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Neutrosophic Sets and Systems

An International Journal in Information Science and Engineering

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Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

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Neutrosophy is a generalization of Hegel’s dialectics (the last one is based on <A> and <antiA> only).

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Neutrosophic Set and Neutrosophic Logic are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (T), a degree of indeterminacy (I), a degree of falsity (F), where T, I, F are standard or non-standard subsets of [0, 1].

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

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Neutrosophic Integer Programming Problem

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Abstract. In this paper, we introduce the integer programming in neutrosophic environment, by considering coefficients of problem as a triangular neutrosophic numbers. The degrees of acceptance, indeterminacy and rejection of objectives are simultaneously considered.

Keywords: Neutrosophic; integer programming; single valued triangular neutrosophic number.

1 Introduction

In linear programming models, decision variables are allowed to be fractional. For example, it is reasonable to accept a solution giving an hourly production of automobiles at $64\frac{1}{2}$, if the model were based upon average hourly production. However, fractional solutions are not realistic in many situations and to deal with this matter, integer programming problems are introduced. We can define integer programming problem as a linear programming problem with integer restrictions on decision variables. When some, but not all decision variables are restricted to be integer, this problem called a mixed integer problem and when all decision variables are integers, it’s a pure integer program. Integer programming plays an important role in supporting managerial decisions. In integer programming problems the decision maker may not be able to specify the objective function and/or constraints functions precisely. In 1995, Smarandache [1-3] introduce neutrosophy which is the study of neutralities as an extension of dialectics. Neutrosophic is the derivative of neutrosophy and it includes neutrosophic set, neutrosophic probability, neutrosophic statistics and neutrosophic logic. Neutrosophic theory means neutrosophy applied in many fields of sciences, in order to solve problems related to indeterminacy. Although intuitionistic fuzzy sets can only handle incomplete information not indeterminate, the neutrosophic set can handle both incomplete and indeterminate information.[4] Neutrosophic sets characterized by three independent degrees as in Fig.1., namely truth-membership degree ($T$), indeterminacy-membership degree ($I$), and falsity-membership degree ($F$), where $T,I,F$ are standard or non-standard subsets of $\{0,1\}$. The decision makers in neutrosophic set want to increase the degree of truth-membership and decrease the degree of indeterminacy and falsity membership.

The structure of the paper is as follows: the next section is a preliminary discussion; the third section describes the formulation of integer programming problem using the proposed model; the fourth section presents some illustrative examples to put on view how the approach can be applied; the last section summarizes the conclusions and gives an outlook for future research.

2 Some Preliminaries

2.1 Neutrosophic Set [4]

Let $X$ be a space of points (objects) and $x \in X$. A neutrosophic set $A$ in $X$ is defined by a truth-membership function ($T_A(x)$), an indeterminacy-membership function ($I_A(x)$) and a falsity-membership function ($F_A(x)$) are real standard or real nonstandard subsets of $\left[0,1\right]$. There is $T_A(x):X \rightarrow \left[0,1\right]$, $I_A(x):X \rightarrow \left[0,1\right]$ and $F_A(x):X \rightarrow \left[0,1\right]$. There is no restriction on the sum of ($x$, $T_A(x)$ and $F_A(x)$, so $0 \leq sup(T_A(x)) \leq sup(I_A(x)) \leq sup(F_A(x)) \leq 3$.}

2.2 Single Valued Neutrosophic Sets (SVNS) [3-4]

Let $X$ be a universe of discourse. A single valued neutrosophic set $A$ over $X$ is an object having the form $A = \{(x, T_A(x), I_A(x), F_A(x)) \mid x \in X\}$, where $T_A(x):X \rightarrow [0,1]$, $I_A(x):X \rightarrow [0,1]$ and $F_A(x):X \rightarrow [0,1]$ with $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$ for all $x \in X$. The intervals $T(x)$,
The truth-membership degree, the indeterminacy-membership degree and the falsity membership degree of $x$ to $A$, respectively.

In the following, we write SVN numbers instead of single valued neutrosophic numbers. For convenience, a SVN number is denoted by $A= (a,b,c)$, where $a,b,c \in [0,1]$ and $a+b+c \leq 3$.

### 2.3 Complement [5]

The complement of a single valued neutrosophic set $A$ is denoted by $c(A)$ and is defined by

$$ T_c(A)(x) = F(A)(x), $$

$$ I_c(A)(x) = 1 - I(A)(x), $$

$$ F_c(A)(x) = T(A)(x) $$

for all $x$ in $X$.

### 2.4 Union [5]

The union of two single valued neutrosophic sets $A$ and $B$ is a single valued neutrosophic set $C$, written as $C = A \cup B$, whose truth-membership, indeterminacy membership and falsity-membership functions are given by

$$ T(C)(x) = \max \left( T(A)(x), T(B)(x) \right), $$

$$ I(C)(x) = \max \left( I(A)(x), I(B)(x) \right), $$

$$ F(C)(x) = \min \left( (A)(x), F(B)(x) \right) $$

for all $x$ in $X$.

### 2.5 Intersection [5]

The intersection of two single valued neutrosophic sets $A$ and $B$ is a single valued neutrosophic set $C$, written as $C = A \cap B$, whose truth-membership, indeterminacy membership and falsity-membership functions are given by

$$ T(C)(x) = \min \left( T(A)(x), T(B)(x) \right), $$

$$ I(C)(x) = \min \left( I(A)(x), I(B)(x) \right), $$

$$ F(C)(x) = \max \left( (A)(x), F(B)(x) \right) $$

for all $x$ in $X$.

### 3 Neutrosophic Integer Programming Problems

Integer programming problem with neutrosophic coefficients (NIPP) is defined as the following:

Maximize $Z= \sum_{j=1}^{n} \tilde{c}_j x_j$

Subject to

$$ \sum_{j=1}^{n} a_{ij}^{-n} x_j \leq b_i \quad i = 1, \ldots, m, $$

$$ x_j \geq 0, \quad j = 1, \ldots, n, $$

$$ x_j \text{ integer for } j \in \{0,1,\ldots,n\}. $$

Where $\tilde{c}_j$, $a_{ij}^{-n}$ are neutrosophic numbers.

The single valued neutrosophic number $(a_{ij}^{-n})$ is donated by $A=(a,b,c)$ where $a,b,c \in [0,1]$ and $a+b+c \leq 3$.

The truth- membership function of neutrosophic number $a_{ij}^{-n}$ is defined as:

$$ T a_{ij}^{-n}(x)= \begin{cases} \frac{x-a_1}{a_2-a_1} & a_1 \leq x \leq a_2 \\ \frac{a_3-x}{a_3-a_2} & a_2 \leq x \leq a_3 \\ 0 & \text{otherwise} \end{cases} $$

The indeterminacy- membership function of neutrosophic number $a_{ij}^{-n}$ is defined as:

$$ I a_{ij}^{-n}(x)= \begin{cases} \frac{x-b_2}{b_3-b_2} & b_2 \leq x \leq b_3 \\ 0 & \text{otherwise} \end{cases} $$

And its falsity- membership function of neutrosophic number $a_{ij}^{-n}$ is defined as:

$$ F a_{ij}^{-n}(x)= \begin{cases} \frac{x-C_1}{C_2-C_1} & C_1 \leq x \leq C_2 \\ \frac{b_2-x}{b_3-b_2} & C_2 \leq x \leq C_3 \\ 1 & \text{otherwise} \end{cases} $$

Then we find the maximum and minimum values of the objective function for truth-membership, indeterminacy and falsity membership as follows:

$$ f^\text{max} = \max\{f(x_i^+)\} \text{ and } f^\text{min} = \min\{f(x_i^-)\} \text{ where } 1 \leq i \leq k $$

$$ f^\text{min} = f^\text{min} \text{ and } f^\text{max} = f^\text{max} - R(f^\text{max} - f^\text{min}) $$
The neutrosophic optimization model can be changed into the following optimization model:

\[
\max (\alpha - \beta - \theta) \\
\text{Subject to} \\
\alpha \leq T(x) \\
\beta \geq F(x) \\
\theta \geq I(x) \\
\alpha \geq \beta \\
\alpha \geq \theta \\
0 \leq \alpha + \beta + \theta \leq 3 \\
x \geq 0 \text{, integer.}
\]

The previous model can be written as:

\[
\min (1 - \alpha)\beta \theta \\
\text{Subject to} \\
\alpha \leq T(x) \\
\beta \geq F(x) \\
\theta \geq I(x) \\
\alpha \geq \beta \\
\alpha \geq \theta \\
0 \leq \alpha + \beta + \theta \leq 3 \\
x \geq 0 \text{, integer.}
\]
5 The Algorithms for Solving Neutrosophic Integer Programming Problem (NIPP)

5.1 Neutrosophic Cutting Plane Algorithm

Step 1: Convert neutrosophic integer programming problem to its crisp model by using the following method:
By defining a method to compare any two single valued triangular neutrosophic numbers which is based on the score function and the accuracy function. Let $\bar{a} = ((a_1, b_1, c_1), w_\alpha, u_\alpha, y_\alpha)$ be a single valued triangular neutrosophic number, then

$$S(\bar{a}) = \frac{1}{16} [a + b + c] \times (2 + \mu_a - v_a - \lambda_a)$$

and

$$A(\bar{a}) = \frac{1}{16} [a + b + c] \times (2 + \mu_a - v_a + \lambda_a)$$

is called the score and accuracy degrees of $\bar{a}$, respectively. The neutrosophic integer programming NIP can be represented by crisp programming model using truth membership, indeterminacy membership, and falsity membership functions and the score and accuracy degrees of $\bar{a}$, at equations (15) or (16).

Step 2: Create the decision set which include the highest degree of truth-membership and the least degree of falsity and indeterminacy memberships.

Step 3: Solve the model as a linear programming problem and ignore integrality.

Step 4: If the optimal solution is integer, then it’s right. Otherwise, go to the next step.

Step 5: Generate a constraint which is satisfied by all integer solutions and add this constraint to the problem.

Step 6: Go to step 1.

5.2 Neutrosophic Branch and Bound Algorithm

Step 1: Convert neutrosophic integer programming problem to its crisp model by using the following method:
By defining a method to compare any two single valued triangular neutrosophic numbers which is based on the score function and the accuracy function. Let $\bar{a} = ((a_1, b_1, c_1), w_\alpha, u_\alpha, y_\alpha)$ be a single valued triangular neutrosophic number, then

$$S(\bar{a}) = \frac{1}{16} [a + b + c] \times (2 + \mu_a - v_a - \lambda_a)$$

and

$$A(\bar{a}) = \frac{1}{16} [a + b + c] \times (2 + \mu_a - v_a + \lambda_a)$$

is called the score and accuracy degrees of $\bar{a}$, respectively. The neutrosophic integer programming NIP can be represented by crisp programming model using truth membership, indeterminacy membership, and falsity membership functions and the score and accuracy degrees of $\bar{a}$, at equations (15) or (16).

Step 2: Create the decision set which include the highest degree of truth-membership and the least degree of falsity and indeterminacy memberships.

Step 3: At the first node let the solution of linear programming model with integer restriction as an upper bound and the rounded-down integer solution as a lower bound.

Step 4: For branching process, we select the variable with the largest fractional part. Two constraints are obtained after the branching process, one for $\leq$ and the other is $\geq$ constraint.

Step 5: Create two nodes for the two new constraints.

Step 6: Solve the model again, after adding new constraints at each node.

Step 7: The optimal integer solution has been reached, if the feasible integer solution has the largest upper bound value of any ending node. Otherwise return to step 4.

The previous algorithm is for a maximization model. For a minimization model, the solution of linear programming problem with integer restrictions are rounded up and upper and lower bounds are reversed.

6 Numerical Examples

To measure the efficiency of our proposed model we solved many numerical examples.

6.1 Illustrative Example #1

$max \quad 5x_1 + 3x_2$

subject to \quad $4x_1 + 3x_2 \leq 12$

$x_1, x_2 \geq 0 \quad and \quad integer$

where

$5 = ((4, 5.6, 0.8, 0.6, 0.4))$

$3 = ((2.5, 3.3.5, 0.75, 0.5, 0.3))$

$4 = ((3.5, 4.4.1), 1, 0.5, 0.0))$

$3 = ((2.5, 3.3.5), 0.75, 0.5, 0.25)$

$1 = ((0.1, 2), 1, 0.5, 0) \quad and \quad integer$

$3 = ((2.8, 3.3.2), 0.75, 0.5, 0.25)$

$12 = ((11, 12, 13), 1, 0.5, 0)$

$6 = ((5.5, 6.7.5), 0.8, 0.6, 0.4)$

Then the neutrosophic model converted to the crisp model by using Eq.15 , Eq.16.as follows :
max \[5.6875x_1 + 3.5968x_2\]
\[4.3125x_1 + 3.625x_2 \leq 14.375\]
subject to \[0.2815x_1 + 3.925x_2 \leq 7.6375\]
\[x_1, x_2 \geq 0 \text{ and integer}\]

The optimal solution of the problem is \(x^* = (3,0)\) with optimal objective value 17.06250.

6.2 Illustrative Example #2
max \[25x_1 + 48x_2\]
\[15x_1 + 30x_2 \leq 45000\]
subject to \[24x_1 + 6x_2 \leq 24000\]
\[21x_1 + 14x_2 \leq 28000\]
\[x_1, x_2 \geq 0 \text{ and integer}\]

where \(25 = ((19,25,33),0.8,0.5,0);\)
\(48 = ((44,48,54),0.9,0.5,0)\)

Then the neutrosophic model converted to the crisp model as:
max \[27.8875x_1 + 55.3x_2\]
\[15x_1 + 30x_2 \leq 45000\]
subject to \[24x_1 + 6x_2 \leq 24000\]
\[21x_1 + 14x_2 \leq 28000\]
\[x_1, x_2 \geq 0 \text{ and integer}\]

The optimal solution of the problem is \(x^* = (500,1250)\) with optimal objective value 83068.75.

7 Conclusions and Future Work
In this paper, we proposed an integer programming model based on neutrosophic environment, simultaneously considering the degrees of acceptance, indeterminacy and rejection of objectives, by proposed model for solving neutrosophic integer programming problems (NIPP). In the model, we maximize the degrees of acceptance and minimize indeterminacy and rejection of objectives. NIPP was transformed into a crisp programming model using truth membership, indeterminacy membership, falsity membership and score functions. We also give numerical examples to show the efficiency of the proposed method. Future research directs to studying the duality theory of integer programming problems based on Neutrosophic.

References

Received: January 6, 2017. Accepted: January 30, 2017.
Multi-Objective Structural Design Optimization using Neutrosophic Goal Programming Technique

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Abstract: This paper develops a multi-objective Neutrosophic Goal Optimization (NSGO) technique for optimizing the design of three bar truss structure with multiple objectives subject to a specified set of constraints. In this optimum design formulation, the objective functions are weight and deflection; the design variables are the cross-sections of the bar; the constraints are the stress in member.

The classical three bar truss structure is presented here in to demonstrate the efficiency of the neutrosophic goal programming approach. The model is numerically illustrated by generalized NSGO technique with different aggregation method. The result shows that the Neutrosophic Goal Optimization technique is very efficient in finding the best optimal solutions.

Keywords: Neutrosophic Set, Single Valued Neutrosophic Set, Generalized Neutrosophic Goal Programming, Arithmetic Aggregation, Geometric Aggregation, Structural Optimization.

1 Introduction

The research area of optimal structural design has been receiving increasing attention from both academia and industry over the past four decades in order to improve structural performance and to reduce design costs. In the real world, uncertainty or vagueness is prevalent in the Engineering Computations. In the context of structural design the uncertainty is connected with lack of accurate data of design factors. This tendency has been changing due to the increase in the use of fuzzy mathematical algorithm for dealing with such kind of problems.

Fuzzy set (FS) theory has long been introduced to deal with inexact and imprecise data by Zadeh [1], Later on the fuzzy set theory was used by Bellman and Zadeh [2] to the decision making problem. A few work has been done as an application of fuzzy set theory on structural design. Several researchers like Wang et al. [3] first applied a-cut method to structural designs where various design levels α were used to solve the non-linear problems. In this regard, a generalized fuzzy number has been used Dey et al. [4] in context of a non-linear structural design optimization. Dey et al. [5] used basic t-norm based fuzzy optimization technique for optimization of structure and Dey et al. [6] developed parameterized t-norm based fuzzy optimization method for optimum structural design.

In such extension, Intuitionistic fuzzy set which is one of the generalizations of fuzzy set theory and was characterized by a membership, a non- membership and a hesitancy function was first introduced by Atanassov [21] (IFS). In fuzzy set theory the degree of acceptance is only considered but in case of IFS it is characterized by degree of membership and non-membership in such a way that their sum is less or equal to one. Dey et al. [7] solved two bar truss non-linear problem by using intuitionistic fuzzy optimization problem.Again Dey et al. [8] used intuitionistic fuzzy optimization technique to solve multi objective structural design. R-x Liang et al. [9] applied interdependent inputs of single valued trapezoidal neutrosophic information on Multi-criteria group decision making problem. P Ji et al. [10], S Yu et al. [11] did so many research study on application based neutrosophic sets and intuitionistic linguistic number. Z-p Tian et al. [12] Simplified neutrosophic linguistic multi-criteria group decision-making approach to green product development. Again J-j Peng et al. [13] introduced multi-valued neutrosophic qualitative flexible approach based on likelihood for multi-criteria decision-making problems. Also, H Zhang et. al. [22] investigates a case study on a novel decision support model for satisfactory restaurants utilizing social information. P Ji et al. [14] developed a projection-based TODIM method under multi-
valued neutrosophic environments and its application in personnel selection. Intuitionistic fuzzy sets consider both truth and falsity membership and can only handle incomplete information but not the information which is connected with indeterminacy or inconsistency.

In neutrosophic sets indeterminacy or inconsistency is quantified explicitly by indeterminacy membership function. Neutrosophic Set (NS), introduced by Smarandache [15] was characterized by truth, falsity and indeterminacy membership so that in case of single valued NS set their sum is less or equal to three. In early [17] Charnes and Cooper first introduced Goal programming problem for a linear model. Usually conflicting goals are presented in a multi-objective goal programming problem. Dey et al. [16] used intuitionistic goal programming on nonlinear structural model. This is the first time NSGO technique is in application to multi-objective structural design. Usually objective goals of existing structural model are considered to be deterministic and a fixed quantity. In a situation, the decision maker can be doubtful with regard to accomplishment of the goal. The DM may include the idea of truth, indeterminacy and falsity bound on objectives goal. The goal may have a target value with degree of truth, indeterminacy as well as degree of falsity. Precisely, we can say a human being that express degree of truth membership of a given element in a fuzzy set, truth and falsity in a intuitionistic fuzzy set, very often does not express the corresponding degree of falsity membership as complement to 3. This fact seems to take the objective goal as a neutrosophic set. The present study investigates computational algorithm for solving multi-objective structural problem by single valued generalized NSGO technique. The results are compared numerically for different aggregation method of NSGO technique. From our numerical result, it has been seen the best result obtained for geometric aggregation method for NSGO technique in the perspective of structural optimization technique.

2 Multi-objective structural model

In the design problem of the structure i.e. lightest weight of the structure and minimum deflection of the loaded joint that satisfies all stress constraints in members of the structure. In truss structure system, the basic parameters (including allowable stress, etc.) are known and the optimization’s target is that identify the optimal bar truss cross-section area so that the structure is of the smallest total weight with minimum nodes displacement in a given load conditions.

The multi-objective structural model can be expressed as

Minimize $WT(A)$

\[
\begin{align*}
\text{subject to } & \sigma(A) \leq [\sigma] \\
& A^{\text{min}} \leq A \leq A^{\text{max}}
\end{align*}
\]

where $A = [A_1, A_2, \ldots, A_n]$ are the design variables for the cross section, $n$ is the number of design variables for the cross section bar, $WT(A) = \sum_{i=1}^{n} \rho_i A_i L_i$ is the total weight of the structure, $\delta(A)$ is the deflection of the loaded joint, where $L_i, A_i$ and $\rho_i$ are the bar length, cross section area and density of the $i^{th}$ group bars respectively. $\sigma(A)$ is the stress constraint and $[\sigma]$ is allowable stress of the group bars under various conditions. $A^{\text{min}}$ and $A^{\text{max}}$ are the lower and upper bounds of cross section area $A$ respectively.

3 Mathematical preliminaries

3.1 Fuzzy set

Let $X$ be a fixed set. A fuzzy set $A$ of $X$ is an object having the form $\hat{A} = \{(x, T_A(x)) : x \in X\}$ where the function $T_A : X \to [0,1]$ defined the truth membership of the element $x \in X$ to the set $A$.

3.2 Intuitionistic fuzzy set

Let a set $X$ be fixed. An intuitionistic fuzzy set or IFS $\hat{A}$ in $X$ is an object of the form

$\hat{A}' = \{(x, T_A(x), I_A(x), F_A(x)) : x \in X\}$

where $T_A : X \to [0,1]$ and $I_A : X \to [0,1]$, $F_A : X \to [0,1]$ define the truth membership and falsity membership respectively, for every element of $x \in X$ $0 \leq T_A + F_A \leq 1$.

3.3 Neutrosophic set

Let a set $X$ be a space of points (objects) and $x \in X$. A neutrosophic set $\hat{A}^*$ in $X$ is defined by a truth membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$ and a falsity membership function $F_A(x)$ and denoted by

$\hat{A}^* = \{(x, T_A(x), I_A(x), F_A(x)) : x \in X\}$.

$T_A(x)$ and $I_A(x)$ and $F_A(x)$ are real standard or non-standard subsets of $[0,1]$.

$T_A(x) : X \to [0,1]$, $I_A(x) : X \to [0,1]$ and $F_A(x) : X \to [0,1]$.

There is no restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$ so $0 \leq \sup T_A(x) + I_A(x) + \sup F_A(x) \leq 3$.

3.4 Single valued neutrosophic set

Let a set $X$ be the universe of discourse. A single valued neutrosophic set $\hat{A}^*$ over $X$ is an object having the
where \( b_i \) is a single valued neutrosophic set and is defined by
\[
0 \leq T_i(x) + I_i(x) + F_i(x) \leq 3 \text{ for all } x \in X.
\]

3.5 Complement of neutrosophic Set

Complement of a single valued neutrosophic set \( A \) is denoted by \( c(A) \) and is defined by \( T_{c(A)}(x) = F_A(x) \),
\[
I_{c(A)}(x) = 1 - F_A(x), \quad F_{c(A)}(x) = T_A(x)
\]

3.6 Union of neutrosophic sets

The union of two single valued neutrosophic sets \( A \) and \( B \) is a single valued neutrosophic set \( C \), written as \( C = A \cup B \), whose truth membership, indeterminacy membership and falsity-membership functions are given by
\[
T_{c(A)}(x) = \max\{T_A(x), T_B(x)\},
\]
\[
I_{c(A)}(x) = \min\{I_A(x), I_B(x)\},
\]
\[
F_{c(A)}(x) = \min\{F_A(x), F_B(x)\} \text{ for all } x \in X.
\]

3.7 Intersection of neutrosophic sets

The intersection of two single valued neutrosophic sets \( A \) and \( B \) is a single valued neutrosophic set \( C \), written as \( C = A \cap B \), whose truth membership, indeterminacy membership and falsity-membership functions are given by
\[
T_{c(A)}(x) = \min\{T_A(x), T_B(x)\},
\]
\[
I_{c(A)}(x) = \max\{I_A(x), I_B(x)\},
\]
\[
F_{c(A)}(x) = \max\{F_A(x), F_B(x)\} \text{ for all } x \in X.
\]

4 Mathematical analysis

4.1 Neutrosophic Goal Programming

Neutrosophic Goal Programming problem is an extension of intuitionistic fuzzy as well as fuzzy goal programming problem in which the degree of indeterminacy of objective(s) and constraints are considered with degree of truth and falsity membership degree.

Goal programming can be written as
Find
\[
x = (x_1, x_2, ..., x_n)^T
\]
(1) to achieve:
\[
z_i = t_i, \quad i = 1, 2, ..., k
\]
Subject to \( x \in X \) where \( t_i \) are scalars and represent the target achievement levels of the objective functions that the decision maker wishes to attain provided, \( X \) is feasible set of constraints.

The nonlinear goal programming problem can be written as
\[
\text{Find } x = (x_1, x_2, ..., x_n)^T
\]
(2) So as to
\[
\text{Minimize } z_i \text{ with target value } t_i \text{ acceptance tolerance } a_i \text{ indeterminacy tolerance } d_i \text{ rejection tolerance } c_i
\]
\[x \in X, \quad g_i(x) \leq b_j, \quad j = 1, 2, ..., m\]
\[x_i \geq 0, \quad i = 1, 2, ..., n\]

This neutrosophic goal programming can be transformed into crisp programming and can be transformed into crisp programming problem model by maximizing the degree of truth and indeterminacy and minimizing the degree of falsity of neutrosophic objectives and constraints. In the above problem (2), multiple objectives are considered as neutrosophic with some relaxed targets. This representation demonstrates that decision maker (DM) is not sure about minimum value of \( z_i, i = 1, 2, ..., k \). DM has some illusive ideas of some optimum values of \( z_i, i = 1, 2, ..., k \). Hence it is quite natural to have desirable values violating the set target. Then question arises that how much bigger the optimum values may be. DM has also specified it with the use of tolerances. The tolerances are set in such a manner that the sum of truth, indeterminacy and falsity membership of objectives \( z_i, i = 1, 2, ..., k \) will lie between 0 and 3. Let us consider the following theorem on membership function:

**Theorem 1.**

For a generalized neutrosophic goal programming problem (2)

The sum of truth, indeterminacy and falsity membership function will lie between 0 and \( w_1 + w_2 + w_3 \).

**Proof:**

Let the truth, indeterminacy and falsity membership functions be defined as membership functions

\[
T_{c_i}(z_i) = \begin{cases} w_1 & \text{if } z_i \leq t_i \\ w_1 \left( t_i + a_i - z_i \right) / a_i & \text{if } t_i \leq z_i \leq t_i + a_i \\ 0 & \text{if } z_i \geq t_i + a_i \end{cases}
\]

\[
I_{c_i}(z_i) = \begin{cases} w_2 & \text{if } z_i \leq t_i \\ w_2 \left( z_i - t_i / d_i \right) & \text{if } t_i \leq z_i \leq t_i + d_i \\ 0 & \text{if } z_i \geq t_i + d_i \end{cases}
\]

\[
F_{c_i}(z_i) = \begin{cases} w_3 & \text{if } z_i \leq t_i \\ w_3 \left( t_i + a_i - z_i / a_i - d_i \right) & \text{if } t_i + a_i \leq z_i \leq t_i + a_i \\ 0 & \text{if } z_i \geq t_i + a_i \end{cases}
\]
From Fig. (1) and definition of generalized single valued neutrosophic set, it is clear that:

\[ 0 \leq T_\alpha (z_i) \leq w_1 , 0 \leq I_\alpha (z_i) \leq w_2 \text{ and } 0 \leq F_\alpha (z_i) \leq w_1 \]
when \( (z_i) \leq t_i \)

\[ T_\alpha (z_i) = w_1 \text{ and } I_\alpha (z_i) = 0 \text{ and } F_\alpha (z_i) = 0 \]

Therefore \( T_\alpha (z_i) + I_\alpha (z_i) + F_\alpha (z_i) = w_1 \leq w_1 + w_2 + w_3 \)
and \( w_1 \geq 0 \) implies that \( T_\alpha (z_i) + I_\alpha (z_i) + F_\alpha (z_i) \geq 0 \)
when \( z_i \in (t_i, t_i + d_i) \) from fig (A) we see that \( T_\alpha (z_i) \) and \( F_\alpha (z_i) \) intersects each other and the point whose coordinate is \( (t_i + d_i, d_i, c_i) \),

where \( d_i = \frac{w_1}{w_1 + w_2} \).

Now in the interval \( z_i \in (t_i, t_i + d_i) \) we see that

\[ T_\alpha (z_i) + I_\alpha (z_i) + F_\alpha (z_i) = w_2 \left( \frac{z_i - t_i}{d_i} \right) \leq w_2 \leq w_1 + w_2 + w_3 \]

Again, in the interval \( z_i \in (t_i + d_i, t_i + a_i) \) we see that

\[ T_\alpha (z_i) + I_\alpha (z_i) + F_\alpha (z_i) = w_2 \left( \frac{t_i + a_i - z_i}{a_i - d_i} \right) \leq w_2 \leq w_1 + w_2 + w_3 \].

Also, for \( t_i \leq z_i \leq t_i + a_i \)
when \( z_i \geq t_i, T_\alpha (z_i) + I_\alpha (z_i) + F_\alpha (z_i) > w_2 \geq 0 \) and
\( T_\alpha (z_i) + I_\alpha (z_i) + F_\alpha (z_i) > w_1 \geq 0 \) and when

\[ z_i \leq t_i + a_i, T_\alpha (z_i) + I_\alpha (z_i) + F_\alpha (z_i) \leq w_1 \frac{a_i}{c_i} < w_i \leq w_1 + w_2 + w_3 \]
(as \( \frac{a_i}{c_i} \leq 1 \)).

In the interval \( z_i \in (t_i + a_i, t_i + c_i) \)
when \( z_i > t_i + a_i, T_\alpha (z_i) + I_\alpha (z_i) + F_\alpha (z_i) > w_1 \frac{a_i}{c_i} > w_2 \geq 0 \)
(as \( \frac{a_i}{c_i} \leq 1 \))
and when \( z_i \leq t_i + c_i, T_\alpha (z_i) + I_\alpha (z_i) + F_\alpha (z_i) \leq w_1 \leq w_1 + w_2 + w_3 \)
for \( z_i > t_i + c_i, T_\alpha (z_i) + I_\alpha (z_i) + F_\alpha (z_i) = w_3 \leq w_1 + w_2 + w_3 \)
and as \( w_2 \geq 0, T_\alpha (z_i) + I_\alpha (z_i) + F_\alpha (z_i) \geq 0 \).

Therefore, combining all the cases we get
\[ 0 \leq T_\alpha (z_i) + I_\alpha (z_i) + F_\alpha (z_i) \leq w_1 + w_2 + w_3 \]
Hence the proof.

4.2. Solution Procedure of Neutrosophic Goal Programming Technique

In fuzzy goal programming, Zimmermann [18] has given a concept of considering all membership functions greater than a single value \( \alpha \) which is to be maximized. Previously many researcher like Bharti and Singh [20], Parvathi and Malathi [19] have followed him in intuitionistic fuzzy optimization. Along with the variable \( \alpha \) and \( \beta, \gamma \) is optimized in neutrosophic goal programming problem.

With the help of generalized truth, indeterminacy, falsity membership function the generalized neutrosophic goal programming problem (2) can be formulated as:

\[ \text{Maximize } T_\alpha (z_i), \ i = 1, 2, \ldots, k \]
(3)

\[ \text{Maximize } I_\alpha (z_i), \ i = 1, 2, \ldots, k \]

\[ \text{Minimize } F_\alpha (z_i), \ i = 1, 2, \ldots, k \]

Subject to
\[ 0 \leq T_\alpha (z_i) + I_\alpha (z_i) + F_\alpha (z_i) \leq w_i + w_2 + w_3, \ i = 1, 2, \ldots, k \]
\[ T_\alpha (z_i) \geq 0, I_\alpha (z_i) \geq 0, F_\alpha (z_i) \geq 0, \ i = 1, 2, \ldots, k \]

\[ T_\alpha (z_i) \geq I_\alpha (z_i), \ i = 1, 2, \ldots, k \]
\[ T_\alpha (z_i) \geq F_\alpha (z_i), \ i = 1, 2, \ldots, k \]
\[ 0 \leq w_1 + w_2 + w_3 \leq 3 \]
\[ w_1, w_2, w_3 \in [0, 1], \ g_j (x) \leq b_j, \ j = 1, 2, \ldots, m \]
\[ x_i \geq 0, \ i = 1, 2, \ldots, n \]
Now the decision set $\hat{D}^*$, a conjunction of Neutrosophic objectives and constraints is defined:

$$\hat{D}^* = \left( \bigcap_{i=1}^{n} z_{ci}^* \right) \cap \left( \bigcap_{j=1}^{m} g_{cj}^* \right) = \{(x, T_{x}^*(x), I_{x}^*(x), F_{x}^*(x)) \}
$$

Here $\alpha = T_{x}^*(x) = \min_{x \in X} \left\{ T_{x}^*(x), T_{x}^*(x), \ldots, T_{x}^*(x) \right\}$ for all $x \in X$

$$\gamma = I_{x}^*(x) = \min_{x \in X} \left\{ I_{x}^*(x), I_{x}^*(x), \ldots, I_{x}^*(x) \right\}
$$

$$\beta = F_{x}^*(x) = \min_{x \in X} \left\{ F_{x}^*(x), F_{x}^*(x), \ldots, F_{x}^*(x) \right\}
$$

where $T_{x}^*(x), I_{x}^*(x), F_{x}^*(x)$ are truth-membership function, indeterminacy membership function, falsity membership function of neutrosophic decision set respectively. Now using the neutrosophic optimization, problem (2) is transformed to the non-linear programming as

Maximize $\alpha$, Maximize $\gamma$, Minimize $\beta$

Subjected to the same constraint as (4).

Now this non-linear programming problem (4 or 5 or 6) can be easily solved by an appropriate mathematical programming to give solution of multi-objective non-linear programming problem (1) by generalized neutrosophic goal optimization approach.

5. Solution of Multi-Objective Structural Optimization Problem (MOSOP) by Generalized Neutrosophic Goal Programming Technique

The multi-objective neutrosophic fuzzy structural model can be expressed as:

$$\text{Minimize } z \left( \begin{array}{c} \alpha_d \left( x, T \right) + \alpha_f \left( x, I \right) + \alpha_i \left( x, F \right) \\ \alpha_d \left( x, T \right) + \alpha_f \left( x, I \right) + \alpha_i \left( x, F \right) \\ \alpha_d \left( x, T \right) + \alpha_f \left( x, I \right) + \alpha_i \left( x, F \right) \\ \end{array} \right)$$

Subjected to the same constraint as (4).

With the help of generalized truth, indeterminacy, falsity membership function the generalized neutrosophic goal programming, based on geometric aggregation operator can be formulated as:

$$\text{Minimize } \sqrt{(1-\alpha) \beta (1-\gamma)}$$

Minimize $WT(A)$ with target value $WT_0$, truth tolerance $\alpha_{WT}$, indeterminacy tolerance $\delta_{WT}$ and rejection tolerance $\beta_{WT}$

Minimize $\delta(A)$ with target value $\delta_0$, truth tolerance $\alpha_{\delta}$, indeterminacy tolerance $\delta_{\delta}$ and rejection tolerance $\beta_{\delta}$

Subject to $\sigma(A) \leq \sigma$

$$A_{\min} \leq A \leq A_{\max}$$

where $A = \left[ A_1, A_2, \ldots, A_n \right]^T$ are the design variables for the cross section, $n$ is the group number of design variables for the cross section bar.

To solve this problem we first calculate truth, indeterminacy and falsity membership function of objective as follows:

$$T_{\min}^n \left( WT(A) \right) = \left\{ \begin{array}{ll}
\frac{w_1}{w_1 + w_2} & \text{if } WT(A) \leq WT_0 \\
0 & \text{if } WT(A) \geq WT_0 + a_{\min}
\end{array} \right.
$$

$$I_{\min}^n \left( WT(A) \right) = \left\{ \begin{array}{ll}
0 & \text{if } WT(A) \leq WT_0 \\
\frac{w_1 \left( WT(A) - WT_0 \right)}{a_{\min}} & \text{if } WT_0 \leq WT(A) \leq WT_0 + a_{\min}
\end{array} \right.
$$

$$F_{\min}^n \left( WT(A) \right) = \left\{ \begin{array}{ll}
0 & \text{if } WT(A) \leq WT_0 \\
\frac{w_1 \left( WT_0 + a_{\min} \right) - WT(A)}{a_{\min}} & \text{if } WT_0 + a_{\min} \leq WT(A) \leq WT_0 + a_{\min}
\end{array} \right.
$$

where $a_{\min} = w_1 \frac{a_{\min}}{w_1 + w_2}$.
and
\[
F_{(A)}^n(\delta(A)) = \left\{ \begin{array}{ll}
0 & \text{if } \delta(A) \leq \delta_0 \\
\frac{c_{\delta}}{w_3} & \text{if } \delta(A) > \delta_0
\end{array} \right.
\]

According to generalized neutrosophic goal optimization technique using truth, indeterminacy and falsity membership function, MOSOP (7) can be formulated as:

**Model I**

Maximize \( \alpha \), Maximize \( \gamma \), Minimize \( \beta \) \hspace{1cm} (8)

\[
WT(A) \leq WT_0 + a_{WT} \left[ 1 - \frac{\alpha}{w_1} \right],
\]

\[
WT(A) \geq WT_0 + \frac{d_{WT}}{w_2} \gamma,
\]

\[
WT(A) \leq WT_0 + a_{WT} - \frac{\gamma}{w_1} (a_{WT} - d_{WT}),
\]

\[
WT(A) \leq WT_0 + \frac{c_{\delta}}{w_3} \beta,
\]

\[
WT(A) \leq WT_0,
\]

\[
\delta(A) \leq \delta_0 + a_{\delta} \left[ 1 - \frac{\alpha}{w_1} \right],
\]

\[
\delta(A) \geq \delta_0 + \frac{d_{\delta}}{w_2} \gamma,
\]

\[
\delta(A) \leq \delta_0 + a_{\delta} - \frac{\gamma}{w_1} (a_{\delta} - d_{\delta}).
\]

\[
\delta(A) \leq \delta_0 + \frac{c_{\delta}}{w_3} \beta, \quad \delta(A) \leq \delta_0,
\]

\[
0 \leq \alpha + \beta + \gamma \leq w_1 + w_2 + w_3;
\]

\[
\alpha \in [0, w_1], \gamma \in [0, w_2], \beta \in [0, w_3];
\]

\[
w_1 \in [0,1], w_2 \in [0,1], w_3 \in [0,1];
\]

\[
0 \leq w_1 + w_2 + w_3 \leq 3;
\]

\[
g_j(x) \leq b_j, j = 1, 2, ..., m
\]

With the help of generalized truth, indeterminacy, falsity membership function the generalized neutrosophic goal programming based on arithmetic aggregation operator can be formulated as:

**Model II**

\[
\text{Minimize } \left[ \frac{(1-\alpha) + \beta + (1-\gamma)}{3} \right]
\]

Subjected to the same constraint as (8)

With the help of generalized truth, indeterminacy, falsity membership function the generalized neutrosophic goal programming based on geometric aggregation operator can be formulated as:

**Model -III**

\[
\text{Minimize } \sqrt{(1-\alpha) \beta (1-\gamma)}
\]

Subjected to the same constraint as (8)

Now these non-linear programming Model-I, II, III can be easily solved through an appropriate mathematical programming to give solution of multi-objective non-linear programming problem (7) by generalized neutrosophic goal optimization approach.

### 6 Numerical illustration

A well-known three bar planer truss is considered in Fig.2 to minimize weight of the structure \( WT(A_1, A_2) \) and minimize the deflection \( \delta(A_1, A_2) \) at a loading point of a statistically loaded three bar planer truss subject to stress constraints on each of the truss members.

![Fig. 2 Design of three bar planer truss](image-url)
The multi-objective optimization problem can be stated as follows:

\[
\text{Minimize} \quad WT(A_i, A_j) = \rho L \left( 2 \sqrt{2} A_i + A_j \right)
\]

\[
\text{Minimize} \quad \delta(A_i, A_j) = \frac{PL}{E(A_i + \sqrt{2} A_j)}
\]

Subject to

\[
\sigma_i(A_i, A_j) = \frac{P \left( \sqrt{2} A_i + A_j \right)}{\left( \sqrt{2} A_i^2 + 2 A_i A_j \right)} \leq \left[ \sigma_i^T \right];
\]

\[
\sigma_2(A_i, A_j) = \frac{PA_i}{A_i + \sqrt{2} A_j} \leq \left[ \sigma_2^T \right];
\]

\[
\sigma_3(A_i, A_j) = \frac{P A_i}{\sqrt{2} A_i^2 + 2 A_i A_j} \leq \left[ \sigma_3^T \right];
\]

\[
A_i^m \leq A_i \leq A_i^m \quad i = 1, 2
\]

where \( P \) = applied load ; \( \rho \) = material density ; \( L \) = length ; \( E \) = Young’s modulus ; \( A_i \) = Cross section of bar-1 and bar-3; \( A_j \) = Cross section of bar-2; \( \delta \) is deflection of loaded joint. \( \left[ \sigma_i^T \right] \) and \( \left[ \sigma_2^T \right] \) are maximum allowable tensile stress for bar 1 and bar 2 respectively, \( \sigma_3^T \) is maximum allowable compressive stress for bar 3. The input data is given in Table 1.

This multi-objective structural model can be expressed as neutrosophic fuzzy model as

\[
\text{Minimize} \quad WT(A_i, A_j) = \rho L \left( 2 \sqrt{2} A_i + A_j \right) \quad \text{with target value} \quad 4 \times 10^7 \text{KN}
\]

\[
\text{indeterminacy tolerance} \quad 2 \times 10^7 \text{KN} \quad \text{with target value} \quad 0.5 \times 10^{-7} \text{KN} \quad \text{and rejection tolerance} \quad 4.5 \times 10^7 \text{KN}
\]

\[
\text{Minimize} \quad \delta(A_i, A_j) = \frac{PL}{E(A_i + \sqrt{2} A_j)} \quad \text{with target value} \quad 2.5 \times 10^{-7} \text{m}
\]

\[
\text{truth tolerance} \quad 2.5 \times 10^{-7} \text{m} \quad \text{indeterminacy tolerance} \quad 2.5 \times 10^{-7} \text{m} \quad \text{with target value} \quad 0.4 \times 10^{-7} \text{m} \quad \text{and rejection tolerance} \quad 4.5 \times 10^{-7} \text{m}
\]

Subject to

\[
\sigma_i(A_i, A_j) = \frac{P \left( \sqrt{2} A_i + A_j \right)}{\left( \sqrt{2} A_i^2 + 2 A_i A_j \right)} \leq \left[ \sigma_i^T \right];
\]

\[
\sigma_2(A_i, A_j) = \frac{PA_i}{A_i + \sqrt{2} A_j} \leq \left[ \sigma_2^T \right];
\]

\[
\sigma_3(A_i, A_j) = \frac{P A_i}{\sqrt{2} A_i^2 + 2 A_i A_j} \leq \left[ \sigma_3^T \right];
\]

\[
A_i^m \leq A_i \leq A_i^m \quad i = 1, 2
\]

According to generalized neutrosophic goal optimization technique using truth, indeterminacy and falsity membership function, MOSOP (12) can be formulated as:

**Model I**

\[
\text{Maximize} \quad \alpha, \text{Maximize} \quad \gamma, \text{Minimize} \quad \beta
\]

\[
(2 \sqrt{2} A_i + A_j) \leq 4 + 2 \left( 1 - \frac{\alpha}{w_1} \right),
\]

\[
(2 \sqrt{2} A_i + A_j) \geq 4 + \frac{w_1}{w_2 (0.5 w_1 + 0.22 w_2)} \gamma,
\]

\[
(2 \sqrt{2} A_i + A_j) \leq 4 + 2 - \gamma \left( 2 - \frac{w_1}{(0.5 w_1 + 0.22 w_2)} \right),
\]

\[
(2 \sqrt{2} A_i + A_j) \leq 4 + \frac{4.5}{w_1} \beta,
\]

\[
(2 \sqrt{2} A_i + A_j) \leq 4,
\]

\[
\frac{20}{A_i + \sqrt{2} A_j} \leq 2.5 + 2.5 \left( 1 - \frac{\alpha}{w_1} \right),
\]

\[
\frac{20}{A_i + \sqrt{2} A_j} \geq 2.5 + \frac{w_1}{w_2 (0.4 w_1 + 0.22 w_2)} \gamma,
\]

\[
\frac{20}{A_i + \sqrt{2} A_j} \leq 2.5 + 2.5 \gamma \left( 2.5 - \frac{w_1}{(0.4 w_1 + 0.22 w_2)} \right),
\]

\[
\frac{20}{A_i + \sqrt{2} A_j} \leq 2.5 + \frac{4.5}{w_1} \beta,
\]

\[
\frac{20}{A_i + \sqrt{2} A_j} \leq 2.5,
\]

\[
0 \leq \alpha + \beta + \gamma \leq w_1 + w_2 + w_3;
\]

\[
\alpha \in [0, w_1], \gamma \in [0, w_2], \beta \in [0, w_1];
\]

\[
w_1 \in [0.1], w_2 \in [0.1], w_3 \in [0.1];
\]

\[
0 \leq w_1 + w_2 + w_3 \leq 3;
\]

\[
20 \left( \sqrt{2} A_i + A_j \right) \leq 20;
\]

\[
\frac{20}{A_i + \sqrt{2} A_j} \leq 20;
\]

\[
\frac{20}{A_i + \sqrt{2} A_j} \leq 20;
\]
\[
\frac{20A_i}{\left(\sqrt{2}A_i^2 + 2AA_i\right)} \leq 15; \\
0.1 \leq A \leq 5 \quad i = 1,2
\]

With the help of generalized truth, indeterminacy, falsity membership function the generalized neutrosophic goal programming problem (12) based on arithmetic aggregation operator can be formulated as:

**Model II**

\[
\text{Minimize } \left\{ \frac{(1-\alpha) + \beta + (1-\gamma)}{3} \right\}
\]

Subjected to the same constraint as (13)

With the help of generalized truth, indeterminacy, falsity membership function the generalized neutrosophic goal programming problem (12) based on geometric aggregation operator can be formulated as:

**Model III**

\[
\text{Minimize } \sqrt[3]{(1-\alpha)\beta(1-\gamma)}
\]

Subjected to the same constraint as (13)

The above problem can be formulated using Model I, II, III and can be easily solved by an appropriate mathematical programming to give solution of multi-objective non-linear programming problem (12) by generalized neutrosophic goal optimization approach and the results are shown in the table 2.

Again, value of membership function in GNGP technique for MOSOP (11) based on different Aggregation is given in Table 3.

---

### Table 1: Input data for crisp model (11)

<table>
<thead>
<tr>
<th>Applied load (P) (KN)</th>
<th>Volume density (\rho) (KN/m³)</th>
<th>Length (L) (m)</th>
<th>Maximum allowable tensile stress (\sigma^t) (KN/m²)</th>
<th>Maximum allowable compressive stress (\sigma^c) (KN/m²)</th>
<th>Young’s modulus (E) (KN/m²)</th>
<th>(A^\min) and (A^\max) of cross section of bars (10⁻⁴ m²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>100</td>
<td>1</td>
<td>20</td>
<td>15</td>
<td>2\times10⁷</td>
<td>(A^\min = 0.1)(A^\max = 5)</td>
</tr>
</tbody>
</table>

### Table 2: Comparison of GNGP solution of MOSOP (11) based on different Aggregation

<table>
<thead>
<tr>
<th>Methods</th>
<th>(A^1\times10^{-4} m^2)</th>
<th>(A^2\times10^{-4} m^2)</th>
<th>(WT(A_1, A_2)\times10^2 KN)</th>
<th>(\delta(A_1, A_2)\times10^{-7} m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Generalized Fuzzy Goal programming (GFGP) (w_i = 0.15)</td>
<td>0.5392616</td>
<td>4.474738</td>
<td>6</td>
<td>2.912270</td>
</tr>
<tr>
<td>Generalized Intuitionistic Fuzzy Goal programming (GIFGP) (w_i = 0.15, w_j = 0.8)</td>
<td>0.5392619</td>
<td>4.474737</td>
<td>6</td>
<td>2.912270</td>
</tr>
<tr>
<td>Generalized Neutrosophic Goal programming (GNGP) (w_i = 0.4, w_j = 0.3, w_k = 0.7)</td>
<td>5</td>
<td>0.4321463</td>
<td>4.904282</td>
<td>3.564332</td>
</tr>
<tr>
<td>Generalized Intuitionistic Fuzzy optimization (GIFGP) based on Arithmetic Aggregation (w_i = 0.15, w_j = 0.8)</td>
<td>0.5392619</td>
<td>4.474737</td>
<td>6</td>
<td>2.912270</td>
</tr>
</tbody>
</table>
Generalized Neutosophic optimization (GNGP) based on Arithmetic Aggregation
\[ w_1 = 0.4, w_2 = 0.3, w_3 = 0.7 \]

Generalized Intuitionistic Fuzzy optimization (GIFGP) based on Geometric Aggregation
\[ w_1 = 0.15, w_2 = 0.8 \]

Here we get best solutions for the different value of \( w_1, w_2, w_3 \) in geometric aggregation method for objective functions. From Table 2 it is clear that Neutrosophic Optimization technique is more fruitful in optimization of weight compare to fuzzy and intuitionistic fuzzy optimization technique.

Moreover it has been seen that more desired value is obtain in geometric aggregation method compare to arithmetic aggregation method in intuitionistic as well as neutrosophic environment in perspective of structural engineering.

Table 3: Value of membership function in GNGP technique for MOSOP (11) based on different Aggregation

<table>
<thead>
<tr>
<th>Methods</th>
<th>( \alpha^* ), ( \beta^* ), ( \gamma^* )</th>
<th>Sum of Truth, Indeterminacy and Falsity Membership Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>Neutrosophic Goal programming (GNGP)</td>
<td>( \alpha^* = 0.1814422 ), ( \beta^* = 0.2191435 ), ( \gamma^* = 0.6013477 )</td>
<td>( T_{w_1} (WT(A_1,A_2)) + I_{w_1} (WT(A_1,A_2)) + F_{w_1} (WT(A_1,A_2)) )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( = 0.1814422 + 0.2191435 + 0.6013477 )</td>
</tr>
<tr>
<td>Generalized Neutrosophic optimization (GNGP) based on Arithmetic Aggregation</td>
<td>( \alpha^* = 0.3075145 ), ( \beta^* = 0.3075145 ), ( \gamma^* = 0.3075145 )</td>
<td>( T_{w_1} (WT(A_1,A_2)) + I_{w_1} (WT(A_1,A_2)) + F_{w_1} (WT(A_1,A_2)) )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( = 0.3075145 + 0.3075145 + 0.3075145 )</td>
</tr>
</tbody>
</table>

From the above table it is clear that all the objective functions attained their goals as well as restriction of truth, indeterminacy and falsity membership function in neutrosophic goal programming problem based on different aggregation operator.

The sum of truth, indeterminacy and falsity membership function for each objective is less than sum of gradiation \((w_1 + w_2 + w_3)\). Hence the criteria of generalized neutrosophic set is satisfied.
7. Conclusions

The research study investigates that neutrosophic goal programming can be utilized to optimize a nonlinear structural problem. The results obtained for different aggregation method of the undertaken problem show that the best result is achieved using geometric aggregation method. The concept of neutrosophic optimization technique allows one to define a degree of truth membership, which is not a complement of degree of falsity; rather, they are independent with degree of indeterminacy. As we have considered a non-linear three bar truss design problem and find out minimum weight of the structure as well as minimum deflection of loaded joint, the results of this study may lead to the development of effective neutrosophic technique for solving other model of nonlinear programming problem in different field.

References


Received: January 10, 2017. Accepted: February 3, 2017.
Topological Manifold Space via Neutrosophic Crisp Set Theory

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Abstract. In this paper, we introduce and study a neutrosophic crisp manifold as a new topological structure of manifold via neutrosophic crisp set. Therefore, we study some new topological concepts and some metric distances on a neutrosophic crisp manifold.

Keywords: neutrosophic crisp manifold, neutrosophic crisp coordinate chart, neutrosophic crisp Haussdorff, neutrosophic crisp countable, neutrosophic crisp basis, neutrosophic crisp Homeomorphism, neutrosophic locally compact.

1 Introduction

Neutrosophics found their places into contemporary research; we have introduced the notions of neutrosophic crisp sets, neutrosophic crisp point and neutrosophic topology on crisp sets.

We presented some new topological concepts and properties on neutrosophic crisp topology. A manifold is a topological space that is locally Euclidean and around every point there is a neighborhood that is topologically the same as the open unit in \(R^n\).

The aim of this paper is to build a new manifold topological structure called neutrosophic crisp manifold as a generalization of manifold topological space by neutrosophic crisp point and neutrosophic crisp topology and present some new topological concepts on a neutrosophic crisp manifold space.

Also, we study some metric distances on a neutrosophic crisp manifold.

The paper is structured as follows: in Section 2, we introduce preliminary definitions of the neutrosophic crisp point and neutrosophic crisp topology; in Section 3, some new topological concepts on neutrosophic crisp topology are presented and defined; in Section 4, we propose some topological concepts on neutrosophic crisp manifold space; Section 5 introduces some metric distances on a neutrosophic crisp manifold. Finally, our future work is presented in conclusion.

2 Terminologies [1, 2, 4]

We recollect some relevant basic preliminaries.

Definition 2.1:

Let \(A = \langle A_1, A_2, A_3 \rangle\) be a neutrosophic crisp set on a set \(X\), then \(p = \langle \{p_1\}, \{p_2\}, \{p_3\} \rangle, p_1 \neq p_2 \neq p_3 \in X\) is called a neutrosophic crisp point.

A NCP \(p = \langle \{p_1\}, \{p_2\}, \{p_3\} \rangle\) belongs to a neutrosophic crisp set \(A = \langle A_1, A_2, A_3 \rangle\) of \(X\) denoted by \(p \in A\) if it defined by: \(\{p_1\} \subseteq A_1, \{p_2\} \subseteq A_2\) and \(\{p_3\} \subseteq A_3\).

Definition 2.2:

A neutrosophic crisp topology (NCT) on a non empty set \(X\) is a family of \(\Gamma\) of neutrosophic crisp subsets in \(X\) satisfying the following axioms:

i. \(\emptyset, X \in \Gamma\)
ii. \(A_1 \cap A_2 \in \Gamma\) for any \(A_1, A_2 \in \Gamma\)
iii. \(\bigcup A_j \in \Gamma \forall \{A_j \in \Gamma\} \subseteq \Gamma\)

Then \((X, \Gamma)\) is called a neutrosophic crisp topological space (NCTS) in \(X\) and the elements in \(\Gamma\) are called neutrosophic crisp open sets (NCOs).
3 Neutrosophic Crisp Topological Manifold
Spaces [2, 5, 4, 7]

We present and study the following new topological concepts about the new neutrosophic crisp topological manifold Space.

Definition 3.1:
A neutrosophic crisp topological space \((X, \Gamma)\) is a neutrosophic crisp Hausdorff (NCH ) if for each two neutrosophic crisp points \(p =< [p_1], [p_2], [p_3] >\) and \(q =< [q_1], [q_2], [q_3] >\) in \(X\) such that \(p \neq q\) there exist neutrosophic crisp open sets \(U =< u_1, u_2, u_3 >\) and \(V =< v_1, v_2, v_3 >\) such that \(p \in U\), \(q \in V\) and \(U \cap V = \phi_x\).

Definition 3.2:
\(\beta\) is collection of neutrosophic crisp open sets in \((X, \Gamma)\) is said to be neutrosophic crisp base of neutrosophic crisp topology (NCT) if \(\Gamma_{NC} = U \beta\).

Definition 3.3:
Neutrosophic crisp topology \((X, \Gamma)\) is countable if it has neutrosophic crisp countable basis for neutrosophic crisp topology, i.e. there exist a countable collection of neutrosophic crisp open set \([U_\alpha]_{\alpha \in \mathbb{N}} = < u_1, u_2, u_3 >, < u_{21}, u_{22}, u_{23} >, \ldots, < u_{n1}, u_{n2}, u_{n3} >\) such that for any neutrosophic crisp open set \(U\) containing a crisp neutrosophic point \(p\) in \(U\), there exist a \(\beta \in \mathbb{N}\) such that \(p \in U_\beta \subseteq U\).

Definition 3.4:
Neutrosophic crisp homeomorphism is a bijective mapping \(f\) of NCTs \((X, \Gamma_1)\) onto NCTs \((Y, \Gamma_2)\) is called a neutrosophic crisp homeomorphism if it is neutrosophic crisp continuous and neutrosophic crisp open.

Definition 3.5:
Neutrosophic crisp topology is neutrosophic crisp Locally Euclidean of dimension \(n\) if for each neutrosophic crisp point \(p = < [p_1], [p_2], [p_3] >\) in \(X\), there exist a neutrosophic crisp open set \(U = < u_1, u_2, u_3 >\) and a map \(\phi: U \rightarrow R^n\) such that \(\phi: U \rightarrow \phi(U)\) which is \(\phi(U) = < \phi(u_1), \phi(u_2), \phi(u_3) >\) is a homeomorphism; in particular \(\phi(U)\) is neutrosophic crisp open set of \(R^n\).

We define a neutrosophic crisp topological manifold (NCM) as follows:

Definition 3.6:

1. (NCM) is a neutrosophic crisp topological manifold space if the following conditions together satisfied
2. (NCM) is satisfying neutrosophic crisp topology axioms.
3. (NCM) is neutrosophic crisp Hausdorff.
4. (NCM) is neutrosophic crisp Locally Euclidean of dimension \(n\).

We give the terminology \((M_{NC})^n\) to mean that it is a neutrosophic crisp manifold of dimension \(n\).

The following graph represents the neutrosophic crisp topological manifold space as a generalization of topological manifold space:

4 Some New Topological Concepts on NCM Space [2, 3, 4, 6, 8]

The neutrosophic crisp set \(U\) and map \(\phi(U)\) in the Definition 3.5 of neutrosophic crisp Locally Euclidean is called a neutrosophic crisp coordinate chart.

Definition 4.1:
A neutrosophic crisp coordinate chart on \((M_{NC})^n\) is a pair \((U, \phi(U))\) where \(U\) in \((M_{NC})^n\) is open and \(\phi: U \rightarrow \phi(U) \subseteq R^n\) is a neutrosophic crisp homeomorphism, and then the neutrosophic crisp set \(U\) is called a neutrosophic crisp coordinate domain or a neutrosophic crisp coordinate neighborhood.

A neutrosophic crisp coordinate chart \((U, \phi(U))\) is centered at \(p\) if \(\phi(p) = 0\) where a neutrosophic crisp coordinate ball \((\phi(U))\) is a ball in \(R^n\).

Definition 4.1.1:
A Ball in neutrosophic crisp topology is an open ball \((r, c, p)\) , \(r\) is radius

\[0 \leq r \leq 1, \ 0 < c < r \text{ and p is NCP.}\]

Theorem 4.1:
Every NCM has a countable basis of coordinate ball.

Theorem 4.2:
In \((M_{NC})^n\) every neutrosophic crisp point \(p = < [p_1], [p_2], [p_3] > \in (M_{NC})^n\) is contained in neutrosophic coordinate ball centered at \(p\) if:
(ϕ⁻¹(ϕ(p)), ϕ(ϕ⁻¹(ϕ(p))))
and then if we compose ϕ with a translating we must get
p = ϕ(p) = 0.

**Proof:** Since (M_NC)^n neutrosophic crisp Locally Euclidean, p must be contained in a coordinate chart (U, ϕ(U)). Since ϕ(U) is a neutrosophic crisp open set containing ϕ(p), by the NCT of R^n there must be an open ball B containing ϕ(p) and contained in ϕ(U). The appropriate coordinate ball is (ϕ⁻¹(ϕ(p)), ϕ(ϕ⁻¹(ϕ(p)))). Compose ϕ with a translation taking ϕ(p) to 0, then p = ϕ(p) = 0, we have completed the proof.

**Theorem 4.3:**
The neutrosophic crisp graph G(f) of a continuous function: U → R^k,
where U is neutrosophic crisp set in R^n, is NCM.
G(f) = {(p, f(p)) in R^n×R^k: p NCP n U }

**Proof:** Obvious.

**Example:** Spheres are NCM. An n-sphere is defined as:
S^n = {p NCP in R^{n+1}: |p|^2 = \sqrt{p_1^2 + p_2^2 + p_3^2} = 1}.

**Definition 4.2:**
Every neutrosophic crisp point p has a neutrosophic crisp neighborhood point p_{NCPbd} contained in an open ball B.

**Definition 4.3:**
Here come the basic definitions first.
Let (X, Γ) be a NCTS.

a) If a family \{< G_{i1}, G_{i2}, G_{i3} >: i \in J\} of NCCSs in X satisfies the condition \bigcup \{< G_{i1}, G_{i2}, G_{i3} >: i \in J\} = X_N then it is called a neutrosophic open cover of X.

b) A finite subfamily of an open cover \{< G_{i1}, G_{i2}, G_{i3} >: i \in J\} on X, which is also a neutrosophic open cover of X is called a neutrosophic finite subcover \{< G_{i1}, G_{i2}, G_{i3} >: i \in J\}.

c) A family \{< K_{i1}, K_{i2}, K_{i3} >: i \in J\} of NCCSs in X satisfies the finite intersection property [FIP] iff every finite subfamily \{< K_{i1}, K_{i2}, K_{i3} >: i = 1, 2, ..., n\} of the family satisfies the condition:
\bigcap \{< K_{i1}, K_{i2}, K_{i3} >: i \in J\} \neq \emptyset.

d) A NCTS (X, Γ) is called a neutrosophic crisp compact iff each crisp neutrosophic open cover of X has a finite subcover.

**Corollary:**
A NCTS (X, Γ) is a neutrosophic crisp compact iff every family \{< G_{i1}, G_{i2}, G_{i3} >: i \in J\} of NCCSs in X having the FIP has non-empty intersection.

**Definition 4.4:**
Every neutrosophic point has a neutrosophic neighborhood contained in a neutrosophic compact set is called neutrosophic locally compact set.

**Corollary:**
Every NCM is neutrosophic locally compact set.

5 Some Metric Distances on a Neutrosophic Crisp Manifold [10, 9]

5.1 Hausdorff Distance between Two Neutrosophic Crisp Sets on NCM:
Let A =< A_1, A_2, A_3 > and B =< B_1, B_2, B_3 > two neutrosophic crisp sets on NCM then the Hausdorff distance between A and B is
\[ d_H(A, B) = \inf \{d(A_i, B_j)\} \text{, } \forall i, j \in J \]

5.2 Modified Hausdorff Distance between Two Neutrosophic Crisp Sets on NCM:
Let A =< A_1, A_2, A_3 > and B =< B_1, B_2, B_3 > two neutrosophic crisp sets on NCM then the Haussdorff distance between A and B is
\[ d_H(A, B) = \frac{1}{n} \sum \{\inf\{d(A_i, B_j)\}\} \text{, } n \text{ is number of NCPs} \]

**Conclusion and Future Work**
In this paper, we introduced and studied the neutrosophic crisp manifold as a new topological structure of manifold via neutrosophic crisp set, and some new topological concepts on a neutrosophic crisp manifold space via neutrosophic crisp set, and also some metric distances on a neutrosophic crisp manifold. Future work will approach neutrosophic fuzzy manifold, a new topological structure of manifold via neutrosophic fuzzy set, and some new topological concepts on a neutrosophic fuzzy manifold space via neutrosophic fuzzy set.

**References**


Received: January 13, 2017. Accepted: February 5, 2017.
Neutrosophic Graphs of Finite Groups

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Abstract: Let $G$ be a finite multiplicative group with identity $e$ and $N(G)$ be the Neutrosophic group with indeterminate $I$. We denote by $Ne(G, I)$, the Neutrosophic graph of $G \cdot N(G)$ and $I$. In this paper, we study the graph $Ne(G, I)$ and its properties. Among the results, it is shown that for any finite multiplicative group $G$, $Ne(G, I)$ is a connected graph of diameter less than or equal to 2. Moreover, for finite group $G$, we obtain a formula for enumerating basic Neutrosophic triangles in $Ne(G, I)$. Furthermore, for every finite groups $G$ and $G'$, we show that $G \cong G'$ if and only if $N(G) \cong N(G')$, and if $N(G) \cong N(G')$, then $Ne(G, I) \cong Ne(G', I')$.

Keywords: Indeterminacy; Finite Multiplicative group; Neutrosophic Group; Basic Neutrosophic triangle; Neutrosophic group and graph isomorphism.

1 Introduction

Most of the real world problems in the fields of philosophy, physics, statistics, finance, robotics, design theory, coding theory, knot theory, engineering, and information science contain subtle uncertainty and inconsistent, which causes complexity and difficulty in solving these problems. Conventional methods failed to handle and estimate uncertainty in the real world problems with near tendency of the exact value. The determinacy of uncertainty in the real world problems have been great challenge for the scientific community, technological people, and quality control of products in the industry for several years. However, different models or methods were presented systematically to estimate the uncertainty of the problems by various incorporated computational systems and algebraic systems. To estimate the uncertainty in any system of the real world problems, first attempt was made by the Lotfi A Zadesh [1] with help of Fuzzy set theory in 1965. Fuzzy set theory is very powerful technique to deal and describe the behavior of the systems but it is very difficult to define exactly. Fuzzy set theory helps us to reduce the errors of failures in modeling and different fields of life. In order to define system exactly, by using Fuzzy set theory many authors were modified, developed and generalized the basic theories of classical algebra and modern algebra. Along with Fuzzy set theory there are other different theories have been study the properties of uncertainties in the real world problems, such as probability theory, intuitionistic Fuzzy set theory, rough set theory, paradoxist set theory [2-5]. Finally, all above theories contributed to explained uncertainty and inconsistency up to certain extent in real world problems. None of the above theories were not studied the properties of indeterminacy of the real world problems in our daily life. To analyze and determine the existence of indeterminacy in various real world problems, the author Smarandache [6] introduced philosophical theory such as Neutrosophic theory in 1990.

Neutrosophic theory is a specific branch of philosophy, which investigates percentage of Truthfulness, falsehood and neutrality of the real world problem. It is a generalization of Fuzzy set theory and intuitionistic Fuzzy set theory. This theory is considered as complete representation of a mathematical model of a real world problem. Consequently, if uncertainty is involved in a problem we use Fuzzy set theory, and if indeterminacy is involved in a problem we essential Neutrosophic theory.

Kandasamy and Smarandache [7] introduced the philosophical algebraic structures, in particular, Neutrosophic algebraic structures with illustrations and examples in 2006 and initiated the new way for the emergence of a new class of structures, namely, Neutrosophic groupoids, Neutrosophic groups, Neutrosophic rings etc. According to these authors, the Neutrosophic algebraic structures $N(I)$ was a nice composition of indeterminate $I$ and the elements of a
In the fourth section, we introduced basic Neutrosophic triangles in the graph \( Ne(G, I) \) and obtained a formula for enumerating basic Neutrosophic triangles in \( Ne(G, I) \) to understand the internal mutual relations between the elements in \( G, I \) and \( N(G) \).

In the last section, all finite isomorphic groups \( G \) and \( G' \) such that \( N(G) \cong N(G') \) and \( Ne(G, I) \cong Ne(G', I') \) are characterized with examples.

Throughout this paper, all groups are assumed to be finite multiplicative groups with identity \( e \). Let \( N(G) \) be a Neutrosophic group generated by \( G \) and \( I \). For classical theorems and notations in algebra and Neutrosophic algebra, the interest reader is refereed to [11] and [8].

Let \( X \) be a graph with vertex set \( V(X) \) and edge set \( E(X) \). The cardinality of \( V(X) \) and \( E(X) \) are denoted by \( |V(X)| \) and \( |E(X)| \), which are order and size of \( X \), respectively. If \( X \) is connected, then there exist a path between any two vertices in \( X \). We denote by \( K_n \) the complete graph of order \( n \). Let \( u \in V(X) \). Then degree of \( u \), \( \deg(u) \) in \( X \) is the number of edges incident at \( u \). If \( \deg(u) = 1 \) then the vertex \( u \) is called pendant. The girth of \( X \) is the length of smallest cycle in \( X \). The girth of \( X \) is infinite if \( X \) has no cycle. Let \( d(x, y) \) be the length of the shortest path from two vertices \( x \) and \( y \) in \( X \), and the diameter of \( X \) denoted by

\[
Diam(X) = \max\{d(x, y) \mid x, y \in V(X)\}.
\]

For further details about graph theory the reader should see [12].

2 Basic Properties of Neutrosophic set and \( GI \)

This section will present a few basic concepts of Neutrosophic set and Neutrosophic group that will then be used repeatedly in further sections, and it will introduce a convenient notations. A few illustrations and examples will appear in later sections.

Neutrosophic set is a mathematical tool for handling real world problems involving imprecise, inconsistent data and indeterminacy; also it generalizes the concept of the classic set, fuzzy set, rough set etc. According to authors Vasantha Kandasamy and Smarandache, the Neutrosophic set is a nice composition of an algebraic set and indeterminate element of the real world problem.

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Let $N$ be a non-empty set and $I$ be an indeterminate. Then the set $N(I) = \{N \cup I\}$ is called a Neutrosophic set generated by $N$ and $I$. If `$\cdot$' is usual multiplication in $N$, then $I$ has the following axioms.

1. $0 \cdot I = 0$
2. $1 \cdot I = I = I \cdot 1$
3. $I^2 = I$
4. $a \cdot I = I \cdot a$, for every $a \in N$.
5. $I^{-1}$ does not exist.

For the definition, notation and basic properties of Neutrosophic group, we refer the reader to Agbool [8]. As treated in [8], we shall denote the finite Neutrosophic group by $N(G)$ for a group $G$.

**Definition 2.1** Let $G$ be any finite group with respect to multiplication. Then the set $GI$ defined as $GI = \{gI : g \in G\} = \{Ig : g \in G\}$.

**Definition 2.2** If a map $f$ from a finite nonempty set $S$ into a finite nonempty set $S'$ is both one-one and onto then there exist a map $g$ from $S'$ into $S$ that is also one-one and onto. In this case we say that the two sets are equivalent, and, abstractly speaking, these sets can be regarded as the same cardinality. We write $S \sim S'$ whenever there is a one-one map of a set $S$ onto $S'$.

Two finite rings $R$ and $R'$ are equivalent if there is a one-one correspondence between $R$ and $R'$. We write $R \sim R'$.

**Definition 2.3** Let $G$ be any finite group with respect to multiplication and let $N(G) = \langle G \cup I \rangle$. Then $\langle N(G), \cdot \rangle$ is called a Neutrosophic group generated by $G$ and $I$ under the binary operation `$\cdot$' on $G$. The Neutrosophic group $N(G)$ has the following properties.

1. $N(G)$ is not a group.
2. $G \subseteq N(G)$.
3. $GI \subseteq N(G)$.
4. $N(G)$ is a specific composition of $G$ and $I$.

**Lemma 2.4** Let $G$ be any finite group with respect to multiplication and $I^2 = I$. Then $G \sim GI$. In particular, $|G| = |GI|$.

**Proof.** For any finite group $G$, we have $G \neq GI$ and $GI \subsetneq G$. Now define a map $f : G \rightarrow GI$ by the relation $f(a) = aI$ for every $a \in I$. Let $a, b \in G$. Then

$a = b \iff a - b = 0 \iff (a - b)I = 0I \iff aI = bI \iff f(a) = f(b)$. This shows that $f$ is a well defined one-one function. Further, we have

$$\text{Range}(f) = \{f(a) \in GI : a \in G\} = \{aI \in GI : a \in G\} = GI.$$ 

This show that for every $aI \in GI$ at least one $a \in G$ such that $f(a) = aI$.

Therefore, $f : G \rightarrow GI$ is one-one correspondence and consequently a bijective function. Hence $G \sim GI$.

**Lemma 2.5** Let $G$ be any finite group with respect to multiplication and let $N(G) = \langle G \cup I \rangle$. Then the order of $N(G)$ is $2|G|$.

**Proof:** We have $GI = \{gI : g \in G\}$. Obviously, $GI \subsetneq G$ and $G \subsetneq GI$ but $GI \subset N(G)$.

It is clear that $N(G)$ is the disjoint union of $G$ and $GI$. That is, $N(G) = G \cup GI$ and $G \cap GI = \emptyset$.

Therefore, $|N(G)| = |G| + |GI| = 2|G|$, since $|G| = |GI|$.

**Lemma 2.6** The set $GI$ is not Neutrosophic group with respect to multiplication of group $G$.

**Proof:** It is obvious, since $GI \neq \langle G \cup I \rangle$.

**Lemma 2.7** The elements in $GI$ satisfies the following properties,

1. $e \cdot gI = gI$
2. $(gI)^2 = g^2I$
3. $gI \cdot gI \ldots gI = g^nI$ for all positive integers $n$.
4. $(gI)^{-1}$ does not exist, since $I^{-1}$ does not exist.
5. $gI = g'I \iff g = g'$.

**Proof:** Directly follows from the results of the group $\langle N(G), \cdot \rangle$. 

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Theorem 2.8 The structure \((GI, \cdot)\) is a monoid under the operation \((al)(bl) = abl\) for all \(a, b\) in the group \((G, \cdot)\) and \(I^2 = I\).

Proof: We know that \(GI = \{gl : g \in G\}\).

Let \(al, bl\) and \(cl\) be any three elements in \(GI\). Then the binary operation \((al)(bl) = abl\) in \((GI, \cdot)\) satisfies the following axioms.

1. \(abI \in GI \Rightarrow (al)(bl) \in GI\).
2. \([(al)(bl)](cl) = [(abI)](cl)\)
   \(= [a(bc)]I = a[(bcI)]I = aI[(bl)(cl)]\)
3. Let \(e\) be the identity element in \((G, \cdot)\). Then \(eI = I = Ie\) and \(I(al) = aI^2 = aI = (al)I\).

Remark 2.9 The structure \((GI, \cdot)\) is never a group because \(I^{-1}\) does not exist.

Here we obtain lower bounds and upper bounds of the order of the Neutrosophic group \(N(G)\). Moreover, these bounds are sharp.

Theorem 2.10 Let \(G\) be a finite group with respect to multiplication. Then,

\[1 \leq |G| \leq n \Leftrightarrow 2 \leq |N(G)| \leq 2n.\]

Proof. We have,

\[|G| = 1 \Leftrightarrow G = \{e\} \Leftrightarrow N(G) = G \cup GI = \{e, I\}\]

\[\Leftrightarrow |N(G)| = 2.\]

This is one extreme of the required inequality. For other extreme, by the Lemma [2.4],

\[|G| > 1 \Leftrightarrow |GI| > 1\]

\[\Leftrightarrow |G| + |GI| > 2 \text{ and } |G| + |GI| \text{ is not odd}\]

\[\Leftrightarrow |G| + |GI| \text{ is even.}\]

\[\Leftrightarrow |N(G)| = |G| + |GI| = 2n.\]

Hence, the theorem.

3 Basic Properties of Neutrosophic Graph

In this section, our aim is to introduce the notion of Neutrosophic graph of finite Neutrosophic group with respect to multiplication and study on its basic and specific properties such as connectedness, completeness, bipartite, order, size, number of pendent vertices, girth and diameter.

Definition 3.1 A graph \(Ne(G, I)\) associated with Neutrosophic group \((N(G), \cdot)\) is an undirected simple graph whose vertex set is \(N(G)\) and two vertices \(x, y\) are adjacent in \(Ne(G, I)\), if and only if \(xy\) is either \(x\) or \(y\).

Theorem 3.2 For any group \((G, \cdot)\), the Neutrosophic graph \(Ne(G, I)\) is connected.

Proof: Let \(e\) be the identity element in \(G\). Then \(e \in N(G)\), since \(G \subseteq N(G)\). Further, \(xe = x\), for every \(x \neq e\) in \(N(G)\). It is clear that the vertex \(e\) is adjacent to all other vertices of the graph \(Ne(G, I)\).

Hence \(Ne(G, I)\) is connected.

Theorem 3.3 Let \(|G| > 1\). Then the graph has at least one cycle of length 3.

Proof: Since \(|G| > 1\) implies \(|N(G)| \geq 4\). So there is at least one vertex \(gl\) of \(N(G)\) such that \(gl\) is adjacent to the vertices \(e\) and \(I\) in \(Ne(G, I)\), since \(eI = I\). \(I(gI) = gl^2 = gl\), and \((gI)e = geI = gl\). Hence we have the cycle \(e - I - gl - e\) of length 3, where \(g \neq e\).

Example 3.4 Since \(N(G_{10}) = \{2, 4, 6, 8, 2I, 4I, 6I, 8I\}\) is the Neutrosophic group of the group \(G_{10} = \{2, 4, 6, 8\}\) with respect to multiplication modulo 10, where \(e = 6\).

The Neutrosophic graph \(Ne(G_{10}, I)\) contains three cycles of length 3, which are listed below.

\[C_1: 6 - I - 2I - 6,\]
\[C_2: 6 - I - 4I - 6,\]
\[C_3: 6 - I - 8I - 8.\]

Theorem 3.5 The Neutrosophic graph \(Ne(G, I)\) is complete if and only if \(|G| = 1\).

Proof: Necessity. Suppose that \(Ne(G, I)\) is complete. If possible assume that \(|G| > 1\), then \(|N(G)| \geq 4\). So without loss of generality we may assume that \(|N(G)| = 4\) and clearly the vertices
e, g, I, gI ∈ V(Ne(G, I)). Therefore the vertex g is not adjacent to the vertex I in Ne(G, I), since gI ≠ g or I for each g ≠ e in G, this contradicts our assumption that Ne(G, I) is complete. It follows that |N(G)| cannot be four. Further, if |N(G)| > 4, then obviously we arrive a contradiction. So our assumption is wrong, and hence |G| = 1.

**Sufficient.** Suppose that |G| = 1. Then, trivially |N(G)| = 2. Therefore, Ne(G, I) ≅ K₂, since eI = 1. Hence, Ne(G, I) is a complete graph.

Recall that |V(Ne(G, I))| is the order and |E(Ne(G, I))| is the size of the Neutrosophic graph Ne(G, I). But,

\[ |V(Ne(G, I))| = |N(G)| = 2|G| \]

and the following theorem shows that the size of Ne(G, I).

**Theorem 3.6** The size of Neutrosophic graph Ne(G, I) is 3|G| - 2.

**Proof:** By the definition of Neutrosophic graph, Ne(G, I) contains \(2(|G| - 1)^2\) non adjacent pairs. But the number of combinations of any two distinct pairs from N(G) is \(\binom{|N(G)|}{2}\). Hence the total number of adjacent pairs in Ne(G, I) is

\[ |E(Ne(G, I))| = \binom{|N(G)|}{2} - 2(|G| - 1)^2 \]

= 3|G| - 2.

**Theorem 3.7** [11] The size of a simple complete graph of order n is \(\frac{1}{2}n(n-1)\).

**Corollary 3.8** The Neutrosophic graph Ne(G, I), |G| > 1 is never complete.

**Proof:** Suppose on contrary that Ne(G, I), |G| > 1 is complete. Then, by the Theorem [3.7], the total number of edges in Ne(G, I) is \(\frac{1}{2}(2|G|(2|G| - 1)) = |G|(2|G| - 1)\), but in view of Theorem [3.6], we arrived a contradiction to the completeness of Ne(G, I).

**Theorem 3.9** The graph Ne(G, I) has exactly |G| - 1 pendent vertices.

**Proof:** Since N(G) = G ∪ GI and G ∩ GI = Ø. Let \(x \in N(G)\). Then either \(x \in G\) or \(x \in GI\). Now consider the following cases on G and GI, respectively.

**Case 1.** If \(x \in GI\), then \(x = gI\) for \(g \in G\). But \(xl = (gI)l = gI^2 = gI = x\) and \(ex = egI = gI = x\). This implies that the vertex \(x\) is adjacent to both the vertices \(e\) and \(l\) in N(G). Hence deg(x) ≠ 1 for every \(x \in GI\).

**Case 2.** If \(x \in G\), then \(ex = x, \text{for every } x \neq e\) and \(egI = gI\), for every \(gI \in GI\). Therefore deg(e) = |N(G)| - 1 ≠ 1. Now show that deg(x) = 1, for every \(x \neq e\) in G. Suppose, deg(x) > 1, for every \(x \neq e\) in G. Then there exist another vertex \(y \neq e\) in G such that either \(xy = x\) or \(y\), this is not possible in G, because G is a finite multiplication group. Thus deg(x) = 1, for \(x \neq e\) in G.

From case (1) and (2), we found the degree of each non identity vertex in G is 1. This shows that each and every non identity element in G is a pendent vertex in Ne(G, I). Hence, the total number of pendent vertices in Ne(G, I) is |G| - 1.

The following result shows that Ne(G, I) is never a traversal graph.

**Corollary 3.10** Let |G| > 1. Then Ne(G, I) is never Eulerian and never Hamiltonian.

**Proof.** It is obvious from the Theorem [3.9].

**Theorem 3.11** [11] A simple graph is bipartite if and only it does not have any odd cycle.

**Theorem 3.12** The Neutrosophic graph Ne(G, I), |G| > 1 is never bipartite.
Proof. Assume that \(|G| > 1\). Suppose, \(\text{Ne}(G, I)\) is a bipartite graph. Then there exist a bipartition \((G, GI)\), since \(N(G) = G \cup GI\) and \(G \cap GI = \emptyset\).

But \(e \in G\) and \(I \in GI\), where \(e \neq I\). So there exist at least one vertex \(gI\) in \(\text{Ne}(G, I)\) such that 
\[ e - I - gl - e \]
is an odd cycle of length 3 because 
\[ eI = I, \quad I(gI) = gI \]
and \((gI)e = gI\).

This violates the condition of the Theorem [3.11].

Hence \(\text{Ne}(G, I)\) is not a bipartite graph.

Theorem 3.13 The girth of a Neutrosophic graph is 3.

Proof. In view of Theorem [3.3], for \(|G| > 1\), we always have a cycle \(e - I - gl - e\) of length 3, for each \(g \neq e\) in \(G\), which is smallest in \(\text{Ne}(G, I)\).

This completes the proof.

Remark 3.14 Let \(G\) be a finite group with respect to multiplication. Then \(\text{gir} (\text{Ne}(G, I)) = \infty\) if \(|G| = 1\), since \(\text{Ne}(G, I)\) is acyclic graph if and only if \(|G| = 1\).

Theorem 3.15 \(\text{Diam} (\text{Ne}(G, I)) \leq 2\).

Proof. Let \(G\) be a finite group with respect to multiplication. Then we consider the following two cases.

Case 1 Suppose \(|G| = 1\). The graph \(\text{Ne}(G, I) \cong K_2\).

It follows that \(\text{Ne}(G, I)\) is complete, so \(\text{diam} (\text{Ne}(G, I)) = 1\).

Case 2 Suppose \(|G| > 1\). Then the vertex \(e\) is adjacent to every vertex of \(\text{Ne}(G, I)\). However the vertex \(al\) is not adjacent to \(bl\) for all \(a \neq b\) in \(G\), so \(d(al, bl) > 1\). But in \(\text{Ne}(G, I)\), there always exist a path \(al - I - bl\), since \((al)I = al\) and \((bl)I = bl\), which gives \(d(al, bl) = 2\), for every \(al, bl \in N(G)\).

Hence, both the cases conclude that:
\(\text{Diam} (\text{Ne}(G, I)) \leq 2\).

4 Enumeration of basic Neutrosophic triangles in \(\text{Ne}(G, I)\)

Since \(\text{Ne}(G, I)\) is triangle free graph for \(|G| = 1\), we will consider \(|G| > 1\) in this section.

Let us denote a triangle by \((x, y, z)\) in \(\text{Ne}(G, I)\) with vertices \(x, y\) and \(z\). Without loss of generality we may assume that our triangles \((e, I, gI)\) have vertices \(e, I\) and \(gI\), where \(g \neq e\) in \(G\). These triangles are called basic Neutrosophic triangles in \(\text{Ne}(G, I)\), which are defined as follows.

Definition 4.1 A triangle in the graph \(\text{Ne}(G, I)\) is said to be basic Neutrosophic if it has the common vertices \(e\) and \(I\).

The following short table illustrates some finite Neutrosophic graphs and their total number of basic Neutrosophic triangles.

<table>
<thead>
<tr>
<th>(\text{Ne}(G, I))</th>
<th>(\text{Ne}(Z^*_p, I))</th>
<th>(\text{Ne}(C_n, I))</th>
<th>(\text{Ne}(G_{2p}, I))</th>
<th>(\text{Ne}(V_4, I))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(T_{el})</td>
<td>(p - 2)</td>
<td>(n - 1)</td>
<td>(p - 2)</td>
<td>(3)</td>
</tr>
</tbody>
</table>

where \(Z^*_p = Z_p - \{0\}\) is a group with respect to multiplication modulo \(p\), a prime, \(C_n = \{1, g, g^2,\ldots, g^{n-1}: g^n = 1\}\) is a cyclic group generated by \(g\) with respect to multiplication, \(G_{2p} = \{0, 2, 4,\ldots, 2(p-1)\}\) is a group with respect to multiplication modulo \(2p\) and \(V_4 = \{e, a, b, c: a^2 = b^2 = c^2 = e\}\) is a Klein 4-group.

Before we continue, it is important to note that the multiplicative identity \(e\) may differ from group to group. However, for simplicity sake we will continue to note that \(e = 1\), and we leave it to reader to understand from context of the group for \(e\).

The following results give information about enumeration of basic and non-basic Neutrosophic triangles in the graph \(\text{Ne}(G, I)\).

First we begin a lemma, which gives a formula for enumerating the number of Neutrosophic triangles in \(\text{Ne}(G, I)\) corresponding to fixed elements \(e\) and \(I\) in the Neutrosophic set \(N(G)\).
This is useful for finding the total number of non-basic Neutrosophic triangles in $Ne(G, I)$.

**Theorem 4.2** Let $|G| > 1$. Then the total number of basic Neutrosophic triangles in $Ne(G, I)$ is $|T_{el}| = |G| - 1$.

**Proof.** Since $N(G) = G \cup GI$ and $G \cap GI = \emptyset$. It is clear that $e \neq I$. For any $aI \in GI$, the traid $(e, I, aI) \in T_{el} \iff (e, I), (e, aI)$, and $(I, aI)$ are edges in $Ne(G, I)$

\[
\iff el = I, e(al) = al, I(al) = al
\]

\[
\iff I, al \in GI, \text{ where } a \neq e \in G.
\]

That is, for fixed vertices $e, I$ and for each $al \in GI$, the traid $(e, I, al)$ exists in $Ne(G, I)$. Further, for any vertex $a \in G$, the vertices $e, I$ and $a$ does not form a triangle in $Ne(G, I)$ because $(I, a)$ is not an edge in $Ne(G, I)$, since $al \neq a$ or $I$ for all $a \neq e$.

So that the total number of triangles having common vertices $e$ and $I$ in $Ne(G, I)$ is

\[
|T_{el}| = |N(G)| - (|G| + 1)
\]

\[
= 2|G| - (|G| + 1) = |G| - 1.
\]

**Theorem 4.3** The total number of non-basic Neutrosophic triangles in $Ne(G, I)$ is zero.

**Proof.** Suppose that two vertices either $x, y$ or $y, z$ or $z, x$ are not equal to $e$ and $I$.

Then the traid $(x, y, z)$ is a non-basic triangle in $Ne(G, I) \iff (x, y, z) \notin T_{el}$

\[
\iff xy = x, yz = y \text{ and } zx = z
\]

\[
\iff \text{either } xyz = x \text{ or } yzx = y
\]

or $xyz = z$.

This is not possible in the Neutrosophic group $N(G)$. Thus there is no any non-basic triangle in the graph $Ne(G, I)$, and hence the total number of non-basic Neutrosophic triangles in $Ne(G, I)$ is zero.

In view of Theorems [3,9] and [4,2], the following theorem is obvious.

**Theorem 4.4** The total number of pendent vertices and basic Neutrosophic triangles in $Ne(G, I)$ is same, which is equal to $|G| - 1$.

---

5 Isomorphic properties of Neutrosophic groups and graphs

In this section we consider important concepts known as isomorphism of groups and Neutrosophic groups. But the notion of isomorphism is common to all aspects of modern algebra [14] and Neutrosophic algebra. An isomorphism of groups and Neutrosophic groups are maps which preserves operations and structures. More precisely we have the following definitions which we make for finite groups and Neutrosophic finite groups.

**Definition 5.1** Two finite groups $G$ and $G'$ are said to be isomorphic if there is a one-one correspondence $f : G \rightarrow G'$ such that $f(ab) = f(a)f(b)$ for all $a, b \in G$ and we write $G \cong G'$.

Now we proceed on to define isomorphism of finite Neutrosophic groups with distinct indeterminate, which can be defined over distinct groups with same binary operation. We can establish two main results.

1. Two groups are isomorphic and their Neutrosophic groups are also isomorphic.
2. If two Neutrosophic groups are isomorphic, then their Neutrosophic graphs are also isomorphic.

**Definition 5.2** Let $(G, \cdot)$ and $(G', \cdot)$ be two finite groups and let $I \neq I'$ be two indeterminates of two distinct real world problems. The Neutrosophic groups $N(G) = \langle (G \cup I), \cdot \rangle$ and $N(G') = \langle (G' \cup I'), \cdot \rangle$ are isomorphic if there exist a group isomorphism $\varphi$ from $G$ onto $G'$ such that $\varphi(I) = I'$ and we write $N(G) \cong N(G')$.

**Definition 5.3** [13] If there is a one-one mapping $a \leftrightarrow a'$ of the elements of a group $G$ onto those a group $G'$ and if $a \leftrightarrow a'$ and $b \leftrightarrow b'$ implies $ab \leftrightarrow a'b'$, then we say that $G$ and $G'$ are isomorphic and write $G \cong G'$. If we put $a' = f(a)$ and $b' = f(b)$ for $a, b \in G$, then $f : G \rightarrow G'$ is a bijection satisfying $f(ab) = a'b' = f(a)f(b)$.

**Lemma 5.4** $G \cong G' \iff N(G) \cong N(G')$.

**Proof. Necessity.** Suppose $G \cong G'$. Then there exist a group isomorphism $\varphi$ from $G$ onto $G'$ such that $\varphi(a) = a'$ for every $a \in G$ and $a' \in G'$. By the definition [12], the relation says that $\varphi$ sends $ab$ onto $a'b'$, where $a' = \varphi(a)$ and $b' = \varphi(b)$ are the elements of.
implies that for and .

Suppose and . Then either or but .

Case 1 Suppose and . Then .

Trivially, , for every and , since . Thus, .

Case 2 Suppose and . Then and . Obviously, is one-one correspondence between and , since .

Further,

Thus is a Neutrosophic group isomorphism from onto . Further, is a Neutrosophic graph isomorphism between and .

Sufficiency. It is similar to necessity, because implies that and under the mapping and , respectively.

Theorem 5.5 If , then . But converse is not true.

Proof. Suppose and . Then be two different Neutrosophic groups generated by and , respectively.

Let be an isomorphism from to . Then is one-one correspondence between the graphs and under the relation for every and . Further to show that preserves the adjacency. For this let and be any two vertices of the graph , then and . This implies that

Hence, and are adjacent in . Similarly, maps non-adjacent vertices to non-adjacent vertices. Thus, is a Neutrosophic graph isomorphism from onto .

The converse of the Theorem [5.5] is not true in general. Let and let and . Clearly, but is not isomorphic to .

This is illustrated in the following figure.

Acknowledgments

The authors express their sincere thanks to Prof. L. Nagamuni Reddy and Prof. S. Vijaya Kumar Varma for his suggestions during the preparation of this paper. My sincere thanks also goes to Dr. B. Jaya Prakash Reddy.

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Received: January 23, 2017. Accepted: February 13, 2017.
A New Similarity Measure Based on Falsity Value between Single Valued Neutrosophic Sets Based on the Centroid Points of Transformed Single Valued Neutrosophic Numbers with Applications to Pattern Recognition

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Abstract. In this paper, we propose some transformations based on the centroid points between single valued neutrosophic numbers. We introduce these transformations according to truth, indeterminacy and falsity value of single valued neutrosophic numbers. We propose a new similarity measure based on falsity value between single valued neutrosophic sets. Then we prove some properties on new similarity measure based on falsity value between single valued neutrosophic sets. Furthermore, we propose similarity measure based on falsity value between single valued neutrosophic sets based on the centroid points of transformed single valued neutrosophic numbers. We also apply the proposed similarity measure between single valued neutrosophic sets to deal with pattern recognition problems.

Keywords: Neutrosophic sets, Single Valued Neutrosophic Numbers, Centroid Points.

1 Introduction

In [1] Atanassov introduced a concept of intuitionistic sets based on the concepts of fuzzy sets [2]. In [3] Smarandache introduced a concept of neutrosophic sets which is characterized by truth function, indeterminacy function and falsity function, where the functions are completely independent. Neutrosophic set has been a mathematical tool for handling problems involving imprecise, indeterminacy and inconsistent data; such as cluster analysis, pattern recognition, medical diagnosis and decision making.In [4] Smarandache et.al introduced a concept of single valued neutrosophic sets. Recently few researchers have been dealing with single valued neutrosophic sets [5-10].

The concept of similarity is fundamentally important in almost every scientific field. Many methods have been proposed for measuring the degree of similarity between intuitionistic fuzzy sets [11-15]. Furthermore, in [13-15] methods have been proposed for measuring the degree of similarity between intuitionistic fuzzy sets based on transformed techniques for pattern recognition. But those methods are unsuitable for dealing with the similarity measures of neutrosophic sets since intuitionistic sets are characterized by only a membership function and a non-membership function. Few researchers dealt with similarity measures for neutrosophic sets [16-22]. Recently, Jun [18] discussed similarity measures on internal neutrosophic sets, Majumdar et al.[17] discussed similarity and entropy of neutrosophic sets, Brouni et.al.[16]discussed several similarity measures of neutrosophic sets, Ye [9] discussed single-valued neutrosophic similarity measures based on cotangent function and their application in the fault diagnosis of steam turbine, Deli et.al.[10] discussed multiple criteria decision making method on single valued bipolar neutrosophic set based on correlation coefficient similarity measure, Ulucay et.al. [21] discussed Jaccard vector similarity measure of bipolar neutrosophic set based on multi-criteria decision making and Ulucay et.al.[22] discussed similarity...
measure of bipolar neutrosophic sets and their application to multiple criteria decision making.

In this paper, we propose methods to transform between single valued neutrosophic numbers based on centroid points. Here, as single valued neutrosophic sets are made up of three functions, to make the transformation functions be applicable to all single valued neutrosophic numbers, we divide them into four according to their truth, indeterminacy and falsity values. While grouping according to the truth values, we take into account whether the truth values are greater or smaller than the indeterminacy and falsity values. Similarly, while grouping according to the indeterminacy/falsity values, we examine the indeterminacy/falsity values and their greatness or smallness with respect to their remaining two values. We also propose a new method to measure the degree of similarity based on falsity values between single valued neutrosophic sets. Then we prove some properties of new similarity measure based on falsity value between single valued neutrosophic sets. When we take this measure with respect to truth or indeterminacy we show that it does not satisfy one of the conditions of similarity measure. We also apply the proposed new similarity measures based on falsity value between single valued neutrosophic sets to deal with pattern recognition problems. Later, we define the method based on falsity value to measure the degree of similarity between single valued neutrosophic set based on centroid points of transformed single valued neutrosophic numbers and the similarity measure based on falsity value between single valued neutrosophic sets.

In section 2, we briefly review some concepts of single valued neutrosophic sets [4] and property of similarity measure between single valued neutrosophic sets. In section 3, we define transformations between the single valued neutrosophic numbers based on centroid points. In section 4, we define the new similarity measures based on falsity value between single valued neutrosophic sets and we prove some properties of new similarity measure between single valued neutrosophic sets. We also apply the proposed method to deal with pattern recognition problems. In section 5, we define the method to measure the degree of similarity based on falsity value between single valued neutrosophic set based on centroid point of transformed single valued neutrosophic number and we apply the measure to deal with pattern recognition problems. Also we compare the traditional and new methods in pattern recognition problems.

2 Preliminaries

Definition 2.1[3] Let $U$ be a universe of discourse. The neutrosophic set $A$ is an object having the form $A = \{\{x: T_A(x), I_A(x), F_A(x)\}, x \in U\}$ where the functions $T, I, F: U \rightarrow [0, 1]$ respectively the degree of membership, the degree of indeterminacy and degree of non-membership of the element $x \in U$ to the set $A$ with the condition:

$$0^- \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+$$

Definition 2.2 [4] Let $U$ be a universe of discourse. The single valued neutrosophic set $A$ is an object having the form $A = \{\{x: T_A(x), I_A(x), F_A(x)\}, x \in U\}$ where the functions $T, I, F: U \rightarrow [0, 1]$ respectively degree of membership, the degree of indeterminacy and degree of non-membership of the element $x \in U$ to the set $A$ with the condition:

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$$

For convenience we can simply use $x = (T, I, F)$ to represent an element $x$ in SVNS, and element $x$ can be called a single valued neutrosophic number.

Definition 2.3 [4] A single valued neutrosophic set $A$ is equal to another single valued neutrosophic set $B$, $A = B$ if $\forall x \in U$,

$$T_A(x) = T_B(x), I_A(x) = I_B(x), F_A(x) = F_B(x).$$

Definition 2.4[4] A single valued neutrosophic set $A$ is contained in another single valued neutrosophic set $B$, $A \subseteq B$ if $\forall x \in U$,

$$T_A(x) \leq T_B(x), I_A(x) \leq I_B(x), F_A(x) \geq F_B(x).$$

Definition 2.5[16] (Axiom of similarity measure)

A mapping $S(A, B): NS(x) \times NS(x) \rightarrow [0, 1]$, where $NS(x)$ denotes the set of all NS $\{x_1, \ldots, x_n\}$ is said to be the degree of similarity between $A$ and $B$ if it satisfies the following conditions:

1. $sP_1) \ 0 \leq S(A, B) \leq 1$
2. $sp_2) S(A, B) = 1$ if and only if $A = B, \forall A, B \in NS$
3. $sP_3) S(A, B) = S(B, A)$
4. $sp_4) \text{If } A \subseteq B \subseteq C \text{ for all } A, B, C \in NS, \text{ then } S(A, B) \geq S(A, C) \text{ and } S(B, C) \geq S(A, C)$.
3 The Transformation Techniques between Single Valued Neutrosophic Numbers

In this section, we propose transformation techniques between a single valued neutrosophic number \((T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)})\) and a single valued neutrosophic number \(C_{A(x_i)}\). Here \((T_{A(x)}, I_{A(x)}, F_{A(x)})\) denote the single valued neutrosophic numbers to represent an element \(x_i\) in the single valued neutrosophic set \(A\), and \(C_{A(x)}\) is the center of a triangle (SLK) which was obtained by the transformation on the three-dimensional \(Z - Y - M\) plane.

First we transform single valued neutrosophic numbers according to their distinct \(T_A\), \(I_A\), \(F_A\) values in three parts.

3.1 Transformation According to the Truth Value

In this section, we group the single valued neutrosophic numbers after the examination of their truth values \(T_A\)’s greatness or smallness against \(I_A\) and \(F_A\) values. We will shift the \(T_{A(x_i)}\) and \(F_{A(x_i)}\) values on the \(Z\) – axis and \(T_{A(x_i)}\) and \(I_{A(x_i)}\) values on the \(Y\) – axis onto each other. We take the \(F_{A(x_i)}\) value on the \(M\) – axis. The shifting on the \(Z\) and \(Y\) planes are made such that we shift the smaller value to the difference of the greater value and 2, as shown in the below figures.

1. First Group

For the single valued neutrosophic numbers \((T_{A(x)}, I_{A(x)}, F_{A(x)})\), if

\[T_{A(x_i)} \leq F_{A(x_i)}\]

and

\[T_{A(x_i)} \leq I_{A(x_i)},\]

as shown in the figure below, we transformed \((T_{A(x)}, I_{A(x)}, F_{A(x)})\) into the single valued neutrosophic number \(C_{A(x)}\), the center of the SKL triangle, where

\[S_{A(x)} = (T_{A(x)}, T_{A(x)}, F_{A(x)})\]

\[K_{A(x)} = (2 - F_{A(x)}, T_{A(x)}, F_{A(x)})\]

\[L_{A(x)} = (T_{A(x)}, 2 - I_{A(x)}, F_{A(x)})\].

Here, as

\[T_{C_{A(x)}} = T_{A(x)} + \frac{(2 - F_{A(x)} - T_{A(x)})}{3} = \frac{2 - F_{A(x)} + 2 T_{A(x)}}{3}\]

\[I_{C_{A(x)}} = T_{A(x)} + \frac{(2 - I_{A(x)} - T_{A(x)})}{3} = \frac{2 - I_{A(x)} + 2 T_{A(x)}}{3}\]

we have

\[C_{A(x)} = \left(\frac{2 - F_{A(x)} + 2 T_{A(x)}}{3}, \frac{2 - I_{A(x)} + 2 T_{A(x)}}{3}, F_{A(x)}\right)\].

2. Second Group

For the single valued neutrosophic numbers \((T_{A(x)}, I_{A(x)}, F_{A(x)})\), if

\[T_{A(x)} \geq F_{A(x)}\]

and

\[T_{A(x)} \geq I_{A(x)}\],

as shown in the figure below, we transformed \((T_{A(x)}, I_{A(x)}, F_{A(x)})\) into the single valued neutrosophic number \(C_{A(x)}\), the center of the SKL triangle, where

\[S_{A(x)} = (T_{A(x)}, T_{A(x)}, F_{A(x)})\]

\[K_{A(x)} = (2 - F_{A(x)}, T_{A(x)}, F_{A(x)})\]

\[L_{A(x)} = (T_{A(x)}, 2 - I_{A(x)}, F_{A(x)})\].
3. Third Group

For the single valued neutrosophic numbers \((T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)})\), if \(I_{A(x_i)} \leq T_{A(x_i)} \leq F_{A(x_i)}\), as shown in the figure below, we transformed \((T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)})\) into the single valued neutrosophic number \(C_{A(x_i)}\), the center of the SKL triangle, where

\[ S_{A(x_i)} = (F_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)}) \]
\[ L_{A(x_i)} = (F_{A(x_i)}, 2 - T_{A(x_i)}, F_{A(x_i)}) \]
\[ K_{A(x_i)} = (2 - T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)}) \]

Here, as

\[ T_{CA(x_i)} = F_{A(x_i)} + \frac{(2 - T_{A(x_i)} - F_{A(x_i)})}{3} = \frac{2 - T_{A(x_i)} + 2 F_{A(x_i)}}{3} \]
\[ I_{CA(x_i)} = I_{A(x_i)} + \frac{(2 - T_{A(x_i)} - I_{A(x_i)})}{3} = \frac{2 - T_{A(x_i)} + 2 I_{A(x_i)}}{3} \]
\[ F_{CA(x_i)} = F_{A(x_i)} \]

we have

\[ C_{A(x_i)} = \left(\frac{2 - T_{A(x_i)} + 2 F_{A(x_i)}}{3}, \frac{2 - T_{A(x_i)} + 2 I_{A(x_i)}}{3}, F_{A(x_i)}\right) \]

4. Fourth Group

For the single valued neutrosophic numbers \((T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)})\), if \(F_{A(x_i)} \leq T_{A(x_i)} \leq I_{A(x_i)}\), as shown in the figure below, we transformed \((T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)})\) into the single valued neutrosophic number \(C_{A(x_i)}\), the center of the SKL triangle, where

\[ S_{A(x_i)} = (T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)}) \]
\[ L_{A(x_i)} = (T_{A(x_i)}, 2 - T_{A(x_i)}, F_{A(x_i)}) \]
\[ K_{A(x_i)} = (2 - F_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)}) \]

Here, as

\[ T_{CA(x_i)} = T_{A(x_i)} + \frac{(2 - F_{A(x_i)} - T_{A(x_i)})}{3} = \frac{2 - F_{A(x_i)} + 2 T_{A(x_i)}}{3} \]
\[ I_{CA(x_i)} = I_{A(x_i)} + \frac{(2 - T_{A(x_i)} - I_{A(x_i)})}{3} = \frac{2 - T_{A(x_i)} + 2 I_{A(x_i)}}{3} \]
\[ F_{CA(x_i)} = F_{A(x_i)} \]

we have

\[ C_{A(x_i)} = \left(\frac{2 - F_{A(x_i)} + 2 T_{A(x_i)}}{3}, \frac{2 - T_{A(x_i)} + 2 I_{A(x_i)}}{3}, F_{A(x_i)}\right) \]
the figure below, we transformed \((T_{A(x)}), I_{A(x)}, F_{A(x)}\) into the single valued neutrosophic number \(C_{A(x)}\), the center of the SKL triangle, where

\[
S_{A(x)} = \left( F_{A(x)}, T_{A(x)}, F_{A(x)} \right) \\
L_{A(x)} = \left( F_{A(x)}, 2 - T_{A(x)}, F_{A(x)} \right) \\
K_{A(x)} = \left( 2 - T_{A(x)}, T_{A(x)}, F_{A(x)} \right) .
\]

Example 3.1.1 Transform the following single valued neutrosophic numbers according to their truth values.

\[
\langle 0.2, 0.5, 0.7 \rangle, \langle 0.9, 0.4, 0.5 \rangle, \langle 0.3, 0.2, 0.5 \rangle, \langle 0.3, 0.2, 0.4 \rangle .
\]

i. \(\langle 0.2, 0.5, 0.7 \rangle\) single valued neutrosophic number belongs to the first group.

The center is calculated by the formula, 

\[
C_{A(x)} = \left( \frac{2 - F_{A(x)} + 2T_{A(x)}}{3}, \frac{2 - I_{A(x)} + 2T_{A(x)}}{3}, \frac{2 - T_{A(x)} + 2I_{A(x)}}{3}, F_{A(x)} \right)
\]

and we have \(C_{A(x)} = (0.566, 0.633, 0.7)\).

ii. \(\langle 0.9, 0.4, 0.5 \rangle\) single valued neutrosophic number is in the second group.

The center for the values of the second group is, 

\[
C_{A(x)} = \left( \frac{2 - T_{A(x)} + 2F_{A(x)}}{3}, \frac{2 - I_{A(x)} + 2T_{A(x)}}{3}, \frac{2 - T_{A(x)} + 2I_{A(x)}}{3}, F_{A(x)} \right)
\]

and for \(\langle 0.9, 0.4, 0.5 \rangle, C_{A(x)} = (0.7, 0.633, 0.5)\).

iii. \(\langle 0.3, 0.2, 0.5 \rangle\) single valued neutrosophic number belongs to the third group.

The formula for the center of \(\langle 0.3, 0.2, 0.5 \rangle\) is

\[
C_{A(x)} = \left( \frac{2 - F_{A(x)} + 2T_{A(x)}}{3}, \frac{2 - I_{A(x)} + 2T_{A(x)}}{3}, \frac{2 - T_{A(x)} + 2I_{A(x)}}{3}, F_{A(x)} \right)
\]

and therefore we have \(C_{A(x)} = (0.7, 0.7, 0.5)\).

iv. \(\langle 0.3, 0.2, 0.4 \rangle\) single valued neutrosophic number is in the third group and the center is calculated to be \(C_{A(x)} = (0.733, 0.7, 0.4)\).

Corollary 3.1.2 The corners of the triangles obtained using the above method need not be single valued neutrosophic number but by definition, trivially their centers are.

Note 3.1.3 As for the single valued neutrosophic number \(\langle 1, 1, 1 \rangle\) there does not exist any transformable triangle in the above four groups, we take its transformation equal to itself.

Corollary 3.1.4 If \(F_{A(x)} = T_{A(x)} = I_{A(x)}\) the transformation gives the same center in all four groups. Also, if \(T_{A(x)} = I_{A(x)} \leq F_{A(x)}\), then the center in the first group is equal to the one in the third group and if \(F_{A(x)} \leq T_{A(x)} = I_{A(x)}\), the center in the second group is equal to the center in the fourth group. Similarly, if \(T_{A(x)} = F_{A(x)} \leq I_{A(x)}\), then the center in the first group is equal to the center in the fourth group and if \(I_{A(x)} \leq T_{A(x)} = F_{A(x)}\), the center in the second group is equal to the one in the third group.

3.2 Transformation According to the Indeterminancy Value

In this section, we group the single valued neutrosophic numbers according to their indeterminacy values, \(I_{A(x)}\)’s greatness or smallness against \(T_{A(x)}\) and \(F_{A(x)}\) values. We will shift the \(I_{A(x)}\) and \(F_{A(x)}\) values on the \(Z\) – axis and \(T_{A(x)}\) and \(I_{A(x)}\) values on the \(Y\) – axis onto each other. We take the \(F_{A(x)}\) value on the \(M\) – axis. The shifting on the \(Z\) and \(Y\) planes are made such that we shift the smaller value to the difference of the greater value and 2, as shown in the below figures.

1. First Group

For the single valued neutrosophic numbers \(\langle T_{A(x)}, I_{A(x)}, F_{A(x)} \rangle\), if

\[
\begin{align*}
&\langle 0.2, 0.5, 0.7 \rangle, \langle 0.9, 0.4, 0.5 \rangle, \langle 0.3, 0.2, 0.5 \rangle, \langle 0.3, 0.2, 0.4 \rangle .
\end{align*}
\]
\[ I_A(x_i) \leq F_A(x_i) \]

and

\[ I_A(x_i) \leq F_A(x_i) , \]

as shown in the figure below, we transformed \( \langle T_A(x_i), I_A(x_i), F_A(x_i) \rangle \) into the single valued neutrosophic number \( C_A(x_i) \), the center of the SKL triangle, where

\[
S_{(A(x_i))} = (I_A(x_i), I_A(x_i), F_A(x_i)) \\
K_{(A(x_i))} = (2 - F_A(x_i), I_A(x_i), F_A(x_i)) \\
L_{(A(x_i))} = (I_A(x_i), 2 - T_A(x_i), F_A(x_i)) .
\]

We transformed the single valued neutrosophic number \( \langle T_A(x_i), I_A(x_i), F_A(x_i) \rangle \) into the center of the SKL triangle, namely \( C_A(x_i) \). Here, as

\[
T_{C_A(x_i)} = I_A(x_i) + \frac{2 - F_A(x_i) - I_A(x_i)}{3} = \frac{2 - F_A(x_i) + 2 I_A(x_i)}{3} \\
I_{C_A(x_i)} = T_A(x_i) + \frac{2 - T_A(x_i) - I_A(x_i)}{3} = \frac{2 - T_A(x_i) + 2 I_A(x_i)}{3} \\
and \\
F_{C_A(x_i)} = F_A(x_i) ,
\]

we have

\[
C_A(x_i) = \left( \frac{2 - F_A(x_i) + 2 I_A(x_i)}{3}, \frac{2 - T_A(x_i) + 2 I_A(x_i)}{3}, F_A(x_i) \right) .
\]

\[ 2. \text{ Second Group} \]

For the single valued neutrosophic numbers \( \langle T_A(x_i), I_A(x_i), F_A(x_i) \rangle \), if

\[ I_A(x_i) \geq F_A(x_i) \]

and

\[ I_A(x_i) \geq F_A(x_i) , \]

as shown in the figure below, we transformed \( \langle T_A(x_i), I_A(x_i), F_A(x_i) \rangle \) into the single valued neutrosophic number \( C_A(x_i) \), the center of the SKL triangle, where

\[
S_{(A(x_i))} = (F_A(x_i), T_A(x_i), F_A(x_i)) \\
K_{(A(x_i))} = (F_A(x_i), 2 - I_A(x_i), F_A(x_i)) \\
L_{(A(x_i))} = (2 - I_A(x_i), T_A(x_i), F_A(x_i)) .
\]

Here, as

\[
T_{C_A(x_i)} = F_A(x_i) + \frac{2 - I_A(x_i) - F_A(x_i)}{3} = \frac{2 - I_A(x_i) + 2 F_A(x_i)}{3} \\
I_{C_A(x_i)} = 2 - T_A(x_i) + \frac{2 - T_A(x_i) - I_A(x_i)}{3} = \frac{2 - T_A(x_i) + 2 I_A(x_i)}{3} \\
and \\
F_{C_A(x_i)} = F_A(x_i) .
\]
\[ I_{CA(x_i)} = T_{A(x_i)} + \frac{(2 - I_{A(x_i)} - T_{A(x_i)})}{3} = \frac{2 - I_{A(x_i)} + 2 T_{A(x_i)}}{3} \]

and

\[ F_{CA(x_i)} = F_{A(x_i)} \]

we have

\[ C_{A(x_i)} = \left( \frac{2 - I_{A(x_i)} + 2 F_{A(x_i)}}{3}, \frac{2 - I_{A(x_i)} + 2 T_{A(x_i)}}{3}, F_{A(x_i)} \right) \]

3. Third Group

For the single valued neutrosophic number \( \langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)} \rangle \), if \( T_{A(x_i)} \leq I_{A(x_i)} \leq F_{A(x_i)} \), as shown in the figure below, we transformed \( \langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)} \rangle \) into the single valued neutrosophic number \( C_{A(x_i)} \), the center of the SKL triangle, where

\[ S_{(A(x_i))} = \left( I_{A(x_i)}, T_{A(x_i)}, F_{A(x_i)} \right) \]

\[ K_{(A(x_i))} = \left( I_{A(x_i)}, 2 - I_{A(x_i)}, F_{A(x_i)} \right) \]

\[ L_{(A(x_i))} = \left( 2 - F_{A(x_i)}, T_{A(x_i)}, F_{A(x_i)} \right) \]

Here as

\[ T_{CA(x_i)} = I_{A(x_i)} + \frac{(2 - F_{A(x_i)} - I_{A(x_i)})}{3} = \frac{2 - F_{A(x_i)} + 2 I_{A(x_i)}}{3} \]

\[ I_{CA(x_i)} = T_{A(x_i)} + \frac{(2 - I_{A(x_i)} - T_{A(x_i)})}{3} = \frac{2 - I_{A(x_i)} + 2 T_{A(x_i)}}{3} \]

and

\[ F_{CA(x_i)} = F_{A(x_i)} \]

we have

\[ C_{A(x_i)} = \left( \frac{2 - F_{A(x_i)} + 2 I_{A(x_i)}}{3}, \frac{2 - I_{A(x_i)} + 2 T_{A(x_i)}}{3}, F_{A(x_i)} \right) \]

4. Fourth Group

For the single valued neutrosophic numbers \( \langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)} \rangle \), if \( F_{A(x_i)} \leq I_{A(x_i)} \leq T_{A(x_i)} \), as shown in the figure below, we transformed \( \langle T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)} \rangle \) into the single valued neutrosophic numbers \( C_{A(x_i)} \), the center of the SKL triangle, where

\[ S_{(A(x_i))} = \left( F_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)} \right) \]

\[ K_{(A(x_i))} = \left( F_{A(x_i)}, 2 - T_{A(x_i)}, F_{A(x_i)} \right) \]

\[ L_{(A(x_i))} = \left( 2 - I_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)} \right) \]
Here, as
\[ T_{c_A(x_i)} = F_{A(x_i)} + \frac{2 - I_{A(x_i)} - F_{A(x_i)}}{3} \]
\[ = \frac{2 - I_{A(x_i)} + 2 F_{A(x_i)}}{3} \]
\[ I_{c_A(x_i)} = I_{A(x_i)} + \frac{2 - T_{A(x_i)} - I_{A(x_i)}}{3} \]
\[ = \frac{2 - T_{A(x_i)} + 2 I_{A(x_i)}}{3} \]
and
\[ F_{c_A(x_i)} = F_{A(x_i)} \]
we have
\[ C_{A(x_i)} = \left( \frac{2 - I_{A(x_i)} + 2 F_{A(x_i)}}{3}, \frac{2 - T_{A(x_i)} + 2 I_{A(x_i)}}{3}, F_{A(x_i)} \right). \]

**Example 3.2.1**: Transform the single neutrosophic numbers of Example 3.1.3.
(0.2, 0.5, 0.7), (0.9, 0.4, 0.5), (0.3, 0.2, 0.5), (0.3, 0.2, 0.4) according to their indeterminacy values.

i. (0.2, 0.5, 0.7) single valued neutrosophic number is in the third group. The center is given by the formula
\[ C_{A(x_i)} = \left( \frac{2 - F_{A(x_i)} + 2 I_{A(x_i)}}{3}, \frac{2 - T_{A(x_i)} + 2 I_{A(x_i)}}{3}, F_{A(x_i)} \right). \]
and so \( C_{A(x)} = (0.766, 0.633, 0.7) \).

ii. (0.9, 0.4, 0.5) single valued neutrosophic number is in the first group.

By
\[ C_{A(x_i)} = \left( \frac{2 - F_{A(x_i)} + 2 I_{A(x_i)}}{3}, \frac{2 - T_{A(x_i)} + 2 I_{A(x_i)}}{3}, F_{A(x_i)} \right), \]
we have \( C_{A(x)} = (0.733, 0.633, 0.5) \).

iii. (0.3, 0.2, 0.5) single valued neutrosophic number belongs to the first group and the center is
\[ C_{A(x_i)} = \left( \frac{2 - I_{A(x_i)} + 2 F_{A(x_i)}}{3}, \frac{2 - T_{A(x_i)} + 2 I_{A(x_i)}}{3}, F_{A(x_i)} \right). \]
so, \( C_{A(x)} = (0.633, 0.9, 0.5) \).

iv. (0.3, 0.2, 0.4) single valued neutrosophic number is in the first group.
Using
\[ C_{A(x_i)} = \left( \frac{2 - F_{A(x_i)} + 2 I_{A(x_i)}}{3}, \frac{2 - T_{A(x_i)} + 2 I_{A(x_i)}}{3}, F_{A(x_i)} \right), \]
we have \( C_{A(x)} = (0.666, 0.7, 0.4) \).

**Corollary 3.2.2** The corners of the triangles obtained using the above method need not be single valued neutrosophic numbers but by definition, trivially their centers are.

**Note 3.2.3** As for the single valued neutrosophic number \((1, 1, 1)\) there does not exist any transformable triangle in the above four groups, we take its transformation equal to itself.

**Corollary 3.2.4** If \( F_{A(x_i)} = T_{A(x_i)} = I_{A(x_i)} \), the transformation gives the same center in all four groups. Also if \( T_{A(x_i)} = l_{A(x_i)} \leq F_{A(x_i)} \), then the center in the first group is equal to the center in the third group, and if \( F_{A(x_i)} \leq T_{A(x_i)} = l_{A(x_i)} \), then the center in the second group is equal to the center in the fourth group. Similarly, if \( F_{A(x_i)} = I_{A(x_i)} \leq T_{A(x_i)} \), then the center in the first group is equal to the center in the fourth and in the case that \( T_{A(x_i)} \leq F_{A(x_i)} = I_{A(x_i)} \), the center in the second group is equal to the center in the third.

### 3.3 Transformation According to the Falsity Value

In this section, we group the single valued neutrosophic numbers after the examination of their indeterminacy values \( F_A \)'s greatness or smallness against \( I_A \) and \( T_A \) values. We will shift the \( I_{A(x_i)} \) and \( F_{A(x_i)} \) values on the \( Z \) – axis and \( T_{A(x_i)} \) and \( F_{A(x_i)} \) values on the \( Y \) – axis onto each other. We take the \( F_{A(x_i)} \) value on the \( M \) – axis. The shifting on the \( Z \) and \( Y \) planes are made such that we shift the smaller value to the difference of the greater value and 2, as shown in the below figures.
1. First Group

For the single valued neutrosophic numbers
\(\langle T_A(x_i), I_A(x_i), F_A(x_i) \rangle\), if
\[ F_A(x_i) \leq T_A(x_i) \]
and
\[ F_A(x_i) \leq I_A(x_i) \]
then
as shown in the figure below, we transformed
\(\langle T_A(x_i), I_A(x_i), F_A(x_i) \rangle\)
into the single valued neutrosophic number \(C_A(x_i)\), the center of the SKL triangle, where
\[
S_{(A(x_i))} = (F_A(x_i), F_A(x_i), F_A(x_i))
\]
\[
K_{(A(x_i))} = (2 - I_A(x_i), F_A(x_i), F_A(x_i))
\]
\[
L_{(A(x_i))} = (F_A(x_i), 2 - T_A(x_i), F_A(x_i))
\]
Here, as
\[
T_{C_A(x_i)} = F_A(x_i) + \frac{(2 - I_A(x_i) - F_A(x_i))}{3} = \frac{2 - I_A(x_i) + 2 F_A(x_i)}{3}
\]
\[
I_{C_A(x_i)} = F_A(x_i) + \frac{(2 - T_A(x_i) - F_A(x_i))}{3} = \frac{2 - T_A(x_i) + 2 F_A(x_i)}{3}
\]
and
\[
F_{C_A(x_i)} = F_A(x_i)
\]
we get
\[
C_A(x_i) = \left( \frac{2 - I_A(x_i) + 2 F_A(x_i)}{3}, \frac{2 - T_A(x_i) + 2 F_A(x_i)}{3}, F_A(x_i) \right).
\]

2. Second Group

For the single valued neutrosophic numbers
\(\langle T_A(x_i), I_A(x_i), F_A(x_i) \rangle\), if
\[ F_A(x_i) \geq T_A(x_i) \]
and
\[ F_A(x_i) \geq I_A(x_i) \]
then
as shown in the figure below, we transformed
\(\langle T_A(x_i), I_A(x_i), F_A(x_i) \rangle\)
into the single valued neutrosophic numbers \(C_A(x_i)\), the center of the SKL triangle, where
\[
S_{(A(x_i))} = (I_A(x_i), T_A(x_i), F_A(x_i))
\]
\[
K_{(A(x_i))} = (I_A(x_i), 2 - F_A(x_i), F_A(x_i))
\]
\[
L_{(A(x_i))} = (2 - F_A(x_i), T_A(x_i), F_A(x_i))
\]
Here, as
\[
T_{C_A(x_i)} = I_A(x_i) + \frac{(2 - F_A(x_i) - I_A(x_i))}{3} = \frac{2 - F_A(x_i) + 2 I_A(x_i)}{3}
\]
\[
I_{C_A(x_i)} = I_A(x_i) + \frac{(2 - T_A(x_i) - F_A(x_i))}{3} = \frac{2 - T_A(x_i) + 2 F_A(x_i)}{3}
\]
\[ I_{C_A(x_i)} = T_{A(x_i)} + \left(\frac{2 - F_{A(x_i)} - T_{A(x_i)}}{3}\right) = \frac{2 - F_{A(x_i)} + 2 T_{A(x_i)}}{3} \]

and
\[ F_{C_A(x_i)} = F_{A(x_i)} \]

we have
\[ C_{A(x_i)} = \left(\frac{2 - F_{A(x_i)} + 2 T_{A(x_i)}}{3}, 2 - F_{A(x_i)} + 2 T_{A(x_i)}, F_{A(x_i)}\right) \].

3. Third Group

For the single valued neutrosophic numbers \( (T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)}) \), if \( I_{A(x_i)} \leq F_{A(x_i)} \leq T_{A(x_i)} \) then as shown in the figure below, we transformed \( (T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)}) \) into the single valued neutrosophic numbers \( C_{A(x_i)} \), the center of the SKL triangle, where
\[ S_{(A(x_i))} = \left( I_{A(x_i)}, F_{A(x_i)}, F_{A(x_i)} \right) \]
\[ K_{(A(x_i))} = \left( I_{A(x_i)}, 2 - T_{A(x_i)}, F_{A(x_i)} \right) \]
\[ L_{(A(x_i))} = \left( 2 - F_{A(x_i)}, F_{A(x_i)}, F_{A(x_i)} \right) \].

Here, as
\[ T_{C_A(x_i)} = \frac{I_{A(x_i)} + \left(\frac{2 - F_{A(x_i)} - I_{A(x_i)}}{3}\right)}{3} = \frac{2 - F_{A(x_i)} + 2 I_{A(x_i)}}{3} \]
\[ I_{C_A(x_i)} = \frac{F_{A(x_i)} + \left(\frac{2 - T_{A(x_i)} - F_{A(x_i)}}{3}\right)}{3} = \frac{2 - T_{A(x_i)} + 2 F_{A(x_i)}}{3} \]
and
\[ F_{C_A(x_i)} = F_{A(x_i)} \]

we have
\[ C_{A(x_i)} = \left(\frac{2 - F_{A(x_i)} + 2 I_{A(x_i)}}{3}, \frac{2 - T_{A(x_i)} + 2 F_{A(x_i)}}{3}, F_{A(x_i)}\right) \].

4. Fourth Group

For the single valued neutrosophic numbers \( (T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)}) \), if \( T_{A(x_i)} \leq F_{A(x_i)} \leq I_{A(x_i)} \), then as shown in the figure below, we transformed \( (T_{A(x_i)}, I_{A(x_i)}, F_{A(x_i)}) \) into the single valued neutrosophic numbers \( C_{A(x_i)} \), the center of the SKL triangle, where
\[ S_{(A(x_i))} = \left( F_{A(x_i)}, T_{A(x_i)}, F_{A(x_i)} \right) \]
\[ K_{(A(x_i))} = \left( F_{A(x_i)}, 2 - F_{A(x_i)}, F_{A(x_i)} \right) \]
\[ L_{(A(x_i))} = \left( 2 - I_{A(x_i)}, T_{A(x_i)}, F_{A(x_i)} \right) \].

Example 3.3.1: Transform the single neutrosophic numbers of Example 3.1.3.
(0.2, 0.5, 0.7), (0.9, 0.4, 0.5), (0.3, 0.2, 0.5), (0.3, 0.2, 0.4) according to their falsity values.

i. (0.2, 0.5, 0.7) single valued neutrosophic number belongs to the second group. So, the center is

\[ C_{A(x)} = \left( \frac{2 - F_{A(x)}}{3}, \frac{2 - F_{A(x)} + 2 T_{A(x)}}{3}, \frac{2 - F_{A(x)} + 2 T_{A(x)}}{3}, \frac{2 - F_{A(x)} + 2 T_{A(x)}}{3}, \frac{2 - F_{A(x)} + 2 T_{A(x)}}{3}, F_{A(x)} \right), \]

and we get \( C_{A(x)} = (0.766, 0.7, 0.7). \)

ii. (0.9, 0.4, 0.5) single valued neutrosophic number is in the third group. Using the formula

\[ C_{A(x)} = \left( \frac{2 - F_{A(x)}}{3}, \frac{2 - F_{A(x)} + 2 T_{A(x)}}{3}, \frac{2 - F_{A(x)} + 2 T_{A(x)}}{3}, \frac{2 - F_{A(x)} + 2 T_{A(x)}}{3}, \frac{2 - F_{A(x)} + 2 T_{A(x)}}{3}, F_{A(x)} \right), \]

we see that \( C_{A(x)} = (0.633, 0.7, 0.5). \)

iii. (0.3, 0.2, 0.5) single valued neutrosophic number is in the second group.

\[ C_{A(x)} = \left( \frac{2 - F_{A(x)}}{3}, \frac{2 - F_{A(x)} + 2 T_{A(x)}}{3}, \frac{2 - F_{A(x)} + 2 T_{A(x)}}{3}, \frac{2 - F_{A(x)} + 2 T_{A(x)}}{3}, \frac{2 - F_{A(x)} + 2 T_{A(x)}}{3}, F_{A(x)} \right), \]

the center of the triangle is \( C_{A(x)} = (0.633, 0.7, 0.5) \).

iv. (0.3, 0.2, 0.4) single valued neutrosophic number belongs to the second group.

\[ C_{A(x)} = \left( \frac{2 - F_{A(x)}}{3}, \frac{2 - F_{A(x)} + 2 T_{A(x)}}{3}, \frac{2 - F_{A(x)} + 2 T_{A(x)}}{3}, \frac{2 - F_{A(x)} + 2 T_{A(x)}}{3}, \frac{2 - F_{A(x)} + 2 T_{A(x)}}{3}, F_{A(x)} \right), \]

and so we have \( C_{A(x)} = (0.666, 0.733, 0.4). \)

Corollary 3.3.2 The corners of the triangles obtained using the above method need not be single valued neutrosophic numbers but by definition, trivially their centers are single valued neutrosophic values.

Note 3.3.3 As for the single valued neutrosophic number (1, 1, 1) there does not exist any transformable triangle in the above four groups, we take its transformation equal to itself.

Corollary 3.3.4 If \( F_{A(x)} = T_{A(x)} = I_{A(x)} \), the transformation gives the same center in all four groups. Also, if \( T_{A(x)} = F_{A(x)} \leq I_{A(x)} \), then the center in the first group is equal to the one in the fourth group, and if \( I_{A(x)} \leq T_{A(x)} = F_{A(x)} \), then the center in the second group is the same as the center in the third. Similarly, if \( I_{A(x)} = \frac{F_{A(x)} \leq T_{A(x)}}{4} \), then the centers in the first and third groups are same and lastly, if \( T_{A(x)} \leq I_{A(x)} = F_{A(x)} \), then the center in the second group is equal to the one in the fourth group.

4. A New Similarity Measure Based on Falsity Value Between Single Valued Neutrosophic Sets

In this section, we propose a new similarity measure based on falsity value between single valued neutrosophic sets.

Definition 4.1 Let \( A \) and \( B \) two single valued neutrosophic sets in \( x = \{x_1, x_2, ..., x_n\} \).

\[ A = \{(x, T_{A(x)}, I_{A(x)}, F_{A(x)})\} \]

and

\[ B = \{(x, T_{B(x)}, I_{B(x)}, F_{B(x)})\}. \]

The similarity measure based on falsity value between the neutrosophic numbers \( A(x) \) and \( B(x) \) is given by

\[ S(A(x), B(x)) = 1 - \left( \frac{2(F_{A(x)} - F_{B(x)}) - (T_{A(x)} - T_{B(x)})}{9} \right) \]

\[ + \frac{2(F_{A(x)} - F_{B(x)}) - (I_{A(x)} - I_{B(x)})}{9} \]

\[ + \frac{3(F_{A(x)} - F_{B(x)})}{9}. \]

Here, we use the values

\[ 2(F_{A(x)} - F_{B(x)}) - (T_{A(x)} - T_{B(x)}), \]

\[ 2(F_{A(x)} - F_{B(x)}) - (I_{A(x)} - I_{B(x)}), \]

\[ 2(F_{A(x)} - F_{B(x)}) + (F_{A(x)} - F_{B(x)}) \]

\[ = 3(F_{A(x)} - F_{B(x)}). \]

Since we use the falsity values \( F_{A(x)} \) in all these three values, we name this formula as “similarity measure based on falsity value between single valued neutrosophic numbers”.

Property 4.2 \( 0 \leq S(A(x), B(x)) \leq 1 \).
Proof: By the definition of Single valued neutrosophic numbers, as

\[ 0 \leq T_{A(x_i)} T_{B(x_i)}, I_{A(x_i)}, I_{B(x_i)}, F_{A(x_i)}, F_{B(x_i)} \leq 1, \]

we have

\[ 0 \leq 2(F_{A(x_i)} - F_{B(x_i)}) - (T_{A(x_i)} + T_{B(x_i)}) \leq 3 \]

\[ 0 \leq 2(F_{A(x_i)} - F_{B(x_i)}) - (I_{A(x_i)} + I_{B(x_i)}) \leq 3 \]

and

\[ 0 \leq 3(F_{A(x_i)}, F_{B(x_i)}) \leq 3. \]

So,

\[ 0 \leq 1 - \left( \frac{2(F_{A(x_i)} - F_{B(x_i)}) - (T_{A(x_i)} - T_{B(x_i)})}{9} \right) \]

\[ + \frac{2(F_{A(x_i)} - F_{B(x_i)}) - (I_{A(x_i)} - I_{B(x_i)})}{9} \]

\[ + \frac{3(F_{A(x_i)} - F_{B(x_i)})}{9} \leq 1. \]

Therefore, \( 0 \leq S(A(x_i), B(x_i)) \leq 1. \)

Property 4.3: \( S(A(x_i), B(x_i)) = 1 \iff A(x_i) = B(x_i) \)

Proof. i) First we show \( A(x_i) = B(x_i) \) when

\( S(A(x_i), B(x_i)) = 1. \)

Let \( (A(x_i), B(x_i)) = 1. \)

\[ S(A(x_i), B(x_i)) = 1 - \left( \frac{2(F_{A(x_i)} - F_{B(x_i)}) - (T_{A(x_i)} - T_{B(x_i)})}{9} \right) \]

\[ + \frac{2(F_{A(x_i)} - F_{B(x_i)}) - (I_{A(x_i)} - I_{B(x_i)})}{9} \]

\[ + \frac{3(F_{A(x_i)} - F_{B(x_i)})}{9} \]

\[ = 1 \]

and thus,

\[ \left( \frac{2(F_{A(x_i)} - F_{B(x_i)}) - (T_{A(x_i)} - T_{B(x_i)})}{9} \right) \]

\[ + \frac{2(F_{A(x_i)} - F_{B(x_i)}) - (I_{A(x_i)} - I_{B(x_i)})}{9} \]

\[ + \frac{3(F_{A(x_i)} - F_{B(x_i)})}{9} \]

\[ = 0 \]

So,

\[ |(F_{A(x_i)} - F_{B(x_i)})| = 0, \]

\[ |(2(F_{A(x_i)} - F_{B(x_i)}) - (T_{A(x_i)} - T_{B(x_i)})| = 0, \]

and

\[ |(2(F_{A(x_i)} - F_{B(x_i)}) - (I_{A(x_i)} - I_{B(x_i)})| = 0. \]

As \( |(F_{A(x_i)} - F_{B(x_i)})| = 0, \) then \( F_{A(x_i)} = F_{B(x_i)}. \)

If \( F_{A(x_i)} = F_{B(x_i)}, \)

\[ |(2(F_{A(x_i)} - F_{B(x_i)}) - (T_{A(x_i)} - T_{B(x_i)})| = 0 \]

and

\[ T_{A(x_i)} = T_{B(x_i)} . \]

When \( F_{A(x_i)} = F_{B(x_i)}, \)

\[ |(2(F_{A(x_i)} - F_{B(x_i)}) - (I_{A(x_i)} - I_{B(x_i)})| = 0 \]

and

\[ I_{A(x_i)} = I_{B(x_i)}. \]

Therefore, \( A(x_i) = B(x_i). \)

ii) Now we show \( A(x_i) = B(x_i), \) then \( S(A(x_i), B(x_i)) = 1. \)

Let \( A(x_i) = B(x_i). \) By Definition 2.3,

\[ T_{A(x_i)} = T_{B(x_i)}, I_{A(x_i)} = I_{B(x_i)}, F_{A(x_i)} = F_{B(x_i)} \]

and we have

\[ T_{A(x_i)} - T_{B(x_i)} = 0, I_{A(x_i)} - I_{B(x_i)} = 0, F_{A(x_i)} - F_{B(x_i)} = 0. \]

So,
Property 4.5: If $A \subseteq B \subseteq C$,

\begin{align*}
S(A(x_i), B(x_i)) & = 1 - \left( \frac{2(F_{A(x_i)} - F_{B(x_i)}) - (T_{A(x_i)} - T_{B(x_i)})}{9} \right) \\
& \quad + \frac{2(F_{A(x_i)} - F_{B(x_i)}) - (I_{A(x_i)} - I_{B(x_i)})}{9} \\
& \quad + \frac{3|F_{A(x_i)} - F_{B(x_i)}|}{9} \\
& = 1 - \left( \frac{2(-F_{A(x_i)} - F_{B(x_i)}) - (-T_{A(x_i)} - T_{B(x_i)})}{9} \right) \\
& \quad + \frac{2(-F_{A(x_i)} - F_{B(x_i)}) - (-I_{A(x_i)} - I_{B(x_i)})}{9} \\
& \quad + \frac{3|-F_{A(x_i)} - F_{B(x_i)}|}{9} \\
& = S(B(x_i), A(x_i)).
\end{align*}

**Proof:**

By the single valued neutrosophic set property, if $A \subseteq B \subseteq C$, then $T_{A(x_i)} \leq T_{B(x_i)} \leq T_{C(x_i)}$

\begin{align*}
T_{A(x_i)} - T_{B(x_i)} & \leq 0, \\
I_{A(x_i)} - I_{B(x_i)} & \leq 0, \\
F_{A(x_i)} - F_{B(x_i)} & \geq 0.
\end{align*}

So,

\begin{align*}
T_{A(x_i)} - T_{B(x_i)} & \leq 0, \\
I_{A(x_i)} - I_{B(x_i)} & \leq 0, \\
F_{A(x_i)} - F_{B(x_i)} & \geq 0 \quad (1) \\
T_{A(x_i)} - T_{C(x_i)} & \leq 0, \\
I_{A(x_i)} - I_{C(x_i)} & \leq 0, \\
F_{A(x_i)} - F_{C(x_i)} & \geq 0 \quad (2)
\end{align*}

\begin{align*}
T_{A(x_i)} - T_{B(x_i)} & \geq T_{A(x_i)} - T_{C(x_i)}, \\
I_{A(x_i)} - I_{B(x_i)} & \geq I_{A(x_i)} - I_{C(x_i)}, \\
F_{A(x_i)} - F_{B(x_i)} & \leq F_{A(x_i)} - F_{C(x_i)} \\
& \quad (3)
\end{align*}

Using (1), we have

\begin{align*}
2(F_{A(x_i)} - F_{B(x_i)}) - (T_{A(x_i)} - T_{B(x_i)}) & \geq 0 \\
2(F_{A(x_i)} - F_{B(x_i)}) - (I_{A(x_i)} - I_{B(x_i)}) & \geq 0
\end{align*}

and

\begin{align*}
3(T_{A(x_i)} - T_{B(x_i)}) & \geq 0.
\end{align*}

Thus, we get

\begin{align*}
S(A(x_i), B(x_i)) & = 1 - \left( \frac{2(F_{A(x_i)} - F_{B(x_i)}) - (T_{A(x_i)} - T_{B(x_i)})}{9} \right) \\
& \quad + \frac{2(F_{B(x_i)} - F_{A(x_i)}) - (I_{B(x_i)} - I_{A(x_i)})}{9} \\
& \quad + \frac{3|F_{A(x_i)} - F_{B(x_i)}|}{9} \\
& = 1 - \left( \frac{7(F_{A(x_i)} - F_{B(x_i)}) - (T_{A(x_i)} - T_{B(x_i)}) - (I_{A(x_i)} - I_{B(x_i)})}{9} \right) \quad (4)
\end{align*}
Similarly, by (2), we have

\[ S(A(x_i), C(x_i)) = 1 - \left( \frac{2(F_{A(x_i)} - F_{C(x_i)}) - (T_{A(x_i)} - T_{C(x_i)})}{9} \right) \]
\[ + \frac{2(F_{A(x_i)} - F_{C(x_i)}) - (I_{A(x_i)} - I_{C(x_i)})}{9} \]
\[ + \frac{3(F_{A(x_i)} - F_{C(x_i)})}{9} \]
\[ = 1 - \frac{7(F_{A(x_i)} - F_{B(x_i)}) - (T_{A(x_i)} - T_{B(x_i)}) - (I_{A(x_i)} - I_{B(x_i)})}{9} \]
\[ + \frac{7(F_{A(x_i)} - F_{B(x_i)}) - (T_{A(x_i)} - T_{B(x_i)}) - (I_{A(x_i)} - I_{B(x_i)})}{9} \]
\[ + \frac{7(F_{A(x_i)} - F_{B(x_i)}) - (T_{A(x_i)} - T_{B(x_i)}) - (I_{A(x_i)} - I_{B(x_i)})}{9} \]
\[ - \frac{(T_{A(x_i)} - T_{C(x_i)}) - (I_{A(x_i)} - I_{B(x_i)}) - (I_{A(x_i)} - I_{C(x_i)})}{9} \]
\[ = S(A(x_i), B(x_i)) - S(A(x_i), C(x_i)) \]

Using (4) and (5) together, we get

\[ S(A(x_i), B(x_i)) - S(A(x_i), C(x_i)) = 1 - \left( \frac{2(F_{A(x_i)} - F_{C(x_i)}) - (T_{A(x_i)} - T_{B(x_i)}) - (I_{A(x_i)} - I_{B(x_i)})}{9} \right) \]
\[ + \frac{2(F_{A(x_i)} - F_{C(x_i)}) - (I_{A(x_i)} - I_{C(x_i)})}{9} \]
\[ + \frac{3(F_{A(x_i)} - F_{C(x_i)})}{9} \]
\[ = 1 - \frac{7(F_{A(x_i)} - F_{B(x_i)}) - (T_{A(x_i)} - T_{B(x_i)}) - (I_{A(x_i)} - I_{B(x_i)})}{9} \]
\[ + \frac{7(F_{A(x_i)} - F_{B(x_i)}) - (T_{A(x_i)} - T_{B(x_i)}) - (I_{A(x_i)} - I_{B(x_i)})}{9} \]
\[ + \frac{7(F_{A(x_i)} - F_{B(x_i)}) - (T_{A(x_i)} - T_{B(x_i)}) - (I_{A(x_i)} - I_{B(x_i)})}{9} \]
\[ - \frac{(T_{A(x_i)} - T_{C(x_i)}) - (I_{A(x_i)} - I_{B(x_i)}) - (I_{A(x_i)} - I_{C(x_i)})}{9} \]
\[ \geq 0, \]

by (1) and (3),

\[ \frac{7(F_{A(x_i)} - F_{B(x_i)}) + 7(F_{A(x_i)} - F_{C(x_i)})}{9} \geq 0, \]
\[ \frac{(T_{A(x_i)} - T_{B(x_i)}) - (T_{A(x_i)} - T_{C(x_i)})}{9} \geq 0, \]
\[ \frac{(I_{A(x_i)} - I_{B(x_i)}) - (I_{A(x_i)} - I_{C(x_i)})}{9} \geq 0, \]
and therefore

\[ S(A(x_i), B(x_i)) - S(A(x_i), C(x_i)) \geq 0, \]

and

\[ S(A(x_i), B(x_i)) \geq S(A(x_i), C(x_i)). \]

**ii.** The proof of the latter part can be similarly done as the first part.

**Corollary 4.6:** Suppose we make similar definitions to Definition 4.1, but this time based on truth values or indeterminacy values. If we define a truth based similarity measure, or namely,

\[ S(A(x_i), B(x_i)) = 1 - \left( \frac{2(T_{A(x_i)} - T_{B(x_i)}) - (F_{A(x_i)} - F_{B(x_i)})}{9} \right) \]
\[ + \frac{2(T_{A(x_i)} - T_{B(x_i)}) - (I_{A(x_i)} - I_{B(x_i)})}{9} \]
\[ + \frac{3(T_{A(x_i)} - T_{B(x_i)})}{9} \]

or if we define a measure based on indeterminacy values like

\[ S(A(x_i), B(x_i)) = 1 - \left( \frac{2(I_{A(x_i)} - I_{B(x_i)}) - (T_{A(x_i)} - T_{B(x_i)})}{9} \right) \]
\[ + \frac{2(I_{A(x_i)} - I_{B(x_i)}) - (F_{A(x_i)} - F_{B(x_i)})}{9} \]
\[ + \frac{3(I_{A(x_i)} - I_{B(x_i)})}{9} \]

these two definitions don’t provide the conditions of Property 4.5. For instance, for the truth value

\[ S(A(x_i), B(x_i)) = 1 - \left( \frac{2(T_{A(x_i)} - T_{B(x_i)}) - (F_{A(x_i)} - F_{B(x_i)})}{9} \right) \]
\[ + \frac{2(T_{A(x_i)} - T_{B(x_i)}) - (I_{A(x_i)} - I_{B(x_i)})}{9} \]
\[ + \frac{3(T_{A(x_i)} - T_{B(x_i)})}{9} \]

when we take the single valued neutrosophic numbers

\[ A(x) = (0, 0.1, 0), \ B(x) = (1, 0.2, 0) \text{ and } C(x) = (1, 0.3, 0), \]
we see \( S(A(x), B(x)) = 0.233 \) and \( S(A(x), C(x)) = 0.244 \). This contradicts with the results of Property 4.5.

Similarly, for the indeterminacy values,
Let \( w_i \) be the weight of element \( w_i \), where \( w_i = \frac{1}{2} \) \( 1 \leq i \leq 2 \).

\[
S_{NS}(\overline{P_1}, \overline{Q}) = 0.711
\]

and

\[
S_{NS}(\overline{P_2}, \overline{Q}) = 0.772.
\]

We can see that \( S_{NS}(\overline{P_2}, \overline{Q}) \) is the largest value among the values of \( S_{NS}(\overline{P_1}, \overline{Q}) \) and \( S_{NS}(\overline{P_2}, \overline{Q}) \).

Therefore, the unknown pattern represented by single valued neutrosophic set \( \overline{Q} \) should be classified into the pattern \( P_2 \).

5. A New Similarity Measure Based on Falsity Measure Between Neutrosophic Sets Based on the Centroid Points of Transformed Single Valued Neutrosophic Numbers

In this section, we propose a new similarity measure based on falsity value between single valued neutrosophic sets based on the centroid points of transformed single valued neutrosophic numbers.

**Definition 5.1:**

\[
S(A(x_i), B(x_i)) = 1 - \left( \frac{2(F(A(x_i)) - F(B(x_i))) - (T(A(x_i)) - T(B(x_i)))}{9} \right)
\]

\[
+ \frac{2(F(A(x_i)) - F(B(x_i))) - (I(A(x_i)) - I(B(x_i)))}{9}
\]

\[
+ \frac{3(F(A(x_i)) - F(B(x_i)))}{9}
\]

Takig the similarity measure as defined in the fourth section, and letting \( C_A(x_i) \) and \( C_B(x_i) \) be the centers of the triangles obtained by the transformation of \( A(x_i) \) and \( B(x_i) \) in the third section respectively, the similarity measure based on falsity value between single valued neutrosophic sets \( A \) and \( B \) based on the centroid points of transformed single valued neutrosophic numbers is

\[
S_{NSC}(A, B) = \sum_{i=1}^{n} \left( w_i \times S(A(x_i), B(x_i)) \right)
\]

where \( w_i \) be the weight of element \( w_i \), where \( w_i = \frac{1}{2} \) \( 1 \leq i \leq 2 \).

\[
S_{NS}(\overline{P_1}, \overline{Q}) = 0.711
\]

and

\[
S_{NS}(\overline{P_2}, \overline{Q}) = 0.772.
\]

We can see that \( S_{NS}(\overline{P_2}, \overline{Q}) \) is the largest value among the values of \( S_{NS}(\overline{P_1}, \overline{Q}) \) and \( S_{NS}(\overline{P_2}, \overline{Q}) \).

Therefore, the unknown pattern represented by single valued neutrosophic set \( \overline{Q} \) should be classified into the pattern \( P_2 \).
\[ A = \{ x : (T_{A(x)}), I_{A(x)}, F_{A(x)} \} \]
\[ B = \{ x : (T_{B(x)}), I_{B(x)}, F_{B(x)} \} . \]

Here again, \( w_i \)'s are the weights of the \( x_i \)'s with the property \( \sum_{i=1}^{n} w_i = 1 . \)

**Examples 5.2**: Let us consider two patterns \( P_1 \) and \( P_2 \) represented by single valued neutrosophic sets \( \bar{P}_1, \bar{P}_2 \) in \( X = \{ x_1, x_2 \} \) respectively in Example 4.8, where
\[
\bar{P}_1 = \{ (x_1,0.2,0.5,0.7), (x_2,0.9,0.4,0.5) \}
\]
and
\[
\bar{P}_2 = \{ (x_1,0.3,0.2,0.5), (x_2,0.3,0.2,0.4) \}
\]
We want to classify an unknown pattern represented by single valued neutrosophic set \( \bar{Q} \) in \( X = \{ x_1, x_2 \} \) into one of the patterns \( \bar{P}_1, \bar{P}_2 \) where
\[
\bar{Q} = \{ (x_1,0.4,0.4,0.1), (x_2,0.6,0.2,0.3) \}
\]
We make the classification using the measure in Definition 5.1, namely
\[
S_{NSC}(A,B) = \sum_{i=1}^{n} (w_i \times S(C_{A(x_i)}, C_{B(x_i)}))
\]

Also we find the \( C_{A(xi)}, C_{B(xi)} \) centers according to the truth values.
Let \( w_i \) be the weight of element \( x_i, w_i = \frac{1}{2}, 1 \leq i \leq 2 \).
\[
\bar{P}_1 x_1 = (0.2,0.5,0.7) \text{transformed based on falsity value in Example 3.1.1}
\]
\[
C_{\bar{P}_1 x_1} = (0.566,0.633,0.7)
\]
\[
\bar{P}_1 x_2 = (0.9,0.4,0.5) \text{transformed based on falsity value in Example 3.1.1}
\]
\[
C_{\bar{P}_1 x_2} = (0.7,0.633,0.5)
\]
\[
\bar{P}_2 x_1 = (0.3,0.2,0.5) \text{transformed based on falsity value in Example 3.1.1}
\]
\[
C_{\bar{P}_2 x_1} = (0.7,0.7,0.5)
\]
\[
\bar{P}_2 x_2 = (0.3,0.2,0.4) \text{transformed based on falsity value in Example 3.1.1}
\]
\[
C_{\bar{P}_2 x_2} = (0.733,0.7,0.4)
\]
\[
\bar{Q}_x_1 = (x_1,0.4,0.4,0.1) \text{transformed based on falsity value in Section 3.1}
\]
\[
C_{\bar{Q}_x_1} = (0.6,0.8,0.1)(\text{second group})
\]
\[
\bar{Q}_x_2 = (x_2,0.6,0.2,0.3) \text{transformed based on truth falsity in Section 3.1}
\]
\[
C_{\bar{Q}_x_2} = (0.666,0.6,0.3)(\text{second group})
\]
\[
S_{NSC}(\bar{P}_1, \bar{Q}) = 0.67592
\]
\[
S_{NSC}(\bar{P}_2, \bar{Q}) = 0.80927
\]
Therefore, the unknown pattern \( Q \), represented by a single valued neutrosophic set based on truth value is classified into pattern \( P_2 \).

**Example 5.3**: Let us consider two patterns \( P_1 \) and \( P_2 \) of example 4.8, represented by single valued neutrosophic sets \( \bar{P}_1, \bar{P}_2 \) in \( X = \{ x_1, x_2 \} \) respectively, where
\[
\bar{P}_1 = \{ (x_1,0.2,0.5,0.7), (x_2,0.9,0.4,0.5) \}
\]
and
\[
\bar{P}_2 = \{ (x_1,0.3,0.2,0.5), (x_2,0.3,0.2,0.4) \}
\]
We want to classify an unknown pattern represented by the single valued neutrosophic set \( \bar{Q} \) in \( X = \{ x_1, x_2 \} \) into one of the patterns \( \bar{P}_1, \bar{P}_2 \), where
\[
\bar{Q} = \{ (x_1,0.4,0.4,0.1), (x_2,0.6,0.2,0.3) \}
\]
We make the classification using the measure in Definition 5.1, namely
\[
S_{NSC}(A,B) = \sum_{i=1}^{n} (w_i \times S(C_{A(xi)}, C_{B(xi)}))
\]
Also we find the \( C_{A(xi)}, C_{B(xi)} \) centers according to the indeterminacy values.
Let \( w_i \) be the weight of element \( x_i, w_i = \frac{1}{2}, 1 \leq i \leq 2 \).
\[
\bar{P}_1 x_1 = (0.2,0.5,0.7) \text{transformed based on falsity value in Example 3.2.1}
\]
\[
C_{\bar{P}_1 x_1} = (0.766,0.633,0.7)
\]
\[ \tilde{P}_1 x_2 = (0.9, 0.4, 0.5) \] transformed based on falsity value in Example 3.2.1
\[ C_{\tilde{P}_1 x_2} = (0.766, 0.633, 0.5) \]

\[ \tilde{P}_2 x_1 = (0.3, 0.2, 0.5) \] transformed based on falsity value in Example 3.2.1
\[ C_{\tilde{P}_2 x_1} = (0.633, 0.9, 0.5) \]

\[ \tilde{P}_2 x_2 = (0.3, 0.2, 0.4) \] transformed based on falsity value in Example 3.2.1
\[ C_{\tilde{P}_2 x_2} = (0.666, 0.7, 0.4) \]

\[ \tilde{Q}_x_1 = (x_1, 0.4, 0.4, 0.1) \] transformed based on falsity value in Section 3.2
\[ C_{\tilde{Q}_x_1} = (0.6, 0.8, 0.1) \]

\[ \tilde{Q}_x_2 = (x_2, 0.6, 0.2, 0.3) \] transformed based on truth falsity in Section 3.2
\[ C_{\tilde{Q}_x_2} = (0.7, 0.666, 0.3) \]

\[ S_{NSC}(\tilde{P}_1, \tilde{Q}) = 0.67592 \]
\[ S_{NSC}(\tilde{P}_2, \tilde{Q}) = 0.80927 \]

Therefore, the unknown pattern \( \tilde{Q} \), represented by a single valued neutrosophic set based on indeterminacy value is classified into pattern \( \tilde{P}_2 \).

**Example 5.4:** Let us consider in example 4.8, two patterns \( \tilde{P}_1 \) and \( \tilde{P}_2 \) represented by single valued neutrosophic sets \( \tilde{P}_1, \tilde{P}_2 \) in \( X = \{x_1, x_2\} \) respectively, where
\[ \tilde{P}_1 = \{(x_1, 0.2, 0.5, 0.7), (x_2, 0.9, 0.4, 0.5)\} \]
and
\[ \tilde{P}_2 = \{(x_1, 0.3, 0.2, 0.5), (x_2, 0.3, 0.2, 0.4)\}. \]

We want to classify an unknown pattern represented by single valued neutrosophic set \( \tilde{Q} \) in \( X = \{x_1, x_2\} \) into one of the patterns \( \tilde{P}_1, \tilde{P}_2 \), where
\[ \tilde{Q} = \{(x_1, 0.4, 0.4, 0.1), (x_2, 0.6, 0.2, 0.3)\}. \]

We make the classification using the measure in Definition 5.1, namely
\[ S_{NSC}(A, B) = \sum_{i=1}^{n} w_i S\left(C_{A(x_i)}, C_{B(x_i)}\right). \]

Also we find the \( C_{A(x_i)}, C_{B(x_i)} \) centers according to the falsity values.

Let \( w_i \) be the weight of element \( x_i \), \( w_i = \frac{1}{2}; 1 \leq i \leq 2 \).

\[ \tilde{P}_1 x_1 = (0.2, 0.5, 0.7) \] transformed based on falsity value in Example 3.3.1
\[ C_{\tilde{P}_1 x_1} = (0.766, 0.7, 0.7) \]

\[ \tilde{P}_2 x_1 = (0.3, 0.2, 0.5) \] transformed based on falsity value in Example 3.3.1
\[ C_{\tilde{P}_2 x_1} = (0.633, 0.7, 0.5) \]

\[ \tilde{P}_2 x_2 = (0.3, 0.2, 0.4) \] transformed based on falsity value in Example 3.3.1
\[ C_{\tilde{P}_2 x_2} = (0.666, 0.733, 0.4) \]

\[ \tilde{Q}_x_1 = (x_1, 0.4, 0.4, 0.1) \] transformed based on falsity value in Section 3.3
\[ C_{\tilde{Q}_x_1} = (0.6, 0.6, 0.1) \]

\[ \tilde{Q}_x_2 = (x_2, 0.6, 0.2, 0.3) \] transformed based on truth falsity in Section 3.3
\[ C_{\tilde{Q}_x_2} = (0.7, 0.666, 0.3) \]

\[ S_{NSC}(\tilde{P}_1, \tilde{Q}) = 0.7091 \]
\[ S_{NSC}(\tilde{P}_2, \tilde{Q}) = 0.8148 \]

Therefore, the unknown pattern \( \tilde{Q} \), represented by a single valued neutrosophic set based on falsity value is classified into pattern \( \tilde{P}_2 \).

In Example 5.2, Example 5.3 and Example 5.4, all measures according to truth, indeterminancy and falsity values give the same exact result.
Conclusion

In this study, we propose methods to transform between single valued neutrosophic numbers based on centroid points. We also propose a new method to measure the degree of similarity based on falsity values between single valued neutrosophic sets. Then we prove some properties of new similarity measure based on falsity value between single valued neutrosophic sets. When we take this measure with respect to truth or indeterminancy we show that it does not satisfy one of the conditions of similarity measure. We also apply the proposed new similarity measures based on falsity value between single valued neutrosophic sets to deal with pattern recognition problems.

References


Received: January 30, 2017. Accepted: February 15, 2017.
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Abstract: In this paper, we have introduced a new concept of multi-dimensional neutrosophic soft sets together with various operations, properties and theorems on them. Then we have proposed an algorithm named $2-DNS$ based on our proposed two-dimensional neutrosophic soft set for solving neutrosophic multi-criteria assignment problems with multiple decision makers. At last, we have applied the $2-DNS$ Algorithm for solving neutrosophic multi-criteria assignment problem in medical science to evaluate the effectiveness of different modalities of treatment of a disease.

Keywords: Assignment, Neutrosophic Multi-Criteria, Multi-Dimensional Neutrosophic Soft Set, $2-DNS$ Algorithm, Application.

1 Introduction
Most of the recent mathematical methods meant for formal modeling, reasoning and computing are crisp, accurate and deterministic in nature. But in ground reality, crisp data is not always the part and parcel of the problems encountered in different fields like economics, engineering, social science, medical science, environment etc. As a consequence various theories viz. theory of probability, theory of fuzzy sets introduced by Zadeh [1], theory of intuitionistic fuzzy sets by Atanassov[2], theory of vague sets by Gau[3], theory of interval mathematics by Gorzalczyan[4], theory of rough sets by Pawlak[5] have been evolved in process. But difficulties present in all these theories have been shown by Molodtsov [6]. The cause of these problems is possibly related to the inadequacy of the parametrization tool of the theories. As a result Molodtsov proposed the concept of soft theory as a new mathematical tool for solving the uncertainties which is free from the above difficulties. Maji et al. [7, 8] have further done various research works on soft set theory. For presence of vagueness Maji et al.[9, 10] have introduced the concept of Fuzzy Soft Set. Then Mitra Basu et al. [14] proposed the mean potentiality approach to get a balanced solution of a fuzzy soft set based decision making problem.

But the intuitionistic fuzzy sets can only handle the incomplete information considering both the truth-membership ( or simply membership ) and falsity-membership ( or non-membership ) values. It does not handle the indeterminate and inconsistent information which exists in belief system. Smarandache [13] introduced the concept of neutrosophic set(NS) which is a mathematical tool for handling problems involving imprecise, indeterminacy and inconsistent data. He showed that NS is a generalization of the classical sets, conventional fuzzy sets, Intuitionistic Fuzzy Sets (IFS) and Interval Valued Fuzzy Sets (IVFS). Then considering the fact that the parameters or criteria ( which are words or sentences ) are mostly neutrosophic set, Maji [11, 12] has combined the concept of soft set and neutrosophic set to make the mathematical model neutrosophic soft set and also given an algorithm to solve a decision making problem. But till now there does not exist any method for solving neutrosophic soft set based assignment problem.

In several real life situations we are encountered with a type of problem which includes in assigning men to offices, jobs to machines, classes in a school to rooms, drivers to trucks, delivery trucks to different routs or problems to different research teams etc in which the assignees depend on some criteria which posses varying degree of efficiency, called cost or effectiveness. The basic assumption of this type of problem is that one person can perform one job at a time. An assignment plan is optimal if it is able to
2 Preliminaries

2.1 Definition: Soft Set [6]
Let \( U \) be an initial universe set and \( E \) be a set of parameters. Let \( P(U) \) denotes the set of all subsets of \( U \). Let \( A \subseteq E \). Then a pair \((F, A)\) is called a soft set over \( U \), where \( F \) is a mapping given by, \( F : A \rightarrow P(U) \).

In other words, a soft set over \( U \) is a parameterized family of subsets of the universe \( U \).

2.2 Definition: NOT Set of a Set of Parameters [9]
Let \( E = \{e_1, e_2, e_3, ..., e_n\} \) be a set of parameters.

The NOT set of \( E \) denoted by \( |E| \) is defined by \( |E| = \{|e_1|, |e_2|, |e_3|,..., |e_n|\} \), where \( |e_i| \) is not \( e_i \) \( \forall i \).

The operator not of an object, say \( k \), is denoted by \( |k| \) and is defined as the negation of the object; e.g., let we have the object \( k = \text{beautiful} \), then \( |k| \) i.e., not \( k \) means \( k \) is not beautiful.

2.3 Definition: Neutrosophic Set [13]
A neutrosophic set \( A \) on the universe of discourse \( X \) is defined as
\[
A = \{x, T_A(x), I_A(x), F_A(x) > x \in X\},
\]
where \( T, I, F : X \rightarrow [0,1] \)
and \( 0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3 \); \( T, I, F \) are called neutrosophic components.

"Neutrosophic" etymologically comes from "neutro-sophy" (French neutre < Latin neuter, neutral and Greek sophia, skill/wisdom) which means knowledge of neutral thought.

From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of \( [0,1] \). The non-standard finite numbers \( 1^+ = 1 + \delta \), where \( 1 \) is the standard part and \( \delta \) is the non-standard part and \( -0 = 0\delta \), where \( 0 \) is its standard part and \( \delta \) is non-standard part. But in real-life application in scientific and engineering problems it is difficult to use neutrosophic set with value from real standard or non-standard subset of \( [0,1] \).

Hence we consider the neutrosophic set which takes the value from the subset of \( [0,1] \).

Any element neutrosophically belongs to any set, due to the percentages of truth/indeterminacy/falsity involved, which varies between 0 and 1 or even less than 0 or greater than 1.

Thus \( x(0.5,0.2,0.3) \) belongs to \( A \) (which means, with a probability of 50 percent \( x \) is in \( A \), with a probability of 30 percent \( x \) is not in \( A \) and the rest is undecidable); or \( y(0,0,1) \) belongs to \( A \) (which normally means \( y \) is not for sure in \( A \)); or \( z(0,1,0) \) belongs to \( A \) (which means one does know absolutely nothing about \( z \)’s affiliation with \( A \)); here \( 0.5 + 0.2 + 0.3 = 1 \); thus \( A \) is a NS and an IFS too.

The subsets representing the appurtenance, indeterminacy and falsity may overlap, say the element \( z(0.3,0.5,0.28) \) and in this case \( 0.3 + 0.5 + 0.28 > 1 \); then \( B \) is a NS but is not
an IFS; we can call it paraconsistent set (from paraconsistent logic, which deals with paraconsistent information).

Or, another example, say the element \( z(0.1,0.3,0.4) \) belongs to the set \( C \), and here \( 0.1 + 0.3 + 0.4 < 1 \); then \( B \) is a NS but is not an IFS; we can call it intuitionistic set (from intuitionistic logic, which deals with incomplete information).

Remarkably, in a NS one can have elements which have paraconsistent information (sum of components \( > 1 \)), or incomplete information (sum of components \( < 1 \)), or consistent information (in the case when the sum of components \( = 1 \)).

### 2.4 Definition: Complement of a Neutrosophic Set [18]

The complement of a neutrosophic set \( S \) is denoted by \( e(S) \) and is defined by \( T_{e(S)}(x) = F_e(x), I_{e(S)}(x) = 1 - I_S(x), F_{e(S)}(x) = T_S(x) \forall x \in X \).

### 2.5 Definition: Neutrosophic Soft Set [12]

Let \( U \) be an initial universe set and \( E \) be a set of parameters. Consider \( A \subseteq E \). Let \( P(U) \) denotes the set of all neutrosophic sets of \( U \). The collection \( (F, A) \) is termed to be the neutrosophic soft set over \( U \), where \( F \) is a mapping given by \( F : A \rightarrow P(U) \).

### 2.6 Traditional Assignment Problems [15]

Sometimes we are faced with a type of problem which consists in assigning men to offices, jobs to machines, classes in a school to rooms, drivers to trucks, delivery trucks to different routs or problems to different research teams etc in which the assignees posses varying degree of efficiency, called cost or effectiveness. The basic assumption of this type of problem is that one person can perform one job at a time with respect to one criterion. An assignment plan is optimal if it optimizes the total effectiveness of performing all the jobs.

**Example 2.1**

Let us consider the assignment problem represented by the following cost matrix (Table-1) in which the elements represent the cost in lacs required by a machine to perform the corresponding job. The problem is to allocate the jobs to the machines so as to minimize the total cost.

<table>
<thead>
<tr>
<th>JOBS</th>
<th>MACHINES</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>M_1,</td>
</tr>
<tr>
<td></td>
<td>M_2,</td>
</tr>
<tr>
<td></td>
<td>M_3,</td>
</tr>
<tr>
<td></td>
<td>M_4,</td>
</tr>
</tbody>
</table>

### Table-1: Cost Matrix

<table>
<thead>
<tr>
<th>JOBS</th>
<th>MACHINES</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>12</td>
</tr>
<tr>
<td></td>
<td>27</td>
</tr>
<tr>
<td></td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>25</td>
</tr>
<tr>
<td>C</td>
<td>18</td>
</tr>
<tr>
<td></td>
<td>17</td>
</tr>
<tr>
<td></td>
<td>14</td>
</tr>
<tr>
<td>D</td>
<td>18</td>
</tr>
<tr>
<td></td>
<td>23</td>
</tr>
<tr>
<td></td>
<td>9</td>
</tr>
</tbody>
</table>

### 3 Neutrosophic Multi-Criteria Assignment Problems With Multiple Decision Makers

Normally in traditional assignment problems one person is assigned for one job with respect to a single criterion but in real life there are different problems in which one person can be assigned for one job with respect to more than one criteria. Such type of problems is known as Multi-Criteria Assignment Problem (MCAP). Moreover in such MCAP all criteria be neutrosophic in nature then the problems will be called Neutrosophic Multi-Criteria Assignment Problem (NMCAP). Now there may be such type of NMCAP in which the criteria matrices are determined by more than one decision maker. In such type of problems there may be more than one matrices associated with a single criterion as the criteria are determined by multiple decision makers. Now we will discuss these new type of NMCAP with more than one decision makers and develop an algorithm to solve such type of problems.

#### 3.1 General Formulation of a Neutrosophic Multi-Criteria Assignment Problem With Multiple Decision Makers

Let \( m \) jobs have to be performed by \( m \) number of machines depending on \( p \) number of criteria (each criterion is neutrosophic in nature) according to \( q \) number of decision makers. Now suppose that to perform \( j \)-th job by \( i \)-th machine it will take the degree of efficiency \( \xi_{kj} \) for the \( k \)-th criterion according to the \( q \)-th decision maker. Then the \( k \)-th \(( k = 1,2,\ldots, p \) ) criteria matrix according to \( q \)-th decision maker will be as given in Table-2.
Suppose that $\mathcal{A} \subseteq \mathcal{B}$ and let $\mathcal{C}$. Now a mapping $\mathcal{D} \subseteq \mathcal{E}$ denotes all $\mathcal{F}$ be $\mathcal{G}$.

$U \subseteq \mathcal{H} \subseteq \mathcal{I}$

Now let $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3$ be the machines $M_1, M_2, M_3$ so as to minimize the total cost and time collectively and simultaneously.

4 The Concept of Multi-Dimensional Neutrosophic Soft Set

4.1 Definition: Multi-Dimensional Neutrosophic Soft Set

Let $U_1, U_2, \ldots, U_n$ be $n$ non-null finite sets of $n$ different type of objects such that, $U_1 = \{O_1, O_2, \ldots, O_{m_1}\}$, $U_2 = \{O_1, O_2, \ldots, O_{m_2}\}$, $\ldots, U_n = \{O_1^{(n-1)}, O_2^{(n-1)}, \ldots, O_{m_n}^{(n-1)}\}$; where $m_1, m_2, \ldots, m_n$ respectively be the cardinalities of $U_1, U_2, \ldots, U_n$ and let $U = U_1 \times U_2 \times \ldots \times U_n$. Now let $E$ be the set of parameters clarifying all types of objects $O_{i_1}, O_{i_2}, \ldots, O_{i_n}^{(n-1)}$, $i_1 = 1, 2, \ldots, m_1; i_2 = 1, 2, \ldots, m_2; \ldots; i_n = 1, 2, \ldots, m_n$ and each parameter is a neutrosophic word or neutrosophic sentence involving neutrosophic words and $A \subseteq E$. Suppose that $N^U$ denotes all neutrosophic sets of $U$. Now a mapping $F$ is defined from the parameter set $A$ to $N^U$, i.e., $F : A \rightarrow N^U$, then the algebraic structure $(F, A)$ is said to be a $n$-Dimensional Neutrosophic soft set over $U$.

Now $n$ may be finite or, infinite. If $n = 1$ then $(F, A)$ will be the conventional neutrosophic soft set, if $n = 2$ then $(F, A)$ is said to be a two-dimensional neutrosophic soft set, if $n = 3$ then

| Table-2: criteria matrix of $k$-th criterion for $q$-th decision maker |
| MACHINES | $M_1$ | $M_2$ | $M_3$ | $\ldots$ | $M_m$ |
|----------|-------|-------|-------|--------|
| $J_1$    | $\tilde{z}_{q_{11}}^k$ | $\tilde{z}_{q_{12}}^k$ | $\tilde{z}_{q_{13}}^k$ | $\ldots$ | $\tilde{z}_{q_{1m}}^k$ |
| $J_2$    | $\tilde{z}_{q_{21}}^k$ | $\tilde{z}_{q_{22}}^k$ | $\tilde{z}_{q_{23}}^k$ | $\ldots$ | $\tilde{z}_{q_{2m}}^k$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ldots$ | $\vdots$ |
| $J_m$    | $\tilde{z}_{q_{m1}}^k$ | $\tilde{z}_{q_{m2}}^k$ | $\tilde{z}_{q_{m3}}^k$ | $\ldots$ | $\tilde{z}_{q_{mm}}^k$ |

If the number of jobs and machines be equal in a criteria matrix then it is called a balanced criteria matrix otherwise it is known as unbalanced criteria matrix. Now the problem is to assign each machine with a unique job in such a way that the total degree of efficiency for an allocation will be optimized for all criteria which is illustrated in the following example.

**Example 3.1**

Let us consider a NMCAP represented by the following cost matrices and time matrix in which the criteria are neutrosophic in nature and the elements of the matrices are representing the degree of cost and time required by a machine to perform the corresponding job according to two decision makers Mr. X and Mr. Y.

<p>| Table-3: Cost Matrix by Mr.X |</p>
<table>
<thead>
<tr>
<th>MACHINES</th>
<th>$M_1$</th>
<th>$M_2$</th>
<th>$M_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J_1$</td>
<td>$(0.8, 0.2, 0.6)$</td>
<td>$(0.2, 0.5, 0.9)$</td>
<td>$(0.6, 0.4, 0.4)$</td>
</tr>
<tr>
<td>$J_2$</td>
<td>$(0.2, 0.6, 0.8)$</td>
<td>$(0.7, 0.2, 0.5)$</td>
<td>$(0.6, 0.3, 0.5)$</td>
</tr>
<tr>
<td>$J_3$</td>
<td>$(0.6, 0.3, 0.5)$</td>
<td>$(0.6, 0.2, 0.7)$</td>
<td>$(0.6, 0.1, 0.5)$</td>
</tr>
</tbody>
</table>

<p>| Table-4: Cost Matrix by Mr.Y |</p>
<table>
<thead>
<tr>
<th>MACHINES</th>
<th>$M_1$</th>
<th>$M_2$</th>
<th>$M_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J_1$</td>
<td>$(0.7, 0.4, 0.3)$</td>
<td>$(0.2, 0.5, 0.9)$</td>
<td>$(0.5, 0.4, 0.6)$</td>
</tr>
<tr>
<td>$J_2$</td>
<td>$(0.3, 0.6, 0.8)$</td>
<td>$(0.7, 0.2, 0.4)$</td>
<td>$(0.6, 0.4, 0.3)$</td>
</tr>
<tr>
<td>$J_3$</td>
<td>$(0.5, 0.3, 0.6)$</td>
<td>$(0.6, 0.3, 0.5)$</td>
<td>$(0.5, 0.2, 0.7)$</td>
</tr>
</tbody>
</table>

<p>| Table-5: Time Matrix by Mr.X and Mr.Y |</p>
<table>
<thead>
<tr>
<th>MACHINES</th>
<th>$M_1$</th>
<th>$M_2$</th>
<th>$M_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J_1$</td>
<td>$(0.3, 0.5, 0.8)$</td>
<td>$(0.7, 0.2, 0.4)$</td>
<td>$(0.5, 0.2, 0.6)$</td>
</tr>
<tr>
<td>$J_2$</td>
<td>$(0.8, 0.3, 0.3)$</td>
<td>$(0.2, 0.5, 0.9)$</td>
<td>$(0.5, 0.3, 0.7)$</td>
</tr>
<tr>
<td>$J_3$</td>
<td>$(0.5, 0.3, 0.6)$</td>
<td>$(0.5, 0.4, 0.5)$</td>
<td>$(0.4, 0.3, 0.7)$</td>
</tr>
</tbody>
</table>
(F, A) is said to be a three-dimensional neutrosophic soft set and so on.

### 4.2 The Features of Multi-Dimensional Neutrosophic Soft Set Compared to Neutrosophic Soft Set

Neutrosophic soft set is just a special type of multi-dimensional neutrosophic soft set where the dimension i.e., the number of the set of objects is one.

A neutrosophic soft set indicates that how a single set of objects is involved with a single set of parameters (or, criteria) where as a n-dimensional neutrosophic soft set(n may be any positive integer) reveals the involvement of n number of sets of different types of objects with a single set of parameters(or, criteria).

So from the perspective of application, multi-dimensional neutrosophic soft set has more vast scope than the conventional neutrosophic soft set.

Now we will discuss the example, operations and properties of two-dimensional neutrosophic soft set and for the higher dimensional neutrosophic soft set they can also be established in the identical manner.

**Example 4.1:** Let U₁ = {J₁, J₂, J₃} and let U₂ be the set of four machines, say, M₁, M₂, M₃, M₄. Now let E = \{ cost requirement, time requirement, troublesome due to transportation \},

\[ E = \{e₁, e₂, e₃\} \]

Let \[ A = \{e₁, e₂\} \]

Now let \[ U = U₁ \times U₂ \] and \[ F : A \rightarrow N^U \] s.t.,

\[ \text{cost requirement} \]

\[ \{(J₁, M₁)/(8,0.3,0.4), (J₂, M₂)/(3,2,8), (J₁, M₃)/(5,4,6), (J₂, M₄)/(7,2,3), (J₁, M₄)/(7,3,4), (J₂, M₂)/(5,5,6), (J₂, M₃)/(3,2,8), (J₃, M₁)/(6,4,6), (J₃, M₂)/(4,2,6), (J₃, M₃)/(7,2,5)\} \]

\[ \text{time requirement} = \{(J₁, M₁)/(2,3,9), (J₂, M₂)/(6,3,5), (J₁, M₃)/(5,6,7), (J₂, M₄)/(6,2,7), (J₁, M₄)/(4,3,7), (J₂, M₂)/(6,4,6), (J₂, M₃)/(4,2,6), (J₃, M₁)/(3,4,8)\} \]

The Tabular Representation of the two-dimensional neutrosophic soft set \((F, A)\) is as follows:

<table>
<thead>
<tr>
<th>Tabular Representation of ((F, A))</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>e₁</strong></td>
</tr>
<tr>
<td>(J₁, M₁)</td>
</tr>
<tr>
<td>(J₁, M₂)</td>
</tr>
<tr>
<td>(J₁, M₃)</td>
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<tr>
<td>(J₁, M₄)</td>
</tr>
<tr>
<td>(J₂, M₁)</td>
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<tr>
<td>(J₂, M₂)</td>
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<tr>
<td>(J₂, M₃)</td>
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<td>(J₂, M₄)</td>
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<tr>
<td>(J₃, M₁)</td>
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<tr>
<td>(J₃, M₂)</td>
</tr>
<tr>
<td>(J₃, M₃)</td>
</tr>
<tr>
<td>(J₃, M₄)</td>
</tr>
</tbody>
</table>

### 4.3 Definition: Choice Value:

According to a decision making problem the parameters of a decision maker’s choice or requirement which forms a subset of the whole
4.4 Definition: Rejection Value:
Rejection value of an object is the sum of the falsity-membership values of that object corresponding to all the choice parameters associated with a decision making problem.

4.5 Definition: Confusion Value:
Confusion value of an object is the sum of the indeterminacy-membership values of that object corresponding to all the choice parameters associated with a decision making problem.

4.6 Definition: Null Two-dimensional Neutrosophic Soft Set:
Let \( U_1 \times U_2 \) be the initial universe set, \( E \) be the universe set of parameters and \( A \subset E \). Then a two-dimensional neutrosophic soft set \((F, A)\) is said to be a null two-dimensional neutrosophic soft set \((\phi_A)\) with respect to the parameter set \( A \) if for each \( e \in A \)
\[
F(e) = \{(O_i, O_j) / 0.0 \} \forall (O_i, O_j) \in U_1 \times U_2
\]

4.7 Definition: Universal Two-dimensional Neutrosophic Soft Set:
Let \( U_1 \times U_2 \) be the initial universe set, \( E \) be the universe set of parameters and \( A \subset E \). Then a two-dimensional neutrosophic soft set \((F, A)\) is said to be a universal two-dimensional neutrosophic soft set \((U_A)\) with respect to the parameter set \( A \) if for each \( e \in A \)
\[
F(e) = \{(O_i, O_j) / 1.0 \} \forall (O_i, O_j) \in U_1 \times U_2
\]

4.8 Definition: Complement of a Two-dimensional Neutrosophic Soft Set
The complement of a two-dimensional neutrosophic soft set \((F, A)\) over the universe \( U \) where
\[
U = U_1 \times U_2; U_1 = \{O_1, O_2, \ldots, O_i\} ,
U_2 = \{O'_1, O'_2, \ldots, O'_j\} ; i, j \in N
\]
over the parameter set \( E \) (where each parameter is a neutrosophic word or neutrosophic sentence involving neutrosophic words) is denoted by \((F, A)^C\) and is defined by
\[
(F, A)^C = (F^C, |A|)
\]
where
\[
F^C : A \rightarrow N^U
\]
where \( |A| \) is the NOT set of the parameter set \( A \).

4.9 Definition: Union
The union of two two-dimensional neutrosophic soft sets \((F, A)\) and \((G, B)\) over the same universe \( U \)
where \( U = U_1 \times U_2; U_1 = \{O_1, O_2, \ldots, O_i\} ,
U_2 = \{O'_1, O'_2, \ldots, O'_j\} ; i, j \in N \)
and over the parameter set \( E \) (where \( A, B \subseteq E \) and each parameter is a neutrosophic word or neutrosophic sentence involving neutrosophic words) is denoted by \((F, A) \widetilde{\cup} (G, B)\) and is defined by
\[
(F, A) \widetilde{\cup} (G, B) = (H, C)
\]
where
\[
H(e) = \begin{cases} 
F(e), & \text{if } e \in (A-B) \\
G(e), & \text{if } e \in (B-A) \\
\{(O_i, O'_j) \cup \max \{\mu_{F(e)}(O_i, O'_j), \mu_{G(e)}(O_i, O'_j)\} \mid (O_i, O'_j) \in U_1 \times U_2 \}, & \text{if } e \in (A \cap B)
\end{cases}
\]
where \( \mu_{F(e)}(O_i, O'_j) \) and \( \mu_{G(e)}(O_i, O'_j) \) denote the membership values of \((O_i, O'_j)\) w.r.t. the functions \( F \) and \( G \) respectively associated with the parameter \( e \).
4.11 Properties:
Let \((F, A), (G, B)\) and \((H, C)\) be three two-dimensional neutrosophic soft sets over the same universe \(U\) and parameter set \(E\). Then we have,

\[(i)(F, A) \sim \cup (G, B) \sim \cap (H, C) = ((F, A) \sim \cup (G, B)) \sim \cap (H, C)\]

\[(ii)(F, A) \sim \cap (G, B) = (G, B) \sim \cap (F, A)\]

\[(iii)((F, A)^C)^C = (F, A)\]

\[(iv)(F, A) \sim \cap (F, A) = (F, A)\]

\[(v)(F, A) \sim \cap (F, A) = (F, A)\]

\[(vi)(F, A) \sim \cap \phi_A = (F, A),\quad \text{where } \phi_A\quad \text{is the null two-dimensional neutrosophic soft set with respect to the parameter set } A.\]

\[(vii)(F, A) \sim \cap \phi_A = \phi_A\]

\[(viii)(F, A) \sim \cap U_A = U_A,\quad \text{where } U_A\quad \text{is the universal two-dimensional neutrosophic soft set with respect to the parameter set } A.\]

\[(ix)(F, A) \sim \cap U_A = (F, A)\]

4.12 De Morgan’s laws in two-dimensional neutrosophic soft set theory:
The well known De Morgan’s type of results hold in two-dimensional neutrosophic soft set theory for the newly defined operations: complement, union and intersection.

**Theorem 4.1**
Let \((F, A)\) and \( (G, B)\) be two two-dimensional neutrosophic soft sets over a common universe \(U\) and parameter set \(E\). Then

\[i)((F, A) \cap (G, B))^C = (F, A)^C \cap (G, B)^C\]

\[ii)((F, A) \cup (G, B))^C = (F, A)^C \cup (G, B)^C\]

5 The Methodology Based On Two-Dimensional Neutrosophic Soft Set For Solving Neutrosophic Multi-Criteria Assignment Problems With Multiple Decision Makers
In many real life problems we have to assign each object of a set of objects to another object in a different set of objects such as assigning men to offices, jobs to machines, classes in a school to rooms, drivers to trucks, delivery trucks to different routes or problems to different research teams etc. in which the assignees possesses varying degree of efficiency, depending on neutrosophic multiple criteria such as cost, time etc. The basic assumption of this type of problem is that one person can perform one job at a time. To solve such type of problems our aim is to make such assignment that optimize the criteria i.e., minimize the degree of cost and time or maximizes the degree of profit. Since in such type of problems the degrees of each criterion (or, parameter) of a set of criteria (or, parameter set) are evaluated with respect to two different types of objects, to solve such problems we can apply two-dimensional neutrosophic soft set and their various operations.

The stepwise procedure to solve such type of problems is given below.

2 — DNS Algorithm:
Step 1: Convert each unbalanced criteria matrix to balanced by adding a fictitious job or machine with zero cost of efficiency.

Step 2: From these balanced criteria matrices construct a two-dimensional neutrosophic soft set \((F_i, E_i)\) according to each decision maker \( d_i; i = 1, 2, ..., q;\quad q\quad \text{be the number of decision makers}.\)

Step 3: Combining the opinions of all the decision makers about the criteria, take the union of all these two-dimensional neutrosophic soft sets \((F_i, E_i); i = 1, 2, ..., q\quad \text{as follows}\)

\[(F, E) = \bigcup_{i=1}^{q} (F_i, E_i)\]

Step 4: Then compute the complement \((F, E)^C\)

Step 5: Construct the tabular representation of \((F, E)^C\) or, \((F, E)^C\) according to maximization or minimization problem with row wise sum of parametric values which is known as choice value \(C_{(J_i, M_j)}\).

Step 6: Now for \(i\)-th job, consider the choice values \(C_{(J_i, M_j)}; \forall j\quad \text{and point out the maximum choice value } C_{(J_i, M_j)}^{\max}\) with a *. 

Step 7: If \(C_{(J_i, M_j)}^{\max}\) holds for all distinct \( j \) ’s then assign \(M_j\) machine for \(J_i\) job and put a tick mark( √ ) beside the choice values corresponding to \(M_j\) to indicate that already \(M_j\) machine has been assigned.

Step 8: If for more than one \( i \), \(C_{(J_i, M_j)}^{\max}\) hold for the same \( j \), i.e., if there is a tie for the assignment of \(M_j\) machine in more than one job then we have to consider the difference value \(V_{(J_i, M_j)}\) between the
maximum and the next to maximum choice values (corresponding to those machines which are not yet assigned). If \( V_{d(J_1,M_j)} < V_{d(J_2,M_j)} \), then \( M_j \) machine will be assigned for the job \( J_{i_2} \). Now if the difference values also be same, i.e., \( V_{d(J_1,M_j)} = V_{d(J_2,M_j)} \) then go to the next step.

**Step 9:** Now for \( i \)-th job, consider the rejection values \( R_{(J_i,M_j)} \) \( \forall j \) and point out the minimum rejection value \( R^{\text{min}}_{(J_i,M_j)} \) with a *.

**Step 10:** If for more than one \( i \), \( R^{\text{min}}_{(J_i,M_j)} \) hold for the same \( j \), consider the difference value (\( V_{dR(J_i,M_j)} \)) between the minimum and the next to minimum rejection values (corresponding to those machines which are not yet assigned). If \( V_{dR(J_1,M_j)} < V_{dR(J_2,M_j)} \) then \( M_j \) machine will be assigned for the job \( J_{i_2} \). Now if the difference values also be same then go to the final step.

**Step 11:** Now for \( i \)-th job, consider the confusion values \( \xi_{(J_i,M_j)} \) \( \forall j \) and point out the minimum confusion value \( \xi^{\text{min}}_{(J_i,M_j)} \) with a *.

**Step 12:** If for more than one \( i \), \( \xi^{\text{min}}_{(J_i,M_j)} \) hold for the same \( j \), consider the difference value (\( V_{d\xi(J_i,M_j)} \)) between the minimum and the next to minimum confusion values (corresponding to those machines which are not yet assigned). If \( V_{d\xi(J_1,M_j)} < V_{d\xi(J_2,M_j)} \) then \( M_j \) machine will be assigned for the job \( J_{i_2} \). Now if the difference values also be same, i.e., \( V_{d\xi(J_1,M_j)} = V_{d\xi(J_2,M_j)} \) then \( M_j \) machine may be assigned to any one of the jobs \( J_{i_1} \) or \( J_{i_2} \).

### 6 Application of 2-DNS Algorithm for Solving Neutrosophic Multi-Criteria Assignment Problems in Medical Science

In medical science there also exist neutrosophic multi-criteria assignment problems and we may apply the 2-DNS Algorithm for solving those problems. Now we will discuss a such type of problem with its solution.

**Problem 1:** In medical science[19] there are different types of diseases and various modalities of treatments in respect to them. On the basis of different aspects of the treatment procedure (such as degree of pain relief, cost and time requirements for treatment etc.) we may measure the degree of effectiveness of the treatment for the disease. Here we consider three common diseases of oral cavity such as dental caries, gum disease and oral ulcer. Now medicinal treatment, extraction and scaling that are commonly executed, have more or less impacts on the treatment of these three diseases. According to the statistics,

( true-membership value, indeterminacy-membership value, falsity-membership value ) of pain relief in case of medicinal treatment on the basis of pain score for dental caries, gum disease, oral ulcer are \((0.7,0.7,0.5),(0.6,0.8,0.5\text{ and }0.9,0.5,0.2)\) respectively; by extraction the degrees of pain relief for dental caries, gum disease and oral ulcer are \((0.8,0.5,0.3)\text{, }0.8,0.7,0.4)\text{ and }0.5,0.7,0.6\) respectively and by scaling the degrees of pain relief for dental caries, gum disease and oral ulcer are \((0.3,0.8,0.8),(0.9,0.4,0.2)\text{ and }0.6,0.7,0.5\) respectively. Now the degree of cost to avail the medicinal treatment, extraction and scaling for both the diseases dental caries, gum disease are \((0.4,0.3,0.8),(0.3,0.2,0.7)\) and \((0.5,0.4,0.6)\) respectively and that for oral ulcer are \((0.3,0.2,0.8),(0.2,0.3,0.9)\text{ and }0.4,0.4,0.7\) respectively. Moreover the degree of time taken to the medicinal treatment, extraction and scaling for gum disease are \((0.6,0.3,0.5),(0.4,0.2,0.8),(0.5,0.5,0.6)\) and for oral ulcer are \((0.6,0.4,0.7),(0.4,0.3,0.8),(0.5,0.5,0.6)\) respectively and that of for dental caries are \((0.6,0.2,0.3),(0.5,0.4,0.7)\text{ and }0.3,0.2,0.9\) respectively. Now the problem is to assign a treatment for each disease so that to maximize the pain relief and minimize the cost and time simultaneously as much as possible.

**Solution By 2-DNS Algorithm**

The set of universe \( U = U_1 \times U_2 \) where

- \( U_1 = \{ \text{dental caries, gum disease, oral ulcer} \} = \{ d_1, d_2, d_3 \} \),
- \( U_2 = \{ \text{medicinal treatment, extraction, scaling} \} = \{ t_1, t_2, t_3 \} \)
and the set of parameters
\[ E = \{ \text{pain score, cost requirement, time requirement} \} \]
\[ = \{ e_1, e_2, e_3 \} \text{(say)} \]
Now from the given data we have the following criteria matrices:

### Table-7 (Pain Score Matrix)

<table>
<thead>
<tr>
<th>TREATMENTS</th>
<th>( t_1 )</th>
<th>( t_2 )</th>
<th>( t_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( d_1 )</td>
<td>(0.5,0.2,0.6)</td>
<td>(0.4,0.3,0.8)</td>
<td>(0.2,0.6,0.9)</td>
</tr>
<tr>
<td>( d_2 )</td>
<td>(0.2,0.5,0.9)</td>
<td>(0.6,0.3,0.5)</td>
<td>(0.5,0.3,0.6)</td>
</tr>
<tr>
<td>( d_3 )</td>
<td>(0.2,0.5,0.9)</td>
<td>(0.6,0.3,0.5)</td>
<td>(0.5,0.3,0.6)</td>
</tr>
</tbody>
</table>

### Table-8 (Cost Matrix)

<table>
<thead>
<tr>
<th>TREATMENTS</th>
<th>( t_1 )</th>
<th>( t_2 )</th>
<th>( t_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( d_1 )</td>
<td>(0.4,0.3,0.8)</td>
<td>(0.3,0.2,0.7)</td>
<td>(0.5,0.4,0.6)</td>
</tr>
<tr>
<td>( d_2 )</td>
<td>(0.4,0.2,0.7)</td>
<td>(0.3,0.3,0.8)</td>
<td>(0.5,0.4,0.6)</td>
</tr>
<tr>
<td>( d_3 )</td>
<td>(0.3,0.2,0.8)</td>
<td>(0.2,0.3,0.9)</td>
<td>(0.4,0.4,0.7)</td>
</tr>
</tbody>
</table>

### Table-9 (Time Matrix)

<table>
<thead>
<tr>
<th>TREATMENTS</th>
<th>( t_1 )</th>
<th>( t_2 )</th>
<th>( t_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( d_1 )</td>
<td>(0.6,0.2,0.3)</td>
<td>(0.5,0.4,0.7)</td>
<td>(0.3,0.2,0.9)</td>
</tr>
<tr>
<td>( d_2 )</td>
<td>(0.6,0.3,0.5)</td>
<td>(0.4,0.2,0.8)</td>
<td>(0.5,0.5,0.6)</td>
</tr>
<tr>
<td>( d_3 )</td>
<td>(0.6,0.4,0.7)</td>
<td>(0.4,0.3,0.8)</td>
<td>(0.5,0.5,0.5)</td>
</tr>
</tbody>
</table>

\[ (F, E) = \{ \text{degree of pain score} \} = \{(d_1, t_1)/(0.5,0.3,0.7), (d_1, t_2)/(0.3,0.2,0.7), (d_1, t_3)/(0.8,0.2,0.3), (d_2, t_1)/(0.5,0.2,0.6), (d_2, t_2)/(0.4,0.3,0.8), (d_2, t_3)/(0.2,0.6,0.9), (d_3, t_1)/(0.6,0.3,0.5), (d_3, t_2)/(0.5,0.3,0.6) \}, \]
\[ \text{degree of cost requirement} = \{(d_1, t_1)/(0.4,0.3,0.8), (d_1, t_2)/(0.3,0.2,0.7), (d_1, t_3)/(0.5,0.4,0.6), (d_2, t_1)/(0.4,0.2,0.7), (d_2, t_2)/(0.3,0.3,0.8), (d_2, t_3)/(0.5,0.4,0.6), (d_3, t_1)/(0.2,0.3,0.9), (d_3, t_2)/(0.4,0.4,0.7) \}, \]
\[ \text{degree of time requirement} = \{(d_1, t_1)/(0.6,0.2,0.3), (d_1, t_2)/(0.5,0.4,0.7), (d_1, t_3)/(0.3,0.2,0.9), (d_2, t_1)/(0.6,0.3,0.5), (d_2, t_2)/(0.4,0.2,0.8), (d_2, t_3)/(0.5,0.5,0.6), (d_3, t_1)/(0.6,0.4,0.7), (d_3, t_2)/(0.4,0.3,0.8), (d_3, t_3)/(0.5,0.5,0.5) \} \}

Here, \[ |E| = \{ \text{pain relief, not requirement of cost, not requirement of time} \} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\} \],
then

\[ (F, E)^\vee = \{ \text{degree of pain relief} \} = \{(d_1, t_1)/(0.7,0.7,0.5), (d_1, t_2)/(0.7,0.8,0.3), (d_1, t_3)/(0.3,0.8,0.8), (d_2, t_1)/(0.6,0.8,0.5), (d_2, t_2)/(0.8,0.2,0.4), (d_2, t_3)/(0.9,0.4,0.2), (d_3, t_1)/(0.8,0.8,0.3), (d_3, t_2)/(0.5,0.7,0.6), (d_3, t_3)/(0.6,0.7,0.5) \}, \]
\[ \text{degree of not requirement of cost} = \{(d_1, t_1)/(0.8,0.7,0.4), (d_1, t_2)/(0.7,0.8,0.3), (d_1, t_3)/(0.6,0.6,0.5), (d_2, t_1)/(0.7,0.8,0.4), (d_2, t_2)/(0.8,0.7,0.3), (d_2, t_3)/(0.6,0.6,0.5), (d_3, t_1)/(0.8,0.8,0.3), (d_3, t_2)/(0.9,0.7,0.2), (d_3, t_3)/(0.7,0.6,0.4) \}, \]
\[ \text{degree of not requirement of time} = \{(d_1, t_1)/(0.3,0.8,0.6), (d_1, t_2)/(0.7,0.6,0.5), (d_1, t_3)/(0.9,0.8,0.3), (d_2, t_1)/(0.5,0.7,0.6), (d_2, t_2)/(0.8,0.8,0.4), (d_2, t_3)/(0.6,0.5,0.5), (d_3, t_1)/(0.7,0.6,0.6), (d_3, t_2)/(0.8,0.7,0.4), (d_3, t_3)/(0.5,0.5,0.5) \} \}

Therefore the tabular representation of \((F, E)^\vee\) is as follows:
Now among the choice values $C_{(d_3,t_j)}; j = 1,2,3$, $C_{(d_3,t_1)}$ is maximum (2.3), which implies that $t_1$ treatment has to be assigned for the disease $d_3$.

But for both the diseases $d_1$ and $d_2$, $C_{(d_j,t_j)}; j = 1,2,3$ take the maximum value at $j = 2$, i.e., for the assignment of $t_2$ treatment there is a tie between the diseases $d_1$ and $d_2$. We have to consider the difference value $V_{d_{(d_j,t_j)}}; i = 1,2; j = 2,3$ between the maximum and the next to maximum choice values (corresponding to those treatments which are not yet assigned).

Now since $V_{d_{(d_1,t_2)}} = 0.3 = V_{d_{(d_2,t_2)}}$ for $j = 2,3$; we have to consider the rejection values. But for both the diseases $d_1$ and $d_2$, $R_{(d_j,t_j)}; j = 1,2,3$ take the minimum value at $j = 2$, therefore we have to consider their confusion values. Now since $\zeta_{(d_2,t_2)}; j = 2,3$ take the minimum value (1.7) at $j = 2$, $t_2$ treatment has to be assigned for the disease $d_2$ and the rest treatment $t_3$ is assigned for the disease $d_1$.

### Table 10

<table>
<thead>
<tr>
<th>$(d_1,t_1)$</th>
<th>$(d_1,t_2)$</th>
<th>$(d_1,t_3)$</th>
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### References


Received: January 31, 2017. Accepted: February 17, 2017.
GRA for Multi Attribute Decision Making in Neutrosophic Cubic Set Environment

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Abstract. In this paper, multi attribute decision making problem based on grey relational analysis in neutrosophic cubic set environment is investigated. In the decision making situation, the attribute weights are considered as single valued neutrosophic sets. The neutrosophic weights are converted into crisp weights. Both positive and negative GRA coefficients, and weighted GRA coefficients are determined.

Keywords: Grey relational coefficient, interval valued neutrosophic set, multi attribute decision making, neutrosophic set, neutrosophic cubic set, relative closeness coefficient

1 Introduction

In management section, banking sector, factory, plant multi attribute decision making (MADM) problems are to be extensively encountered. In a MADM situation, the most appropriate alternative is selecting from the set of alternatives based on highest degree of acceptance. In a decision making situation, decision maker (DM) considers the efficiency of each alternative with respect to each attribute. In crisp MADM, there are several approaches [1, 2, 3, 4, 5] in the literature. The weight of each attribute and the elements of decision matrix are presented by crisp numbers. But in real situation, DMs may prefer to use linguistic variables like ‘good’, ‘bad’, ‘hot’, ‘cold’, ‘tall’, etc. So, there is an uncertainty in decision making situation which can be mathematically explained by fuzzy set [6]. Zadeh [6] explained uncertainty mathematically by defining fuzzy set (FS). Bellman and Zadeh [7] studied decision making in fuzzy environment. Atanassov [8, 9] narrated uncertainty by introducing non-membership as independent component and defined intuitionistic fuzzy set (IFS). Degree of indeterminacy (hesitancy) is not independent.

maximizing deviation method for neutrosophic MADM with incomplete weight information. Ye [27] studied bidirectional projection method for MADM with neutrosophic numbers of the form $a + bi$, where $i$ is characterized by indeterminacy. Biswas et al. [28] presented value and ambiguity index based ranking method of single-valued trapezoidal neutrosophic numbers and its application to MADM. Dey et al. [29] studied extended projection-based models for solving MADM problems with interval-valued neutrosophic information.


Several neutrosophic hybrid sets have been recently proposed in the literature, such as neutrosophic soft set proposed by Maji [39], single valued soft expert set proposed by Broumi and Smarandache [40], rough neutrosophic set proposed by Broumi, et al. [41], neutrosophic bipolar set proposed by Deli et al. [42], rough bipolar neutrosophic set proposed by Pramanik and Mondal [43], neutrosophic cubic set proposed by Jun et al. [44] and Ali et al. [45]. Jun et al. [44] presented the concept of neutrosophic cubic set by extending the concept of cubic set proposed by Jun et al. [46] and introduced the notions of truth-internal (indeterminacy-internal, falsity-internal) neutrosophic cubic sets and truth-external (indeterminacy-external, falsity-external) and investigated related properties. Ali et al. [45] presented concept of neutrosophic cubic set by extending the concept of cubic set [46] and defined internal neutrosophic cubic set (INCS) and external neutrosophic cubic set (ENCS). In their study, Ali et al.[45] also introduced an adjustable approach to neutrosophic cubic set based decision making.

GRA based MADM/ MCDM problems have been proposed for various neutrosophic hybrid environments [47, 48, 49, 50]. MADM with neutrosophic cubic set is yet to appear in the literature. It is an open area of research in neutrosophic cubic set environment.

The present paper is devoted to develop GRA method for MADM in neutrosophic cubic set environment. The attribute weights are described by single valued neutrosophic sets. Positive and negative grey relational coefficients are determined. We define ideal grey relational coefficients and relative closeness coefficients in neutrosophic cubic set environment. The ranking of alternatives is made in descending order.

The rest of the paper is designed as follows: In Section 2, some relevant definitions and properties are recalled. Section 3 presents MADM in neutrosophic cubic set environment based on GRA. In Section 4, a numerical example is solved to illustrate the proposed approach. Section 5 presents conclusions and future scope of research.

2 Preliminaries

In this section, we recall some established definitions and properties which are connected in the present article.

2.1 Definition (Fuzzy set) [6]

Let W be a universal set. Then a fuzzy set F over W can be defined by $F = \{ w, \mu_F(w) : w \in W \} \text{ where } \mu_F(w) : W \rightarrow [0, 1]$ is called membership function of F and $\mu_F(w)$ is the degree of membership to which w belongs.

2.2 Definition (Interval valued fuzzy set) [52]

Let W be a universal set. Then, an interval valued fuzzy set F over W is defined by $F = \{ [F^-(w), F^+(w)] : w \in W \}$ where $F^-(w)$ and $F^+(w)$ are referred to as the lower and upper degrees of membership w respectively.

0 $\leq F^-(w) + F^+(w) \leq 1$, respectively.

2.3 Definition (Cubic set) [46]

Let W be a non-empty set. A cubic set C in W is of the form $c = \{ w, F(w), \lambda(w) / w \in W \}$ where F is an interval valued fuzzy set in W and $\lambda$ is a fuzzy set in W.

2.4 Definition (Neutrosophic set (NS)) [10]

Let W be a space of points (objects) with generic element w in W. A neutrosophic set N in W is denoted by $N = \{ w, T_N(w), I_N(w), F_N(w) : w \in W \}$ where $T_N$, $I_N$, $F_N$ represent membership, indeterminacy and non-membership function respectively. $T_N$, $I_N$, $F_N$ can be defined as follows:

$T_N : W \rightarrow ]0, 1[$

$I_N : W \rightarrow ]0, 1[$

$F_N : W \rightarrow ]0, 1[$

Here, $T_N(w), I_N(w), F_N(w)$ are the real standard and non-standard subset of $]0, 1[$ and

$0 \leq T_N(w) + I_N(w) + F_N(w) \leq 3$.

2.5 Definition (Complement of neutrosophic set) [10]

The complement of a neutrosophic set N is denoted by $N'$ and defined as
2.6 Definition (Containment) [10, 20]

A neutrosophic set P is contained in the other neutrosophic set Q, P ⊆ Q, if and only if

\[
\inf(T_P) \leq \inf(T_Q), \sup(T_P) \leq \sup(T_Q), \\
\inf(I_P) \geq \inf(I_Q), \sup(I_P) \geq \sup(I_Q), \\
\inf(F_P) \geq \inf(F_Q), \sup(F_P) \geq \sup(F_Q).
\]

2.7 Definition (Union) [10]

The union of two neutrosophic sets P and Q is a neutrosophic set R, written as R = P ∪ Q, whose truth-membership, indeterminacy-membership and falsity-membership functions are related to those of P and Q by

\[
T_R(w) = T_P(w) + T_Q(w) - T_P(w) \times T_Q(w), \\
I_R(w) = I_P(w) + I_Q(w) - I_P(w) \times I_Q(w), \\
F_R(w) = F_P(w) + F_Q(w) - F_P(w) \times F_Q(w),
\]

for all w ∈ W.

2.8 Definition (Intersection) [10]

The intersection of two neutrosophic sets P and Q is a neutrosophic set R, written as R = P ∩ Q, whose truth-membership, indeterminacy-membership and falsity-membership functions are related to those of P and Q by

\[
T_R(w) = T_P(w) \times T_Q(w), \\
I_R(w) = I_P(w) \times I_Q(w), \\
F_R(w) = F_P(w) \times F_Q(w),
\]

for all w ∈ W.

2.9 Definition (Hamming distance) [20, 53]

Let P = \{< w_i : T_P(w_i), I_P(w_i), F_P(w_i), i = 1, 2, ..., n >\} and Q = \{< w_i : T_Q(w_i), I_Q(w_i), F_Q(w_i), i = 1, 2, ..., n >\} be any two neutrosophic sets. Then the Hamming distance between P and Q can be defined as follows:

\[
d(P, Q) = \sum_{i=1}^{n} \left| T_P(w_i) - T_Q(w_i) \right| + \left| I_P(w_i) - I_Q(w_i) \right| + \left| F_P(w_i) - F_Q(w_i) \right|
\]

2.10 Definition (Normalized Hamming distance)

The normalized Hamming distance between two SVNSs, A and B can be defined as follows:

\[
N(d(P, Q) = \frac{1}{3n} \sum_{i=1}^{n} \left| T_P(w_i) - T_Q(w_i) \right| + \left| I_P(w_i) - I_Q(w_i) \right| + \left| F_P(w_i) - F_Q(w_i) \right|
\]

2.11 Definition (Interval neutrosophic set) [51]

Let W be a non-empty set. An interval neutrosophic set (INS) P in W is characterized by the truth-membership function T_P, the indeterminacy-membership function I_P and the falsity-membership function F_P. For each point w ∈ W, \(P_1(w), P_2(w), P_1^i(w)\) ∈ [0, 1]. Here P can be presented as follows:

\[
P = \{< w, [P_1^e(w), P_2^e(w)], [P_1^i(w), P_2^i(w)] : w \in W\}.
\]

2.12 Definition (Neutrosophic cubic set) [44, 45]

Let W be a set. A neutrosophic cubic set (NCS) in W is a pair (P, λ) where P = \{< w, P_1(w), P_2(w), P_3(w) > : w ∈ W\} is an interval neutrosophic set in W and

\[
\lambda = \{< w, \lambda_1(w), \lambda_2(w), \lambda_3(w) > : w \in W\} is a neutrosophic set in W.
\]

3 GRA for MADM in neutrosophic cubic set environment

We consider a MADM problem with r alternatives \{A_1, A_2, ..., A_n\} and s attributes \{C_1, C_2, ..., C_s\}. Every attribute is not equally important to decision maker. Decision maker provides the neutrosophic weights for each attribute. Let W = \{w_1, w_2, ..., w_n\} be the neutrosophic weights of the attributes.

Step 1 Construction of decision matrix

Step 1. The decision matrix (see Table 1) is constructed as follows:

<table>
<thead>
<tr>
<th>A</th>
<th>C_1</th>
<th>C_2</th>
<th>...</th>
<th>C_s</th>
</tr>
</thead>
<tbody>
<tr>
<td>A_1</td>
<td>\lambda_{11}</td>
<td>\lambda_{12}</td>
<td>...</td>
<td>\lambda_{1s}</td>
</tr>
<tr>
<td>A_2</td>
<td>\lambda_{21}</td>
<td>\lambda_{22}</td>
<td>...</td>
<td>\lambda_{2s}</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>...</td>
<td>\vdots</td>
</tr>
<tr>
<td>A_n</td>
<td>\lambda_{n1}</td>
<td>\lambda_{n2}</td>
<td>...</td>
<td>\lambda_{ns}</td>
</tr>
</tbody>
</table>

Here \(\lambda_{ij} = (A_{ij}, \lambda_{ij})\), \(A_{ij} = (T_{ij}, I_{ij}, F_{ij})\), \(\lambda_{ij}\) means the rating of alternative A_i with respect to the attribute C_j. Each weight component \(w_j\) of attribute C_j has been taken as neutrosophic set and

\[w_j = (T_j, I_j, F_j), \quad A_{ij} = (T^i_{ij}, I^i_{ij}, F^i_{ij})\]

are interval neutrosophic set and \(A_{ij} = (T^i_{ij}, I^i_{ij}, F^i_{ij})\) is a neutrosophic set.
Step 2 Crispification of neutrosophic weight

Let \( w_j = \left( T_j, I_j, F_j \right) \) be the \( j \)-th neutrosophic weight for the attribute \( C_j \). The equivalent crisp weight of \( C_j \) is defined as follows:

\[
w_j^c = \frac{\sqrt{T_j^2 + I_j^2 + F_j^2}}{\sum_{j=1}^{n} T_j^2 + I_j^2 + F_j^2} \quad \text{and} \quad \sum_{j=1}^{n} w_j^c = 1.
\]

Step 3 Conversion of interval neutrosophic set into neutrosophic set decision matrix

In the decision matrix (1), each \( A_{ij} = \{T_{ij}, I_{ij}, F_{ij}\} \) is an INS. Taking mid value of each interval decision matrix reduces to single valued neutrosophic decision matrix (See Table 2).

Table 2: Neutrosophic decision matrix

\[
M_{ij} = \left\{ M_{ij}^{c}(m)_{ij}, C_i, C_j \right\} = \begin{bmatrix}
A_1 & \left( M_{11}^{c}(m), A_{11} \right) & \left( M_{12}^{c}(m), A_{12} \right) & \ldots & \left( M_{1n}^{c}(m), A_{1n} \right) \\
A_2 & \left( M_{21}^{c}(m), A_{21} \right) & \left( M_{22}^{c}(m), A_{22} \right) & \ldots & \left( M_{2n}^{c}(m), A_{2n} \right) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
A_r & \left( M_{r1}^{c}(m), A_{r1} \right) & \left( M_{r2}^{c}(m), A_{r2} \right) & \ldots & \left( M_{rn}^{c}(m), A_{rn} \right)
\end{bmatrix}
\]

where each \( m_i = \{M_i, \Lambda_i\} \) and

\[
M_i = \left( \frac{T_{ij}^m + T_{ij}^l}{2}, \frac{I_{ij}^m + I_{ij}^l}{2}, \frac{F_{ij}^m + F_{ij}^l}{2} \right) = (T_{ij}^{m*}, I_{ij}^{m*}, F_{ij}^{m*}).
\]

Step 4 Some definitions of GRA method for MADM with NCS

The GRA method for MADM with NCS can be presented in the following steps:

Step 4.1 Definition:

The ideal neutrosophic estimates reliability solution (INERS) can be denoted as \( \{M^*, \Lambda^*\} = \{M_1, \Lambda_1\}, \{M_2, \Lambda_2\}, \ldots, \{M_n, \Lambda_n\} \)

and defined as \( M_j^* = (T_j^*, I_j^*, F_j^*) \) where \( T_j^* = \max \{T_j^n\} \), \( I_j^* = \min \{I_j^n\} \), \( F_j^* = \min \{F_j^n\} \) and \( \Lambda_j^* = (T_j^*, I_j^*, F_j^*) \)

where \( T_j^* = \max T_{ij}^n, I_j^* = \min I_{ij}^n, F_j^* = \min F_{ij}^n \) in the neutrosophic cubic decision matrix \( M_4(\{m_{ij}\})_{ij}, \quad i = 1, 2, \ldots, r \) and \( j = 1, 2, \ldots, s. \)

Step 4.2 Definition:

The ideal neutrosophic estimates unreliability solution (INEURS) can be denoted as \( \{M^*, \Lambda^*\} = \{M_1, \Lambda_1\}, \{M_2, \Lambda_2\} \ldots, \{M_n, \Lambda_n\} \)

and defined as \( M_j^* = (T_j^m, I_j^m, F_j^m) \) where \( T_j^m = \min T_{ij}^n \), \( I_j^m = \max I_{ij}^n \), \( F_j^m = \max F_{ij}^n \) and \( \Lambda_j^* = (T_j^*, I_j^*, F_j^*) \)

where \( T_j^m = \min T_{ij}^n, I_j^m = \max I_{ij}^n, F_j^m = \max F_{ij}^n \) in the neutrosophic cubic decision matrix \( M_4(\{m_{ij}\})_{ij}, \quad i = 1, 2, \ldots, r \) and \( j = 1, 2, \ldots, s. \)

Step 4.3 Definition:

The grey relational coefficients of each alternative from INERS can be defined as:

\[
\left( \eta_{ij}^r, \xi_{ij}^r \right) = \begin{cases} 
\min \min \delta_{ij}^r + \lambda \max \max \delta_{ij}^r, & \min \min \Omega_{ij}^r + \lambda \max \max \Omega_{ij}^r \\
\delta_{ij}^r + \lambda \max \max \delta_{ij}^r, & \Omega_{ij}^r + \lambda \max \max \Omega_{ij}^r 
\end{cases}
\]

Here,

\[
\delta_{ij}^r = d\left( M_j^*, M_{ij} \right) = \sum_{i=1}^{r} \left( |T_{ij}^m - T_{ij}^n| + |I_{ij}^m - I_{ij}^n| + |F_{ij}^m - F_{ij}^n| \right)
\]

and

\[
\Omega_{ij}^r = d\left( \Lambda_j^*, \Lambda_{ij} \right) = \sum_{i=1}^{r} \left( |T_{ij}^* - T_{ij}^n| + |I_{ij}^* - I_{ij}^n| + |F_{ij}^* - F_{ij}^n| \right),
\]

\( i = 1, 2, \ldots, r \) and \( j = 1, 2, \ldots, s, \lambda \in [0,1] \).

We call \( \left( \eta_{ij}^r, \xi_{ij}^r \right) \) as positive grey relational coefficient.

Step 4.4 Definition:

The grey relational coefficient of each alternative from INEURS can be defined as:

\[
\left( \eta_{ij}^r, \xi_{ij}^r \right) = \begin{cases} 
\min \min \delta_{ij}^r + \lambda \max \max \delta_{ij}^r, & \min \min \Omega_{ij}^r + \lambda \max \max \Omega_{ij}^r \\
\delta_{ij}^r + \lambda \max \max \delta_{ij}^r, & \Omega_{ij}^r + \lambda \max \max \Omega_{ij}^r 
\end{cases}
\]

Here,

\[
\delta_{ij}^r = d\left( M_j^*, M_{ij} \right) = \sum_{i=1}^{r} \left( |T_{ij}^m - T_{ij}^n| + |I_{ij}^m - I_{ij}^n| + |F_{ij}^m - F_{ij}^n| \right)
\]

and:

\[
\Omega_{ij}^r = d\left( \Lambda_j^*, \Lambda_{ij} \right) = \sum_{i=1}^{r} \left( |T_{ij}^* - T_{ij}^n| + |I_{ij}^* - I_{ij}^n| + |F_{ij}^* - F_{ij}^n| \right),
\]

\( i = 1, 2, \ldots, r \) and \( j = 1, 2, \ldots, s, \lambda \in [0,1] \).

We call \( \left( \eta_{ij}^r, \xi_{ij}^r \right) \) as negative grey relational coefficient.

\( \lambda \) is called distinguishable coefficient or identification coefficient and it is used to reflect the range of comparison environment that controls the level of differences of the grey relational coefficient. \( \lambda = 0 \) indicates comparison environment disappears and \( \lambda = 1 \) indicates comparison environment is unaltered. Generally, \( \lambda = 0.5 \) is assumed for decision making.
Step 4.5 Calculation of weighted grey relational coefficients for MADM with NCS

We can construct two $r \times s$ order matrices namely $M_{c1} = \left( \eta_{ij}, \bar{\eta}_{ij} \right)_{rs}$ and $M_{c2} = \left( \xi_{ij}, \bar{\xi}_{ij} \right)_{rs}$. The crisp weight is to be multiplied with the corresponding elements of $M_{c1}$ and $M_{c2}$ to obtain weighted matrices $w M_{c1}$ and $w M_{c2}$ and defined as:

$$w M_{c1} = \left( w_{ij} \eta_{ij}, w_{ij} \bar{\eta}_{ij} \right)_{rs}$$

and

$$w M_{c2} = \left( w_{ij} \xi_{ij}, w_{ij} \bar{\xi}_{ij} \right)_{rs}$$

Step 4.6

From the definition of grey relational coefficient, it is clear that grey relational coefficients of both types must be less than equal to one. This claim is going to be proved in the following theorems.

Theorem 1

The positive grey relational coefficient is less than unity i.e. $\eta_i \leq 1$, and $\xi_i \leq 1$.

Proof:

From the definition

$$\eta_i = \frac{\min_j \delta_j + \lambda \max_j \delta_j}{\delta_i + \lambda \max_j \delta_j} \leq 1$$

Now, $\min_j \delta_j \leq \delta^*$

$$\Rightarrow \min_i \delta_i + \lambda \max_j \delta_j \leq \delta^* + \lambda \max_j \delta_j$$

$$\Rightarrow \frac{\min_i \delta_i + \lambda \max_j \delta_j}{\delta_i + \lambda \max_j \delta_j} \leq 1$$

$$\Rightarrow \eta_i \leq 1$$

Again, from the definition, we can write:

$$\xi_i = \frac{\min_j \Omega_j + \lambda \max_j \Omega_j}{\Omega_i + \lambda \max_j \Omega_j} \leq 1$$

Now, $\min_j \Omega_j \leq \Omega^*$

$$\Rightarrow \min_i \Omega_i + \lambda \max_j \Omega_j \leq \Omega^* + \lambda \max_j \Omega_j$$

$$\Rightarrow \frac{\min_i \Omega_i + \lambda \max_j \Omega_j}{\Omega_i + \lambda \max_j \Omega_j} \leq 1$$

$$\Rightarrow \xi_i \leq 1$$

The negative grey relational coefficient is less than unity i.e. $\eta_i \leq 1$, $\xi_i \leq 1$.

Proof:

From the definition, we can write

$$\eta_i = \frac{\min_j \delta_j + \lambda \max_j \delta_j}{\delta_i + \lambda \max_j \delta_j}$$

Now, $\min_j \delta_j \leq \delta^*$

$$\Rightarrow \min_j \delta_j + \lambda \max_j \delta_j \leq \delta^* + \lambda \max_j \delta_j$$

$$\Rightarrow \frac{\min_j \delta_j + \lambda \max_j \delta_j}{\delta_i + \lambda \max_j \delta_j} \leq 1$$

$$\Rightarrow \eta_i \leq 1$$

Again, from the definition

$$\xi_i = \frac{\min_j \Omega_j + \lambda \max_j \Omega_j}{\Omega_i + \lambda \max_j \Omega_j}$$

Now, $\min_j \Omega_j \leq \Omega^*$

$$\Rightarrow \min_j \Omega_j + \lambda \max_j \Omega_j \leq \Omega^* + \lambda \max_j \Omega_j$$

$$\Rightarrow \frac{\min_j \Omega_j + \lambda \max_j \Omega_j}{\Omega_i + \lambda \max_j \Omega_j} \leq 1$$

$$\Rightarrow \xi_i \leq 1$$

Note 1:

i. Since $\eta_i \leq 1$, $w_i \leq 1$ then $\eta_i w_i \leq 1$ \Rightarrow $\eta_i \leq 1$

ii. Since $\eta_i \leq 1$, $w_i \leq 1$ then $\eta_i w_i \leq 1$ \Rightarrow $\eta_i \leq 1$

iii. Since $\xi_i \leq 1$, $w_i \leq 1$ then $\xi_i w_i \leq 1$ \Rightarrow $\xi_i \leq 1$

iv. Since $\xi_i \leq 1$, $w_i \leq 1$ then $\xi_i w_i \leq 1$ \Rightarrow $\xi_i \leq 1$

Step 4.7

We define the ideal or standard grey relational coefficient as (1, 1). Then we construct ideal grey relational coefficient matrix of order $r \times s$ (see Table 3).

Table 3: Ideal grey relational coefficient matrix of order $r \times s$

<table>
<thead>
<tr>
<th>(1,1) (1,1) (1,1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,1) (1,1) (1,1)</td>
</tr>
<tr>
<td>..................</td>
</tr>
<tr>
<td>(1,1) (1,1) (1,1)</td>
</tr>
</tbody>
</table>

Step 5 Determination of Hamming distances

We find the distance $d_i$ between the corresponding elements of i-th row of I and $w M_{c1}$ by employing Hamming
distance. Similarly, $d_i^+$ can be determined between 1 and $M_{ca}$ by employing Hamming distance as follows:

\[
d_i^+ = \frac{1}{2s} \left( \sum_{j=1}^{r} \left[ \left| \bar{v}_{y_j} - \bar{v}_{y_j} \right| + \left| \bar{v}_{y_j} - \bar{v}_{y_j} \right| \right] \right), \quad i = 1, 2, ..., r.
\]

\[
d_i^- = \frac{1}{2s} \left( \sum_{j=1}^{r} \left[ \left| \bar{v}_{y_j} - \bar{v}_{y_j} \right| + \left| \bar{v}_{y_j} - \bar{v}_{y_j} \right| \right] \right), \quad i = 1, 2, ..., r.
\]

**Step 6 Determination of relative closeness coefficient**

The relative closeness coefficient can be calculated as:

\[\Delta_i = \frac{d_i^-}{d_i^+ + d_i^-}, \quad i = 1, 2, ..., r.\]

**Step 7 Ranking the alternatives**

According to the relative closeness coefficient, the ranking order of all alternatives is determined. The ranking order is made according to descending order of relative closeness coefficients.

### 4 Numerical example

Consider a hypothetical MADM problem. The problem consists of single decision maker, three alternatives with three attributes \{A_1, A_2, A_3\} and four attributes \{C_1, C_2, C_3, C_4\}. The solution of the problem is presented using the following steps:

**Step 1. Construction of neutrosophic cubic decision matrix**

The decision maker forms the decision matrix which is displayed in the Table 4, at the end of article.

**Step 2. Crispification of neutrosophic weight set**

The neutrosophic weights of the attributes are taken as:

\[W = \{0.5, 0.2, 0.1, 0.6, 0.1, 0.1\}, \{0.9, 0.2, 0.1\}, \{0.6, 0.3, 0.4\}\]

The equivalent crisp weights are

\[W^+ = \{0.1907, 0.2146, 0.3228\}, \{0.2719\}\]

**Step 3 Conversion of interval neutrosophic set into neutrosophic set in decision matrix**

Taking the mid value of INS in the Table 4, the new decision matrix is presented in the following Table 5, at the end of article.

**Step 4 Some Definitions of GRA method for MADM with NCS**

The ideal neutrosophic estimates reliability solution (INERS) \(M^* \cdot \Lambda^*\) and the ideal neutrosophic estimates unreliability solution (INEURS) \(M^- \cdot \Lambda^-\) are presented in the Table 6, at the end of article.

\[\delta^- = (\delta^-) = (d(M^*, M^-)) \forall i, j\]

The \(\Omega^- = (\delta^-) = (d(M^*, M^-)) \forall i, j\) is presented as below:

\[
\delta^- = \begin{pmatrix}
0.85 & 0.95 & 0.05 & 0.15 \\
0.65 & 0 & 0.7 & 0.25 \\
0.05 & 0.15 & 0.25 & 0.45
\end{pmatrix}
\]

The \(\Omega^+ = (\delta^+) = (d(M^*, M^-)) \forall i, j\) is presented as below:

\[
\delta^+ = \begin{pmatrix}
0.45 & 1.2 & 0.4 & 0.15 \\
0.05 & 0.5 & 0.2 & 0.2 \\
0.25 & 0.3 & 0.2 & 0.5
\end{pmatrix}
\]

The \(\Omega^- = (\delta^-) = (d(M^*, M^-)) \forall i, j\) is presented as below:

\[
\delta^- = \begin{pmatrix}
0.25 & 0.3 & 0.7 & 0.55 \\
0.45 & 1.2 & 0 & 0.45 \\
1.05 & 0.65 & 0.5 & 0.25
\end{pmatrix}
\]

The \(\Omega^+ = (\delta^+) = (d(M^*, M^-)) \forall i, j\) is presented as below:

\[
\delta^+ = \begin{pmatrix}
0.844966 & 0.83845625, \\
0.82444375, & 0.85328875, \\
0.82368675, & 0.852777
\end{pmatrix}
\]

**Step 5 Determination of Hamming distances**

Hamming distances are calculated as follows:

\[d_i^- = 0.844966, \quad d_i^+ = 0.83845625, \quad d_i^- = 0.82444375, \quad d_i^- = 0.85328875, \quad d_i^- = 0.82368675, \quad d_i^- = 0.852777.
\]

**Step 6 Determination of relative closeness coefficient**

The relative closeness coefficients are calculated as:

\[\Delta_i = \frac{d_i^-}{d_i^+ + d_i^-} = 0.501932\]

\[\Delta_i = \frac{d_i^-}{d_i^+ + d_i^-} = 0.491403576\]

\[\Delta_i = \frac{d_i^-}{d_i^+ + d_i^-} = 0.49132\]

**Step 7 Ranking the alternatives**

The ranking of alternatives is made according to descending order of relative closeness coefficients. The ranking order is shown in the Table 11 below.
### Alternatives | Ranking order
--- | ---
A₃ | 1
A₂ | 2
A₁ | 3

**Conclusion**
This paper develops GRA based MADM in neutrosophic cubic set environment. This is the first approach of GRA in MADM in neutrosophic cubic set environment. The proposed approach can be applied to other decision making problems such as pattern recognition, personnel selection, etc. The proposed approach can be applied for decision making problem described by internal NCSs and external NCSs. We hope that the proposed approach will open up a new avenue of research in newly developed neutrosophic cubic set environment.

**References**


Received: February 1, 2017. Accepted: February 20, 2017.
**Table 4: Construction of neutrosophic cubic decision matrix**

\[ A = \begin{pmatrix} \mathcal{C}_1 \\
(0.2,0.3,0.5), (0.3,0.2,0.3)) \\
& (0.1,0.3), (0.2,0.4,0.3,0.6), (0.2,0.5,0.4)) \\
(0.6,0.9), (0.1,0.2,0.2,0.3), (0.4,0.5,0.1)) \\
& (0.4,0.7), (0.1,0.3,0.2,0.3), (0.7,0.3,0.2)) \\
\end{pmatrix} \]

**Table 5: Construction of neutrosophic decision matrix**

\[ M = \begin{pmatrix} \mathcal{C}_1 \\
(0.25,0.4,0.35), (0.3,0.2,0.3)) \\
& (0.2,0.3,0.45), (0.2,0.5,0.4)) \\
(0.75,0.15,0.1), (0.4,0.5,0.1)) \\
& (0.55,0.2,0.25), (0.7,0.3,0.2)) \\
\end{pmatrix} \]

**Table 6: The ideal neutrosophic estimates reliability solution (INERS) \((M^+, \Lambda^-)\) and the ideal neutrosophic estimates unreliability solution (INEURS) \((M^-, \Lambda^-)\)**

\[
\begin{array}{|c|c|c|}
\hline
(M^+, \Lambda^-) & (0.7,0.25,0.1), (0.5,0.15,0.1) & (0.85,0.25,0.2), (0.8,0.1,0.2) & (0.75,0.15,0.1), (0.5,0.1,0.1) & (0.7,0.2,0.25), (0.7,0.1,0.1) \\
\hline
(M^-, \Lambda^-) & (0.25,0.5,0.5), (0.25,0.2,0.3) & (0.2,0.55,0.5), (0.2,0.5,0.4) & (0.6,0.5,0.4), (0.4,0.5,0.3) & (0.45,0.6,0.3), (0.5,0.3,0.4) \\
\hline
\end{array}
\]

**Table 7: The positive grey relational coefficient \(M_{ca} = (\eta_{ca}, \xi_{ca})\)**

\[
M_{ca} = \begin{pmatrix}
(0.3585,0.6190) & (0.333,0.3611) & (0.9048,0.65) & (0.76,0.7222) \\
(0.4222,1) & (1,0.5909) & (0.4042,0.8125) & (0.6552,0.8125) \\
(0.9048,0.7647) & (0.76,0.7222) & (0.6552,0.8125) & (0.5135,0.5909)
\end{pmatrix}
\]
Table 8: The negative grey relational coefficient $M_{\text{GR}} = (\eta, \xi)$

\[
M_{\text{GR}} = \begin{pmatrix}
(0.7059, 0.5454) & (0.6667, 1) & (0.4615, 0.75) & (0.5217, 0.6) \\
(0.5714, 0.5714) & (0.3333, 0.4286) & (1, 0.5454) & (0.5714, 0.5454) \\
(0.3636, 0.7059) & (0.48, 0.3333) & (0.5, 0.75) & (0.7059, 0.75)
\end{pmatrix}
\]

Table 9: Weighted matrix $w^{M_{\text{GR}}}$

\[
w^{M_{\text{GR}}} = \begin{pmatrix}
(0.06836, 0.11804) & (0.07153, 0.07749) & (0.29207, 0.20982) & (0.20664, 0.19637) \\
(0.08051, 0.1907) & (0.2146, 0.12681) & (0.13048, 0.26228) & (0.17815, 0.22092) \\
(0.17252, 0.14583) & (0.163096, 0.15498) & (0.21150, 0.26228) & (0.13962, 0.16066)
\end{pmatrix}
\]

Table 10: Weighted matrix $w^{M_{\text{GR}}}$

\[
w^{M_{\text{GR}}} = \begin{pmatrix}
(0.13461, 0.10401) & (0.14307, 0.2146) & (0.14897, 0.2421) & (0.14185, 0.16314) \\
(0.10896, 0.10896) & (0.07153, 0.08173) & (0.3228, 0.17606) & (0.15536, 0.14829) \\
(0.06934, 0.13461) & (0.10301, 0.07153) & (0.1614, 0.2421) & (0.19193, 0.20392)
\end{pmatrix}
\]
Bipolar Neutrosophic Projection Based Models for Solving Multi-attribute Decision Making Problems

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Abstract. Bipolar neutrosophic sets are the extension of neutrosophic sets and are based on the idea of positive and negative preferences of information. Projection measure is a useful apparatus for modelling real life decision making problems. In the paper, we define projection, bidirectional projection and hybrid projection measures between bipolar neutrosophic sets. Three new methods based on the proposed projection measures are developed for solving multi-attribute decision making problems. In the solution process, the ratings of performance values of the alternatives with respect to the attributes are expressed in terms of bipolar neutrosophic values. We calculate projection, bidirectional projection, and hybrid projection measures between each alternative and ideal alternative with bipolar neutrosophic information. All the alternatives are ranked to identify the best alternative. Finally, a numerical example is provided to demonstrate the applicability and effectiveness of the developed methods. Comparison analysis with the existing methods in the literature in bipolar neutrosophic environment is also performed.

Keywords: Bipolar neutrosophic sets; projection measure; bidirectional projection measure; hybrid projection measure; multi-attribute decision making.

1 Introduction

For describing and managing indeterminate and inconsistent information, Smarandache [1] introduced neutrosophic set which has three independent components namely truth membership degree (T), indeterminacy membership degree (I) and falsity membership degree (F) where T, I, and F lie in [0, 1]. Later, Wang et al. [2] proposed single valued neutrosophic set (SVNS) to deal real decision making problems where T, I, and F lie in [0, 1].

Zhang [3] grounded the notion of bipolar fuzzy sets by extending the concept of fuzzy sets [4]. The value of membership degree of an element of bipolar fuzzy set belongs to [-1, 1]. With reference to a bipolar fuzzy set, the membership degree zero of an element reflects that the element is irrelevant to the corresponding property, the membership degree belongs to (0, 1] of an element reflects that the element somewhat satisfies the property, and the membership degree belongs to [-1, 0) of an element reflects that the element somewhat satisfies the implicit counter-property.

Deli et al. [5] extended the concept of bipolar fuzzy set to bipolar neutrosophic set (BNS). With reference to a bipolar neutrosophic set Q, the positive membership degrees $T^+_Q(x)$, $I^+_Q(x)$, and $F^+_Q(x)$ represent respectively the truth membership, indeterminate membership and falsity membership of an element $x \in X$ corresponding to the bipolar neutrosophic set $Q$ and the negative membership degrees $T^-_Q(x)$, $I^-_Q(x)$, and $F^-_Q(x)$ denote respectively the truth membership, indeterminate membership and false membership degree of an element $x \in X$ to some implicit counter-property corresponding to the bipolar neutrosophic set $Q$.

Projection measure is a useful decision making device as it takes into account the distance as well as the included angle for measuring the closeness degree between two objects [6, 7]. Yue [6] and Zhang et al. [7] studied projection based multi-attribute decision making (MADM) in crisp environment i.e. projections are defined by ordinary numbers or crisp numbers. Yue [8] further investigated a new multi-attribute group decision making (MAGDM) method based on determining the weights of the decision makers by employing projection technique with interval data. Yue and Jia [9] established a methodology for MAGDM based on a new normalized projection measure, in which the attribute values are provided by decision makers in hybrid form with crisp values and interval data.


In neutrosophic environment, Chen and Ye [18] developed projection based model of neutrosophic numbers and presented MADM method to select clay-bricks in construction field. Bidirectional projection measure [19, 20] considers the distance and included angle between two vectors x, y. Ye [19] defined bidirectional projection measure as an improvement of the general projection measure of SVNSs to overcome the drawback of the general projection measure. In the same study, Ye [19] developed MADM method for selecting problems of mechanical design schemes under a single-valued neutrosophic environment. Ye [20] also presented bidirectional projection method for MAGDM with neutrosophic numbers.

Ye [21] defined credibility – induced interval neutrosophic weighted arithmetic averaging method and credibility – induced interval neutrosophic weighted geometric averaging operator and developed the projection measure based ranking method for MADM problems with interval neutrosophic information and credibility information. Dey et al. [22] proposed a new approach to neutrosophic soft MADM using grey relational projection method. Dey et al. [23] defined weighted projection measure with interval neutrosophic assessments and applied the proposed concept to solve MAGDM problems with interval valued neutrosophic information. Pramanik et al. [24] defined projection and bidirectional projection measures between rough neutrosophic sets and proposed two new multi-criteria decision making (MCDM) methods based on projection and bidirectional projection measures in rough neutrosophic set environment.

In the field of bipolar neutrosophic environment, Deli et al. [5] defined score, accuracy, and certainty functions in order to compare BNSs and developed bipolar neutrosophic weighted average (BNWA) and bipolar neutrosophic weighted geometric (BNWG) operators to obtain collective bipolar neutrosophic information. In the same study, Deli et al. [5] also proposed a MCDM approach on the basis of score, accuracy, and certainty functions and BNWA, BNWG operators. Deli and Subas [25] presented a single valued bipolar neutrosophic MCDM through correlation coefficient similarity measure. Şahin et al. [26] provided a MCDM method based on Jaccard similarity measure of BNS. Ulucaý et al. [27] defined Dice similarity, weighted Dice similarity, hybrid vector similarity, weighted hybrid vector similarity measures under BNSs and developed MCDM methods based on the proposed similarity measures. Dey et al. [28] defined Hamming and Euclidean distance measures to compute the distance between BNSs and investigated a TOPSIS approach to derive the most desirable alternative.

In this study, we define projection, bidirectional projection and hybrid projection measures under bipolar neutrosophic information. Then, we develop three methods for solving MADM problems with bipolar neutrosophic assessments. We organize the rest of the paper in the following way. In Section 2, we recall several useful definitions concerning SVNSs and BNSs. Section 3 defines projection, bidirectional projection and hybrid projection measures between BNSs. Section 4 is devoted to present three models for solving MADM under bipolar neutrosophic environment. In Section 5, we solve a decision making problem with bipolar neutrosophic information on the basis of the proposed measures. Comparison analysis is provided to demonstrate the feasibility and flexibility of the proposed methods in Section 6. Finally, Section 7 provides conclusions and future scope of research.

2 Basic Concepts Regarding SVNSs and BNSs
In this Section, we provide some basic definitions regarding SVNSs, BNSs which are useful for the construction of the paper.

2.1 Single valued neutrosophic sets [2]
Let X be a universal space of points with a generic element of X denoted by x, then a SVNS P is characterized by a truth membership function $T_p(x)$, an indeterminate membership function $I_p(x)$ and a falsity membership function $F_p(x)$. A SVNS P is expressed in the following way:

$$P = \{x, T_p(x), I_p(x), F_p(x)\} \mid x \in X$$

where, $T_p(x)\cup I_p(x)\cup F_p(x) : X \rightarrow [0, 1]$ and $0 \leq T_p(x) + I_p(x) + F_p(x) \leq 3$ for each point $x \in X$.

2.2 Bipolar neutrosophic set [5]
Consider X be a universal space of objects, then a BNS Q in X is presented as follows:

$$Q = \{x, T_Q^+(x), I_Q^+(x), F_Q^+(x), T_Q^-(x), I_Q^-(x), F_Q^-(x)\} \mid x \in X$$,
where \( T^+_Q (x) \), \( I^+_Q (x) \), \( F^+_Q (x) : X \rightarrow [0, 1] \) and \( T^-_Q (x) \), \( I^-_Q (x) \), \( F^-_Q (x) : X \rightarrow [-1, 0] \). The positive membership degrees \( T^+_Q (x) \), \( I^+_Q (x) \), \( F^+_Q (x) \) denote the truth membership, indeterminate membership, and falsity membership functions of an element \( x \in X \) corresponding to a BNS \( Q \) and the negative membership degrees \( T^-_Q (x) \), \( I^-_Q (x) \), \( F^-_Q (x) \) denote the truth membership, indeterminate membership, and falsity membership of an element \( x \in X \) to several implicit counterparty associated with a BNS \( Q \). For convenience, a bipolar neurospheric value (BNSV) is presented as \( \tilde{q} = < T^+_Q, I^+_Q, F^+_Q, T^-_Q, I^-_Q, F^-_Q >. \)

**Definition 1 [5]**

Let, \( Q_1 = \{x, \{T^+_Q (x), I^+_Q (x), F^+_Q (x), T^-_Q (x), I^-_Q (x), F^-_Q (x)\} \mid x \in X\} \) and \( Q_2 = \{x, \{T^+_Q (x), I^+_Q (x), F^+_Q (x), T^-_Q (x), I^-_Q (x), F^-_Q (x)\} \mid x \in X\} \) be any two BNSVs. Then \( Q_1 \subseteq Q_2 \) if and only if \( T^+_Q (x) \leq T^+_Q (x), I^+_Q (x) \leq I^+_Q (x), F^+_Q (x) \geq F^+_Q (x) \); \( T^-_Q (x) \geq T^-_Q (x), I^-_Q (x) \geq I^-_Q (x), F^-_Q (x) \leq F^-_Q (x) \) for all \( x \in X \).

**Definition 2 [5]**

Let, \( Q_1 = \{x, \{T^+_Q (x), I^+_Q (x), F^+_Q (x), T^-_Q (x), I^-_Q (x), F^-_Q (x)\} \mid x \in X\} \) and \( Q_2 = \{x, \{T^+_Q (x), I^+_Q (x), F^+_Q (x), T^-_Q (x), I^-_Q (x), F^-_Q (x)\} \mid x \in X\} \) be any two BNSVs. Then \( Q_1 = Q_2 \) if and only if \( T^+_Q (x) = T^+_Q (x), I^+_Q (x) = I^+_Q (x), F^+_Q (x) = F^+_Q (x) \); \( T^-_Q (x) = T^-_Q (x), I^-_Q (x) = I^-_Q (x), F^-_Q (x) = F^-_Q (x) \) for all \( x \in X \).

**Definition 3 [5]**

Let, \( Q = \{x, \{T^+_Q (x), I^+_Q (x), F^+_Q (x), T^-_Q (x), I^-_Q (x), F^-_Q (x)\} \mid x \in X\} \) be a BNS. The complement of \( Q \) is represented by \( Q^c \) and is defined as follows:

\[
T^+_Q (x) = \{1^+\} - T^-_Q (x), I^+_Q (x) = \{1^+\} - I^-_Q (x), F^+_Q (x) = \{1^+\} - F^-_Q (x);
\]
\[
T^-_Q (x) = \{1^-\} - T^+_Q (x), I^-_Q (x) = \{1^-\} - I^+_Q (x), F^-_Q (x) = \{1^-\} - F^+_Q (x).
\]

**Definition 4 [5]**

Let, \( Q_1 = \{x, \{T^+_Q (x), I^+_Q (x), F^+_Q (x), T^-_Q (x), I^-_Q (x), F^-_Q (x)\} \mid x \in X\} \) and \( Q_2 = \{x, \{Q^c (x), I^c (x), F^c (x), T^c (x), I^c (x), F^c (x)\} \mid x \in X\} \) be any two BNSVs. Their union \( Q_1 \cup Q_2 \) is defined as follows:

\[
\begin{align*}
Q_1 \cup Q_2 &= \{\max (T^+_Q (x), T^-_Q (x)), \min (I^+_Q (x), I^-_Q (x)), \min (F^+_Q (x), F^-_Q (x)), \max (I^+_Q (x), I^-_Q (x)), \max (F^+_Q (x), F^-_Q (x)) \}, \forall x \in X.
\end{align*}
\]

Their intersection \( Q_1 \cap Q_2 \) is defined as follows:

\[
\begin{align*}
Q_1 \cap Q_2 &= \{\min (T^+_Q (x), T^-_Q (x)), \max (I^+_Q (x), I^-_Q (x)), \\
&\min (F^+_Q (x), F^-_Q (x)), \max (I^+_Q (x), I^-_Q (x)), \max (F^+_Q (x), F^-_Q (x)) \}, \forall x \in X.
\end{align*}
\]

**Definition 5 [5]**

Let \( \tilde{q}_1 = < T^+_Q, I^+_Q, F^+_Q, T^-_Q, I^-_Q, F^-_Q > \) and \( \tilde{q}_2 = < T^+_Q, I^+_Q, F^+_Q, T^-_Q, I^-_Q, F^-_Q > \) be any two BNSVs, then

i. \( \beta \cdot \tilde{q}_1 = < 1 - (1 - T^+_Q)^\beta, (I^+_Q)^\beta, (F^+_Q)^\beta, -(T^-_Q)^\beta, -1 - (1 - F^-_Q)^\beta >; \)

ii. \( (\tilde{q}_1)\beta = < (T^+_Q)^\beta, 1 - (1 - I^+_Q)^\beta, 1 - (1 - F^+_Q)^\beta, 1 - (1 - (F^-_Q)^\beta >; \)

iii. \( \tilde{q}_1 + \tilde{q}_2 = < T^+_Q + T^-_Q, T^-_Q + T^-_Q, I^+_Q + I^-_Q, I^-_Q + F^+_Q, F^-_Q + F^-_Q, -T^+_Q - T^-_Q, -I^+_Q - I^-_Q, -F^+_Q - F^-_Q >; \)

iv. \( \tilde{q}_1 \cdot \tilde{q}_2 = < T^+_Q \cdot T^-_Q, T^-_Q \cdot T^+_Q, I^+_Q \cdot I^-_Q, I^-_Q \cdot F^+_Q, F^-_Q \cdot F^-_Q, -T^+_Q \cdot T^-_Q, -I^+_Q \cdot I^-_Q, -F^+_Q \cdot F^-_Q >; \)

where \( \beta > 0 \).

**3 Projective, bidirectional projection and hybrid projection measures of BNSs**

This Section proposes a general projection, a bidirectional projection and a hybrid projection measures for BNSs.

**Definition 6**

Assume that \( X = (x_1, x_2, ..., x_m) \) be a finite universe of discourse and \( Q \) be a BNS in \( X \), then modulus of \( Q \) is defined as follows:

\[
\|Q\| = \sqrt{\sum_{i=1}^{m} a_i^2} = \sqrt{\sum_{i=1}^{m} (T^+_Q (x_i))^2 + (I^+_Q (x_i))^2 + (F^+_Q (x_i))^2 + (T^-_Q (x_i))^2 + (I^-_Q (x_i))^2 + (F^-_Q (x_i))^2}
\]

where \( a_i = \{T^+_Q (x_i), I^+_Q (x_i), F^+_Q (x_i), T^-_Q (x_i), I^-_Q (x_i), F^-_Q (x_i)\}, \)

\( j = 1, 2, ..., m. \)
Definition 7 [10, 29]
Assume that \( u = (u_1, u_2, ..., u_n) \) and \( v = (v_1, v_2, ..., v_n) \) be two vectors, then the projection of vector \( u \) onto vector \( v \) can be defined as follows:

\[
\text{Proj} (u)_v = \| u \| \cos (\alpha, v) = \frac{\sum_{i=1}^{n} u_i v_i}{\sqrt{\sum_{i=1}^{n} u_i^2} \sqrt{\sum_{i=1}^{n} v_i^2}}
\]

(2)

where, \( \text{Proj} (u) \), represents the closeness of \( u \) and \( v \) in magnitude.

Definition 8
Assume that \( X = (x_1, x_2, ..., x_n) \) be a finite universe of discourse and \( R, S \) be any two BNSs in \( X \), then

\[
\text{Proj} (R)_S = \| R \| \cos (R, S) = \frac{1}{\| S \|} (R, S)
\]

(3)

is called the projection of \( R \) on \( S \), where

\[
\| R \| = \sqrt{\sum_{i=1}^{n} (T^R_i(x_i) + I^R_i(x_i) + F^R_i(x_i) + (T^R_i(x_i) + I^R_i(x_i) + F^R_i(x_i)))}
\]

\[
\| S \| = \sqrt{\sum_{i=1}^{n} (T^S_i(x_i) + I^S_i(x_i) + F^S_i(x_i) + (T^S_i(x_i) + I^S_i(x_i) + F^S_i(x_i)))}
\]

and

\[
R \cdot S = \sum_{i=1}^{n} (T^R_i(x_i)I^S_i(x_i) + I^R_i(x_i)F^S_i(x_i) + F^R_i(x_i)T^S_i(x_i) + T^R_i(x_i)I^S_i(x_i) + I^R_i(x_i)F^S_i(x_i) + F^R_i(x_i)T^S_i(x_i))
\]

\[
\text{Example 1.} \quad \text{Suppose that } R = [0.5, 0.3, 0.2, -0.2, -0.1, -0.05] \quad \text{and } S = [0.7, 0.3, 0.1, -0.4, -0.2, -0.3]
\]

\[
\text{Then the projection of } R \text{ on } S \text{ is obtained as follows:}
\]

\[
\text{Proj} (R)_S = \frac{1}{\| R \|} (R, S) = \frac{0.5 \cdot 0.7 + 0.3 \cdot 0.3 + 0.2 \cdot 0.1 + (-0.2) \cdot (-0.4) + (-0.1) \cdot (-0.2) + (-0.05) \cdot (-0.3)}{\sqrt{0.5^2 + 0.3^2 + 0.2^2 + (-0.2)^2 + (-0.1)^2 + (-0.05)^2}}
\]

\[
= 0.612952
\]

The bigger value of \( \text{Proj} (R)_S \) reflects that \( R \) and \( S \) are closer to each other.

However, in single valued neutrosophic environment, Ye [20] observed that the general projection measure cannot describe accurately the degree of \( \alpha \) close to \( \beta \). We also notice that the general projection incorporated by Xu [11] is not reasonable in several cases under bipolar neutrosophic setting, for example let, \( \alpha = \beta < a, a, -a, -a \) and \( \gamma < 2a, a, -2a, -2a, -2a > \), then \( \text{Proj} (\alpha, \beta) = 2.449499 \) \( \| a \| \) and \( \text{Proj} (\gamma)_{\beta} = 4.8989979 \) \( \| a \| \). This shows that \( \beta \) is much closer to \( \gamma \) than \( \alpha \) which is not true because \( \alpha = \beta \). Ye [20] opined that \( \alpha \) is equal to \( \beta \) whenever \( \text{Proj} (\alpha, \beta) \) and \( \text{Proj} (\beta, \alpha) \) should be equal to 1. Therefore, Ye [20] proposed an alternative method called bidirectional projection measure to overcome the limitation of general projection measure as given below.

Definition 9 [20]
Consider \( x \) and \( y \) be any two vectors, then the bidirectional projection between \( x \) and \( y \) is defined as follows:

\[
\text{B-proj} (x, y) = \frac{1}{1+|x|} \frac{1}{1+|y|} \|| x \|| \|| y \||
\]

(4)

where \( |x| \), \( |y| \) denote the moduli of \( x \) and \( y \) respectively, and \( x, y \) is the inner product between \( x \) and \( y \). Here, \( \text{B-proj} (x, y) = 1 \) if and only if \( x = y \) and \( 0 \leq \text{B-proj} (x, y) \leq 1 \), i.e. bidirectional projection is a normalized measure.

Definition 10
Consider \( R = \{ T^R_k(x), I^R_k(x), F^R_k(x), T^-R_k(x), I^-R_k(x), F^-R_k(x) \} \) and \( S = \{ T^S_k(x), I^S_k(x), F^S_k(x), T^-S_k(x), I^-S_k(x), F^-S_k(x) \} \) be any two BNSs \( x = (x_1, x_2, ..., x_n) \), then the bidirectional projection measure between \( R \) and \( S \) is defined as follows:

\[
\text{B-proj} (R, S) = \frac{1}{1+|R|} \frac{1}{1+|S|} \|| R \|| \|| S \||
\]

(5)

where

\[
| R | = \sqrt{\sum_{i=1}^{n} (T^R_i(x_i) + I^R_i(x_i) + F^R_i(x_i) + (T^R_i(x_i) + I^R_i(x_i) + F^R_i(x_i)))}
\]

\[
| S | = \sqrt{\sum_{i=1}^{n} (T^S_i(x_i) + I^S_i(x_i) + F^S_i(x_i) + (T^S_i(x_i) + I^S_i(x_i) + F^S_i(x_i)))}
\]

and

\[
R \cdot S = \sum_{i=1}^{n} (T^R_i(x_i)I^S_i(x_i) + I^R_i(x_i)F^S_i(x_i) + F^R_i(x_i)T^S_i(x_i) + T^R_i(x_i)I^S_i(x_i) + I^R_i(x_i)F^S_i(x_i) + F^R_i(x_i)T^S_i(x_i))
\]

\[
\text{Proposition 1.} \quad \text{Let } \text{B-proj} (R)_S \text{ be a bidirectional projection measure between any two BNSs } R \text{ and } S, \text{ then}
\]

1. \( 0 \leq \text{B-proj} (R, S) \leq 1 \);
2. \( \text{B-proj} (R, S) = \text{B-proj} (S, R) \);
3. \( \text{B-proj} (R, S) = 1 \) for \( R = S \).

\textbf{Proof.}
1. For any two non-zero vectors \( R \) and \( S \),

\[
\frac{1}{1+|R|} \frac{1}{1+|S|} > 0 \quad \Rightarrow \quad \frac{1}{1+|x|} > 0, \quad \text{when } x > 0
\]
\[ \text{B-Proj} (R, S) = 0 \text{ if and only if } \|R\| = 0 \text{ or } \|S\| = 0 \]

i.e. when either \( R = (0, 0, 0, 0, 0, 0) \) or \( S = (0, 0, 0, 0, 0, 0) \) which is trivial case.

\[ \text{B-Proj} (R, S) \geq 0 . \]

For two non-zero vectors \( R \) and \( S \),
\[ \| R \| \| S \| + \| R \| - \| S \| \leq \| R \| \| S \| \leq \| R \| \| S \| + \| R \| - \| S \| \leq \| R \| \| S \| \]

\[ \text{B-Proj} (R, S) \leq 1. \]

\[ 0 \leq \text{B-Proj} (R, S) \leq 1; \]

2. From definition, \( RS = SR \), therefore,
\[ \text{B-Proj} (R, S) = \frac{\| R \| \| S \|}{\| R \| \| S \| + \| R \| - \| S \| \| RS \|} = \text{Proj} (S, R). \]

This completes the proof.

**Example 2.** Assume that \( R = < 0.5, 0.3, 0.2, -0.2, -0.1, -0.05 > \), \( S = < 0.7, 0.3, 0.1, -0.4, -0.2, -0.3 > \) be the two BNSs in \( X \), then the bidirectional projection measure between \( R \) on \( S \) is computed as given below.

\[ \text{B-Proj} (R, S) = \frac{(0.6576473)(0.9380832)}{(0.6576473)(0.9380832) + |0.9380832 - 0.6576473|} = 0.7927845 \]

**Definition 11**

Let \( R = \{ T_R^+(x_i), I_R^+(x_i), F_R^+(x_i), T_R^-(x_i), I_R^-(x_i), F_R^-(x_i) \} \) and \( S = \{ T_S^+(x_i), I_S^+(x_i), F_S^+(x_i), T_S^-(x_i), I_S^-(x_i), F_S^-(x_i) \} \) be any two BNSs in \( X = (x_1, x_2, ..., x_n) \), then the hybrid projection measure is defined as the combination of projection measure and bidirectional projection measure. The hybrid projection measure between \( R \) and \( S \) is represented as follows:

\[ \text{Hyb-Proj} (R, S) = \rho \text{ Proj} (R, S) + (1 - \rho) \text{ B-Proj} (R, S) \]

where

\[ \rho \]

\[ \| R \| = \sqrt{\sum_{i=1}^{n} (T_R^+(x_i))^2 + (I_R^+(x_i))^2 + (F_R^+(x_i))^2 + (T_R^-(x_i))^2 + (I_R^-(x_i))^2 + (F_R^-(x_i))^2} \]

\[ \| S \| = \sqrt{\sum_{i=1}^{n} (T_S^+(x_i))^2 + (I_S^+(x_i))^2 + (F_S^+(x_i))^2 + (T_S^-(x_i))^2 + (I_S^-(x_i))^2 + (F_S^-(x_i))^2} \]

\[ \| RS \| = \sqrt{\sum_{i=1}^{n} (T_R^+(x_i))^2 + (T_S^+(x_i))^2 + (I_R^+(x_i))^2 + (I_S^+(x_i))^2 + (F_R^+(x_i))^2 + (F_S^+(x_i))^2 + (T_R^-(x_i))^2 + (I_R^-(x_i))^2 + (F_R^-(x_i))^2 + (T_S^-(x_i))^2 + (I_S^-(x_i))^2 + (F_S^-(x_i))^2} \]

where \( 0 \leq \rho \leq 1 \).

**Proposition 2**

Let \( \text{Hyb-Proj} (R, S) \) be a hybrid projection measure between any two BNSs \( R \) and \( S \), then

1. \( 1.0 \leq \text{Hyb-Proj} (R, S) \leq 1; \)
2. \( \text{Hyb-Proj} (R, S) = \text{B-Proj} (S, R); \)
3. \( \text{Hyb-Proj} (R, S) = 1 \text{ for } R = S. \)

**Example 3.** Assume that \( R = < 0.5, 0.3, 0.2, -0.2, -0.1, -0.05 > \), \( S = < 0.7, 0.3, 0.1, -0.4, -0.2, -0.3 > \) be the two BNSs, then the hybrid projection measure between \( R \) on \( S \) with \( \rho = 0.7 \) is calculated as given below.

\[ \text{Hyb-Proj} (R, S) = (0.7)(0.612952) + (1 - 0.7)(0.7927845) = 0.6669018 \]

**4 Projection, bidirectional projection and hybrid projection based decision making methods for MADM problems with bipolar neutrosophic information**

In this section, we develop projection based decision making models to MADM problems with bipolar neutrosophic assessments. Consider \( E = \{ E_1, E_2, ..., E_m \} \), \( (m \geq 2) \) be a discrete set of \( m \) feasible alternatives, \( F = \{ F_1, F_2, ..., F_n \} \), \( (n \geq 2) \) be a set of attributes under consideration and \( w = (w_1, w_2, ..., w_n)^T \) be the weight vector of the attributes such that \( 0 \leq w_j \leq 1 \) and \( \sum_{j=1}^{n} w_j = 1 \). Now, we present three algorithms for MADM problems involving bipolar neutrosophic information.

**4.1 Method 1**

Step 1. The rating of evaluation value of alternative \( E_i \) \( (i = 1, 2, ..., m) \) for the predefined attribute \( F_j \) \( (j = 1, 2, ..., n) \) is presented by the decision maker in terms of bipolar neutrosophic values and the bipolar neutrosophic decision matrix is constructed as given below.

\[ \begin{pmatrix}
q_{11} & q_{12} & \cdots & q_{1n} \\
q_{21} & q_{22} & \cdots & q_{2n} \\
\vdots & \ddots & \ddots & \ddots \\
q_{m1} & q_{m2} & \cdots & q_{mn}
\end{pmatrix} \]
where \( q_{ij} = \langle T_{ij}, I_{ij}, F_{ij}, T_{ij}^-, I_{ij}^-, F_{ij}^- \rangle \) with \( T_{ij}^+, I_{ij}^+, F_{ij}^+ \), \( T_{ij}^-, I_{ij}^-, F_{ij}^- \in [0, 1] \) and \( 0 \leq T_{ij}^+ + I_{ij}^+ + F_{ij}^+ - T_{ij}^- - I_{ij}^- - F_{ij}^- \leq 6 \) for \( i = 1, 2, \ldots , m \), \( j = 1, 2, \ldots , n \).

Step 2. We formulate the bipolar weighted decision matrix by multiplying weights \( w_j \) of the attributes as follows:

\[
\begin{pmatrix}
\bar{z}_{i1} & \ldots & \bar{z}_{im} \\
\bar{z}_{21} & \ldots & \bar{z}_{2m} \\
\vdots & \ddots & \vdots \\
\bar{z}_{m1} & \ldots & \bar{z}_{mn}
\end{pmatrix}
\]

where \( \bar{z}_{ij} = w_j q_{ij} = (1 - (1 - T_{ij}^+)) \cdot (1 - I_{ij}^-) \cdot (1 - (1 - F_{ij}^-)) \cdot (-T_{ij}^+) \cdot (-I_{ij}^+) \cdot (-F_{ij}^-) \in [0, 1] \) and \( \bar{z}_{ij} = -\bar{z}_{ij} \) with \( \mu_{i1}^+, v_{i1}^+, \omega_{i1}^+, \mu_{i1}^-, v_{i1}^-, \omega_{i1}^- \) for \( \mu_{i1}^+, v_{i1}^+, \omega_{i1}^+ \in [0, 1] \) and \( 0 \leq \mu_{i1}^+ + v_{i1}^+ + \omega_{i1}^+ - \mu_{i1}^- - v_{i1}^- - \omega_{i1}^- \leq 6 \) for \( i = 1, 2, \ldots , m \), \( j = 1, 2, \ldots , n \).

Step 3. We identify the bipolar neutrosophic positive ideal solution (BNPIS) [27, 28] as follows:

\[
z_{\text{PIS}} = \left\{ (e_j^+ f_j^+, g_j^+, e_j^+ f_j^+, g_j^+) \right\} \leq \left\{ \left\{ \min_i (\mu_{ij}^+) \right\} \left\{ \min_i (v_{ij}^+) \right\} \left\{ \min_i (\omega_{ij}^+) \right\} \left\{ \max_i (\mu_{ij}^-) \right\} \left\{ \min_i (v_{ij}^-) \right\} \left\{ \min_i (\omega_{ij}^-) \right\} \right\}
\]

Step 4. Determine the projection measure between \( z_{\text{PIS}} \) and \( Z = \left\{ z_{ij} \right\} \) for all \( i = 1, 2, \ldots , m \) and \( j = 1, 2, \ldots , n \) by using the following Eq.

\[
\text{Proj} (Z^i)_{\text{PIS}} = \frac{\sum_{j=1}^{n} \left( \mu_{ij}^+ e_j^+ + \mu_{ij}^- f_j^- + v_{ij}^+ g_j^+ + v_{ij}^- g_j^- + \omega_{ij}^+ e_j^+ + \omega_{ij}^- f_j^- \right)}{\sqrt{n} \left( \sum_{j=1}^{n} (e_j^+ f_j^+)^2 + (g_j^+)^2 + (e_j^- f_j^-)^2 + (g_j^-)^2 \right)}
\]

Step 5. Rank the alternatives in a descending order based on the projection measure \( \text{Proj} (Z^i)_{\text{PIS}} \) for \( i = 1, 2, \ldots , m \) and bigger value of \( \text{Proj} (Z^i)_{\text{PIS}} \) determines the best alternative.

4.2. Method 2

Step 1. Give the bipolar neutrosophic decision matrix \( \left\{ q_{ij} \right\}_{i=1}^{m} \), \( i = 1, 2, \ldots , m \).

Step 2. Construct weighted bipolar neutrosophic decision matrix \( \left\{ z_{ij} \right\}_{i=1}^{m} \), \( i = 1, 2, \ldots , m \).

Step 3. Determine \( z_{\text{PIS}} = \left\{ (e_1^+, f_1^+, g_1^+, e_1^+ f_1^+, g_1^+) \right\} \) for \( i = 1, 2, \ldots , n \).

Step 4. Compute the bidirectional projection measure between \( z_{\text{PIS}} \) and \( Z = \left\{ z_{ij} \right\} \) for all \( i = 1, 2, \ldots , m \) and \( j = 1, 2, \ldots , n \) using the Eq. as given below.

\[
\text{B-Proj} (Z^i, z_{\text{PIS}}) = \frac{\| Z' \| \| z_{\text{PIS}} \| + \| Z' \| - \| z_{\text{PIS}} \| \| Z' - z_{\text{PIS}} \|}{\| Z' \| \| z_{\text{PIS}} \|}
\]

where \( \| Z'^i \| = \sum_{j=1}^{n} (\mu_{ij}^+)^2 + (v_{ij}^+)^2 + (\omega_{ij}^+)^2 + (\mu_{ij}^-)^2 + (v_{ij}^-)^2 + (\omega_{ij}^-)^2 \) and \( Z':z_{\text{PIS}} = \frac{1}{\sqrt{n} \sum_{j=1}^{n} (e_j^+)^2 + (g_j^+)^2 + (e_j^-)^2 + (g_j^-)^2 + (e_j^- f_j^-)^2 + (g_j^-)^2)} \) and

Step 5. According to the bidirectional projection measure \( \text{B-Proj} (Z^i, z_{\text{PIS}}) \) for \( i = 1, 2, \ldots , m \) the alternatives are ranked and highest value of \( \text{B-Proj} (Z^i, z_{\text{PIS}}) \) reflects the best option.

4.3. Method 3

Step 1. Construct the bipolar neutrosophic decision matrix \( \left\{ q_{ij} \right\}_{i=1}^{m} \), \( i = 1, 2, \ldots , m \).

Step 2. Formulate the weighted bipolar neutrosophic decision matrix \( \left\{ z_{ij} \right\}_{i=1}^{m} \), \( i = 1, 2, \ldots , m \).

Step 3. Identify \( z_{\text{PIS}} = \left\{ (e_1^+, f_1^+, g_1^+, e_1^+ f_1^+, g_1^+) \right\} \) for \( j = 1, 2, \ldots , n \).

Step 4. By combining projection measure \( \text{Proj} (Z^i)_{\text{PIS}} \) and bidirectional projection measure \( \text{B-Proj} (Z^i, z_{\text{PIS}}) \), we calculate the hybrid projection measure between \( z_{\text{PIS}} \) and \( Z^i = \left\{ z_{ij} \right\} \) for all \( i = 1, 2, \ldots , m \), \( j = 1, 2, \ldots , n \) as follows.

Surapati Pramanik, Partha Pratim Dey, Bibhas C. Giri, Florentin Smarandache, Bipolar Neutrosophic Projection Based Models for Solving Multi-attribute Decision Making Problems
Hyb-Proj (Z', z^PIS) = \rho \ Proj (Z') + (1 - \rho) B-Proj (Z', z^PIS) =
\rho \frac{\sum_{j=1}^{m} (\mu_j^e) + (\nu_j^e) + (\alpha_j^e) + (\mu_j^f) + (\nu_j^f) + (\alpha_j^f)}{\sum_{j=1}^{m} (\mu_j^e) + (\nu_j^e) + (\alpha_j^e) + (\mu_j^f) + (\nu_j^f) + (\alpha_j^f)}, \quad i = 1, 2, ..., m,
where \|Z'\| = \sqrt{\sum_{j=1}^{m} [(\mu_j^e)^2 + (\nu_j^e)^2 + (\alpha_j^e)^2 + (\mu_j^f)^2 + (\nu_j^f)^2 + (\alpha_j^f)^2]}, \quad i = 1, 2, ..., m,
\|z^PIS\| = \sqrt{\sum_{j=1}^{m} [(\mu_j^e)^2 + (\nu_j^e)^2 + (\alpha_j^e)^2 + (\mu_j^f)^2 + (\nu_j^f)^2 + (\alpha_j^f)^2]}, \quad Z' = z^PIS
(9)

Step 5. We rank all the alternatives in accordance with the hybrid projection measure Hyb-Proj (Z', z^PIS) and greater value of Hyb-Proj (Z', z^PIS) indicates the better alternative.

5 A numerical example

We solve the MADM studied in [5, 28] where a customer desires to purchase a car. Suppose four types of car (alternatives) E_i (i = 1, 2, 3, 4) are taken into consideration in the decision making situation. Four attributes namely Fuel economy (F_1), Aerod (F_2), Comfort (F_3) and Safety (F_4) are considered to evaluate the alternatives. Assume the weight vector [5] of the attribute is given by w = (w_1, w_2, w_3, w_4) = (0.5, 0.25, 0.125, 0.125).

Method 1: The proposed projection measure based decision making with bipolar neutrosophic information for car selection is presented in the following steps:

Step 1: Construct the bipolar neutrosophic decision matrix

The bipolar neutrosophic decision matrix \{d_{ij}\}_{m \times n} is presented by the decision maker as given below (see Table 1).

Table 1. The bipolar neutrosophic decision matrix

<table>
<thead>
<tr>
<th></th>
<th>F_1</th>
<th>F_2</th>
<th>F_3</th>
<th>F_4</th>
</tr>
</thead>
<tbody>
<tr>
<td>E_1</td>
<td>&lt;0.293, 0.837, 0.447, 0.837, 0.0818, 0.182&gt;</td>
<td>&lt;0.121, 0.795, 0.841, 0.915, 0.106&gt;</td>
<td>&lt;0.140, 0.956, 0.917, 0.972, 0.108&gt;</td>
<td>&lt;0.013, 0.917, 0.956, 0.917, 0.108&gt;</td>
</tr>
<tr>
<td>E_2</td>
<td>&lt;0.684, 0.837, 0.707, 0.837, 0.083, 0.051&gt;</td>
<td>&lt;0.126, 0.680, 0.946, 0.915, 0.012&gt;</td>
<td>&lt;0.250, 0.892, 0.938, 0.975, 0.005&gt;</td>
<td>&lt;0.183, 0.918, 0.956, 0.917, 0.025&gt;</td>
</tr>
<tr>
<td>E_3</td>
<td>&lt;0.165, 0.632, 0.447, 0.774, 0.548, 0.452&gt;</td>
<td>&lt;0.054, 0.669, 0.915, 0.795, 0.120&gt;</td>
<td>&lt;0.250, 0.917, 0.938, 0.975, 0.028&gt;</td>
<td>&lt;0.140, 0.917, 0.860, 0.918, 0.028&gt;</td>
</tr>
<tr>
<td>E_4</td>
<td>&lt;0.648, 0.837, 0.447, 0.774, 0.548, 0.452&gt;</td>
<td>&lt;0.085, 0.841, 0.669, 0.915, 0.054&gt;</td>
<td>&lt;0.083, 0.892, 0.917, 0.975, 0.028&gt;</td>
<td>&lt;0.062, 0.818, 0.972, 0.917, 0.028&gt;</td>
</tr>
</tbody>
</table>

Step 3. Selection of BNPRIS

The BNPRIS \( z^PIS \) = \( \{e_{i1}^+, f_{i1}^+, g_{i1}^+, e_{i2}^-, f_{i2}^-, g_{i2}^-\} \), (j = 1, 2, 3, 4) is computed from the weighted decision matrix as follows:

\( e_{i1}^+ \cdot f_{i2}^+ \cdot f_{i1}^- \cdot f_{i2}^- \cdot g_{i1}^+ \cdot g_{i2}^- < 0.684, 0.632, 0.447, -0.894, -0.548, -0.051 \);

\( e_{i2}^+ \cdot f_{i2}^+ \cdot g_{i1}^+ \cdot e_{i1}^- \cdot f_{i2}^- \cdot g_{i2}^- < 0.25, 0.892, 0.917, -0.972, -0.917, -0.028 \);

\( e_{i1}^+ \cdot f_{i1}^+ \cdot g_{i1}^+ \cdot e_{i2}^- \cdot f_{i1}^- \cdot g_{i2}^- < 0.14, 0.818, 0.86, -0.917, -0.75, -0.028 \).
Step 4. Determination of weighted projection measure
The projection measure between positive ideal bipolar neutrosophic solution $z_{PIS}^{\text{Proj}}$ and each weighted decision matrix $\{z_\phi\}_{\text{m} \times \text{n}}$ can be obtained as follows:

$$\text{Proj} (Z^1)_{\text{cm}} = 3.4214, \text{Proj} (Z^2)_{\text{cm}} = 3.4972, \text{Proj} (Z^3)_{\text{cm}} = 3.1821, \text{Proj} (Z^4)_{\text{cm}} = 3.3904.$$  

Step 5. Rank the alternatives
We observe that $\text{Proj} (Z^3)_{\text{cm}} > \text{Proj} (Z^1)_{\text{cm}} > \text{Proj} (Z^4)_{\text{cm}} > \text{Proj} (Z^2)_{\text{cm}}$. Therefore, the ranking order of the cars is $E_2 > E_1 > E_4 > E_3$. Hence, $E_2$ is the best alternative for the customer.

Method 2: The proposed bidirectional projection measure based decision making for car selection is presented as follows:

Step 1. Same as Method 1  
Step 2. Same as Method 1  
Step 3. Same as Method 1  
Step 4. Calculation of bidirectional projection measure  
The bidirectional projection measure between positive ideal bipolar neutrosophic solution $z_{PIS}^{\text{Proj}}$ and each weighted decision matrix $\{z_\phi\}_{\text{m} \times \text{n}}$ can be determined as given below.

$$B-\text{Proj} (Z^1, z_{PIS}^{\text{Proj}}) = 0.8556, B-\text{Proj} (Z^2, z_{PIS}^{\text{Proj}}) = 0.8101, B-\text{Proj} (Z^3, z_{PIS}^{\text{Proj}}) = 0.9503, B-\text{Proj} (Z^4, z_{PIS}^{\text{Proj}}) = 0.8969.$$  

Step 5. Ranking the alternatives
Here, we notice that $B-\text{Proj} (Z^3, z_{PIS}^{\text{Proj}}) > B-\text{Proj} (Z^4, z_{PIS}^{\text{Proj}}) > B-\text{Proj} (Z^1, z_{PIS}^{\text{Proj}}) > B-\text{Proj} (Z^2, z_{PIS}^{\text{Proj}})$ and therefore, the ranking order of the alternatives is obtained as $E_3 > E_4 > E_1 > E_2$. Hence, $E_3$ is the best choice among the alternatives.

Method 3: The proposed hybrid projection measure based MADM with bipolar neutrosophic information is provided as follows:

Step 1. Same as Method 1  
Step 2. Same as Method 1  
Step 3. Same as Method 1  
Step 4. Computation of hybrid projection measure  
The hybrid projection measures for different values of $\rho \in [0, 1]$ and the ranking order are shown in the Table 3.

### Table 3. Results of hybrid projection measure for different values of $\rho$

<table>
<thead>
<tr>
<th>Similarity measure</th>
<th>$\rho$</th>
<th>Measure values</th>
<th>Ranking order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{Hyb-Proj}<em>{\text{PIS}} (Z, z</em>{PIS}^{\text{Proj}})$</td>
<td>0.25</td>
<td>$\text{Hyb-Proj} (Z^1, z_{PIS}^{\text{Proj}}) = 3.4214$</td>
<td>$E_2 &gt; E_1 &gt; E_4$</td>
</tr>
<tr>
<td></td>
<td>0.50</td>
<td>$\text{Hyb-Proj} (Z^2, z_{PIS}^{\text{Proj}}) = 3.4972$</td>
<td>$E_2 &gt; E_1 &gt; E_4$</td>
</tr>
<tr>
<td></td>
<td>0.75</td>
<td>$\text{Hyb-Proj} (Z^3, z_{PIS}^{\text{Proj}}) = 3.1821$</td>
<td>$E_2 &gt; E_1 &gt; E_4$</td>
</tr>
<tr>
<td></td>
<td>0.90</td>
<td>$\text{Hyb-Proj} (Z^4, z_{PIS}^{\text{Proj}}) = 3.3904$</td>
<td>$E_2 &gt; E_1 &gt; E_4$</td>
</tr>
</tbody>
</table>

### 6 Comparative analysis

In the Section, we compare the results obtained from the proposed methods with the results derived from other existing methods under bipolar neutrosophic environment to show the effectiveness of the developed methods.

Dey et al. [28] assume that the weights of the attributes are not identical and weights are fully unknown to the decision maker. Dey et al. [28] formulated maximizing deviation model under bipolar neutrosophic assessment to compute unknown weights of the attributes as $w = (0.2585, 0.2552, 0.2278, 0.2585)$. By considering $w = (0.2585, 0.2552, 0.2278, 0.2585)$, the proposed projection measures are shown as follows:

$$\text{Proj} (Z^1)_{\text{cm}} = 3.3954, \text{Proj} (Z^2)_{\text{cm}} = 3.3872, \text{Proj} (Z^3)_{\text{cm}} = 3.1625, \text{Proj} (Z^4)_{\text{cm}} = 3.2567.$$  

Since, $\text{Proj} (Z^1)_{\text{cm}} > \text{Proj} (Z^2)_{\text{cm}} > \text{Proj} (Z^3)_{\text{cm}} > \text{Proj} (Z^4)_{\text{cm}}$, therefore the ranking order of the four alternatives is given by $E_1 > E_2 > E_4 > E_3$. Thus, $E_1$ is the best choice for the customer.

Now, by taking $w = (0.2585, 0.2552, 0.2278, 0.2585)$, the bidirectional projection measures are calculated as given below.

$$B-\text{Proj} (Z^1, z_{PIS}^{\text{Proj}}) = 0.8113, B-\text{Proj} (Z^2, z_{PIS}^{\text{Proj}}) = 0.8111, B-\text{Proj} (Z^3, z_{PIS}^{\text{Proj}}) = 0.9854, B-\text{Proj} (Z^4, z_{PIS}^{\text{Proj}}) = 0.9974.$$  

Since, $B-\text{Proj} (Z^2, z_{PIS}^{\text{Proj}}) > B-\text{Proj} (Z^3, z_{PIS}^{\text{Proj}}) > B-\text{Proj} (Z^4, z_{PIS}^{\text{Proj}}) > B-\text{Proj} (Z^1, z_{PIS}^{\text{Proj}})$, consequently the ranking...
order of the four alternatives is given by $E_4 \succ E_3 \succ E_2 \succ E_1$. Hence, $E_4$ is the best option for the customer.

Also, by taking $w = (0.2585, 0.2552, 0.2278, 0.2585)$, the proposed hybrid projection measures for different values of $\rho \in [0, 1]$ and the ranking order are revealed in the Table 4.

Del et al. [5] assume the weight vector of the attributes as $w = (0.5, 0.25, 0.125, 0.125)$ and the ranking order based on score values is presented as follows:

$$E_3 \succ E_4 \succ E_2 \succ E_1$$

Thus, $E_3$ was the most desirable alternative.

Dey et al. [28] employed maximizing deviation method to find unknown attribute weights as $w = (0.2585, 0.2552, 0.2278, 0.2585)$. The ranking order of the alternatives is presented based on the relative closeness coefficient as given below.

$$E_3 \succ E_2 \succ E_4 \succ E_1$$

Obviously, $E_3$ is the most suitable option for the customer.

Dey et al. [28] also consider the weight vector of the attributes as $w = (0.5, 0.25, 0.125, 0.125)$, then using TOPSIS method, the ranking order of the cars is represented as follows:

$$E_4 \succ E_2 \succ E_3 \succ E_1$$

So, $E_4$ is the most preferable alternative for the buyer. We observe that different projection measure provides different ranking order and the projection measure is weight sensitive. Therefore, decision maker should choose the projection measure and weights of the attributes in the decision making context according to his/her needs, desires and practical situation.

### Conclusion

In this paper, we have defined projection, bidirectional projection measures between bipolar neutrosophic sets. Further, we have defined a hybrid projection measure by combining projection and bidirectional projection measures. Through these projection measures we have developed three methods for multi-attribute decision making models under bipolar neutrosophic environment. Finally, a car selection problem has been solved to show the flexibility and applicability of the proposed methods. Furthermore, comparison analysis of the proposed methods with the other existing methods has also been demonstrated.

The proposed methods can be extended to interval bipolar neutrosophic set environment. In future, we shall apply projection, bidirectional projection, and hybrid projection measures of interval bipolar neutrosophic sets for group decision making, medical diagnosis, weaver selection, pattern recognition problems, etc.

### Table 4. Results of hybrid projection measure for different values of $\rho$

<table>
<thead>
<tr>
<th>Similarity measure</th>
<th>$\rho$</th>
<th>Measure values</th>
<th>Ranking order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{Hyb-Prj}$ $(Z, z^{\text{PIS}})$</td>
<td>0.25</td>
<td>$\text{Hyb-Prj}(Z, z^{\text{PIS}})_1 = 1.4970$</td>
<td>$E_4 \succ E_3 \succ E_2 \succ E_1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\text{Hyb-Prj}(Z, z^{\text{PIS}})_2 = 1.4819$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\text{Hyb-Prj}(Z, z^{\text{PIS}})_3 = 1.5082$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\text{Hyb-Prj}(Z, z^{\text{PIS}})_4 = 1.5203$</td>
<td></td>
</tr>
<tr>
<td>$\text{Hyb-Prj}$ $(Z, z^{\text{PIS}})$</td>
<td>0.50</td>
<td>$\text{Hyb-Prj}(Z, z^{\text{PIS}})_1 = 2.1385$</td>
<td>$E_4 \succ E_3 \succ E_2 \succ E_1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\text{Hyb-Prj}(Z, z^{\text{PIS}})_2 = 2.1536$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\text{Hyb-Prj}(Z, z^{\text{PIS}})_3 = 2.0662$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\text{Hyb-Prj}(Z, z^{\text{PIS}})_4 = 2.1436$</td>
<td></td>
</tr>
<tr>
<td>$\text{Hyb-Prj}$ $(Z, z^{\text{PIS}})$</td>
<td>0.75</td>
<td>$\text{Hyb-Prj}(Z, z^{\text{PIS}})_1 = 2.7800$</td>
<td>$E_4 \succ E_3 \succ E_2 \succ E_1$</td>
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<tr>
<td></td>
<td></td>
<td>$\text{Hyb-Prj}(Z, z^{\text{PIS}})_2 = 2.8254$</td>
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<tr>
<td></td>
<td></td>
<td>$\text{Hyb-Prj}(Z, z^{\text{PIS}})_3 = 2.6241$</td>
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<tr>
<td></td>
<td></td>
<td>$\text{Hyb-Prj}(Z, z^{\text{PIS}})_4 = 2.7670$</td>
<td></td>
</tr>
<tr>
<td>$\text{Hyb-Prj}$ $(Z, z^{\text{PIS}})$</td>
<td>0.90</td>
<td>$\text{Hyb-Prj}(Z, z^{\text{PIS}})_1 = 3.1648$</td>
<td>$E_4 \succ E_3 \succ E_2 \succ E_1$</td>
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<tr>
<td></td>
<td></td>
<td>$\text{Hyb-Prj}(Z, z^{\text{PIS}})_2 = 3.2285$</td>
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<td></td>
<td>$\text{Hyb-Prj}(Z, z^{\text{PIS}})_3 = 2.9589$</td>
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<td></td>
<td>$\text{Hyb-Prj}(Z, z^{\text{PIS}})_4 = 3.1410$</td>
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### References


Received: February 3, 2017. Accepted: February 21, 2017.
Abstract

Uncertainty and indeterminacy are two major problems in data analysis these days. Neutrosophy is a generalization of the fuzzy theory. Neutrosophic system is based on indeterminism and falsity of concepts in addition to truth degrees. Any neutrosophy variable or concept is defined by membership, indeterminacy and non-membership functions. Finding efficient and accurate definition for neutrosophic variables is a challenging process. This paper presents a framework of Ant Colony Optimization and entropy theory to define a neutrosophic variable from concrete data.

Keywords


1. Introduction

These days, Indeterminacy is the key idea of the information in reality issues. This term alludes to the obscure some portion of the information representation. The fuzzy logic [1][2][3] serves the piece of information participation degree. Thus, the indeterminacy and non-participation ideas of the information ought to be fittingly characterized and served. The neutrosophic [4][16] theory characterizes the informational index in mix with their membership, indeterminacy and non-membership degrees. Thus, the decisions could be practically figured out from these well defined information.

Smarandache in [5][13][14], and Salama et al. in[4], [9],[10][11][12][12][16] present the mathematical base of neutrosophic system and principles of neutrosophic data. Neutrosophy creates the main basics for a new mathematics field through adding indeterminacy concept to traditional and fuzzy theories[1][2][3][15].

Handling neutrosophic system is a new, moving and appealing field for scientists. In literature, neutrosophic toolbox implementation using object oriented programming operations and formulation is introduced in[18]. Moreover, a data warehouse utilizing neutrosophic methodologies and sets is applied in [17]. Also, the problem of optimizing membership functions using Particle Swarm Optimization was introduced in [24]. This same mechanism could be generalized to model neutrosophic variable.
The neutrosophic framework depends actually on the factors or variables as basics. The neutrosophic variable definition is without a doubt the base in building a precise and productive framework. The neutrosophic variable is made out of a tuple of value, membership, indeterminacy and non-membership. Pronouncing the elements of participation, indeterminacy and non-enrollment and map those to the variable values would be an attainable arrangement or solution for neutrosophic variable formulation.

Finding the subsets boundary points of membership and non-membership functions within a variable data would be an interesting optimization problem. Ant Colony Optimization (ACO)[19][20] is a meta-heuristic optimization and search procedure[22] inspired by ants lifestyle in searching for food. ACO initializes a population of ants in the search space traversing for their food according to some probabilistic transition rule. Ants follow each other basing on rode pheromone level and ant desirability to go through a specific path. The main issue is finding suitable heuristic desirability which should be based on the information conveyed from the variable itself. Information theory measures [6][20][21], [23] collect information from concrete data. The entropy definition is the measure of information conveyed in a variable. Whereas, the mutual information is the measure of data inside a crossing point between two nearby subsets of a variable. These definitions may help in finding limits of a membership function of neutrosophic variable subsets depending on the probability distribution of the data as the heuristic desirability of ants.

In a similar philosophy, the non-membership of a neutrosophic variable might be characterized utilizing the entropy and mutual information basing on the data probability distribution complement. Taking the upsides of the neutrosophic set definition; the indeterminacy capacity could be characterized from the membership and non-membership capacities.

This paper exhibits an incorporated hybrid search model amongst ACO and information theory measures to demonstrate a neutrosophic variable. The rest of this paper is organized as follows. Section 2 shows the hypotheses and algorithms. Section 3 announces the proposed integrated framework. Section 4 talks about the exploratory outcomes of applying the framework on a general variable and demonstrating the membership, indeterminacy and non-membership capacities. Conclusion and future work is displayed in section 5.

2. Theory overview

2.1 Parameters of a neutrosophic variable

In the neutrosophy theory[5][13][14], every concept is determined by rates of truth \( \mu_A(x) \), indeterminacy \( \sigma_A(x) \), and negation \( \nu_A(x) \) in various partitions. Neutrosophy is a generalization of the fuzzy hypothesis[1][2][3] and an extension of the regular set. Neutrosophic is connected to concepts identified with indeterminacy. Neutrosophic data is defined by three main concepts to manage uncertainty. These concepts are joined together in the triple:

\[
A = (\mu_A(x), \sigma_A(x), \nu_A(x))
\]

Where

\( \mu_A(x) \) is the membership degree,

\( \sigma_A(x) \) is the indeterminacy degree,

\( \nu_A(x) \) is the falsity degree.

These three terms form the fundamental concepts and they are independent and explicitly quantified. In neutrosophic set [7], each value \( x \in X \) in set A defined by Eq. 1 is constrained by the following conditions:

\[
0^- \leq \mu_A(x), \sigma_A(x), \nu_A(x) \leq 1^+
\]
Whereas, Neutrosophic intuitionistic set of type 1 \[8\] is subjected to the following:

\[0 \leq \mu_A(x), \sigma_A(x), \nu_A(x) \leq 1^+ \] (4)

\[\mu_A(x) \land \sigma_A(x) \land \nu_A(x) \leq 0.5 \] (5)

\[0 \leq \mu_A(x) + \sigma_A(x) + \nu_A(x) \leq 3^+ \] (6)

Neutrosophic intuitionistic set of type 2 [5] is obliged by to the following conditions:

\[0.5 \leq \mu_A(x), \sigma_A(x), \nu_A(x) \] (7)

\[\mu_A(x) \land \sigma_A(x) \leq 0.5, \mu_A(x) \land \nu_A(x) \leq 0.5, \sigma_A(x) \land \nu_A(x) \leq 0.5 \] (8)

\[0 \leq \mu_A(x) + \sigma_A(x) + \nu_A(x) \leq 2^+ \] (9)

### 2.2 Ant Colony Optimization (ACO)

The ACO [19][20] is an efficient search algorithm used to find feasible solutions for complex and high dimension problems. The intelligence of the ACO is based on a population of ants traversing the search workspace for their food. Each ant follows a specific path depending on information left previously from other ants. This information is characterized by the probabilistic transition rule Eq. 10.

\[
p_j^m(t) = \frac{[\eta_j \times g(t)]}{\sum_{i \in \text{path}} [\eta_i \times \tau_{ij}(t)]} \tag{10}
\]

Where

\[\eta_j\] is the heuristic desirability of choosing node \(j\) and

\[\tau_{ij}\] is the amount of virtual pheromone on edge \((i, j)\).

The pheromone level guides the ant through its journey. This guide is a hint of the significance level of a node (exhibited by the ants went to the nodes some time recently). The pheromone level is updated by the algorithm using the fitness function.

\[
\tau_{ij}(t + 1) = (1 - \rho). \tau_{ij}(t) + \Delta \tau_{ij}(t) \tag{11}
\]

Where \(0 < \rho < 1\) is a decay constant used to estimate the evaporation of the pheromone from the edges. \(\Delta \tau_{ij}(t)\) is the amount of pheromone deposited by the ant.

The heuristic desirability \(\eta_j\) describes the association between a node \(j\) and the problem solution or the fitness function of the search. If a node has a heuristic value for a certain path then the ACO will use this node in the solution of the problem. The algorithm of ACO is illustrated in figure 1.

\[
\eta_j = \text{objective function} \tag{12}
\]
2.3 Entropy and Mutual Information

Information theory measures [6][20][23] collect information from raw data. The entropy of a random variable is a function which characterizes the unexpected events of a random variable. Consider a random variable \( X \) expressing the number on a roulette wheel or the number on a fair 6-sided die.

\[
H(X) = \sum_{x \in X} -P(x) \log P(x)
\]  

(13)

Joint entropy is the entropy of a joint probability distribution, or a multi-valued random variable. For example, consider the joint entropy of a distribution of mankind (\( X \)) defined by a characteristic (\( Y \)) like age or race or health status of a disease.

\[
I(X; Y) = \sum_{x,y} p(x,y) \log \frac{p(x,y)}{p(x)p(y)}
\]  

(14)

3. The proposed framework

An Integrated hybrid model of ACO and information theory measures (entropy and mutual information) as the objective function is presented. The ACO[19][20] is a heuristic searching algorithm used to locate the ideal segments of the membership and non-membership functions of a neutrosophic variable. The indeterminacy function is calculated by the membership and non-membership functions basing on the definitions of neutrosophic set illustrated in section 2. The objective function is the amount of information conveyed from various partitions in the workspace. Therefore, the total entropy [21] is used as the objective function on the variables workspace. Total entropy calculates amount of information of various partitions and intersections between these partitions. Best points in declaring the membership function are the boundaries of the partitions. The ants are designed to form the membership and non-membership partitions as illustrated in figure 2. A typical triangle membership function would take the shape of figure 2.

The triangle function of a variable partition is represented by parameters (\( L \), (\( L+U \))/2, \( U \)). Finding best values of \( L \) and \( U \) for all partitions would optimize the membership (non-membership) function definition. Figure 3 give a view of the ant with \( n \) partitions for each fuzzy variable.

One of the main difficulties in designing optimization problem using ACO is finding the heuristic desirability which formulates the transition rule. The amount of information deposited by neutrosophic variable inspires the ACO to calculate the transition rule and find parameters of membership, indeterminism and non-membership declarations. The membership function subsets are declared by ant parameters in figure 2. The histogram of a variable shows the data distribution of the different values. Therefore, the set of parameters are mapped to the histogram of a given variable data (Fig. 4).

The objective function is set as the total entropy of partitions[23]. By enhancing partition's parameters to optimize the total entropy of the histogram subsets, the optimal membership design of the variable is found.
Figure 4: Flow chart for the modelling neutrosophic variable using ACO
Input: \( pd, N, \) variable_datafile

% number of decision variables in particle, \( N \) iteration, Present position in the search universe \( X_{id}, \) \( \rho \) is the decay rate of phermone.

Output: membership, non-membership and indeterminacy function, conversion rate.

1: \( X \leftarrow \text{Initialize_Ants}(); \) \% Each ant is composed of \( pd \) decision variables for fuzzy partitions
2: \( \text{Att} \leftarrow \text{Read_data(variable_datafile)} \)
3: Objective_mem \( \leftarrow \text{Evaluate\_Objective\_of\_Particles}(X, P(\text{Att})); \) \% According to entropy and Mutual information
4: Objective_non_mem \( \leftarrow \text{Evaluate\_Objective\_of\_Particles}(X, 1-P(\text{Att})); \) \% According to entropy and Mutual information
5: While \( \text{(num\_of\_Iterations}\lt\text{Max\_iter}) \)
6: foreach Ant
7: \( \eta_j \leftarrow H = \sum_{i=1}^{n} H(i) - \sum_{j=1}^{n-1} I(j, j + 1) \)
8: \( p^m_j(t) \leftarrow \frac{[\eta_j] \times [\eta(o)]}{\sum_{l=1}^{m} [\eta_l] \times [\tau_{lj}(t)]} \)
9: \( \tau_{ij}(t + 1) = (1 - \rho) \cdot \tau_{ij}(t) + \Delta \tau_{ij}(t) \)
10: end foreach
11: Best_sol_mem \( \leftarrow \max(\eta_j) \) \% Best found value until iteration \( t \)
12: foreach Ant
13: \( \eta_j \leftarrow H = \sum_{i=1}^{n} H(i) - \sum_{j=1}^{n-1} I(j, j + 1) \)
14: \( p^n_j(t) \leftarrow \frac{[\eta_j] \times [\eta(o)]}{\sum_{l=1}^{m} [\eta_l] \times [\tau_{lj}(t)]} \)
15: \( \tau_{ij}(t + 1) = (1 - \rho) \cdot \tau_{ij}(t) + \Delta \tau_{ij}(t) \)
16: end foreach
17: Best_sol_non-mem \( \leftarrow \max(\eta_j) \) \% Best found value until iteration \( t \)
18: End While
19: Best_mem \( \leftarrow \) Best_sol_mem
20: Best_non-mem \( \leftarrow \) Best_sol_non-mem
21: indeterminacy \( \leftarrow \text{calculate-ind}(\text{Best\_mem, Best\_non-mem}); \)
22: Draw(Best_mem, Best_non-mem, indeteminary)
23: Draw_conversions_rate()
24: Output membership, non-membership and indeterminacy function, conversion rate.

Function calculate-ind(\( \mu_A(x), \nu_A(x) \))

1: Input(\( \mu_A(x), \nu_A(x) \))
2: Output: indeterminacy
3: \( 0^- - [\mu_A(x) + \nu_A(x)] \leq \sigma_A(x) \leq 3^+ - [\mu_A(x) + \nu_A(x)] \)
4: indeterminacy \( \leftarrow \text{Normalize}(\sigma_A(x)); \)
5: Return indeterminacy
5: End Fun

Figure 5: Algorithm for the modelling neutrosophic variable using ACO
To model (n) membership functions, variable histogram is partitioned into n overlapped subsets that produce n-1 intersections. Every joint partition corresponds to joint entropy and each overlap is modelled by mutual information. Eq. 15 shows the total entropy which is assigned to the heuristic desirability of ants.

$$\eta_j = H = \sum_{i=1}^{n} H(i) - \sum_{j=1}^{n-1} I(j, j+1)$$

(15)

Where n is the number of partitions or subsets in the fuzzy variable, H is the total entropy, H(i) is the entropy of subset i, I is the mutual information between intersecting partitions(i,j).

In membership function modelling, the total entropy function Eq. 13, 14 and 15 are calculated by the probability distribution \( P(x) \) of the variable data frequency in various partitions and the intersecting between them. The complement of probability distribution \( 1 - P(x) \) is utilized to measure the non-membership of variable data in different partitions. Therefore, the non-membership objective function will compute Eq. 13, 14 and 15 with the variable data frequency complement in different partitions and overlapping.

Accorrding to Eq. 3 & 6, the summation of the membership, non membership and indeterminacy values for the same instance is in the interval \([0^-, 2^+]\). Hence, the indeterminacy function is defined as Eq. 17.

$$0^- - [\mu_A(x) + \nu_A(x)] \leq \sigma_A(x) \leq 2^+ - [\mu_A(x) + \nu_A(x)]$$

(17)

By finding the membership and non-membership definition of \( x \), the indeterminacy function \( \sigma_A(x) \) could be driven easily from Eq. 15 or 16. The value of the indeterminacy function should be in the interval \([0^- 1^+]\), hence the \( \sigma_A(x) \) function is normalized according to Eq. 18.

$$\text{Normalized}_{\sigma_A}(x_i) = \frac{\sigma_A(x_i) - \min(\sigma_A(x))}{\max(\sigma_A(x)) - \min(\sigma_A(x))}$$

(18)

Where \( \sigma_A(x_i) \) is the indeterminacy function for the value \( x_i \). The flow chart and algorithm of the integrated framework is illustrated in figure 5 and 6 respectively.

4. Experimental Results

The present reality issues are brimming with vulnerability and indeterminism. The neutrosophic field is worried by picking up information with degrees of enrollment, indeterminacy and non-participation. Neutrosophic framework depends on various neutrosophic factors or variables. Unfortunately, the vast majority of the informational indexes accessible are normal numeric qualities or unmitigated characteristics. Henceforth, creating approaches for characterizing a neutrosophic set from the current informational indexes is required.

The membership capacity function of a neutrosophy variable, similar to the fuzzy variable, can take a few sorts. Triangle membership is very popular due to its simplicity and accuracy. Triangle function is characterized by various overlapping partitions. These subsets are characterized by support, limit and core parameters. The most applicable parameter to a specific subset is the support which is the space of characterizing...
the membership degree. Finding the start and closure of a support over the universe of a variable could be an intriguing search issue suitable for optimization. Meta-heuristic search methodologies [22] give an intelligent procedure for finding ideal arrangement of solutions is any universe. ACO is a well defined search procedure that mimics ants in discovering their sustenance. Figure 3 presents the ant as an individual in a population for upgrading a triangle membership function through the ACO procedure. The ACO utilizes the initial ant population and emphasizes to achieve ideal arrangement.

Table 1: Parameters of ACO

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximum Number of Iterations</td>
<td>50</td>
</tr>
<tr>
<td>Population Size (number of ants)</td>
<td>10</td>
</tr>
<tr>
<td>Decaying rate</td>
<td>0.1</td>
</tr>
</tbody>
</table>

The total entropy given by Eq. 15 characterizes the heuristic desirability which affects the probabilistic transition rule of ants in the ACO algorithm. The probability distribution $P(x)$ presented in Eq. 13, 14 and 15 is used to calculate the total entropy function. The ACO parameters like Maximum Number of Iterations, Population Size, and pheromone decaying rate are presented in table 1.

The non-membership function means the falsity degree in the variables values. Hence, the complement of a data probability distribution $1 - P(x)$ is utilized to create the heuristic desirability of the ants in designing the non-membership function Eq. 13, 14 and 15.

The indeterminacy capacity of variable data is created by both membership and non-membership capacities of the same data using neutrosophic set declaration in section 2 and Eq. 16 or 17. Afterwards, Eq. 18 is used to normalize the indeterminacy capacity of the data. Through simulation, the ACO is applied by MATLAB, PC with Intel(R) Core (TM) CPU and 4 GB RAM. The simulation are implemented on the temperature variable from the Forest Fires data set created by Paulo Cortez and Anbal Morais (Univ. Minho) [25]. The histogram of a random collection of the temperature data is shown in figure 7.

![Figure 6: Temperature Variable Histogram](image)

Figures 8: a, b and c presents the resulting membership, non-membership and indeterminacy capacities produced by applying the ACO on a random collection of the temperature data.

![Figure 7: a. Membership Function b. Non-membership Function c. Indeterminacy Function](image)
5. Conclusion

A proposed framework utilizing the ant colony optimization and the total entropy measure for mechanizing the design of neutrosophic variable is exhibited. The membership, non-membership and indeterminacy capacities are utilized to represent the neutrosophy idea. The enrollment or truth of subset could be conjured from total entropy measure. The fundamental system aggregates the total entropy to the participation or truth subsets of a neutrosophic concept. The ant colony optimization is a meta-heuristic procedure which seeks the universe related to variable X to discover ideal segments or partitions parameters. The heuristic desirability of ants, for membership generation, is the total entropy based on the probability density function of random variable X. Thusly, the probability density complement is utilized to design non-membership capacity. The indeterminacy capacity is identified, as indicated by neutrosophic definition, by the membership and non-membership capacities. The results in light of ACO proposed system are satisfying. Therefore, the technique can be utilized as a part of data preprocessing stage within knowledge discovery system. Having sufficient data gathering, general neutrosophic variable outline for general data can be formulated.

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Research(IJMCAR), 3(1), (2013), 171 - 178.


Received: February 6, 2017. Accepted: February 22, 2017.
Neutrosophic Modal Logic

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Abstract: We introduce now for the first time the neutrosophic modal logic. The Neutrosophic Modal Logic includes the neutrosophic operators that express the modalities. It is an extension of neutrosophic predicate logic and of neutrosophic propositional logic.

Keywords: neutrosophic operators, neutrosophic predicate logic, neutrosophic propositional logic, neutrosophic epistemology, neutrosophic mereology.

1 Introduction.

The paper extends the fuzzy modal logic [1, 2, and 4], fuzzy environment [3] and neutrosophic sets, numbers and operators [5 – 12], together with the last developments of the neutrosophic environment {including (t, i, f)-neutrosophic algebraic structures, neutrosophic triplet structures, and neutrosophic overset / underset / offset} [13 - 15] passing through the symbolic neutrosophic logic [16], ultimately to neutrosophic modal logic.

All definitions, sections, and notions introduced in this paper were never done before, neither in my previous work nor in other researchers’.

Therefore, we introduce now the Neutrosophic Modal Logic and the Refined Neutrosophic Modal Logic.

Then we can extend them to Symbolic Neutrosophic Modal Logic and Refined Symbolic Neutrosophic Modal Logic, using labels instead of numerical values.

There is a large variety of neutrosophic modal logics, as actually happens in classical modal logic too. Similarly, the neutrosophic accessibility relation and possible neutrosophic worlds have many interpretations, depending on each particular application. Several neutrosophic modal applications are also listed.

Due to numerous applications of neutrosophic modal logic (see the examples throughout the paper), the introduction of the neutrosophic modal logic was needed.

Neutrosophic Modal Logic is a logic where some neutrosophic modalities have been included.

Let \( \mathcal{P} \) be a neutrosophic proposition. We have the following types of neutrosophic modalities:

A) Neutrosophic Alethic Modalities (related to truth) has three neutrosophic operators:
   i. Neutrosophic Possibility: It is neutrosophically possible that \( \mathcal{P} \).
   ii. Neutrosophic Necessity: It is neutrosophically necessary that \( \mathcal{P} \).
   iii. Neutrosophic Impossibility: It is neutrosophically impossible that \( \mathcal{P} \).

B) Neutrosophic Temporal Modalities (related to time)
   It was the neutrosophic case that \( \mathcal{P} \).
   It will neutrosophically be that \( \mathcal{P} \).
   And similarly:
   It has always neutrosophically been that \( \mathcal{P} \).
   It will always neutrosophically be that \( \mathcal{P} \).

C) Neutrosophic Epistemic Modalities (related to knowledge):
   It is neutrosophically known that \( \mathcal{P} \).

D) Neutrosophic Doxastic Modalities (related to belief):
   It is neutrosophically believed that \( \mathcal{P} \).

E) Neutrosophic Deontic Modalities:
   It is neutrosophically obligatory that \( \mathcal{P} \).
   It is neutrosophically permissible that \( \mathcal{P} \).

2 Neutrosophic Alethic Modal Operators

The modalities used in classical (alethic) modal logic can be neutrosophicated by inserting the indeterminacy. We insert the degrees of possibility and degrees of necessity, as refinement of classical modal operators.

3 Neutrosophic Possibility Operator

The classical Possibility Modal Operator \( \Box P \) meaning “It is possible that \( P \)” is extended to Neutrosophic Possibility Operator: \( \Box_{\mathcal{N}} \mathcal{P} \) meaning
«It is (t, i, f)-possible that \( P \)», using Neutrosophic Probability, where «(t, i, f)-possible» means \( t \% \) possible (chance that \( P \) occurs), \( i \% \) indeterminate (indeterminate-chance that \( P \) occurs), and \( f \% \) impossible (chance that \( P \) does not occur).

If \( P(t_p, i_p, f_p) \) is a neutrosophic proposition, with \( t_p, i_p, f_p \) subsets of \([0, 1]\), then the neutrosophic truth-value of the neutrosophic possibility operator is:

\[
\diamond N P = \left( \sup(t_p), \inf(i_p), \inf(f_p) \right),
\]

which means that if a proposition \( P \) is \( t \) true, \( i \) indeterminate, and \( f \) false, then the value of the neutrosophic possibility operator \( \diamond N P \) is: \( \sup(t_p) \) possibility, \( \inf(i_p) \) indeterminate-possibility, and \( \inf(f_p) \) impossibility.

**For example.**

Let \( P = «\text{It will be snowing tomorrow}» \). According to the meteorological center, the neutrosophic truth-value of \( P \) is:

\[
P:\{(0.5, 0.6], (0.2, 0.4), (0.3, 0.5)\},
\]

i.e. \([0.5, 0.6]\) true, \((0.2, 0.4)\) indeterminate, and \((0.3, 0.5)\) false.

Then the neutrosophic possibility operator is:

\[
\diamond N P = \left( \sup(0.5, 0.6], \inf(0.2, 0.4), \inf(0.3, 0.5) \right) = (0.6, 0.2, 0.3),
\]

i.e. 0.6 possible, 0.2 indeterminate-possibility, and 0.3 impossible.

### 4 Neutrosophic Necessity Operator

The classical Necessity Modal Operator «\( \square P \)» meaning «it is necessary that \( P \)» is extended to **Neutrosophic Necessity Operator**: \( \square N P \) meaning «it is \((t, i, f)\)-necessary that \( P \)», using again the Neutrosophic Probability, where similarly «(t, i, f)-necessity» means \( t \% \) necessary (surety that \( P \) occurs), \( i \% \) indeterminate (indeterminate-surety that \( P \) occurs), and \( f \% \) unnecessary (unsurely that \( P \) occurs).

If \( P(t_p, i_p, f_p) \) is a neutrosophic proposition, with \( t_p, i_p, f_p \) subsets of \([0, 1]\), then the neutrosophic truth-value of the neutrosophic necessity operator is:

\[
\square N P = \left( \inf(t_p), \sup(i_p), \sup(f_p) \right),
\]

which means that if a proposition \( P \) is \( t \) true, \( i \) indeterminate, and \( f \) false, then the value of the neutrosophic necessity operator \( \square N P \) is: \( \inf(t_p) \) necessary, \( \sup(i_p) \) indeterminate-necessity, and \( \sup(f_p) \) unnecessary.

Taking the previous example:

\[
P = «\text{It will be snowing tomorrow}», \quad \text{with} \quad P:\{(0.5, 0.6], (0.2, 0.4), (0.3, 0.5)\},
\]

then the neutrosophic necessity operator is:

\[
\square N P = (\inf(0.5, 0.6], \sup(0.2, 0.4), \sup(0.3, 0.5)) = (0.5, 0.4, 0.5),
\]
i.e. 0.5 necessary, 0.4 indeterminate-necessity, and 0.5 unnecessary.

### 5 Connection between Neutrosophic Possibility Operator and Neutrosophic Necessity Operator

In classical modal logic, a modal operator is equivalent to the negation of the other:

\[
\diamond P \leftrightarrow \neg \square \neg P,
\]

\[
\square P \leftrightarrow \neg \diamond \neg P.
\]

In neutrosophic logic one has a class of neutrosophic negation operators. The most used one is:

\[
\neg N P(t, i, f) = \overline{P}(f, 1 - i, t),
\]

where \( t, i, f \) are real subsets of the interval \([0, 1]\).

Let’s check what’s happening in the neutrosophic modal logic, using the previous example.

One had:

\[
P:\{(0.5, 0.6], (0.2, 0.4), (0.3, 0.5)\},
\]

then

\[
\neg N P = \overline{P}((0.3, 0.5), 1 - (0.2, 0.4), [0.5, 0.6]) = \overline{P}((0.3, 0.5), (0.6, 0.8), [0.5, 0.6]).
\]

Therefore, denoting by \( \leftrightarrow \) the neutrosophic equivalence, one has:

\[
\neg \square N P:\{(0.5, 0.6], (0.2, 0.4), (0.3, 0.5)\} \leftrightarrow \nabla N N P:\{(0.5, 0.6], (0.2, 0.4), (0.3, 0.5)\}
\]

\[
\neg N N P:\{(0.3, 0.5), (0.6, 0.8), [0.5, 0.6]\} \leftrightarrow \nabla N P:\{(0.3, 0.5), (0.6, 0.8), [0.5, 0.6]\}
\]

\[
\neg N P:\{(0.3, 0.5), (0.6, 0.8), [0.5, 0.6]\} \leftrightarrow \nabla N N P:\{(0.3, 0.5), (0.6, 0.8), [0.5, 0.6]\}
\]

\[
\neg N P:\{(0.5, 0.6], (0.2, 0.4), (0.3, 0.5)\} \leftrightarrow \nabla N N P:\{(0.5, 0.6], (0.2, 0.4), (0.3, 0.5)\} = (0.6, 0.2, 0.3).
Let’s check the second neutrosophic equivalence.

\[ \neg \phi^n \neg N^N \mathcal{P}([0.5, 0.6], (0.2, 0.4), (0.3, 0.5)) \]

\[ \neg N^N \mathcal{P}(\{0.3, 0.5\}, (0.6, 0.8), [0.5, 0.6]) \]

\[ \neg N \mathcal{P}(\{0.3, 0.5\}, (0.6, 0.8), [0.5, 0.6]) \]

\[ \neg \mathcal{P}(\{0.3, 0.5\}, (0.6, 0.8), (0.3, 0.5)) \]

\[ \neg N^N \mathcal{P}(\{0.5, 0.6\}, [0.2, 0.4), (0.3, 0.5)) \]

\[ \neg \mathcal{P}(\{0.5, 0.6\}, [0.2, 0.4), (0.3, 0.5)) \]

\[ \mathcal{P}(\{0.5, 0.6\}, [0.2, 0.4), (0.3, 0.5)) = (0.6, 0.2, 0.3). \]

6 Neutrosophic Modal Equivalences

Neutrosophic Modal Equivalences hold within a certain accuracy, depending on the definitions of neutrosophic possiblity operator and neutrosophic necessity operator, as well as on the definition of the neutrosophic negation – employed by the experts depending on each application. Under these conditions, one may have the following neutrosophic modal equivalences:

\[ \phi^n \mathcal{P}(\{t_p, i_p, f_p\}) \]

\[ \Box^n \mathcal{P}(\{t_p, i_p, f_p\}) \]

For example, other definitions for the neutrosophic modal operators may be:

\[ \phi^n \mathcal{P}(\{t_p, i_p, f_p\}) = \left( \sup(t_p), \sup(i_p), \inf(f_p) \right), \text{ or} \]

\[ \phi^n \mathcal{P}(\{t_p, i_p, f_p\}) = \left( \sup(t_p), \frac{i_p}{2}, \inf(f_p) \right) \text{ etc.,} \]

while

\[ \Box^n \mathcal{P}(\{t_p, i_p, f_p\}) = \left( \inf(t_p), \inf(i_p), \sup(f_p) \right), \text{ or} \]

\[ \Box^n \mathcal{P}(\{t_p, i_p, f_p\}) = \left( \inf(t_p), 2i_p \cap [0,1], \sup(f_p) \right) \text{ etc.} \]

7 Neutrosophic Truth Threshold

In neutrosophic logic, first we have to introduce a neutrosophic truth threshold, \( TH = \{T_{th}, I_{th}, F_{th}\} \), where \( T_{th}, I_{th}, F_{th} \) are subsets of \([0, 1]\). We use uppercase letters \( T, I, F \) in order to distinguish the neutrosophic components of the threshold from those of a proposition in general.

We can say that the proposition \( \mathcal{P}(t_p, i_p, f_p) \) is neutrosophically true if:

\[ \inf(t_p) \geq \inf(T_{th}) \text{ and } \sup(t_p) \geq \sup(T_{th}); \]

\[ \inf(i_p) \leq \inf(I_{th}) \text{ and } \sup(i_p) \leq \sup(I_{th}); \]

\[ \inf(f_p) \leq \inf(F_{th}) \text{ and } \sup(f_p) \leq \sup(F_{th}). \]

For the particular case when all \( T_{th}, I_{th}, F_{th} \) and \( t_p, i_p, f_p \) are single-valued numbers from the interval \([0, 1]\), then one has:

The proposition \( \mathcal{P}(t_p, i_p, f_p) \) is neutrosophically true if:

\[ t_p \geq T_{th}; \]

\[ i_p \leq I_{th}; \]

\[ f_p \leq F_{th}. \]

The neutrosophic truth threshold is established by experts in accordance to each application.

8 Neutrosophic Semantics

Neutrosophic Semantics of the Neutrosophic Modal Logic is formed by a neutrosophic frame \( G_N \), which is a non-empty neutrosophic set, whose elements are called possible neutrosophic worlds, and a neutrosophic binary relation \( R_N \), called neutrosophic accessibility relation, between the possible neutrosophic worlds. By notation, one has:

\( (G_N, R_N). \)

A neutrosophic world \( w_N \) that is neutrosophically accessible from the neutrosophic world \( w_N \) is symbolized as:

\( w_N R_N w'_N. \)

In a neutrosophic model each neutrosophic proposition \( \mathcal{P} \) has a neutrosophic truth-value \( \{t_{w_N}, i_{w_N}, f_{w_N}\} \) respectively to each neutrosophic world \( w_N \in G_N \), where \( t_{w_N}, i_{w_N}, f_{w_N} \) are subsets of \([0, 1]\).

A neutrosophic actual world can be similarly noted as in classical modal logic as \( w_N^*. \)

Formalization.

Let \( S_N \) be a set of neutrosophic propositional variables.

9 Neutrosophic Formulas

1) Every neutrosophic propositional variable \( \mathcal{P} \in S_N \) is a neutrosophic formula. 

2) If \( A, B \) are neutrosophic formulas, then \( \neg A, A \lor B, A \land B, A \rightarrow B, \) and \( \phi A, \Box A, \) are also neutrosophic formulas, where \( \neg, \lor, \land, \rightarrow, \phi \) and \( \Box, \phi \)
represent the neutrosophic negation, neutrosophic intersection, neutrosophic union, neutrosophic implication, neutrosophic equivalence, and neutrosophic possibility operator, neutrosophic necessity operator respectively.

10 Accessibility Relation in a Neutrosophic Theory

Let $G_N$ be a set of neutrosophic worlds $w_N$ such that each $w_N$ characterizes the propositions (formulas) of a given neutrosophic theory $\tau$.

We say that the neutrosophic world $w'_N$ is accessible from the neutrosophic world $w_N$, and we write: $w_N \mathcal{R}_N w'_N$ or $\mathcal{R}_N(w_N, w'_N)$, if for any proposition (formula) $\mathcal{P} \in w_N$, meaning the neutrosophic truth-value of $\mathcal{P}$ with respect to $w_N$ is

$$\mathcal{P}(t^w_N, i^w_N, f^w_N),$$

one has the neutrosophic truth-value of $\mathcal{P}$ with respect to $w'_N$

$$\mathcal{P}(t^{w'_N}, i^{w'_N}, f^{w'_N}),$$

where

$$\inf(t^w_N) \geq \inf(t^{w'_N})$$

and $\sup(t^w_N) \geq \sup(t^{w'_N})$;

$$\inf(i^w_N) \leq \inf(i^{w'_N})$$

and $\sup(i^w_N) \leq \sup(i^{w'_N})$;

$$\inf(f^w_N) \leq \inf(f^{w'_N})$$

and $\sup(f^w_N) \leq \sup(f^{w'_N})$.

(in the general case when $t_p^{w_N}, i_p^{w_N}, f_p^{w_N}$ and $t_p^{w'_N}, i_p^{w'_N}, f_p^{w'_N}$ are subsets of the interval $[0, 1]$).

But in the instant of $t_p^{w_N}, i_p^{w_N}, f_p^{w_N}$ and $t_p^{w'_N}, i_p^{w'_N}, f_p^{w'_N}$ as single-values in $[0, 1]$, the above inequalities become:

$$t_p^{w_N} \geq t_p^{w'_N},$$

$$i_p^{w_N} \leq i_p^{w'_N},$$

$$f_p^{w_N} \leq f_p^{w'_N}.$$  

11 Applications

If the neutrosophic theory $\tau$ is the Neutrosophic Mereology, or Neutrosophic Gnosiology, or Neutrosophic Epistemology etc., the neutrosophic accessibility relation is defined as above.

12 Neutrosophic n-ary Accessibility Relation

We can also extend the classical binary accessibility relation $\mathcal{R}$ to a neutrosophic $n$-ary accessibility relation $\mathcal{R}^{(n)}_N$, for $n$ integer $\geq 2$.

Instead of the classical $R(w, w')$, which means that the world $w'$ is accesible from the world $w$, we generalize it to:

$$\mathcal{R}^{(n)}_N(w_1, w_2, \ldots, w_n, w'_n),$$

which means that the neutrosophic world $w'_n$ is accesible from the neutrosophic worlds $w_1, w_2, \ldots, w_n$ all together.

13 Neutrosophic Kripke Frame

$k_N = (G_N, R_N)$ is a neutrosophic Kripke frame, since:

(i) $G_N$ is an arbitrary non-empty neutrosophic set of neutrosophic worlds, or neutrosophic states, or neutrosophic situations.

(ii) $R_N \subseteq G_N \times G_N$ is a neutrosophic accessibility relation of the neutrosophic Kripke frame. Actually, one has a degree of accessibility, degree of indeterminacy, and a degree of non-accessibility.

14 Neutrosophic (t, i, f)-Assignment

The Neutrosophic (t, i, f)-Assignment is a neutrosophic mapping

$$v_N: S_N \times G_N \rightarrow [0, 1] \times [0, 1] \times [0, 1]$$

where, for any neutrosophic proposition $\mathcal{P} \in S_N$ and for any neutrosophic world $w_N$, one defines:

$$v_N(\mathcal{P}, w_N) = (t^w_N, i^w_N, f^w_N) \in [0, 1] \times [0, 1] \times [0, 1]$$

which is the neutrosophical logical truth value of the neutrosophic proposition $\mathcal{P}$ in the neutrosophic world $w_N$.

15 Neutrosophic Deducibility

We say that the neutrosophic formula $\mathcal{P}$ is neutrosophically deducible from the neutrosophic Kripke frame $k_N$, the neutrosophic (t, i, f)-assignment $v_N$, and the neutrosophic world $w_N$, and we write as:

$$k_N, v_N, w_N \models \mathcal{P}.$$  

Let’s make the notation:

$$\alpha_N(\mathcal{P}; k_N, v_N, w_N)$$

denotes the neutrosophic logical value that the formula $\mathcal{P}$ takes with respect to the neutrosophic Kripke frame $k_N$, the neutrosophic (t, i, f)-assignment $v_N$, and the neutrosophic world $w_N$.

We define $\alpha_N$ by neutrosophic induction:

1. $\alpha_N(\mathcal{P}; k_N, v_N, w_N) = v_N(\mathcal{P}, w_N)$ if $\mathcal{P} \in S_N$ and $w_N \in G_N$.

2. $\alpha_N(\mathcal{P}; k_N, v_N, w_N) = \max_\mathcal{N}[\alpha_N(\mathcal{P}; k_N, v_N, w_N)].$

3. $\alpha_N(\mathcal{P}; k_N, v_N, w_N) = [\alpha_N(\mathcal{P}; k_N, v_N, w_N)]^\mathcal{N}$.  

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4. \( a_N(\mathcal{P} \sqcup_N Q; k_N, v_N, w_N) \) \( \overset{\text{def}}{=} \)
   \( [a_N(\mathcal{P}; k_N, v_N, w_N)]^N \cup \left( a_N(\mathcal{Q}; k_N, v_N, w_N) \right) \)

5. \( a_N(\mathcal{P} \sqcap_N Q; k_N, v_N, w_N) \) \( \overset{\text{def}}{=} \)
   \( [a_N(\mathcal{P}; k_N, v_N, w_N)]^N \cap \left( a_N(\mathcal{Q}; k_N, v_N, w_N) \right) \)

6. \( a_N(\mathcal{P}^\circ; k_N, v_N, w_N) \) \( \overset{\text{def}}{=} \)
   \( \langle \sup, \inf, \inf \rangle \left( a_N(\mathcal{P}; k_N, v_N, w_N) \right), w' \in \mathcal{G}_N \) and \( w_N R_N w' \).

7. \( a_N(\mathcal{P}^\star; k_N, v_N, w_N) \) \( \overset{\text{def}}{=} \)
   \( \langle \inf, \sup, \sup \rangle \left( a_N(\mathcal{P}; k_N, v_N, w_N) \right), w' \in \mathcal{G}_N \) and \( w_N R_N w' \).

8. \( \mathcal{P} \overset{\mathbb{N}}{\Rightarrow} \mathcal{P} \) if and only if \( w_N \neq \mathcal{P} \) (a formula \( \mathcal{P} \) is neutrosophically deducible if and only if \( \mathcal{P} \) is neutrosophically deducible in the actual neutrosophic world).

We should remark that \( a_N \) has a degree of truth \( \langle t_{ax} \rangle \), a degree of indeterminacy \( \langle i_{ax} \rangle \), and a degree of falsehood \( \langle f_{ax} \rangle \), which are in the general case subsets of the interval \([0, 1]\).

Applying \( \langle \sup, \inf, \inf \rangle \) to \( a_N \) is equivalent to calculating:

\( \langle \sup(\langle t_{ax} \rangle), \inf(\langle i_{ax} \rangle), \inf(\langle f_{ax} \rangle) \rangle \),

and similarly

\( \langle \inf(\langle t_{ax} \rangle), \sup(\langle i_{ax} \rangle), \sup(\langle f_{ax} \rangle) \rangle \).

16 Refined Neutrosophic Modal Single-Valued Logic

Using neutrosophic \( (t, i, f) \) - thresholds, we refine for the first time the neutrosophic modal logic as:

a) Refined Neutrosophic Possibility Operator.

\( \overset{\mathbb{N}}{\Phi}^1_{(t, i, f)} = \langle \text{It is very little possible (degree of possibility } t_1) \rangle \) \( \mathcal{P}_N \), corresponding to the threshold \( \langle t_1, i_1, f_1 \rangle \), i.e. \( 0 \leq t \leq t_1, i \geq i_1, f \geq f_1 \), for \( t_1 \) a very little number in \([0, 1]\);

\( \overset{\mathbb{N}}{\Phi}^2_{(t, i, f)} = \langle \text{It is little possible (degree of possibility } t_2) \rangle \) \( \mathcal{P}_N \), corresponding to the threshold \( \langle t_2, i_2, f_2 \rangle \), i.e. \( t_1 \leq t \leq t_2, i \geq i_2 \geq i_1, f \geq f_2 \geq f_1 \);

\( \cdots \cdots \)

and so on;

\( \overset{\mathbb{N}}{\Phi}^m_{(t, i, f)} = \langle \text{It is possible (with a degree of possibility } t_m) \rangle \) \( \mathcal{P}_N \), corresponding to the threshold \( \langle t_m, i_m, f_m \rangle \), i.e. \( t_{m-1} < t \leq t_m, i \geq i_m > i_{m-1}, f \geq f_m > f_{m-1} \).

b) Refined Neutrosophic Necessity Operator.

\( \overset{\mathbb{N}}{\Psi}^1_{(t, i, f)} = \langle \text{It is a small necessity (degree of necessity } t_{m+1}) \rangle \) \( \mathcal{P}_N \), i.e. \( t \leq t < t_{m+1}, i \geq i_{m+1} > i_m, f \geq f_{m+1} > f_m \);

\( \cdots \cdots \)

\( \overset{\mathbb{N}}{\Psi}^m_{(t, i, f)} = \langle \text{It is a very high necessity (degree of necessity } t_{m+k}) \rangle \) \( \mathcal{P}_N \), i.e. \( t \leq t_m, i \geq i_k > i_{m+k-1}, f \geq f_{m+k} > f_{m+k-1} \).

17 Application of the Neutrosophic Threshold

We have introduced the term of \((t, i, f)\)-physical law, meaning that a physical law has a degree of truth \((t)\), a degree of indeterminacy \((i)\), and a degree of falsehood \((f)\). A physical law is \(100\%\) true, \(0\%\) indeterminate, and \(0\%\) false in perfect (ideal) conditions only, maybe in laboratory.

But our actual world \((w_N)\) is not perfect and not steady, but continuously changing, varying, fluctuating.

For example, there are physicists that have proved a universal constant \((c)\) is not quite universal (i.e. there are special conditions where it does not apply, or its value varies between \((c - \varepsilon, c + \varepsilon)\), for \(\varepsilon > 0\) that can be a tiny or even a bigger number).

Thus, we can say that a proposition \( \mathcal{P} \) is neutrosophically nomological necessary, if \( \mathcal{P} \) is neutrosophically true at all possible neutrosophic worlds that obey the \((t, i, f)\)-physical laws of the actual neutrosophic world \(w_N\).

In other words, at each possible neutrosophic world \(w_N\), neutrosophically accessible from \(w_N\), one has:

\( \mathcal{P}(t_W^N, i_W^N, f_W^N) \geq \mathcal{P}(T_{th}, I_{th}, F_{th}), \)

i.e. \( t_W^N \geq T_{th}, i_W^N \leq I_{th}, \) and \( f_W^N \geq F_{th} \).

18 Neutrosophic Mereology

Neutrosophic Mereology means the theory of the neutrosophic relations among the parts of a whole, and the neutrosophic relations between the parts and the whole.

A neutrosophic relation between two parts, and similarly a neutrosophic relation between a part and the whole, has a degree of connectibility \((t)\), a degree of indeterminacy \((i)\), and a degree of disconnectibility \((f)\).

19 Neutrosophic Mereological Threshold

Neutrosophic Mereological Threshold is defined as:

\[ TH_M = (\min(t_M), \max(i_M), \max(f_M)) \]

where \(t_M\) is the set of all degrees of connectibility between the parts, and between the parts and the whole;
i_M is the set of all degrees of indeterminacy between the parts, and between the parts and the whole;

f_M is the set of all degrees of disconnectibility between the parts, and between the parts and the whole.

We have considered all degrees as single-valued numbers.

20 Neutrosophic Gnosisology

Neutrosophic Gnosisology is the theory of (t, i, f)-knowledge, because in many cases we are not able to completely (100%) find whole knowledge, but only a part of it (t %), another part remaining unknown (f%), and a third part indeterminate (unclear, vague, contradictory) (i %), where t, i, f are subsets of the interval [0, 1].

21 Neutrosophic Gnosisological Threshold

Neutrosophic Gnosisological Threshold is defined, similarly, as:

$$TH_G = \left( \min(t_G), \max(i_G), \max(f_G) \right)$$

where t_G is the set of all degrees of knowledge of all theories, ideas, propositions etc.,

i_G is the set of all degrees of indeterminate-knowledge of all theories, ideas, propositions etc.,

f_G is the set of all degrees of non-knowledge of all theories, ideas, propositions etc.

We have considered all degrees as single-valued numbers.

22 Neutrosophic Epistemology

And Neutrosophic Epistemology, as part of the Neutrosophic Gnosisology, is the theory of (t, i, f)-scientific knowledge.

Science is infinite. We know only a small part of it (t %), another big part is yet to be discovered (f %), and a third part is yet to be discovered (i %).

Of course, t, i, f are subsets of [0, 1].

23 Neutrosophic Epistemological Threshold

It is defined as:

$$TH_E = \left( \min(t_E), \max(i_E), \max(f_E) \right)$$

where t_E is the set of all degrees of scientific knowledge of all scientific theories, ideas, propositions etc.,

i_E is the set of all degrees of indeterminate scientific knowledge of all scientific theories, ideas, propositions etc.,

f_E is the set of all degrees of non-scientific knowledge of all scientific theories, ideas, propositions etc.

We have considered all degrees as single-valued numbers.

24 Conclusions

We have introduced for the first time the Neutrosophic Modal Logic and the Refined Neutrosophic Modal Logic.

Symbolic Neutrosophic Logic can be connected to the neutrosophic modal logic too, where instead of numbers we may use labels, or instead of quantitative neutrosophic logic we may have a qualitative neutrosophic logic. As an extension, we may introduce Symbolic Neutrosophic Modal Logic and Refined Symbolic Neutrosophic Modal Logic, where the symbolic neutrosophic modal operators (and the symbolic neutrosophic accessibility relation) have qualitative values (labels) instead on numerical values (subsets of the interval [0, 1]).

Applications of neutrosophic modal logic are to neutrosophic modal metaphysics. Similarly to classical modal logic, there is a plethora of neutrosophic modal logics. Neutrosophic modal logics is governed by a set of neutrosophic axioms and neutrosophic rules. The neutrosophic accessibility relation has various interpretations, depending on the applications. Similarly, the notion of possible neutrosophic worlds has many interpretations, as part of possible neutrosophic semantics.

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Received: February 10, 2017. Accepted: February 24, 2017.
Information about the journal:

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The neutrosophics website at UNM is: http://fs.gallup.unm.edu/neutrosophy.htm

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