** | high | very high | high | low | mean | indeterminacy | neg_mean | low | low | low |
| **D8** | mean | very high | high | mean | low | indeterminacy | neg_very high | neg_low | very low | mean |
| **D9** | mean | high | very high | mean | low | very low | low | neg_very low | low | low |
| **D10** | indeterminacy | low | very high | low | low | mean | very low | neg_highest | low | low |
| **D11** | high | very high | high | low | mean | low | neg_very high | low | mean | low |

Table 5. Represents the prediction process through relations between decisions and prediction concepts, used by PM_NCM model.

| | **F1** | **F2** | **F3** | **F4** | **F5** | **F6** | **F7** | **F8** |
|---|---|---|---|---|---|---|---|---|
| **P1** | very low | low | low | very low | low | low | very low | low |
| **P2** | none | very low | very low | low | very low | very low | very low | very low |
| **P3** | none | very low | very low | very low | very low | very low | very low | very low |
| **P4** | very low | mean | very low | low | very low | very low | very low | very low |
| **P5** | very low | very low | very low | low | low | low | low | low |
| **P6** | highest | highest | low | very high | mean | mean | low | low |
| **P7** | low | very high | very high | high | very high | high | high | high |
| **P8** | very low | high | high | low | mean | mean | very high | very high |
| **P9** | highest | very high | mean | highest | high | low | high | mean |
| **P10** | high | highest | very high | highest | high | mean | mean | mean |

Table 6. Represents the prediction process through relations between prediction and diagnosis concepts.

The second map "PM_NCM" is based on a neutrosophic cognitive map NCM for decision making in project management. The Map was constructed by means of the concepts and relations illustrated in tables 2.3.5.6, and according to the algorithms (1) and (2) explained previously, see Figure 6.

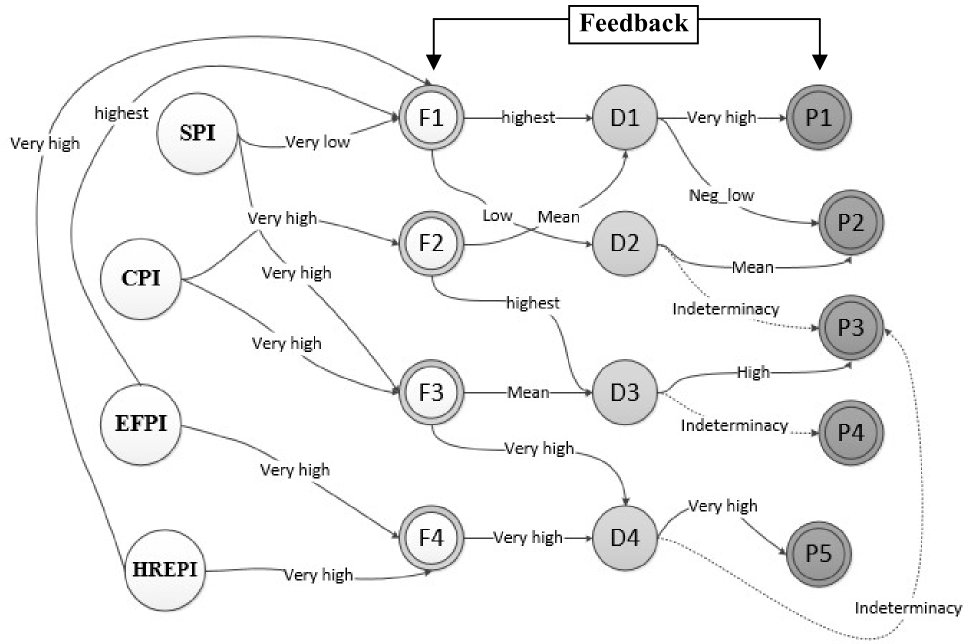


Figure 6. Partial representation of the aggregated neutrosophic cognitive map "PM_NCM".

The relationships between the maps' concepts are shown in the following tables: Table 2 represents the initial diagnosis process of both maps, which connects the indicator and diagnosis nods. Table 3 represents the decision process of both maps through the connection between diagnosis and decision nods. Tables 4 represents the prediction processes in PM_FCM through the connection between decision and prediction nods whereas table 5 represents the same process in PM_NCM, in which the indeterminacy relations were considered, see Figure 6. Table 6 represents the prediction process of both maps through the connection between prediction and diagnosis nods, in which the feedback relationships is expressed.

The two models were compared considering efficiency and efficacy. Concerning efficiency, PM_FCM obtained better results, PM_FCM was 6.9 times faster than PM_NCM. The model PM_FCM evaluated the 6115 records in 9.8 sec as an average, while PM_NCM evaluated the same records in 63.22 sec.

In regards to the indeterminacy relations between concepts, the simulation results showed that PM_NCM have the capability to represent efficiently the indeterminacy relations, in contrast with PM_FCM. Hence, PM_NCM is better than PM_FCM when it comes to dealing with uncertainty, missing values, and incomplete information.

In terms of efficacy, the authors of this paper introduced a metric success *C* to evaluate the two models as follows: Let $\rho$ be a model, a metric success *C* is defined in (14) as a percentage of records classified as experts do, where *n* represents the number of records. This equation was applied in order to evaluate the result of diagnosis, decisions, and prediction of both PM_FCM and PM_NCM.

$$C(\rho) = 100\,\frac{\sum_{i=1}^{n} S_i(\rho)}{n} \qquad (14) \qquad\qquad S_i(\rho) = \begin{cases} 0 & \text{if not coincidence with experts} \\ 1 & \text{if coincidence with experts} \end{cases}$$

The authors of this paper consider model's efficacy metrics to be the capacity to detect true indeterminacy records. In this sense, this paper redefines precision *P* (15) and recalls R (16) metrics to evaluate the models capacity to detect the indeterminacy as follows:

$$P(\rho) = 100\,\frac{|Z(\rho) \cap I|}{Z(\rho)} \qquad (15) \qquad\qquad R(\rho) = 100\,\frac{|Z(\rho) \cap I|}{I} \qquad (16)$$

Let $\rho$ be a model, $Z(\rho)$ is the set of records that a model $\rho$ detects with high indeterminacy and $I$ is the set of records with true high indeterminacy.

| Algorithm | PM_FCM | PM_NCM |
|---|---|---|
| Successful diagnosis | 85% | 95,33 |
| Successful decision | 93% | 100% |
| Successful prediction | 84,1% | 94,45% |
| Precision on indeterminacy detection (diagnosis) | - | 60% |
| Precision on indeterminacy detection (decisions) | - | 75% |
| Precision on indeterminacy detection (prediction) | - | 72% |
| Recall on indeterminacy detection (diagnosis) | - | 100% |
| Recall on indeterminacy detection (decisions) | - | 100% |
| Recall on indeterminacy detection (prediction) | - | 95% |

Table 7. The comparison results between PM_FCM and PM_NCM.

Respecting successful evaluation in diagnosis, decision, and prediction, PM_NCM reported better results than PM_FCM. Regarding precision on indeterminacy detection ($P$) and recall of indeterminacy detection ($R$), PM_NCM reported good results. PM_FCM did not report any results since it does not consider the indeterminacy relations between concepts.

## 5. Conclusions

In this paper, we proposed a new decision-making model based on Neutrosophic Cognitive Maps (NCMs) for making comprehensive decisions from multi-objective approach (diagnosis, decisions, and prediction), considering an adequate balance between time, cost, and quality, during the execution of many projects simultaneously. The suggested model overcomes the drawback of not representing indeterminacy relations between concepts presented in traditional Fuzzy Cognitive Maps FCMs. Besides, the NCM model constitutes a more realistic and robust tool to decision support through considering all aspects of the decision-making process and dealing efficiently with uncertainty, missing values, and incomplete information. The suggested model was compared with a traditional FCM-based model, showing its superiority regarding successful evaluation in diagnosis, decision, and prediction; and when it comes to dealing with uncertainty, missing values, and incomplete information. The traditional FCM-based model provided better results in respect to efficiency. In the future, we will extend the application of the proposed model to other disciplines, mainly, in medical diagnosis.

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# Single Valued Neutrosophic Hesitant Fuzzy Computational Algorithm for Multiobjective Nonlinear Optimization Problem

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**Abstract:** In many real-life situations, it is often observed that the degree of indeterminacy (neutrality) plays an important role along with the satisfaction and dissatisfaction levels of the decision maker(s) (DM(s)) in any decision making process. Due to some doubt or hesitation, it may necessary for DM(s) to take opinions from experts which leads towards a set of conflicting values regarding satisfaction, indeterminacy and dis-satisfaction level of DM(s). In order to highlight the above-mentioned insight, we have developed an effective framework which reflects the reality involved in any decision-making process. In this study, a multiobjective nonlinear programming problem (MO-NLPP) has been formulated in the manufacturing system. A new algorithm, neutrosophic hesitant fuzzy programming approach (NHFPA), based on single-valued neutrosophic hesitant fuzzy decision set has been proposed which contains the concept of indeterminacy hesitant degree along with truth and falsity hesitant degrees of different objectives. In order to show the validity and applicability of the proposed approach, a numerical example has been presented. The superiority of the proposed approach has been shown by comparing with other existing approaches. Based on the present work, conclusions and future scope have been presented.

**Keywords:** Indeterminacy hesitant membership function, Neutrosophic hesitant fuzzy programming, Multiobjective nonlinear programming problem.

# 1 Introduction

Many decision-making processes inherently involved different conflicting objectives which are to be optimized (maximize/minimize) under given circumstances. In the present competitive era, it is indispensable for decision maker(s) (DM(s)) to obtain better possible outcomes/results when dealing with multiple objectives at a time. Although, it is quite difficult to have an optimal solution which satisfies all the objectives efficiently a compromise solution is possible which is accepted by DM(s) up to some extent. Literature reveals various approaches for multiobjective optimization problem and continuous effort have been made to obtain the best compromise solution. It is often observed that the modeling and formulation of the problem arising in agriculture production planning, manufacturing system etc. takes the form of nonlinear programming problem with multiple objective which is realistic in nature. Thus, multiobjective nonlinear programming problem (MO-NLPP) is also a challenging problem due to its local and global optimal concept, unlike multiobjective linear programming problem.

Bellman and Zadeh [5] introduced fuzzy set (FS) and based on that set Zimmermann [27] proposed fuzzy programming approach (FPA) for multiobjective optimization problems. The FPA deals only degree of belongingness but sometimes it may necessary to deal with non-membership function (non-belongingness) in order to obtain the results in the more realistic way. To overcome the above fact, Atanassov [4] introduced the intuitionistic fuzzy set (IFS) which is the extension of the FS. The IFS is based on more intuition as compared to FS because it also deals with the non-membership function (non-belongingness) of the element in the set. Based on IFS, intuitionistic fuzzy programming approach (IFPA) gained its own popularity among the existing multiobjective optimization techniques. Angelov [3] first used the optimization technique under intuitionistic fuzzy environment. Mahmoodirad et al. [15] proposed a new approach for the balanced transportation problem by considering all parameters and variables are of triangular intuitionistic fuzzy values and pointed out some shortcomings of existing approaches. Singh and Yadav [19] discussed multiobjective nonlinear programming problem in the manufacturing system and solved by using three approaches namely; *Zimmerman's technique, $\gamma$- operator* and *Min. bounded sum operator* with intuitionistic fuzzy parameters. Bharati and Singh [6] also proposed a new computational algorithm for multiobjective linear programming problem in the interval-valued intuitionistic fuzzy environment.

In recent years, the extensions or generalizations of FS and IFS have been presented with the fact that indeterminacy degree exists in real life and as a result, a set named neutrosophic set came in existence. Smarandache [20] introduced the concept of the neutrosophic set (NS). The term neutrosophic is the combination of two words, *neutre* from French meaning, neutral, and *sophia* from Greek meaning, skill/wisdom. Thus neutrosophic literally means *knowledge of neutral thoughts* which well enough differentiate it from FS and IFS. The neutrosophic set involves three membership functions, namely; maximization of truth (belongingness), indeterminacy (belongingness to some extent) and minimization of falsity (non-belongingness) in an efficient manner. Based on NS, neutrosophic programming approach (NPA) came into existence and extensively used in real life applications. Abdel-Basset et al. [1] proposed a novel approach to solving fully neutrosophic linear programming problem and applied to production planning problem. Rizk-Allah et al. [16] solved the MO-TPs under neutrosophic environment and compared the obtained results with the existing approach by measuring the ranking degree using TOPSIS approach. Ye et al. [23] formulated neutrosophic number nonlinear programming problem (NN-NPP) and proposed an effective method to solve the problem under neutrosophic number environments. Liu and You [12] extended Muirhead mean to interval neutrosophic set and developed some new operator named as interval neutrosophic Muirhead mean operators which have been further applied to multi-attribute decision making (MADM) problem. Liu et al. [14] have combined the power average operator with Herorian mean operator which results in linguistic neutrosophic power Herorian aggregation operator and extended them for neutrosophic information process. Ahmad and Adhami [2] have also solved the nonlinear transportation problem with fuzzy parameters using neutrosophic programming approach and compared the solution results with other existing approaches. Liu and Shi [10] have introduced the valued neutrosophic uncertain linguistic set and developed some operators which have been further used to multi-attribute group decision making (MAGDM) problem. Liu and Teng [11] have proposed some normal neutrosophic operator based on normal neutrosophic numbers and developed an MADM method based on neutrosophic number generalized weighted power averaging operator. Zhang et al. [25] have proposed some new MAGDM methods in which the attributes are interactive in the form of the interval-valued hesitant uncertain linguistic number. Liu and Zhang [13] have extended the Maclaurian symmetric mean operator to single-valued trapezoidal neutrosophic numbers and developed a method to deal with MAGDM problem based on single-valued trapezoidal neutrosophic weighted Maclaurian symmetric mean operator.

Sometimes, the DM(s) is(are) not sure about the single specific value of the parameters in the set due to doubt or incomplete information but a set of different conflicting values may possible to represent the membership degree for any element to the set. In order to deal with the above fact, Torra and Narukawa [21] introduced the concept of the hesitant fuzzy set (HFS). The HFS is the generalization of fuzzy set and is very useful tools by ensuring the active involvement of different experts' opinions in the decision-making process. Based on HFS, hesitant fuzzy programming approach (HFPA) has been developed which incontinently allows the DM(s) to collaborate with experts in order to collect their incompatible opinions. Bharati [7] developed the hesitant computational algorithm for multiobjective linear programming problem and applied to production planning problem. Zhang et al. [24] developed a hesitant fuzzy programming technique to deal with multi-criteria decision-making problems within the hesitant fuzzy elements environment. Zhou and Xu [26] proposed new portfolio selection and risk investment approaches under hesitant fuzzy environment. All the above-discussed sets have its own limitations regarding the existence of each element in the set. In brief, FS deals only the membership degree of the element in the set whereas IFS considers both membership and non-membership degree of the element in the set simultaneously. NS is the generalization of FS and IFS because it allows the DM(s) to implement the thoughts of neutrality which gives the indeterminacy membership degree for an element to the set. Furthermore, HFS is also an extension of FS as its membership is represented by a set of different conflicting values in the set. Based on the above-mentioned sets, various optimization techniques such as fuzzy optimization techniques, intuitionistic fuzzy optimization techniques, neutrosophic optimization techniques, and hesitant fuzzy optimization techniques have been developed and widely used to solve multiobjective optimization problem which usually exists in real life.

In real life, hesitancy is the most trivial issue in the decision-making process. To deal with it, HFS may be used as an appropriate tool by assigning a set of different membership degree for an element in the set. The limitation of HFS is that it only represents the truth hesitant membership degree and does not deals with indeterminacy hesitant membership degree and falsity hesitant membership degree for an element in the set which arises due to inconsistent, imprecise, inappropriate and incomplete information. On the other hand, a single-valued neutrosophic set (SVNS) is a special case of NS which provides an additional opportunity to the DM(s) by incorporating the thoughts of neutrality. It is only confined to the truth, indeterminacy and a falsity membership degree for an element to the set. It can not ensure the interference of a set of membership values due to doubt and consequently the involvement of different experts' opinions in the decision-making process. The crucial situation arises when the two aspects namely; hesitations and neutral thoughts exist simultaneously in the decision-making process. In this case, HFS and SVNS may not be an appropriate tool to represent the situation in an efficient and effective manner. Thus, this kind of situations are beyond the scope of FS, IFS, SVNS, and HFS and consequently beyond the scope of FPA, IFPA, NPA, and HFPA to decision making process respectively. Therefore, truth, indeterminacy and the falsity situations under hesitant uncertainty is more practical terminology in real life optimization problems.

To get rid of the above limitations, Ye [22] investigated a new set named single-valued neutrosophic hesitant fuzzy set (SVNHFS) which is the combination of HFS and SVNS respectively. The SVNHFS contemplate over truth hesitant fuzzy membership, indeterminacy hesitant fuzzy membership and the falsity hesitant fuzzy membership degrees for an element to the set. Biswas et al. [8] discussed multi-attribute decision-making problems in which the rating values are expressed with single-valued neutrosophic hesitant fuzzy set information and proposed grey relational analysis method for multi-attribute decision making. Şahin and Liu [17] investigated correlation and correlation coefficient of SVNHFSs and discussed its applications in the decision-making process. Biswas et al. [9] proposed a variety of distance measures for single-valued neutrosophic sets and applied these measures to multi-attribute decision-making problems. In this present study, a new computational method, neutrosophic hesitant fuzzy programming approach (NHFPA) has been proposed to obtain the best possible solution of MO-NLPP which is based on SVNHFS. The proposed NHFPA involves the three membership function, namely; maximization of truth hesitant fuzzy (belongingness), indeterminacy hesitant fuzzy (belongingness to some extent) and minimization of falsity hesitant fuzzy (non-belongingness) in an emphatic manner.

To best of our knowledge, no such method has been proposed in the literature to solve the MO-NLPP. The proposed method covers different aspects of impreciseness, vagueness, inaccuracy, the incompleteness that are often encountered in real life optimization problems and provides flexibility in the decision-making process. The remarkable point is that the proposed approach actively seeks opinions from different experts under the neutrosophic environment which is more practical in real life situations and strongly concerned with the involvement of distinguished experts in order to make the fruitful decision. The neutral/indeterminacy hesitant fuzzy concept involved in single-valued neutrosophic hesitant fuzzy set leads towards the future research scope in this domain.

The rest of the paper has been summarized as follows:
In section 2, the preliminaries regarding neutrosophic set, hesitant fuzzy set, and single-valued neutrosophic hesitant fuzzy set have been discussed while section 3 represents the problem formulation and development of the proposed neutrosophic hesitant fuzzy programming approach (NHFPA). In section 4, a numerical study has been presented in order to show the applicability and validity of the proposed approach. A comparative study has also done with other existing approaches. Finally, conclusions and future scope have been discussed based on the present work in section 5.

# 2 Preliminaries

## 2.1 Neutrosophic Set (NS)

**Definition 2.1.1:** [20] Let X be a universe discourse such that x ∈ X, then a neutrosophic set A in X is defined by three membership functions namely, truth $T_A(x)$, indeterminacy $I_A(x)$ and a falsity $F_A(x)$ and is denoted by the following form:

$$A = \{< x, T_A(x), I_A(x), F_A(x) > | x \in X\} \tag{1}$$

where $T_A(x), I_A(x)$ and $F_A(x)$ are real standard or non-standard subsets belong to $]0^-, 1^+[$, also given as, $T_A(x) : X \rightarrow ]0^-, 1^+[$, $I_A(x) : X \rightarrow ]0^-, 1^+[$, and $F_A(x) : X \rightarrow ]0^-, 1^+[$. There is no restriction on the sum of $T_A(x), I_A(x)$ and $F_A(x)$, so we have,

$$0^- \leq \; sup\, T_A(x) + I_A(x) + \; sup\, F_A(x) \leq 3^+ \tag{2}$$

**Definition 2.1.2:** [20] A single valued neutrosophic set $A$ over universe of discourse $X$ is defined as

$$A = \{< x, T_A(x), I_A(x), F_A(x) > | x \in X\} \tag{3}$$

where $T_A(x), I_A(x)$ and $F_A(x) \in [0,1]$ and $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$ for each $x \in X$.

## 2.2 Hesitant Fuzzy Set (HFS)

**Definition 2.2.1:** [21] Let there be a fixed set $X$; a hesitant fuzzy set $A$ on $X$ is defined in terms of a function $h_A(x)$ that when applied to $X$ returns a finite subset of [0,1] and mathematically can be represented as follows:

$$A = \{< x, h_A(x) > | x \in X\} \tag{4}$$

where $h_A(x)$ is a set of some different values in [0,1], denoting the possible membership degrees of the element $x \in X$ to $A$. Also, we call $h_A(x)$ a hesitant fuzzy element.

**Definition 2.2.2:** [21] For a given hesitant fuzzy element $h$, its lower and upper bounds are defined as $h^-(x) = min\ h(x)$ and $h^+(x) = max\ h(x)$, respectively.

## 2.3 Single Valued Neutrosophic Hesitant Fuzzy Set (SVNHFS)

**Definition 2.3.1:** [22] Let there be a fixed set $X$; an SVNHFS on $X$ is defined as follows:

$$N_h = \{< x, T_h(x), I_h(x), F_h(x) > | x \in X\} \tag{5}$$

where $T_h(x)$, $I_h(x)$ and $F_h(x)$ are three sets of some values in [0,1], denoting the possible truth hesitant membership degree, indeterminacy hesitant membership degree and the falsity hesitant membership degree of the element $x \in X$ to the set $N_h$, respectively, with the conditions $0 \leq \alpha,\ \beta,\ \gamma \leq 1$ and $0 \leq \alpha^+,\ \beta^+,\ \gamma^+ \leq 3$, where $\alpha \in T_h(x)$, $\beta \in I_h(x)$, $\gamma \in F_h(x)$ with $\alpha^+ \in T_h^+(x) = \cup_{\alpha \in T_h(x)} max\{\alpha\}$, $\beta^+ \in I_h^+(x) = \cup_{\beta \in I_h(x)} max\{\beta\}$ and $\gamma^+ \in F_h^+(x) = \cup_{\gamma \in F_h(x)} max\{\gamma\}$ for all $x \in X$.
For simplicity, the three-tuple $N_h(x) = \{T_h(x), I_h(x), F_h(x)\}$ is called a single-valued neutrosophic hesitant fuzzy element (SVNHFE) or triple hesitant fuzzy element.

From Definition 2.3.1, it is clear that the SVNHFS comprises three different kinds of membership functions, namely; truth hesitant membership function, indeterminacy hesitant membership function and the falsity hesitant membership function, which consequently results in a more reliable framework and provides pliable access to assign values for each element in the domain, and can deal with three kind of hesitancy in this situation at a time. Thus, classical sets, including fuzzy sets, intuitionistic fuzzy sets, single-valued neutrosophic sets, hesitant fuzzy sets, can be considered as special cases of SVNHFSs (see [22]). Fig. 1 shows the graphical representation of classical sets to SVNHFSs.

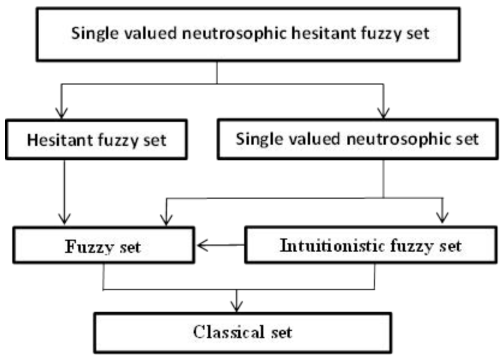


Figure 1: Diagrammatic coverage of classical sets to SVNHFSs.

**Definition 2.3.2:** [22] Let $N_{h_1}$ and $N_{h_2}$ be two SVNHFSs in a fixed set $X$; then their union can be defined as follows:

$$\begin{aligned} N_{h_1} \cup N_{h_2} = &\{T_h \in (T_{h_1} \cup T_{h_2}) | T_h \geq max\ (min\ \{T_{h_1} \cup T_{h_2}\}), \\ &I_h \in (I_{h_1} \cup I_{h_2}) | I_h \leq min\ (max\ \{I_{h_1} \cup I_{h_2}\}), \\ &F_h \in (F_{h_1} \cup F_{h_2}) | F_h \leq min\ (max\ \{F_{h_1} \cup F_{h_2}\})\} \end{aligned}$$

**Definition 2.3.3:** [22] Let $N_{h_1}$ and $N_{h_2}$ be two SVNHFSs in a fixed set $X$; then their intersection can be defined as follows:

$$\begin{aligned} N_{h_1} \cap N_{h_2} = &\{T_h \in (T_{h_1} \cap T_{h_2}) | T_h \leq min\ (max\ \{T_{h_1} \cap T_{h_2}\}), \\ &I_h \in (I_{h_1} \cap I_{h_2}) | I_h \geq max\ (min\ \{I_{h_1} \cap I_{h_2}\}), \\ &F_h \in (F_{h_1} \cap F_{h_2}) | F_h \geq max\ (min\ \{F_{h_1} \cap F_{h_2}\})\} \end{aligned}$$

# 3 Problem formulation and solution algorithm

## 3.1 General mathematical model of multiobjective nonlinear programming problem (MO-NLPP)

Generally, a mathematical programming problem is said to be nonlinear programming problem (NLPP) if either objective function, constraints or both are real-valued nonlinear functions. The objective function(s) is (are) to be optimized (minimize or maximize) under the given constraints. The classical multiobjective nonlinear programming problem (MO-NLPP) is represented in $\mathbf{M}_1$.

$$\begin{aligned}\mathbf{M}_1 :\ & Optimize\ Z_k(x),\ \ k = 1, 2, ..., K,\\ & s.t\ \ g_j(x) \leq d_j,\ \ j = 1, 2, ..., m_1,\\ & g_j(x) \geq d_j, \qquad j = m_1 + 1, m_1 + 2, ..., m_2,\\ & g_j(x) = d_j, \qquad j = m_2 + 1, m_2 + 2, ..., m,\\ & x \geq 0.\end{aligned}$$

where, either $Z_k,\ (k = 1, 2, ..., K), g_j,\ (j = 1, 2, ..., m)$ or both may be real valued nonlinear functions. $x = (x_1, x_2, ..., x_q)$ is a set of decision variables.

## 3.2 Development of proposed neutrosophic hesitant fuzzy programming approach (NHFPA)

In this study, a new approach based on single-valued neutrosophic hesitant fuzzy set to solve MO-NLPP has been investigated. The proposed approach is based on the hybrid combination of the two sets, namely; neutrosophic set (Smarandache [20]) and hesitant fuzzy set (Torra and Narukawa [21]) respectively. The proposed neutrosophic hesitant fuzzy programming approach (NHFPA) introduces more realistic aspects in dealing with the indeterminacy hesitation present in the decision-making problem. The interesting point is that the proposed NHFPA also considers the conflicting opinions of different experts regarding some parameters in real life problem which enables the DM(s) to obtain the adequate results under neutrosophic environment.
According to Bellman and Zadeh [5], the fuzzy set includes three concepts, namely; fuzzy decision (D), fuzzy goal (G) and fuzzy constraints (C) and incorporated these concepts in many real-life applications of decision-making under fuzzy environment. So, the fuzzy decision set is defined as follows:

$$D = G \cap C \tag{6}$$

Consequently, the neutrosophic hesitant fuzzy decision set $D_h^N$, with neutrosophic hesitant objectives and constraints, is defined as follows:

$$\begin{aligned}D_h^N = G \cap C &= (\cap_{k=1}^K D_k)(\cap_{i=1}^m C_i)\\ &= \{x,\ T_D(x),\ I_D(x),\ F_D(x)\}\\ &= \{T_D \in (T_{G_h} \cap T_{C_h}) \mid T_D \leq min\ (max\ \{T_{G_h} \cap T_{C_h}\}),\\ &\quad I_D \in (I_{G_h} \cap I_{C_h}) \mid I_D \geq max\ (min\ \{I_{G_h} \cap I_{C_h}\}),\\ &\quad F_D \in (F_{G_h} \cap F_{C_h}) \mid F_D \geq max\ (min\ \{F_{G_h} \cap F_{C_h}\})\}\end{aligned}$$

Where, $T_D(x),\ I_D(x)$ and $F_D(x)$ are a set of degree of acceptance of neutrosophic hesitant fuzzy decision solution under single-valued neutrosophic hesitant fuzzy decision set. Fig.2 shows the neutrosophic hesitant fuzzy membership degree for the objective function.
On solving each objective function individually, we have $k$ solutions set, $X^1, X^2, ..., X^k$, after that the obtained solutions are substituted in each objective function to determine the lower and upper bound for each objective as given below:

$$U_k = max[Z_k(X^k)]\ \ and\ \ L_k = min[Z_k(X^k)]\ \ \forall\, k = 1, 2, 3, ..., K. \tag{7}$$

Now, we can define the different hesitant membership function more elaborately under neutrosophic hesitant fuzzy environment as follows:

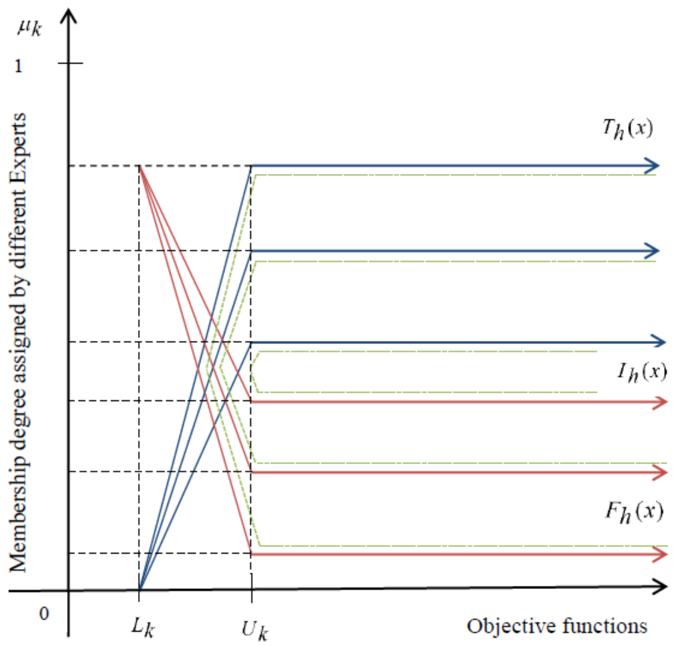


Figure 2: Graphical representation of neutrosophic hesitant fuzzy membership of objective function.

$\mathbf{C}ase - I$ : For maximization type objective function.

The truth hesitant-membership functions:

$$T^{E_1}_{h^+}(Z_k(x)) = \begin{cases} 0 & if \ \ Z_k(x) < L_k \\ \alpha_1 \ \frac{(Z_k(x))^t-(L_k)^t}{(U_k)^t-(L_k)^t} & if \ \ L_k \leq Z_k(x) \leq U_k \\ 1 & if \ \ Z_k(x) > U_k \end{cases} \tag{8}$$

$$T^{E_2}_{h^+}(Z_k(x)) = \begin{cases} 0 & if \ \ Z_k(x) < L_k \\ \alpha_2 \ \frac{(Z_k(x))^t-(L_k)^t}{(U_k)^t-(L_k)^t} & if \ \ L_k \leq Z_k(x) \leq U_k \\ 1 & if \ \ Z_k(x) > U_k \end{cases} \tag{9}$$

$$\begin{array}{cc} . & ... \\ . & ... \\ . & ... \end{array}$$

$$T^{E_n}_{h^+}(Z_k(x)) = \begin{cases} 0 & if \ \ Z_k(x) < L_k \\ \alpha_n \ \frac{(Z_k(x))^t-(L_k)^t}{(U_k)^t-(L_k)^t} & if \ \ L_k \leq Z_k(x) \leq U_k \\ 1 & if \ \ Z_k(x) > U_k \end{cases} \tag{10}$$

The indeterminacy hesitant-membership functions:

$$I^{E_1}_{h^+}(Z_k(x)) = \begin{cases} 0 & if \ \ Z_k(x) < L_k \\ \beta_1 \ \frac{(Z_k(x))^t-(L_k)^t}{(s_k)^t} & if \ \ L_k \leq Z_1(x) \leq L_k + s_k \\ 1 & if \ \ Z_k(x) > L_k + s_k \end{cases} \tag{11}$$

$$I^{E_2}_{h^+}(Z_k(x)) = \begin{cases} 0 & if \ \ Z_k(x) < L_k \\ \beta_2 \ \frac{(Z_k(x))^t-(L_k)^t}{(s_k)^t} & if \ \ L_k \leq Z_k(x) \leq L_k + s_k \\ 1 & if \ \ Z_k(x) > L_k + s_k \end{cases} \tag{12}$$

$$\begin{array}{cc} . & ... \\ . & ... \\ . & ... \end{array}$$

$$I^{E_n}_{h^+}(Z_k(x)) = \begin{cases} 0 & if \ \ Z_k(x) < L_k \\ \beta_n \ \frac{(Z_k(x))^t-(L_k)^t}{(s_k)^t} & if \ \ L_k \leq Z_k(x) \leq L_k + s_k \\ 1 & if \ \ Z_k(x) > L_k + s_k \end{cases} \tag{13}$$

The falsity hesitant-membership functions:

$$F^{E_1}_{h^+}(Z_k(x)) = \begin{cases} 1 & if \ \ Z_k(x) < L_k + t_k \\ \gamma_1 \ \frac{(U_k)^t-(Z_k(x))^t-(t_k)^t}{(U_k)^t-(L_k)^t-(t_k)^t} & if \ \ L_k + t_k \leq Z_k(x) \leq U_k \\ 0 & if \ \ Z_k(x) > U_k \end{cases} \tag{14}$$

$$F^{E_2}_{h^+}(Z_k(x)) = \begin{cases} 1 & if \ \ Z_k(x) < L_k + t_k \\ \gamma_2 \ \frac{(U_k)^t-(Z_k(x))^t-(t_k)^t}{(U_k)^t-(L_k)^t-(t_k)^t} & if \ \ L_k + t_k \leq Z_k(x) \leq U_k \\ 0 & if \ \ Z_k(x) > U_k \end{cases} \tag{15}$$

$$\begin{array}{ccc} . & .. & . \\ . & .. & . \\ . & .. & . \end{array}$$

$$F^{E_n}_{h^+}(Z_k(x)) = \begin{cases} 1 & if \ \ Z_k(x) < L_k + t_k \\ \gamma_n \ \frac{(U_k)^t-(Z_k(x))^t-(t_k)^t}{(U_k)^t-(L_k)^t-(t_k)^t} & if \ \ L_k + t_k \leq Z_k(x) \leq U_k \\ 0 & if \ \ Z_k(x) > U_k \end{cases} \tag{16}$$

where parameter $t > 0$ and $s_k, t_k \in (0, 1)\ \forall k$, are indeterminacy and falsity tolerance values, which is assigned by DM(s) and $h^+$ represents the maximization type hesitant objective function.
$T^{E_1}_{h^+}(Z_k(x)),\ I^{E_1}_{h^+}(Z_k(x)),\ F^{E_1}_{h^+}(Z_k(x))$ are truth, indeterminacy and the falsity-hesitant-membership degrees assigned by $1^{st}$ expert.
$T^{E_2}_{h^+}(Z_k(x)),\ I^{E_2}_{h^+}(Z_k(x)),\ F^{E_2}_{h^+}(Z_k(x))$ are truth, indeterminacy and the falsity-hesitant-membership degrees assigned by $2^{nd}$ expert.
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……
$T^{E_n}_{h^+}(Z_k(x)),\ I^{E_n}_{h^+}(Z_k(x)),\ F^{E_n}_{h^+}(Z_k(x))$ are truth, indeterminacy and the falsity-hesitant-membership degrees assigned by $n^{th}$ expert.

$\mathbf{C}ase - II$ : For minimization type objective function.
The truth hesitant-membership functions:

$$T^{E_1}_{h^-}(Z_k(x)) = \begin{cases} 1 & if \ \ Z_k(x) < L_k \\ \alpha_1 \ \frac{(U_k)^t-(Z_k(x))^t}{(U_k)^t-(L_k)^t} & if \ \ L_k \leq Z_k(x) \leq U_k \\ 0 & if \ \ Z_k(x) > U_k \end{cases} \tag{17}$$

$$T^{E_2}_{h^-}(Z_k(x)) = \begin{cases} 1 & if \ \ Z_k(x) < L_k \\ \alpha_2 \ \frac{(U_k)^t-(Z_k(x))^t}{(U_k)^t-(L_k)^t} & if \ \ L_k \leq Z_k(x) \leq U_k \\ 0 & if \ \ Z_k(x) > U_k \end{cases} \tag{18}$$

$$\begin{matrix} . & .. & . \\ . & .. & . \\ . & .. & . \end{matrix}$$

$$T_{h^-}^{E_n}(Z_k(x)) = \begin{cases} 1 & if \;\; Z_k(x) < L_k \\ \alpha_n \; \frac{(U_k)^t - (Z_k(x))^t}{(U_k)^t - (L_k)^t} & if \;\; L_k \leq Z_k(x) \leq U_k \\ 0 & if \;\; Z_k(x) > U_k \end{cases} \tag{19}$$

The indeterminacy hesitant-membership functions:

$$I_{h^-}^{E_1}(Z_k(x)) = \begin{cases} 1 & if \;\; Z_k(x) < U_k - s_k \\ \beta_1 \; \frac{(U_k)^t - (Z_k(x))^t}{(s_k)^t} & if \;\; U_k - s_k \leq Z_k(x) \leq U_k \\ 0 & if \;\; Z_k(x) > U_k \end{cases} \tag{20}$$

$$I_{h^-}^{E_2}(Z_k(x)) = \begin{cases} 1 & if \;\; Z_k(x) < U_k - s_k \\ \beta_2 \; \frac{(U_k)^t - (Z_k(x))^t}{(s_k)^t} & if \;\; U_k - s_k \leq Z_k(x) \leq U_k \\ 0 & if \;\; Z_k(x) > U_k \end{cases} \tag{21}$$

$$\begin{matrix} . & ... \\ . & ... \\ . & ... \end{matrix}$$

$$I_{h^-}^{E_n}(Z_k(x)) = \begin{cases} 1 & if \;\; Z_k(x) < U_k - s_k \\ \beta_n \; \frac{(U_k)^t - (Z_k(x))^t}{(s_k)^t} & if \;\; U_k - s_k \leq Z_k(x) \leq U_k \\ 0 & if \;\; Z_k(x) > U_k \end{cases} \tag{22}$$

The falsity hesitant-membership functions:

$$F_{h^-}^{E_1}(Z_k(x)) = \begin{cases} 0 & if \;\; Z_k(x) < L_k + t_k \\ \gamma_1 \; \frac{(Z_k(x))^t - (L_k)^t - (t_k)^t}{(U_k)^t - (L_k)^t - (t_k)^t} & if \;\; L_k + t_k \leq Z_k(x) \leq U_k \\ 1 & if \;\; Z_k(x) > U_k \end{cases} \tag{23}$$

$$F_{h^-}^{E_2}(Z_k(x)) = \begin{cases} 0 & if \;\; Z_k(x) < L_k + t_k \\ \gamma_2 \; \frac{(Z_k(x))^t - (L_k)^t - (t_k)^t}{(U_k)^t - (L_k)^t - (t_k)^t} & if \;\; L_k + t_k \leq Z_k(x) \leq U_k \\ 1 & if \;\; Z_k(x) > U_k \end{cases} \tag{24}$$

$$\begin{matrix} . & ... \\ . & ... \\ . & ... \end{matrix}$$

$$F_{h^-}^{E_n}(Z_k(x)) = \begin{cases} 0 & if \;\; Z_k(x) < L_k + t_k \\ \gamma_n \; \frac{(Z_k(x))^t - (L_k)^t - (t_k)^t}{(U_k)^t - (L_k)^t - (t_k)^t} & if \;\; L_k + t_k \leq Z_k(x) \leq U_k \\ 1 & if \;\; Z_k(x) > U_k \end{cases} \tag{25}$$

where parameter $t > 0$ and $s_k, t_k \in (0, 1) \; \forall k$, are indeterminacy and falsity tolerance values, which is assigned by DM(s) and $h^-$ represents the minimization type hesitant objective function.
$T_{h^-}^{E_1}(Z_k(x)),\; I_{h^-}^{E_1}(Z_k(x)),\; F_{h^-}^{E_1}(Z_k(x))$ are truth, indeterminacy and the falsity-hesitant-membership degrees assigned by $1^{st}$ expert.
$T_{h^-}^{E_2}(Z_k(x)),\; I_{h^-}^{E_2}(Z_k(x)),\; F_{h^-}^{E_2}(Z_k(x))$ are truth, indeterminacy and the falsity-hesitant-membership degrees assigned by $2^{nd}$ expert.
......
......
$T_{h^-}^{E_n}(Z_k(x)),\; I_{h^-}^{E_n}(Z_k(x)),\; F_{h^-}^{E_n}(Z_k(x))$ are truth, indeterminacy and the falsity-hesitant-membership degrees assigned by $n^{th}$ expert.

Let $T_h^{E_n} = min\; (T_{h^+}^{E_n},\; T_{h^-}^{E_n})$, $I_h^{E_n} = min\; (I_{h^+}^{E_n},\; I_{h^-}^{E_n})$ and $F_h^{E_n} = max\; (F_{h^+}^{E_n},\; F_{h^-}^{E_n}) \; \forall \; k = 1, 2, ..., K$. Now, the motive is to determine the highest degree of satisfaction for DM(s) by establishing a balance between objectives and constraints.
The neutrosophic hesitant fuzzy model for MO-NLPP ($\mathbf{M}_1$) can be represented as follows:

$$\begin{aligned} \mathbf{M}_2 :& Max \;\; min_{k=1,2,3,...,K} \;\; T_h^{E_n}(Z_k(x)) \\ & Max \;\; min_{k=1,2,3,...,K} \;\; I_h^{E_n}(Z_k(x)) \\ & Min \;\; max_{k=1,2,3,...,K} \;\; F_h^{E_n}(Z_k(x)) \\ & s.t \;\; g_j(x) \leq d_j, \;\; j = 1, 2, ..., m_1, \\ & g_j(x) \geq d_j, \qquad j = m_1 + 1, m_1 + 2, ..., m_2, \\ & g_j(x) = d_j, \qquad j = m_2 + 1, m_2 + 2, ..., m, \\ & x \geq 0. \end{aligned}$$

With the help of auxiliary parameters, model $\mathbf{M}_2$ can be transformed into the following form $\mathbf{M}_3$.

$$\begin{aligned}
\mathbf{M}_3 :& Max\ \frac{\sum \alpha_n}{n} \\
& Max\ \frac{\sum \beta_n}{n} \\
& Min\ \frac{\sum \gamma_n}{n} \\
& s.t.\ T_{h^+}^{E_n}(Z_k(x)) \geq \alpha_n,\ \ I_{h^+}^{E_n}(Z_k(x)) \geq \beta_n,\ \ F_{h^+}^{E_n}(Z_k(x)) \leq \gamma_n \\
& T_{h^-}^{E_n}(Z_k(x)) \geq \alpha_n,\ \ I_{h^-}^{E_n}(Z_k(x)) \geq \beta_n,\ \ F_{h^-}^{E_n}(Z_k(x)) \leq \gamma_n \\
& g_j(x) \leq d_j,\ \ j = 1, 2, ..., m_1, \\
& g_j(x) \geq d_j,\ \ j = m_1 + 1, m_1 + 2, ..., m_2, \\
& g_j(x) = d_j,\ \ j = m_2 + 1, m_2 + 2, ..., m, \\
& x \geq 0, \quad \alpha_n, \beta_n, \gamma_n \in (0, 1) \\
& \alpha_n + \beta_n + \gamma_n \leq 3, \quad \alpha_n \geq \beta_n, \quad \alpha_n \geq \gamma_n, \quad \forall\ n.
\end{aligned}$$

Using linear membership function, model $\mathbf{M}_3$ can be written as in $\mathbf{M}_4$.

$$\begin{aligned}
\mathbf{M}_4 :& Max\ \ \chi = \frac{\alpha_1 + \alpha_2 + ... + \alpha_n}{n} + \frac{\beta_1 + \beta_2 + ... + \beta_n}{n} - \frac{\gamma_1 + \gamma_2 + ... + \gamma_n}{n} \\
& s.t.\ T_{h^+}^{E_1}(Z_k(x)) \geq \alpha_1,\ T_{h^+}^{E_2}(Z_k(x)) \geq \alpha_2,\ ...,\ T_{h^+}^{E_n}(Z_k(x)) \geq \alpha_n \\
& I_{h^+}^{E_1}(Z_k(x)) \geq \beta_1,\ I_{h^+}^{E_2}(Z_k(x)) \geq \beta_2,\ ...,\ I_{h^+}^{E_n}(Z_k(x)) \geq \beta_n \\
& F_{h^+}^{E_1}(Z_k(x)) \leq \gamma_1,\ F_{h^+}^{E_2}(Z_k(x)) \leq \gamma_2,\ ...,\ F_{h^+}^{E_n}(Z_k(x)) \leq \gamma_n \\
& T_{h^-}^{E_1}(Z_k(x)) \geq \alpha_1,\ T_{h^-}^{E_2}(Z_k(x)) \geq \alpha_2,\ ...,\ T_{h^-}^{E_n}(Z_k(x)) \geq \alpha_n \\
& I_{h^-}^{E_1}(Z_k(x)) \geq \beta_1,\ I_{h^-}^{E_2}(Z_k(x)) \geq \beta_2,\ ...,\ I_{h^-}^{E_n}(Z_k(x)) \geq \beta_n \\
& F_{h^-}^{E_1}(Z_k(x)) \leq \gamma_1,\ F_{h^-}^{E_2}(Z_k(x)) \leq \gamma_2,\ ...,\ F_{h^-}^{E_n}(Z_k(x)) \leq \gamma_n \\
& g_j(x) \leq d_j, \qquad j = 1, 2, ..., m_1, \\
& g_j(x) \geq d_j, \qquad j = m_1 + 1, m_1 + 2, ..., m_2, \\
& g_j(x) = d_j, \qquad j = m_2 + 1, m_2 + 2, ..., m, \\
& x \geq 0,\ \ 0 \leq \alpha_1, \alpha_2, ..., \alpha_n \leq 1,\ \ 0 \leq \beta_1, \beta_2, ..., \beta_n \leq 1 \\
& 0 \leq \gamma_1, \gamma_2, ..., \gamma_n \leq 1,\ \ \alpha_n \geq \beta_n,\ \ \alpha_n \geq \gamma_n, \\
& \alpha_n + \beta_n + \gamma_n \leq 3,\ \ \forall\ n.
\end{aligned}$$

Finally, model $\mathbf{M}_4$ gives the compromise solution to MO-NLPP.

## 3.3 Proposed NHFPA algorithm for MO-NLPP

The whole procedure from problem formulation to final solvable model $\mathbf{M}_4$ discussed in section 3 is summarized as step-wise algorithm.
**Step-1.** Formulate the multiobjective nonlinear programing problems as in $\mathbf{M}_1$.
**Step-2.** Determine the bounds $U_k$ and $L_k$, for each objective by using equation (7).
**Step-3.** By using $U_k$ and $L_k$, define the upper and lower bound for truth hesitant, indeterminacy hesitant and falsity hesitant membership functions as given in equation (8)-(25).
**Step-4.** Ask for the truth hesitant, indeterminacy hesitant and the falsity hesitant membership degrees from different experts or DM(s).
**Step-5.** Formulate MO-NLPP under neutrosophic hesitant fuzzy environment defined in $\mathbf{M}_4$.
**Step-6.** Solve the multiobjective nonlinear programing problem in order to obtain the compromise solution using suitable techniques or some optimizing software packages.

# 4 Experimental study

In order to show the efficiency and validity of the proposed method, we adopted the numerical example of the manufacturing system discussed by Singh and Yadav [19]. The DM(s) of the company intends to maximize the total profit incurred over products and minimize the total time required for each product. Also, assumed that the DM(s) seeks three experts' opinion in the decision-making process. Therefore, the crisp multiobjective non-linear programming problem formulation [19] is given as follows:

$$\begin{aligned}
\mathbf{M}_1 :\ & Max\ Z_1(x) = 99.875x_1^{\frac{1}{2}} - 8x_1 + 119.875x_2^{\frac{1}{2}} - 10.125x_2 + 95.125x_3^{\frac{1}{3}} - 8x_3 \\
& Min\ Z_2(x) = 3.875x_1 + 5.125x_2 + 5.9375x_3 \\
& s.t\ \ 2.0625x_1 + 3.875x_2 + 2.9375x_3 \leq 333.125 \\
& 3.875x_1 + 2.0625x_2 + 2.0625x_3 \leq 365.625 \\
& 2.9375x_1 + 2.0625x_2 + 2.9375x_3 \geq 360 \\
& x_1,\ x_2,\ x_3 \geq 0.
\end{aligned}$$

On solving each objective function individually given in $(\mathbf{M}_1)$, we get the following individual best solution, lower and upper bound for each objective.
$X^1 = (57.82,\ 13.09,\ 55.53)$, $X^2 = (62.26,\ 0,\ 60.28)$ along with $L_1 = 180.72$, $U_1 = 516.70$, $L_2 = 599.23$ and $U_2 = 620.84$.

Since, the first objective $Z_1(x)$ is of maximization type and the satisfaction level of Experts or DMs increases if the values of objective function tends towards its upper bound. Therefore the truth hesitant membership, indeterminacy hesitant membership and falsity hesitant membership functions of upper bound can be represented as follows:

For $Z_1$: The upper and lower bound for first objective and its membership functions.

$$T_{h^+}^{E_1}(Z_1(x)) = \begin{cases} 0 & if \;\; Z_1(x) < 180.72 \\ 0.98\ \frac{(99.875x_1^{\frac{1}{2}}-8x_1+119.875x_2^{\frac{1}{2}}-10.125x_2+95.125x_3^{\frac{1}{3}}-8x_3)^t-(180.72)^t}{(516.70)^t-(180.72)^t} & if \;\; 180.72 \le Z_1(x) \le 516.70 \\ 1 & if \;\; Z_1(x) > 516.70 \end{cases} \tag{26}$$

$$T_{h^+}^{E_2}(Z_1(x)) = \begin{cases} 0 & if \;\; Z_1(x) < 180.72 \\ 0.99\ \frac{(99.875x_1^{\frac{1}{2}}-8x_1+119.875x_2^{\frac{1}{2}}-10.125x_2+95.125x_3^{\frac{1}{3}}-8x_3)^t-(180.72)^t}{(516.70)^t-(180.72)^t} & if \;\; 180.72 \le Z_1(x) \le 516.70 \\ 1 & if \;\; Z_1(x) > 516.70 \end{cases} \tag{27}$$

$$T_{h^+}^{E_3}(Z_1(x)) = \begin{cases} 0 & if \;\; Z_1(x) < 180.72 \\ \frac{(99.875x_1^{\frac{1}{2}}-8x_1+119.875x_2^{\frac{1}{2}}-10.125x_2+95.125x_3^{\frac{1}{3}}-8x_3)^t-(180.72)^t}{(516.70)^t-(180.72)^t} & if \;\; 180.72 \le Z_1(x) \le 516.70 \\ 1 & if \;\; Z_1(x) > 516.70 \end{cases} \tag{28}$$

$$I_{h^+}^{E_1}(Z_1(x)) = \begin{cases} 0 & if \;\; Z_1(x) < 180.72 \\ 0.98\ \frac{(99.875x_1^{\frac{1}{2}}-8x_1+119.875x_2^{\frac{1}{2}}-10.125x_2+95.125x_3^{\frac{1}{3}}-8x_3)^t-(180.72)^t}{(s_1)^t} & if \;\; 180.72 \le Z_1(x) \le 180.72+s_1 \\ 1 & if \;\; Z_1(x) > 180.72+s_1 \end{cases} \tag{29}$$

$$I_{h^+}^{E_2}(Z_1(x)) = \begin{cases} 0 & if \;\; Z_1(x) < 180.72 \\ 0.99\ \frac{(99.875x_1^{\frac{1}{2}}-8x_1+119.875x_2^{\frac{1}{2}}-10.125x_2+95.125x_3^{\frac{1}{3}}-8x_3)^t-(180.72)^t}{(s_1)^t} & if \;\; 180.72 \le Z_1(x) \le 180.72+s_1 \\ 1 & if \;\; Z_1(x) > 180.72+s_1 \end{cases} \tag{30}$$

$$I_{h^+}^{E_3}(Z_1(x)) = \begin{cases} 0 & if \;\; Z_1(x) < 180.72 \\ \frac{(99.875x_1^{\frac{1}{2}}-8x_1+119.875x_2^{\frac{1}{2}}-10.125x_2+95.125x_3^{\frac{1}{3}}-8x_3)^t-(180.72)^t}{(s_1)^t} & if \;\; 180.72 \le Z_1(x) \le 180.72+s_1 \\ 1 & if \;\; Z_1(x) > 180.72+s_1 \end{cases} \tag{31}$$

$$F_{h^+}^{E_1}(Z_1(x)) = \begin{cases} 1 & if \;\; Z_1(x) < 180.72 \\ 0.98\ \frac{(516.70)^t-(t_1)^t-(99.875x_1^{\frac{1}{2}}-8x_1+119.875x_2^{\frac{1}{2}}-10.125x_2+95.125x_3^{\frac{1}{3}}-8x_3)^t}{(516.70)^t-(180.72)^t-(t_1)^t} & if \;\; 180.72 \le Z_1(x) \le 516.70-t_1 \\ 0 & if \;\; Z_1(x) > 516.70-t_1 \end{cases} \tag{32}$$

$$F_{h^+}^{E_2}(Z_1(x)) = \begin{cases} 1 & if \;\; Z_1(x) > 516.70 \\ 0.99\ \frac{(516.70)^t-(t_1)^t-(99.875x_1^{\frac{1}{2}}-8x_1+119.875x_2^{\frac{1}{2}}-10.125x_2+95.125x_3^{\frac{1}{3}}-8x_3)^t}{(516.70)^t-(180.72)^t-(t_1)^t} & if \;\; 180.72+t_1 \le Z_1(x) \le 516.70 \\ 0 & if \;\; Z_1(x) < 180.72+t_1 \end{cases} \tag{33}$$

$$F_{h^+}^{E_3}(Z_1(x)) = \begin{cases} 1 & if \;\; Z_1(x) > 516.70 \\ \frac{(516.70)^t-(t_1)^t-(99.875x_1^{\frac{1}{2}}-8x_1+119.875x_2^{\frac{1}{2}}-10.125x_2+95.125x_3^{\frac{1}{3}}-8x_3)^t}{(516.70)^t-(180.72)^t-(t_1)^t} & if \;\; 180.72+t_1 \le Z_1(x) \le 516.70 \\ 0 & if \;\; Z_1(x) < 180.72+t_1 \end{cases} \tag{34}$$

Similarly, the second objective $Z_2(x)$ is of minimization type and the satisfaction level of Experts or DMs increases if the values of objective function tends towards its lower bound. Thus the truth hesitant membership, indeterminacy hesitant membership and falsity hesitant membership functions of lower bound can be represented as follows:

For $Z_2$: The upper and lower bound for second objective and its membership functions.

$$T_{h^-}^{E_1}(Z_2(x)) = \begin{cases} 1 & if \;\; Z_2(x) < 599.23 \\ 0.98\ \frac{(620.84)^t-(3.875x_1+5.125x_2+5.9375x_3)^t}{(620.84)^t-(599.23)^t} & if \;\; 599.23 \le Z_2(x) \le 620.84 \\ 0 & if \;\; Z_2(x) > 620.84 \end{cases} \tag{35}$$

$$T_{h^-}^{E_2}(Z_2(x)) = \begin{cases} 1 & if \;\; Z_2(x) < 599.23 \\ 0.99\ \frac{(620.84)^t-(3.875x_1+5.125x_2+5.9375x_3)^t}{(620.84)^t-(599.23)^t} & if \;\; 599.23 \le Z_2(x) \le 620.84 \\ 0 & if \;\; Z_2(x) > 620.84 \end{cases} \tag{36}$$

$$T_{h^-}^{E_3}(Z_2(x)) = \begin{cases} 1 & if \;\; Z_2(x) < 599.23 \\ \frac{(620.84)^t-(3.875x_1+5.125x_2+5.9375x_3)^t}{(620.84)^t-(599.23)^t} & if \;\; 599.23 \le Z_2(x) \le 620.84 \\ 0 & if \;\; Z_2(x) > 620.84 \end{cases} \tag{37}$$

$$I_{h^-}^{E_1}(Z_2(x)) = \begin{cases} 1 & if \;\; Z_2(x) < 620.84-s_2 \\ 0.98\ \frac{(620.84)^t-(3.875x_1+5.125x_2+5.9375x_3)^t}{(s_2)^t} & if \;\; 620.84-s_2 \le Z_2(x) \le 620.84 \\ 0 & if \;\; Z_2(x) > 620.84 \end{cases} \tag{38}$$

$$I_{h^-}^{E_2}(Z_2(x)) = \begin{cases} 1 & if \;\; Z_2(x) < 620.84-s_2 \\ 0.99\ \frac{(620.84)^t-(3.875x_1+5.125x_2+5.9375x_3)^t}{(s_2)^t} & if \;\; 620.84-s_2 \le Z_2(x) \le 620.84 \\ 0 & if \;\; Z_2(x) > 620.84 \end{cases} \tag{39}$$

$$I^{E_3}_{h^-}(Z_2(x)) = \begin{cases} 1 & if \ Z_2(x) < 620.84 - s_2 \\ \frac{(620.84)^t - (3.875x_1 + 5.125x_2 + 5.9375x_3)^t}{(s_2)^t} & if \ 620.84 - s_2 \leq Z_2(x) \leq 620.84 \\ 0 & if \ Z_2(x) > 620.84 \end{cases} \tag{40}$$

$$F^{E_1}_{h^-}(Z_2(x)) = \begin{cases} 0 & if \ Z_2(x) < 599.23 + t_2 \\ 0.98 \ \frac{(3.875x_1 + 5.125x_2 + 5.9375x_3)^t - (599.23)^t - (t_2)^t}{(620.84)^t - (599.23)^t - (t_2)^t} & if \ 599.23 + t_2 \leq Z_2(x) \leq 620.84 \\ 1 & if \ Z_2(x) > 620.84 \end{cases} \tag{41}$$

$$F^{E_2}_{h^-}(Z_2(x)) = \begin{cases} 0 & if \ Z_2(x) < 599.23 + t_2 \\ 0.99 \ \frac{(3.875x_1 + 5.125x_2 + 5.9375x_3)^t - (599.23)^t - (t_2)^t}{(620.84)^t - (599.23)^t - (t_2)^t} & if \ 599.23 + t_2 \leq Z_2(x) \leq 620.84 \\ 1 & if \ Z_2(x) > 620.84 \end{cases} \tag{42}$$

$$F^{E_3}_{h^-}(Z_2(x)) = \begin{cases} 0 & if \ Z_2(x) < 599.23 + t_2 \\ \frac{(3.875x_1 + 5.125x_2 + 5.9375x_3)^t - (599.23)^t - (t_2)^t}{(620.84)^t - (599.23)^t - (t_2)^t} & if \ 599.23 + t_2 \leq Z_2(x) \leq 620.84 \\ 1 & if \ Z_2(x) > 620.84 \end{cases} \tag{43}$$

The final solution model is given as follows:

$$\begin{aligned}
&\mathbf{M}_4 : Max \ \ \chi = \frac{\alpha_1 + \alpha_2 + \alpha_3}{3} + \frac{\beta_1 + \beta_2 + \beta_3}{3} - \frac{\gamma_1 + \gamma_2 + \gamma_3}{3} \\
&s.t. \ 0.98 \ \frac{(99.875x_1^{\frac{1}{2}} - 8x_1 + 119.875x_2^{\frac{1}{2}} - 10.125x_2 + 95.125x_3^{\frac{1}{3}} - 8x_3)^t - (180.72)^t}{(516.70)^t - (180.72)^t} \geq \alpha_1 \\
&0.99 \ \frac{(99.875x_1^{\frac{1}{2}} - 8x_1 + 119.875x_2^{\frac{1}{2}} - 10.125x_2 + 95.125x_3^{\frac{1}{3}} - 8x_3)^t - (180.72)^t}{(516.70)^t - (180.72)^t} \geq \alpha_2 \\
&\frac{(99.875x_1^{\frac{1}{2}} - 8x_1 + 119.875x_2^{\frac{1}{2}} - 10.125x_2 + 95.125x_3^{\frac{1}{3}} - 8x_3)^t - (180.72)^t}{(516.70)^t - (180.72)^t} \geq \alpha_3 \\
&0.98 \ \frac{(99.875x_1^{\frac{1}{2}} - 8x_1 + 119.875x_2^{\frac{1}{2}} - 10.125x_2 + 95.125x_3^{\frac{1}{3}} - 8x_3)^t - (180.72)^t}{(s_1)^t} \geq \beta_1 \\
&0.99 \ \frac{(99.875x_1^{\frac{1}{2}} - 8x_1 + 119.875x_2^{\frac{1}{2}} - 10.125x_2 + 95.125x_3^{\frac{1}{3}} - 8x_3)^t - (180.72)^t}{(s_1)^t} \geq \beta_2 \\
&\frac{(99.875x_1^{\frac{1}{2}} - 8x_1 + 119.875x_2^{\frac{1}{2}} - 10.125x_2 + 95.125x_3^{\frac{1}{3}} - 8x_3)^t - (180.72)^t}{(s_1)^t} \geq \beta_3 \\
&0.98 \ \frac{(516.70)^t - (t_1)^t - (99.875x_1^{\frac{1}{2}} - 8x_1 + 119.875x_2^{\frac{1}{2}} - 10.125x_2 + 95.125x_3^{\frac{1}{3}} - 8x_3)^t}{(516.70)^t - (180.72)^t - (t_1)^t} \leq \gamma_1 \\
&0.99 \ \frac{(516.70)^t - (t_1)^t - (99.875x_1^{\frac{1}{2}} - 8x_1 + 119.875x_2^{\frac{1}{2}} - 10.125x_2 + 95.125x_3^{\frac{1}{3}} - 8x_3)^t}{(516.70)^t - (180.72)^t - (t_1)^t} \leq \gamma_2 \\
&\frac{(516.70)^t - (t_1)^t - (99.875x_1^{\frac{1}{2}} - 8x_1 + 119.875x_2^{\frac{1}{2}} - 10.125x_2 + 95.125x_3^{\frac{1}{3}} - 8x_3)^t}{(516.70)^t - (180.72)^t - (t_1)^t} \leq \gamma_3 \\
&0.98 \ \frac{(620.84)^t - (3.875x_1 + 5.125x_2 + 5.9375x_3)^t}{(620.84)^t - (599.23)^t} \geq \alpha_1 \\
&0.99 \ \frac{(620.84)^t - (3.875x_1 + 5.125x_2 + 5.9375x_3)^t}{(620.84)^t - (599.23)^t} \geq \alpha_2 \\
&\frac{(620.84)^t - (3.875x_1 + 5.125x_2 + 5.9375x_3)^t}{(620.84)^t - (599.23)^t} \geq \alpha_3 \\
&0.98 \ \frac{(620.84)^t - (3.875x_1 + 5.125x_2 + 5.9375x_3)^t}{(s_2)^t} \geq \beta_1 \\
&0.99 \ \frac{(620.84)^t - (3.875x_1 + 5.125x_2 + 5.9375x_3)^t}{(s_2)^t} \geq \beta_2 \\
&\frac{(620.84)^t - (3.875x_1 + 5.125x_2 + 5.9375x_3)^t}{(s_2)^t} \geq \beta_3 \\
&0.98 \ \frac{(3.875x_1 + 5.125x_2 + 5.9375x_3)^t - (599.23)^t - (t_2)^t}{(620.84)^t - (599.23)^t - (t_2)^t} \leq \gamma_1 \\
&0.99 \ \frac{(3.875x_1 + 5.125x_2 + 5.9375x_3)^t - (599.23)^t - (t_2)^t}{(620.84)^t - (599.23)^t - (t_2)^t} \leq \gamma_2 \\
&\frac{(3.875x_1 + 5.125x_2 + 5.9375x_3)^t - (599.23)^t - (t_2)^t}{(620.84)^t - (599.23)^t - (t_2)^t} \leq \gamma_3 \\
&2.0625x_1 + 3.875x_2 + 2.9375x_3 \leq 333.125 \\
&3.875x_1 + 2.0625x_2 + 2.0625x_3 \leq 365.625 \\
&2.9375x_1 + 2.0625x_2 + 2.9375x_3 \geq 360
\end{aligned}$$

$$x_1,\ x_2,\ x_3 \geq 0,\quad 0 \leq \alpha_1, \alpha_2, \alpha_3 \leq 1,\ \ 0 \leq \beta_1, \beta_2, \beta_3 \leq 1,$$
$$0 \leq \gamma_1, \gamma_2, \gamma_3 \leq 1,\ \ 0 \leq s_1, t_1 \leq 1,\ \ 0 \leq s_2, t_2 \leq 1,$$
$$\alpha_n \geq \beta_n,\ \ \alpha_n \geq \gamma_n,\ \ \alpha_n + \beta_n + \gamma_n \leq 3,\quad \forall\ n = 1, 2, 3.$$

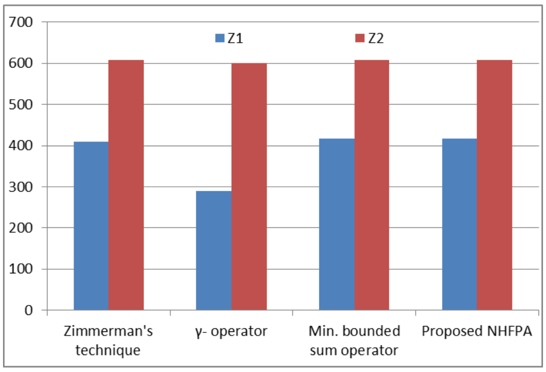
The multiobjective nonlinear programming problem $\mathbf{M}_4$ has been written in AMPL language and solved using solvers available on NEOS server online facility provided by Wisconsin Institutes for Discovery at the University of Wisconsin in Madison for solving Optimization problems, see (Server [18]).
At $t = 2$, the optimal solution of the multiobjective nonlinear programming problem by using the proposed neutrosophic hesitant fuzzy programming approach (NHFPA) is $x = (60.48,\ 5.26,\ 58.37)$, $Z_1 = 416.58,\ Z_2 = 607.88$ with the degree of satisfaction $\chi = 1.20$ respectively.

## 4.1 Comparative study

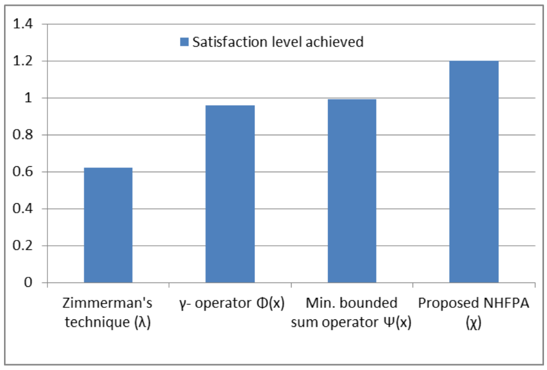
The multiobjective nonlinear programming problem of manufacturing system with conflicting objectives have been solved by using proposed neutrosophic hesitant fuzzy programming approach (NHFPA). The solution results obtained by proposed method and with other existing approaches discussed in [19] have been summarized in Table-1. From the table, it is clear that the minimum deviation from ideal solution of each objective function is 100.12 and 0.41 by using proposed NHFPA and $\gamma$- operator respectively. Furthermore, the highest satisfaction level has been attained by proposed approach i.e; $\chi$=1.20, which reveals the superiority of proposed NHFPA over other existing approaches in terms of satisfactory degree of DM(s). Fig-3 shows the graphical representation of the objective functions and satisfaction level obtained by different approaches.

Table 1: Comparison of results with existing methods.

| Solution method | Objective values | | Deviations from ideal solutions | | Satisfaction level |
|---|---|---|---|---|---|
| | Max. $Z_1$ | Min. $Z_2$ | $(U_1 - Z_1)$ | $(Z_2 - L_2)$ | |
| Zimmerman's technique [19] | 409.70 | 607.28 | 107 | 8.05 | $\lambda$= 0.62 |
| $\gamma$- operator [19] | 288.86 | 599.64 | 227.84 | 0.41(min.) | $\phi(x)$= 0.96 |
| Min. bounded sum operator [19] | 416.58 | 607.88 | 100.12 | 8.65 | $\psi(x)$= 0.99 |
| Proposed NHFPA | 416.58 | 607.88 | 100.12(min.) | 8.65 | $\chi$= 1.20 (max.) |




(a) Objective functions obtained by different approaches.




(b) Satisfaction level achieved by different approaches.

Figure 3: Comparison of results with proposed NHFPA and different existing approaches.

# 5 Conclusions

In this study, a new approach has been suggested to solve the multiobjective nonlinear programming problem in the neutrosophic hesitant fuzzy environment. The proposed neutrosophic hesitant fuzzy programming approach (NHFPA) comprises three different membership functions, namely; truth hesitant, indeterminacy hesitant and a falsity hesitant membership function which contains a set of different values between 0 and 1. The proposed approach provides the more realistic framework and considers various aspects of the DM's neutral thoughts with hesitations in the decision-making process. The main contribution by introducing the proposed approach is that it allows the DM(s) to express his/her(their) degree of hesitation and neutral thoughts according to the need of adverse situations in a convenient manner. In order to show the superiority of proposed NHFPA, it is applied to solve multiobjective nonlinear programming problem in the manufacturing system. To best of our knowledge, no such approach is suggested in the literature to solve MO-NLPP in such an efficient and effective manner.

Therefore, the proposed NHFPA will be very helpful in such a typical situation when the DM(s) have some neutral thoughts and also with a set some hesitation values in the decision-making process. In future, the proposed approach may be applied to the multiobjective fractional programming problem, bi-level nonlinear programming problem, multilevel fractional programming problem etc.

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# Extensions to Linguistic Summaries Indicators based on Neutrosophic Theory, Applications in Project Management Decisions

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**Abstract**. The quick development of the markets and companies, especially those that apply information technology, has made it easy to store a large volume of digital information. Nevertheless, the extraction of potentially useful knowledge is difficult; also could not be easily understandable by humans. One of the techniques applied to the solution to this problem is the linguistic data summarizations, whose objective is to discover knowledge to extract patterns from databases, from which are generated explicit and concise summaries. Another important element of the linguistic summaries is the indicators (T) for their evaluation proposed by Zadeh when including linguistic terms evaluation in fuzzy sets. However, these indicators not include the analysis in indeterminate sets. In this paper, it is discussed the use of linguistic data summarization in project management environments and new T indicators are proposed including neutrosophic sets with single value neutrosophic numbers. Authors evaluate *T*-values proposed by Zadeh and *T*-values based on neutrosophic theory in the evaluation of linguistic summaries recovered.

**Keywords:** neutrosophic sets, single value neutrosophic numbers, linguistic data summarization, project management.

## 1 Introduction

The market growth, even in the digital world, has led to the availability of a large volume of data, in different formats and from various sources. Unfortunately, while greater are data volumes, the more difficult is interpretation. The important information of those data is non-trivial dependencies, which are encoded. These dependencies usually are hidden; their discovery requires some intelligence.

In general, many companies have limitations in data analysis that affect their decisions. Making decision problems can be classified in structured and not structured. Structured decision-making problems have defined methods for solutions and they are supported by procedures and rules. In another hand, not structured decision making, resolve low frequency problems that need specific solutions. Examples of not structure problems are alternative selection [1] [2] [3], diagnostics, prediction [4] [5] [6], prognosis, a classification, machine learning and data mining [7]. In the context of this paper, authors focus in a data mining problem by using linguistic data summarization (LDS) techniques.

Frequently, companies have large databases, that contains heterogeneous data and difficult to understand. In this context, Kacprzyk [8] and other authors develop linguistic data summarization algorithms. This technique is oriented to produce linguistic summaries in natural language from numeric data. Besides, it will help the organization to solve the dilemma rich data poor information for making decisions.

About linguistic data summarization, Kacprzyk and Zadrożny [8] said "data summarization is one of the basic capabilities that is now needed by any "intelligent" system that is mean to operate in real life". They define a set of six protoforms that describe the structure of the linguistic summaries and queries for their search [9], see Table 1. All summaries are represented in the following two basic structures:

(i) First: summaries without filters Qy's are S, representing relationships such as:
T(Most of projects are renegotiated) = 0.8
(ii) Second: summaries with filters QRy´s are S, which describes relationships such as:
T(Most of the projects with low performance of human resources are renegotiated) = 0.7
Where:

(a) "Q" represents quantifiers such: most, some, a few, etc.
(b) "R" represents filters for example: "high planned material resources", or concepts that influence the objects recovery.
(c) The objects "y" represents the object of study for example "outlier projects".
(d) The "S" represents summarizer such as: "very high "amount of human resources" and concepts under which the "y" objects are grouped in the query.
(e) The "T" represents measures to evaluate the linguistic summaries quality, the most of authors use the T values [9][10].

**Table 1:** Classification of protoforms of LDS, taken from **[11]**.

| Type | Protoform | Given | Sought |
|---|---|---|---|
| 0 | QRy's are S | All | Validity T |
| 1 | Qy's are S | S | Q |
| 2 | QRy's are S | S and R | Q |
| 3 | Qy's are S | Q and structure of S | Linguistic values in S |
| 4 | QRy's are S | Q, R and structure of S | Linguistic values in S |
| 5 | QRy's are S | Nothing | S, R and Q |

In order to evaluate summaries, authors develop different *T* measures [8][11], proposed: degree of truth, degree of imprecision, degree of coverage and the length of the summary.

- Degree of truth (T1): evaluates the truth of summary based on the object's membership to the summary and to summary's quantifier.
- Degree of imprecision (T2): calculates the summary vagueness degree and summary imprecision´s degree by considering the alternative values for each summarizer.
- Degree of coverage (T3): calculates the objects relative frequency that belongs to summarizer´s fuzzy set and to the filter´s fuzzy sets.
- Degree of appropriateness (T4): measures the usefulness of the summary, combining the coincidence relative frequency between the objects, and the summary with the degree of coverage. This measure reports low values with high values of coverage degrees.
- Length of summary (T5): measure to get the summary length based on variables number implicated on it. Very large summaries with a high length usually are incompressible.
- Degree of validity (T6): measure to combine the rest of T values based on OWA aggregators.

But some of these measures fail when in a database there are objects with a high vagueness that such as objects with a high level of neutrality respect to the specific fuzzy sets memberships.

Traditional linguistic data summarization techniques do not consider the neutrality on data. The creation of new measures by considering the linguistic summaries neutrality can help to select the best summaries to make decisions. Neutrosophic numbers theory extends the fuzzy logic theory and helps improve neutrality treatment.

Neutrosophy was introduced by Smarandache in 1995 [12] and in this theory is essentially the definition of neutrosophic sets defined by Smarandache and Wang et al. in [13]. The use of neutrosophic theory in linguistic data summarization techniques allows the introduction of concepts of indetermination; also, improve the interpretability of the summaries [14].

The aim of this work extends measures to evaluate linguistic summaries by considering different elements of neutrosophic theory. In the work are applied the new measures and traditional measures to evaluate linguistic summaries on project management environment.

The remaining of the paper is structured as follows. Section 2 describes preliminary concepts and notation of linguistic data summarization techniques and neutrosophic theory. In section 3, the authors present different extensions to traditional measures proposed by Zadeh; they introduce neutrosophic sets and other concepts into the computation of quality´s measures of the linguistic summaries. In section 4, authors compare the results of measures in a project management environment. Finally, the paper ends with conclusions and further work recommendations in Section 5.

## 2 Preliminary concepts and notation

In this section, authors present preliminary concepts associated with this work. The first subsection discusses in details the measures proposed for Zadeh to evaluate linguistic summaries. In second subsection authors present preliminary concepts associated to neutrosophic theory useful in linguistic data summarization environments.

### 2.2 Measures to evaluate the linguistic summaries quality, based on traditional fuzzy sets

Different authors have been proposed measures to evaluate linguistic summaries quality. In this sense, a set of measures proposed by Zadeh are well known. In this section, the measures proposed by Zadeh [15] are described in details.

The degree of truth (T1) is a measure of how much data supports a linguistic summary.

For summaries with the structure "*Qy's are S"* can be used equation (1), while for summaries with structure "*QRy's are S*" when *R* is a filter, can be used equation (2).

$$\mathrm{T}(\mathrm{Qy's\ are\ S}) = \mu_{\mathrm{Q}}\left[\frac{1}{\mathrm{n}}\sum_{\mathrm{i}=1}^{n}\mu_S(\mathrm{y_i})\right] \quad (1)$$

$$\mathrm{T}(\mathrm{QRy's\ are\ S}) = \mu_{\mathrm{Q}}\,[\mathrm{r}] \quad (2)$$

Where

$$\mathrm{r} = \frac{\sum_{\mathrm{i}=1}^{n}\mu_R(\mathrm{y_i}) \cap \mu_S(\mathrm{y_i})}{\sum_{\mathrm{i}=1}^{n}\mu_R(\mathrm{y_i})} \quad (3)$$

Degree of imprecision (T2) is a useful validity criterion. Basically, a vague linguistic summary has a T2 with a very high degree of truth, but it is not a relevant summary (for example, on almost projects with low-performance indicators are bad evaluated).

$$\mathrm{T_2} = 1 - \sqrt[\mathrm{m}]{\prod_{j=1,\dots,m} in(S_j)} \quad (4)$$

Where *m* is the implicated summarizers number in the summary and *in(S$_j$)* is defined as:

$$\mathrm{in}(\mathrm{S_j}) = \frac{\mathrm{card}\{\mathrm{x} \in \mathrm{X_j} : \mu_{S_j}(x) > \varepsilon\}}{cardX_j} \quad (5)$$

Equation (5) measure the cardinality of the corresponding set and all $X_j$ domains. That is, the more "flat" the diffuse $S_j$ set is, the higher the value of in ($S_j$).

The degree of imprecision T2 depends on the summary form; its calculation does not involve all records on the database, for this reason, does not require searching the database.

The degree of coverage (T3) is defined by:

$$\mathrm{T_3} = \frac{\sum_{\mathrm{i}=1}^{n}\mathrm{t_i}}{\sum_{\mathrm{i}=1}^{n}\mathrm{h_i}} \quad (6)$$

Where:

$$\mathrm{t_i} = \begin{cases} 1 & \text{if} \quad \mu_S(\mathrm{y_i}) > \varepsilon \text{ and } \mu_{w_g}(\mathrm{V_g}(\mathrm{y_i})) > \varepsilon \\ 0 & \text{in another case} \end{cases} \quad (7)$$

$$h_i = \begin{cases} 1 & \text{if} \quad \mu_{w_g}(V_g(y_i)) > \varepsilon \\ 0 & \text{in another case} \end{cases} \quad (8)$$

The degree of appropriateness, T4, describes how relevant is the summary for the particular environment represented by objects in the database and is defined as:

$$T_4 = \text{abs}\left( \prod_{j=1,\ldots,m} r_j - T_3 \right) \quad (9)$$

Where:

$$r_j = \frac{\sum_{i=1}^{n} h_i}{n} \qquad j = 1,\ldots, n$$

$$h_j = \begin{cases} 1 & \text{if} \quad S_j(y_i) > \varepsilon \\ 0 & \text{in another case} \end{cases}$$

The length of the summary (T5) can be defined as:

$$T_5 = 2\left(0.5^{\text{card}\,S}\right) \quad (10)$$

Finally, the total degree of validity, (T6), could be calculated by using different operators of aggregation, for example in [15], this indicator is defined as the weighted average of the previous 5 degrees of validity, i.e.

$$T_{LS} = \sum_{i=1}^{k} w_i T_i \quad (11)$$

Total validity of a linguistic overview, where:

- $k$ is the quantity of T that is calculated, in this case, there are five, that is, from T1 to T5.
- $w$ is weight assigned for the aggregation of the T, therefore $i = [1,\ldots,5]$. Each weight is a values between [0,1].

The combination of $T$ values is very useful to detect the most relevant summaries. To find the optimal summary for an $S^* \in \{S\}$ would be:

$$S^* = \arg\left( \max_{LS=1,\ldots,k} T_{LS} \right) \quad (12)$$

Where $k$ is the total number of language overviews generated and *arg* is a function that returns the language summary obtained as a result of the operation.

Art state study of "linguistic data summarization" led to the following partial conclusions:

- The T values proposed does not consider the indeterminacy or the falsity of objects respect to different fuzzy sets memberships. In this sense objects with high indeterminacy could be considered with the same weight during calculation than objects with high membership value and low indeterminacy.
- In many T values calculation, are consider all elements with memberships value greater than 0. But this condition is not so good because this approach considers objects with very low memberships as the same relevance as objects with the highest membership. Authors of this paper consider as necessary to limit the calculation just for objects with membership values greater than an epsilon value.
- In particular, there are different scenarios with high vagueness and ambiguous concepts where is necessary taking into account elements as neutrality and the uncertainty of concept to making decisions process. For example, in project management some time the experts do not have a definitive opinion about a decision and they have to take neutral positions before a definitive decision.
- Authors of this paper consider that the aggregation of different $T$ values proposed in T6 (total degree of validity), in some cases create a noise in the selection of summaries, and recommend the use of Pareto approach [16].

Next subsection presents preliminary concepts of neutrosophic theory necessaries to introduce the extensions to traditional T values.

### 2.3 Preliminary concepts about the neutrosophic theory

In [17][18] Smarandache introduced the concept of neutrosophic set and neutrosophic logic, which allows handling efficiently the indeterminate and inconsistent information. Neutrosophic set is a generalization of the theory of fuzzy set [19], intuitionistic fuzzy sets, interval-valued fuzzy sets [20] [21] and interval-valued intuitionistic fuzzy sets [6]. A neutrosophic set has the three following degrees: truth-membership degree, indeterminacy-membership degree, and a falsity-membership degree. All these degrees are in the interval [-0, 1+].

However, the neutrosophic theory is difficult to be directly applied in real scientific and engineering areas. For this reason, Smarandache [17] proposed the neutrosophic set theory, which is the more general form of intuitionistic fuzzy logic, whose functions of truth, indeterminacy, and falsity lie in [0, 1]. Since then, publications on neutrosophic set theory and its applications in several fields have been increasing in recent years; this is evidenced by the works presented in [22] [23] [24] [25] [26] [27].

For this work is particularly important the definition 1 of neutrosophic sets defined by Smarandache and Wang et al. in [13], [12], [27].

**Definition 1.** Let $M$ a neutrosophic set in universe $X$ characterized by a triple $(Label, X, \mu_M(x), \tau_M(x), \sigma_M(x))$ where: *Label* is a linguistic term which represents the name of set, X represents the universe of discourse, $\mu_M(x) \in [0,1]$ represents a membership function, $\tau_M(x) \in [0,1]$ represents a indeterminacy-membership function and $\sigma_M(x) \in [0,1]$ represents a falsity-membership function, where $0 \leq \mu_M(x) + \tau_M(x) + \sigma_M(x) \leq 3$.

This definition implies that each value of the domain $x \in X$ when evaluated in neutrosophic set $M$, such that $M(x)$ returns the value $(\mu_M(x), \tau_M(x), \sigma_M(x))$ where the first component represents the membership degree of the value x to the set M, the second component represents the indetermination degree of the value x to the set $M$ and the third component means the non-membership degree of the value $x$ to the set $M$.

$$\mu_A(x) = \begin{cases} (x-a)u_A/(b-a) & (a \leq x < b) \\ u_A & (x = b) \\ (c-x)u_A/(c-b) & (b < x \leq c) \\ 0 & otherwise \end{cases}$$

$$\tau_A(x) = \begin{cases} \big(b - x + v_A(x-a)\big)/(b-a) & (a \leq x < b) \\ r_A & (x = b) \\ \big(x - b + v_A(c-x)\big)/(c-b) & (b < x \leq c) \\ 1 & otherwise \end{cases}$$

$$\sigma_A(x) = \begin{cases} \big(b - x + f_A(x-a)\big)/(b-a) & (a \leq x < b) \\ f_A & (x = b) \\ \big(x - b + f_A(c-x)\big)/(c-b) & (b < x \leq c) \\ 1 & otherwise \end{cases}$$

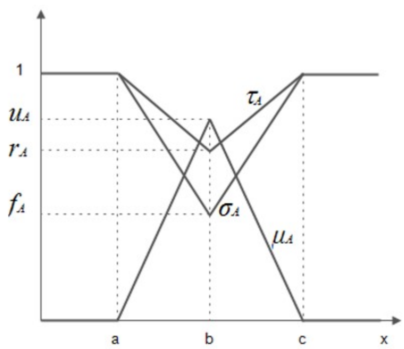


**Figure 1:** Representation of a fuzzy set incorporating the truth-membership ($\mu_A$), indeterminacy-membership ($\tau_A$) and falsity-membership ($\sigma_A$) functions.

Single Valued Neutrosophic Set (SVNS) concept permits the application of neutrosophic set theories on real scientific and engineering applications [13], see definition 2. Many studies have been done on this theory and have been used in many application fields. In this theory, the values of truth, falsity, and indeterminacy of a situation are considered. Many uncertainties and complex situations arise in decision-making applications.

**Definition 2.** Let $X$ be a set of objects and $x \in X$ represents a single valued neutrosophic number (SVN) and is characterized by a vector $(V, I, F)$ where $V$ indicates truth-value, $I$ indeterminacy-value and $F$ falsity-value.

Other important group of definitions are proposed by Subas [28]. He defines a single valued triangular neutrosophic number $x = ((a, b, c), \mu_A(x), \tau_A(x), \sigma_A(x))$ where:

$a, b, c \in \mathfrak{R}^+$ if $x$ is a positive single valued triangular neutrosophic number and

$a, b, c \in \mathfrak{R}$ if $x$ is a negative single valued triangular neutrosophic number.

In neutrosophic theory, different authors define operations between single value neutrosophic numbers [29] [30] [18] as follows:

Let $A$ be a variable represented by number $((a_1, a_2, a_3); u_A, r_A, f_A)$ and $B$ is represented by number $((b_1, b_2, b_3); u_B, r_B, f_B)$

Sum: $A (+) B= ((a_1+b_1, a_2+b_2, a_3+b_3); T(u_A, u_B), S_1(r_A, r_B), S_2(f_A, f_B))$ (13)

Difference: $A (-) B= ((a_1-b_3, a_2-b_2, b_3-a_1), T(u_A, u_B), S_1(r_A, r_B), S_2(f_A, f_B))$ (14)

Product: lets $\pounds = \{a_1b_1, a_3b_3, a_1b_3, a_3b_1\}$
*where* $\lambda_1$ : is the minimum value of $\pounds$, $\lambda_2$ : be the largest of $\pounds$
$A\ (*)\ B = ((a_1b_1, a_1a_2, b_1bc_2); T(u_A, u_B), S_1(r_A, r_B), S_2(f_A, f_B))$ (15)
Division: lets $\pounds = \{a_1/b_1, a_3/b_3, a_1/b_3, a_3/b_1\}$
*where* $\lambda_1$ : is the minimum value of $\pounds$, $\lambda_2$ : be the largest of $\pounds$
$A\ (/)\ B = ((\lambda_1, a_1b_2, \lambda_2); T(u_A, u_B), S_1(r_A, r_B), S_2(f_A, f_B))$ (16)

Let x = (a, b, c, *u, r, f*) be a single valued triangular neutrosophic number [5] [6][7]:

Score: $sc(x) = a+1- b+1- c;$ (17)

Certainty: $ac(x) = a - c;$ (18)

Let x = (a1, a2, a3) and y = (b1, b2, b3) be two single valued neutrosophic numbers, the comparison approach can be defined as follows [5]:

If sc(x)>sc(y), then x is greater than y and denoted x ≻y.

If sc(x)=sc(y) and ac(x)>ac(y), then x is greater than y and denoted x ≻y.

If sc(x)=sc(y) and ac(x)=ac(y), then x is equal to y and denoted by x~y.

T-Norm function TNorm: [0,1] x [0,1] → [0,1] Example (min) (19)
$T(a, b) = T(b, a)$ Commutativity
$T(T(a, b), c) = T(a, T(b, c))$ Associativity
Si $a \geq b$ y $c \geq d$ then $T(a, c) \geq T(b, d)$ Monotony
$T(a, 1) = a$ Neutral element
Conorma function SNorm: [0,1] x [0,1] → [0,1] Example (max) (20)
$S(a, b) = S(b, a)$ Commutativity
$S(S(a, b), c) = S(a, S(b, c))$ Associativity
Si $a \geq b$ y $c \geq d$ entonces $S(a, c) \geq S(b, d)$ Monotony
$S(a, 0) = a$ Neutral element

All these operations are necessary to extend the measures to evaluate the quality of linguistic summaries.

## 3 Extensions to T-values to evaluate linguistic summaries based on neutrosophic numbers

In this section, different extensions to traditional T–values are proposed.
Inspired in rough sets theory [31] the authors of this work propose the following equations and notation. Lets $Y_{A*}$ the set of objects with memberships to neutrosophic set *A* (definition 2) greater than alpha-cut: where $\alpha \in [0,1]$ and $Y_A^*$ the set of objects with memberships to neutrosophic set *A* greater than 0.

For summaries, *A* with structure "*Qy's are S*" see equation (21), (22), (23), (24), (25) and (26)

$$\mu_{SUMMARY}(y_i) = \text{TNorm}\left(\mu_{S_j}(y_i)\right) \quad (21)$$

$$\tau_{SUMMARY}(y_i) = \text{TNorm}\left(\tau_{S_j}(y_i)\right) \quad (22)$$

$$\sigma_{SUMMARY}(y_i) = \text{TNorm}\left(\sigma_{S_j}(y_i)\right) \quad (23)$$

$$Y_A{*} = \{\, y_i : \mu_{SUMMARY}(y_i) \geq 0 \,\} \quad (24)$$

$$Y_{A*} = \{\, y_i : \mu_{SUMMARY}(y_i) \geq \alpha \,\} \quad (25)$$

For summaries, *A* with structure "*QRy's are S*" see equation (26), (27), (28), (29) and (30)

$$\mu_{SUMMARY}(y_i) = \text{TNorm}\left(\mu_R(y_i), \mu_S(y_i)\right) \quad (26)$$

$$\tau_{SUMMARY}(y_i) = \text{TNorm}\left(\tau_R(y_i), \tau_S(y_i)\right) \quad (27)$$

$$\sigma_{SUMMARY}(\mathrm{y_i}) = \mathrm{TNorm}(\sigma_R(\mathrm{y_i}), \sigma_S(\mathrm{y_i})) \quad (28)$$

$$\mathrm{Y_A}^* = \{\mathrm{y_i} : \mu_{\mathrm{SUMMARY}}(\mathrm{y_i}) \geq 0\} \quad (29)$$

$$\mathrm{Y_{A*}} = \{\mathrm{y_i} : \mu_{\mathrm{SUMMARY}}(\mathrm{y_i}) \geq \alpha\} \quad (30)$$

Then T1 applied to summary *A* is extended $\mathrm{Te_{1a}}$ as equation (31):

$$\mathrm{Te_{1a}} = \mu_{\mathrm{Q}}\left[\frac{1}{\mathrm{n}} \sum_{\mathrm{y_i} \in Y_{A*}} \mu_{SUMMARY}(\mathrm{y_i})\right] \quad (31)$$

Another T1 extension on summary *A* is $\mathrm{Te_{1b}}$ as equation (32):

$$\mathrm{Te_{1b}} = \mu_{\mathrm{Q}}\left[\frac{|\mathrm{Y_{A*}}|}{\mathrm{n}}\right] \quad (32)$$

To complement T1 extension a new metric called precision is introduced, $\mathrm{Te_{1c}}$ as equation (33)

$$\mathrm{Te_{1c}} = \frac{|Y_{A*}|}{|Y_A{}^*|} \quad (33)$$

T2 extension on summary *A* is $\mathrm{Te_{2a}}$ imprecision degree as equation (34):

$$\mathrm{Te_{2a}} = 1 - \frac{1}{1 + \left(\dfrac{\sum_{y_i \in Y_{A*}} \tau_{SUMMARY}(y_i) + \sum_{y_i \in Y_{A*}} \sigma_{SUMMARY}(y_i)}{1 + \sum_{y_i \in Y_{A*}} \mu_{SUMMARY}(y_i)}\right)^2} \quad (34)$$

To complement $\mathrm{Te_{2a}}$ authors introduce a metric degree of indeterminacy called $Te_{2b}$ as equation (35)

$$\mathrm{Te_{2b}} = \frac{\sum_{y_i \in Y_{A*}} \tau_{SUMMARY}(y_i)}{|Y_{A*}|} \quad (35)$$

To complement $\mathrm{Te_{2a}}$ authors introduce a metric degree of falsity called $Te_{2c}$ as equation (36)

$$\mathrm{Te_{2c}} = \frac{\sum_{y_i \in Y_{A*}} \sigma_{SUMMARY}(y_i)}{|Y_{A*}|} \quad (36)$$

T3 extension on summary *A* is $\mathrm{Te_{3a}}$ as equation (37): $\mathrm{Te_{3a}} = \frac{|\mathrm{Y_{A*}}|}{\mathrm{n}}$ (37)

T4 extension on summary $A$ is $\mathrm{Te}_{4a}$ appropriateness degree as equation (38):

$$\mathrm{Te}_{4a} = \frac{1}{\frac{\left|Y_A^{*}\right|}{1+\left|Y_{A^*}\right|} + \left(\frac{\sum_{y_i \in Y_{A^*}} \tau_{SUMMARY}(y_i) + \sum_{y_i \in Y_{A^*}} \sigma_{SUMMARY}(y_i)}{1+\sum_{y_i \in Y_{A^*}} \mu_{SUMMARY}(y_i)}\right)^2} \quad (38)$$

Introducing a new T called certainty on summary $A$ is $\mathrm{Te}_{4b}$ certainty degree as equation (39)

$$\mathrm{Te}_{4b} = \frac{1}{1 + \left(\frac{\sum_{y_i \in Y_{A^*}} \tau_{SUMMARY}(y_i) + \sum_{y_i \in Y_{A^*}} \sigma_{SUMMARY}(y_i)}{1+\sum_{y_i \in Y_{A^*}} \mu_{SUMMARY}(y_i)}\right)^2} \quad (39)$$

T5 extension on summary $A$ is $\mathrm{Te}_{5a}$ as equation (40):

$$\mathrm{Te}_{5a} = \frac{1}{1+\left(\frac{|R|+|S|-0.15m}{0.2m}\right)^2} \quad (40)$$

Where $m$ is the number of variables, |R| is the cardinality of filters of linguistic summary and |S| is the cardinality of summarizer of a linguistic summary.

In order to select the best summaries, the authors of this work recommend using the Pareto approach, combining all the proposed extensions in the selection. The best summaries will be those has the best value regarding the combination of the proposed T values.

## 6 Results and discussion

To proof the extensions proposed in this work, data related to project management was used. From them, linguistic summaries were generated to make decisions in the project management environment. This environment is characterized by the following elements:

- There are different information systems with a lot of heterogeneous data.
- There are different project management schools [32] that develop good practices through standards as PMBok [33], ISO 21500 [34] and CMMI v1.3 [35, p. 1], but persist difficulties in projects.
- Different studies develop by the Standish Group [36] shows that there are numerous difficulties in projects associated with TIC technologies. Approximately 52% of the projects are renegotiated while just around 33 % corresponds with successful projects.
- In particular, TIC projects are affected by numerous risks due to their high dependence on the creativity and skills of its human resources.
- Among the fundamental causes of project failure are: poor management, inadequacies in planning, control and monitoring processes [37]. These causes can be mitigated if techniques are available for knowledge discovery and analysis of data historical summaries in linguistic form. From this, decision-makers would have easily understandable information to facilitate tasks such as decision analysis, prediction or forecasting [38].
- Different authors point to causes of this phenomenon: poor management and insufficient planning, control and monitoring processes [37]. In this scenario, it is useful to have techniques for linguistic summaries discovery that allow the complex interrelationships between variables to be presented in natural language [39][40][41].

In this scenario, learning from the mistakes and successes contained in project history data becomes a necessity. On the other hand, in the project management scenario, most decision makers are not experts in data mining and require understandable information for decision-making.

In this context, the techniques for linguistic summarization of data are applied as one of the descriptive knowledge discovery techniques, with a promising and interesting approach to producing linguistic summaries from heterogeneous data using natural language.

For all reasons explained, authors of this paper select a Database for Project Management Evaluation (DPME01) from the Research Database Repository of the Project Management Research Group [42]. This database contains:

- 8430 records with 24 attributes represented by real values in [0, 1] interval,
- 4254 projects evaluated as "bad performance",
- 1021 project evaluated as "regular performance" and
- 3155 projects evaluated as "correct performance".

Table 2 contains attributes associated with the database.

**Table 2:** Variables in DPME01 data set.

| Variables | Description |
|---|---|
| cant_comp_alta, cant_comp_media, cant_comp_baja, | Variables to calculate amount of persons by competence levels. |
| cant_rrhh_eval_b, cant_rrhh_eval_m, cant_rrhh_eval_r | Variables to calculate amount of persons for each performance level. |
| time_availability, time_plan, time_real, tptp, tptr, trtr | Variables associated with the availability of time planned and real-time dedicated for human resources. |
| icd, iref (quality) ie, ire, irp (control of time) irl (procurements) irha, irhe, irhf, irht, irrh (human resources) | Main indicators evaluated during the cut of the project, these indicators are associated to control of scheduling. |
| eval_fuzzysystem_advanced_01 | Variable to calculate evaluation of the project. |

The summaries were generated by using the algorithm AprioriUnificatorLDS [43] based on the combination of the apriori algorithm and fuzzy logic techniques. This algorithm generates 79 linguistic summaries that were evaluated by 7 experts and preferences of these experts were consider in final results.

In order to evaluate the summaries, each expert provides his preferences through a vector $X = (x_j^{ki}, x_j^{ki}, \ldots, x_j^{k})$, where $x_j^{ki}$ represents the preference of expert $e_i$ about summary $j$ and considering the criterion $c_k$. Later the preferences of experts are aggregated by using the computing with words technique 2-tuples [44]. The criteria used to evaluate the summaries were: level of novelty, complexity, simplicity, relevance for making decision. In this work authors use specifically, weighted average operator, to combine the preferences of experts. The summaries with high relevance for experts were:

**First summary (O1):** Many "projects" with (Around 50% "quantity of human resources with high competences") or (Around 50% "quantity of human resources with low competences") or (High "quantity of human resources with bad evaluation ") or (Mean "quantity of human resources with good evaluation ") or (Mean "quantity of human resources with Regular evaluation ") have Bad "Performance indicator".

Zadeh quality of summary: T(0.872, 0.29, 0.748, 0.04, 1, 0.74)
T extended quality of summary: T(0.976, 0.68, 0.706, 0.56, 0.589, 0.503, 0.618, 0.205, 0.062, 0.915)

This summary indicates that human resource performance has a high influence on project evaluation. Summary explains that there are many projects with bad evaluation having a bad performance of its human resources too. Organizations that develop these projects have to improve human resource control.

About its T-values: The *T*-values proposed by Zadeh presents this summary with low appropriateness than *T*-values based on neutrosophic. In this case, the project manager's preferences are closer to *T*-values based on neutrosophic than Zadeh´s *T*-values. The truth values in *T*-values, based on neutrosophic, report better results than Zadeh´s *T*-values respect to nearness to project managers' preferences.

**Second summary (O2):** Around 50% of "projects" with (High "quantity of "human resources with bad evaluation") have (Perfect "real time of real work").

Zadeh quality of summary: T(0.610, 0.83, 0.209, 0.04, 1, 0.42)
T extended quality of summary: T(0.979, 0.57, 0.880, 0.441, 0.309, 0.049, 0.504, 0.925, 0.458, 0.915)

This summary has a high degree of truth and it explains that in Around 50% of projects with "High" quantity of human resources bad evaluated to have a "real time of real work" "Perfect". Besides, this summary states that in these projects there are false statements of real-time dedicated. The Project Management Office (PMO) that control these projects have to improve the control of time declared by human resources and to analyze with Project managers and team leaders the false declarations.

**Third summary (O3):** Around 50% of "projects" with (Regular "useful performance of human resources ") or (Low "quantity of human resources well evaluated") or (Regular "performance of human resources ") or (Mean "time availability ") have (Bad "efficacy").

Zadeh quality of summary: T(0.863, 0.29, 0.802, 0.09, 1, 0.76)
T extended quality of summary: T(0.965, 0.56, 0.655, 0.637, 0.609, 0.574, 0.539, 0.189, 0.062, 0.915)

This summary indicates that human resources have a high influence on project quality. In situations with low performance of human resources or with low available time, then it affected the quality of the project frequently. In this case, the project manager has to check the dedicated time of human resources and the quality of the project.

**Four summary (O4):** Around 50% of "projects" with (Bad "performance indicator ") or (Low "quantity of human resources well evaluated ") or (Regular "Project evaluation ") or (Low "quantity of human resources regular evaluated") have (Bad "production on process of project ").

Zadeh quality of summary: T(0.664, 0.96, 0.127, 0.09, 1, 0.41)
T extended quality of summary: T(0.903, 0.686, 0.833, 0.622, 0.259, 0.243, 0.053, 0.972, 0.277, 0.915)

This summary shows that in 50% of projects with a bad evaluation, presents difficulties with its production on a process. This situation should be attending quickly because of would trigger conflicts with clients in the future.

**Five summaries (O5):** Around 50% of "projects" with (Mean "quantity of human resources bad evaluated ") have (Bad "efficacy").

Zadeh quality of summary: T(0.976, 0.98, 0.011, 0.01, 1, 0.5)
T extended quality of summary: T(0.931, 0.766, 0.808, 0.516, 0.242, 0.227, 0.059, 0.976, 0.383, 0.915)

This summary means that projects with bad performance of human resources present serious problems in the quality of the project. Besides; PMO that controls these projects have to elevate the control of quality. In order to elevate the levels of quality, they should have decisions such as: to increase the rewards to human resources, penalize the bad performance of human resources or to contract new workers with better competencies.

**Six summaries (O6):** Few "projects" with (Perfect "dedicated time") have (Around 50% "human resources with low competence").

Zadeh quality of summary: T(0.728, 0.89, 0.103, 0.01, 1, 0.43)
T extended quality of summary: T(0.761, 0.234, 0.764, 0.513, 0.244, 0.076, 0.130, 0.959, 0.486, 0.915)

This summary shows high dependence between human competences and efficiency. In this case, project managers should keep this work to improve the competencies.

In order to compare the results of two methods, authors, create three ranking list of linguistic summaries:

- The first ranking called *"ideal ranking"* represents the order of summaries by considering the preferences of project managers implicated on validations.
- The second ranking contains the order of summaries by considering the T-values from Zadeh.
- The third ranking of summaries represents the order of summaries by considering the T-values based on neutrosophic theory, proposed in this work.

Authors calculate deviations between *"ideal ranking"* with respect to the others by using the least squares method, see equation (41).

$$\text{D(ideal, output)} = \sum_{i=1}^{z} (ideal_i\text{-output}_{\text{i}})^2 \qquad (41)$$

Where $z$ represents the number of summaries obtained and $ideal_i$ $output_i$ represents the position of summary on ranking. The method with a low deviation to "*ideal ranking*" represents the method with better results. The results of the T exposed above of the six summaries analyzed are presented in table 3.

**Table 3:** Comparisons of rankings, in algorithms outputs.

| Ideal ranking, considering project managers | Ranking based on T-values of Zadeh | Ranking based on neutrosophic T-values |
|---|---|---|
| 1 | 2 | 2 |
| 2 | 5 | 1 |
| 3 | 3 | 4 |
| 4 | 6 | 5 |
| 5 | 1 | 3 |
| 6 | 4 | 6 |
| Least squares measure of deviation | 5.83 | 2.83 |

Results of comparison permit to identify that ranking obtained from proposed *T*-values is closer to *"ideal ranking"* than ranking based on *T*-values of Zadeh. The *T*-values proposed permits to evaluate the indeterminacy and the falsity of the membership of objects to linguistic summaries while *T*-values of Zadeh does not permits to evaluate these values.

The *T*-values based on neutrosophic evaluate more dimensions of summaries and report more data useful to select the relevant summaries. For example, the combination of $Te_{1a}$, $Te_{1b}$, $Te_{1c}$ (equations 31, 32, and 33 respectively) reports more information than *T1* proposed by Zadeh associated with the truth of summary.

The length of the summary called *T5* proposed by Zadeh does not consider the number of variables implicated on search while the $Te_5$ (equations 40) consider the number of variables. Indicator $Te_5$ is represented by a bell function with a better behavior than the exponential function proposed by Zadeh.

## Conclusion

The use of neutrosophic theory in linguistic data summarization techniques permits the introduction of the indeterminacy concepts on linguistic summaries and permits to improve the interpretability of summaries.

The incorporation of neutrosophic sets in *T* certainty calculation allows having a fairer notion about the certainty of the objects of the summary.

Ranking of summaries obtained from *T*-values based on neutrosophic is closer to "ideal ranking" than ranking based on *T*-values of Zadeh.

The *T*-values proposed permits to evaluate the indeterminacy and the falsity of the membership of objects to linguistic summaries while *T*-values of Zadeh do not permits to evaluate these values.

The *T*-values based on neutrosophic evaluate more dimensions of summaries and report more data useful to select the relevant summaries.

The incorporation of the alpha-cut value avoids recovering objects with low influence in summaries into the T-values calculation.

Experts consider neutrosophic T-values more expressiveness than traditional T-values. The application of linguistic data summarization techniques combined with neutrosophic numbers in the project management environment reports good results and should be applied in future works too.

Summaries obtained permit to project managers and to PMO personal improve the decisions. The use of neutrosophic theory combined with linguistic data summarization techniques constitutes a new area of investigations.

Summaries show high dependence between human competences and efficiency. Summaries obtained permit detection in some projects of false declarations of "real time dedicated" indicator and permit to increase the control of projects with these difficulties.

Summaries obtained help to detect project with serious problems in the indicator "production on process" and to emit alerts to PMO personal about these projects.

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# Gaussian single-valued neutrosophic numbers and its application in multi-attribute decision making

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**Abstract:**The fuzzy set and intuitionistic fuzzy set are two useful mathematical tool for dealing with impression and uncertainty. However sometimes these theories may not suffice to model indeterminate and inconsistent information encountered in real world. To overcome this insufficiency, neutrosophic set theory and single-valued neutrosophic set (SVNS) theory which is useful in practical applications, were proposed. Many researchers have studied on single-valued triangular neutrosophic numbers and single-valued trapezoidal neutrosophic numbers. In this paper, concepts of Gaussian single-valued neutrosophic number (GSVNN), $\alpha$-cut of a GSVNN and parametric form of a GSVNN are defined, and based on $\alpha$-cuts of GSVNNs, arithmetic operations for GSVNNs are defined. Also, some results are obtained related to arithmetic operations of GSVNNs. Furthermore, a decision making algorithm is developed by using GSVNNs operations, and its an application in medical diagnosis is given.

**Keywords:** Neutrosophic set, Single-valued neutrosophic number, Gaussian single-valued neutrosophic number, $\alpha-$cut, decision making

# 1 Introduction

The concept of fuzzy set was defined by Zadeh [38] in 1965. A fuzzy set $A$ on a fixed set $X$ is character-ized by membership function denoted by $\mu_A$ such that $\mu_A : A \to [0, 1]$. In 1976, Sanchez [32] proposed a method to solve basic fuzzy relational equations, and in [33] he gave a method for medical diagnosis based on composition of fuzzy relations. In 2013, ¸Celik and Yamak [11] applied the fuzzy soft set theory to Sanchezs approach for medical diagnosis by using fuzzy arithmetic operations, and presented a hypothetical case study to illustrate process of proposed method. Concept of Gaussian fuzzy number and its $\alpha-$cuts were defined by Dutta and Ali [13]. Garg and Singh [15] suggested the numerical solution for fuzzy system of equations by using the Gaussian membership function to the fuzzy numbers considering in its parametric form. In 2017, Dutta and Limboo [14] introduced a new concept called Bell-shaped fuzzy soft set, and gave some applications of this set in medical diagnosis based on C¸ elik and Yamak's work [11].

The concept of neutrosophic set, which is a generalization of fuzzy sets [38], intuitionistic fuzzy sets [1], was introduced by Smarandache [34] to overcome problems including indeterminate and inconsistent information. A neutrosophic set is characterized by three functions called truth-membership function $(T(x))$, indeterminacy-membership function $(I(x))$ and falsity membership function $(F(x))$. These functions are real standard or nonstandard subsets of $]^{-}0, 1^{+}[$. In some areas such as engineering and real scientific fields, modeling of some problems is difficult with real standard or nonstandard subsets of $]^{-}0, 1^{+}[$. To make a success of this difficulties, concepts of single-valued neutrosophic set (SVNS) and interval neutrosophic set (INS) were in-troduced by Wang et al. in [35] and [36]. Recently, many researchers have studied on concept of single-valued neutrosophic numbers, which are a special case of SVNS, and is very important tool for multi criteria decision making problems. For example, Liu et al. [19] proposed some new aggregation operators and presented some new operational laws for neutrosophic numbers (NNs) based on Hamacher operations and studied their prop-erties. Then, they proposed the generalized neutrosophic number Hamacher weighted averaging (GNNHWA) operator, generalized neutrosophic number Hamacher ordered weighted averaging (GNNHOWA) operator, and generalized neutrosophic number Hamacher hybrid averaging (GNNHHA) operator, and explored some properties of these operators and analyzed some special cases of them. Biswas et al. [4] studied on trape-zoidal fuzzy neutrosophic numbers and its application in multi-attribute decision making Deli and Subaş[16] defined single-valued triangular neutrosophic numbers (SVTrNN) and proposed some new geometric operators for SVTrNNs. They also gave MCDM under SVTrN information based on geometric operators of SVTrNN. In [17], Deli and Subaş defined $\alpha-$cut of SVNNs to apply the single-valued trapezoidal neutrosophic num-bers (SVTNNs) and SVTrNNs, then they used these new concepts to solve a MCDM problem. Also, many researchers studied on applications in decision making and group decision making of neutrosophic sets and their some extensions and subclasses, based on similarity measures [37, 23, 24, 25, 21, 27], TOPSIS method [26, 5, 7, 10, 39], grey relational analysis [2, 12], distance measure [8], entropy [28], correlation coefficient [18] and special problem in real life [3, 6, 9, 30, 31, 20, 22].

The SVTrNNs and SVTNNs are useful tool indeterminate and inconsistent information. However, in some cases obtained data may not be SVTrN or SVTN. Therefore, in this study, a new kind of SVNNs called Gaussian single-valued neutrosophic numbers (GSVNNs) is introduced. Also, $\alpha-$cut, parametric form of GSVNNs, and arithmetic operations of GSVNNs by using $\alpha-$cuts of GSVNNs are defined, and some results are obtained related to $\alpha-$cut of GSVNNs. Furthermore, based on Çelik and Yamak's work in [11] and Dutta and Limboo's work in [14], a decision making method is proposed for medical diagnosis problem. Finally, a hypothetical case study is given to illustrate processing of the proposed method.

# 2 Preliminaries

## 2.1 Single-valued neutrosophic sets

A neutrosophic set $\tilde{a}$ on the universe of discourse $X$ is defined as follows:

$$\tilde{a} = \left\{ \langle x, a_t(x), a_i(x), a_f(x)) \rangle : x \in X \right\}$$

where $a_t$, $a_i$, $a_f$: $X \to ]^{-}0, 1^{+}[$ and $^{-}0 \leq a_t(x) + a_i(x) + a_f(x) \leq 3^{+}$ [34]. From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]^{-}0, 1^{+}[$. In some real life applications, modeling of problems by using real standard or nonstandard subsets of $]^{-}0, 1^{+}[$ may not be easy sometimes. Therefore concept of single valued neutrosophic set (SVN-set) was defined by Wang et al. [36] as follow:

Let $X \neq \emptyset$, with a generic element in $X$ denoted by $x$. A single-valued neutrosophic set ($SVNS$) $\tilde{a}$ is characterized by three functions called truth- membership function $a_t(x)$, indeterminacy-membership function $a_i(x)$ and falsity-membership function $a_f(x)$ such that $a_t(x)$, $a_i(x)$, $a_f(x) \in [0, 1]$ for all $x \in X$.

If $X$ is continuous, a $SVNS$ $\tilde{a}$ can be written as follows:

$$\tilde{a} = \int_X \langle a_t(x), a_i(x), a_f(x) \rangle / x, \quad \text{for all } x \in X.$$

If $X$ is crisp set, a $SVNS$ $\tilde{a}$ can be written as follows:

$$\tilde{a} = \sum_x \langle a_t(x), a_i(x), a_f(x) \rangle / x, \quad \text{for all } x \in X.$$

Here $0 \leq a_t(x) + a_i(x) + a_f(x) \leq 3$ for all $x \in X$. For convenience, a SVNN is denoted by $\tilde{a} = \langle a_t, a_i, a_f \rangle$.

**Definition 2.1. (Gaussian fuzzy number)** A fuzzy number is said to be Gaussian fuzzy number $GFN(\mu, \sigma)$ whose membership function is given as follows:

$$f(x) = \exp(-\frac{1}{2}(\frac{x - \overline{\mu}}{\sigma})^2), -\infty < x < \infty,$$

where $\overline{\mu}$ denotes the mean and $\sigma$ denotes standard deviations of the distribution.

**Definition 2.2. $\alpha$-cut of Gaussian fuzzy number:** Let membership function for Gaussian fuzzy number is given as follows:

$$f(x) = \exp(-\frac{1}{2}(\frac{x - \overline{\mu}}{\sigma})^2).$$

Then, $\alpha$-cut is given $A_\alpha = \left[\overline{\mu} - \sigma\sqrt{-2\log\alpha}, \overline{\mu} + \sigma\sqrt{-2\log\alpha}\right]$

# 3 Gaussian SVN-number:

**Definition 3.1.** A SVN-number is said to be Gaussian SVN-number $GSVNN\Big((\overline{\mu}_t, \sigma_t), (\overline{\mu}_i, \sigma_i), (\overline{\mu}_f, \sigma_f)\Big)$ whose truth-membership function, indeterminacy-membership function and falsity-membership function are given as follows:

$$\varphi(x_t) = \exp\left(-\frac{1}{2}(\frac{x_t - \overline{\mu}_t}{\sigma_t})^2\right),$$

$$\varphi(x_i) = 1 - \left(\exp(-\frac{1}{2}(\frac{x_i - \overline{\mu}_i}{\sigma_i})^2\right),$$

$$\varphi(x_f) = 1 - \left(\exp(-\frac{1}{2}(\frac{x_f - \overline{\mu}_f}{\sigma_f})^2\right),$$

respectively. Here $\overline{\mu}_t$ $(\overline{\mu}_i, \overline{\mu}_f)$ denotes mean of truth-membership (indeterminacy-membership, falsity-membership) value. $\sigma_t$ $(\sigma_i, \sigma_f)$ denotes standard deviation of the distribution of truth-membership (indeterminacy-membership, falsity-membership) value.

**Example 3.2.** Let $\tilde{A} = GSVNN\Big((0.4, 0.2), (0.6, 0.3), (0.3, 0.1)\Big)$ be Gaussian SVN-number. Then graphics of truth-membership function, indeterminacy-membership function and falsity-membership function of GSVNN $\tilde{A}$ are depicted in Fig 1.

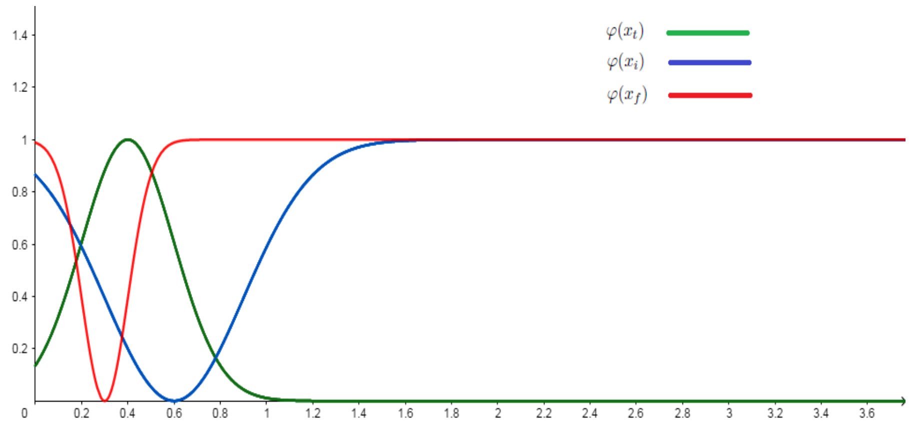


Figure 1: GSVNN $\tilde{A}$

**Definition 3.3.** **$\alpha$-cut of Gaussian SVN-number:** Truth-membership function, indeterminacy-membership function and falsity-membership function for Gaussian SVN-number $\tilde{A}$ are given as follows:

$$\varphi(x_t) = \exp(-\frac{1}{2}(\frac{x_t - \overline{\mu}_t}{\sigma_t})^2),$$

$$\varphi(x_i) = 1 - \exp(-\frac{1}{2}(\frac{x_i - \overline{\mu}_i}{\sigma_i})^2),$$

$$\varphi(x_f) = 1 - \exp(-\frac{1}{2}(\frac{x_f - \overline{\mu}_f}{\sigma_f})^2)$$

respectively.

Then $\alpha$-cuts of them are as follows:

$$A_{t_\alpha} = \Big[\overline{\mu}_t - (\sigma_t\sqrt{-2\log\alpha}), \overline{\mu}_t + (\sigma_t\sqrt{-2\log\alpha})\Big],$$

$$A_{i_\alpha} = \Big[\overline{\mu}_i - (\sigma_i\sqrt{-2\log(1-\alpha)}), \overline{\mu}_i + (\sigma_i\sqrt{-2\log(1-\alpha)})\Big],$$

$$A_{f_\alpha} = \Big[\overline{\mu}_f - (\sigma_f\sqrt{-2\log(1-\alpha)}), \overline{\mu}_f + (\sigma_f\sqrt{-2\log(1-\alpha)})\Big],$$

respectively.

## 3.1 Arithmetic operations of Gaussian SVN-numbers

Let $\tilde{A} = GSVNN\Big((\overline{\mu}_{A_t}, \sigma_{A_t}), (\overline{\mu}_{A_i}, \sigma_{A_i}), (\overline{\mu}_{A_f}, \sigma_{A_f})\Big)$ and $\tilde{B} = GSVNN\Big((\overline{\mu}_{B_t}, \sigma_{B_t}), (\overline{\mu}_{B_i}, \sigma_{B_i}), (\overline{\mu}_{B_f}, \sigma_{B_f})\Big)$ be two Gaussian SVN-numbers. Then their $\alpha$-cuts $(0 < \alpha < 1)$ of these numbers are as follows:

$$A_{t_\alpha} = \Big[\overline{\mu}_{A_t} - (\sigma_{A_t}\sqrt{-2\log\alpha}), \overline{\mu}_{A_t} + (\sigma_{A_t}\sqrt{-2\log\alpha})\Big],$$

$$A_{i_\alpha} = \Big[\overline{\mu}_{A_i} - (\sigma_{A_i}\sqrt{-2\log(1-\alpha)}), \overline{\mu}_{A_i} + (\sigma_{A_i}\sqrt{-2\log(1-\alpha)})\Big],$$

$$A_{f_\alpha} = \Big[\overline{\mu}_{A_f} - (\sigma_{A_f}\sqrt{-2\log(1-\alpha)}), \overline{\mu}_{A_f} + (\sigma_{A_f}\sqrt{-2\log(1-\alpha)})\Big]$$

and

$$B_{t_\alpha} = \Big[\overline{\mu}_{B_t} - (\sigma_{B_t}\sqrt{-2\log\alpha}), \overline{\mu}_{B_t} + (\sigma_{B_t}\sqrt{-2\log\alpha})\Big],$$

$$B_{i_\alpha} = \Big[\overline{\mu}_{B_i} - (\sigma_{B_i}\sqrt{-2\log(1-\alpha)}), \overline{\mu}_{B_i} + (\sigma_{B_i}\sqrt{-2\log(1-\alpha)})\Big],$$

$$B_{f_\alpha} = \Big[\overline{\mu}_{B_f} - (\sigma_{B_f}\sqrt{-2\log(1-\alpha)}), \overline{\mu}_{B_f} + (\sigma_{B_f}\sqrt{-2\log(1-\alpha)})\Big],$$

respectively.

Based on $\alpha$-cuts of $\tilde{A}$ and $\tilde{B}$, arithmetic operations between GSVNN $\tilde{A}$ and GSVNN $\tilde{B}$ are defined as follows:

1. Addition:
$A_{t_\alpha} + B_{t_\alpha} = \Big[(\overline{\mu}_{A_t} + \overline{\mu}_{B_t}) - (\sigma_{A_t} + \sigma_{B_t})\sqrt{-2\log\alpha}, \overline{\mu}_{A_t} + (\sigma_{A_t} + \sigma_{B_t})\sqrt{-2\log\alpha}\Big]$
$A_{i_\alpha} + B_{i_\alpha} = \Big[(\overline{\mu}_{A_i} + \overline{\mu}_{B_i}) - (\sigma_{A_i} + \sigma_{B_i})\sqrt{-2\log(1-\alpha)}, (\mu_{A_i} + \mu_{B_i}) + (\sigma_{A_i} + \sigma_{B_i})\sqrt{-2\log(1-\alpha)}\Big]$,
$A_{f_\alpha} + B_{f_\alpha} = \Big[(\overline{\mu}_{A_f} + \overline{\mu}_{B_f}) - (\sigma_{A_f} + \sigma_{B_f})\sqrt{-2\log(1-\alpha)}, (\mu_{A_f} + \mu_{B_f}) + (\sigma_{A_f} + \sigma_{B_f})\sqrt{-2\log(1-\alpha)}\Big]$,
Truth-membership function, indeterminacy-membership function and falsity-membership function of addition of GSVNNs $\tilde{A}$ and $\tilde{B}$ are as follows:

$$\varphi_{(A+B)}(x_t) = \exp\Big(-\frac{1}{2}(\frac{x_t - (\overline{\mu}_{A_t} + \overline{\mu}_{B_t})}{\sigma_{A_t} + \sigma_{B_t}})^2\Big),$$

$$\varphi_{(A+B)}(x_i) = 1 - \exp\Big(-\frac{1}{2}(\frac{x_i - (\overline{\mu}_{A_i} + \overline{\mu}_{B_i})}{\sigma_{A_i} + \sigma_{B_i}})^2\Big),$$

and

$$\varphi_{(A+B)}(x_f) = 1 - \exp\Big(-\frac{1}{2}(\frac{x_f - (\overline{\mu}_{A_f} + \overline{\mu}_{B_f})}{\sigma_{A_f} + \sigma_{B_f}})^2\Big),$$

respectively.

2. Substraction:
$A_{t_\alpha} - B_{t_\alpha} = \Big[(\overline{\mu}_{A_t} - \overline{\mu}_{B_t}) - (\sigma_{A_t} + \sigma_{B_t})\sqrt{-2\log\alpha}, \overline{\mu}_{A_t} - (\sigma_{A_t} + \sigma_{B_t})\sqrt{-2\log\alpha}\Big]$
$A_{i_\alpha} - B_{i_\alpha} = \Big[(\overline{\mu}_{A_i} - \overline{\mu}_{B_i}) - (\sigma_{A_i} + \sigma_{B_i})\sqrt{-2\log(1-\alpha)}, (\mu_{A_i} - \mu_{B_i}) + (\sigma_{A_i} + \sigma_{B_i})\sqrt{-2\log(1-\alpha)}\Big]$,
$A_{f_\alpha} - B_{f_\alpha} = \Big[(\overline{\mu}_{A_f} - \overline{\mu}_{B_f}) - (\sigma_{A_f} + \sigma_{B_f})\sqrt{-2\log(1-\alpha)}, (\mu_{A_f} - \sigma_{B_f}) + (\sigma_{A_f} + \sigma_{B_f})\sqrt{-2\log(1-\alpha)}\Big]$,
Truth-membership function, indeterminacy-membership function and falsity-membership function of substraction of GSVNNs $\tilde{A}$ and $\tilde{B}$ are as follows:

$$\varphi_{(A-B)}(x_t) = \exp\Big(-\frac{1}{2}(\frac{x_t - (\overline{\mu}_{A_t} - \overline{\mu}_{B_t})}{\sigma_{A_t} + \sigma_{B_t}})^2\Big),$$

$$\varphi_{(A-B)}(x_i) = 1 - \exp\Big(-\frac{1}{2}(\frac{x_i - (\overline{\mu}_{A_i} - \overline{\mu}_{B_i})}{\sigma_{A_i} + \sigma_{B_i}})^2\Big),$$

and

$$\varphi_{(A-B)}(x_f) = 1 - \exp\Big(-\frac{1}{2}(\frac{x_f - (\overline{\mu}_{A_f} - \overline{\mu}_{B_f})}{\sigma_{A_f} + \sigma_{B_f}})^2\Big),$$

respectively.

3. Multiplication:
$A_{t_\alpha}.B_{t_\alpha} = \Big[(\overline{\mu}_{A_t} - \sigma_{A_t}\sqrt{-2\log\alpha})\cdot(\overline{\mu}_{B_t} - \sigma_{B_t}\sqrt{-2\log\alpha}), (\overline{\mu}_{A_t} + \sigma_{A_t}\sqrt{-2\log\alpha})\cdot(\overline{\mu}_{B_t} + \sigma_{B_t}\sqrt{-2\log\alpha})\Big]$
$A_{i_\alpha}.B_{i_\alpha} = \Big[(\overline{\mu}_{A_i} - \sigma_{A_i}\sqrt{-2\log(1-\alpha)})\cdot(\overline{\mu}_{B_i} - \sigma_{B_i}\sqrt{-2\log(1-\alpha)}), (\overline{\mu}_{A_i} + \sigma_{A_i}\sqrt{-2\log(1-\alpha)})\cdot (\overline{\mu}_{B_i} + \sigma_{B_i}\sqrt{-2\log(1-\alpha)})\Big]$
$A_{f_\alpha}.B_{f_\alpha} = \Big[(\overline{\mu}_{A_f} - \sigma_{A_f}\sqrt{-2\log(1-\alpha)})\cdot(\overline{\mu}_{B_f} - \sigma_{B_f}\sqrt{-2\log(1-\alpha)}), (\overline{\mu}_{A_f} + \sigma_{A_f}\sqrt{-2\log(1-\alpha)})\cdot (\overline{\mu}_{B_f} + \sigma_{B_f}\sqrt{-2\log(1-\alpha)})\Big]$

4. Division:
$\frac{A_{t_\alpha}}{B_{t_\alpha}} = \Big[\frac{\overline{\mu}_{A_t} - (\sigma_{A_t}\sqrt{-2\log\alpha})}{\overline{\mu}_{B_t} + (\sigma_{B_t}\sqrt{-2\log\alpha})}, \frac{\overline{\mu}_{A_t} + (\sigma_{A_t}\sqrt{-2\log\alpha})}{\overline{\mu}_{B_t} - (\sigma_{B_t}\sqrt{-2\log\alpha})}\Big],$

$\frac{A_{i_\alpha}}{B_{i_\alpha}} = \Big[\frac{\overline{\mu}_{A_i} - (\sigma_{A_i}\sqrt{-2\log(1-\alpha)})}{\overline{\mu}_{B_i} + (\sigma_{B_i}\sqrt{-2\log(1-\alpha)})}, \frac{\overline{\mu}_{A_i} + (\sigma_{A_i}\sqrt{-2\log(1-\alpha)})}{\overline{\mu}_{B_i} - (\sigma_{B_i}\sqrt{-2\log(1-\alpha)})}\Big],$

$\frac{A_{f_\alpha}}{B_{f_\alpha}} = \Big[\frac{\overline{\mu}_{A_f} - (\sigma_{A_f}\sqrt{-2\log(1-\alpha)})}{\overline{\mu}_{B_f} + (\sigma_{B_f}\sqrt{-2\log(1-\alpha)})}, \frac{\overline{\mu}_{A_f} + (\sigma_{A_f}\sqrt{-2\log(1-\alpha)})}{\overline{\mu}_{B_f} - (\sigma_{B_f}\sqrt{-2\log(1-\alpha)})}\Big].$

## 3.2 Parametric Form of SVN-numbers

A SVN $\tilde{n}$ in parametric form is a triple of pairs $((\underline{n}_t(x), \overline{n}_t(x)), (\underline{n}_i(x), \overline{n}_i(x)), (\underline{n}_f(x), \overline{n}_f(x)))$ of the functions $\underline{n}_t(x), \overline{n}_t(x), \underline{n}_i(x), \overline{n}_i(x), \underline{n}_f(x)$ and $\overline{n}_f(x)$ for $0 \leq x \leq 1$ which satisfies the following conditions.

1. (a) $\underline{n}_t(x)$ is bounded and monotonic increasing left continuous function,
   (b) $\overline{n}_t(x)$ is bounded and monotonic decreasing right continuous function,
   (c) $\underline{n}_t(x) \leq \overline{n}_t(x)$ for $0 \leq x \leq 1$.
2. (a) $\underline{n}_i(x)$ is bounded and monotonic decreasing left continuous function,
   (b) $\overline{n}_i(x)$ is bounded and monotonic increasing right continuous function,
   (c) $\underline{n}_i(x) \leq \overline{n}_i(x)$ for $0 \leq x \leq 1$.
3. (a) $\underline{n}_f(x)$ is bounded and monotonic decreasing left continuous function,
   (b) $\overline{n}_f(x)$ is bounded and monotonic increasing right continuous function,
   (c) $\underline{n}_f(x) \leq \overline{n}_f(x)$ for $0 \leq x \leq 1$.

A SVN-number $\tilde{\alpha} = \langle \alpha_t, \alpha_i, \alpha_f \rangle$ is simply represented by $\underline{n}_t(x) = \overline{n}_t(x) = \alpha_t$, $\underline{n}_i(x) = \overline{n}_i(x) = \alpha_i$ and $\underline{n}_f(x) = \overline{n}_f(x) = \alpha_f$, $0 \leq x \leq 1$. For $\tilde{n} = ((\underline{n}_t(x), \overline{n}_t(x)), (\underline{n}_i(x), \overline{n}_i(x)), (\underline{n}_f(x), \overline{n}_f(x)))$ and $\tilde{m} =$

$((\underline{m}_t(x), \overline{m}_t(x)), (\underline{m}_i(x), \overline{m}_i(x)), (\underline{m}_f(x), \overline{m}_f(x)))$, we may define addition and scalar multiplication as

$$\underline{(n+m)}_t(x) = \underline{(n)}_t(x) + \underline{(m)}_t(x), \quad \underline{(n+m)}_i(x) = \underline{(n)}_i(x) + \underline{(m)}_i(x), \quad \underline{(n+m)}_f(x) = \underline{(n)}_f(x) + \underline{(m)}_f(x)$$

$$\overline{(n+m)}_t(x) = \overline{(n)}_t(x) + \overline{(m)}_t(x), \quad \overline{(n+m)}_i(x) = \overline{(n)}_i(x) + \overline{(m)}_i(x), \quad \overline{(n+m)}_f(x) = \overline{(n)}_f(x) + \overline{(m)}_f(x)$$

and

$$\underline{(cn)}_t(x) = c\underline{n}_i(x), \underline{(cn)}_i(x) = c\underline{n}_i(x), \underline{(cn)}_f(x) = c\underline{n}_f(x), c \neq 0$$
$$\overline{(cn)}_t(x) = c\overline{n}_i(x), \overline{(cn)}_i(x) = c\overline{n}_i(x), \overline{(cn)}_f(x) = c\overline{n}_f(x), c \neq 0,$$
$$\underline{(cn)}_t(x) = c\overline{n}_i(x), \underline{(cn)}_i(x) = c\overline{n}_i(x), \underline{(cn)}_f(x) = c\overline{n}_f(x), c \leq 0$$
$$\overline{(cn)}_t(x) = c\underline{n}_i(x), \overline{(cn)}_i(x) = c\underline{n}_i(x), \overline{(cn)}_f(x) = c\underline{n}_f(x), c \leq 0.$$

If $c = \langle c_t, c_i, c_f \rangle$ is a SVN-value, then $c\tilde{n}$ is defined as follows:

$$\underline{(cn)}_t(x) = c_t\underline{n}_t(x), \underline{(cn)}_i(x) = c_i\underline{n}_i(x), \underline{(cn)}_f(x) = c_f\underline{n}_f(x),$$
$$\overline{(cn)}_t(x) = c_t\overline{n}_t(x), \overline{(cn)}_i(x) = c_i\overline{n}_i(x), \overline{(cn)}_f(x) = c_f\overline{n}_f(x).$$

Let $\tilde{A}$ be a GSVNN as $\varphi_A(x_t) = \exp(-\frac{1}{2}(\frac{x_t-\overline{\mu}_t}{\sigma_t})^2), \varphi_A(x_i) = -\exp(-\frac{1}{2}(\frac{x_i-\overline{\mu}_i}{\sigma_i})^2) + 1$ and $\varphi_A(x_f) = -\exp(-\frac{1}{2}(\frac{x_f-\overline{\mu}_f}{\sigma_f})^2) + 1$. Then, parametric form of GSVNN $\tilde{A}$ can be transformed as

$$\begin{aligned}((\underline{n}_t(x), \overline{n}_t(x)), (\underline{n}_i(x), \overline{n}_i(x)), (\underline{n}_f(x), \overline{n}_f(x))) \;=\; & \Big((\overline{\mu}_t - \sigma_t\sqrt{-2\log\alpha}), \overline{\mu}_t + \sigma_t\sqrt{-2\log\alpha}), \\ & (\overline{\mu}_i - \sigma_i\sqrt{-2\log(1-\alpha)}, \overline{\mu}_i + \sigma_i\sqrt{-2\log(1-\alpha)}), \\ & (\overline{\mu}_f - \sigma_f\sqrt{-2\log(1-\alpha)}, \overline{\mu}_f + \sigma_f\sqrt{-2\log(1-\alpha)})\Big)\end{aligned}$$

**Example 3.4.** Let us consider $(0.5, 0.2, 0.8) = GSVNN((0.5, 0.02), (0.2, 0.05), (0.8, 0.01))$ to be a SVN-number with Gaussian membership functions. Then, its parametric form is as follows:

$$\Big((0.5 - 0.02\sqrt{-2ln(\alpha)}, 0.5 + 0.02\sqrt{-2ln(\alpha)}), (0.2 - 0.05\sqrt{-2ln(1-\alpha)}, 0.2 + 0.05\sqrt{-2ln(1-\alpha)}), (0.8 - 0.01\sqrt{-2ln(1-\alpha)}, 0.8 + 0.01\sqrt{-2ln(1-\alpha)})\Big)$$

**Proposition 3.5.** *Addition of two GSVNNs is a GSVNN. Namely;*

$$(\tilde{A} + \tilde{B})(x_t) = \Big(\mu_A(x_t) + \mu_B(x_t) - (\sigma_{A_t} + \sigma_{B_t})\sqrt{-2\ln(\alpha)}, \mu_A(x_t) + \mu_B(x_t) + (\sigma_{A_t} + \sigma_{B_t})\sqrt{-2\ln(\alpha)}\Big),$$
$$(\tilde{A} + \tilde{B})(x_i) = \Big(\mu_A(x_i) + \mu_B(x_i) - (\sigma_{A_i} + \sigma_{B_i})\sqrt{-2\ln(1-\alpha)}, \mu_A(x_i) + \mu_B(x_i) + (\sigma_{A_i} + \sigma_{B_i})\sqrt{-2\ln(1-\alpha)}\Big),$$
$$(\tilde{A} + \tilde{B})(x_f) = \Big(\mu_A(x_f) + \mu_B(x_f) - (\sigma_{A_f} + \sigma_{B_f})\sqrt{-2\ln(1-\alpha)}, \mu_A(x_f) + \mu_B(x_f) + (\sigma_{A_f} + \sigma_{B_f})\sqrt{-2\ln(1-\alpha)}\Big).$$

*Proof.* The proof is obvious from definition. □

Let us consider GSVNNs $A = GSVNN((0.3, 0.5), (0.4, 0.2), (0.7, 0.1))$ and $B = GSVNN((0.6, 0.2), (0.5, 0.1), (0.4, 0.3))$. Their graphics are shown in Figs (2) and (3)

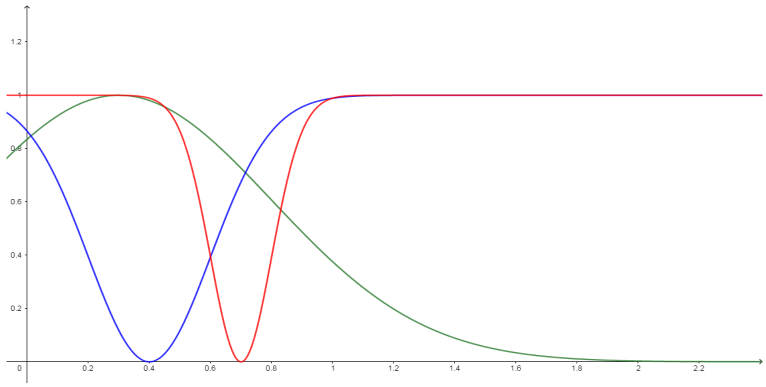

Figure 2: GSVNN $\tilde{A}$

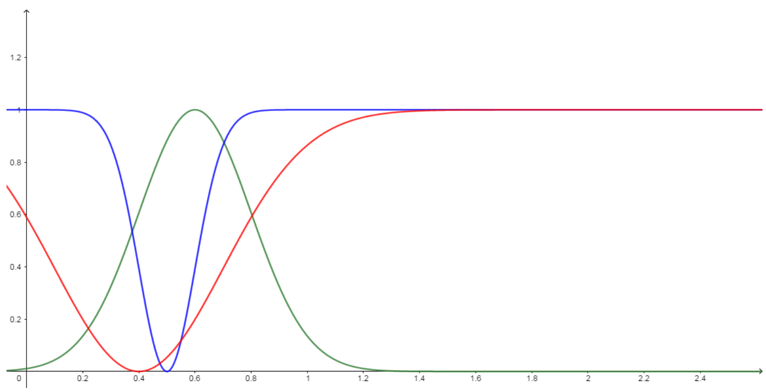

Figure 3: GSVNN $\tilde{B}$

Truth-membership, indeterminacy-membership and falsity-membership function of GSVNN $A+B$ and $A-B$ are as follows:

$$\varphi_{(A+B)}(x_t) = \exp\Big(-\frac{1}{2}(\frac{x-0.9}{0.7})^2\Big),$$
$$\varphi_{(A+B)}(x_i) = 1-\exp\Big(-\frac{1}{2}(\frac{x-0.9}{0.3})^2\Big),$$
$$\varphi_{(A+B)}(x_f) = 1-\exp\Big(-\frac{1}{2}(\frac{x-1.1}{0.4})^2\Big),$$

and

$$\varphi_{(A-B)}(x_t) = \exp\Big(-\frac{1}{2}(\frac{x-0.3}{0.7})^2\Big),$$
$$\varphi_{(A-B)}(x_i) = 1-\exp\Big(-\frac{1}{2}(\frac{x-0.1}{0.3})^2\Big),$$
$$\varphi_{(A-B)}(x_f) = 1-\exp\Big(-\frac{1}{2}(\frac{x-0.3}{0.4})^2\Big).$$

Then, figures of GSVNNs $\tilde{A}+\tilde{B}$ and $\tilde{A}-\tilde{B}$ are as in Figs (4) and (5).

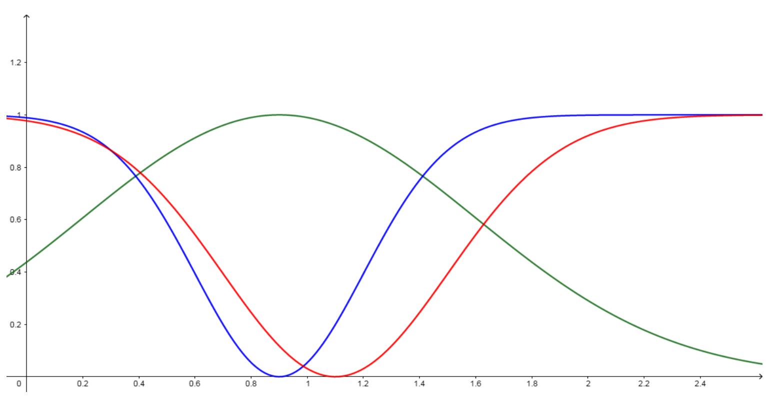

Figure 4: GSVNN $\tilde{A}+\tilde{B}$

# 4 Application of Gaussian SVN-numbers in Medical Diagnosis

Let us consider the decision-making problem adapted from [11].

## 4.1 Method and Algorithm

Let $P=\{p_1,p_2,...,p_p\}$ be a set of patients, $S=\{s_1,s_2,...,s_s\}$ be set of symptoms and $D=\{d_1,d_2,...,d_d\}$ be a set of diseases. Patients $p_i(i=1,2,...,p)$ are evaluated by experts by using Table 1 for each symptom $s_j(j=1,2,...,s)$, and patient-symptom $(PS)$ matrix is given as follows:

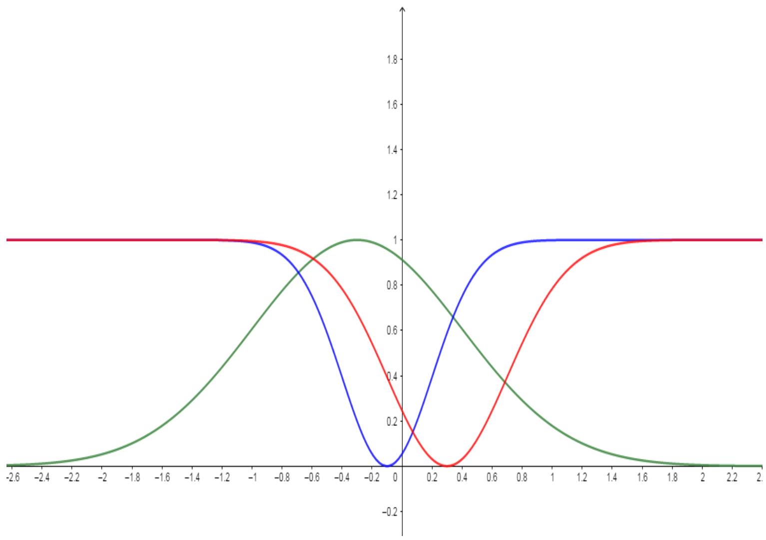

Figure 5: GSVNN $\tilde{A} - \tilde{B}$

$$PS = \begin{pmatrix} \tilde{m}_{11} & \tilde{m}_{12} & \cdots & \tilde{m}_{1s} \\ \tilde{m}_{21} & \tilde{m}_{22} & \cdots & \tilde{m}_{2s} \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{m}_{p1} & \tilde{m}_{p2} & \cdots & \tilde{m}_{ps} \end{pmatrix}$$

Here $\tilde{m}_{ij} = \langle m_{t_{ij}}, m_{i_{ij}}, m_{f_{ij}} \rangle$ denotes SVN-value of patient $p_i$ related to symptom $s_j$.

Table 1: SVN-numbers for linguistic terms

| Linguistic terms | Linguistic values of SVN-numbers |
|---|---|
| Absolutely low(AL) | $\langle 0.05, 0.95, 0.95 \rangle$ |
| Low(L) | $\langle 0.20, 0.75, 0.80 \rangle$ |
| Fairly low(FL) | $\langle 0.35, 0.60, 0.65 \rangle$ |
| Medium(M) | $\langle 0.50, 0.50, 0.50 \rangle$ |
| Fairly high(FH) | $\langle 0.65, 0.40, 0.35 \rangle$ |
| High(H) | $\langle 0.80, 0.25, 0.20 \rangle$ |
| Absolutely high(AH) | $\langle 0.95, 0.10, 0.05 \rangle$ |

Symptoms $s_i (i = 1, 2, ..., s)$ are evaluated with Gaussian SVN-numbers for each disease $d_k (j = 1, 2, ..., d)$, and symptoms-disease $(SD)$ matrix is given as follows:

$$SD = \begin{pmatrix} \tilde{n}_{11} & \tilde{n}_{12} & \cdots & \tilde{n}_{1d} \\ \tilde{n}_{21} & \tilde{n}_{22} & \cdots & \tilde{n}_{2d} \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{n}_{s1} & \tilde{n}_{s2} & \cdots & \tilde{n}_{sd} \end{pmatrix}.$$

Here $\tilde{n}_{jk} = GSVNN\langle (n_{t_{jk}}, \sigma_t), (n_{i_{jk}}, \sigma_i), (n_{f_{jk}}, \sigma_f) \rangle$ denotes Gaussian SVN-value of symptom $s_j$ related to disease $d_k$ $(k = 1, 2, ..., d)$.

Decision matrix $(PD)$ is defined by using composition of matrices $PS$ and $SD$ as follows:

$$PD = \begin{pmatrix} \tilde{q}_{11} & \tilde{q}_{12} & \cdots & \tilde{q}_{1d} \\ \tilde{q}_{21} & \tilde{q}_{22} & \cdots & \tilde{q}_{2d} \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{q}_{p1} & \tilde{q}_{s2} & \cdots & \tilde{q}_{pd} \end{pmatrix} = \begin{pmatrix} \tilde{m}_{11} & \tilde{m}_{12} & \cdots & \tilde{m}_{1s} \\ \tilde{m}_{21} & \tilde{m}_{22} & \cdots & \tilde{m}_{2s} \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{m}_{p1} & \tilde{m}_{p2} & \cdots & \tilde{m}_{ps} \end{pmatrix} \circ \begin{pmatrix} \tilde{n}_{11} & \tilde{n}_{12} & \cdots & \tilde{n}_{1d} \\ \tilde{n}_{21} & \tilde{n}_{22} & \cdots & \tilde{n}_{2d} \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{n}_{s1} & \tilde{n}_{s2} & \cdots & \tilde{n}_{sd} \end{pmatrix}.$$

Here $\tilde{q}_{ik}$ $(i = 1, 2, ..., p; k = 1, 2, ..., d)$ is calculated by

$$(\langle \tilde{u}_t, \tilde{u}_i, \tilde{u}_f \rangle GSVNN\langle (n_{t_{jk}}, \sigma_t), (n_{i_{jk}}, \sigma_i), (n_{f_{jk}}, \sigma_f)\rangle)(x) =$$

$$\begin{aligned} &\langle (u_t n_{t_{jk}} - u_t\sigma_t\sqrt{-2ln(\alpha)}, u_t n_{t_{jk}} + u_t\sigma_t\sqrt{-2ln(\alpha)}), \\ &(u_i n_{i_{jk}} - u_i\sigma_i\sqrt{-2ln(1-\alpha)}, u_i n_{i_{jk}} + u_i\sigma_i\sqrt{-2ln(1-\alpha)}), \\ &(u_f n_{f_{jk}} - u_f\sigma_f\sqrt{-2ln(1-\alpha)}, u_f n_{f_{jk}} + u_f\sigma_f\sqrt{-2ln(1-\alpha)})\rangle. \end{aligned}$$

For the sake of shortness, $(\langle \tilde{u}_t, \tilde{u}_i, \tilde{u}_f \rangle GSVNN\langle (n_{t_{jk}}, \sigma_t), (n_{i_{jk}}, \sigma_i), (n_{f_{jk}}, \sigma_f)\rangle)(x)$ will be denoted by $\langle (\underline{a}, \overline{a}), (\underline{b}, \overline{b}), (\underline{c}, \overline{c})\rangle$.

For obtained parametric forms of Gaussian SVN-numbers, score functions are defined as follows:

$$S_{q_{ik}} = \frac{4 - (\underline{a} - \underline{b} - \underline{c}) + (\overline{a} - \overline{b} - \overline{c})}{6} \tag{4.1}$$

If $\max S_{q_{ik}} = S_{q_{it}}$ for $1 \leq t \leq k$, then it is said that patient $p_i$ suffers from disease $d_t$. In case $\max S_{q_{ik}}$ occurs for more than one value, for $1 \leq t \leq k$, then symptoms can be reassessed.

**Algorithm 1**

Input: The matrix PS (patient-symptom) obtained according to opinion of expert (decision maker)
Output: Diagnosis of disease

# algorithmic

1. Construct matrix $PS$ according to opinions of experts by using Table 1.
2. Construct matrix $SD$ by using GSVNNs.
3. Calculate decision matrix $PD$.
4. Compute score values of elements of decision matrix $PD$.
5. Find $t$ for which $\max S_{q_{ik}} = S_{q_{it}}$ for $1 \leq t \leq k$

# 5 Hypothetical case study

In this section, a hypothetical case study is given to illustrate processing of the proposed method.

There are three patients $p_1, p_2, p_3, p_4$ and $p_5$ who it is considered that they suffer from $d_1$ =viral fever, $d_2$ =tuberculosis, $d_3$ =typhoid, $d_4$ =throat disease or $d_5$ =malaria. In these diseases, common symptoms are $s_1$ =temperature, $s_2$ =cough, $s_3$ =throat pain, $s_4$ =headache, $s_5$ =body pain.
**Step 1:** In the results of observation made by an expert, suppose that matrix $PS$ is as follows:

$$
PS = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{array}
\begin{array}{c}
\begin{array}{ccccc} \quad s_1 \quad & \quad s_2 \quad & \quad s_3 \quad & \quad s_4 \quad & \quad s_5 \quad \end{array} \\
\begin{pmatrix}
\langle 0.95, 0.10, 0.05\rangle & \langle 0.65, 0.40, 0.35\rangle & \langle 0.20, 0.75, 0.80\rangle & \langle 0.20, 0.75, 0.80\rangle & \langle 0.80, 0.25, 0.20\rangle \\
\langle 0.35, 0.60, 0.65\rangle & \langle 0.65, 0.40, 0.35\rangle & \langle 0.50, 0.50, 0.50\rangle & \langle 0.80, 0.25, 0.20\rangle & \langle 0.95, 0.10, 0.05\rangle \\
\langle 0.65, 0.40, 0.35\rangle & \langle 0.50, 0.50, 0.50\rangle & \langle 0.80, 0.25, 0.20\rangle & \langle 0.20, 0.75, 0.80\rangle & \langle 0.80, 0.25, 0.20\rangle \\
\langle 0.20, 0.75, 0.80\rangle & \langle 0.80, 0.25, 0.20\rangle & \langle 0.95, 0.10, 0.05\rangle & \langle 0.35, 0.60, 0.65\rangle & \langle 0.20, 0.75, 0.80\rangle \\
\langle 0.80, 0.25, 0.20\rangle & \langle 0.35, 0.65, 0.65\rangle & \langle 0.05, 0.10, 0.95\rangle & \langle 0.80, 0.25, 0.20\rangle & \langle 0.20, 0.250.80\rangle
\end{pmatrix}
\end{array}
$$

**Step 2:** Suppose that matrix $SD$ is as follows:

$$
SD = \begin{array}{c|ccccc}
 & d_1 & d_2 & d_3 & d_4 & d_5 \\
s_1 & \begin{matrix}\langle(.30, .10),\\(.50, .01),\\(.20, .03)\rangle\end{matrix} & \begin{matrix}\langle(.25, .03),\\(.70, .02),\\(.50, .01)\rangle\end{matrix} & \begin{matrix}\langle(.25, .15),\\(.25, .015),\\(.50, .02)\rangle\end{matrix} & \begin{matrix}\langle(.60, .03),\\(.40, .18),\\(.20, .1)\rangle\end{matrix} & \begin{matrix}\langle(.71, .04),\\(.52, .05),\\(.45, .02)\rangle\end{matrix} \\ \hline
s_2 & \begin{matrix}\langle(.65, .12),\\(.48, .02),\\(.25, .04)\rangle\end{matrix} & \begin{matrix}\langle(.25, .03),\\(.32, .02),\\(.60, .01)\rangle\end{matrix} & \begin{matrix}\langle(.60, .10),\\(.64, .06),\\(.18, .018)\rangle\end{matrix} & \begin{matrix}\langle(.60, .09),\\(.80, .001),\\(.40, .13)\rangle\end{matrix} & \begin{matrix}\langle(.60, .05),\\(.30, .05),\\(.20, .02)\rangle\end{matrix} \\ \hline
s_3 & \begin{matrix}\langle(.40, .1),\\(.23, .08),\\(.50, .01)\rangle\end{matrix} & \begin{matrix}\langle(.50, .06),\\(.35, .02),\\(.32, .03)\rangle\end{matrix} & \begin{matrix}\langle(.45, .12),\\(.40, .15),\\(.56, .03)\rangle\end{matrix} & \begin{matrix}\langle(.45, .10),\\(.90, .015),\\(.50, .18)\rangle\end{matrix} & \begin{matrix}\langle(.88, .02),\\(.60, .07),\\(.40, .02)\end{matrix} \\ \hline
s_4 & \begin{matrix}\langle(.90, .35),\\(.43, .05),\\(.80, .07)\rangle\end{matrix} & \begin{matrix}\langle(.25, .01),\\(.12, .09),\\(.44, .04)\rangle\end{matrix} & \begin{matrix}\langle(.60, .22),\\(.30, .19),\\(.13, .022)\rangle\end{matrix} & \begin{matrix}\langle(.65, .14),\\(.23, .012),\\(.41, .20)\rangle\end{matrix} & \begin{matrix}\langle(.90, .06),\\(.30, .20),\\(.65, .01)\rangle\end{matrix} \\ \hline
s_5 & \begin{matrix}\langle(.50, .09),\\(.32, .021),\\(.44, .06)\rangle\end{matrix} & \begin{matrix}\langle(.33, .02),\\(.70, .08),\\(.60, .07)\rangle\end{matrix} & \begin{matrix}\langle(.75, .03),\\(.50, .11),\\(.25, .02)\rangle\end{matrix} & \begin{matrix}\langle(.48, .12),\\(.43, .02),\\(.41, .04)\rangle\end{matrix} & \begin{matrix}\langle(.28, .02),\\(.63, .20),\\(.50, .08)\rangle\end{matrix}
\end{array}
$$

**Step 3:** Elements of decision matrix $PD = PS \circ SD$ are obtained as follows:
For the sake of shortness, some annotations are adapted as follows:

$x = \sqrt{-2ln(\alpha)})$ and $y = \sqrt{-2ln(1-\alpha)})$

$$\begin{aligned}
\tilde{q}_{11} &= \Big((0.285 - 0.095x, 0.285 + 0.095x), (0.050 - 0.001y, 0.050\\
&+ 0.001y), (0.010 - 0.002y, 0.010 + 0.002y)\Big) + \Big((0.423\\
&- 0.078x, 0.423 + 0.078x), (0.192 - 0.008y, 0.192\\
&+ 0.008y), (0.088 - 0.014y, 0.088 + 0.014y)\Big)\\
&+ \Big((0.08 - 0.02x, 0.08 + 0.02x), (0.173 - 0.06y, 0.173\\
&+ 0.06y), (0.40 - 0.008y, 0.40 + 0.008y)\Big)\\
&+ \Big((0.18 - 0.07x, 0.18 + 0.07x), (0.323 - 0.038y, 0.323\\
&+ 0.038y), (0.64 - 0.056y, 0.64 + 0.056y)\Big)\\
&+ \Big((0.40 - 0.072x, 0.40 + 0.072x), (0.080 - 0.005y, 0.080\\
&+ 0.005y), (0.088 - 0.012y, 0.088 + 0.012y)\Big)\\
&= \Big((1.368 - 0.335x, 1.368 + 0.335x), (0.817 - 0.112y,\\
&\quad 0.817 + 0.112y), (1.226 - 0.092y, 1.226 + 0.092y)\Big)
\end{aligned}$$

By similar way, we have $\tilde{q}_{12} = \Big((0.814 - 0.078x, 0.814 + 0.078x), (0.726 - 0.113y, 0.726 + 0.113y), (0.963 - 0.074y, 0.963 + 0.074y)\Big)$

$\tilde{q}_{13} = \Big((1.438 - 0.300x, 1.438 + 0.300x), (0.931 - 0.308y, 0.931 + 0.308y), (0.690 - 0.053y, 0.690 + 0.053y)\Big)$

$\tilde{q}_{14} = \Big((1.564 - 0.231x, 1.564 + 0.231x), (1.314 - 0.044y, 1.314 + 0.044y), (0.960 - 0.363y, 0.960 + 0.363y)\Big)$

$\tilde{q}_{15} = \Big((1.645 - 0.103x, 1.645 + 0.103x), (1.005 - 0.278y, 1.005 + 0.278y), (1.033 - 0.048y, 1.033 + 0.048y)\Big)$

$\tilde{q}_{21} = \Big((1,923 - 0.529x, 1.923 + 0.529x), (0.747 - 0.069y, 0.747 + 0.069y), (0.650 - 0,056y, 0.650 + 0.056y)\Big)$

$\tilde{q}_{22} = \Big((1.014 - 0.087x, 1.014 + 0.087x), (0.823 - 0.061y, 0.823 + 0.061y), (0.813 - 0.037y, 0.813 + 0.037y)\Big)$

$\tilde{q}_{23} = \Big((1.895 - 0.382x, 1.895 + 0.382x), (0.731 - 0.167y, 0.731 + 0.167y), (0.707 - 0.040y, 0.707 + 0.040y)\Big)$

$\tilde{q}_{24} = \Big((1.801 - 0.345x, 1.801 + 0.345x), (1.111 - 0.121y, 1.111 + 0.121y), (0.623 - 0.243y, 0.623 + 0.243y)\Big)$

$\tilde{q}_{25} = \Big((2.065 - 0.124x, 2.065 + 0.124x), (0.870 - 0.155y, 0.870 + 0.155y), (0.718 - 0.036y, 0.718 + 0.036y)\Big)$

$\tilde{q}_{31} = \Big((1,420 - 0.347x, 1.429 + 0.347x), (0.900 - 0.077y, 0.900 + 0.077y), (1.023 - 0,101y, 1.023 + 0.101y)\Big)$

$\tilde{q}_{32} = \Big((1.002 - 0.101x, 1.002 + 0.101x), (0.793 - 0.111y, 0.793 + 0.111y), (1.011 - 0.061y, 1.011 + 0.061y)\Big)$

$\tilde{q}_{33} = \Big((1.543 - 0.312x, 1.543 + 0.312x), (0.870 - 0.244y, 0.870 + 0.244y), (0.531 - 0.044y, 0.531 + 0.044y)\Big)$

$\tilde{q}_{34} = \Big((1.564 - 0.269x, 1.564 + 0.269x), (1.065 - 0.090y, 1.065 + 0.090y), (0.780 - 0.304y, 0.780 + 0.304y)\Big)$

$$\tilde{q}_{35} = \Big((1.870 - 0.095x, 1.870 + 0.095x), (0.891 - 0.263y, 0.891 + 0.263y), (0.958 - 0.045y, 0.958 + 0.045y)\Big)$$

$$\tilde{q}_{41} = \Big((1.375 - 0.352x, 1.375 + 0.352x), (1.016 - 0.066y, 1.016 + 0.066y), (1.107 - 0,126y, 1.107 + 0.126y)\Big)$$
$$\tilde{q}_{42} = \Big((0.879 - 0.095x, 0.879 + 0.095x), (1.237 - 0.136y, 1.237 + 0.136y), (1.302 - 0.094y, 1.302 + 0.094y)\Big)$$
$$\tilde{q}_{43} = \Big((1.318 - 0.307x, 1.318 + 0.307x), (0.943 - 0.238y, 0.943 + 0.238y), (0.749 - 0.051y, 0.749 + 0.051y)\Big)$$
$$\tilde{q}_{44} = \Big((1.423 - 0.264x, 1.423 + 0.264x), (0.986 - 0.156y, 0.986 + 0.156y), (0.798 - 0.271y, 0.798 + 0.271y)\Big)$$
$$\tilde{q}_{45} = \Big((1.829 - 0.092x, 1.829 + 0.092x), (1.178 - 0.327y, 1.178 + 0.327y), (1.243 - 0.092y, 1.243 + 0.092y)\Big)$$

$$\tilde{q}_{51} = \Big((1.308 - 0.425x, 1.308 + 0.425x), (0.648 - 0.041y, 0.648 + 0.041y), (1.190 - 0.104y, 1.190 + 0.104y)\Big)$$
$$\tilde{q}_{52} = \Big((0.579 - 0.050x, 0.579 + 0.050x), (0.623 - 0.063y, 0.623 + 0.063y), (1.362 - 0.101y, 1.362 + 0.101y)\Big)$$
$$\tilde{q}_{53} = \Big((1.063 - 0.343x, 1.063 + 0.343x), (0.719 - 0.133y, 0.719 + 0.133y), (0.975 - 0.065y, 0.975 + 0.065y)\Big)$$
$$\tilde{q}_{54} = \Big((1.329 - 0.197x, 1.329 + 0.197x), (0.875 - 0.055y, 0.875 + 0.055y), (1.185 - 0.348y, 1.185 + 0.348y)\Big)$$
$$\tilde{q}_{55} = \Big((1.589 - 0.103x, 0.509 + 0.103x), (0.618 - 0.152y, 0.618 + 0.152y), (1.130 - 0.102y, 1.130 + 0.102y)\Big)$$

**Step 4:** By using Eq. (4.1), scores of elements of decision matrix $PD$ are obtained as follows:

$$SM = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{array} \begin{array}{c} \begin{array}{ccccc} d_1 & d_2 & d_3 & d_4 & d_5 \end{array} \\ \begin{pmatrix} 0.442 & 0.375 & \mathbf{0.606} & 0.430 & 0.536 \\ \mathbf{0.842} & 0.459 & 0.819 & 0.689 & 0.826 \\ 0.499 & 0.399 & \mathbf{0.714} & 0.573 & 0.674 \\ 0.417 & 0.113 & 0.542 & \mathbf{0.546} & 0.470 \\ 0.490 & 0.198 & 0.456 & 0.423 & \mathbf{0.617} \end{pmatrix} \end{array}$$

**Step 5:** According to score matrix SM, we say that patient 1 suffer from typhoid, patient 2 suffer from viral fever, patient 3 suffer from typhoid, patient 4 suffer from threat disease and patient 5 suffer from malaria.

# 6 Conclusion

In this paper, some new concepts and operations was defined such as GSVNNs, $\alpha-$cuts of GSVNNs, parametric forms of GSVNNs and arithmetic operations of GSVNNs. Also, based on operations between parametric forms of GSVNNs and composition of matrices, a decision making method was proposed and presented an application in medical diagnosis based on hypothetical data. In future, Cauchy single-valued neutrosophic numbers may be defined and its properties can be investigated. Also, this study can be extended for other distributions in mathematical statistics. Furthermore, decision making methods can be developed for proposed new SVNNs.

# Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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# VIKOR based MAGDM Strategy with Trapezoidal Neutrosophic Numbers

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**Abstract**. VIseKriterijumska Optimizacija I Kompromisno Resenje (VIKOR) is a popular strategy for multi- attribute decision making (MADM). We extend the VIKOR strategy for MAGDM problems in trapezoidal neutrosophic number environment. In decision making situation, single-valued trapezoidal neutrosophic numbers are employed to express the attribute values. Then we develop an extended VIKOR strategy to deal with MAGDM in single-valued trapezoidal neutrosophic number environment. The influence of decision-making mechanism coefficient is presented. To illustrate and validate the proposed VIKOR strategy, an illustrative numerical example of MAGDM problem is solved in trapezoidal neutrosophic number environment.

**Keywords:** Neutrosophic set, Trapezoidal neutrosophic fuzzy number, Multi-attribute decision making, VIKOR strategy..

## 1. Introduction:

Smarandache [1] poineered the neutrosophic set based on neutrosophy in 1998. In 2010, Wang et al. [2] proposed single valued neutrosophic set (SVNS). SVNS has been successfully applied to solve decision making problems [3-31], image processing [32-35], conflict resolution [36], educational problem [37, 38], social problem [39, 40], etc. Broumi et al. [41] presented an overview neutrosophic sets. Recently, Peng and Dai [42] presented a bibliometric analysis of neutrosophic sets for last two decades.

Single valued trapezoidal neutrosophic number (SVTrNN) [43] is an extension of SVNS. Every element of SVTrNN is a trapezoidal number with three membership degress namely, truth, indeterminacy and falsity membership degrees. Deli and Subhas [10] developed a ranking strategy of SVTrNNs. Biswas et al. [7] established value and ambiguity based ranking strategy for SVTrNN and employed the strategy to deal with MADM problem. Biswas et al. [44] developed TOPSIS strategy for MADM with trapezoidal neutrosophic numbers (TrNNs). Biswas et al. [45] presented distance measure based MADM strategy with interval trapezoidal neutrosophic numbers (ITrNNs). For simplicity, we call SVTrNN as TrNN.

In 1998, Opricovic [46] first introduced the VIKOR strategy in the literature to deal with multi criteria group decision making (MCGDM) with conflicting objectives[47,48].

Using TOPSIS and VIKOR, Pouresmaeil et al.[49] proposed an MAGDM strategy in SVNS environment. Bausys and Zavadskas[50] proposed the VIKOR strategy in interval neutrosophic set (INS) environment. Huang et al.[51] developed a VIKOR strategy in INS environment. Liu and Zhang[52] studied VIKOR strategy in neutrosophic hesitant fuzzy set environment. In 2017, Hu et al.[53] developed a projection based VIKOR strategy for doctor selection problem in INS environment. In 2017, Selvakumari et al. [54] studied VIKOR strategy using octagonal neutrosophic soft matrix. Pramanik et al. [55] developed a VIKOR based MAGDM strategy in bipolar neutrosophic set environment. In 2018, Pramanik et al. [56] introduced a VIKOR strategy for MAGDM in neutrosophic cubic set environment. Dalapati and Pramanik [57] further revisited VIKOR based MAGDM strategy [56] to make it more comprehensive.

VIKOR strategy in trapezoidal neutrosophic number (TrNN) environment is not studied in the literature.
To fill up this research gap, we propose a VIKOR strategy to deal with MAGDM problems in TrNN environment. Also, we solve an MAGDM problem based on VIKOR strategy in trapezoidal neutrosophic number.

The rest of the paper is develpoed as follows. In section 2, we briefly describe definitions of trapezoidal fuzzy number, TrNN, trapezoidal neutrosophic weighted arithmetic averaging (TrNWAA) operator, Hamming distance between two TrNNs. In section 3, we briefly describe extended VIKOR strategy. Thereafter in section 4, we present a VIKOR strategy in TrNN environment. In section 5, we solve an MAGDM problem using the proposed

VIKOR strategy. In section 6, we present the sensitivity analysis. We represent conclusion and scope of future research in section 7.

## 2. Preliminaries

We present some fundamental definitions of fuzzy sets, neutrosophic set, SVNS, and TrNN.

**Definition 2.1** [58] Let $\ddot{Y}$ be a universal set . Then, a fuzzy set F is presented as:

$$F = \{< \dot{y}, \nu_F(\dot{y}) : \dot{y} \in \ddot{Y} >\} \quad (1)$$

where $\nu_F(\dot{y})$ is the degree of membership which maps $\ddot{Y}$ to [0,1] or we can express by $\nu_F : \ddot{Y} \to [0,1]$.

**Definition 2.2**[1]Let $\ddot{Y}$ be an universal set . A neutrosophic set N can be presented of the form:

$$N = \{< z : T_N(\dot{y}), I_N(\dot{y}), F_N(\dot{y}) > \dot{y} \in \ddot{Y}\} \quad (2)$$

where the functions $T, I, F : \ddot{Y} \to ]^-0, 1^+[$ define repectively the degree of truth membership, the degree of indeterminacy, and the degree of non-membership or falsity of the component $\dot{y} \in \ddot{Y}$ and satisfy the condition,

$$^-0 \le T_N(\dot{y}) + F_N(\dot{y}) + I_N(\dot{y}) \le 3^+ \quad (3)$$

**Definition 2.3** [2] Let $\ddot{Y}$ be a universal set. An SVNS N in $\ddot{Y}$ is described by

$$N = \{< \dot{y} : T_N(\dot{y}), I_N(\dot{y}), F_N(\dot{y}) > \dot{y} \in \ddot{Y}\}$$

where $T_N(\dot{y}) : \ddot{Y} \to [0,1]$ , $I_N(\dot{y}) : \ddot{Y} \to [0,1]$ and $F_N(\dot{y}) : \ddot{Y} \to [0,1]$ with the condition $0 \le T_N(\dot{y}) + F_N(\dot{y}) + I_N(\dot{y}) \le 3$ for all $\dot{y} \in \ddot{Y}$ .The functions $T_N(\dot{y}), I_N(\dot{y})$ and $F_N(\dot{y})$ are respectively, the truth membership function, the indeterminacy membership function and the falsity membership function of the element $\dot{y}$ to the set *N*.

**Definition 2.4[59]** A generalized trapezoidal fuzzy number $\dot{T}$ denoted by $\dot{T} = (\dot{b}_1, \dot{b}_2, \dot{b}_3, \dot{b}_4; v)$ is described as a fuzzy subset of the real number R with membership function $\underset{\cdot}{\kappa}$ which is defined by

$$\underset{\cdot}{\kappa}_{\dot{T}}(x) = \begin{cases} \dfrac{(x - \dot{b}_1)v}{(\dot{b}_2 - \dot{b}_1)}, \dot{b}_1 \le x < \dot{b}_2 \\ v, \dot{b}_2 \le x \le \dot{b}_3 \\ \dfrac{(\dot{b}_4 - x)v}{(\dot{b}_4 - \dot{b}_3)}, \dot{b}_3 < x \le \dot{b}_4 \\ 0, otherwise \end{cases} \quad (4)$$

where $\dot{b}_1, \dot{b}_2, \dot{b}_3, \dot{b}_4$ are real number satisfying $\dot{b}_1 \le \dot{b}_2 \le \dot{b}_3 \le \dot{b}_4$ and v is the membership degree.

**Definition 2.5**[43, 44] Let *x* be a TrNN. Then, its truth membership, indeterminacy membership, and falsity membership functions are presented respectively as:

$$T_x(z) = \begin{cases} \dfrac{(z - b_1')t_x}{(b_2' - b_1')}, b_1' \le z < b_2' \\ t_x, b_2' \le z \le b_3' \\ \dfrac{(b_4' - z)t_x}{(b_4' - b_3')}, b_3' \le z \le b_4' \\ 0, otherwise \end{cases} \quad (5)$$

$$I_x(z) = \begin{cases} \dfrac{(b_2' - z) + (z - b_1')i_x}{(b - a)}, b_1' \le z < b_2' \\ i_x, b_2' \le z \le b_3' \\ \dfrac{z - b_3' + (b_4' - z)i_x}{b_4' - b_3'}, b_3' < z \le b_4' \\ 0, otherwise \end{cases} \quad (6)$$

$$F_x(z)=\begin{cases}\dfrac{b_2'-z+(z-b_1')f_x}{b_2'-b_1'}, b_1'\le z<b_2' \\ f_x, b_2'\le z\le b_3' \\ \dfrac{z-b_3'+(b_4'-z)f_x}{b_4'-b_3'}, b_3'<z\le b_4' \\ 0, otherwise\end{cases} \tag{7}$$

Here $0\le T_x(z)\le 1, 0\le I_x(z)\le 1$ and $0\le F_x(z)\le 1$ and $0\le T_x(z)+I_x(z)+F_x(z)\le 3; b_1', b_2', b_3', b_4' \in R.$ Then $x=([b_1', b_2', b_3', b_4']: t_x, i_x, f_x)$ is called a TrNN.

**Definition 2.6** [43] Let $m_i =<(b_{1i}, b_{2i}, b_{3i}, b_{4i}); T_{m_i}, I_{m_i}, F_{m_i}>$ (i = 1, 2, …., $n'$) be a group of TrNNs, then a trapezoidal neutrosophic weighted arithmetic averaging (TrNWAA) operator is defined as follows:

$$\mathrm{TrNWAA}(m_1, m_2, ..., m_n)=\sum_{i=1}^{n'}\dot{w}_i m_i \tag{8}$$

where, $\dot{w}_i$ is the weight of $m_i$ (i = 1, 2, …, n) such that $\dot{w}_i$ >0 and $\sum_{i=1}^{n'}\dot{w}_i=1$ .Specially, when $\dot{w}_i=1/n'$ for i=1,2,…, $n'$ the TrNWAA operator transform into the trapezoidal neutrosophic arithmetic averaging (TrNAA) operator.

**Definition 2.7[44]** Let $m_1=([\ddot{p}_1, \ddot{q}_1, \ddot{r}_1, \ddot{s}_1]; t_{m_1}, i_{m_1}, f_{m_1})$ and $m_2=([\ddot{p}_2, \ddot{q}_2, \ddot{r}_2, \ddot{s}_2]; t_{m_2}, i_{m_2} f_{m_2})$ be any two TrNNs. The normalized Hamming distance between $m_1$ and $m_2$ is defined as:

$$d(m_1, m_2)=\frac{1}{12}\begin{pmatrix}\left|\ddot{p}_1(2+t_{m_1}-i_{m_1}-f_{m_1})-\ddot{p}_2(2+t_{m_2}-i_{m_2}-f_{m_2})\right|+\left|\ddot{q}_1(2+t_{m_1}-i_{m_1}-f_{m_1})-\ddot{q}_2(2+t_{m_2}-i_{m_2}-f_{m_2})\right| \\ +\left|\ddot{r}_1(2+t_{m_1}-i_{m_1}-f_{m_1})-\ddot{r}_2(2+t_{m_2}-i_{m_2}-f_{m_2})\right|+\left|\ddot{s}_1(2+t_{m_1}-i_{m_1}-f_{m_1})-\ddot{s}_2(2+t_{m_2}-i_{m_2}-f_{m_2})\right|\end{pmatrix} \tag{9}$$

**2.8. Standardize the decision matrix[44]**

Let $D=(b_{ij})_{p\times n}$ be a neutrosophic matrix, where $b_{ij}=([b_{ij}^1, b_{ij}^2, b_{ij}^3, b_{ij}^4]; t_{b_{ij}}, i_{b_{ij}}, f_{b_{ij}})$ is the rating value of the alternative $x_i$ with respect to attribute $y_i$. To remove the effect of several physical dimensions, we standardize the decision matrix $(b_{ij})_{p\times n}$ for benefit type and cost type attributes.

We denote the standardized decision matrix by $D^*=(s_{ij})_{p'\times n'}$

1. For benefit type attribute

$$b_{ij}^*=([\frac{b_{ij}^1}{v_j^+}, \frac{b_{ij}^2}{v_j^+}, \frac{b_{ij}^3}{v_j^+}, \frac{b_{ij}^4}{v_j^+}]; t_{b_{ij}}, i_{b_{ij}}, f_{b_{ij}}) \tag{10}$$

2.For cost type attribute:

$$b_{ij}^*=([\frac{v_j^-}{b_{ij}^4}, \frac{v_j^-}{b_{ij}^3}, \frac{v_j^-}{b_{ij}^2}, \frac{v_j^-}{b_{ij}^1}]; t_{b_{ij}}, i_{b_{ij}}, f_{b_{ij}}) \tag{11}$$

Here $v_j^+=\max\{b_{ij}^4 : i=1, 2, ..., p'\}$ and $v_j^-=\min\{b_{ij}^1 : i=1, 2, ..., p'\}$ for j = 1, 2, ..., $n'$

Hence, we obtain standardized matrix $D^*$ as:

$$D^*=(s_{ij})_{p'\times n'}=\begin{pmatrix} & b_1 & b_2 & ... & b_{n'} \\ a_1 & s_{11} & s_{12} & ... & s_{1n'} \\ a_2 & s_{21} & s_{22} & ... & s_{2n'} \\ ... & ... & ... & ... & ... \\ a_{p'} & s_{p'1} & s_{p'2} & ... & s_{p'n'}\end{pmatrix} \tag{12}$$

## 3.VIKOR Strategy for MADM

Assume that $B_1, B_2, ..., B_s$ are the s alternatives. For the alternative $B_{i'}$, assume that the rating of the $j'$ th criterion is $h_{i'j'}$, i.e. $h_{i'j'}$ is the value of $j'$ th criterion for the alternative $B_i$; the number of criteria is assumed to be r. Development of the extended VIKOR strategy is started with the following form of LP - metric:

$$L_{i'q} = \{\sum_{i=1}^{r}[(h_{i'}^{+} - h_{i'j'}) / (\mathrm{h}_{i'}^{+} - h_{i'}^{-})]^{q}\}^{\frac{1}{q}}, 1 \le q \le \infty; i' = 1, 2, ..., \mathrm{s} \tag{13}$$

To formulate ranking measure, $L_{1,i'}$ (as $S_{i'}$ ) and $L_{\infty,i'}$ (as $R_{i'}$) are employed. The solution obtained by $\min S_{i'}$ reflects a maximum group utility (''majority'' rule), and the solution obtained by $\min R_{i'}$ reflects a minimum individual regret of the ''opponent''.

VIKOR stratgey is presented using the following steps:

(a) Evaluate the best $\mathrm{h}_{\mathrm{j}'}^{+}$ and the worst $\mathrm{h}_{\mathrm{j}'}^{-}$ values of all criteria $j' = 1, 2, ..., n'$.

$$h_{j'}^{+} = \max_{i'} h_{i'j'}, \mathrm{h}_{j'}^{-} = \min_{i'} h_{i'j'}, \text{for benefit criterion,} \tag{14}$$

$$h_{j'}^{+} = \min_{i'} h_{i'j'}, \mathrm{h}_{j'}^{-} = \max_{i'} h_{i'j'}, \text{for cost criterion} \tag{15}$$

(a) Calculate the values $F_{i'}$ and $G_{i'}$; $i'$ = 1, 2, ..., $m'$, by these relations:

$$F_{i'} = \sum_{j'=1}^{n'} \frac{w_{j'}(\mathrm{h}_{j'}^{+} - h_{i'j'})}{(\mathrm{h}_{j'}^{+} - h_{j'}^{-})} \tag{16}$$

$$G_{i'} = \max_{j'}\left(\frac{w_{j'}(\mathrm{h}_{j'}^{+} - h_{i'j'})}{(\mathrm{h}_{j'}^{+} - h_{j'}^{-})}\right) \tag{17}$$

where $w_{j'}$ ( $j'$ =1, 2, …, r) represent the weights of criteria.

(b) Evaluate the value $K_{i'}$; $i'$ = 1, 2, ..., s, using following relation:

$$K_{i'} = \frac{v(\mathrm{F}_{i'} - F^{+})}{(\mathrm{F} - F^{+})} + \frac{(1-v)(\mathrm{G}_{i'} - G^{+})}{(\mathrm{G}^{-} - G^{+})} \tag{18}$$

where

$$F^{+} = \min_{i'} F_{i'}, \mathrm{F}^{-} = \max_{i'} F_{i'}$$

$$G^{+} = \min_{i'} G_{i'}, \mathrm{G}^{-} = \max_{i'} G_{i'}$$

Here, v indicates the weight of the technique of ''the majority of criteria'' (or ''the maximum group utility'') . Here we set v = 0.5.

(c) Sorting by the values F, G and K in decreasing order, we rank the alternatives.

(d) Propose the alternative $\mathrm{B}^1$ as a compromise solution that is ranked the best by the measure K (minimal) subject to the conditions A1 and A2:

A1. Acceptable advantage:

$$\mathrm{K}(\mathrm{B}^2) - \mathrm{K}(\mathrm{B}^1) \ge \mathrm{DK}$$

where $B^2$ is second alternative in the ranking list by K; DK= 1/(s-1); s = the number of alternatives.

A2. Acceptable stability in decision making:

Using F or/and G, we must have alternative $B^1$ as the best ranked. We say the compromise solution as stable subject to

i. ''voting by majority rule'' (when v > 0.5 is needed),
ii. or ''by consensus'' v $\approx$ 0.5,
iii. or ''with veto'' (v < 0.5).

v refelcts the weight of the decision making strategy of ''the majority of criteria'' (or ''the maximum group utility'').

- $B^1$ and $B^2$ are compromise solutions if A2 is not satisfied, Or
- $B^1, B^2, ...., B^M$ are compromise solutions if A1 is not satisfied.

  Evaluate $\mathrm{K}(\mathrm{B}^2) - \mathrm{K}(\mathrm{B}^1) \ge \mathrm{DK}$ to determine $\mathrm{B}^{\mathrm{M}}$ for maxium M (the positions of these alternatives are ''in closeness'').

The minimal value of K determines the best alternative.

**4.VIKOR strategy for solving MAGDM problem in TrNN environment:**

Consider an MADGM problem consisitng of r alternatives and t attributes. The alternatives and attributes are presented by $\alpha' = \{\alpha'_1, \alpha'_2, ..., \alpha'_r\}$ and $\beta' = \{\beta'_1, \beta'_2, ..., \beta'_t\}$ respectively. Assume that $\tau = \{\tau_1, \tau_2, ..., \tau_t\}$ is the set of weights of the attributes, where $\tau_{i'} \geq 0$ and $\sum_{i'=1}^{t} \tau_{i'} = 1$. Assume that $B = \{B_1, B_2, ..., B_K\}$ be the set of K decision makers and $\dot{\sigma} = \{\dot{\sigma}_1, \dot{\sigma}_2, ..., \dot{\sigma}_K\}$ be the set of weights of the decision makers, where $\dot{\sigma}_K \geq 0$ and $\sum_{i'=1}^{K} \dot{\sigma}_{i'} = 1$. The rating values offfered by the experts are presented in terms of Trnn

The MAGDM strategy is described as follows:

Step-1: Let $D = (p_{i'j'}^{N'})$ ($N'$ = 1, 2, ..., s) be the $N'$-th decision matrix where $\alpha'_{i'}$ is alternative with respect to attribute $\beta'_{i'}$. The $N'$-th decision matrix denoted by $D^{N'}$ is presented as:

$$D^M = \begin{pmatrix} & \beta'_1 & \beta'_2 & \cdots & \beta'_t \\ \alpha'_1 & p_{11}^{N'} & p_{12}^{N'} & \cdots & p_{1t}^{N'} \\ \alpha'_2 & p_{21}^{N'} & p_{22}^{N'} & \cdots & p_{2t}^{N'} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha'_r & p_{r1}^{N'} & p_{r2}^{N'} & \cdots & p_{rt}^{N'} \end{pmatrix} \tag{19}$$

where $N'$ = 1, 2, ....., s; $i'$ = 1,2, ..., r; $j'$ = 1, 2, ..., t.

Step-2: To standardize the benefit criterion, we use the equation (10) and for cost criterion, we use (11). After standardizing, the decision matrix reduces to

$$D^M = \begin{pmatrix} & \beta'_1 & \beta'_2 & \cdots & \beta'_t \\ \alpha'_1 & p_{11}^{*N'} & p_{12}^{*N'} & \cdots & p_{1t}^{*N'} \\ \alpha'_2 & p_{21}^{*N'} & p_{22}^{*N'} & \cdots & p_{2t}^{*N'} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha'_r & p_{r1}^{*N'} & p_{r2}^{*N'} & \cdots & p_{rt}^{*N'} \end{pmatrix}$$

$N'$=1, 2, ..., s; $i'$=1, 2, ..., r; $j'$ = 1, 2, ..., t..

Step-3:To obtain aggregate decision matrix, we use trapezoidal neutrosophic weighted arithmetic operator(TrNWAA) which is presented below:

$$p_{ij} = \text{TrNWAA}(p_{ij}^1, p_{ij}^2, ..., p_{ij}^M)$$

$$= \sum_{q=1}^{M} \sigma_q p_{ij}^q \tag{20}$$

Therefore, we obtain the aggregated decision matrix as:

$$D^{N'} = \begin{pmatrix} & \beta'_1 & \beta'_2 & \cdots & \beta'_t \\ \alpha'_1 & \ddot{p}_{11} & \ddot{p}_{12} & \cdots & \ddot{p}_{1t} \\ \alpha'_2 & \ddot{p}_{21} & \ddot{p}_{22} & \cdots & \ddot{p}_{2t} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha'_r & \ddot{p}_{r1} & \ddot{p}_{r2} & \cdots & \ddot{p}_{rt} \end{pmatrix}$$

Step -4: Define the positive ideal solution (PIS) $S^+$ and negative ideal solution (NIS) $S^-$

$$S^+ = ([\dot{b}_1^+, \dot{b}_2^+, \dot{b}_3^+, \dot{b}_4^+]; t_{\dot{b}}^+, i_{\dot{b}}^-, f_{\dot{b}}^-) = ([\dot{b}_1^+, \dot{b}_2^+, \dot{b}_3^+, \dot{b}_4^+]; \max t_{\dot{b}}, \min i_{\dot{b}}, \min f_{\dot{b}}) \tag{21}$$

$$S^- = ([\dot{b}_1^-, \dot{b}_2^-, \dot{b}_3^-, \dot{b}_4^-]; t_{\dot{b}}^-, i_{\dot{b}}^+, f_{\dot{b}}^+) = ([\dot{b}_1^-, \dot{b}_2^-, \dot{b}_3^-, \dot{b}_4^-]; \max t_{\dot{b}}, \min i_{\dot{b}}, \min f_{\dot{b}}) \tag{22}$$

Step 5:Compute

$$\Gamma_m = \sum_{j=1}^{t} \frac{\tau_m \times d(([\dot{b}_1^+, \dot{b}_2^+, \dot{b}_3^+, \dot{b}_4^+]; t_{\dot{b}}^+, i_{\dot{b}}^-, f_{\dot{b}}^-), ([\dot{b}_{j1}, \dot{b}_{j2}, \dot{b}_{j3}, \dot{b}_{j4}]; t_{nj}, i_{nj}, f_{nj}))}{d(([\dot{b}_1^+, \dot{b}_2^+, \dot{b}_3^+, \dot{b}_4^+]; t_{\dot{b}}^+, i_{\dot{b}}^-, f_{\dot{b}}^-), ([\dot{b}_1^-, \dot{b}_2^-, \dot{b}_3^-, \dot{b}_4^-]; t_{\dot{b}}^-, i_{\dot{b}}^+, f_{\dot{b}}^+))} \tag{23}$$

$$Z_m = \max \sum_{j=1}^{t} \frac{\tau_m \times d(([\dot{b}_1^+, \dot{b}_2^+, \dot{b}_3^+, \dot{b}_4^+]; t_{\dot{b}}^+, i_{\dot{b}}^-, f_{\dot{b}}^-), ([\dot{b}_{j1}, \dot{b}_{j2}, \dot{b}_{j3}, \dot{b}_{j4}]; t_{nj}, i_{nj}, f_{nj}))}{d(([\dot{b}_1^+, \dot{b}_2^+, \dot{b}_3^+, \dot{b}_4^+]; t_{\dot{b}}^+, i_{\dot{b}}^-, f_{\dot{b}}^-), ([\dot{b}_1^-, \dot{b}_2^-, \dot{b}_3^-, \dot{b}_4^-]; t_{\dot{b}}^-, i_{\dot{b}}^+, f_{\dot{b}}^+))} \tag{24}$$

where $\tau_m$ is the weight of $\beta'_m$.

Using equation (9), we obtain

$$d(([\dot{b}_1^+, \dot{b}_2^+, \dot{b}_3^+, \dot{b}_4^+]; t_{\dot{b}}^+, i_{\dot{b}}^-, f_{\dot{b}}^-), ([\dot{b}_{j1}, \dot{b}_{j2}, \dot{b}_{j3}, \dot{b}_{j4}]; t_{nj}, i_{nj}, f_{nj})$$
$$= \frac{1}{12}\left( \begin{array}{l} \left|\dot{b}_1^+(2 + t_{\dot{b}}^+ - i_{\dot{b}}^- - f_{\dot{b}}^-) - \dot{b}_{j1}(2 + t_{nj} - i_{nj} - f_{nj})\right| + \left|\dot{b}_2^+(2 + t_{\dot{b}}^+ - i_{\dot{b}}^- - f_{\dot{b}}^-) - \dot{b}_{j2}(2 + t_{nj} - i_{nj} - f_{nj})\right| \\ + \left|\dot{b}_3^+(2 + t_{\dot{b}}^+ - i_{\dot{b}}^- - f_{\dot{b}}^-) - \dot{b}_{j3}(2 + t_{nj} - i_{nj} - f_{nj})\right| + \left|\dot{b}_4^+(2 + t_{\dot{b}}^+ - i_{\dot{b}}^- - f_{\dot{b}}^-) - \dot{b}_{j4}(2 + t_{nj} - i_{nj} - f_{nj})\right| \end{array} \right)$$

and

$$d(([\dot{b}_1^+, \dot{b}_2^+, \dot{b}_3^+, \dot{b}_4^+]; t_{\dot{b}}^+, i_{\dot{b}}^-, f_{\dot{b}}^-), ([\dot{b}_1^-, \dot{b}_2^-, \dot{b}_3^-, \dot{b}_4^-]; t_{\dot{b}}^-, i_{\dot{b}}^+, f_{\dot{b}}^+))$$
$$= \frac{1}{12}\left( \begin{array}{l} \left|\dot{b}_1^+(2 + t_{\dot{b}}^+ - i_{\dot{b}}^- - f_{\dot{b}}^-) - \dot{b}_1^-(2 + t_{\dot{b}}^- - i_{\dot{b}}^+ - f_{\dot{b}}^+))\right| + \left|\dot{b}_2^+(2 + t_{\dot{b}}^+ - i_{\dot{b}}^- - f_{\dot{b}}^-) - \dot{b}_2^-(2 + t_{\dot{b}}^- - i_{\dot{b}}^+ - f_{\dot{b}}^+))\right| \\ + \left|\dot{b}_3^+(2 + t_{\dot{b}}^+ - i_{\dot{b}}^- - f_{\dot{b}}^-) - \dot{b}_3^-(2 + t_{\dot{b}}^- - i_{\dot{b}}^+ - f_{\dot{b}}^+))\right| + \left|\dot{b}_4^+(2 + t_{\dot{b}}^+ - i_{\dot{b}}^- - f_{\dot{b}}^-) - \dot{b}_4^-(2 + t_{\dot{b}}^- - i_{\dot{b}}^+ - f_{\dot{b}}^+))\right| \end{array} \right)$$

Step 6: Compute the $\Theta$ by the following formula:

$$\Theta_m = \Psi \frac{(\Gamma_m - \Gamma''_m)}{(\Gamma_m^+ - \Gamma''_m)} + (1 - \Psi)\frac{(Z_m - Z''_m)}{(Z_m^+ - Z''_m)} \tag{25}$$

where $\Gamma''_m = \min_m \Gamma_m$, $\Gamma_m^+ = \max_m \Gamma_m$ (26)

$Z''_m = \min_m Z_m$ , $Z_m^+ = \max_m Z_m$ (27)

Here, $\Psi$ denotes "decision-making mechanism coefficient".

i. $\Theta$ is the minimal if $\Psi \le 0.5$

ii. $\Theta$ is the "maximum group utility" if $\Psi \ge 0.5$ ,

iii. $\Theta$ is both the minimal and the "maximum group utility" if $\Psi = 0.5$ .

Step-7: Ranking the alternative by $\Gamma_m, Z_m$, and $\Theta_m$ .

**Step-8: Determine the compromise solution**

Obtain alternative $\alpha'^1$ as a compromise solution, that is ranked as the best by the measure $\alpha'$ (minimal) if the A1 and A2 are satisfied:

**A1.** Acceptable stability:

$$\Theta(\alpha'^2) - \Theta(\alpha'^1) \ge \frac{1}{r - 1} \tag{28}$$

where $\alpha'^1$, $\alpha'^2$ are the alternatives with 1st and 2nd positions in the ranking by $\Theta$ **;** r = the number of alternatives.

**A2.** Acceptable stability in decision making:

Alternative $\alpha'^1$ must also be the best ranked by $\Gamma$ or/and Z. This compromise solution is stable within whole decision making process.

- $\alpha'^1$ and $\alpha'^2$ are compromise solutions if A2 is not satisfied, or
- $\alpha'^1, \alpha'^2, ...., \alpha'^r$ are compromise solutions if A1 is not satisfied and $\alpha'^r$ is decided by constraint $\Theta(\alpha'^2) - \Theta(\alpha'^1) \le \frac{1}{r-1}$ for maximum r.

The minimal value of $\Theta$ determines the best alternative.

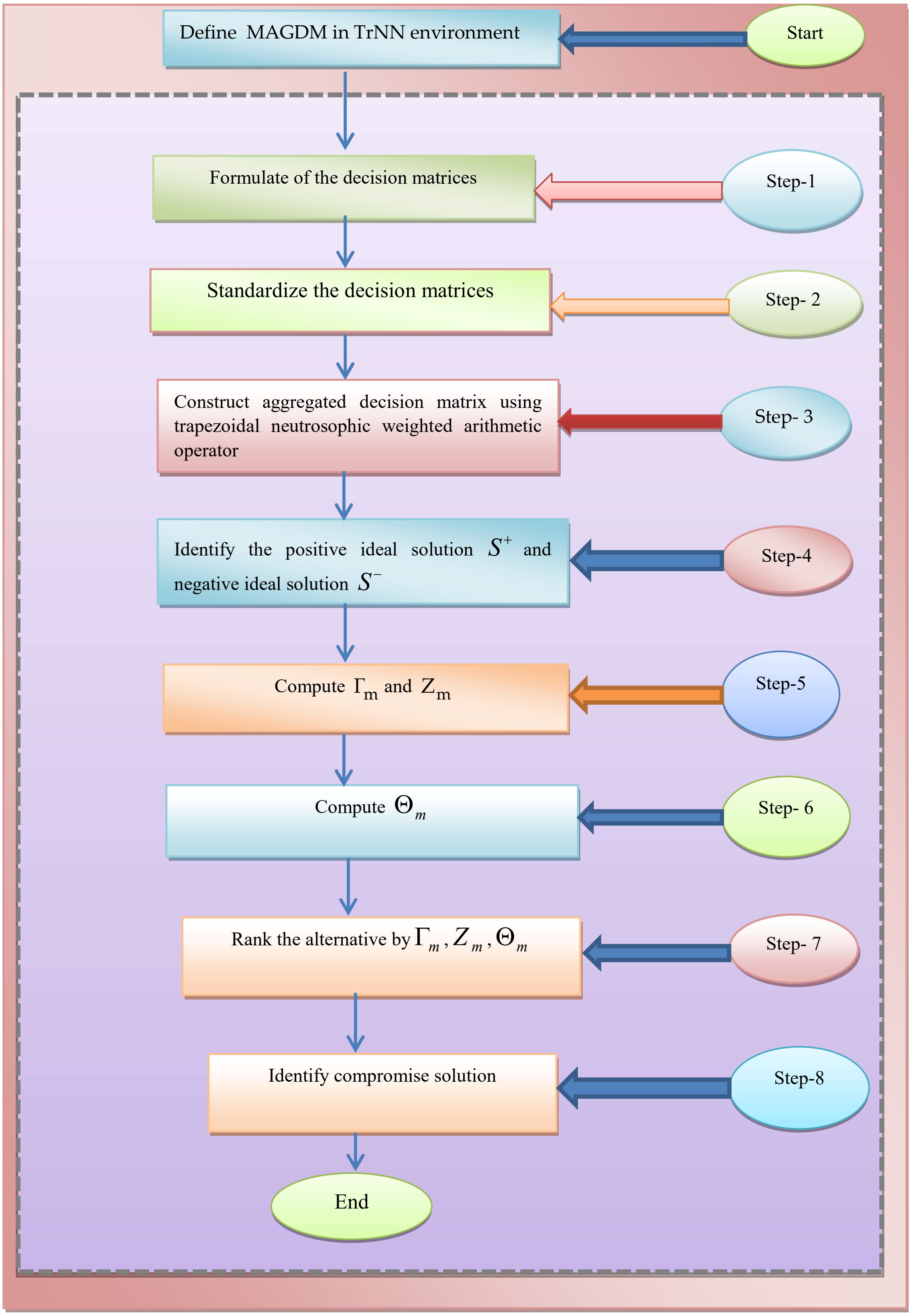


Figure 1. VIKOR based MAGDM strategy in trapezoidal neutrosophic number environment

**5. Numerical example**

To illustrate the developed VIKOR strategy, we consider an MAGDM problem adapted from [57]. The considered MAGDM problem is described as follows:

An investment company constitutes a decision making board with three experts to invest certain amount of money in the best alternative. The experts evaluate the four alternatives and three attributes which are described below :

Alteranatives:

1. Car company( $\alpha_1'$ )
2. Food company( $\alpha_2'$ )
3. Computer company( $\alpha_3'$ )
4. Arms company( $\alpha_4'$ )

Attributes:

1. Risk factor( $\beta_1'$ )
2. Growth factor( $\beta_2'$ )
3. Environment impact( $\beta_3'$ )

Suppose, $\tau = (0.30, 0.42, 0.28)$ be the set of weigts of the decision makers and $\sigma = (0.33, 0.39, 0.28)$ be the set of weights of the attributes.

Step-1: In this step,we construct the decision matrix in TrNNs form

Decision matrix $D^1$

$$\begin{array}{cccc} & \beta_1' & \beta_2' & \beta_3' \\ \alpha_1' & <(0.5,0.6,0.7,0.8);0.1,0.4,0.7> & <(0.1,0.1,0.2,0.3);0.6,0.7,0.5> & <(0.1,0.2,0.2,0.3);0.7,0.2,0.4> \\ \alpha_2' & <(0.3,0.4,0.5,0.5);0.4,0.5,0.2> & <(0.1,0.2,0.2,0.4);0.1,0.4,0.3> & <(0.1,0.1,0.2,0.3);0.5,0.3,0.5> \\ \alpha_3' & <(0.3,0.3,0.30.3);0.1,0.2,0.3> & <(0.2,0.3,0.4,0.4);0.8,0.2,0.5> & <(0.6,0.7,0.8,0.9);0.4,0.3,0.1> \\ \alpha_4' & <(0.7,0.8,0.8,0.9);0.3,0.3,0.2> & <(0.1,0.2,0.3,0.3);0.6,0.5,0.2> & <(0.2,0.2,0.2,0.2);0.5,0.2,0.2> \end{array} \tag{29}$$

Decision matrix $D^2$

$$\begin{array}{cccc} & \beta_1' & \beta_2' & \beta_3' \\ \alpha_1' & <(0.1,0.1,0.2,0.3);0.2,0.5,0.1> & <(0.2,0.2,0.3,0.4);0.2,0.5,0.1> & <(0.4,0.5,0.6,0.7);0.5,0.7,0.2> \\ \alpha_2' & <(0.2,0.3,0.4,0.5);0.3,0.3,0.2> & <(0.1,0.1,0.2,0.3);0.3,0.3,0.4> & <(0.2,0.2,0.3,0.3);0.4,0.5,0.2> \\ \alpha_3' & <(0.1,0.2,0.2,0.3);0.2,0.5,0.6> & <(0.2,0.3,0.3,0.4);0.2,0.2,0.1> & <(0.4,0.5,0.6,0.6);0.8,0.1,0.1> \\ \alpha_4' & <(0.5,0.6,0.7,0.7);0.5,0.2,0.1> & <(0.2,0.2,0.2,0.2);0.3,0.4,0.5> & <(0.1,0.1,0.2,0.2);0.3,0.7,0.4> \end{array} \tag{30}$$

Decision matrix $D^3$

$$\begin{array}{cccc} & \beta_1' & \beta_2' & \beta_3' \\ \alpha_1' & <(0.3,0.4,0.4,0.5);0.5,0.1,0.1> & <(0.1,0.2,0.2,0.3);0.5,0.1,0.1> & <(0.2,0.2,0.3,0.4);0.6,0.2,0.1> \\ \alpha_2' & <(0.2,0.2,0.2,0.2);0.3,0.2,0.7> & <(0.1,0.1,0.1,0.1);0.4,0.4,0.1> & <(0.6,0.7,0.8,0.8);0.4,0.1,0.1> \\ \alpha_3' & <(0.2,0.3,0.4,0.5);0.4,0.5,0.3> & <(0.2,0.3,0.3,0.4);0.5,0.4,0.3> & <(0.3,0.4,0.4,0.5);0.5,0.2,0.3)> \\ \alpha_4' & <(0.3,0.4,0.4,0.5);0.5,0.2,0.1> & <(0.1,0.2,0.2,0.3);0.5,0.1,0.1> & <(0.1,0.2,0.3,0.4);0.2,0.2,0.5)> \end{array} \tag{31}$$

Step-2: We do not need to standardize the defining matrix as all the criteria are profit type.

Step-3:Using TrNWAA operator of equation (20),we get aggregate decision matrix of (29), (30), and (31) which is presented below:

$$\begin{array}{cccc} & \beta_1' & \beta_2' & \beta_3' \\ \alpha_1' & <(0.276,0.334,0.406,0.506);0.273,0.298,0.179> & <(0.142,0.17,0.242,0.342);.431,0.352,0.162> & <(0.254,0.326,0.396,0.496);0.633,0.294,0.203> \\ \alpha_2' & <(0.23,0.302,0.374,0.416);0.332,0.312,0.284> & <(0.1,0.13,0.172,0.274);0.277,0.354,0.249> & <(0.282,0.31,0.41,0.44);0.564,0.242,0.217> \\ \alpha_3' & <(0.188,0.258,0.286,0.356);0.235,0.380,0.401> & <(0.2,0.3,0.33,0.372);0.537,0.242,0.162> & <(0.432,0.532,0.604,0.662);0.640,0.169,0.136> \\ \alpha_4' & <(0.504,0.604,0.646,0.704);0.447,0.226,0.123> & <(0.142,0.2,0.23,0.258);0.461,0.290,0.242> & <(0.13,0.158,0.228,0.256);0.343,0.338,0.346> \end{array}$$

step-4:Here we define positive ideal soluton and negative solution by employing equations (21) and (22)

The positive ideal solution $R^+$ is presented as:

$$\begin{array}{ccc} \beta_1' & \beta_2' & \beta_3' \\ <(0.504,0.604,0.646,0.704);0.447,0.226,0.123> & <(0.2,0.3,0.33,0.372);0.537,0.242,0.162> & <(0.432,0.532,0.604,0.662);0.640,0.169,0.136> \end{array}$$

The negative ideal solution $R^-$ is presented as:

$$\begin{array}{ccc} \beta_1' & \beta_2' & \beta_3' \\ <(0.188,0.258,0.286,0.356);0.235,0.380,0.401> & <(0.1,0.13,0.172,0.274);0.277,0.354,0.249> & <(0.13,0.158,0.228,0.256);0.343,0.338,0.346> \end{array}$$

Step-5:Using equations (23) and (24), we compute $\Gamma_m$ and $Z_m$ which are presented as:

$$\Gamma_1 = \left(\frac{0.33\times0.202}{0.294}\right)+\left(\frac{0.39\times0.070}{0.121}\right)+\left(\frac{0.28\times0.172}{0.327}\right) = 0.601,$$

$$\Gamma_2 = \left(\frac{0.33\times0.238}{0.294}\right)+\left(\frac{0.39\times0.119}{0.121}\right)+\left(\frac{0.28\times0.181}{0.327}\right) = 0.805,$$

$$\Gamma_3 = \left(\frac{0.33\times0.298}{0.294}\right)+\left(\frac{0.39\times0}{0.121}\right)+\left(\frac{0.28\times0}{0.327}\right) = 0.334,$$

$$\Gamma_4 = \left(\frac{0.33\times0}{0.294}\right)+\left(\frac{0.39\times0.080}{0.121}\right)+\left(\frac{0.28\times0.284}{0.327}\right) = 0.501.$$

Here we use Hamming distance to measure the distantance between two TrNN .
And

$$Z_1 = \max\left\{\left(\frac{0.33\times0.202}{0.294}\right)+\left(\frac{0.39\times0.070}{0.121}\right)+\left(\frac{0.28\times0.172}{0.327}\right)\right\} = 0.227,$$

$$Z_2 = \max\left\{\left(\frac{0.33\times0.238}{0.294}\right)+\left(\frac{0.39\times0.119}{0.121}\right)+\left(\frac{0.28\times0.181}{0.327}\right)\right\} = 0.383,$$

$$Z_3 = \max\left\{\left(\frac{0.33\times0.298}{0.294}\right)+\left(\frac{0.39\times0}{0.121}\right)+\left(\frac{0.28\times0}{0.327}\right)\right\} = 0.334,$$

$$Z_4 = \max\left\{\left(\frac{0.33\times0}{0.294}\right)+\left(\frac{0.39\times0.080}{0.121}\right)+\left(\frac{0.28\times0.284}{0.327}\right)\right\} = 0.258.$$

Step-6:Using (25),(26), and (27) we calculate $\Theta_i$

$\Theta_1 = 0.283, \Theta_2 = 1, \Theta_3 = 0.342, \Theta_4 = 0.274$

Step -7:The ranking order of alternatives is

$\Theta_4 \le \Theta_1 \le \Theta_3 \le \Theta_2$

**Table 1.** Preference ranking order and compromise solution based on $\Gamma$ , Z and $\Theta$

| | $\alpha'_1$ | $\alpha'_2$ | $\alpha'_3$ | $\alpha'_4$ | Ranking | Compromise solution |
|---|---|---|---|---|---|---|
| $\Gamma$ | 0.6 | 0.805 | 0.334 | 0.501 | $\alpha'_3 < \alpha'_4 < \alpha'_1 < \alpha'_2$ | $\alpha'_3$ |
| Z | 0.228 | 0.383 | 0.334 | 0.258 | $\alpha'_1 < \alpha'_3 < \alpha'_3 < \alpha'_2$ | $\alpha'_1$ |
| $\Theta(\Psi = 0.5)$ | 0.282 | 1 | 0.342 | 0.274 | $\alpha'_4 < \alpha'_1 < \alpha'_3 < \alpha'_2$ | $\alpha'_4$ |

**Step 8: Determine the compromise solution**

If we rank $\Theta$ in decreasing order, the best position alternative is $\alpha'_4$ with $\Theta(\alpha'_4)$ =0.274, and the 2nd best position $\alpha'^1$ with $\Theta(\alpha'^1)$=0.283. Therefore, $\Theta(\alpha'_1)$- $\Theta(\alpha'_4)$=O.008<0.33(since r = 4;1/(r-1)=0.33) , which does not satisfy the condition 1( $\Theta(\alpha'^2)$- $\Theta(\alpha'^1) \ge \frac{1}{r-1}$ ).

Here $\alpha'_4$ is ranked best by $\Gamma$ and Z and satisfies the condition 2.

So, the compromise solution as follows:

$\Theta(\alpha'_1)$- $\Theta(\alpha'_4)$=0.008<0.33,

$\Theta(\alpha'_2)$- $\Theta(\alpha'_4)$=0.726>0.33,

$\Theta(\alpha_3')$- $\Theta(\alpha_4')$=0.05<0.33,

Therefore, $\alpha_1', \alpha_3'$ and $\alpha_4'$ are compromise solutions.

### 6.1 The impact of parameter $\Psi$

Table 2 demonstrates how the different values of $\Psi$ impact the ranking order of the alternatives $\alpha_{i'}'$.

**Table 2.** For different values of $\Psi$, ranking the value of $\alpha_{i'}'$ $(i'$ = 1, 2, 3, 4) .

| Values of $\Psi$ | Values of $\alpha_{i'}'$ | Preference order |
|---|---|---|
| $\Psi = 0.1$ | $\alpha_1'$ =0.057, $\alpha_2'$ =1, $\alpha_3'$ =0.615 , $\alpha_4'$ =0.209 | $\alpha_1' < \alpha_4' < \alpha_3' < \alpha_2'$ |
| $\Psi = 0.2$ | $\alpha_1'$ =0.113 , $\alpha_2'$ =1, $\alpha_3'$ =0.547, $\alpha_4'$ =0.225 | $\alpha_1' < \alpha_4' < \alpha_3' < \alpha_2'$ |
| $\Psi = 0.3$ | $\alpha_1'$ =0.170 , $\alpha_2'$ =1, $\alpha_3'$ =0.479, $\alpha_4'$ =0.241 | $\alpha_1' < \alpha_4' < \alpha_3' < \alpha_2'$ |
| $\Psi = 0.4$ | $\alpha_1'$ =0.227 , $\alpha_2'$ =1, $\alpha_3'$ =0.410 , $\alpha_4'$ =0.257 | $\alpha_1' < \alpha_4' < \alpha_3' < \alpha_2'$ |
| $\Psi = 0.5$ | $\alpha_1'$ =0.282 , $\alpha_2'$ =1, $\alpha_3'$ =0.342 , $\alpha_4'$ =0.274 | $\alpha_4' < \alpha_1' < \alpha_3' < \alpha_2'$ |
| $\Psi = 0.6$ | $\alpha_1'$ =0.340 , $\alpha_2'$ =1, $\alpha_3'$ =0.274, $\alpha_4'$ =0.290 | $\alpha_3' < \alpha_4' < \alpha_1' < \alpha_2'$ |
| $\Psi = 0.7$ | $\alpha_1'$ =0.370, $\alpha_2'$ =1, $\alpha_3'$ =0.205, $\alpha_4'$ =0.306 | $\alpha_3' < \alpha_4' < \alpha_1' < \alpha_2'$ |
| $\Psi = 0.8$ | $\alpha_1'$ =0.454 , $\alpha_2'$ =1, $\alpha_3'$ =0.137 , $\alpha_4'$ =0.399 | $\alpha_3' < \alpha_4' < \alpha_1' < \alpha_2'$ |
| $\Psi = 0.9$ | $\alpha_1'$ =0.510 , $\alpha_2'$ =1, $\alpha_3'$ =0.068 , $\alpha_4'$ =0.338 | $\alpha_3' < \alpha_4' < \alpha_1' < \alpha_2'$ |

## 7. Conclusions

Extended VIKOR strategy for MAGDM in trapezoidal neutrosophic number environment is presented in the paper. TrNWAA operator and Hamming distance are employed to develop the VIKOR strategy for MAGDM. Finally, an MAGDM problem is solved to demonstrate the proposed VIKOR strategy. Here, a sensitivity analysis is performed to demonstrate the impact of different values of the "decision making mechanism coefficient'' on ranking system. The proposed extended VIKOR strategy for MAGDM problems can be used to deal with decision making problems such as brick selection [60, 61], logistics center selection [62], teacher selection [63], weaver selection [64], etc.

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# Operators on Single Valued Trapezoidal Neutrosophic Numbers and SVTN-Group Decision Making

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**Abstract:** In this paper, we first introduce single valued trapezoidal neutrosophic (SVTN) numbers with their properties. We then define some operations and distances of the SVTN-numbers. Based on these new operations, we also define some aggregation operators, including SVTN-ordered weighted geometric operator, SVTN-hybrid geometric operator, SVTN-ordered weighted arithmetic operator and SVTN-hybrid arithmetic operator. We then examine the properties of these SVTN-information aggregation operators. By using the SVTN-weighted geometric operator and SVTN-hybrid geometric operator, we also define a multi attribute group decision making method, called SVTN-group decision making method. We finally give an illustrative example and comparative analysis to verify the developed method and to demonstrate its practicality and effectiveness.

**Keywords:** Single valued neutrosophic sets, neutrosophic numbers, trapezoidal neutrosophic numbers, SVTN-numbers, SVTN-group decision making.

## 1 Introduction

In real decision making, there usually are many multiple attribute group decision making (MAGDM) problems. Due to the ambiguity of people's thinking and the complexity of objective things, the attribute values of the MAGDM problems cannot always be expressed by exact and crisp values and it may be easier to describe them by neutrosophic information. Zadeh [77] initiated fuzzy set theory. It is one of the most effective tools for processing fuzzy information which has only one membership, and is unable to express non-membership. Therefore, Atanassov [3] presented the intuitionistic fuzzy sets by adding a nonmembership function. Also, Atanassov and Gargov [4] proposed the interval-valued intuitionistic fuzzy set by extending the membership function and nonmembership function to the interval numbers. These sets can only handle incomplete information, not the indeterminate information and inconsistent information. For this reason, Smarandache [53, 54, 55] introduced a new concept that is called neutrosophic set by adding an independent indeterminacy-membership on the basis of intuitionistic fuzzy sets from philosophical point of view, which is a generalization of the concepts of classical sets, probability sets, rough sets [43], fuzzy sets [77, 23], intuitionistic fuzzy sets [3], paraconsistent sets, dialetheist sets, paradoxist sets and tautological sets. In theory of neutrosophic sets, truth-membership, indeterminacy-membership and falsity-membership are represented independently. Also, Wang et al. [62] proposed the interval neutrosophic sets by extending the truth-membership, indeterminacy-membership, and falsity-membership functions to interval numbers. After

Smarandache, Broumi et al. [5, 6, 7], Biswas et al.[8, 9, 10, 11, 12, 13, 14, 15], Kahraman and Otay [32], Mondal et al. [35, 36, 37, 38, 39, 40] and Pramanik et al. [44, 45, 46, 47] studied on some decision making problems based on neutrosophic information. Recently, fuzzy and neutrosophic models have been studied by many authors, such as [1, 2, 19, 20, 28, 29, 30, 48, 49, 50, 52, 57, 58, 62, 63, 80, 81, 82, 83].

Gani et al. [27] presented a method called weighted average rating method for solving group decision making problem by using an intuitionistic trapezoidal fuzzy hybrid aggregation operator. Wan et al. [65] investigated MAGDM problems, in which the ratings of alternatives are expressed with triangular intuitionistic fuzzy numbers. Wei [66, 67], introduced some new group decision making methods by developing aggregation operators with intuitionistic fuzzy information. Xu and Yager [60], presented some new geometric aggregation operators, such as intuitionistic fuzzy weighted geometric operator, intuitionistic fuzzy ordered weighted geometric operator, and intuitionistic fuzzy hybrid geometric operator. Wu and Cao [68] developed some geometric aggregation operators with intuitionistic trapezoidal fuzzy numbers and examined their desired properties. Power average operator of real numbers is extended to four kinds of power average operators of trapezoidal intuitionistic fuzzy numbers by Wan [64]. Farhadinia and Ban [25] initiated a novel method to extend a similarity measure of generalized trapezoidal fuzzy numbers to similarity measures of generalized trapezoidal intuitionistic fuzzy numbers and generalized interval-valued trapezoidal fuzzy numbers. Ye [71] proposed an extended technique for order preference by similarity to ideal solution method for group decision making with interval-valued intuitionistic fuzzy numbers to solve the partner selection problem under incomplete and uncertain information environment. Recently, some intuitionistic models with intuitionistic values have been studied by many authors. For example, on intuitionistic fuzzy sets [26, 59, 76], on interval-valued intuitionistic fuzzy sets [16, 26], interval-valued intuitionistic trapezoidal fuzzy numbers [26, 69], on triangular intuitionistic fuzzy number [17, 24, 26, 33, 34, 61, 78], on trapezoidal intuitionistic fuzzy numbers [18, 26, 31, 34, 41, 42, 51, 72, 75, 79], on generalized trapezoidal fuzzy numbers, on generalized trapezoidal intuitionistic fuzzy numbers and generalized interval-valued trapezoidal fuzzy numbers [25].

A neutrosophic set can handle a incomplete, indeterminate and inconsistent information from philosophical point of view. Ye [74] and ¸Subas¸ [56] introduced single valued neutrosophic numbers, which is a generaliza-tion of fuzzy numbers and intuitionistic fuzzy numbers. The neutrosophic numbers are special single valued neutrosophic sets on the real number sets, which are useful to deal with ill-known quantities in decision data and decision making problems themselves. Then, Ye [73] and Deli and ¸Subas¸ [21, 22] developed new methods on single valued neutrosophic numbers based on multi-criteria decision making problem. But, multi-criteria group decision making problem has not yet been studied.

The paper is organized as follows. In the next section, we give some basic definitions and properties of single valued trapezoidal neutrosophic (SVTN) numbers. In Section 3, some operations for SVTN-numbers and distance between two SVTN-number are presented. In Section 4, we introduce some new geometric aggregation operators, including SVTN-ordered weighted geometric operator, SVTN-hybrid geometric operator, SVTN-ordered weighted arithmetic operator and SVTN-hybrid arithmetic operator. In Section 5, we develope a group decision making method, so called SVTN-group decision making method to solve MAGDM problems based on the SVTN-weighted geometric operator and the SVTN-hybrid geometric operators. We then present an illustrative example to verify the developed method and to demonstrate its practicality. In Section 6 we give a comparative analysis. In Section 7, we conclude the paper and give some remarks.

## 2 Preliminary

In this section, some basic concepts and definitions on fuzzy sets, intuitionistic fuzzy sets, neutrosophic sets, single valued neutrosophic sets and single valued neutrosophic numbers are given.

**Definition 2.1.** [77] Let $E$ be a universe. Then, a fuzzy set $X$ over $E$ is defined by

$$X = \{(\mu_X(x)/x) : x \in E\}$$

where $\mu_X : E \to [0.1]$ is called membership function of $X$. For each $x \in E$, the value $\mu_X(x)$ represents the degree of $x$ belonging to the fuzzy set $X$.

**Definition 2.2.** [3] Let $E$ be a universe. Then, an intuitionistic fuzzy set $K$ over $E$ is defined by

$$K = \{< x, \mu_K(x), \gamma_K(x) >: \ x \in E\}$$

where $\mu_K : E \to [0, 1]$ and $\gamma_K : E \to [0, 1]$ such that $0 \leq \mu_K(x) + \gamma_K(x) \leq 1$ for any $x \in E$. For each $x \in E$, the values $\mu_K(x)$ and $\gamma_K(x)$ are the degree of membership and degree of non-membership of $x$, respectively.

**Definition 2.3.** [54] Let $E$ be a universe. Then, a neutrosophic set $A$ over $E$ is defined by

$$A = \{< x, (T_A(x), I_A(x), F_A(x)) >: x \in E\}.$$

where $T_A(x)$, $I_A(x)$ and $F_A(x)$ are called truth-membership function, indeterminacy-membership function and falsity-membership function, respectively. They are respectively defined by $T_A : E \to ]^-0, 1^+[, \quad I_A : E \to ]^-0, 1^+[, \quad F_A : E \to ]^-0, 1^+[$ such that $0^- \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+$.

**Definition 2.4.** [63] Let $E$ be a universe. Then, a single valued neutrosophic set over E is a neutrosophic set over $E$, but the truth-membership function, indeterminacy-membership function and falsity-membership function are respectively defined by

$$T_A : E \to [0, 1], \quad I_A : E \to [0, 1], \quad F_A : E \to [0, 1]$$

such that $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

**Definition 2.5.** [22, 56] A single valued trapezoidal neutrosophic number $\tilde{a} = \langle (a_1, b_1, c_1, d_1); w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}} \rangle$ is a special neutrosophic set on the real number set $R$, whose truth-membership, indeterminacy-membership, and a falsity-membership are given as follows:

$$\mu_{\tilde{a}}(x) = \begin{cases} (x - a_1)w_{\tilde{a}}/(b_1 - a_1), & (a_1 \leq x < b_1) \\ w_{\tilde{a}}, & (b_1 \leq x \leq c_1) \\ (d_1 - x)w_{\tilde{a}}/(d_1 - c_1), & (c_1 < x \leq d_1) \\ 0, & otherwise, \end{cases}$$

$$\nu_{\tilde{a}}(x) = \begin{cases} (b_1 - x + u_{\tilde{a}}(x - a_1))/(b_1 - a_1), & (a_1 \leq x < b_1) \\ u_{\tilde{a}}, & (b_1 \leq x \leq c_1) \\ (x - c_1 + u_{\tilde{a}}(d_1 - x))/(d_1 - c_1), & (c_1 < x \leq d_1) \\ 1, & otherwise \end{cases}$$

and

$$\lambda_{\tilde{a}}(x)=\begin{cases}(b_1-x+y_{\tilde{a}}(x-a_1))/(b_1-a_1), & (a_1\leq x<b_1)\\ y_{\tilde{a}}, & (b_1\leq x\leq c_1)\\ (x-c_1+y_{\tilde{a}}(d_1-x))/(d_1-c_1), & (c_1<x\leq d_1)\\ 1, & otherwise\end{cases}$$

respectively.

Sometimes, we use the $\tilde{a}_i=\langle(a_i,b_i,c_i,d_i);w_i,u_i,y_i\rangle$, instead of $\tilde{a}_i=\langle(a_i,b_i,c_i,c_i);w_{\tilde{a}_i},u_{\tilde{a}_i},y_{\tilde{a}_i}\rangle$.

Note that the single valued trapezoidal neutrosophic number is abbreviated as SVTN-number and the set of all SVTN-numbers on $R$ will be denoted by $\Omega$.

# 3 Operations and Distances of SVTN-Numbers

In this section, we give operations and distances of SVTN-numbers and investigate their related properties.

**Definition 3.1.** [73] Let $\tilde{a}=\langle(a_1,b_1,c_1,d_1);w_{\tilde{a}},u_{\tilde{a}},y_{\tilde{a}}\rangle$, $\tilde{b}=\langle(a_2,b_2,c_2,d_2);w_{\tilde{b}},u_{\tilde{b}},y_{\tilde{b}}\rangle\in\Omega$ and $\gamma\geq 0$ be any real number. Then,

1. $\tilde{a}\oplus\tilde{b}=\langle(a_1+a_2,b_1+b_2,c_1+c_2,d_1+d_2);w_{\tilde{a}}+w_{\tilde{b}}-w_{\tilde{a}}w_{\tilde{b}},u_{\tilde{a}}u_{\tilde{b}},y_{\tilde{a}}y_{\tilde{b}}\rangle$
2. $\tilde{a}\otimes\tilde{b}=\langle(a_1a_2,b_1b_2,c_1c_2,d_1d_2);w_{\tilde{a}}w_{\tilde{b}},u_{\tilde{a}}+u_{\tilde{b}}-u_{\tilde{a}}u_{\tilde{b}},y_{\tilde{a}}+y_{\tilde{b}}-y_{\tilde{a}}y_{\tilde{b}}\rangle$
3. $\gamma\tilde{a}=\langle(\gamma a_1,\gamma b_1,\gamma c_1,\gamma d_1);1-(1-w_{\tilde{a}})^\gamma,u_{\tilde{a}}^\gamma,y_{\tilde{a}}^\gamma\rangle$
4. $\tilde{a}^\gamma=\langle(a_1^\gamma,b_1^\gamma,c_1^\gamma,d_1^\gamma);w_{\tilde{a}}^\gamma,1-(1-u_{\tilde{a}})^\gamma,1-(1-y_{\tilde{a}})^\gamma\rangle$

**Theorem 3.2.** *Let $\tilde{a}=\langle(a_1,b_1,c_1,d_1);w_1,u_1,y_1\rangle$, $\tilde{b}=\langle(a_2,b_2,c_2,d_1);w_2,u_2,y_2\rangle\in\Omega$. Then, $\tilde{a}\oplus\tilde{b}$, $\tilde{a}\otimes\tilde{b}$, $\gamma\tilde{a}$ and $\tilde{a}^\gamma$ are also SVTN-numbers.*

***Proof:*** It is easy from Definition 3.1.

**Theorem 3.3.** *Let $\tilde{a}=\langle(a_1,b_1,c_1,d_1);w_1,u_1,y_1\rangle$, $\tilde{b}=\langle(a_2,b_2,c_2,d_2);w_2,u_2,y_2\rangle$, $\tilde{c}=\langle(a_3,b_3,c_3,d_3);w_3,u_3,y_3\rangle\in\Omega$ and $\gamma$, $\gamma_1$, $\gamma_2$ be positif real numbers. Then, the followings are valid.*

1. $\tilde{a}\oplus\tilde{b}=\tilde{b}\oplus\tilde{a}$
2. $\tilde{a}\otimes\tilde{b}=\tilde{b}\otimes\tilde{a}$
3. $(\tilde{a}\otimes\tilde{b})\otimes\tilde{c}=\tilde{a}\otimes(\tilde{b}\otimes\tilde{c})$
4. $(\tilde{a}\oplus\tilde{b})\oplus\tilde{c}=\tilde{a}\oplus(\tilde{b}\oplus\tilde{c})$
5. $\tilde{a}\otimes(\tilde{b}\oplus\tilde{c})=(\tilde{a}\otimes\tilde{b})\oplus(\tilde{a}\otimes\tilde{c})$
6. $(\tilde{a}\otimes\tilde{b})^\gamma=\tilde{b}^\gamma\otimes\tilde{a}_1^\gamma$
7. $\tilde{a}^{\gamma_1}\otimes\tilde{a}^{\gamma_2}=\tilde{a}^{(\gamma_1+\gamma_2)}$ *or* $\tilde{b}^{\gamma_2}\otimes\tilde{b}^{\gamma_2}=\tilde{b}^{(\gamma_1+\gamma_2)}$

***Proof:*** It is easy from Definition 3.1.

**Definition 3.4.** Let $\tilde{a} = \langle (a_1, b_1, c_1, d_1); w_1, u_1, y_1 \rangle$, $\tilde{b} = \langle (a_2, b_2, c_2, d_2);\, w_2, u_2, y_2 \rangle \in \Omega$. Then, the distance between $\tilde{a}$ and $\tilde{b}$ is defined by

$$\begin{aligned} d_h(\tilde{a}, \tilde{b}) = \; & \tfrac{1}{6}\big( |(1 + w_1 - u_1 - y_1)a_1 - (1 + w_2 - u_2 - y_2)a_2| + \\ & |(1 + w_1 - u_1 - y_1)b_1 - (1 + w_2 - u_2 - y_2)b_2| + \\ & |(1 + w_1 - u_1 - y_1)c_1 - (1 + w_2 - u_2 - y_2)c_2| + \\ & |(1 + w_1 - u_1 - y_1)d_1 - (1 + w_2 - u_2 - y_2)d_2| \quad \big) \end{aligned}$$

**Example 3.5.** Assume that $\tilde{a} = \langle (1, 4, 5, 6); 0.3, 0.4, 0.7 \rangle$, $\tilde{b} = \langle (1, 2, 5, 7); 0.7, 0.5, 0.1 \rangle \in \Omega$. Then, the distance of $\tilde{a}$ and $\tilde{b}$ is computed by

$$\begin{aligned} d_h(\tilde{a}, \tilde{b}) = \; & \tfrac{1}{6}\big( |(1 + 0.3 - 0.4 - 0.7)1 - (1 + 0.7 - 0.5 - 0.1)1| + \\ & |(1 + 0.3 - 0.4 - 0.7)4 - (1 + 0.7 - 0.5 - 0.1)2| + \\ & |(1 + 0.3 - 0.4 - 0.7)5 - (1 + 0.7 - 0.5 - 0.1)5| + \\ & |(1 + 0.3 - 0.4 - 0.7)6 - (1 + 0.7 - 0.5 - 0.1)7| \big) \\ & \cong 7.78 \end{aligned}$$

**Theorem 3.6.** *Let $\tilde{a} = \langle (a_1, b_1, c_1, d_1); w_1, u_1, y_1 \rangle$, $\tilde{b} = \langle (a_2, b_2, c_2, d_2);\, w_2, u_2, y_2 \rangle \in \Omega$. Then, $d_h(\tilde{a}, \tilde{b})$ meet the nonnegative, symmetric and triangle inequality (or metric).*

***Proof:*** Clearly, the $d_h(\tilde{a}, \tilde{b})$ meet the nonnegative, symmetric properties. For $\tilde{a} = \langle (a_1, b_1, c_1, d_1); w_1, u_1, y_1 \rangle$, $\tilde{b} = \langle (a_2, b_2, c_2, d_2);\, w_2, u_2, y_2 \rangle$, $\tilde{c} = \langle (a_3, b_3, c_3, d_3);\, w_3, u_3, y_3 \rangle \in \Omega$, to prove the triangle inequality, since

$$\begin{aligned} & |(1 + w_1 - u_1 - y_1)a_1 - (1 + w_2 - u_2 - y_2)a_2| + \\ & |(1 + w_2 - u_2 - y_2)a_2 - (1 + w_3 - u_3 - y_3)a_3| \\ & \geq |(1 + w_1 - u_1 - y_1)a_1 - (1 + w_3 - u_3 - y_3)a_3| \end{aligned}$$

$$\begin{aligned} & |(1 + w_1 - u_1 - y_1)b_1 - (1 + w_2 - u_2 - y_2)b_2| + \\ & |(1 + w_2 - u_2 - y_2)b_2 - (1 + w_3 - u_3 - y_3)b_3| \\ & \geq |(1 + w_1 - u_1 - y_1)b_1 - (1 + w_3 - u_3 - y_3)b_3| \end{aligned}$$

$$\begin{aligned} & |(1 + w_1 - u_1 - y_1)c_1 - (1 + w_2 - u_2 - y_2)c_2| + \\ & |(1 + w_2 - u_2 - y_2)c_2 - (1 + w_3 - u_3 - y_3)c_3| \\ & \geq |(1 + w_1 - u_1 - y_1)c_1 - (1 + w_3 - u_3 - y_3)c_3| \end{aligned}$$

$$\begin{aligned} & |(1 + w_1 - u_1 - y_1)d_1 - (1 + w_2 - u_2 - y_2)d_2| + \\ & |(1 + w_2 - u_2 - y_2)d_2 - (1 + w_3 - u_3 - y_3)d_3| \\ & \geq |(1 + w_1 - u_1 - y_1)d_1 - (1 + w_3 - u_3 - y_3)d_3| \end{aligned}$$

we have

$$\begin{aligned}
&|(1+w_1-u_1-y_1)a_1-(1+w_2-u_2-y_2)a_2|+\\
&|(1+w_2-u_2-y_2)a_2-(1+w_3-u_3-y_3)a_3|\\
&+|(1+w_1-u_1-y_1)b_1-(1+w_2-u_2-y_2)b_2|+\\
&|(1+w_2-u_2-y_2)b_2-(1+w_3-u_3-y_3)b_3|\\
&+|(1+w_1-u_1-y_1)c_1-(1+w_2-u_2-y_2)c_2|+\\
&|(1+w_2-u_2-y_2)c_2-(1+w_3-u_3-y_3)c_3|\\
&+|(1+w_1-u_1-y_1)d_1-(1+w_2-u_2-y_2)d_2|+\\
&|(1+w_2-u_2-y_2)d_2-(1+w_3-u_3-y_3)d_3|\\
&\geq |(1+w_1-u_1-y_1)a_1-(1+w_3-u_3-y_3)a_3|+\\
&|(1+w_1-u_1-y_1)b_1-(1+w_3-u_3-y_3)b_3|\\
&+|(1+w_1-u_1-y_1)c_1-(1+w_3-u_3-y_3)c_3|+\\
&|(1+w_1-u_1-y_1)d_1-(1+w_3-u_3-y_3)d_3|
\end{aligned}$$

and then,

$$d_h(\tilde{a},\tilde{b})+d_h(\tilde{b},\tilde{c})\geq d_h(\tilde{a},\tilde{c})$$

**Definition 3.7.** [56] Let $\tilde{a}=\langle(a_1,b_1,c_1,d_1);w_1,u_1,y_1\rangle\in\Omega$. Then, a normalized SVTN-number of $\tilde{a}$ is defined by

$$\langle(\tfrac{a_1}{a_1+b_1+c_1+d_1},\ \tfrac{b_1}{a_1+b_1+c_1+d_1},\tfrac{c_1}{a_1+b_1+c_1+d_1},\tfrac{d_1}{a_1+b_1+c_1+d_1});w_1,u_1,y_1\rangle$$

**Example 3.8.** Assume that $\tilde{a}=\langle(1,4,5,10);0.3,0.4,0.7\rangle\in\Omega$. Then, a normalized SVTN-number of $\tilde{a}$ is computed as

$$\langle(0.05,0.2,0.25,0.5);0.3,0.4,0.7\rangle$$

**Definition 3.9.** The SVTN-numbers $\tilde{a}^+=\langle(1,1,1,1);1,0,0\rangle$, $\tilde{a}_s^+=\langle(1,1,1,1);1,1,0\rangle$, $\tilde{a}^-=\langle(0,0,0,0);0,1,1\rangle$ and $\tilde{a}_s^-=\langle(0,0,0,0);0,0,1\rangle$ are called SVTN-positive ideal solution, strongly SVTN-positive ideal solution, SVTN-negative ideal solution and strongly SVTN-negative ideal solution, respectively.

**Definition 3.10.** Let $\tilde{a}_i=\langle(a_i,b_1,c_i,d_i);w_i,u_i,y_i\rangle\in\Omega$ for all $i=1,2$ and $\tilde{a}^+$, $\tilde{a}_s^+$, $\tilde{a}^-$ and $\tilde{a}_s^-$ be SVTN-positive ideal solution, strongly SVTN-positive ideal solution, SVTN-negative ideal solution and strongly SVTN-negative ideal solution, respectively. Then, the distance between $\tilde{a}_i$ and $\tilde{a}^+$, $\tilde{a}_s^+$, $\tilde{a}^-$, $\tilde{a}_s^-$ are denoted as $d_h(\tilde{a}_i,\tilde{a}^+)$, $d_h(\tilde{a}_i,\tilde{a}_s^+)$, $d_h(\tilde{a}_i,\tilde{a}^-)$, $d_h(\tilde{a}_i,\tilde{a}_s^-)$ for all $i=1,2$, respectively. Then,

1. If $d_h(\tilde{a}_1,\tilde{a}^+)<d_h(\tilde{a}_2,\tilde{a}^+)$, then $\tilde{a}_2$ is smaller than $\tilde{a}_1$, denoted by $\tilde{a}_1>\tilde{a}_2$
2. If $d_h(\tilde{a}_1,\tilde{a}^+)=d_h(\tilde{a}_2,\tilde{a}^+)$;
   (a) If $d_h(\tilde{a}_1,\tilde{a}_s^+)<d_h(\tilde{a}_2,\tilde{a}_s^+)$, then $\tilde{a}_2$ is smaller than $\tilde{a}_1$, denoted by $\tilde{a}_1>\tilde{a}_2$
   (b) If $d_h(\tilde{a}_1,\tilde{a}_s^+)=d_h(\tilde{a}_2,\tilde{a}_s^+)$;
      i. If $d_h(\tilde{a}_1,\tilde{a}^-)<d_h(\tilde{a}_2,\tilde{a}^-)$, then $\tilde{a}_1$ is smaller than $\tilde{a}_2$, denoted by $\tilde{a}_1<\tilde{a}_2$
      ii. If $d_h(\tilde{a}_1,\tilde{a}^-)=d_h(\tilde{a}_2,\tilde{a}^-)$;
         A. If $d_h(\tilde{a}_1,\tilde{a}_s^-)<d_h(\tilde{a}_2,\tilde{a}_s^-)$, then $\tilde{a}_1$ is smaller than $\tilde{a}_2$, denoted by $\tilde{a}_1<\tilde{a}_2$
         B. If $d_h(\tilde{a}_1,\tilde{a}_s^-)=d_h(\tilde{a}_2,\tilde{a}_s^-)$; $\tilde{a}_1$ and $\tilde{a}_2$ are the same, denoted by $\tilde{a}_1=\tilde{a}_2$

**Example 3.11.** Assume that $\tilde{a_1} = \langle(2, 3, 5, 6); 0.3, 0.4, 0.7\rangle$, $\tilde{a_2} = \langle(1, 3, 6, 7); 0.7, 0.5, 0.1\rangle$, $\tilde{a}^+ = \langle(1, 1, 1, 1); 1, 0, 0\rangle \in \Omega$. Then,

$$\begin{array}{rcl} d_h(\tilde{a}_1, \tilde{a}^+) & = & \frac{1}{6}\big(|(1 + 0.3 - 0.4 - 0.7)2 - |(1 + 1 - 0.0 - 0.0)1|+ \\ & & |(1 + 0.3 - 0.4 - 0.7)3 - |(1 + 1 - 0.0 - 0.0)1|+ \\ & & |(1 + 0.3 - 0.4 - 0.7)5 - |(1 + 1 - 0.0 - 0.0)1|+ \\ & & |(1 + 0.3 - 0.4 - 0.7)6 - |(1 + 1 - 0.0 - 0.0)1|\big) \\ & = & \frac{7}{60} \end{array}$$

and

$$\begin{array}{rcl} d_h(\tilde{a}_2, \tilde{a}^+) & = & \frac{1}{6}\big(|(1 + 0.7 - 0.5 - 0.1)1 - |(1 + 1 - 0.0 - 0.0)1|+ \\ & & |(1 + 0.7 - 0.5 - 0.1)3 - |(1 + 1 - 0.0 - 0.0)1|+ \\ & & |(1 + 0.7 - 0.5 - 0.1)6 - |(1 + 1 - 0.0 - 0.0)1|+ \\ & & |(1 + 0.7 - 0.5 - 0.1)7 - |(1 + 1 - 0.0 - 0.0)1|\big) \\ & = & \frac{65}{60} \end{array}$$

Since $d_h(\tilde{a}_1, \tilde{a}^+) < d_h(\tilde{a}_2, \tilde{a}^+)$, $\tilde{a}_2$ is smaller than $\tilde{a}_1$ (or $\tilde{a}_1 > \tilde{a}_2$).

From now on we use $I_n = \{1, 2, ..., n\}$ $I_m = \{1, 2, ..., m\}$ and $I_t = \{1, 2, ..., t\}$ as an index set for $n \in N$, $m \in N$ and $t \in N$, respectively.

# 4 SVTN-Weighted Operators

In this section, we present some arithmetic and geometric operators including SVTN-weighted geometric operator, SVTN-ordered weighted geometric operator, SVTN-hybrid geometric operator, SVTN-weighted arithmetic operator, SVTN-ordered weighted arithmetic operator and SVTN-hybrid arithmetic operator with their properties.

## 4.1 SVTN-Weighted Geometric Operators

In this subsection, we introduce some SVTN-weighted geometric operators on the SVTN-numbers.

**Definition 4.1.** [73] Let $\tilde{a}_j = \langle(a_j, b_j, c_j, d_j); w_{\tilde{a}_j}, u_{\tilde{a}_j}, y_{\tilde{a}_j}\rangle \in \Omega$ for all $j \in I_n$. Then, SVTN-weighted geometric operator, denoted by $S_{go}$, is defined by $S_{go} : \Omega^n \to \Omega$,

$$S_{go}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \tilde{a}_1^{w_1} \otimes \tilde{a}_2^{w_2} \otimes \cdots \otimes \tilde{a}_n^{w_n}$$

where $w = (w_1, w_2, ..., w_n)^T$ is a weight vector of $\tilde{a}_j$ for every $j \in I_n$ such that $w_j \in [0, 1]$ and $\sum_{j=1}^n w_j = 1$.

**Theorem 4.2.** *[73] Let $\tilde{a}_j = \langle(a_j, b_j, c_j, d_j); w_{\tilde{a}_j}, u_{\tilde{a}_j}, y_{\tilde{a}_j}\rangle \in \Omega$ for $j \in I_n$ and $S_{go}$ be the SVTN-weighted geometric operator. Then, their aggregated value by using $S_{go} : \Omega^n \to \Omega$, operator is also a SVTN-number and*

$$\begin{array}{rcl} S_{go}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) & = & \prod_{j=1}^n \tilde{a}_j^{w_j} \\ & = & \langle(\prod_{j=1}^n a_j^{w_j}, \prod_{j=1}^n b_j^{w_j}, \prod_{j=1}^n c_j^{w_j}, \prod_{j=1}^n d_j^{w_j}); \\ & & \prod_{j=1}^n w_{\tilde{a}_j}^{w_j}, 1 - \prod_{j=1}^n (1 - u_{\tilde{a}_j})^{w_j}, 1 - \prod_{j=1}^n (1 - y_{\tilde{a}_j})^{w_j}\rangle \end{array}$$

*where $w = (w_1, w_2, ..., w_n)^T$ is a weight vector of $\tilde{a}_j$ for all $j \in I_n$ such that $w_j \in [0, 1]$ and $\sum_{j=1}^n w_j = 1$.*

**Theorem 4.3.** *[73] Let $a\tilde{}_j = \langle (a_j, b_j, c_j, d_j); w_{a\tilde{}_j}, u_{a\tilde{}_j}, y_{a\tilde{}_j}\rangle \in \Omega$ for $j \in I_n$. Then,*

1. *If $\tilde{a}_j = \tilde{a}$, for all $j \in I_n$, then $S_{go}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \tilde{a}$,*
2. *$min_{j\in I}\{\tilde{a}_j\} \leq S_{go}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) \leq max_{j\in I}\{\tilde{a}_j\}$,*
3. *If $\tilde{a}^*_j = \langle (a^*_j, b^*_j, c^*_j, d_j)^*; w^*_{\tilde{a}_j}, u^*_{\tilde{a}_j}, y^*_{\tilde{a}_j}\rangle \in \Omega$ and $\tilde{a}_j \leq \tilde{a_j}^*$ for all $j \in I_n$, then $S_{go}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) \leq S_{go}(\tilde{a}^*_1, \tilde{a}^*_2, ..., \tilde{a}^*_n)$.*

**Definition 4.4.** Let $\tilde{a}_j = \langle (a_j, b_j, c_j, d_j); w_{\tilde{a}_j}, u_{\tilde{a}_j}, y_{\tilde{a}_j}\rangle \in \Omega$ for all $j \in I_n$. Then, an SVTN-ordered weighted geometric operator, denoted by $S_{ogo}$, is defined by $S_{ogo} : \Omega^n \to \Omega$,

$$S_{ogo}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \tilde{a}^{w_1}_{\sigma(1)} \otimes \tilde{a}^{w_2}_{\sigma(2)} \otimes \cdots \otimes \tilde{a}^{w_n}_{\sigma(n)}$$

where $w = (w_1, w_2, ..., w_n)^T$ is a weight vector of $\tilde{a}_j$ for every $j \in I$ such that $w_j \in [0, 1]$ and $\sum_{j=1}^n w_j = 1$.

Here, $(\sigma(1), \sigma(2), ..., \sigma(n))$ is a permutation of $(1, 2, ..., n)$ such that $a_{\sigma(j-1)} \geq a_{\sigma(j)}$ for all $j \in I_n$.

**Theorem 4.5.** *Let $\tilde{a}_j = \langle (a_j, b_j, c_j, d_j); w_{\tilde{a}_j}, u_{\tilde{a}_j}, y_{\tilde{a}_j}\rangle \in \Omega$ for $j \in I_n$ and $S_{ogo}$ be an SVTN-ordered weighted geometric operator. Then, their aggregated value by using $S_{ogo}$ operator is also a SVTN-number an*

$$\begin{aligned} S_{ogo}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \prod_{j=1}^n \tilde{a}^{w_j}_{\sigma(j)} &= \langle (\prod_{j=1}^n a^{w_j}_{\sigma(j)}, \prod_{j=1}^n b^{w_j}_{\sigma(j)}, \prod_{j=1}^n c^{w_j}_{\sigma(j)}, \prod_{j=1}^n d^{w_j}_{\sigma(j)}); \\ &\prod_{j=1}^n w^{w_j}_{\tilde{a}_{\sigma(j)}}, 1 - \prod_{j=1}^n (1 - u_{\tilde{a}_{\sigma(j)}})^{w_j}, 1 - \prod_{j=1}^n (1 - y_{\tilde{a}_{\sigma(j)}})^{w_j}\rangle \end{aligned}$$

*where $w = (w_1, w_2, ..., w_n)^T$ is a weight vector of $\tilde{a}_j$ for all $j \in I_n$ such that $w_j \in [0, 1]$ and $\sum_{j=1}^n w_j = 1$.*

**Theorem 4.6.** *Let $\tilde{a}_j = \langle (a_j, b_j, c_j, d_j); w_{\tilde{a}_j}, u_{\tilde{a}_j}, y_{\tilde{a}_j}\rangle \in \Omega$ for $j \in I_n$. Then,*

1. *If $\tilde{a}_j = \tilde{a}$, then $S_{ogo}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \tilde{a}$.*
2. *$min_j\{\tilde{a}_j\} \leq S_{ogo}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) \leq max_j\{\tilde{a}_j\}$*
3. *If $\tilde{a}^*_j = \langle (a^*_j, b^*_j, c^*_j, d_j)^*; w^*_{\tilde{a}_j}, u^*_{\tilde{a}_j}, y^*_{\tilde{a}_j}\rangle \in \Omega$ and $\tilde{a_j}^{\leq}\tilde{a_j}^*$, then $S_{ogo}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) \leq S_{ogo}(\tilde{a}^*_1, \tilde{a}^*_2, ..., \tilde{a}^*_n)$*
4. *If $\tilde{\acute{a}}_j \in \Omega$, then $S_{ogo}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = S_{ogo}(\tilde{\acute{a}}_1, \tilde{\acute{a}}_2, ..., \tilde{\acute{a}}_n)$ where $(\tilde{\acute{a}}_1, \tilde{\acute{a}}_2, ..., \tilde{\acute{a}}_n)$ is any permutation of $(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n)$.*

**Theorem 4.7.** *Let $\tilde{a}_j = \langle (a_j, b_j, c_j, d_j); w_{\tilde{a}_j}, u_{\tilde{a}_j}, y_{\tilde{a}_j}\rangle \in \Omega$ for $j \in I_n$ and $S_{ogo}$ be the SVTN-geometric averaging operator. Then, for all $j \in I_n$,*

1. *If $w = (1, 0, ..., 0)^T$, then $S_{ogo}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = max_j\{\tilde{a}_j\}$.*
2. *If $w = (0, 0, ..., 1)^T$, then $S_{ogo}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = min_j\{\tilde{a}_j\}$.*

**Definition 4.8.** Let $\tilde{a}_j = ((a_j, b_j, c_j, d_j); w_{\tilde{a}_j}, u_{\tilde{a}_j}, y_{\tilde{a}_j}) \in \Omega$ for $j \in I_n$. Then, an SVTN-hybrid geometric operator, denoted by $S^{\tilde{s}}_{hgo}$, is defined by

$$S^{\tilde{s}}_{hgo} : \Omega^n \to \Omega, \quad S^{\tilde{s}}_{hgo}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \tilde{\tilde{a}}^{\tilde{s}_1}_{\sigma(1)} \otimes \tilde{\tilde{a}}^{\tilde{s}_2}_{\sigma(2)} \otimes \cdots \otimes \tilde{\tilde{a}}^{\tilde{s}_n}_{\sigma(n)}$$

where for $j \in I_n$, $\tilde{\tilde{a}}_{\sigma(j)}$ is the jth largest of the weighted SVTN-numbers $\tilde{\tilde{a}}_j$, $\tilde{\tilde{a}}_j = \tilde{a}_j^{nw_j}$, $w = (w_1, w_2, ..., w_n)^T$ is a weight vector of $\tilde{a}_j$ such that $w_j \in [0, 1]$ and $\sum_{j=1}^n w_j = 1$, and $\tilde{s} = (\tilde{s}_1, \tilde{s}_2, ..., \tilde{s}_n)^T$ is a vector associated with the $S^{\tilde{s}}_{hgo}$ such that $\tilde{s}_j \in [0, 1]$ and $\sum_{j=1}^n \tilde{s}_j = 1$.

**Theorem 4.9.** *Let $\tilde{a}_j = ((a_j, b_j, c_j, d_j); w_{\tilde{a}_j}, u_{\tilde{a}_j}, y_{\tilde{a}_j}) \in \Omega$ for $j \in I_n$ and $S^{\tilde{s}}_{hgo}$ be the SVTN-hybrid geometric operator. Then, their aggregated value by using $S^{\tilde{s}}_{hgo}$ operator is also a SVTN-number and*

$$\begin{array}{rcl} S^{\tilde{s}}_{hgo}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \prod_{j=1}^n \tilde{\bar{a}}^{\tilde{s}_j}_{\sigma(j)} & = & \langle (\prod_{j=1}^n \bar{a}^{\tilde{s}_j}_{\sigma(j)}, \prod_{j=1}^n \bar{b}^{\tilde{s}_j}_{\sigma(j)} \prod_{j=1}^n \bar{c}^{\tilde{s}_j}_{\sigma(j)}, \prod_{j=1}^n \bar{d}^{\tilde{s}_j}_{\sigma(j)}); \\ & & \prod_{j=1}^n w^{\tilde{s}_j}_{\tilde{\bar{a}}_{\sigma(j)}}, 1 - \prod_{j=1}^n (1 - u_{\tilde{\bar{a}}_{\sigma(j)}})^{\tilde{s}_j}, 1 - \prod_{j=1}^n (1 - y_{\tilde{\bar{a}}_{\sigma(j)}})^{\tilde{s}_j} \rangle \end{array}$$

*where for $j \in I_n$, $\tilde{\bar{a}}_{\sigma(j)}$ is the jth largest of the weighted SVTN-numbers $\tilde{\bar{a}}_j$, $\tilde{\bar{a}}_j = \tilde{a}_j^{nw_j}$, $w = (w_1, w_2, ..., w_n)^T$ is a weight vector of $\tilde{a}_j$ such that $w_j \in [0, 1]$ and $\sum_{j=1}^n w_j = 1$, and $\tilde{s} = (\tilde{s}_1, \tilde{s}_2, ..., \tilde{s}_n)^T$ is a vector associated with the $S^{\tilde{s}}_{hgo}$ such that $\tilde{s}_j \in [0, 1]$ and $\sum_{j=1}^n \tilde{s}_j = 1$.*

**Corollary 4.10.** *Let $\tilde{a}_j = ((a_j, b_j, c_j, d_j); w_{\tilde{a}_j}, u_{\tilde{a}_j}, y_{\tilde{a}_j}) \in \Omega$ for $j \in I_n$. Then, SVTN-weighted geometric operator $S_{go}$ and SVTN-ordered weighted geometric operator $S_{ogo}$ operator is a special case of the SVTN-hybrid geometric operator $S^{\tilde{s}}_{hgo}$.*

## 4.2 SVTN-Weighted arithmetic Operators

In this subsection, we introduce some SVTN-weighted arithmetic operators on the SVTN-numbers.

**Definition 4.11.** [73] Let $a\tilde{\ }_j = \langle (a_j, b_j, c_j, d_j); w_{a\tilde{\ }_j}, u_{a\tilde{\ }_j}, y_{a\tilde{\ }_j} \rangle \in \Omega$ for all $j \in I_n$. Then, SVTN-weighted arithmetic operator, denoted by $S_{ao} : \Omega^n \to \Omega$, is defined by

$$S_{ao}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = w_1\tilde{a}_1 \oplus w_2\tilde{a}_2 \oplus \cdots \oplus w_n\tilde{a}_n$$

where $w = (w_1, w_2, ..., w_n)^T$ is a weight vector of $\tilde{a}_j$ for every $j \in I_n$ such that $w_j \in [0, 1]$ and $\sum_{j=1}^n w_j = 1$.

**Theorem 4.12.** *[73] Let $a\tilde{\ }_j = \langle (a_j, b_j, c_j, d_j); w_{a\tilde{\ }_j}, u_{a\tilde{\ }_j}, y_{a\tilde{\ }_j} \rangle \in \Omega$ for $j \in I_n$ and $S_{ao}$ be the SVTN-weighted arithmetic operator. Then, their aggregated value by using $S_{ao}$ operator is also a SVTN-number and*

$$\begin{array}{rcl} S_{ao}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) & = & \sum_{j=1}^n w_j\tilde{a}_j = \langle (\sum_{j=1}^n w_j a_j, \sum_{j=1}^n w_j b_j, \sum_{j=1}^n w_j c_j, \sum_{j=1}^n w_j d_j); \\ & = & 1 - \prod_{j=1}^n (1 - w_{\tilde{a}_j})^{w_j}, \prod_{j=1}^n u^{w_j}_{\tilde{a}_j}, \prod_{j=1}^n y^{w_j}_{\tilde{a}_j} \rangle \end{array}$$

*where $w = (w_1, w_2, ..., w_n)^T$ is a weight vector of $a\tilde{\ }_j$ for all $j \in I_n$ such that $w_j \in [0, 1]$ and $\sum_{j=1}^n w_j = 1$.*

**Theorem 4.13.** *[73] Let $a\tilde{\ }_j = \langle (a_j, b_j, c_j, d_j); w_{a\tilde{\ }_j}, u_{a\tilde{\ }_j}, y_{a\tilde{\ }_j} \rangle \in \Omega$ for $j \in I_n$. Then,*

1. *If $\tilde{a}_j = \tilde{a}$, for all $j \in I_n$, then $S_{ao}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \tilde{a}$,*
2. *$min_j\{\tilde{a}_j\} \leq S_{ao}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) \leq max_j\{\tilde{a}_j\}$,*
3. *If $\tilde{a}^*_j = \langle (a^*_j, b^*_j, c^*_j, d_j)^*; w^*_{\tilde{a}_j}, u^*_{\tilde{a}_j}, y^*_{\tilde{a}_j} \rangle \in \Omega$ and $\tilde{a}_j \leq \tilde{a}_j{}^*$ for all $j \in I_n$, then $S_{ao}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) \leq S_{ao}(\tilde{a}^*_1, \tilde{a}^*_2, ..., \tilde{a}^*_n)$.*

**Definition 4.14.** Let $\tilde{a}_j = \langle (a_j, b_j, c_j, d_j); w_{\tilde{a}_j}, u_{\tilde{a}_j}, y_{\tilde{a}_j} \rangle \in \Omega$ for all $j \in I_n$. Then, an SVTN-ordered weighted arithmetic operator, denoted by $S_{oao} : \Omega^n \to \Omega$, is defined by

$$S_{oao}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = w_1\tilde{a}_{\sigma(1)} \oplus w_2\tilde{a}_{\sigma(2)} \oplus \cdots \oplus w_n\tilde{a}_{\sigma(n)}$$

where $w = (w_1, w_2, ..., w_n)^T$ is a weight vector of $\tilde{a}_j$ for every $j \in I$ such that $w_j \in [0, 1]$ and $\sum_{j=1}^n w_j = 1$.
Here, $(\sigma(1), \sigma(2), ..., \sigma(n))$ is a permutation of $(1, 2, ..., n)$ such that $a_{\sigma(j-1)} \geq a_{\sigma(j)}$ for all $j \in I_n$.

**Theorem 4.15.** *Let $\tilde{a}_j = \langle (a_j, b_j, c_j, d_j); w_{\tilde{a}_j}, u_{\tilde{a}_j}, y_{\tilde{a}_j} \rangle \in \Omega$ for $j \in I_n$ and $S_{oao}$ be an SVTN-ordered weighted arithmetic operator. Then, their aggregated value by using $S_{oao}$ operator is also a SVTN-number and*

$$\begin{aligned} S_{oao}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) &= \textstyle\sum_{j=1}^n w_j \tilde{a}_{\sigma(j)} = \langle (\sum_{j=1}^n w_j a_{\sigma(j)}, \sum_{j=1}^n w_j b_{\sigma(j)}, \sum_{j=1}^n w_j c_{\sigma(j)}, \sum_{j=1}^n w_j d_{\sigma(j)}); \\ &= 1 - \textstyle\prod_{j=1}^n (1 - w_{\tilde{a}_{\sigma(j)}})^{w_j}, \prod_{j=1}^n u_{\tilde{a}_{\sigma(j)}}^{w_j}, \prod_{j=1}^n y_{\tilde{a}_{\sigma(j)}}^{w_j} \rangle \end{aligned}$$

*where $w = (w_1, w_2, ..., w_n)^T$ is a weight vector of $\tilde{a}_j$ for all $j \in I_n$ such that $w_j \in [0, 1]$ and $\sum_{j=1}^n w_j = 1$.*

**Theorem 4.16.** *Let $\tilde{a}_j = \langle (a_j, b_j, c_j, d_j); w_{\tilde{a}_j}, u_{\tilde{a}_j}, y_{\tilde{a}_j} \rangle \in \Omega$ for $j \in I_n$ and $S_{oao}$ be the SVTN-arithmetic averaging operator. Then, for all $j \in I_n$,*

1. *If $\tilde{a}_j = \tilde{a}$, then $S_{oao}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \tilde{a}$.*
2. *$min_j\{\tilde{a}_j\} \leq S_{oao}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) \leq max_j\{\tilde{a}_j\}$*
3. *If $\tilde{a}_j^* = \langle (a_j^*, b_j^*, c_j^*, d_j)^*; w_{\tilde{a}_j}^*, u_{\tilde{a}_j}^*, y_{\tilde{a}_j}^* \rangle \in \Omega$ and $\tilde{a}_j \overset{\leq}{} \tilde{a}_j{}^*$, then $S_{oao}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) \leq S_{oao}(\tilde{a}_1^*, \tilde{a}_2^*, ..., \tilde{a}_n^*)$*
4. *If $\tilde{\acute{a}}_j \in \Omega$, then $S_{oao}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = S_{oao}(\tilde{\acute{a}}_1, \tilde{\acute{a}}_2, ..., \tilde{\acute{a}}_n)$ where $(\tilde{\acute{a}}_1, \tilde{\acute{a}}_2, ..., \tilde{\acute{a}}_n)$ is any permutation of $(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n)$.*

**Theorem 4.17.** *Let $\tilde{a}_j = \langle (a_j, b_j, c_j, d_j); w_{\tilde{a}_j}, u_{\tilde{a}_j}, y_{\tilde{a}_j} \rangle \in \Omega$ for $j \in I_n$ and $S_{oao}$ be the SVTN-arithmetic averaging operator. Then, for all $j \in I_n$,*

1. *If $w = (1, 0, ..., 0)^T$, then $S_{oao}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = max_j\{\tilde{a}_j\}$.*
2. *If $w = (0, 0, ..., 1)^T$, then $S_{oao}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = min_j\{\tilde{a}_j\}$.*

**Definition 4.18.** Let $\tilde{a}_j = ((a_j, b_j, c_j, d_j); w_{\tilde{a}_j}, u_{\tilde{a}_j}, y_{\tilde{a}_j}) \in \Omega$ for $j \in I_n$. Then, an SVTN-hybrid arithmetic operator, denoted by $S_{hao}^{\tilde{s}} : \Omega^n \to \Omega$,, is defined by

$$S_{hao}^{\tilde{s}}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \tilde{s}_1 \bar{\tilde{a}}_{\sigma(1)} \oplus \tilde{s}_2 \bar{\tilde{a}}_{\sigma(2)} \oplus \cdots \oplus \tilde{s}_n \bar{\tilde{a}}_{\sigma(n)}$$

where for $j \in I_n$, $\bar{\tilde{a}}_{\sigma(j)}$ is the jth largest of the weighted SVTN-numbers $\bar{\tilde{a}}_j$, $\bar{\tilde{a}}_j = \tilde{a}_j^{nw_j}$, $w = (w_1, w_2, ..., w_n)^T$ is a weight vector of $\tilde{a}_j$ such that $w_j \in [0, 1]$ and $\sum_{j=1}^n w_j = 1$, and $\tilde{s} = (\tilde{s}_1, \tilde{s}_2, ..., \tilde{s}_n)^T$ is a vector associated with the $S_{hao}^{\tilde{s}}$ such that $\tilde{s}_j \in [0, 1]$ and $\sum_{j=1}^n \tilde{s}_j = 1$.

**Theorem 4.19.** *Let $\tilde{a}_j = ((a_j, b_j, c_j, d_j); w_{\tilde{a}_j}, u_{\tilde{a}_j}, y_{\tilde{a}_j}) \in \Omega$ for $j \in I_n$ and $S_{hao}^{\tilde{s}}$ be the SVTN-hybrid arithmetic operator. Then, their aggregated value by using $S_{hao}^{\tilde{s}}$ operator is also a SVTN-number and*

$$\begin{aligned} S_{hao}^{\tilde{s}}(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) &= \textstyle\sum_{j=1}^n \bar{\tilde{a}}_{\sigma(j)} \tilde{s}_j = \langle (\sum_{j=1}^n \bar{a}_{\sigma(j)}^{\tilde{s}_j}, \sum_{j=1}^n \bar{b}_{\sigma(j)}^{\tilde{s}_j} \sum_{j=1}^n \bar{c}_{\sigma(j)}^{\tilde{s}_j}, \sum_{j=1}^n \bar{d}_{\sigma(j)}^{\tilde{s}_j}); \\ &= 1 - \textstyle\prod_{j=1}^n (1 - w_{\bar{\tilde{a}}_{\sigma(j)}} \tilde{s}_j), \prod_{j=1}^n u_{\bar{\tilde{a}}_{\sigma(j)}} \tilde{s}_j, \prod_{j=1}^n y_{\bar{\tilde{a}}_{\sigma(j)}} \tilde{s}_j \rangle \end{aligned}$$

*where for $j \in I_n$, $\bar{\tilde{a}}_{\sigma(j)}$ is the jth largest of the weighted SVTN-numbers $\bar{\tilde{a}}_j$, $\bar{\tilde{a}}_j = \tilde{a}_j^{nw_j}$, $w = (w_1, w_2, ..., w_n)^T$ is a weight vector of $\tilde{a}_j$ such that $w_j \in [0, 1]$ and $\sum_{j=1}^n w_j = 1$, and $\tilde{s} = (\tilde{s}_1, \tilde{s}_2, ..., \tilde{s}_n)^T$ is a vector associated with the $S_{hao}^{\tilde{s}}$ such that $\tilde{s}_j \in [0, 1]$ and $\sum_{j=1}^n \tilde{s}_j = 1$.*

**Corollary 4.20.** *Let $\tilde{a}_j = ((a_j, b_j, c_j, d_j); w_{\tilde{a}_j}, u_{\tilde{a}_j}, y_{\tilde{a}_j}) \in \Omega$ for $j \in I_n$. Then, SVTN-weighted arithmetic operator $S_{ao}$ and SVTN-weighted arithmetic operator $S_{oao}$ operator is a special case of the SVTN-hybrid arithmetic operator $S_{hao}^{\tilde{s}}$.*

# 5 SVTN-Group Decision Making Method

In this section, by using the $S^{\tilde{s}}_{hgo}$ and $S_{go}$ operators we define a multi attribute group decision making method called SVTN-group decision making method.

**Definition 5.1.** Let $B = \{B_1, B_2, ..., B_m\}$ be a set of alternatives, $U = \{u_1, u_2, ..., u_n\}$ be a set of attributes, $D = \{d_1, d_2, ..., d_t\}$ be a set of decision makers, $\tilde{s} = (\tilde{s}_1, \tilde{s}_2, ..., \tilde{s}_n)^T$ be a weighting vector of the attributes where $\tilde{s}_j \in [0, 1]$ for $j \in I_n$ and $\sum_{j=1}^{n} \tilde{s}_j = 1$, and $w = (w_1, w_2, ..., w_t)^T$ be a weighting vector of the decision makers such that $w_j \in [0, 1]$ for $j \in I_n$ and $\sum_{j=1}^{t} w_j = 1$. If $\tilde{a}^k_{ij} = \langle (a^k_{ij}, b^k_{ij}, c^k_{ij}, d^k_{ij}); w^k_{ij}, u^k_{ij}, y^k_{ij} \rangle \in \Omega$, then

$$[\tilde{a}^k_{ij}]_{m\times n} = \begin{array}{c} \\ B_1 \\ B_2 \\ \vdots \\ B_m \end{array} \begin{array}{c} \begin{array}{cccc} u_1 & u_2 & \cdots & u_n \end{array} \\ \begin{pmatrix} \tilde{a}^k_{11} & \tilde{a}^k_{12} & \cdots & \tilde{a}^k_{1n} \\ \tilde{a}^k_{21} & \tilde{a}^k_{22} & \cdots & \tilde{a}^k_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}^k_{m1} & \tilde{a}^k_{m2} & \cdots & \tilde{a}^k_{mn} \end{pmatrix} \end{array}$$

is called an SVTN-group decision making matrix of the decision maker $d_k$ for each $k \in I_t$. The matrix is also written shortly as

$$[\tilde{a}^k_{ij}]_{m\times n} = \langle (a^k_{ij}, b^k_{ij}, c^k_{ij}, d^k_{ij}); w^k_{ij}, u^k_{ij}, y^k_{ij} \rangle$$

Now, we can give an algorithm of the SVTN-group decision making method as follows;

***Algorithm:***

**Step 1.** Construct

$[\tilde{a}^k_{ij}]_{m\times n} = \langle (a^k_{ij}, b^k_{ij}, c^k_{ij}, d^k_{ij}); w^k_{ij}, u^k_{ij}, y^k_{ij} \rangle$ of $d_k$ for each $k \in I_t$.

**Step 2.** Compute $\tilde{a}^k_i = S_{go}(\tilde{a}^k_{i1}, \tilde{a}^k_{i2}, ..., \tilde{a}^k_{in}) = \prod_{j=1}^{n} (\tilde{a}^k_{ij})^{w_j}$ for each $k \in I_t$ and $i \in I_m$ to derive the individual overall preference SVTN-values $\tilde{a}^k_i$ of the alternative $B_i$.

**Step 3.** Compute $\tilde{a}_i = S^{\tilde{s}}_{hgo}(\tilde{a}^1_i, \tilde{a}^2_i, ..., \tilde{a}^t_i) =< (a_i, b_i, c_i); w_{\tilde{a}_i}, u_{\tilde{a}_i}, y_{\tilde{a}_i} >$ for each $i \in I_m$ to derive the collective overall preference SVTN-values $\tilde{a}_i$ of the alternative $B_i$.

**Step 4.** Compute $d_h(\tilde{a}_i, \tilde{a}^+)$ for each $i \in I_m$.

**Step 5.** Rank all alternatives $B_i$ according to the $d_h(\tilde{a}_i, \tilde{a}^+)$ for each $i \in I_m$.

**Example 5.2.** (It's adopted from [70]) Let us suppose there is a risk investment company, which wants to invest a sum of money in the best option. There is a panel with five possible alternatives (engineer construction projects) to invest the money. The risk investment company must take a decision according to four attributes: $u_1$ = "risk analysis", $u_2$ = "growth analysis", $u_3$ = "social-political impact analysis", $u_4$ = "environmental impact analysis". The five possible alternatives $B_i\,(i = 1, 2, ..., 5)$ are to be evaluated using the SVTN-numbers by the four decision makers (whose weighting vector $w = (0.2, 0.4, 0.1, 0.3)^T$) under the above four attributes (whose weighting vector $\tilde{s} = (0.25, 0.25, 0.25, 0.25)^T$), and construct, respectively,

**Step 1.** For each $k = 1, 2, 3, 4$, the decision maker $d_k$ construct own decision matrices $[\tilde{a}^k_{ij}]_{5x4}$ as Table 1:

**Step 2.** For each $k = 1, 2, 3, 4$ and $i = 1, 2, 3, 4, 5$ compute $\tilde{a}_i^k = S_{go}(\tilde{a}_{i1}^k, \tilde{a}_{i2}^k, ..., \tilde{a}_{in}^k)$ as follows:

$$\begin{array}{l}
\tilde{a}_1^1 = \big((0.170, 0.411, 0.606, 0.814); 0.442, 0.749, 0.409\big) \\
\tilde{a}_2^1 = \big((0.194, 0.342, 0.517, 0.800); 0.534, 0.543, 0.302\big) \\
\tilde{a}_3^1 = \big((0.224, 0.259, 0.517, 0.628); 0.237, 0.513, 0.281\big) \\
\tilde{a}_4^1 = \big((0.214, 0.332, 0.464, 0.774); 0.460, 0.518, 0.407\big) \\
\tilde{a}_5^1 = \big((0.139, 0.209, 0.401, 0.580); 0.186, 0.587, 0.332\big) \\
\tilde{a}_1^2 = \big((0.226, 0.278, 0.459, 0.763); 0.540, 0.423, 0.500\big) \\
\tilde{a}_2^2 = \big((0.285, 0.388, 0.592, 0.728); 0.379, 0.686, 0.522\big) \\
\tilde{a}_3^2 = \big((0.476, 0.581, 0.700, 0.814); 0.394, 0.349, 0.300\big) \\
\tilde{a}_4^2 = \big((0.230, 0.332, 0.613, 0.738); 0.564, 0.714, 0.346\big) \\
\tilde{a}_5^2 = \big((0.132, 0.147, 0.355, 0.531); 0.293, 0.396, 0.635\big) \\
\tilde{a}_1^3 = \big((0.115, 0.155, 0.459, 0.599); 0.275, 0.806, 0.674\big) \\
\tilde{a}_2^3 = \big((0.298, 0.375, 0.592, 0.806); 0.309, 0.387, 0.679\big) \\
\tilde{a}_3^3 = \big((0.107, 0.112, 0.150, 0.513); 0.491, 0.537, 0.670\big) \\
\tilde{a}_4^3 = \big((0.200, 0.310, 0.565, 0.673); 0.500, 0.346, 0.693\big) \\
\tilde{a}_5^3 = \big((0.164, 0.176, 0.355, 0.650); 0.426, 0.527, 0.519\big) \\
\tilde{a}_1^4 = \big((0.182, 0.302, 0.537, 0.781); 0.275, 0.627, 0.527\big) \\
\tilde{a}_2^4 = \big((0.154, 0.305, 0.428, 0.693); 0.225, 0.568, 0.617\big) \\
\tilde{a}_3^4 = \big((0.000, 0.232, 0.504, 0.675); 0.354, 0.551, 0.513\big) \\
\tilde{a}_4^4 = \big((0.200, 0.300, 0.417, 0.660); 0.509, 0.481, 0.342\big) \\
\tilde{a}_5^4 = \big((0.000, 0.182, 0.374, 0.625); 0.282, 0.424, 0.270\big)
\end{array}$$

$$\begin{array}{l}
\tilde{a}_1^1 = \big((0.170, 0.411, 0.606, 0.814); 0.442, 0.749, 0.409\big) \\
\tilde{a}_2^1 = \big((0.194, 0.342, 0.517, 0.800); 0.534, 0.543, 0.302\big) \\
\tilde{a}_3^1 = \big((0.224, 0.259, 0.517, 0.628); 0.237, 0.513, 0.281\big) \\
\tilde{a}_4^1 = \big((0.214, 0.332, 0.464, 0.774); 0.460, 0.518, 0.407\big) \\
\tilde{a}_5^1 = \big((0.139, 0.209, 0.401, 0.580); 0.186, 0.587, 0.332\big) \\
\tilde{a}_1^2 = \big((0.226, 0.278, 0.459, 0.763); 0.540, 0.423, 0.500\big) \\
\tilde{a}_2^2 = \big((0.285, 0.388, 0.592, 0.728); 0.379, 0.686, 0.522\big) \\
\tilde{a}_3^2 = \big((0.476, 0.581, 0.700, 0.814); 0.394, 0.349, 0.300\big) \\
\tilde{a}_4^2 = \big((0.230, 0.332, 0.613, 0.738); 0.564, 0.714, 0.346\big) \\
\tilde{a}_5^2 = \big((0.132, 0.147, 0.355, 0.531); 0.293, 0.396, 0.635\big) \\
\tilde{a}_1^3 = \big((0.115, 0.155, 0.459, 0.599); 0.275, 0.806, 0.674\big) \\
\tilde{a}_2^3 = \big((0.298, 0.375, 0.592, 0.806); 0.309, 0.387, 0.679\big) \\
\tilde{a}_3^3 = \big((0.107, 0.112, 0.150, 0.513); 0.491, 0.537, 0.670\big) \\
\tilde{a}_4^3 = \big((0.200, 0.310, 0.565, 0.673); 0.500, 0.346, 0.693\big) \\
\tilde{a}_5^3 = \big((0.164, 0.176, 0.355, 0.650); 0.426, 0.527, 0.519\big) \\
\tilde{a}_1^4 = \big((0.182, 0.302, 0.537, 0.781); 0.275, 0.627, 0.527\big) \\
\tilde{a}_2^4 = \big((0.154, 0.305, 0.428, 0.693); 0.225, 0.568, 0.617\big) \\
\tilde{a}_3^4 = \big((0.000, 0.232, 0.504, 0.675); 0.354, 0.551, 0.513\big) \\
\tilde{a}_4^4 = \big((0.200, 0.300, 0.417, 0.660); 0.509, 0.481, 0.342\big) \\
\tilde{a}_5^4 = \big((0.000, 0.182, 0.374, 0.625); 0.282, 0.424, 0.270\big)
\end{array}$$

**Step 3.** Assume that $w = (0.2, 0.4, 0.1, 0.3)^T$ and $\tilde{s} = (0.25, 0.25, 0.25)^T$. We can compute

$$\tilde{a}_i = S_{hgo}^{\tilde{s}}(\tilde{a}_i^1, \tilde{a}_i^2, ..., \tilde{a}_i^t) =< (a_i, b_i, c_i); w_{\tilde{a}_i}, u_{\tilde{a}_i}, y_{\tilde{a}_i} >$$

for each $i = 1, 2, 3, 4, 5$ as follows:

$$\begin{aligned}
\tilde{a}_1 &= \big((0.187, 0.291, 0.509, 0.760); 0.396, 0.569, 0.513\big)\\
\tilde{a}_2 &= \big((0.219, 0.351, 0.523, 0.738); 0.340, 0.584, 0.536\big)\\
\tilde{a}_3 &= \big((0.000, 0.298, 0.524, 0.698); 0.352, 0.451, 0.414\big)\\
\tilde{a}_4 &= \big((0.214, 0.320, 0.512, 0.714); 0.519, 0.553, 0.404\big)\\
\tilde{a}_5 &= \big((0.000, 0.171, 0.370, 0.579); 0.274, 0.450, 0.479\big)
\end{aligned}$$

**Step 4.** Compute $d_h(\tilde{a}_i, \tilde{a}_i^+)$ for each alternative $B_i$, $i = 1, 2, 3, 4, 5$, as follows:

$$\begin{aligned}
&d_h(\tilde{a}_1, \tilde{a}^+) = 1.242, d_h(\tilde{a}_2, \tilde{a}^+) = 1.266,\\
&d_h(\tilde{a}_3, \tilde{a}^+) = 1.210, d_h(\tilde{a}_4, \tilde{a}^+) = 1.169,\\
&d_h(\tilde{a}_5, \tilde{a}^+) = 1.269
\end{aligned}$$

Then we get the rank,

$$d_h(\tilde{a}_5, \tilde{a}^+) > d_h(\tilde{a}_2, \tilde{a}^+) > d_h(\tilde{a}_1, \tilde{a}^+) > d_h(\tilde{a}_3, \tilde{a}^+) > d_h(\tilde{a}_4, \tilde{a}^+)$$

**Step 5.** Therefore, we can rank all alternatives $B_i$ according to the $d_h(\tilde{a}_i, \tilde{a}_i^+)$ for each $i = 1, 2, 3, 4, 5$.

$$B_5 < B_2 < B_1 < B_3 < B_4$$

and thus the most desirable alternative is $B_4$.

# 6 Comparative Analysis and Discussion

In this section, a comparative study is presented to show the flexibility and feasibility of the introduced SVTN-group decision making method. Different methods used to solve the same SVTN-group decision making problem with SVTN-information is given by Ye [73]. The ranking results obtained by different methods are summarized in Table 2.

From the results presented in Table 2, the best alternative in proposed method and Ye's method [73] with geometric operator is $B_4$, whilst the worst one is $B_5$. In contrast, by using the methods in the proposed method and Ye's method [73] with arithmetic operator, the best is $B_3$, whilst the worst is $B_5$. There are a number of reasons why differences exist between the final rankings of the methods. First, the author uses a score and accurate function in Ye's method [73] with arithmetic operator and Ye's method [73] with geometric opera-tor. Moreover, different aggregation operators, which is arithmetic and geometric operator, lead to different rankings because the operators emphasize the decision makers judgments differently. The proposed method is different in that it contains two major phrases. First, the proposed method uses both SVTN-weighted geo-metric operator and the SVTN-hybrid geometric operator to aggregate the SVTN-numbers. Second, based on distance measure, the method uses SVTN-positive ideal solution and SVTN-negative ideal solution to rank the SVTN-information. Finally, the ranking of the proposed method is similar to other methods. Therefore, the proposed method is flexible and feasible.

# 7 Conclusion

Due to the ambiguity of people's thinking and the complexity of objective things, the attribute values of the MAGDM problems cannot always be expressed by exact and crisp values and it may be easier to escribe them by neutrosophic information. This paper introduced an MAGDM in which the attribute values are expressed with the SVTN-numbers, which are solved by developing a new decision method based on geometric aggregation operators of SVTN-numbers. The proposed method with SVTN-numbers is more suitable for real scientific and engineering applications, because the proposed decision-making method includes much more information and can deal with indeterminate and inconsistent decision-making problems. In the future, we shall further develop more aggregation operators for SVTN-numbers and apply them to solve practical applications in areas such as group decision making, expert system, information fusion system, fault diagnoses, medical diagnoses and so on.

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**Table 1.** The decision matrices of decision maker $d_k$.

$$[\tilde{a}^1_{ij}]_{5x4} = \begin{pmatrix} \big((0.5,0.7,0.8,0.9);0.5,0.6,0.7\big) & \big((0.1,0.3,0.4,0.7);0.4,0.9,0.3\big) & \big((0.1,0.1,0.8,0.9);0.7,0.5,0.3\big) & \big((0.2,0.7,0.8,0.9);0.4,0.5,0.3\big) \\ \big((0.4,0.5,0.7,0.8);0.3,0.2,0.3\big) & \big((0.2,0.3,0.5,0.8);0.7,0.4,0.2\big) & \big((0.3,0.4,0.7,0.8);0.7,0.4,0.6\big) & \big((0.1,0.3,0.4,0.8);0.5,0.8,0.3\big) \\ \big((0.5,0.6,0.7,0.8);0.2,0.7,0.2\big) & \big((0.1,0.1,0.4,0.5);0.1,0.4,0.3\big) & \big((0.2,0.3,0.5,0.7);0.4,0.7,0.3\big) & \big((0.4,0.5,0.6,0.7);0.7,0.4,0.3\big) \\ \big((0.4,0.5,0.6,0.7);0.2,0.5,0.4\big) & \big((0.2,0.3,0.4,0.9);0.5,0.5,0.6\big) & \big((0.1,0.3,0.4,0.7);0.5,0.8,0.2\big) & \big((0.2,0.3,0.5,0.7);0.7,0.4,0.1\big) \\ \big((0.3,0.5,0.6,0.7);0.2,0.5,0.4\big) & \big((0.1,0.2,0.5,0.8);0.1,0.8,0.3\big) & \big((0.3,0.4,0.4,0.8);0.1,0.5,0.4\big) & \big((0.1,0.1,0.2,0.3);0.5,0.1,0.3\big) \end{pmatrix}$$

$$[\tilde{a}^2_{ij}]_{5x4} = \begin{pmatrix} \big((0.1,0.1,0.2,0.5);0.8,0.4,0.7\big) & \big((0.2,0.3,0.4,0.8);0.8,0.4,0.3\big) & \big((0.1,0.1,0.8,0.9);0.3,0.3,0.3\big) & \big((0.6,0.7,0.8,0.9);0.3,0.5,0.6\big) \\ \big((0.3,0.4,0.7,0.8);0.2,0.4,0.8\big) & \big((0.3,0.4,0.7,0.9);0.7,0.9,0.3\big) & \big((0.6,0.7,0.7,0.8);0.8,0.5,0.7\big) & \big((0.2,0.3,0.4,0.5);0.2,0.2,0.4\big) \\ \big((0.6,0.7,0.8,0.9);0.5,0.1,0.3\big) & \big((0.6,0.7,0.8,0.9);0.3,0.4,0.3\big) & \big((0.2,0.3,0.5,0.7);0.7,0.4,0.3\big) & \big((0.4,0.5,0.6,0.7);0.4,0.4,0.3\big) \\ \big((0.4,0.5,0.6,0.7);0.6,0.4,0.5\big) & \big((0.2,0.3,0.7,0.8);0.7,0.8,0.3\big) & \big((0.2,0.3,0.4,0.7);0.3,0.9,0.3\big) & \big((0.2,0.3,0.6,0.7);0.5,0.6,0.3\big) \\ \big((0.1,0.1,0.4,0.8);0.3,0.4,0.2\big) & \big((0.2,0.2,0.5,0.6);0.2,0.3,0.1\big) & \big((0.10.3,0.4,0.8);0.5,0.4,0.9\big) & \big((0.1,0.1,0.2,0.3);0.4,0.5,0.9\big) \end{pmatrix}$$

$$[\tilde{a}^3_{ij}]_{5x4} = \begin{pmatrix} \big((0.1,0.1,0.2,0.5);0.5,0.9,0.9\big) & \big((0.1,0.2,0.4,0.8);0.1,0.6,0.3\big) & \big((0.4,0.5,0.8,0.9);0.8,0.7,0.3\big) & \big((0.1,0.1,0.8,0.4);0.5,0.9,0.8\big) \\ \big((0.3,0.4,0.7,0.8);0.5,0.1,0.3\big) & \big((0.4,0.4,0.7,0.9);0.5,0.3,0.9\big) & \big((0.3,0.5,0.7,0.8);0.5,0.4,0.3\big) & \big((0.2,0.3,0.4,0.7);0.1,0.6,0.3\big) \\ \big((0.1,0.1,0.1,0.4);0.4,0.7,0.8\big) & \big((0.1,0.1,0.1,0.4);0.8,0.6,0.8\big) & \big((0.2,0.3,0.7,0.9);0.1,0.1,0.3\big) & \big((0.1,0.1,0.2,0.7);0.5,0.4,0.3\big) \\ \big((0.4,0.5,0.8,0.9);0.5,0.4,0.3\big) & \big((0.2,0.3,0.7,0.8);0.5,0.4,0.9\big) & \big((0.4,0.5,0.8,0.9);0.5,0.4,0.3\big) & \big((0.1,0.2,0.3,0.4);0.5,0.2,0.4\big) \\ \big((0.3,0.3,0.4,0.8);0.5,0.2,0.3\big) & \big((0.2,0.2,0.5,0.8);0.5,0.7,0.3\big) & \big((0.1,0.2,0.4,0.8);0.1,0.5,0.3\big) & \big((0.1,0.1,0.2,0.4);0.5,0.4,0.8\big) \end{pmatrix}$$

$$[\tilde{a}^4_{ij}]_{5x4} = \begin{pmatrix} \big((0.5,0.7,0.8,0.9);0.1,0.9,0.3\big) & \big((0.2,0.3,0.5,0.8);0.5,0.6,0.6\big) & \big((0.1,0.2,0.4,0.5);0.5,0.4,0.3\big) & \big((0.1,0.2,0.5,0.8);0.2,0.3,0.6\big) \\ \big((0.2,0.4,0.5,0.6);0.5,0.4,0.3\big) & \big((0.2,0.3,0.4,0.8);0.1,0.4,0.8\big) & \big((0.1,0.2,0.5,0.8);0.5,0.4,0.3\big) & \big((0.1,0.3,0.4,0.6);0.3,0.8,0.5\big) \\ \big((0.1,0.2,0.3,0.4);0.1,0.2,0.3\big) & \big((0.0,0.1,0.6,0.7);0.5,0.7,0.3\big) & \big((0.0,0.1,0.4,0.7);0.4,0.7,0.2\big) & \big((0.4,0.5,0.8,0.9);0.5,0.4,0.8\big) \\ \big((0.2,0.3,0.4,0.7);0.5,0.6,0.3\big) & \big((0.2,0.3,0.4,0.5);0.5,0.2,0.4\big) & \big((0.2,0.3,0.6,0.7);0.6,0.9,0.3\big) & \big((0.2,0.3,0.4,0.9);0.5,0.4,0.3\big) \\ \big((0.0,0.1,0.2,0.8);0.5,0.4,0.3\big) & \big((0.1,0.3,0.7,0.9);0.5,0.4,0.3\big) & \big((0.4,0.5,0.7,0.8);0.2,0.6,0.5\big) & \big((0.0,0.1,0.2,0.3);0.1,0.4,0.1\big) \end{pmatrix}$$

**Table 2.** The ranking results of different methods.

| Methods | Ranking results |
|---|---|
| *The proposed method with arithmetic operator* | $B_5 < B_2 < B_1 < B_4 < B_3$ |
| *The proposed method with geometric operator* | $B_5 < B_2 < B_1 < B_3 < B_4$ |
| *Ye's method* [73] *with geometric operator* | $B_5 < B_2 < B_3 < B_1 < B_4$ |
| *Ye's method* [73] *with arithmetic operator* | $B_5 < B_2 < B_1 < B_4 < B_3$ |

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# TOPSIS Method for MADM based on Interval Trapezoidal Neutrosophic Number

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**Abstract:** TOPSIS (Technique for Order Preference by Similarity to Ideal Solution) is a very common method for Multiple Attribute Decision Making (MADM) problem in crisp as well as uncertain environment. The interval trapezoidal neutrosophic number can handle incomplete, indeterminate and inconsistent information which are generally occurred in uncertain environment. In this paper, we propose TOPSIS method for MADM, where the rating values of the attributes are interval trapezoidal neutrosophic numbers and the weight information of the attributes are known or partially known or completely unknown. We develop optimization models to obtain weights of the attributes with the help of maximum deviation strategy for partially known and completely unknown cases. Finally, we provide a numerical example to illustrate the proposed approach and make a comparative analysis.

**Keywords:** Interval trapezoidal neutrosophic number, Multi-attribute decision making, TOPSIS, Unknown weight information.

# 1 Introduction

Multi-attribute decision making (MADM) is a popular field of study in decision analysis. MADM refers to making choice of the best alternative from a finite set of decision alternatives in terms of multiple, usually conflicting criteria. The decision maker uses the rating value of the attribute in terms of fuzzy sets [1], intuitionistic fuzzy sets [2], hesitant fuzzy sets [3], and neutrosophic sets [4].

In classical MADM methods, the ratings and weights of the criteria are known precisely. TOPSIS [5] is one of the classical methods among many MADM techniques like Preference Ranking Organization METHod for Enrichment of Evaluations (PROMETHEE) [6], Vlse Kriterijuska OptimizacijaI Komoromisno Resenje (VIKOR) [7], ELimination Et Choix Traduisant la REalit (ELECTRE) [8], Analytic Hierarchy Process (AHP) [9], etc. MADM problem has also been studied in fuzzy environment [10–14] and intuitionistic fuzzy environment [15–18]. Researchers have extended the TOPSIS method to deal with MADM problems in different environment. Chen [19] extended the concept of TOPSIS method to develop a methodology for MADM problem in fuzzy environment. Boran et al. [20] extended the TOPSIS method for MADM in intutionistic fuzzy sets. Zhao [21] proposed TOPSIS method under interval intutionistic fuzzy number. Liu [22] proposed TOPSIS method for MADM under trapezoidal intuitionistic fuzzy environment with partial and unknown attribute weight information.

Compared to fuzzy set and intutionistic fuzzy set, neutrosophic set [4] has the potential to deal with MADM problem because it can effectively handle indeterminate and incomplete information. Hybrids of neutrosophic set have been applied in MADM and multi-attribute group decision making (MAGDM) problems [23–30]. Biswas et al. [31] developed TOPSIS strategy for MAGDM with single valued neutrosophic sets [32]. Chi and Liu [33] extended TOPSIS method for MADM problem based on interval neutrosophic set [4]. Zhang and Wu [34] developed TOPSIS strategy in single valued neutrosophic set where weights of the attributes are completely unknown. Biswas et al. [35] introduced Neutrosophic TOPSIS with group decision making. Pramanik et al. [36] extended TOPSIS approach for MAGDM in refined neutrosophic environment. Dey et al. [37] introduced TOPSIS for solving MADM problems under bi-polar neutrosophic environment. Mondal et al. [38] developed Rough neutrosophic TOPSIS for MAGDM problem. Ye [39] proposed TOPSIS method for MAGDM problem based on single valued neutrosophic linguistic number. Dey et al. [40] developed generalized neutrosophic soft MADM based on TOPSIS. Pramanik et al. [41] extended TOPSIS method for MADM based on soft expert set environment. Biswas et al. [42] developed TOPSIS strategy for MADM with trapezoidal neutrosophic numbers.

Interval trapezoidal neutrosophic number (ITrNN) [43] is a generalization of single valued trapezoidal neutrosophic number (SVTrNN). Ye [44] and Subhaş [45] introduced the SVTrNN where each element is expressed by trapezoidal number that has truth, indeterminacy and falsity membership degrees which are single valued. However, decision makers may face difficulties to express their opinions in terms of single valued truth, indeterminacy and falsity membership degrees. In interval trapezoidal neutrosophic number truth, indeterminacy and falsity membership degrees are interval valued. Therefore, decision makers can express their opinion throughout this number in a flexible way to face such difficulties.

The above literature review reflects that the TOPSIS method has not been studied earlier based on interval trapezoidal neutrosophic number, even though this number can play effective role with indeterminate and uncertain information in MADM problem. To fill this research gap, our objectives in this paper are as follows:

- To propose TOPSIS method for MADM problem based on interval valued trapezoidal neutrosophic number.
- To develop the model where the rating values of the attributes are ITrNN and weight information is known, partially known and completely unknown.

We organise the paper as follows: Section 2 describes the preliminaries of fuzzy sets, trapezoidal fuzzy number, neutrosophic sets, SVTrNN, ITrNN, and Hamming distance between ITrNNs. Section 3 briefly presents classical TOPSIS method. Section 4 presents TOPSIS method for MADM based on ITrNN. An application example with comparative analysis is given in Section 5. Finally, Section 6 presents some conclusions and future scopes of research.

# 2 Preliminaries

In this section, we briefly review the definition of fuzzy sets, single-valued neutrosophic sets, single-valued trapezoidal neutrosophic number, and interval trapezoidal neutrosophic numbers.

**Definition 1.** [1] Let $X$ be a universe of discourse. Then a fuzzy set $A$ is defined by

$$A = \{\langle x, \mu_A(x)\rangle | \ x \in X\}, \tag{1}$$

which is characterized by a membership function $\mu_A : X \to [0, 1]$, where $\mu_A(x)$ is the degree of membership of the element $x$ to the set $A$.

**Definition 2.** [46,47] A generalized trapezoidal fuzzy number $A$ denoted by $A = (a, b, c, d; w)$ is described as a fuzzy subset of a real number $\mathbb{R}$ with membership function $\mu_A$ which is defined by

$$\mu_A(x) = \begin{cases} \dfrac{(x-a)w}{b-a}, & a \leq x < b \\ w, & b \leq x \leq c \\ \dfrac{(d-x)w}{d-c}, & c < x \leq d \\ 0, & otherwise. \end{cases}$$

where $a, b, c, d \in \mathbb{R}$ and $w$ is a membership degree.

**Definition 3.** [32] Let $X$ be universe of discourse. Then a single-valued neutrosophic set $A$ is defined as $A = \{< x, T_A(x), F_A(x), I_A(x) >: x \in X\}$ which is characterized by a truth-membership function $T_A(x) : X \rightarrow [0, 1]$, falsity membership function $F_A : X \rightarrow [0, 1]$, and an indeterminacy membership function $I_A : X \rightarrow [0, 1]$ of the element $x$ to the set $A$, and the condition $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3 \; \forall \; x \in X$.

**Definition 4.** [44,45] Let $\alpha$ be a single-valued neutrosophic trapezoidal number (SVNTrN). Then its membership functions are given by

$$T_\alpha(x) = \begin{cases} \dfrac{(x-a)t_\alpha}{b-a}, & a \leq x < b \\ t_\alpha, & b \leq x \leq c \\ \dfrac{(d-x)t_\alpha}{d-c}, & c < x \leq d \\ 0, & otherwise. \end{cases}$$

$$I_\alpha(x) = \begin{cases} \dfrac{b-x+(x-a)i_\alpha}{b-a}, & a \leq x < b \\ i_\alpha, & b \leq x \leq c \\ \dfrac{x-c+(d-x)i_\alpha}{d-c}, & c < x \leq d \\ 0, & otherwise. \end{cases}$$

$$F_\alpha(x) = \begin{cases} \dfrac{b-x+(x-a)f_\alpha}{b-a}, & a \leq x < b \\ f_\alpha, & b \leq x \leq c \\ \dfrac{x-c+(d-x)f_\alpha}{d-c}, & c < x \leq d \\ 0, & otherwise. \end{cases}$$

where $T_\alpha$ is truth membership function, $I_\alpha$ is indeterminancy membership function and $F_\alpha$ is falsity membership function, and they all lie between 0 and 1 and satisfy the condition $0 \leq T_\alpha(x) + I_\alpha(x) + F_\alpha(x) \leq 3$ where $a, b, c, d$ are real numbers. Then $\alpha = ([a, b, c, d]; t_\alpha, i_\alpha, f_\alpha)$ is called a neutrosophic trapezoidal number.

**Definition 5.** [43] Let $\tilde{\alpha}$ be trapezoidal neutrosophic number. Then its membership functions are given by

$$T_{\tilde{\alpha}}(x) = \begin{cases} \dfrac{(x-a)t_{\tilde{\alpha}}}{b-a}, & a \leq x < b \\ t_{\tilde{\alpha}}, & b \leq x \leq c \\ \dfrac{(d-x)t_{\tilde{\alpha}}}{d-c}, & c < x \leq d \\ 0, & otherwise. \end{cases}$$

$$I_{\tilde{\alpha}}(x) = \begin{cases} \dfrac{b-x+(x-a)i_{\tilde{\alpha}}}{b-a}, & a \leq x < b \\ i_{\tilde{\alpha}}, & b \leq x \leq c \\ \dfrac{x-c+(d-x)i_{\tilde{\alpha}}}{d-c}, & c < x \leq d \\ 0, & otherwise. \end{cases}$$

$$F_{\tilde{\alpha}}(x) = \begin{cases} \dfrac{b-x+(x-a)f_{\tilde{\alpha}}}{b-a}, & a \leq x < b \\ f_{\tilde{\alpha}}, & b \leq x \leq c \\ \dfrac{x-c+(d-x)f_{\tilde{\alpha}}}{d-c}, & c < x \leq d \\ 0, & otherwise. \end{cases}$$

where $T_{\tilde{\alpha}}$ is truth membership function, $I_{\tilde{\alpha}}$ is indeterminancy membership function and $F_{\tilde{\alpha}}$ is falsity membership function and $t_{\tilde{\alpha}}, i_{\tilde{\alpha}}, f_{\tilde{\alpha}}$ are subsets of [0,1] and $0 \leq \sup(t_{\tilde{\alpha}}) + \sup(i_{\tilde{\alpha}}) + \sup(f_{\tilde{\alpha}}) \leq 3$. Then $\alpha$ is called an interval trapezoidal neutrosophic number and it is denoted by $\tilde{\alpha} = ([a, b, c, d]; t_{\tilde{\alpha}}, i_{\tilde{\alpha}}, f_{\tilde{\alpha}})$. We take $t_{\tilde{\alpha}} = [\underline{t}, \bar{t}], i_{\tilde{\alpha}} = [\underline{i}, \bar{i}]$ and $f_{\tilde{\alpha}} = [\underline{f}, \bar{f}]$

**Definition 6.** [43] An interval trapezoidal neutrosophic number (ITrNN) $\tilde{\alpha} = ([a, b, c, d]; [\underline{t}, \bar{t}], [\underline{i}, \bar{i}], [\underline{f}, \overline{f}])$ is said to be positive ITrNN if $a \geq 0$ and one of the four values of $a, b, c, d$ is not equal to zero.

**Definition 7.** Let $\tilde{\alpha} = ([a_1, b_1, c_1, d_1]; [\underline{t}_1, \bar{t}_1], [\underline{i}_1, \bar{i}_1], [\underline{f}_1, \bar{f}_1])$ and $\tilde{\beta} = ([a_2, b_2, c_2, d_2]; [\underline{t}_2, \bar{t}_2], [\underline{i}_2, \bar{i}_2], [\underline{f}_2, \bar{f}_2])$ be two ITrNNs. Then the following operations are valid:

1. $\tilde{\alpha} \bigoplus \tilde{\beta} = \left( \begin{array}{c} [a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2]; \\ [\underline{t}_1 + \underline{t}_2 - \underline{t}_1\underline{t}_2, \bar{t}_1 + \bar{t}_2 - \bar{t}_1\bar{t}_2], [\underline{i}_1\underline{i}_2, \bar{i}_1\bar{i}_2], [\underline{f}_1\underline{f}_2, \bar{f}_1\bar{f}_2] \end{array} \right);$

2. $\tilde{\alpha} \bigotimes \tilde{\beta} = \left( \begin{array}{c} ([a_1a_2, b_1b_2, c_1c_2, d_1d_2]; [\underline{t}_1\underline{t}_2, \bar{t}_1\bar{t}_2], \\ [\underline{i}_1 + \underline{i}_2 - \underline{i}_1\underline{i}_2, \bar{i}_1 + \bar{i}_2 - \bar{i}_1\bar{i}_2], \\ [\underline{f}_1 + \underline{f}_2 - \underline{f}_1\underline{f}_2, \bar{f}_1 + \bar{f}_2 - \bar{f}_1\bar{f}_2] \end{array} \right);$

3. $\lambda\tilde{\alpha} = \left( \begin{array}{c} \Big[\lambda a_1, \lambda b_1, \lambda c_1, \lambda d_1\Big]; \Big[1 - (1 - \underline{t}_1)^\lambda, 1 - (1 - \bar{t}_1)^\lambda\Big] \\ \Big[(\underline{i}_1)^\lambda, (\bar{i}_1)^\lambda\Big], \Big[(\underline{f}_1)^\lambda, (\bar{f}_1)^\lambda\Big] \end{array} \right), \lambda \geq 0;$

4. $(\tilde{\alpha})^\lambda = \left( \begin{array}{c} \Big[(a_1)^\lambda, (b_1)^\lambda, (c_1)^\lambda, (d_1)^\lambda\Big]; \Big[(\underline{t}_1)^\lambda, (\bar{t}_1)^\lambda\Big], \\ \Big[1 - (1 - \underline{i}_1)^\lambda, 1 - (1 - \bar{i}_1)^\lambda\Big], \\ \Big[1 - (1 - \underline{f}_1)^\lambda, 1 - (1 - \bar{f}_1)^\lambda\Big] \end{array} \right), \lambda \geq 0.$

**Definition 8.** [43] Let $\tilde{\alpha} = ([a_1, b_1, c_1, d_1]; [\bar{t}_1, \underline{t}_1], [\bar{i}_1, \underline{i}_1], [\bar{f}_1, \underline{f}_1])$ and $\tilde{\beta} = ([a_2, b_2, c_2, d_2]; [\bar{t}_2, \underline{t}_2], [\bar{i}_2, \underline{i}_2], [\bar{f}_2, \underline{f}_2])$ be two ITrNNs. Then the distance between two numbers is defined as

$$\begin{aligned} d(\tilde{\alpha}, \tilde{\beta}) =& \frac{1}{24}\left( \left| \begin{array}{l} a_1(2+\underline{t}_1-\underline{i}_1-\underline{f}_1)+a_1(2+\bar{t}_1-\bar{i}_1-\bar{f}_1) \\ -a_2(2+\underline{t}_2-\underline{i}_2-\underline{f}_2)-a_2(2+\bar{t}_2-\bar{i}_2-\bar{f}_2) \end{array} \right| \right. \\ &+ \left| \begin{array}{l} b_1(2+\underline{t}_1-\underline{i}_1-\underline{f}_1)+b_1(2+\bar{t}_1-\bar{i}_1-\bar{f}_1) \\ -b_2(2+\underline{t}_2-\underline{i}_2-\underline{f}_2)-b_2(2+\bar{t}_2-\bar{i}_2-\bar{f}_2) \end{array} \right| \\ &+ \left| \begin{array}{l} c_1(2+\underline{t}_1-\underline{i}_1-\underline{f}_1)+c_1(2+\bar{t}_1-\bar{i}_1-\bar{f}_1) \\ -c_2(2+\underline{t}_2-\underline{i}_2-\underline{f}_2)-c_2(2+\bar{t}_2-\bar{i}_2-\bar{f}_2) \end{array} \right| \\ &+ \left. \left| \begin{array}{l} d_1(2+\underline{t}_1-\underline{i}_1-\underline{f}_1)+d_1(2+\bar{t}_1-\bar{i}_1-\bar{f}_1) \\ -d_2(2+\underline{t}_2-\underline{i}_2-\underline{f}_2)-d_2(2+\bar{t}_2-\bar{i}_2-\bar{f}_2) \end{array} \right| \right) \end{aligned} \tag{2}$$

This distance is called normalized Hamming distance.

# 3 TOPSIS method for MADM

TOPSIS [5] method is based on the concept that the chosen alternative should have the shortest geometric distance from the positive ideal solution and the longest geometric distance from the negative ideal solution.

Let $A$= $\{A_i | i = 1, 2, \ldots, m\}$ be the set of alternatives, $C$= $\{C_j | j = 1, 2, \ldots, n\}$ be the set of criteria and $D$= $\{d_{ij} | i = 1, 2, \ldots, m : j = 1, 2, \ldots, n\}$ be the performance ratings with the criteria weight vector $W$= $\{w_j | j = 1, 2, \ldots, n\}$. The idea of classical TOPSIS method can be expressed in a series of following steps:

**Step 1.** Normalize the decision matrix.

The normalized value $\bar{d}_{ij}$ is calculated as follows:

$$\bar{d}_{ij} = \frac{d_{ij}}{\sqrt{\sum_{i=1}^{m} (d_{ij})^2}}, \ i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n.$$

**Step 2.** Calculate the weighted normalized decision matrix.

In the weighted normalized decision matrix, the modified ratings are calculated as given below:

$$v_{ij} = w_j \times \bar{d}_{ij} \text{ for } i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n. \tag{3}$$

where $w_j$ is the weight of the $j$-th attribute such that $w_j \geq 0$ for $j = 1, 2, \ldots, n$ and $\sum_{j=1}^{n} w_j = 1$.

**Step 3.** Determine the positive and the negative ideal solutions.

The positive ideal solution (PIS) and the negative ideal solution (NIS) are determined as follows:

$$\begin{aligned} \text{PIS} = A^{+} &= \left\{v_1^{+}, v_2^{+}, \ldots v_n^{+},\right\} \\ &= \left\{\left(\max_j v_{ij} | j \in J_1\right), \left(\min_j v_{ij} | j \in J_2\right) | j = 1, 2, \ldots, n\right\}; \\ \text{NIS} = A^{-} &= \left\{v_1^{-}, v_2^{-}, \ldots v_n^{-},\right\} \\ &= \left\{\left(\min_j v_{ij} | j \in J_1\right), \left(\max_j v_{ij} | j \in J_2\right) | j = 1, 2, \ldots, n\right\}, \end{aligned} \tag{4}$$

where $J_1$ and $J_2$ are the benefit type and the cost type attributes, respectively.

**Step 4.** Calculate the separation measures for each alternative from the PIS and the NIS.

The separation values for the PIS can be measured using the n-dimensional Euclidean distance measure as follows:

$$D_i^{+} = \sqrt{\sum_{j=1}^{n} \left(v_{ij} - v_j^{+}\right)^2} \quad i = 1, 2, \ldots m. \tag{5}$$

Similarly, separation values for the NIS can be measured as

$$D_i^{-} = \sqrt{\sum_{j=1}^{n} \left(v_{ij} - v_j^{-}\right)^2} \quad i = 1, 2, \ldots m. \tag{6}$$

**Step 5.** Calculate the relative closeness coefficient to the positive ideal solution.

The relative closeness coefficient for the alternative $A_i$ with respect to $A^{+}$ is calculated as

$$C_i = \frac{D_i^{-}}{D_i^{+} + D_i^{-}} \text{ for } \quad i = 1, 2, \ldots m. \tag{7}$$

**Step 6.** Rank the alternatives.

According to relative closeness coefficient to the ideal alternative, the larger value of $C_i$ reflects the better alternative $A_i$.

# 4 TOPSIS for multi-attribute decision making based on ITrNN

In this section, we put forward a framework for determining the attribute weights and the ranking orders for all the alternatives with incomplete weight information under neutrosophic environment.

For a multi-attribute decision making problem, let $A = (A_1, A_2, ....., A_n)$ be a discrete set of alternatives and $C = (C_1, C_2, ...., C_n)$ be a discrete set of attributes. Suppose that $D = [\tilde{a}_{ij}]$ is the decision matrix, where $\tilde{a}_{ij} = ([a_{ij}^1, a_{ij}^2, a_{ij}^3, a_{ij}^4]; \tilde{t}_{ij}, \tilde{i}_{ij}, \tilde{f}_{ij})$ is ITrNN for alternative $A_i$ with respect to attribute $C_j$ and $\tilde{t}_{ij}, \tilde{i}_{ij}$ and $\tilde{f}_{ij}$ are subsets of $[0, 1]$ and $0 \leq \sup \tilde{t}_{ij} + \sup \tilde{i}_{ij} + \sup \tilde{f}_{ij} \leq 3$ for $i = 1, 2, ..., m$ and $j = 1, 2, ...., n$. Here $\tilde{t}_{ij}$ denotes interval truth membership function, $\tilde{i}_{ij}$ denotes interval indeterminate membership function, $\tilde{f}_{ij}$

denotes interval falsity membership function. Then we have the following decision matrix:

$$D = (\tilde{a}_{ij})_{m\times n} = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \dots & C_n \end{array} \\ \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \dots & \tilde{a}_{1n} \\ \tilde{a}_{21} & \tilde{a}_{22} & \dots & \tilde{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{m1} & \tilde{a}_{m2} & \dots & \tilde{a}_{mn} \end{pmatrix} \end{array} \tag{8}$$

Now, we develop this method when attribute weights are completely known, partially known and completely unknown. The steps of the ranking are as follows:

**Step 1:** Standardize the decision matrix.

This step transforms various attribute dimensions into non-dimensional attributes which allow comparison across criteria because various criteria are usually measured in various units. In general, there are two types of attribute. One is benefit type attribute and another one is cost type attribute. Let $D = (a_{ij})_{m\times n}$ be a decision matrix where the ITrNN $\tilde{a}_{ij} = ([a^1_{ij}, a^2_{ij}, a^3_{ij}, a^4_{ij}]; \tilde{t}_{ij}, \tilde{i}_{ij}, \tilde{f}_{ij})$ is the rating value of the alternative $A_i$ with respect to the attribute $C_j$.

In order to eliminate the influence of attribute type, we consider the following technique and obtain the standardize matrix $R = (\tilde{r}_{ij})_{m\times n}$, where $\tilde{r}_{ij} = ([r^1_{ij}, r^2_{ij}, r^3_{ij}, r^4_{ij}]; [\underline{t}_{ij}, \bar{t}_{ij}], [\underline{f}_{ij}, \bar{f}_{ij}], [\underline{f}_{ij}, \bar{f}_{ij}])$ is ITrNN. Then we have

$$\tilde{r}_{ij} = \Big(\Big[\frac{a^1_{ij}}{u^+_j}, \frac{a^2_{ij}}{u^+_j}, \frac{a^3_{ij}}{u^+_j}, \frac{a^4_{ij}}{u^+_j}\Big]; [\underline{t}_{ij}, \bar{t}_{ij}], [\underline{f}_{ij}, \bar{f}_{ij}], [\underline{f}_{ij}, \bar{f}_{ij}]\Big), \text{ for benefit type attribute} \tag{9}$$

$$\tilde{r}_{ij} = \Big(\Big[\frac{u^-_j}{a^4_{ij}}, \frac{u^-_j}{a^3_{ij}}, \frac{u^-_j}{a^2_{ij}}, \frac{u^-_j}{a^1_{ij}}\Big]; [\underline{t}_{ij}, \bar{t}_{ij}], [\underline{f}_{ij}, \bar{f}_{ij}], [\underline{f}_{ij}, \bar{f}_{ij}]\Big), \text{ for cost type attribute} \tag{10}$$

where $u^+_j = max\{a^4_{ij} : i = 1, 2, ...., m\}$ and $u^-_j = min\{a^1_{ij} : i = 1, 2, ...., m\}$ for $j = 1, 2, ..., n$.

**Step 2:** Calculate the attribute weight.

The attribute weights may be completely known, partially known or completely unknown. So we need to determine the attribute weights by maximum deviation method which is proposed by Wang [48]. If the attributes have larger deviation, smaller deviation and no deviation then we assign larger weight, smaller weight and zero weight, respectively.

For MADM problem, the deviation values of alternative $A_i$ to the other alternatives under the attribute $C_j$ can be defined as follows:

$$d_{ij}(w) = \sum_{k=1}^{m} d(\tilde{a}_{ij}, \tilde{a}_{kj})w_j,\ i = 1, 2, ...., m;\ \ j = 1, 2, ....., n, \text{where}$$

$$\begin{aligned}
d(\tilde{a}_{ij},\tilde{a}_{kj}) =& \frac{1}{24}\left(\left| \begin{array}{l} a^1_{ij}(2+\underline{t}_{ij}-\underline{i}_{ij}-\underline{f}_{ij})+a^1_{ij}(2+\bar{t}_{ij}-\bar{i}_{ij}-\bar{f}_{ij}) \\ -a^1_{kj}(2+\underline{t}_{kj}-\underline{i}_{kj}-\underline{f}_{kj})-a^1_{kj}(2+\bar{t}_{kj}-\bar{i}_{kj}-\bar{f}_{kj}) \end{array}\right|\right. \\
&+\left| \begin{array}{l} a^2_{ij}(2+\underline{t}_{ij}-\underline{i}_{ij}-\underline{f}_{ij})+a^2_{ij}(2+\bar{t}_{ij}-\bar{i}_{ij}-\bar{f}_{ij}) \\ -a^2_{kj}(2+\underline{t}_{kj}-\underline{i}_{kj}-\underline{f}_{kj})-a^2_{kj}(2+\bar{t}_{kj}-\bar{i}_{kj}-\bar{f}_{kj}) \end{array}\right| \\
&+\left| \begin{array}{l} a^3_{ij}(2+\underline{t}_{ij}-\underline{i}_{ij}-\underline{f}_{ij})+a^3_{ij}(2+\bar{t}_{ij}-\bar{i}_{ij}-\bar{f}_{ij}) \\ -a^3_{kj}(2+\underline{t}_{kj}-\underline{i}_{kj}-\underline{f}_{kj})-a^3_{kj}(2+\bar{t}_{kj}-\bar{i}_{kj}-\bar{f}_{kj}) \end{array}\right| \\
&\left.+\left| \begin{array}{l} a^4_{ij}(2+\underline{t}_{ij}-\underline{i}_{ij}-\underline{f}_{ij})+a^4_{ij}(2+\bar{t}_{ij}-\bar{i}_{ij}-\bar{f}_{ij}) \\ -a^4_{kj}(2+\underline{t}_{kj}-\underline{i}_{kj}-\underline{f}_{kj})-a^4_{kj}(2+\bar{t}_{kj}-\bar{i}_{kj}-\bar{f}_{kj}) \end{array}\right|\right) \\
=& \frac{1}{24}\sum_{p=1}^{4}\left(\left| \begin{array}{l} a^p_{ij}(2+\underline{t}_{ij}-\underline{i}_{ij}-\underline{f}_{ij})+a^p_{ij}(2+\bar{t}_{ij}-\bar{i}_{ij}-\bar{f}_{ij}) \\ -a^p_{kj}(2+\underline{t}_{kj}-\underline{i}_{kj}-\underline{f}_{kj})-a^p_{kj}(2+\bar{t}_{kj}-\bar{i}_{kj}-\bar{f}_{kj}) \end{array}\right|\right)
\end{aligned}$$

The deviation values of all the alternatives to other alternatives for the attributes $C_j$ can be defined as

$$\begin{aligned}
D_j(w) =& \sum_{i=1}^{m} d_{ij}(w) = \sum_{i=1}^{m}\sum_{k=1}^{m} d(\tilde{a}_{ij},\tilde{a}_{kj})w_j \\
=& \sum_{i=1}^{m}\sum_{k=1}^{m}\Big(\frac{1}{24}\sum_{p=1}^{4}\left| \begin{array}{l} a^p_{ij}(2+\underline{t}_{ij}-\underline{i}_{ij}-\underline{f}_{ij})+a^p_{ij}(2+\bar{t}_{ij}-\bar{i}_{ij}-\bar{f}_{ij}) \\ -a^p_{kj}(2+\underline{t}_{kj}-\underline{i}_{kj}-\underline{f}_{kj})-a^p_{kj}(2+\bar{t}_{kj}-\bar{i}_{kj}-\bar{f}_{kj}) \end{array}\right|\Big)w_j
\end{aligned}$$

Therefore, the total deviation value $D(w)=\sum_{j=1}^{n} D_j(w)$.
In the following, we develop three cases:
**Case 1.** When the attribute weights are completely known.
In this case, the attribute weights $w_1, w_2, ........, w_n$ are known in advance and $\sum_{j=1}^{n} w_j = 1, w_j \geq 0,\ for\ j = 1, 2, ...., n$.
**Case 2.** When attributes weights are partially known.
In this case, we assume a non-linear programming model. This model maximizes all deviation values of the attributes.

$$\textbf{Model 1}\left\{ \begin{array}{l} max\ D(w) \\ = \dfrac{1}{24}\sum\limits_{j=1}^{n}\sum\limits_{i=1}^{m}\sum\limits_{k=1}^{m}\sum\limits_{p=1}^{4}\left(\left| \begin{array}{l} a^p_{ij}(2+\underline{t}_{ij}-\underline{i}_{ij}-\underline{f}_{ij})+a^p_{ij}(2+\bar{t}_{ij}-\bar{i}_{ij}-\bar{f}_{ij}) \\ -a^p_{kj}(2+\underline{t}_{kj}-\underline{i}_{kj}-\underline{f}_{kj})-a^p_{kj}(2+\bar{t}_{kj}-\bar{i}_{kj}-\bar{f}_{kj}) \end{array}\right|\right)w_j \\ \text{subject to } w\in\Delta,\quad \sum\limits_{j=1}^{n} w = 1, w_j \geq 0,\ \text{for } j = 1, 2, ......, n. \end{array}\right.$$

Here, the incomplete attribute weight information $\Delta$ is taken in the following form ([49, 50]):

1. A weak ranking:$\{w_i \geq w_j\}, i \neq j$;
2. A strict ranking:$\{w_i - w_j \geq \epsilon_i(> 0)\}, i \neq j$;
3. A ranking of difference:$\{w_i - w_j \geq w_k - w_p\}, i \neq j \neq k \neq p$;
4. A ranking with multiples:$\{w_i \geq \alpha_i w_j\}, 0 \leq \alpha_i \leq 1, i \neq j$;

5. An interval form: $\{\beta_i \leq w_i \leq \beta_i + \epsilon_i(> 0)\}, 0 \leq \beta_i \leq \beta_i + \epsilon_i \leq 1.$

Solving this model, we get the optimal solution which is to be used as the weight vector.
**Case 3.** When attribute weights are completely unknown:
In this case, we can establish the following programming model:

$$\textbf{Model 2}\begin{cases} max\ D(w) \\ = \dfrac{1}{24}\sum\limits_{j=1}^{n}\sum\limits_{i=1}^{m}\sum\limits_{k=1}^{m}\sum\limits_{p=1}^{4}\left(\left|\begin{matrix} a_{ij}^{p}(2+\underline{t}_{ij}-\underline{i}_{ij}-\underline{f}_{ij})+a_{ij}^{p}(2+\bar{t}_{ij}-\bar{i}_{ij}-\bar{f}_{ij}) \\ -a_{kj}^{p}(2+\underline{t}_{kj}-\underline{i}_{kj}-\underline{f}_{kj})-a_{kj}^{p}(2+\bar{t}_{kj}-\bar{i}_{kj}-\bar{f}_{kj})\end{matrix}\right|\right)w_j \\ \text{subject to } w\in\Delta,\quad \sum\limits_{j=1}^{n} w_j^2=1, w_j\geq 0,\ \text{for } j=1,2,......,n. \end{cases}$$

To solve this model, we construct the Lagrangian function:

$$L(w,\xi)=\frac{1}{24}\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m}\sum_{p=1}^{4}\left(\left|\begin{matrix} a_{ij}^{p}(2+\underline{t}_{ij}-\underline{i}_{ij}-\underline{f}_{ij})+a_{ij}^{p}(2+\bar{t}_{ij}-\bar{i}_{ij}-\bar{f}_{ij}) \\ -a_{kj}^{p}(2+\underline{t}_{kj}-\underline{i}_{kj}-\underline{f}_{kj})-a_{kj}^{p}(2+\bar{t}_{kj}-\bar{i}_{kj}-\bar{f}_{kj})\end{matrix}\right|\right)w_j + \frac{\xi}{48}\Big(\sum_{j=1}^{n} w_j^2-1\Big) \tag{11}$$

where $\xi \in \mathbb{R}$ is Lagrange multiplier.

Now, we calculate the partial derivatives of $L$ with respect to $w_j(j = 1, 2, ....n)$ and $\xi$:

$$\frac{\partial L}{\partial w_j}=\sum_{i=1}^{m}\sum_{k=1}^{m}\sum_{p=1}^{4}\left(\left|\begin{matrix} a_{ij}^{p}(2+\underline{t}_{ij}-\underline{i}_{ij}-\underline{f}_{ij})+a_{ij}^{p}(2+\bar{t}_{ij}-\bar{i}_{ij}-\bar{f}_{ij}) \\ -a_{kj}^{p}(2+\underline{t}_{kj}-\underline{i}_{kj}-\underline{f}_{kj})-a_{kj}^{p}(2+\bar{t}_{kj}-\bar{i}_{kj}-\bar{f}_{kj})\end{matrix}\right|\right)+\xi w_j=0 \tag{12}$$

$$\frac{\partial L}{\partial \xi}=\sum_{j=1}^{n} w_j^2-1=0 \tag{13}$$

From Eq. (12), we get

$$w_j=\frac{-\sum\limits_{i=1}^{m}\sum\limits_{k=1}^{m}\sum\limits_{p=1}^{4}\left(\left|\begin{matrix} a_{ij}^{p}(2+\underline{t}_{ij}-\underline{i}_{ij}-\underline{f}_{ij})+a_{ij}^{p}(2+\bar{t}_{ij}-\bar{i}_{ij}-\bar{f}_{ij}) \\ -a_{kj}^{p}(2+\underline{t}_{kj}-\underline{i}_{kj}-\underline{f}_{kj})-a_{kj}^{p}(2+\bar{t}_{kj}-\bar{i}_{kj}-\bar{f}_{kj})\end{matrix}\right|\right)}{\xi},\quad j=1,2,.....,n \tag{14}$$

Putting this value in Eq.(13), we get

$$\xi^2=\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\sum_{k=1}^{m}\sum_{p=1}^{m}\left(\left|\begin{matrix} a_{ij}^{p}(2+\underline{t}_{ij}-\underline{i}_{ij}-\underline{f}_{ij})+a_{ij}^{p}(2+\bar{t}_{ij}-\bar{i}_{ij}-\bar{f}_{ij}) \\ -a_{kj}^{p}(2+\underline{t}_{kj}-\underline{i}_{kj}-\underline{f}_{kj})-a_{kj}^{p}(2+\bar{t}_{kj}-\bar{i}_{kj}-\bar{f}_{kj})\end{matrix}\right|\right)\right)^2 \tag{15}$$

$$\Rightarrow \xi=-\sqrt{\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\sum_{k=1}^{m}\sum_{p=1}^{m}\left(\left|\begin{matrix} a_{ij}^{p}(2+\underline{t}_{ij}-\underline{i}_{ij}-\underline{f}_{ij})+a_{ij}^{p}(2+\bar{t}_{ij}-\bar{i}_{ij}-\bar{f}_{ij}) \\ -a_{kj}^{p}(2+\underline{t}_{kj}-\underline{i}_{kj}-\underline{f}_{kj})-a_{kj}^{p}(2+\bar{t}_{kj}-\bar{i}_{kj}-\bar{f}_{kj})\end{matrix}\right|\right)\right)^2}\ \text{for } \xi<0 \tag{16}$$

From Eq. (14) and Eq. (16), we get the formula for determining attribute weights for $C_j (j = 1, 2, ..., n)$ :

$$w_j = \frac{\sum_{i=1}^{m}\sum_{k=1}^{m}\sum_{p=1}^{m}\left(\left|\begin{array}{l} a_{ij}^{p}(2+\underline{t}_{ij}-\underline{i}_{ij}-\underline{f}_{ij})+a_{ij}^{p}(2+\bar{t}_{ij}-\bar{i}_{ij}-\bar{f}_{ij}) \\ -a_{kj}^{p}(2+\underline{t}_{kj}-\underline{i}_{kj}-\underline{f}_{kj})-a_{kj}^{p}(2+\bar{t}_{kj}-\bar{i}_{kj}-\bar{f}_{kj}) \end{array}\right|\right)}{\sqrt{\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\sum_{k=1}^{m}\sum_{p=1}^{m}\left(\left|\begin{array}{l} a_{ij}^{p}(2+\underline{t}_{ij}-\underline{i}_{ij}-\underline{f}_{ij})+a_{ij}^{p}(2+\bar{t}_{ij}-\bar{i}_{ij}-\bar{f}_{ij}) \\ -a_{kj}^{p}(2+\underline{t}_{kj}-\underline{i}_{kj}-\underline{f}_{kj})-a_{kj}^{p}(2+\bar{t}_{kj}-\bar{i}_{kj}-\bar{f}_{kj}) \end{array}\right|\right)\right)^{2}}} \tag{17}$$

Now, we can get the normalized attribute weight as

$$\bar{w}_j = \frac{w_j}{\sum_{j=1}^{n} w_j} = \frac{\sum_{i=1}^{m}\sum_{k=1}^{m}\sum_{p=1}^{m}\left(\left|\begin{array}{l} a_{ij}^{p}(2+\underline{t}_{ij}-\underline{i}_{ij}-\underline{f}_{ij})+a_{ij}^{p}(2+\bar{t}_{ij}-\bar{i}_{ij}-\bar{f}_{ij}) \\ -a_{kj}^{p}(2+\underline{t}_{kj}-\underline{i}_{kj}-\underline{f}_{kj})-a_{kj}^{p}(2+\bar{t}_{kj}-\bar{i}_{kj}-\bar{f}_{kj}) \end{array}\right|\right)}{\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m}\sum_{p=1}^{m}\left(\left|\begin{array}{l} a_{ij}^{p}(2+\underline{t}_{ij}-\underline{i}_{ij}-\underline{f}_{ij})+a_{ij}^{p}(2+\bar{t}_{ij}-\bar{i}_{ij}-\bar{f}_{ij}) \\ -a_{kj}^{p}(2+\underline{t}_{kj}-\underline{i}_{kj}-\underline{f}_{kj})-a_{kj}^{p}(2+\bar{t}_{kj}-\bar{i}_{kj}-\bar{f}_{kj}) \end{array}\right|\right)} \tag{18}$$

Therefore, we get the normalized weight vector $\bar{w} = \{\bar{w}_1, \bar{w}_2, \dots, \bar{w}_n\}$.
**Step 3:** Determine the positive and negative ideal solutions.
The normalized decision matrix $R = (\tilde{r}_{ij})_{m\times n}$ in the interval trapezoidal neutroshopic number, the positive and negative ideal solutions are defined as follows:
$\tilde{r}^{+} = (\tilde{r}_1^{+}, \tilde{r}_2^{+}, \dots, \tilde{r}_n^{+})$ and $\tilde{r}^{-} = (\tilde{r}_1^{-}, \tilde{r}_2^{-}, \dots, \tilde{r}_n^{-})$ where,

$$\begin{aligned} \tilde{r}_j^{+} =& ([r_j^{1+}, r_j^{2+}, r_j^{3+}, r_j^{4+}]; [\underline{t}_j^{+}, \bar{t}_j^{+}], [\underline{i}_j^{+}, \bar{i}_j^{+}], [\underline{f}_j^{+}, \bar{f}_j^{+}]) \\ =& \Big( [\max_i(r_{ij}^{1}), \max_i(r_{ij}^{2}), \max_i(r_{ij}^{3}), \max_i(r_{ij}^{4})]; \\ & [\max_i(\underline{t}_{ij}), \max_i(\bar{t}_{ij})][\min_i(\underline{i}_{ij}), \min_i(\bar{i}_{ij})], [\min_i(\underline{f}_{ij}), \min_i(\bar{f}_{ij})] \Big) \end{aligned} \tag{19}$$

$$\begin{aligned} \tilde{r}_j^{-} =& ([r_j^{1-}, r_j^{2-}, r_j^{3-}, r_j^{4-}]; [\underline{t}_j^{-}, \bar{t}_j^{-}], [\underline{i}_j^{-}, \bar{i}_j^{-}], [\underline{f}_j^{-}, \bar{f}_j^{-}]) \\ =& \Big( [\min_i(r_{ij}^{1}), \min_i(r_{ij}^{2}), \min_i(r_{ij}^{3}), \min_i(r_{ij}^{4})]; \\ & [\min_i(\underline{t}_{ij}), \min_i(\bar{t}_{ij})][\max_i(\underline{i}_{ij}), \max_i(\bar{i}_{ij})], [\max_i(\underline{f}_{ij}), \max_i(\bar{f}_{ij})] \Big) \end{aligned} \tag{20}$$

The global positive and negative ideal solutions for ITrNN can be considered as $\tilde{r}_j^{+} = ([1, 1, 1, 1]; [1, 1], [0, 0], [0, 0])$ and $\tilde{r}_j^{-} = ([0, 0, 0, 0]; [0, 0], [1, 1], [1, 1])$.
**Step 4:** Calculate the separation measure from ideal solutions.
Now, using Eqs.(2), (19) and (20), we calculate separation measure $d_i^{+}$ from positive ideal solution and $d_i^{-}$ from negative ideal solution as

$$\begin{aligned} d_i^{+} =& \sum_{j=1}^{n} w_j d(\tilde{r}_{ij}, \tilde{r}_j^{+}) \\ =& \frac{1}{24}\sum_{j=1}^{n} w_j \sum_{p=1}^{4}\left(\left|\begin{array}{l} r_{ij}^{p}(2+\underline{t}_{ij}-\underline{i}_{ij}-\underline{f}_{ij})+r_{ij}^{p}(2+\bar{t}_{ij}-\bar{i}_{ij}-\bar{f}_{ij}) \\ -r_j^{p+}(2+\underline{t}_j^{+}-\underline{i}_j^{+}-\underline{f}_j^{+})-r_j^{p+}(2+\bar{t}_j^{+}-\bar{i}_j^{+}-\bar{f}_j^{+}) \end{array}\right|\right), \quad i = 1, 2, ...., m. \end{aligned} \tag{21}$$

$$
\begin{aligned}
d_i^- &= \sum_{j=1}^{n} w_j d(\tilde{r}_{ij}, \tilde{r}_j^-) \\
&= \frac{1}{24} \sum_{j=1}^{n} w_j \sum_{p=1}^{4} \left( \left| \begin{array}{l} r_{ij}^p (2 + \underline{t}_{ij} - \underline{i}_{ij} - \underline{f}_{ij}) + r_{ij}^p (2 + \bar{t}_{ij} - \bar{i}_{ij} - \bar{f}_{ij}) \\ - r_j^{p-} (2 + \underline{t}_j^- - \underline{i}_j^- - \underline{f}_j^-) - r_j^{p-} (2 + \bar{t}_j^- - \bar{i}_j^- - \bar{f}_j^-) \end{array} \right| \right), \quad i = 1, 2, ..., m.
\end{aligned}
\tag{22}
$$

**Step 5:** Calculate the relative closeness co-efficient.
We calculate the relative closeness co-efficient of an alternative $A_i$ with respect to the ideal alternative $A^+$ as

$$
RCC(A_i) = \frac{d_i^-}{d_i^+ + d_i^-}, \; for \; i = 1, 2, ...., n, \tag{23}
$$

where $0 \leq RCC(A_i) \leq 1$. We then rank the best alternative according to $RCC$.
**Step 6:** End.

# 5 An illustrative example

In order to demonstrate the proposed method, we consider the following MADM problem. Suppose that a person wants to buy a laptop. Let there be four companies $A_1, A_2, A_3, A_4$ and laptop of each company has three attributes such as cost, warranty, and quality. We consider $C_1$ for cost, $C_2$ for warranty and $C_3$ for quality type of attribute.
The person evaluates the rating values of the alternatives $A_i$ $(i = 1, 2, 3, 4)$ with respect to attributes $C_j(j = 1, 2, 3)$. Then we get the neutrosophic decision matrix $D = (\tilde{a}_{ij})_{4 \times 3} =$

| | $C_1$ |
|---|---|
| $A_1$ | $([50, 60, 70, 80]; [0.1, 0.2], [0.2, 0.3], [0.4, 0.5])$ |
| $A_2$ | $([30, 40, 50, 60]; [0.3, 0.4], [0.2, 0.3], [0.1, 0.2])$ |
| $A_3$ | $([70, 80, 90, 100]; [0.6, 0.7], [0.2, 0.3], [0.4, 0.5])$ |
| $A_4$ | $([40, 50, 60, 70]; [0.4, 0.5], [0.6, 0.7], [0.2, 0.3])$ |
| | $C_2$ |
| $A_1$ | $([30, 40, 50, 60]; [0.2, 0.3], [0.4, 0.5], [0.6, 0.7])$ |
| $A_2$ | $([10, 20, 30, 40]; [0.1, .2], [0.3, 0.4], [0.6, 0.7])$ |
| $A_3$ | $([50, 60, 70, 80]; [0.1, 0.2], [0.3, 0.4], [0.6, 0.7])$ |
| $A_4$ | $([70, 80, 90, 100]; [0.2, 0.3], [0.4, 0.5], [0.6, 0.8])$ |
| | $C_3$ |
| $A_1$ | $([40, 50, 60, 70]; [0.4, 0.5], [0.6, 0.7], [0.7, 0.8])$ |
| $A_2$ | $([20, 30, 40, 50]; [0.1, 0.2], [0.3, 0.4], [0.8, 0.9])$ |
| $A_3$ | $([70, 80, 90, 100]; [0.3, 0.5], [0.4, 0.6], [0.7, 0.8])$ |
| $A_4$ | $([30, 40, 50, 60]; [0.4, 0.5], [0.6, 0.7], [0.7, 0.8])$ |

Now, with the help of the proposed method, we find the best alternative following the steps given below:
**Step 1:** Standardize the decision matrix.
In the decision matrix, the first column represents the cost type attribute, and the second and the third columns represent benefit type attribute. Then, using Eqs. $(9)$ and $(10)$, we get the following standardize decision

matrix $R_{ij}$ =

| | $C_1$ |
|---|---|
| $A_1$ | $([0.38, 0.43, 0.50, 0.60]; [0.1, 0.2], [0.2, 0.3], [0.4, 0.5])$ |
| $A_2$ | $([0.50, 0.60, 0.75, 1.0]; [0.3, 0.4], [0.2, 0.3], [0.1, 0.2])$ |
| $A_3$ | $([0.30, 0.33, 0.38, 0.43]; [0.6, 0.7], [0.2, 0.3], [0.4, 0.5])$ |
| $A_4$ | $([0.43, 0.50, 0.60, 0.75]; [0.4, 0.5], [0.6, 0.7], [0.2, 0.3])$ |
| | $C_2$ |
| $A_1$ | $([0.30, 0.40, 0.50, 0.60]; [0.2, .3], [0.4, 0.5], [0.6, 0.7])$ |
| $A_2$ | $([0.10, 0.20, 0.30, 0.40]; [0.1, 0.2], [0.3, 0.4], [0.6, 0.7])$ |
| $A_3$ | $([0.50, 0.60, 0.70, 0.80]; [0.1, 0.2], [0.3, 0.4], [0.6, 0.7])$ |
| $A_4$ | $([0.70, 0.80, 0.90, 1.0]; [0.2, 0.3], [0.4, .5], [0.6, 0.8])$ |
| | $C_3$ |
| $A_1$ | $([0.40, 0.50, 0.60, 0.70]; [0.4, 0.5], [0.6, 0.7], [0.7, 0.8])$ |
| $A_2$ | $([0.20, 0.30, 0.40, 0.50]; [0.1, 0.2], [0.3, 0.4], [0.8, 0.9])$ |
| $A_3$ | $([0.70, 0.80, 0.90, 1.0]; [0.3, 0.5], [0.4, 0.6], [0.7, 0.8])$ |
| $A_4$ | $([0.30, 0.40, 0.50, 0.60]; [0.4, .5], [0.6, 0.7], [0.7, 0.8])$ |

**Step 2:** Calculate the attribute weight.

Here we assume three cases for the attribute weight.

**Case 1** : When the attribute weights are completely known, let the weight vector be $\bar{w} = (0.25, 0.55, 0.20)$.

**Case 2** : When the attribute weights are partially known, we select the weight information as follows:

$$\Delta = \begin{cases} 0.35 \leq w_1 \leq 0.75 \\ 0.25 \leq w_2 \leq 0.60 \\ 0.30 \leq w_3 \leq 0.45 \\ \text{and } w_1 + w_2 + w_3 = 1 \end{cases}$$

Using Model 1, we develop the single objective programming problem as

$$\begin{cases} max(D) = 45.92w_1 + 109.56w_2 + 98.20w_3 \\ \text{subject to } w \in \Delta \text{ and } \sum_{j=1}^{3} w_j = 1,\ w_j > 0\ for\ j = 1, 2, 3. \end{cases}$$

Solving this problem with optimization software LINGO 11, we get the optimal weight vector as

$\bar{w} = (0.35, 0.35, 0.30)$.

**Case 3** : When the attribute weights are completely unknown, we use Model 2 and Eqn. (18) and obtain the following weight vector:

$\bar{w} = (0.18, 0.43, 0.39)$.

**Step 3:** Determine the positive and negative ideal solutions.

Since the cost of the laptop is cost type attribute, and warranty and quality are benefit type attributes, therefore, using Eqs.(19) and (20), we get the following neutrosophic positive and negative ideal solutions:

$$A^+ = \begin{pmatrix} \left([0.30,0.33,0.38,0.43];[0.10,0.20],[0.20,0.30],[0.20,0.30]\right) \\ \left([0.70,0.80,0.90,1.0];[0.20,0.30],[0.40,0.50],[0.60,0.70]\right) \\ \left([0.70,0.80,0.90,1.0];[0.40,0.50],[0.60,0.70],[0.80,0.90]\right) \end{pmatrix}$$

$$A^- = \begin{pmatrix} \left([0.50,0.60,0.75,1.0];[0.60,0.70],[0.40,0.50],[0.40,0.50]\right) \\ \left([0.10,0.20,0.30,0.40];[0.10,0.20],[0.30,0.40],[0.60,0.70]\right) \\ \left([0.20,0.30,0.40,0.50];[0.10,0.20],[0.30,0.40],[0.70,0.80]\right) \end{pmatrix}$$

**Step 4** : Calculate the separation measure from ideal solutions.

**Case 1** : From Eq. (21), we get the separation measure $d_i^+$ of each $A_i$ from $A^+$:

$d_1^+ = d(A_1, A^+) = 0.179, d_2^+ = d(A_2, A^+) = 0.425, d_3^+ = d(A_3, A^+) = 0.106, d_4^+ = d(A_4, A^+) = 0.325$

From Eq. (22), we get the separation measure $d_i^-$ of each $A_i$ from $A^-$:

$d_1^- = d(A_1, A^-) = 0.304, d_2^- = d(A_2, A^-) = 0.083, d_3^- = d(A_3, A^-) = 0.485, d_4^- = d(A_4, A^-) = 0.503$

**Case 2** : From Eq. (21), we get the separation measure $d_i^+$ of each $A_i$ from $A^+$:

$d_1^+ = d(A_1, A^+) = 0.185, d_2^+ = d(A_2, A^+) = 0.434, d_3^+ = d(A_3, A^+) = 0.141, d_4^+ = d(A_4, A^+) = 0.335$

From Eq. (22), we get the separation measure $d_i^-$ of each $A_i$ from $A^-$:

$d_1^- = d(A_1, A^-) = 0.299, d_2^- = d(A_2, A^-) = 0.084, d_3^- = d(A_3, A^-) = 0.479, d_4^- = d(A_4, A^-) = 0.381$

**Case 3** : From Eq. (21), we get the separation measure $d_i^+$ of each $A_i$ from $A^+$:

$d_1^+ = d(A_1, A^+) = 0.167, d_2^+ = d(A_2, A^+) = 0.429, d_3^+ = d(A_3, A^+) = 0.126, d_4^+ = d(A_4, A^+) = 0.307$

From Eq. (22), we get the separation measure $d_i^-$ of each $A_i$ from $A^-$:

$d_1^- = d(A_1, A^-) = 0.604, d_2^- = d(A_2, A^-) = 0.094, d_3^- = d(A_3, A^-) = 0.554, d_4^- = d(A_4, A^-) = 0.467$

**Step 5:** Calculate the relative closeness co-efficient.

In this step, using Eq. (22), we calculate the relative closeness coefficient of the alternatives $A_1, A_2, A_3, A_4$ and obtain the following results (see Table 1):

Table 1: Relative closeness co-efficient

| $RCC(A_i)$ | Case 1 | Case 2 | Case 3 |
|---|---|---|---|
| $RCC(A_1)$ | 0.629 | 0.618 | 0.783 |
| $RCC(A_2)$ | 0.163 | 0.162 | 0.180 |
| $RCC(A_3)$ | 0.819 | 0.773 | 0.814 |
| $RCC(A_4)$ | 0.607 | 0.532 | 0.603 |

From the above table, we see that $RCC(A_3) \geq RCC(A_1) \geq RCC(A_4) \geq RCC(A_2)$ in all cases. Therefore, we conclude that

$$A_3 \succ A_1 \succ A_4 \succ A_2$$

where $A_3$ is the best alternative.

**Step 6:** End.

## 5.1 Comparative analysis

The study made by Liu [22] presents TOPSIS method for MADM based on trapezoidal intuitionistic fuzzy number and does not include indeterminate type information in the decision making process. The preference value considered in our paper is interval trapezoidal neutrosophic number, which deals with indeterminate type information effectively along with truth and falsity type information. The method presented by Ye [44] and

Subaş [45] discusses some aggregation operators of trapezoidal neutrosophic number and the decision making method proposed by Biswas et al. [42] presents trapezoidal neutrosophic number based TOPSIS method for MADM with partially known, and completely unknown weight information. We know that interval trapezoidal neutrosophic number is a generalization of trapezoidal neutrosophic number. The approach provided by Biswas et al. [43] discusses ITrNN based MADM with known weight information, whereas our proposed model develops ITrNN based MADM model with known, partially known, and completely unknown weight information. Furthermore, the methods suggested by Biswas et al. [42], Ye [44], and Subaş [45] are not suitable for the decision making problem in this paper. In Table 2, we compare our results with those obtained by the method given by Biswas et al. [43].

Table 2: A comparison of the results

| Method | Type of weight information | Ranking result |
|---|---|---|
| Biswas et al.'s method [43] | Partially known | Not Applicable |
| | Completely unknown | Not Applicable |
| Proposed method | Partially known | $A_3 \succ A_1 \succ A_4 \succ A_2$ |
| | Completely unknown | $A_3 \succ A_1 \succ A_4 \succ A_2$ |

Therefore, our proposed method is more general than the existing methods because the existing methods cannot deal with ITrNN based MADM with partially known, and completely unknown weight information.

# 6 Conclusions

TOPSIS method is a very popular method for MADM problem and this method has been extended under different environments like fuzzy sets, intuitionistic fuzzy sets and neutrosophic sets. In this paper, we have extended TOPSIS method based on ITrNN. First, we have developed an optimization model to calculate the attribute weight with the help of maximum deviation strategy when the weight information is partially known. We have also developed another model by using Lagrangian function to determine attributes' weights for unknown weight information case. With these weights we have solved MADM problem by TOPSIS method. Finally we have provided a numerical example of MADM problem and compared with existing methods. The proposed strategy can be extended to multi-attribute group decision making problem with ITrNN. This model can be used in various selection problems like weaver selection problem [51, 52], data mining [53], teacher selection problem [54], brick field selection problem [55], center location selection problem [56,57], etc. under ITrNN environment.

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# Extension of Soft Set to Hypersoft Set, and then to Plithogenic Hypersoft Set

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**Abstract**. In this paper, we generalize the soft set to the hypersoft set by transforming the function F into a multi-attribute function. Then we introduce the hybrids of Crisp, Fuzzy, Intuitionistic Fuzzy, Neutrosophic, and Plithogenic Hypersoft Set.

**Keywords:** Plithogeny; Plithogenic Set; Soft Set; Hypersoft Set; Plithogenic Hypersoft Set; Multi-argument Function.

## 1 Introduction

We generalize the soft set to the **hypersoft set** by transforming the function $F$ into a **multi-argument function**.

Then we make the distinction between the types of Universes of Discourse: **crisp**, **fuzzy**, **intuitionistic fuzzy**, **neutrosophic**, and respectively **plithogenic**.

Similarly, we show that a hypersoft set can be **crisp**, **fuzzy**, **intuitionistic fuzzy**, **neutrosophic**, or **plithogenic**.

A detailed numerical example is presented for all types.

## 2 Definition of Soft Set [1]

Let $\mathcal{U}$ be a universe of discourse, $\mathcal{P}(\mathcal{U})$ the power set of $\mathcal{U}$, and $A$ a set of attributes. Then, the pair $(F, \mathcal{U})$, where

$$F: A \longrightarrow \mathcal{P}(\mathcal{U}) \tag{1}$$

is called a **Soft Set** over $\mathcal{U}$.

## 3 Definition of Hypersoft Set

Let $\mathcal{U}$ be a universe of discourse, $\mathcal{P}(\mathcal{U})$ the power set of $\mathcal{U}$.

Let $a_1, a_2, \dots, a_n$, for $n \geq 1$, be $n$ distinct attributes, whose corresponding attribute values are respectively the sets $A_1, A_2, \dots, A_n$, with $A_i \cap A_j = \emptyset$, for $i \neq j$, and $i, j \in \{1, 2, \dots, n\}$.

Then the pair $(F, A_1 \times A_2 \times \dots \times A_n)$, where:

$$F: A_1 \times A_2 \times \dots \times A_n \longrightarrow \mathcal{P}(\mathcal{U}) \tag{2}$$

is called a **Hypersoft Set** over $\mathcal{U}$.

## 4 Particular case

For $n = 2$, we obtain the Γ–Soft Set [2].

## 5 Types of Universes of Discourses

**5.1.** A **Universe of Discourse** $\mathcal{U}_C$ is called **Crisp** if $\forall x \in \mathcal{U}_C$, $x$ belongs 100% to $\mathcal{U}_C$, or $x$'s membership $(T_x)$ with respect to $\mathcal{U}_C$ is 1. Let's denote it $x(1)$.

**5.2.** A **Universe of Discourse** $\mathcal{U}_F$ is called **Fuzzy** if $\forall x \in \mathcal{U}_c$, $x$ partially belongs to $\mathcal{U}_F$, or $T_x \subseteq [0, 1]$, where $T_x$ may be a subset, an interval, a hesitant set, a single-value, etc. Let's denote it by $x(T_x)$.

**5.3.** A **Universe of Discourse** $\mathcal{U}_{IF}$ is called **Intuitionistic Fuzzy** if $\forall x \in \mathcal{U}_{IF}$, $x$ partially belongs $(T_x)$ and partially doesn't belong $(F_x)$ to $\mathcal{U}_{IF}$, or $T_x, F_x \subseteq [0, 1]$, where $T_x$ and $F_x$ may be subsets, intervals, hesitant sets, single-values, etc. Let's denote it by $x(T_x, F_x)$.

**5.4.** A **Universe of Discourse** $\mathcal{U}_N$ is called **Neutrosophic** if $\forall x \in \mathcal{U}_N$, $x$ partially belongs $(T_x)$, partially its membership is indeterminate $(I_x)$, and partially it doesn't belong $(F_x)$ to $\mathcal{U}_N$, where $T_x, I_x, F_x \subseteq [0, 1]$, may be subsets, intervals, hesitant sets, single-values, etc. Let's denote it by $x(T_x, I_x, F_x)$.

**5.5.** A **Universe of Discourse** $\mathcal{U}_P$ **over a set** $V$ **of attributes' values**, where $V = \{v_1, v_2, \dots, v_n\}, n \geq 1$, is called Plithogenic, if $\forall x \in \mathcal{U}_P$, $x$ belongs to $\mathcal{U}_P$ in the degree $d_x^0(v_i)$ with respect to the attribute value $v_i$, for all

$i \in \{1, 2, \ldots, n\}$. Since the degree of membership $d_x^0(v_i)$ may be crisp, fuzzy, intuitionistic fuzzy, or neutrosophic, the Plithogenic Universe of Discourse can be Crisp, Fuzzy, Intuitionistic Fuzzy, or respectively Neutrosophic.

Consequently, a Hypersoft Set over a Crisp / Fuzzy / Intuitionistic Fuzzy / Neutrosophic / or Plithogenic Universe of Discourse is respectively called **Crisp / Fuzzy / Intuitionistic Fuzzy / Neutrosophic /** or **Plithogenic Hypersoft Set**.

## 6 Numerical Example

Let $\mathcal{U} = \{x_1, x_2, x_3, x_4\}$ and a set $\mathcal{M} = \{x_1, x_3\} \subset \mathcal{U}$.

Let the attributes be: $a_1 =$ size, $a_1 =$ color, $a_1 =$ gender, $a_1 =$ nationality, and their attributes' values respectively:

Size = $A_1$ ={small, medium, tall},
Color = $A_2$ ={white, yellow, red, black},
Gender = $A_3$ ={male, female},
Nationality = $A_4$ ={American, French, Spanish, Italian, Chinese}.

Let the function be:

$$F: A_1 \times A_2 \times A_3 \times A_4 \longrightarrow \mathcal{P}(\mathcal{U}). \tag{3}$$

Let's assume:

$F(\{\text{tall, white, female, Italian}\}) = \{x_1, x_3\}$.

With respect to the set $\mathcal{M}$, one has:

### 6.1 Crisp Hypersoft Set

$$F(\{\text{tall, white, female, Italian}\}) = \{x_1(1), x_3(1)\}, \tag{4}$$

which means that, with respect to the attributes' values $\{\text{tall, white, female, Italian}\}$ *all together*, $x_1$ belongs 100% to the set $\mathcal{M}$; similarly $x_3$.

### 6.2 Fuzzy Hypersoft Set

$$F(\{\text{tall, white, female, Italian}\}) = \{x_1(0.6), x_3(0.7)\}, \tag{5}$$

which means that, with respect to the attributes' values $\{\text{tall, white, female, Italian}\}$ *all together*, $x_1$ belongs 60% to the set $\mathcal{M}$; similarly, $x_3$ belongs 70% to the set $\mathcal{M}$.

### 6.3 Intuitionistic Fuzzy Hypersoft Set

$$F(\{\text{tall, white, female, Italian}\}) = \{x_1(0.6, 0.1), x_3(0.7, 0.2)\}, \tag{6}$$

which means that, with respect to the attributes' values $\{\text{tall, white, female, Italian}\}$ *all together*, $x_1$ belongs 60% and 10% it does not belong to the set $\mathcal{M}$; similarly, $x_3$ belongs 70% and 20% it does not belong to the set $\mathcal{M}$.

### 6.4 Neutrosophic Hypersoft Set

$$F(\{\text{tall, white, female, Italian}\}) = \{x_1(0.6, 0.2, 0.1), x_3(0.7, 0.3, 0.2)\}, \tag{7}$$

which means that, with respect to the attributes' values $\{\text{tall, white, female, Italian}\}$ all together, $x_1$ belongs 60% and its indeterminate-belongness is 20% and it doesn't belong 10% to the set $\mathcal{M}$; similarly, $x_3$ belongs 70% and its indeterminate-belongness is 30% and it doesn't belong 20%.

### 6.5 Plithogenic Hypersoft Set

$$F(\{\text{tall, white, female, Italian}\}) = \left\{\begin{matrix} x_1\left(d_{x_1}^0(\text{tall}), d_{x_1}^0(\text{white}), d_{x_1}^0(\text{female}), d_{x_1}^0(\text{Italian})\right), \\ x_2\left(d_{x_2}^0(\text{tall}), d_{x_2}^0(\text{white}), d_{x_2}^0(\text{female}), d_{x_2}^0(\text{Italian})\right) \end{matrix}\right\}, \tag{8}$$

where $d_{x_1}^0(\alpha)$ means the degree of appurtenance of element $x_1$ to the set $\mathcal{M}$ with respect to the attribute value α; and similarly $d_{x_2}^0(\alpha)$ means the degree of appurtenance of element $x_2$ to the set $\mathcal{M}$ with respect to the attribute value α; where $\alpha \in \{\text{tall, white, female, Italian}\}$.

Unlike the Crisp / Fuzzy / Intuitionistic Fuzzy / Neutrosophic Hypersoft Sets [where the degree of appurtenance of an element *x* to the set $\mathcal{M}$ is *with respect to all attribute values* ***tall, white, female, Italian*** *together* (as a whole), therefore a degree of appurtenance *with respect to a set of attribute values*], the Plithogenic Hypersoft Set is a **refinement** of Crisp / Fuzzy / Intuitionistic Fuzzy / Neutrosophic Hypersoft Sets [since the degree of appurtenance of an element *x* to the set $\mathcal{M}$ is with respect to each single attribute value].

But the Plithogenic Hypersoft St is also **combined** with each of the above, since the degree of degree of appurtenance of an element *x* to the set $\mathcal{M}$ with respect to each single attribute value may be: crisp, fuzzy, intuitionistic fuzzy, or neutrosophic.

## 7 Classification of Plithogenic Hypersoft Sets

### 7.1 Plithogenic Crisp Hypersoft Set

It is a plithogenic hypersoft set, such that the degree of appurtenance of an element *x* to the set $\mathcal{M}$, with respect to each attribute value, is **crisp**:

$d_x^0(\alpha) = 0$ (nonappurtenance), or 1 (appurtenance).

In our example:

$$F(\{\text{tall, white, female, Italian}\}) = \{x_1(1,1,1,1), x_3(1,1,1,1)\}. \quad (9)$$

### 7.2 Plithogenic Fuzzy Hypersoft Set

It is a plithogenic hypersoft set, such that the degree of appurtenance of an element *x* to the set $\mathcal{M}$, with respect to each attribute value, is **fuzzy**:

$d_x^0(\alpha) \in \mathcal{P}([0,1])$, power set of $[0,1]$,

where $d_x^0(\cdot)$ may be a subset, an interval, a hesitant set, a single-valued number, etc.

In our example, for a single-valued number:

$$F(\{\text{tall, white, female, Italian}\}) = \{x_1(0.4, 0.7, 0.6, 0.5), x_3(0.8, 0.2, 0.7, 0.7)\}. \quad (10)$$

### 7.3 Plithogenic Intuitionistic Fuzzy Hypersoft Set

It is a plithogenic hypersoft set, such that the degree of appurtenance of an element *x* to the set $\mathcal{M}$, with respect to each attribute value, is **intuitionistic fuzzy**:

$d_x^0(\alpha) \in \mathcal{P}([0,1]^2)$, power set of $[0,1]^2$,

where similarly $d_x^0(\alpha)$ may be: a Cartesian product of subsets, of intervals, of hesitant sets, of single-valued numbers, etc.

In our example, for single-valued numbers:

$$F(\{\text{tall, white, female, Italian}\}) = \begin{Bmatrix} x_1(0.4,0.3)(0.7,0.2)(0.6,0.0)(0.5,0.1) \\ x_3(0.8,0.1)(0.2,0.5)(0.7,0.0)(0.7,0.4) \end{Bmatrix}. \quad (11)$$

### 7.4 Plithogenic Neutrosophic Hypersoft Set

It is a plithogenic hypersoft set, such that the degree of appurtenance of an element *x* to the set $\mathcal{M}$, with respect to each attribute value, is **neutrosophic**:

$d_x^0(\alpha) \in \mathcal{P}([0,1]^3)$, power set of $[0,1]^3$,

where $d_x^0(\alpha)$ may be: a triple Cartesian product of subsets, of intervals, of hesitant sets, of single-valued numbers, etc.

In our example, for single-valued numbers:

$$F(\{\text{tall, white, female, Italian}\}) = \begin{Bmatrix} x\_1\ [(0.4,0.1,0.3)(0.7,0.0,0.2)(0.6,0.3,0.0)(0.5,0.2,0.1)] \\ x\_3\ [(0.8,0.1,0.1)(0.2,0.4,0.5)(0.7,0.1,0.0)(0.7,0.5,0.4)] \end{Bmatrix}. \quad (12)$$

## Conclusion & Future Research

For all types of plithogenic hypersoft sets, the aggregation operators (union, intersection, complement, inclusion, equality) have to be defined and their properties found.

Applications in various engineering, technical, medical, social science, administrative, decision making and so on, fields of knowledge of these types of plithogenic hypersoft sets should be investigated.

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# On Single Valued Neutrosophic Signed Digraph and its Applications

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**Abstract:** The development of the theory of the single valued neutrosophic (SVN) digraph is done in this paper. Also this paper introduces the concept of SVN signed digraph. Some basic terminologies and operations of SVN digraphs and SVN signed digraphs have been defined. Finally classification problem in a signed network system is solved with the help of SVN signed digraphs.

**Keywords:** SVN set, SVN digraph, SVN signed digraph, Classification Problem.

## 1 Introduction

Uncertainty is something that we cannot be sure about. It is a common phenomenon of our daily existence, because our world is full of uncertainties. There are many situations and complex physical processes, where we encounter uncertainties of different types and often face many problems due to it. Therefore it is natural for us to understand and try to model these uncertain situations prevailing in those physical processes. From centuries, the Science, whether Physics or Biology, or in Philosophy, i.e. every domain of knowledge has strived to understand the manifestations and features of uncertainty. Perhaps that is the main reason behind the development of Probability theory and Stochastic techniques which started in early eighteenth century, which has the ability to model uncertainties arising due to randomness. But the traditional view of Science, especially Mathematics was to worship certainty and to avoid uncertainty by all possible means. Therefore the classical mathematics failed to model many complex physical phenomena such as complex chemical processes or biological systems where uncertainty was unavoidable. Again probabilistic techniques cannot also model all kinds of uncertain situations. Natural language processing is an example of such problem where the above method fails. Thus the need for a fundamentally different approach to study such problems, where uncertainty plays a key role, was felt and that stimulated new developments in Mathematics.

Recently a new theory has been introduced and which is known as neutrosophic logic and sets. The term neutro-sophy means knowledge of neutral thought and this neutral represents the main distinction between fuzzy and intuitionistic fuzzy logic and set. Neutrosophic logic was introduced by Florentin Smarandache in 1995. It is a logic in which each proposition is estimated to have a degree of truth (T), a degree of indeterminacy (I) and a degree of falsity (F). A Neutrosophic set is a set where each element of the universe has a degree of truth, indeterminacy and falsity respectively and which lies between, the non-standard unit interval. Unlike in intuitionistic fuzzy sets, where the incorporated uncertainty is dependent of the degree of belongingness and degree of non belongingness, here the uncertainty present, i.e. indeterminacy factor, is independent of truth and falsity values. In 2005, Wang et. Al. introduced an instance of neutrosophic set known as single valued neutrosophic sets which were motivated from the practical point of view and that can be used in real scientific and engineering applications. The single valued neutrosophic set is a generalization of classical set, fuzzy set, intuitionistic fuzzy set and paraconsistent sets etc.

The recently proposed notion of neutrosophic sets is a general formal framework for studying uncertainties arising due to indeterminacy factors. From the philosophical point of view, it has been shown that a neutrosophic set generalizes a classical set, fuzzy set, interval valued fuzzy set, intuitionistic fuzzy set etc. Also single valued neutrosophic (SVN) set can be used in modeling real scientific and engineering problems. The SVN set is a generalization of classical fuzzy set [54], intuitionstic fuzzy set [7] etc. Therefore the study of neutrosophic sets and its properties have a considerable significance in the the sense of applications as well as in understanding the fundamentals of uncertainty [See [2, 3, 4, 5, 8, 10, 15, 16, 28, 29, 32, 33, 35, 36, 37, 38, 39, 40, 55]]. This new topic is very sophisticated and only a handful of papers have been published till date but it has immense possibilities which are to be explored.

Graphs and Digraphs play an important role to solve many pratical problems in algebra, analysis, geometry etc. A couple of researchers are continuously engaged in research on fuzzy graph theory, fuzzy digraph theory, intuitionstic fuzzy graphs, soft digraphs [17, 22, 23, 24, 49]. However Neutrosophic graphs, SVN graphs concept have been defined by Samarandache and Broumi et al. in their papers [12, 45]. We have defined the SVN digraphs in our previous paper [50]

In this paper we have developed the notion of SVN digraphs. Some preliminaries regarding SVN sets, graph theory etc. are discussed in Section 2. In section 3, we have defined the some terminologies regarding SVN digraph with examples. We have discussed SVN signed digraphs for the first time in Section 4. In section 5, we have solved a real life networking problem by using SVN signed digraph. Section 6 concludes the paper.

## 2 Preliminaries

Neutrosophic sets play an important role in decision making under uncertain environment of Mathematics. Most of the preliminary ideas regarding Neutrosophic sets and its possible applications can be easily found in any standard reference say [30, 43, 45, 46]. However we will discuss some definitions and terminologies regarding neutrosophic sets which will be used in the rest of the paper. Also we have added some new definitions and results on SVN digraphs in this section.

**Definition 1** *[30] Let $X$ be a universal set. A neutrosophic set $A$ on $X$ is characterized by a truth membership function $t_A$, an indeterminacy function $i_A$ and a falsity function $f_A$, where $t_A, i_A, f_A : X \to [0, 1]$, are functions and $\forall\ x \in X$, $x = x(t_A(x), i_A(x), f_A(x)) \in A$ is a single valued neutrosophic element of $A$. A single valued neutrosophic set (SVNS) $A$ over a finite universe $X = \{x_1, x_2, \ldots, x_n\}$ is represented as below:*

$$A = \sum_{i=1}^{n} \frac{x_i}{\langle t_A(x_i), i_A(x_i), f_A(x_i)\rangle}$$

**Definition 2** *[1] Let* $A = \{\langle x; t_A(x); i_A(x); f_A(x)\rangle; x \in X\}$ *be a single-valued neutrosophic set of the set* $X$*. For* $\alpha \in [0,1]$*, the* $\alpha$*-cut of* $A$ *is the crisp set* $A_\alpha$

$$A_\alpha = \{x \in X : \textit{either}(t_A(x); i_A(x)) \geq \alpha \textit{ or } f_A(x) < 1 - \alpha\}.$$

*Let* $B = \{\langle (x,y); t_B(x,y); i_B(x,y); f_B(x,y)\rangle\}$ *be a neutrosophic set on* $E \subseteq X \times X$ *. For* $\alpha \in [0,1]$*, the* $\alpha$*-cut is the crisp set* $B_\alpha$ *defined by,*

$$B_\alpha = \{(x,y) \in E : \textit{either } (t_B(x,y); i_B(x,y) \geq \alpha) \textit{ or } \\ f_B(x,y) \leq 1 - \alpha\}.$$

**Definition 3** *[30] Suppose* $N(X)$ *be the collection of all SVN sets on* $X$ *and* $A, B \in N(X)$*. A similarity measure between two SVN sets* $A$ *and* $B$ *is a function* $S : N(X) \times N(X) \to [0,1]$ *which satisfies the following condition:*

(i) $0 \leq S(A,B) \leq 1$,

(ii) $S(A,B) = 1$ *if and only if* $A = B$.

(iii) $S(A,B) = S(B,A)$

(iv) *If* $A \subseteq B \subseteq C$*, then* $S(A,C) \leq S(A,B)$ *and* $S(A,C) \leq S(B,C)$ *for all* $A, B, C \in N(X)$.

Note that here (i)-(iii) are essential for any similarity measure and (iv) is a desirable property although not mandatory.

**Definition 4** *The entropy of SVNS* $A$ *is defined as a function* $E : N(X) \to [0,1]$ *which satisfies the following axioms:*

(i) $E(A) = 0$ *if* $A$ *is a crisp set.*

(ii) $E(A) = 1$ *if* $(t_A(x), i_A(x), f_A(x)) = (0.5, 0.5, 0.5)$ $\forall x \in X$.

(iii) $E(A) \geq E(B)$ *if* $A$ *is more uncertain than* $B$ *i.e.* $t_A(x) + f_A(x) \leq t_B(x) + f_B(x)$ *and* $|i_A(x) - i_{A^c}(x)| \leq |i_B(x) - i_{B^c}(x)|$ $\forall\, x \in X \;\; A, B \in X$.

(iv) $E(A) = E(A^c)$ $\forall A \in N(X)$*, where* $N(X)$ *is the collection of all SVNS over* $X$.

**Example 5** *An entropy measure of an element* $x_1$ *of a SVNS* $A$ *can be calculated as follows:*

$$E_1(x_1) = 1 - (t_A(x_1) + f_A(x_1)) \times |i_A(x_1) - i_{A^c}(x_1)|.$$

Graph theory are widely used in different areas of neutrosophic theory. Many authors have used different types of graphs in neutrosophic theory. Consider a SVN set $V_D = \{(v_i, \langle t_{V_D}(v_i), i_{V_D}(v_i), f_{V_D}(v_i)\rangle), i = 1, \ldots, n\}$ over a finite universal set $X$.

**Definition 6** *[50] A SVN digraph* $D$ *is of the form* $D = (V_D, A_D)$ *where,*

(i) $V_D = \{v_1, v_2, v_3, \ldots, v_n\}$ *and the functions* $t_{V_D} : V_D \to [0,1]$, $i_{V_D} : V_D \to [0,1]$, $f_{V_D} : V_D \to [0,1]$ *denote the truth-membership function, a indeterminacy-membership function and falsity-membership function of the element* $v_i \in V_D$ *respectively such that* $0 \leq t_{V_D}(v_i) + i_{V_D}(v_i) + f_{V_D}(v_i) \leq 3$, $\forall v_i \in V_D, i = 1, 2, \ldots, n$.

(ii) $A_D = \{(v_i, v_j); (v_i, v_j) \in V_D \times V_D\}$ *provided* $0 < E(v_i) - E(v_j) \leq 0.5$ *and the functions* $t_{A_D} : A_D \to [0,1]$, $i_{A_D} : A_D \to [0,1]$, $f_{A_D} : A_D \to [0,1]$ *are defined by*

$$t_{A_D}(\{v_i, v_j\}) \leq min[t_{V_D}(v_i), t_{V_D}(v_j)],$$
$$i_{A_D}(\{v_i, v_j\}) \geq max[i_{V_D}(v_i), i_{V_D}(v_j)],$$
$$f_{A_D}(\{v_i, v_j\}) \geq max[f_{V_D}(v_i), f_{V_D}(v_j)]$$

*where* $t_{A_D}, i_{A_D}, f_{A_D}$ *denotes the truth-membership function, a indeterminacy-membership function and falsity-membership function of the arc* $(v_i, v_j) \in A_D$ *respectively where* $0 \leq t_{A_D}(v_i, v_j) + i_{A_D}(v_i, v_j) + f_{A_D}(v_i, v_j) \leq 3$, $\forall (v_i, v_j) \in A_D,\ i, j \in \{1, 2, \ldots n\}$.

*We call* $V_D$ *as the vertex set of* $D$*,* $A_D$ *as the arc set of* $D$ *where* $E(v)$ *is the entropy of the vertex* $v$*. Please note that if* $E(v_i) = E(v_j)$*, then* $\{(v_i, v_j), (v_j, v_i)\} \in A_D$*. Since for a vertex* $v \in V_D$ *of a SVN digraph* $D$ *we have* $E(v) = E(v)$*, thus every vertex of a SVN digraph* $D$ *contains a loop* $(v, v)$ *at* $v$*. On the other hand, if* $E(v_i) - E(v_j) > 0.5$*, we define that there exists no arc between the vertices* $v_i$ *and* $v_j$.

**Example 7** *Consider the SVN digraph* $D_0 = (V_{D_0}, A_{D_0})$ *in Figure 1 with vertex set* $V_{D_0} = \{v_1, v_2, v_3\}$ *and arc set* $A_{D_0} = \{(v_2, v_1), (v_1, v_3), (v_2, v_3)\}$ *with one loop at each vertex as follows:*

$$\begin{bmatrix} & v_1 & v_2 & v_3 \\ t_{V_D} & 0.4 & 0.4 & 0.5 \\ i_{V_D} & 0.1 & 0.3 & 0.2 \\ f_{V_D} & 0.2 & 0.1 & 0.5 \\ E & 0.52 & 0.8 & 0.4 \end{bmatrix},$$

$$\begin{bmatrix} & (v_2, v_1) & (v_2, v_3) & (v_1, v_3) \\ t_{A_D} & 0.3 & 0.2 & 0.4 \\ i_{A_D} & 0.4 & 0.3 & 0.3 \\ f_{A_D} & 0.2 & 0.6 & 0.4 \end{bmatrix}.$$

*It is clear that the* $D_0$ *is a SVN digraph.*

**Definition 8** *Suppose* $D = (V_D, A_D)$ *and* $H = (V_H, A_H)$ *be two SVN digraphs with* $|V_D| = |V_H|$ *corresponding to the SVNS* $V_D$ *and* $V_H$ *over an universal set* $X$*. Then the cartesian product of two SVN digraphs* $D$ *and* $H$ *is defined as a SVN digraph* $C = (V_C, A_C)$ *in which the following holds:*

(i) $V_C = V_D \times V_H$,

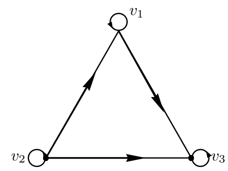


Figure 1: The SVN Digraph $D_0$

(ii) $t_{V_C}(v_1,v_2) \le min(t_{V_D}(v_1), t_{V_H}(v_2))$; $i_{V_C}(v_1,v_2) \le min(i_{V_D}(v_1), i_{V_H}(v_2))$; $f_{V_C}(v_1,v_2) \ge max(f_{V_D}(v_1), f_{V_H}(v_2))$; $\forall\ (v_1,v_2) \in V_D \cap V_H$ *and*,

(iii) $A_C = \{((v_i,v_j),(v_k,v_l)); ((v_i,v_j),(v_k,v_l)) \in V_C \times V_C\}$ *provided* $0 < E(v_i,v_j) - E(v_k,v_l) \le 0.5$.

**Definition 9** *The degree and the total degree of a vertex $v_i$ of a SVN digraph $D=(V,A)$ are denoted by*

$$\begin{aligned} d_D(v_i) &= (d_t(v_i), d_i(v_i), d_f(v_i)) \\ &= \Big(\sum_{j,i\neq j} t_A(v_i,v_j), \sum_{j,i\neq j} i_A(v_i,v_j), \sum_{j,i\neq j} f_A(v_i,v_j)\Big), \\ Td_D(v_i) &= \Big(\sum_{j,i\neq j} t_A(v_i,v_j) + t_V(v_i), \sum_{j,i\neq j} i_A(v_i,v_j) + \\ &\quad i_V(v_i), \sum_{j,i\neq j} f_A(v_i,v_j) + f_V(v_i)\Big). \end{aligned}$$

**Example 10** *The degree and total degree of the vertex $v_2$ of the digraph $D_0$ in Example 7 are $d_D(v_2) = (0.5, 0.7, 0.8)$ and $Td_D(v_2) = (0.9, 1, 0.9)$.*

**Definition 11** *A SVN digraph $D=(V_D,A_D)$ is called a k-regular SVN digraph if $d_D(v_i) = (k,k,k)\ \forall v_i \in V_D$.*

**Definition 12** *A SVN digraph $D=(V_D,A_D)$ is called a totally regular SVN digraph of degree $(k_1,k_2,k_3)$ if $Td_D(v_i) = (k_1,k_2,k_3)\ \forall v_i \in V_D$.*

It is quite clear that the concept of a regular SVN digraph and totally regular SVN digraph are completely different. We have seen that the arc set $A_D$ in $D$ forms a SVN set [50]. Now we consider the concept of degree and total degree of an arc of a SVN digraph in the next definition.

**Definition 13** *The degree and the total degree of an arc $(u,v)$ of a SVN digraph are denoted by $d_D(u,v) = (d_t(u,v), d_i(u,v), d_f(u,v))$ and $Td_D(u,v) = (Td_t(u,v), Td_i(u,v), Td_f(u,v))$, respectively and are defined as follows:*

$$\begin{aligned} d_D(u,v) &= d_D(u) + d_D(v) - \frac{1}{2}(t_A(u,v), i_A(u,v), f_A(u,v)), \\ Td_D(u,v) &= d_D(u,v) + (t_A(u,v), i_A(u,v), f_A(u,v)) \end{aligned}$$

**Example 14** *Consider the SVN digraph $D_0$ in Figure 1. Here the degree and total degree of the vertices $\{v_1,v_2,v_3\}$ of $D_0$ as follows:*

$$\begin{aligned} d_D(v_1) &= (0.4, 0.3, 0.4), Td_D(v_1) = (0.8, 0.4, 0.6), \\ d_D(v_2) &= (0.5, 0.7, 0.8), Td_D(v_2) = (0.9, 1, 0.9), \\ d_D(v_3) &= (0, 0, 0), Td_D(v_3) = (0.5, 0.2, 0.5). \end{aligned}$$

*Now we calculate the degree and total degree of each arc of $A_{D_0}$ of $D_0$ as follows:*

$$\begin{aligned} d_D(v_2,v_1) &= (0.85, 0.8, 1.1), Td_D(v_2,v_1) = (0.6, 0.6, 0.1), \\ d_D(v_2,v_3) &= (0.4, 0.55, 0.5), Td_D(v_2,v_3) = (0.3, 0.4, 0.2), \\ d_D(v_1,v_3) &= (0.2, 0.15, 0.2), Td_D(v_1,v_3) = (0, 0, 0). \end{aligned}$$

**Definition 15** *The maximum degree of a SVN digraph $D=(V_D,A_D)$ is defined as $\Delta(D) = (\Delta_t(D), \Delta_i(D), \Delta_f(D))$ where*

$$\begin{aligned} \Delta_t(D) &= max\{d_t(v) : v \in V_D\}, \\ \Delta_i(D) &= max\{d_i(v) : v \in V_D\}, \\ \Delta_f(D) &= max\{d_f(v) : v \in V_D\}, \end{aligned}$$

**Definition 16** *The minimum degree of a SVN digraph $D=(V_D,A_D)$ is defined as $\delta(D) = (\delta_t(D), \delta_i(D), \delta_f(D))$ where*

$$\begin{aligned} \delta_t(D) &= min\{d_t(v) : v \in V_D\}, \\ \delta_i(D) &= min\{d_i(v) : v \in V_D\}, \\ \delta_f(D) &= min\{d_f(v) : v \in V_D\}, \end{aligned}$$

**Example 17** *For the SVN digraph $D_0$ in Figure 1, we have $\Delta(D) = (0.5, 0.7, 0.8)$ and $\delta(D) = (0, 0, 0, )$.*

**Definition 18** *Suppose $D = (V_D, A_D)$ be a SVN digraph corresponding to a SVN set $V_D$. Then $D$ is said to be*

(i) *arc regular SVN digraph if every arc in $D$ has the same degree $(k_1, k_2, k_3)$.*

(ii) *equally arc regular SVN digraph if $k_1 = k_2 = k_3$.*

(iii) *totally arc regular SVN digraph if every arc in $D$ has the same total degree $(k_1, k_2, k_3)$.*

It is also quite clear that the above three concepts are completely different to each other.

# 3 SVN Signed Digraph

In this paper, we will define the SVN signed digraph for the first time.

**Definition 19** *Suppose $D = (V_D, A_D)$ be a SVN digraph over a single valued neutrosophic set $V_D$. A signing of a SVN digraph $D$ is an assignment of a sign ($+$ or $-$) to each arc of the digraph; the sign of arc $(v, w)$ is denoted $sgn(v, w)$. The result of a signing of $D$ is called a SVN signed digraph.*

*However to assign the sign of the arcs, we will follow some rules. For this, we will consider the $\alpha$-level subdigraph $D_1$ of a SVN digraph $D$. Then we will assign $+$ sign only to those arcs of $D$ which are also the arcs of $D_1$. For the rest of arcs of $D$, we will assign $-$ sign.*

**Example 20** *Consider the SVN digraph $D_1 = (V_{D_1}, A_{D_1})$ in Figure 2 with vertex set $V_{D_1} = \{v_1, v_2, v_3, v_4\}$ and arc set $A_{D_1} = \{(v_2, v_1), (v_1, v_3), (v_2, v_3), (v_2, v_4), (v_3, v_4), (v_4, v_1)\}$ with one loop at each vertex as follows:*

$$\left[\begin{array}{ccccc} & v_1 & v_2 & v_3 & v_4 \\ t_{V_{D_1}} & 0.4 & 0.4 & 0.5 & 0.2 \\ i_{V_{D_1}} & 0.1 & 0.3 & 0.2 & 0.5 \\ f_{V_{D_1}} & 0.2 & 0.1 & 0.5 & 0.3 \\ E & 0.52 & 0.8 & 0.4 & 1 \end{array}\right],$$

$$\left[\begin{array}{cccccc} & (v_2, v_1) & (v_2, v_3) & (v_4, v_1) & (v_1, v_3) & (v_4, v_2) \\ t_{A_{D_1}} & 0.3 & 0.2 & 0.1 & 0.4 & 0.2 \\ i_{A_{D_1}} & 0.4 & 0.3 & 0.5 & 0.3 & 0.5 \\ f_{A_{D_1}} & 0.2 & 0.6 & 0.3 & 0.4 & 0.4 \end{array}\right].$$

*We take $\alpha = 0.5$. In this case, the vertices $\{v_1, v_2, v_4\}$ of $D_1$ are $\alpha$-level vertices and the arcs $\{(v_2, v_1), (v_4, v_1), (v_4, v_2)\}$ are the $\alpha$-level arcs. Thus we will assign the sign as follows to the arcs of $D_1$*

$$sgn(v_2, v_1) = sgn(v_4, v_1) = sgn(v_4, v_2) = +$$
$$sgn(v_1, v_1) = sgn(v_2, v_2) = sgn(v_4, v_4) = +,$$
$$sgn(v_2, v_3) = sgn(v_1, v_3) = sgn(v_3, v_3) = -$$

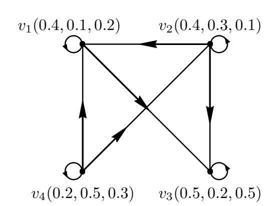


Figure 2: The SVN Digraph $D_1$

**Remark 21** *Throughout this paper, we have taken the value of $\alpha$ is 0.5. However, for different values of $\alpha$ we will get different signed SVN digraphs. Also, by $K_n$, we denote the complete SVN digraph of $n$-vertices.*

**Definition 22** *The sets of positive and negative arcs of a SVN signed digraph $D$ are respectively denoted by $D^+$ and $D^-$. Thus $D = D^+ \cup D^-$.*

**Definition 23** *A SVN signed digraph is said to be homogeneous if all of its arcs have either positive sign or negative sign, otherwise heterogeneous.*

**Definition 24** *The sign of a SVN signed digraph is defined as the product of signs of its arcs.A SVN signed digraph is said to be positive (negative) if its sign is positive (negative) i.e., it contains an even (odd) number of negative arcs. A signed digraph is said to be all-positive (respectively, all negative) if all its arcs are positive (negative).*

**Example 25** *It is clear that the sign of the SVN digraph $D_1$ in Example 20 is negative. It is clear that the SVN digraph $D_1$ is neither all positive nor all negative.*

**Definition 26** *A SVN signed digraph is said to be cycle balanced if each of its cycles is positive, otherwise non cycle balanced.*

**Definition 27** *A SVN signed digraph is symmetric if* $(u, v) \in D^+$ $(or D^-)$ *then* $(v, u) \in D^+$ $(or D^-)$ *where* $u, v \in V_D$.

**Definition 28** *The adjacency matrix of a SVN signed digraph* $D$ *is the square matrix* $M = (a_{ij})$ *whose* $(i, j)$ *entry* $a_{ij}$ *is* $+1$ *if arc* $(v_i, v_j)$ *in* $D$ *has a* $+$ *sign,* $-1$ *if arc* $(v_i, v_j)$ *in* $D$ *has a* $-$ *sign, and* 0 *if arc* $(v_i, v_j)$ *is not in* $D$.

**Example 29** *The adjacency matrix* $M$ *of the SVN signed digraph* $D_1$ *in Figure 2 is as following:*

$$M = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}.$$

**Definition 30** *The characteristic polynomial* $\delta(t) = |tI - M|$ *of the adjacency matrix* $M$ *of a SVN signed digraph* $D$ *is called the characteristic polynomial of* $D$ *and it is denoted by* $p(t)$. *The eigenvalues of* $M$ *are called the spectral of the digraph* $D$.

**Example 31** *The characteristic polynomial* $\delta(t)$ *of the SVN signed digraph* $D_1$ *in Figure 2 is* $\delta(t) = (t-1)^3(t+1)$ *and spectral values are* $1, 1, 1, -1$.

**Definition 32** *Suppose* $D = (V_D, A_D)$ *be a SVN signed digraph over a single valued neutrosophic set* $V_D = \{v_1, v_2, \ldots, v_n\}$. *Consider the complement SVN digraph* $D^C$ *corresponding to the complement SVN set* $V_D^C$. *The digraph* $V_D^C$ *with a signing of arcs is called the signed complement of the SVN signed digraph* $V_D$.

Here we choose the same value of $\alpha$ as of $D$ and also consider the $\alpha$-arcs and $\alpha$-vertices. According to the $\alpha$-arcs and $\alpha$-vertices we assign signs to the arcs of $V_D^c$.

**Example 33** *Consider the SVN complement digraph* $D^c$ *of the SVN digraph* $D_1$ *in Figure 2.*

$$\begin{aligned} sgn(v_2, v_1) &= sgn(v_4, v_1) = sgn(v_4, v_2) = -, \\ sgn(v_1, v_1) &= sgn(v_2, v_2) = sgn(v_4, v_4) = +, \\ sgn(v_2, v_3) &= sgn(v_1, v_3) = sgn(v_3, v_3) = +. \end{aligned}$$

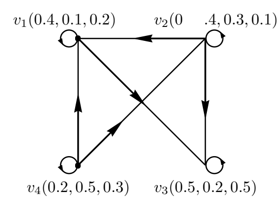


Figure 3: The SVN Digraph $D^C$

# 4 Some important results of a SVN Signed Digraph

In this section we will discuss some results regarding SVN signed Neutrosophic digraphs. Like wise a SVN digraph $D$, we define the terminologies of a SVN signed digraph. However, the *order* of a SVN signed digraph $D$, denoted by $|D|$, is the number of vertices of $D$. The *size* of a SVN signed digraph $D$, is the number of arcs of $D$ i.e. $|A_D|$.

**Theorem 34** *A SVN (signed) digraph* $D \neq K_n$ *of order* $\geq 3$ *is always acyclic.*

**Proof 35** *Suppose there exist a cyclic SVN (signed) digraph* $D = (V_D, A_D)$ *has vertex set* $V_D = \{v_1, v_2, v_3, \ldots, v_n\}$. *Without loss of generality, let* $D$ *has a cycle of length k, where* $k \geq 3$ *say* $\langle v_1, v_2, \ldots, v_k \rangle$. *Then we have* $E(v_1) > E(v_2) > \ldots,$ $E(v_k) > E(v_1)$*- which is impossible. Hence* $D$ *does not have a cycle of length* $k$.

**Corollary 36** *Any asymmetric SVN signed digraph of order* $\geq 3$ *is not balanced.*

**Theorem 37** *Any asymmetric SVN (signed) digraph* $H$ *of order* $\geq 3$ *is not strongly connected.*

**Proof 38** *Since there does not exists any SVN (signed) digraph with a cycle of length* $\geq 3$*, hence the results follows.*

**Theorem 39** *In any complete symmetric SVN digraph* $D = (V_D, A_D)$*, where* $V_D = \{v_1, v_2, \ldots, v_n\}$*,*

$$\begin{aligned} &\sum d_D(v_i) = (d_t(v_i), d_i(v_i), d_f(v_i)) \\ &= \Big( \sum_{j, i \neq j} t_A(v_i, v_j), \sum_{j, i \neq j} i_A(v_i, v_j), \sum_{j, i \neq j} f_A(v_i, v_j) \Big), \end{aligned}$$

$\forall v_i \in V_D$.

**Proof 40** *For any symmetric complete SVN digraph $D = (V_D, A_D)$, where $V_D = \{v_1, v_2, \ldots, v_n\}$, we have,* $\sum d_D(v_i) = (\sum d_t(v_i), \sum d_i(v_i), \sum d_f(v_i))$
$= (d_t(v_1), d_i(v_1), d_f(v_1)) + \ldots + (d_t(v_n), d_i(v_n), d_f(v_n))$
$= \big(\sum_{j,1\neq j} t_A(v_1, v_j), \sum_{j,1\neq j} i_A(v_1, v_j), \sum_{j,1\neq j} f_A(v_1, v_j)\big) + \ldots +$
$\big(\sum_{j,n\neq j} t_A(v_n, v_j), \sum_{j,n\neq j} i_A(v_n, v_j), \sum_{j,n\neq j} f_A(v_n, v_j)\big)$
$= 2\big(\sum_{j,i\neq j} t_A(v_i, v_j), \sum_{j,i\neq j} i_A(v_i, v_j), \sum_{j,i\neq j} f_A(v_i, v_j)\big)$.

However the converse of the Theorem 39 is not true for a asymmetric and incomplete SVN digraph which can be followed from the Example 10. The SVN digraph $D_0$ is asymmetric as well as incomplete. Clearly the Theorem 39 does not hold.

**Theorem 41** *Suppose $D = (V_D, A_D)$ be a SVN symmetric digraph which has a cycle $C$ on p-vertices, say $\{v_1, v_2, \ldots, v_p\}$. Then,*
$\sum d_D(v_i)$

$$= \sum d_D(v_i, v_j) + \frac{3}{2}\big( \sum_{j,i\neq j} t_A(v_i, v_j), \sum_{j,i\neq j} i_A(v_i, v_j), \sum_{j,i\neq j} f_A(v_i, v_j)\big)$$

*where $(v_i, v_j) \in C, i \neq j$.*

**Proof 42** *We have,* $\sum d_D(v_i, v_j) = d_D(v_1, v_2) + \ldots + d_D(v_p, v_1)$
$= d_D(v_1) + d_D(v_2) - \frac{1}{2}(t_A(v_1, v_2), i_A(v_1, v_2), f_A(v_1, v_2)) + \ldots +$
$d_D(v_p) + d_D(v_1) - \frac{1}{2}(t_A(v_p, v_1), i_A(v_p, v_1), f_A(v_p, v_1))$,
$= 2\sum_{v_i\in C} d_D(v_i) -$
$\frac{1}{2}\big\{ \sum_{j,i\neq j} t_A(v_i, v_j), \sum_{j,i\neq j} i_A(v_i, v_j), \sum_{j,i\neq j} f_A(v_i, v_j)\big\}$
$= \sum_{v_i\in C} d_D(v_i) +$,
$2\big\{ \sum_{j,i\neq j} t_A(v_i, v_j), \sum_{j,i\neq j} i_A(v_i, v_j), \sum_{j,i\neq j} f_A(v_i, v_j)\big\}$
$-\frac{1}{2}\big( \sum_{j,i\neq j} t_A(v_i, v_j), \sum_{j,i\neq j} i_A(v_i, v_j), \sum_{j,i\neq j} f_A(v_i, v_j)\big)$,
$= \sum d_D(v_i, v_j) +$,
$\frac{3}{2}\big( \sum_{j,i\neq j} t_A(v_i, v_j), \sum_{j,i\neq j} i_A(v_i, v_j), \sum_{j,i\neq j} f_A(v_i, v_j)\big)$.

**Theorem 43** *The maximum value of the degree of any vertex in a complete SVN digraph $D$ with $n$ vertices is $(n-1, n-1, n-1)$.*

**Proof 44** *Suppose $D = (V_D, A_D)$ be a complete SVN digraph. Then the maximum truth-membership value given to an arc is 1 and the number of arcs incident on a vertex can be at most $n-1$. Hence the maximum truth-membership degree of any vertex in a complete SVN-digraph with $n$ vertices is $n-1$. Similar argument can be done for indeterminacy-membership degree and falsity-membership degree of any vertex. Hence the result follows.*

The following remarks are quite natural for a SVN signed digraph:

**Remark 45** (i) *A single valued neutrosophic signed digraph is a single valued neutrosophic positive signed digraph if every even length cycles having all negative signed arcs.*

(ii) *Odd length cycle having all negative signed arcs is always a negative signed digraph.*

(iii) *An odd length single valued signed neutrosophic cycle is balanced if and only if it contains at least one positive arcs or odd number of positive arcs.*

# 5 Applications of a SVN Signed digraph

The applications of SVN sets in solving real life problems under uncertainty has been shown by many authors. In this section we have shown the application of our SVN signed digraphs in solving two problems namely a classification problem and a decision making problem.

## 5.1 Classification problem

Consider the SVN set $V(D) = \{v_1, v_2, v_3.v_4\}$ in Example 20 and the corresponding SVN signed digraph $D = (V_D, A_D)$ in Figure 2. To draw SVN signed digraph, we have taken $\alpha = 0.5$. Based on this $\alpha$, we find that the vertices $\{v_1, v_2, v_4\}$ of $D_1$ as $\alpha$-level vertices and the arcs $\{(v_2, v_1), (v_4, v_1), (v_4, v_2)\}$ as the $\alpha$-level arcs. Then we assign the signs to the arcs of $D$ as follows:

$$sgn(v_2, v_1) = sgn(v_4, v_1) = sgn(v_4, v_2) = +,$$
$$sgn(v_1, v_1) = sgn(v_2, v_2) = sgn(v_4, v_4) = +,$$
$$sgn(v_2, v_3) = sgn(v_1, v_3) = sgn(v_3, v_3) = -.$$

Hence, we can form a partition of two sets namely $P, Q$, where $P = \{v_1, v_2, v_4\}$ and $Q = \{v_3\}$ from the elements of a SVN set $V(D)$. The partition is done on the basis of signing of the $\alpha$-level vertices. Thus by drawing SVN signed digraph of a SVN set, we can get a 2-point classification of a SVN set.

## 5.2 Algorithm for 2-point classification of a SVN set

One can attempt for 2-point classification of a SVN set by using the following algorithm:

(i) Consider a SVN set $V(D)$.

(ii) Draw a SVN digraph $D = (V(D), A(D))$, where $V(D), A(D)$ are the vertex set and arc set of $D$ respectively.

(iii) Choose the value of $\alpha$ and find out $\alpha$ level vertices of $D$. The choice of the value of the $\alpha$ is completely depend on the programmer.

(iv) Assign the positive sign with the $\alpha$ level vertices, arcs and negative sign to rest of the vertices, arcs of $D$. In that case $D$ turns into a SVN signed digraph.

(v) Finally consider two sets $P, Q$ s.t $P$ consists the positive vertices and $Q$ contains the negative vertices. Hence a partition of the SVN sets $V(D)$ is done consisting of two sets $P$ and $Q$ respectively.

### 5.3 A Decision Making Problem

Suppose $A, B, C$ be three nations willing to explore the possibility trade between them. Considering various situations in there countries like, political stability, case of doing business, human resource, trade laws etc. Each country was assigned grades of positive factors, indeterminacy and negative factors as follows:

$$A(0.4, 0.3, 0.2), B(0.4, 0.1, 0.2), C(0.5, 0.2, 0.4).$$

In these way, we can characterize the three country $A, B, C$ respectively. We must to find the possibility of trade between them. For this, we consider $A, B, C$ as the three vertices $v_1, v_2, v_3$ respectively as a vertex set $V_{D_4}$ of a proposed SVN digraph $D_4 = (V_{D_4}, A_{D_4})$. Now we draw the SVN digraph $D_4$ as follows:

$$\begin{bmatrix} & v_1 & v_2 & v_3 \\ t_{V_D} & 0.4 & 0.4 & 0.5 \\ i_{V_D} & 0.3 & 0.1 & 0.2 \\ f_{V_D} & 0.2 & 0.2 & 0.4 \\ E & 0.76 & 0.52 & 0.46 \end{bmatrix}, \begin{bmatrix} & (v_1, v_2) & (v_1, v_3) & (v_2, v_3) \\ t_{A_D} & 0.3 & 0.2 & 0.4 \\ i_{A_D} & 0.4 & 0.3 & 0.3 \\ f_{A_D} & 0.2 & 0.6 & 0.4 \end{bmatrix}.$$

Here, we have seen that $A_{D_4} = \{(v_1, v_2), (v_1, v_3), (v_2, v_3)\}$. So we can say that, there is a good transport communication between the country pair $(A, B), (A, C), (B, C)$

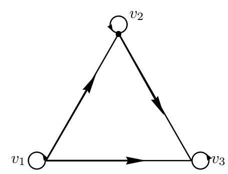


Figure 4: The SVN Digraph $D_4$

respectively. Now consider $\alpha = 0.3$. Here, the vertices $\{v_1, v_2\}$ of $D_4$ are $\alpha$-level vertices and the arcs $\{(v_1, v_2)\}$ is the only $\alpha$-level arcs. Thus we will assign the sign as follows to the arcs of $D_4$

$$\begin{aligned} sgn(v_1, v_2) &= sgn(v_1, v_1) = sgn(v_2, v_2) = +, \\ sgn(v_2, v_3) &= sgn(v_1, v_3) = sgn(v_3, v_3) = - \end{aligned}$$

From this SVN signed digraph $D_4$ we can conclude that both $A$ and $B$ have a common enemy $C$. Hence although there is a good communication between two country $(A, C)$ and $(B, C)$, it is not possible to do business between them due to their political situation. Hence a cyclic triple SVN signed digraph $D_4$ with one positive arcs can evaluate the real networks.

## 6 Conclusion

F. Smarandache introduced the neutrosophic set theory in his paper [43] as a generalization of fuzzy intuitionistic set theory. After that many researchers have developed the neutrosophic set theory, SVN theory, neutrosophic graph theory etc. and have applied those theories in solving many practical problems ([1, 6, 10, 12, 13, 14, 19, 20, 21, 26, 31, 41, 42, 48, 50, 51, 52, 53] etc.). We have developed earlier SVN digraph theory corresponding to a SVN set in our paper [50]. In this paper we have further developed the SVN digraph theory and introduced the notion of SVN signed digraphs and studied some of its important properties and applied it in a decision making problem. In future, one may study the decision making problems using SVN signed digraphs. The study of deeper properties of SVN signed digraphs and solution of more real life problems will be done in our subsequent papers.

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# On single-valued co-neutrosophic graphs

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**Abstract:** In this paper, we introduce the notion of a single-valued co-neutrosophic graphs and study some methods of construction of new single-valued co-neutrosophic graphs. We compute degree of a vertex, strong single-valued co-neutrosophic graphs and complete single-valued co-neutrosophic graphs. We also introduce and give properties of regular and totally regular single-valued co-neutrosophic graphs.

**Keywords:** Single-valued neutrosophic graphs; degree of a vertex; strong single-valued co-neutrosophic graphs; complete single-valued co-neutrosophic graphs; regular and totally regular single-valued co-neutrosophic graphs.

# 1 Introduction and preliminaries

Zadeh [21] introduced the concepts of fuzzy set theory as a generalized concept of crisp set theory. The concept of fuzzy graph theory as a generalization of Eulers graph theory was first introduced by Rosenfeld [17] in 1975. Later, Bhattacharya [5] gave some remarks on fuzzy graphs. The concept of cofuzzy graphs by M. Akram [1]. The concepts of intuitionistic cofuzzy graph by Dhavaseelan [9]. Smarandache [20] introduced the concept of neutrosophic sets. Certain types of neutrosophic graphs were introduced by R. Dhavaseelan et al. [10].Some more work in single valued neutrosophic set,interval valued neutrosophic set and their application may be found in Karaaslan,et .al., [13], Hamidi,et .al., [11, 14], Broumi, et.al., [6–8, 15] and Shimaa Fathi,et.al [18]. Kandasamy, et.al [12], introduced the new dimension of neutrosophic graph.

In this paper, we introduce the notion of a single-valued co-neutrosophic graphs and study some methods of construction of new single-valued co-neutrosophic graphs. We compute degree of a vertex, strong single-valued co-neutrosophic graphs and complete single-valued co-neutrosophic graphs. We also introduce and give properties of regular and totally regular single-valued co-neutrosophic graphs.

**Definition 1.1.** [19] Let X be a space of points. A neutrosophic set A in X is characterized by a truth-membership function $T_A(x)$, an indeterminacy membership function $I_A(x)$ and a falsity membership function $F_A(x)$. The functions $T_A(x), I_A(x)$and $F_A(x)$ are real standard or non standard subsets of $]0^-, 1^+[$. That is,

$T_A(x) : X \to ]0^-, 1^+[$, $I_A(x) : X \to ]0^-, 1^+[$,$F_A(x) : X \to ]0^-, 1^+[$ and $0^- \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+$. From philosophical point view, the neutrosophic set takes the value from real standard or non standard subsets of $]0^-, 1^+[$. In real life applications in scientific and engineering problems, it is difficult to use neutrosophic set with value from real standard or non standard subset of $]0^-, 1^+[$.

**Definition 1.2.** $[2, 4]$ A single-valued neutrosophic graph is a pair $G = (A, B)$, where $A : V \to [0, 1]$ is single-valued neutrosophic set in V and $B : V \times V \to [0, 1]$ is single-valued neutrosophic relation on V such that

$$T_B(xy) \leq \min\{T_A(x), T_A(y)\}\ I_B(xy) \leq \min\{I_A(x), I_A(y)\}\ F_B(xy) \geq \max\{F_A(x), F_A(y)\}$$

for all $x, y \in V$. A is called single-valued neutrosophic vertex set of G and B is called single-valued neutrosophic edge set of G, respectively. We note that B is symmetric single-valued neutrosophic relation on A. If B is not symmetric single-valued neutrosophic relation on A, then $G = (A, B)$ is called a single-valued neutrosophic directed graph.

# 2 Single-valued co-neutrosophic graphs

**Definition 2.1.** A single-valued co-neutrosophic graph is a pair $G = (A, B)$, where $A : V \to [0, 1]$ is a single-valued co-neutrosophic set in V and $B : V \times V \to [0, 1]$ is a single-valued co-neutrosophic relation on V such that

$$T_B(xy) \geq \max\{T_A(x), T_A(y)\}$$
$$I_B(xy) \geq \max\{I_A(x), I_A(y)\}$$
$$F_B(xy) \leq \min\{F_A(x), F_A(y)\}$$

for all $x, y \in V$. $A$ and $B$ are called the single-valued co-neutrosophic vertex set of $G$ and the single-valued co-neutrosophic edge set of $G$, respectively. We note that B is a symmetric single-valued co-neutrosophic relation on A. If B is not a symmetric single-valued co-neutrosophic relation on A, then $G = (A, B)$ is called a single-valued co-neutrosophic directed graph.

**Notation 2.1.** The triples $\langle T_A(x), I_A(x), F_A(x)\rangle$ denotes the degree of membership, an indeterminacy membership and nonmembership of vertex $x$, The triples $\langle T_B(xy), I_B(xy), F_B(xy)\rangle$ denote the degree of membership, an indeterminacy membership and nonmembership of edge relation $xy = (x, y)$ on $V$.

**Definition 2.2.** A partial single-valued co-neutrosophic subgraph of single-valued co-neutrosophic graph $G = (A, B)$ is a single-valued co-neutrosophic graph $H = (V', E')$ such that

(i) $V' \subseteq V$, where $T'_A(v_i) \leq T_A(v_i), I'_A(v_i) \leq I_A(v_i), F'_A(v_i) \geq F_A(v_i)$ for all $v_i \in V$.

(ii) $T_B(xy)' \leq T_B(xy)$;$I_B(xy)' \leq I_B(xy)$; $F_B(xy)' \geq F_B(xy)$ for every x and y

**Definition 2.3.** A single-valued co-neutrosophic graph $H = \langle A', B'\rangle$ is said to be a single-valued co-neutrosophic subgraph of the single-valued co-neutrosophic graph $G = \langle A, B\rangle$ if $A' \subseteq A$ and $B' \subseteq B$. In other words if $T_A{}'(x) = T_A(x)$;$I_A{}'(x) = I_A(x)$; $F'_A(x) = F_A(x)$ and $T_B(xy)' = T_B(xy)$;$I_B(xy)' = I_B(xy)$; $F_B(xy)' = F_B(xy)$ for every x and y

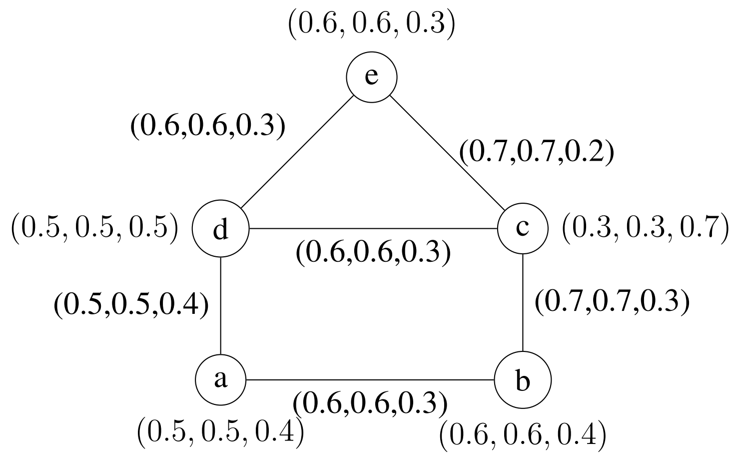



Figure 1: G : Single-valued co-neutrosophic graph

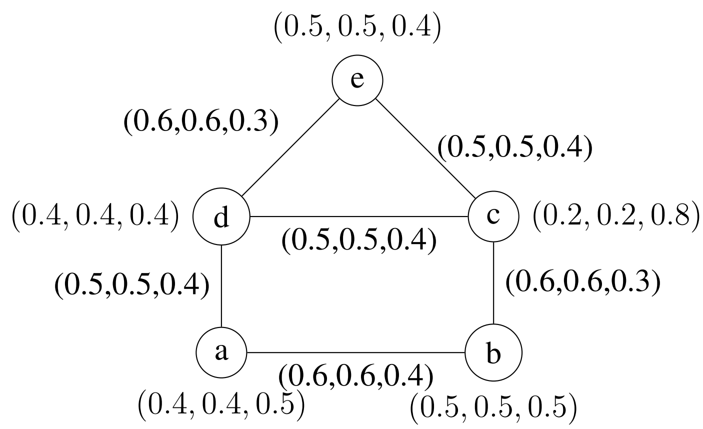



Figure 2: H : Single-valued co-neutrosophic partial subgraph ($H \subseteq G$)

**Definition 2.4.** A single-valued co-neutrosophic graph $G = \langle A, B \rangle$ is said to be strong single-valued co-neutrosophic graph if $T_B(xy) = max(T_A(x), T_A(y))$,$I_B(xy) = max(I_A(x), I_A(y))$ and $F_B(xy) = min(F_A(x), F_A(y))$, for all $(xy) \in E$.

**Definition 2.5.** A single-valued co-neutrosophic graph $G = \langle A, B \rangle$ is said to be complete single-valued co-neutrosophic graph if $T_B(xy) = max(T_A(x), T_A(y))$, $I_B(xy) = max(I_A(x), I_A(y))$ and $F_B(xy) = min(F_A(x), F_A(y))$, for every $x, y \in V$.

**Definition 2.6.** Let $G = \langle A, B \rangle$ be a single-valued co-neutrosophic graph. Then the degree of a vertex v is defined by $d(v) = (d_T(v), d_I(v), d_F(v))$, where $d_T(v) = \sum_{u \neq v} T_B(u, v)$, $d_I(v) = \sum_{u \neq v} I_B(u, v)$ and $d_F(v) = \sum_{u \neq v} F_B(u, v)$

**Definition 2.7.** The minimum degree of G is $\delta(G) = (\delta_T(G), \delta_I(G), \delta_F(G))$, where $\delta_T(G) = min\{d_T(v)|v \in V\}$, $\delta_I(G) = min\{d_I(v)|v \in V\}$ and $\delta_F(G) = max\{d_F(v)|v \in V\}$

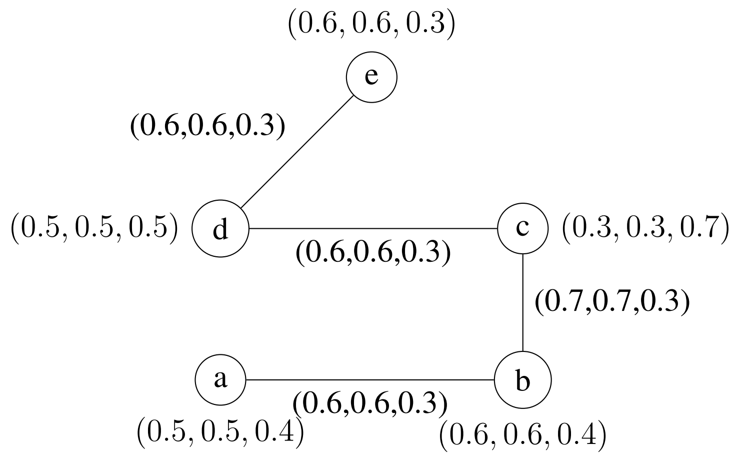



Figure 3: H : Single-valued co-neutrosophic subgraph

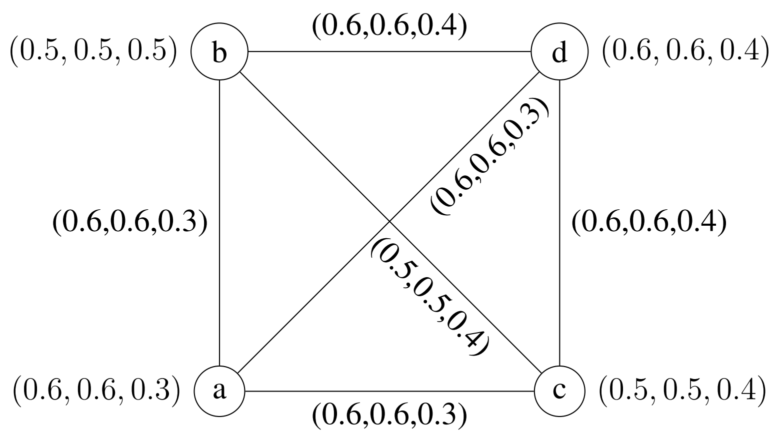


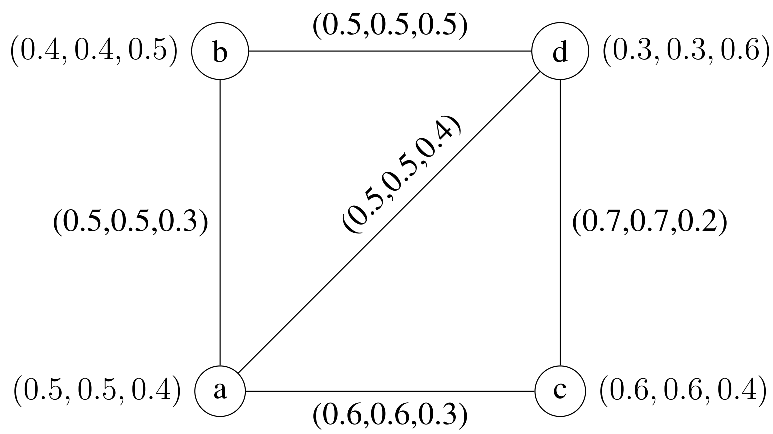

Figure 4: Complete single-valued co-neutrosophic graph

**Definition 2.8.** The maximum degree of G is $\Delta(G) = (\Delta_T(G), \Delta_I(G), \Delta_F(G))$, where $\Delta_T(G) = max\{d_T(v)|v \in V\}$, $\Delta_I(G) = max\{d_I(v)|v \in V\}$ and $\Delta_F(G) = min\{d_F(v)|v \in V\}$

**Example 2.1.** Let $G = \langle A, B \rangle$ be a single-valued co-neutrosophic graph. Draw as below

The degrees are $d_T(a) = 1.6, d_I(a) = 1.6, d_F(a) = 1.0, d_T(c) = 1.3, d_I(c) = 1.3, d_F(c) = 0.5$, $d_T(d) = 1.7, d_I(d) = 1.7, d_F(d) = 1.1, d_T(b) = 1.0, d_I(b) = 1.0, d_F(b) = 0.8$.
Minimum degree of a graph is $\delta_T(G) = 1.0, \delta_I(G) = 1.0, \delta_F(G) = 1.1$
Maximum degree of a graph is $\Delta_T(G) = 1.7, \Delta_I(G) = 1.7, \Delta_F(G) = 0.5$

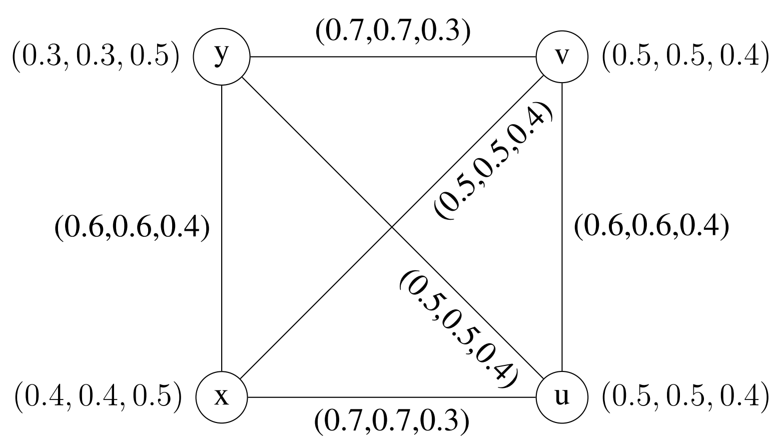
**Definition 2.9.** Let $G = \langle A, B \rangle$ be a single-valued co-neutrosophic graph. The total degree of a vertex $v \in V$ is defined as :
$Td(v) = Td_T(v)+Td_I(v)+Td_F(v)$, where $Td_T(v) = \sum_{(u,v)\in E} T_B(u, v)+T_A(v)$, $Td_I(v) = \sum_{(u,v)\in E} I_B(u, v)+ I_A(v)$ and $Td_F(v) = \sum_{(u,v)\in E} F_B(u, v) + F_A(v)$.




If each vertex of G has the same total degree $(r_1, r_2, r_3)$, then G is said to be an $(r_1, r_2, r_3)$ totally regular single-valued co-neutrosophic graph.

**Definition 2.10.** Let $G = \langle A, B\rangle$ be a single-valued co-neutrosophic graph. If each vertex has same degree $(r, s, t)$, then G is called $(r, s, t)$ regular single-valued co-neutrosophic graph. Thus $r = d_T(v), s = d_I(v), t = d_F(v)$;for $v \in V$.

**Example 2.2.** Let $G = \langle A, B\rangle$ be a single-valued co-neutrosophic graph. Draw as below




$d(y) = (1.8, 1.8, 1.1), d(v) = (1.8, 1.8, 1.1), d(u) = (1.8, 1.8, 1.1), d(x) = (1.8, 1.8, 1.1)$. So, G is a regular single-valued co-neutrosophic graph. But $G$ is not totally regular single-valued co-neutrosophic graph. Since $Td(y) = 5.8 \neq 6.1 = Td(v)$.

**Remark 2.1.** (a) For a single-valued co-neutrosophic graph, H = (A,B) to be both regular & totally regular, the number of vertices in each edge must be same.

(b) And also each vertex lies in exactly same number of edges.

**Proposition 2.1.** Let $G = \langle A, B\rangle$ be a single-valued co-neutrosophic graph. Then $T_A : V \to [0, 1], I_A : V \to [0, 1], F_A : V \to [0, 1]$ is a constant function iff following are equivalent.

(1) G is a regular single-valued co-neutrosophic graph,

(2) G is a totally regular single-valued co-neutrosophic graph.

*Proof.* suppose that $(T_A, I_A, F_A)$ is a constant function. Let $T_A(v_i) = k_1, I_A(v_i) = k_2, F_A(v_i) = k_3$ for all $v_i \in V$. Assume that $G$ is a $(r_1, r_2, r_3)$ regular single-valued co-neutrosophic graph. Then $d_T(v_i) = r_1, d_I(v_i) = r_2, d_F(v_i) = r_3$ for all $v_i \in V$. So $Td(v_i) = Td_T(v_i) + Td_I(v_i) + Td_F(v_i)$

$$\begin{aligned} Td_T(v_i) &= d_T(v_i) + T_A(v_i), \text{ for all } v_i \in V \\ &= r_1 + k_1 = c_1. \\ Td_I(v_i) &= d_I(v_i) + I_A(v_i), \text{ for all } v_i \in V \\ &= r_2 + k_2 = c_2. \\ Td_F(v_i) &= d_F(v_i) + F_A(v_i), \text{ for all } v_i \in V \\ &= r_3 + k_3 = c_3. \end{aligned}$$

Hence $G$ is totally regular single-valued co-neutrosophic graph. Thus $(1) \Rightarrow (2)$ is proved.

Now, suppose that G is a $(t_1, t_2, t_3)$ totally regular single-valued co-neutrosophic graph, then $Td_T(v_i) = t_1, Td_T(v_i) = t_2, Td_F(v_i) = t_3$ for all $v_i \in V$.

$$Td_T(v_i) = d_T(v_i) + T_A(v_i) = t_1,$$
$$\Rightarrow d_T(v_i) = t_1 - T_A(v_i) = t_1 - k_1, \text{ for all } v_i \in V.$$

Similarly, $Td_I(v_i) = d_I(v_i) + I_A(v_i) = t_2$,
$\Rightarrow d_I(v_i) = t_2 - I_A(v_i) = t_2 - k_2$, for all $v_i \in V$.

$Td_F(v_i) = d_F(v_i) + F_A(v_i) = t_3$,
$\Rightarrow d_F(v_i) = t_3 - F_A(v_i) = t_3 - k_3$, for all $v_i \in V$. So, G is a regular single-valued co-neutrosophic graph. Thus $(2) \Rightarrow (1)$ is proved. Hence $(1)$ and $(2)$ are equivalent. □

**Proposition 2.2.** If a single-valued co-neutrosophic graph is both regular and totally regular, then $(T_A, I_A, F_A)$ is constant function.

*Proof.* Let $G$ be a $(r, s, t)$ regular and $(k_1, k_2, k_3)$ totally regular single-valued co-neutrosophic graphs. So, $d_T(v_1) = r, d_I(v_1) = s, d_F(v_1) = t$ for $v_1 \in V$ and $Td_T(v_1) = k_1, Td_I(v_1) = k_2, Td_F(v_1) = k_3$ for all $v_1 \in V$. Now,

$$\begin{aligned} Td_T(v_1) &= k_1, \text{ for all } v_1 \in V, \\ d_T(v_1) + T_A &= k_1, \text{ for all } v_1 \in V, \\ r + T_A(v_1) &= k_1, \text{ for all } v_1 \in V, \\ T_A(v_1) &= k_1 - r, \text{ for all } v_1 \in V. \end{aligned}$$

Hence $T_A(v_1)$ is a constant function.

Similarly, $I_A(v_1) = k_2 - s$ for all $v_1 \in V$ and $F_A(v_1) = k_3 - t$ for all $v_1 \in V$. Hence $(T_A, I_A, F_A)$ is a constant. □

# 3 Conclusion

In this paper, we introduced the notion of a single-valued co-neutrosophic graphs and study some methods of construction of new single-valued co-neutrosophic graphs. We computed degree of a vertex, strong single-valued co-neutrosophic graphs and complete single-valued co-neutrosophic graphs. Properties of regular and totally regular single-valued co-neutrosophic graphs are discussed. In future, we are introduce and discuss the energy of Single-valued co-neutrosophic graphs.

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FOLDOC Dictionary of Computing (1, 2), Webster Dictionary (1, 2), Wordnik (1), Dictionary.com, The Free Dictionary (1),Wiktionary (2), YourDictionary (1, 2), OneLook Dictionary (1, 2), Dictionary / Thesaurus (1), Online Medical Dictionary (1, 2), and Encyclopedia (1, 2) have included these scientific neologisms.

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