Multi-Objective Portfolio Selection Model with Diversification by Neutrosophic Optimization Technique

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Abstract. In this paper, we first consider a multi-objective Portfolio Selection model and then we add another entropy objective function and next we generalized the model. We solve the problems using Neutrosophic optimization technique. The models are illustrated with numerical examples.

Keywords: Portfolio Optimization, Multi-objective Model, Entropy, Neutrosophic set, Neutrosophic optimization method.

1. Introduction:

Markowitz [5] first introduced the theory of mean-variance efficient portfolios and also gave his critical line method for finding these. He combined probability and optimization theory. Roll [2] gave an analytical method to find modified mean-variance efficient portfolios where he allowed short sales. Single objective portfolio optimization method using fuzzy decision theory, possibilistic and interval programming are given by Wang et. al.[7].Inuiguchi and Tanino [3] proposed a new approach to the possibilistic portfolio selection problem.

Very few authors discussed entropy based multi-objective portfolio selection method. Here entropy is acted as a measure of dispersal. The entropy maximization model has attracted a good deal of attention in urban and regional analysis as well as in other areas. Usefulness of entropy optimization models in portfolio selection based problems are illustrated in two well-known books ([4],[6]).

Zadeh [1] first introduced the concept of fuzzy set theory. Zimmermann [13] used Bellman and Zadeh's [14] fuzzy decision concept. Zimmermann applied the fuzzy set theory concept with some suitable membership functions to solve linear programming problem with several objective functions. In traditional fuzzy sets, one real value $\mu_A(x) \in [0,1]$ represents the truth membership function of fuzzy set defined on universe of discourse X. But sometimes we have problems due to uncertainty of $\mu_A(x)$ itself. It is very hard to find a crisp value then. To avoid the problem, the concept of interval valued fuzzy sets was proposed. In real life problem, we should consider the truth membership function supported by the evident as well as the falsity membership function against by the evident. So, Atanassov ([8],[10]) introduced the intuitionistic fuzzy sets in 1986. The intuitionistic fuzzy sets consider both truth and falsity membership functions. But it can only effective for incomplete information. Intuitionistic fuzzy sets cannot handle when we have indeterminate information and inconsistent information. In decision making theory, decision makers can make a decision, cannot make a decision or can hesitate to make a decision. We cannot use intuitionistic fuzzy sets in this situation. Then Neutrosophy was introduced by Smarandache [11] in 1995. Realising the difference between absolute truth and relative truth or between absolute falsehood and relative falsehood, Smarandache started to use non-standard analysis. Then he combined the non-standard analysis with logic, set, probability theory and philosophy. Neutrosophic theory has various fields like Neutrosophic Set, Neutrosophic Logic, Neutrosophic Probability, Neutrosophic Statistics, Neutrosophic Precalculus and Neutrosophic Calculus. In neutrosophic sets we have truth membership, indeterminacy membership and falsity membership functions which are independent. In Neutrosophic logic, a proposition has a degree of truth(T), degree of indeterminacy(I) and a degree of falsity(F), where T,I,F are standard or non-standard subsets of $]0^-,1^+[$. Wang, Smarandache, Zhang and Sunderraman [12] discussed about single valued neutrosophic sets, multispace and multistructure. S. Pramanik ([15], [16]) and Abdel-Baset, Hezam & Smarandache ([18], [19]) used Neutrosophic theory in multi-objective linear programming, linear goal programming. Sahidul Islam, Tanmay Kundu [20] applied Neutrosophic optimization technique to solve multi- objective Reliability problem. M. Sarkar, T. K. Roy [17] used Neutrosophic optimization technique in optimization of welded beam structure. Pintu Das, T.K.Roy [9] applied Neutrosophic optimization technique in Riser design problem.

Our objective in this paper is to give a computational algorithm for solving multi-objective portfolio selection problem with diversification by single valued neutrosophic optimization technique. We also take different weights on objective functions. The models are illustrated with numerical examples.

2. Mathematical Model:

Suppose that a prosperous individual has an opportunity to invest an asset (i.e. a fixed amount of money) in n different bonds and stocks. Let x =(x_1, x_2,……..,x_n)^T , where x_j is the proportion of his assets invested in the j-th security. The vector x is called portfolio. Clearly, a physically realizable portfolio must satisfy $x_j \geq 0, (j = 1,2, \dots, n), \sum_{j=1}^{n} x_j$=1. The agents are assumed to strike balance between maximizing the return and minimizing the risk of their investment decision. Return is quantified by the mean, and risk is characterized by the variance, of a portfolio assets. The return R_j for the j-th security, (j =1,2,…,n), is a random variable, with expected return r_j= E(R_j). Let R = (R_1, R_2…… R_n)^T , r = (r_1, r_2…… r_n)^T . The return for the portfolio is thus R^Tx=$\sum_{j=1}^{n} R_j x_j$and expected return Er(x)=E(R^Tx)=$\sum_{j=1}^{n} r_j x_j$.
Let $(\sigma_{ij})_{n\times n}$be the covariance matrix of a random vector R, the variance of the portfolio is Vr(x) = Var(R^Tx) =$\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} x_i x_j$ where

$$\sigma_{ij} = \sigma_i^2, \quad i = j$$
$$= \rho_{ij}\sigma_i\sigma_j, \quad i > j \text{ or } j > i.$$

σ_j^2 is the variance of R_j and ρ_{ij} is the correlation coefficient between R_j and R_i, $\forall i \text{ and } j = 1,2, \dots, n$.

2.1 Portfolio Selection problem (PSP):

The two objectives of an investor are thus to maximize the expected value of return and minimize the variance subject to a constraint of a Portfolio. So the Portfolio Selection Problem (PSP) is:

$$\text{Maximize } Er(x) = \sum_{j=1}^{n} r_j x_j \qquad (2.1)$$
$$\text{Minimize } Vr(x) = \sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} x_i x_j$$
$$\text{subject to}$$
$$\sum_{j=1}^{n} x_j = 1,$$
$$\text{and } x_j \geq 0, \; j = 1,2, \dots, n.$$

Markowitz's mean variance criterion simply states that an investor should always choose an efficient portfolio.

2.2 Entropy:

In physics, the word entropy has important physical implications as the amount of "disorder" of a system but in mathematics, we use more abstract definition. The (Shannon) entropy of a variable X is defined as
$EN(X) = -\sum_{j=1}^{n} x_j \ln x_j$, where $x_j$is the probabilty that X is in the state x, and $x_j \ln x_j$ is defined as 0 if x_j= 0.

2.3 Portfolio Selection problem with Diversification (PSPD):

In real life problem, we introduce another entropy objective function in problem (2.1) which is a Portfolio Selection Problem with Diversification (PSPD) and it is written as

$$\text{Maximize } EN(X) = -\sum_{j=1}^{n} x_j \ln x_j \qquad (2.2)$$
$$\text{Maximize } Er(x) = \sum_{j=1}^{n} r_j x_j$$
$$\text{Minimize } Vr(x) = \sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} x_i x_j$$
$$\text{subject to}$$
$$\sum_{j=1}^{n} x_j = 1,$$
$$\text{and } x_j \geq 0, \; j = 1,2, \dots, n.$$

2.4 Generalized Portfolio Selection problem with Diversification (GPSPD):

For generalization of the above model, an investor can construct a portfolio based on m potential market scenarios from an investment universe of n assets. Let R_j^k $(j = 1,2, \dots, n, \; k = 1,2, \dots, m)$denotes the return of the $j-th$ asset and let $R_k(x) = \sum_{j=1}^{n} R_j^k x_j$ denotes the portfolio return with expected return $Er_k(x) = \sum_{j=1}^{n} r_j^k x_j$ and

$$\sigma_{ij}^k = (\sigma_i^k)^2, \quad i = j$$

$\sigma_{ij}{}^{k} = \rho_{ij}{}^{k}\sigma_i{}^{k}\sigma_j{}^{k},\ i > j \text{ or } j > i.$

Where $(\sigma_i{}^{k})^2$ is the variance of $R_i{}^{k}$ and $\rho_{ij}{}^{k}$ is the correlation coefficient between $R_i{}^{k}$ and $R_j{}^{k}$ ($\forall i, j = 1,2,\ldots,n$) for the k-th market scenario at the end of investment period, then $Vr_k(x) = \sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}{}^{k}x_ix_j$ denote the risk for the $k-th$ scenario. So Generalized Portfolio Selection Problem with Diversification (GPSPD) can be stated as follows:

$$
\begin{aligned}
&\text{Maximize } EN(X) = -\sum_{i=1}^{n} x_i \ln x_i \qquad (2.3)\\
&\text{Maximize } Er_1(x) = \sum_{i=1}^{n} r_i{}^{1} x_i,\\
&\text{Maximize } Er_2(x) = \sum_{i=1}^{n} r_i{}^{2} x_i,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
&\text{Maximize } Er_m(x) = \sum_{i=1}^{n} r_i{}^{m} x_i,\\
&\text{Minimize } Vr_1(x) = \sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}{}^{1}x_ix_j,\\
&\text{Minimize } Vr_2(x) = \sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}{}^{2}x_ix_j,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
&\text{Minimize } Vr_m(x) = \sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}{}^{m}x_ix_j,\\
&\quad\text{Subject to}\\
&\qquad \sum_{i=1}^{n} x_i = 1\\
&\qquad x_j \geq 0,\ j = 1,2,\ldots,n.
\end{aligned}
$$

3. Preliminaries:

3.1 Fuzzy Set:

Fuzzy set was introduced by Zadeh [1] in 1965. A fuzzy set $\tilde{A}$ in a universe of discourse X is defined as $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$. Here $\mu_{\tilde{A}} : X \to [0,1]$ is a mapping which is called the membership function of the fuzzy set $\tilde{A}$ and $\mu_{\tilde{A}}(x)$ is called the membership value of $x \in X$ in the fuzzy set $\tilde{A}$. The larger $\mu_{\tilde{A}}(x)$ is the stronger the grade of membership form in $\tilde{A}$.

3.2 Neutrosophic Set:

Let X be a universe of discourse. A neutrosophic set $\tilde{A}^n$ in X is defined by a Truth-membership function $\mu_A(x)$, an indeterminacy-membership function $\sigma_A(x)$ and a falsity-membership function $v_A(x)$ having the form $\tilde{A}^n = \{< X, \mu_A(x), \sigma_A(x), v_A(x) >: x \in X\}$.

Where, $\mu_A(x) : X \to]0^-, 1^+[$

$\sigma_A(x) : X \to]0^-, 1^+[$

$v_A(x) : X \to]0^-, 1^+[$ and there is no restriction on the sum of $\mu_A(x)$, $\sigma_A(x)$ and $v_A(x)$.

So, $0^- \leq \sup\mu_A(x) + \sup\sigma_A(x) + \sup v_A(x) \leq 3^+$.

3.3 Single valued Neutrosophic Set:

Let X be a universe of discourse. A single valued neutrosophic set $\tilde{A}^n$ over X is an object with the form $\tilde{A}^n = \{< X, \mu_A(x), \sigma_A(x), v_A(x) >: x \in X\}$, where

$\mu_A(x) : X \to [0,1]$

$\sigma_A(x) : X \to [0,1]$

$v_A(x) : X \to [0,1]$

with $0 \leq \mu_A(x) + \sigma_A(x), + v_A(x) \leq 3, \forall x \in X$.

3.4 Complement of Single valued Neutrosophic Set:

Let X be a universe of discourse. The complement of a single valued neutrosophic set $\tilde{A}^n$ is denoted by c($\tilde{A}^n$) and is defined by

$\mu_{c(A)}(x) = v_A(x)$

$\sigma_{c(A)}(x) = 1 - \sigma_A(x)$

$v_{c(A)}(x) = \mu_A(x), \forall x \in X.$

3.5 Union of two Single valued Neutrosophic Sets:

The union of two single valued neutrosophic sets $\tilde{A}^n$ and $\tilde{B}^n$ is a single valued neutrosophic set $\tilde{C}^n$, where $\tilde{C}^n = \tilde{A}^n \cup \tilde{B}^n$ and

$\mu_c(x) = \max(\mu_A(x), \mu_B(x))$

$\sigma_c(x) = \max(\sigma_A(x), \sigma_B(x))$

$v_c(x) = \min(v_A(x), v_B(x)), \forall x \in X.$

3.6 Intersection of two Single valued Neutrosophic Sets:

The union of two single valued neutrosophic sets $\tilde{A}^n$ and $\tilde{B}^n$ is a single valued neutrosophic set $\tilde{C}^n$, where $\tilde{C}^n = \tilde{A}^n \cap \tilde{B}^n$ and

$$\mu_c(x) = \min\,(\mu_A(x), \mu_B(x))$$
$$\sigma_c(x) = \min\,(\sigma_A(x), \sigma_B(x))$$
$$\nu_c(x) = \max\,(\nu_A(x), \nu_B(x)),\ \forall x \in X.$$

4. Neutrosophic Optimization Method to solve minimization type multi-objective non-linear programming problem.

A minimization type multi-objective non-linear problem is of the form

$$Min\,\{f_1(x), f_2(x), \ldots\ldots, f_p(x)\} \qquad (4.1)$$
$$g_i(x) \le b_i,\ i = 1,2,\ldots\ldots,q$$

We define the decision set $\tilde{D}^n$ which is a conjunction of neutrosophic objectives and constraints and is defined by

$$\tilde{D}^n = \left(\cap_{k=1}^{p} \tilde{G}_k^{\,n}\right) \cap \left(\cap_{i=1}^{q} \tilde{C}_i^{\,n}\right) = \{(x, \mu_{\tilde{D}^n}(x), \sigma_{\tilde{D}^n}(x), \nu_{\tilde{D}^n}(x))\}, \text{ where}$$
$$\mu_{\tilde{D}^n}(x) = \min\{\mu_{\tilde{G}_1^n}(x), \mu_{\tilde{G}_2^n}(x), \ldots\ldots, \mu_{\tilde{G}_p^n}(x); \mu_{\tilde{C}_1^n}(x), \mu_{\tilde{C}_2^n}(x), \ldots\ldots, \mu_{\tilde{C}_q^n}(x)\}, \forall x \in X.$$
$$\sigma_{\tilde{D}^n}(x) = \min\{\sigma_{\tilde{G}_1^n}(x), \sigma_{\tilde{G}_2^n}(x), \ldots\ldots, \sigma_{\tilde{G}_p^n}(x); \sigma_{\tilde{C}_1^n}(x), \sigma_{\tilde{C}_2^n}(x), \ldots\ldots, \sigma_{\tilde{C}_q^n}(x)\}, \forall x \in X.$$
$$\nu_{\tilde{D}^n}(x) = \max\{\nu_{\tilde{G}_1^n}(x), \nu_{\tilde{G}_2^n}(x), \ldots\ldots, \nu_{\tilde{G}_p^n}(x); \nu_{\tilde{C}_1^n}(x), \nu_{\tilde{C}_2^n}(x), \ldots\ldots, \nu_{\tilde{C}_q^n}(x)\}, \forall x \in X.$$

Here $\mu_{\tilde{D}^n}(x), \sigma_{\tilde{D}^n}(x), \nu_{\tilde{D}^n}(x)$ are Truth-membership function, indeterminacy-membership function and falsity-membership function of neutrosophic decision set respectively.

Now the transformed non-linear programming problem of the problem (4.1) can be written as

$$Max\ \alpha \qquad (4.2)$$
$$Max\ \gamma$$
$$Min\ \beta$$

With $\mu_{\tilde{G}_k^n}(x) \ge \alpha$

$$\mu_{\tilde{C}_j^n}(x) \ge \alpha$$
$$\sigma_{\tilde{G}_k^n}(x) \ge \gamma$$
$$\sigma_{\tilde{C}_j^n}(x) \ge \gamma$$
$$\nu_{\tilde{G}_k^n}(x) \le \beta$$
$$\nu_{\tilde{C}_j^n}(x) \le \beta,\ (k = 1,2,\ldots,p;\ \ j = 1,2,\ldots,q)$$
$$\alpha + \beta + \gamma \le 3$$
$$\alpha \ge \beta$$
$$\alpha \ge \gamma$$
$$\alpha, \beta, \gamma \in [0,1]$$

5. Computational Algorithm:

Step-1: First we convert all the objective functions of the problem (2.3) into minimization type. So the problem (2.3) becomes

$$\text{Minimize } -EN(X) = \sum_{i=1}^{n} x_i \ln x_i \qquad (5.1)$$
$$\text{Minimize } -\text{Er}_1(\text{x}) = -\sum_{i=1}^{n} r_i^{\,1} x_i$$
$$\text{Minimize } -\text{Er}_2(\text{x}) = -\sum_{i=1}^{n} r_i^{\,2} x_i$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$\text{Minimize } -\text{Er}_m(\text{x}) = -\sum_{i=1}^{n} r_i^{\,m} x_i$$
$$\text{Minimize } \text{Vr}_1(\text{x}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij}^{\,1} x_j x_i$$
$$\text{Minimize } \text{Vr}_2(\text{x}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij}^{\,2} x_j x_i$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$\text{Minimize } \text{Vr}_m(\text{x}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij}^{\,m} x_j x_i$$

Subject to

$$\sum_{j=1}^{n} x_j = 1$$
$$x_j \ge 0,\ j = 1,2,\ldots,n.$$

Let us rename the above $(2m+1)$ objective functions as $f_1(x), f_2(x), \ldots\ldots, f_{2m+1}(x)$ respectively. Now solve the problem (5.1) as a single objective non-linear programming problem using only one objective at a time and ignoring the others. These solutions are known as ideal solutions.

Step-2: From the results of step 1, determine the corresponding values for every objective at each solution derived. With the values of all objectives at each ideal solution, pay-off matrix can be formulated as follows:

$$\begin{pmatrix} f_1^*(x^1) & f_2(x^1) & \dots & f_{2m+1}(x^1) \\ f_1(x^2) & f_2^*(x^2) & \dots & f_{2m+1}(x^2) \\ \dots\dots & \dots\dots\dots & \dots & \dots\dots \\ f_1(x^{2m+1}) & f_2(x^{2m+1}) & \dots & f_{2m+1}(x^{2m+1}) \end{pmatrix}$$

Step-3: For each objective $f_k(x), (k = 1,2,\dots,2m+1)$, we now find lower and upper bounds L_k^{μ} and U_k^{μ} respectively for truth-membership of objectives.
$L_k^{\mu} = \min f_k(x^{r*})$ and $U_k^{\mu} = \max f_k(x^{r*})$,where $r = 1,2,\dots,2m+1$; $k = 1,2,\dots,2m+1$.
The upper and lower bounds for indeterminacy and falsity membership of objectives can be calculated as follows:
$U_k^{v} = U_k^{\mu}$ and $L_k^{v} = L_k^{\mu} + t(U_k^{\mu} - L_k^{\mu})$.
$U_k^{\sigma} = L_k^{\mu} + s(U_k^{\mu} - L_k^{\mu})$ and $L_k^{\sigma} = L_k^{\mu}$.
Here t and s are predetermined real number in $(0,1)$.
Step-4: We define Truth-membership function, indeterminacy-membership function and falsity-membership function as follows:
$\mu_k(f_k(x)) = 1 \quad for\ f_k(x) \le L_k^{\mu}$
$\mu_k(f_k(x)) = (U_k^{\mu} - f_k(x))/(U_k^{\mu} - L_k^{\mu}),\ for\ L_k^{\mu} \le f_k(x) \le U_k^{\mu}$
$\mu_k(f_k(x)) = 0 \quad for\ f_k(x) \ge U_k^{\mu}$

$\sigma_k(f_k(x)) = 1 \quad for\ f_k(x) \le L_k^{\sigma}$
$\sigma_k(f_k(x)) = (U_k^{\sigma} - f_k(x))/(U_k^{\sigma} - L_k^{\sigma}),\ for\ L_k^{\sigma} \le f_k(x) \le U_k^{\sigma}$
$\sigma_k(f_k(x)) = 0 \quad for\ f_k(x) \ge U_k^{\sigma}$

$v_k(f_k(x)) = 0 \quad for\ f_k(x) \le L_k^{v}$
$v_k(f_k(x)) = (f_k(x) - L_k^{v})/(U_k^{v} - L_k^{v}),\ for\ L_k^{v} \le f_k(x) \le U_k^{v}$
$v_k(f_k(x)) = 1 \quad for\ f_k(x) \ge U_k^{v}$

Step-5: Now by using neutrosophic optimization method, we can write the problem (4.2)as:
$Max\ \alpha - \beta + \gamma$ (5.2)
Such that
$\mu_k(f_k(x)) \ge \alpha$
$\sigma_k(f_k(x)) \ge \gamma$
$v_k(f_k(x)) \le \beta$, for $k = 1,2,\dots,2m+1$
$\alpha + \beta + \gamma \le 3$
$\alpha \ge \beta$
$\alpha \ge \gamma$
$\alpha, \beta, \gamma \in [0,1]$
$g_i(x) \le b_i,\ i = 1,2,\dots\dots,q$,
$x \ge 0$
Again we reduce the problem (5.2) to equivalent non-linear programming problem as:

$Max\ \alpha - \beta + \gamma$ (5.3)
Such that $f_k(x) + (U_k^{\mu} - L_k^{\mu}).\alpha \le U_k^{\mu}$
$f_k(x) + (U_k^{\sigma} - L_k^{\sigma}).\gamma \le U_k^{\sigma}$
$f_k(x) - (U_k^{v} - L_k^{v}).\beta \le L_k^{v}$,
for $k = 1,2,\dots,2m+1$
$\alpha + \beta + \gamma \le 3$
$\alpha \ge \beta$
$\alpha \ge \gamma$
$\alpha, \beta, \gamma \in [0,1]$
$g_i(x) \le b_i,\ i = 1,2,\dots\dots,q$,
$x \ge 0$

So the problem (5.1) is reduced to equivalent non-linear programming problem as:

Model-A:

$$Max\ \alpha - \beta + \gamma \tag{5.4}$$

$$-EN(X) + (U_{-EN}{}^{\mu} - L_{-EN}{}^{\mu}).\alpha \le U_{-EN}{}^{\mu}$$
$$-EN(X) + (U_{-EN}{}^{\sigma} - L_{-EN}{}^{\sigma}).\gamma \le U_{-EN}{}^{\sigma}$$
$$-EN(X) - (U_{-EN}{}^{v} - L_{-EN}{}^{v}).\beta \le L_{-EN}{}^{v}$$
$$-Er_i(x) + \left(U_{-Er_i}{}^{\mu} - L_{-Er_i}{}^{\mu}\right).\alpha \le U_{-Er_i}{}^{\mu}$$
$$-Er_i(x) + \left(U_{-Er_i}{}^{\sigma} - L_{-Er_i}{}^{\sigma}\right).\gamma \le U_{-Er_i}{}^{\sigma}$$
$$-Er_i(x) - \left(U_{-Er_i}{}^{v} - L_{-Er_i}{}^{v}\right).\beta \le L_{-Er_i}{}^{v}$$
$$Vr_i(x) + \left(U_{Vr_i}{}^{\mu} - L_{Vr_i}{}^{\mu}\right).\alpha \le U_{Vr_i}{}^{\mu}$$
$$Vr_i(x) + \left(U_{Vr_i}{}^{\sigma} - L_{Vr_i}{}^{\sigma}\right).\gamma \le U_{Vr_i}{}^{\sigma}$$
$$Vr_i(x) - \left(U_{Vr_i}{}^{v} - L_{Vr_i}{}^{v}\right).\beta \le L_{Vr_i}{}^{v},\ j = 1,2,\ldots\ldots,m.$$
$$\alpha + \beta + \gamma \le 3$$
$$\alpha \ge \beta$$
$$\alpha \ge \gamma$$
$$\alpha,\beta,\gamma \in [0,1]$$
$$\sum_{j}^{n} x_j = 1$$
$$x_j \ge 0,\ j = 1,2,\ldots,n.$$

If we take

$Max\alpha$

$Min\gamma$

$Min\beta$

in problem (4.2), then it reduced to equivalent non-linear programming problem as:

Model-B:

$$Max\ \alpha - \beta - \gamma \tag{5.5}$$

And same constraints as problem (5.4).

Now, positive weights W_k reflect the decision maker's preferences regarding the relative importance of each objective goal $f_k(x)$ for $k = 1,2,\ldots,2m+1$.

These weights can be normalized by taking $\sum W_k = 1$. If we take weights w for $-EN(X)$, w_{ei} for $-Er_i(x)$ and w_{vi} for $Vr_i(x)$, where , $j = 1,2,\ldots\ldots,m$ and $w + \sum_{i=1}^{m} w_{ei} + \sum_{i=1}^{m} w_{vi} = 1$.

Then the problem (5.4) becomes:

$$Max\ \alpha - \beta + \gamma \tag{5.6}$$

$$-wEN(X) + (U_{-EN}{}^{\mu} - L_{-EN}{}^{\mu}).\alpha \le wU_{-EN}{}^{\mu}$$
$$-wEN(X) + (U_{-EN}{}^{\sigma} - L_{-EN}{}^{\sigma}).\gamma \le wU_{-EN}{}^{\sigma}$$
$$-wEN(X) - (U_{-EN}{}^{v} - L_{-EN}{}^{v}).\beta \le wL_{-EN}{}^{v}$$
$$-w_{ei}Er_i(x) + \left(U_{-Er_i}{}^{\mu} - L_{-Er_i}{}^{\mu}\right).\alpha \le w_{ei}U_{-Er_i}{}^{\mu}$$
$$-w_{ei}Er_i(x) + \left(U_{-Er_i}{}^{\sigma} - L_{-Er_i}{}^{\sigma}\right).\gamma \le w_{ei}U_{-Er_i}{}^{\sigma}$$
$$-w_{ei}Er_i(x) - \left(U_{-Er_i}{}^{v} - L_{-Er_i}{}^{v}\right).\beta \le w_{ei}L_{-Er_i}{}^{v}$$
$$w_{vi}Vr_i(x) + \left(U_{Vr_i}{}^{\mu} - L_{Vr_i}{}^{\mu}\right).\alpha \le w_{vi}U_{Vr_i}{}^{\mu}$$
$$w_{vi}Vr_i(x) + \left(U_{Vr_i}{}^{\sigma} - L_{Vr_i}{}^{\sigma}\right).\gamma \le w_{vi}U_{Vr_i}{}^{\sigma}$$
$$w_{vi}Vr_i(x) - \left(U_{Vr_i}{}^{v} - L_{Vr_i}{}^{v}\right).\beta \le w_{vi}L_{Vr_i}{}^{v},\ j = 1,2,\ldots\ldots,m.$$
$$\alpha + \beta + \gamma \le 3$$
$$\alpha \ge \beta$$
$$\alpha \ge \gamma$$
$$\alpha,\beta,\gamma \in [0,1]$$
$$\sum_{j}^{n} x_j = 1$$
$$x_j \ge 0,\ j = 1,2,\ldots,n.$$
$$w + \sum_{i=1}^{m} w_{ei} + \sum_{i=1}^{m} w_{vi} = 1.$$

6. Numerical Examples

6.1 Numerical Examples (for PSP and PSPD):

Let us consider the three-security problems with expected returns vector and covariance matrix given by

$(r_1, r_2, r_3) = (0.073, 0.165, 0.133)$ and

$$\sigma_1{}^2 = 0.0152\ \rho_{12}\sigma_1\sigma_2 = 0.0211,\ \rho_{13}\sigma_1\sigma_3 = 0.0197$$
$$\sigma_2{}^2 = 0.0678\ ,\ \sigma_3{}^2 = 0.0294,\ \rho_{23}\sigma_2\sigma_3 = 0.0256$$

Let x $=(x_1, x_2, x_3)^T$, where x_1, x_2, x_3 is the proportion of an asset invested in the 1-st, 2-nd and 3-rd security respectively .

So model-1 (PSP) is

$$\text{Maximize } Er(x) = 0.073\, x_1 + 0.165\, x_2 + 0.133\, x_3 \tag{6.1}$$

$$\text{Minimize } Vr(x) = 0.0152\, x_1^2 + 0.0678\, x_2^2 + 0.0294\, x_3^2 + 2(0.0211 x_1 x_2 + 0.0197\, x_1 x_3 + 0.0256\, x_2 x_3)$$

Subject to

$$x_1 + x_2 + x_3 = 1,$$

and $x_1, x_2, x_3 \geq 0$.

And Model-II (PSPD) is

$$\text{Maximize } En(x) = -(x_1 \ln x_1 + x_2 \ln x_2 + x_3 \ln x_3) \tag{6.2}$$

$$\text{Maximize } Er(x) = 0.073\, x_1 + 0.165\, x_2 + 0.133\, x_3$$

$$\text{Minimize } Vr(x) == 0.0152\, x_1^2 + 0.0678\, x_2^2 + 0.0294\, x_3^2 + 2(0.0211 x_1 x_2 + 0.0197\, x_1 x_3 + 0.0256\, x_2 x_3)$$

subject to

$$x_1 + x_2 + x_3 = 1,$$

and $x_1, x_2, x_3 \geq 0$.

Converting problem (6.2) into minimization problem, we have

$$\text{Minimize } -En(x) = (x_1 \ln x_1 + x_2 \ln x_2 + x_3 \ln x_3) \tag{6.3}$$

$$\text{Minimize } -Er(x) = -(0.073\, x_1 + 0.165\, x_2 + 0.133\, x_3)$$

$$\text{Minimize } Vr(x) = 0.0152\, x_1^2 + 0.0678\, x_2^2 + 0.0294\, x_3^2 + 2(0.0211 x_1 x_2 + 0.0197\, x_1 x_3 + 0.0256\, x_2 x_3)$$

subject to

$$x_1 + x_2 + x_3 = 1,$$

and $x_1, x_2, x_3 \geq 0$.

Here

$L_{-EN}{}^{\mu} = -1.0986,$

$L_{-EN}{}^{\nu} = -1.0986 + (-1.0986)t,$

$L_{-EN}{}^{\sigma} = -1.0986$

$U_{-EN}{}^{\mu} = 0,\ U_{-EN}{}^{\nu} = 0,\ U_{-EN}{}^{\sigma} == -1.0986 + (-1.0986)s$

$L_{-Er}{}^{\mu} = -0.165,$

$L_{-Er}{}^{\nu} = -0.165 + 0.092.t,$

$L_{-Er}{}^{\sigma} = -0.165$

$U_{-Er}{}^{\mu} = -0.073,$

$U_{-Er}{}^{\sigma} = -0.165 + 0.092.s,$

$U_{-Er}{}^{\nu} = -0.073$

$L_{Vr}{}^{\mu} = 0.0152,$

$L_{Vr}{}^{\nu} = 0.0152 + 0.0526.t,$

$L_{Vr}{}^{\sigma} = 0.0152$

$U_{Vr}{}^{\mu} = 0.0678,$

$U_{Vr}{}^{\sigma} = 0.0152 + 0.0526.s,$

$U_{Vr}{}^{\nu} = 0.0678$

We take $t = 0.4, s = 0.6$ in all the examples which are considered in this paper.

So optimal solutions for model-1 (PSP) and Model-II (PSPD) are given below (Table-1):

Model	x_1	x_2	x_3	Er(x*)	Vr(x*)	En(x*)
Model- I (PSP)	0	0.20398	0.79602	0.13953	0.02976	—
Model- II (PSPD)	0.05328	0.2850	0.66172	0.13892	0.03011	0.78721

Table-1: Optimal solutions of Model-I and Model-II.

We see that model-I has one variable x_1 with zero value whereas there is no non-zero value of x_1, x_2, x_3 in Model-II. Here entropy is acted as a measure of dispersal of assets investment with small changes of Er(x), Vr(x). If an investor wishes to distribute his asset in various bonds, the PSPD (Model-II) will be more realistic for him.

Comparison of Model-A & Model-B are given below (Table-2):

Model	α	β	γ	x_1^*	x_2^*	x_3^*	Er(x*)	Vr(x*)	En(x*)
Model-A	0.717	0	0.528	0.053	0.285	0.662	0.139	0.03	0.787
Model-B	0.717	0	0	0.053	0.285	0.662	0.139	0.03	0.787

Table-2: Optimal solutions of Model-A and Model-B.

In Model-A (where we maximize γ), we see that there is an indeterminacy but in Model-B (where we minimize γ), there is no indeterminacy condition. So we can conclude that Model-B is no longer neutrosophic set, it becomes intuitionistic set. The result is only for this particular model which we considered in this paper. We verify this by taking different problems of Portfolio model and we get same results except the value of γ in each problem. In Model-A, we have positive value of γ and in Model-B we get γ as 0.

For using different weights, optimal solution of Model-II is given below (Table-3):

Weights	x_1	x_2	x_3	Er(x*)	Vr(x*)	En(x*)	Type
$w_e = 1/3$ $w_v = 1/3$ $w = 1/3$	0.05328	0.28499	0.66173	0.13892	0.03011	0.78721	I
$w_e = 0.26$ $w_v = 0.17$ $w = 0.57$	0.109	0.16616	0.72484	0.13178	0.02754	0.77307	II
$w_e = 0.12$ $w_v = 0.48$ $w = 0.4$	0	0.47698	0.52302	0.14826	0.03624	0.69209	III

Table-3: Optimal solutions of Model-II .

Here, results have been presented for model-II with the different weights to the objectives. Types-I, II and III , give respectively, the results with equal importance of the objectives, more importance of the expected return and more importance of the risk.

6.2 Numerical example of GPSPD :

Consider the three-security problems with expected returns vector and covariance matrix given by

$(r_1^{1}, r_2^{1}, r_3^{1})$ = (0.073, 0.165, 0.133) and

$(\sigma_1^{\,1})^2 = 0.0152$, $\rho_{12}{}^{1}\sigma_1{}^{1}\sigma_2{}^{1} = 0.0211$, $\rho_{13}{}^{1}\sigma_1{}^{1}\sigma_3{}^{1} = 0.0197$
$(\sigma_2^{\,1})^2 = 0.0678$ $(\sigma_3^{\,1})^2 = 0.0294$, $\rho_{23}{}^{1}\sigma_2{}^{1}\sigma_3{}^{1} = 0.0256$

$(r_1^{2}, r_2^{2}, r_3^{2})$ = (0.104, 0.187, 0.077) and
$(\sigma_1^{\,2})^2 = 0.0685$, $\rho_{12}{}^{2}\sigma_1{}^{2}\sigma_2{}^{2} = 0.0171$,
$(\sigma_2^{\,2})^2 = 0.0327$, $(\sigma_3^{\,2})^2 = 0.0843$, $\rho_{23}{}^{2}\sigma_2{}^{2}\sigma_3{}^{2} = 0.0121$

$(r_1^{3}, r_2^{3}, r_3^{3})$ = (0.082, 0.106, 0.128) and
$(\sigma_1^{\,3})^2 = 0.0273$, $\rho_{12}{}^{3}\sigma_1{}^{3}\sigma_2{}^{3} = 0.0133$, $\rho_{13}{}^{3}\sigma_1{}^{3}\sigma_3{}^{3} = 0.0152$
$(\sigma_2^{\,3})^2 = 0.0726$, $(\sigma_3^{\,3})^2 = 0.0147$, $\rho_{23}{}^{3}\sigma_2{}^{3}\sigma_3{}^{3} = 0.0116$

So the optimal solutions of GPSPD is

x_1= 0.08699 , x_2 = 0.54754, x_3 = 0.36548,
$Er_1(x)$ = 0.1453, $Er_2(x)$= 0.139578,
$Er_3(x)$ = 0.111953
$Vr_1(x)$ = 0.0378765, $Vr_2(x)$= 0.028769,
$Vr_3(x)$ = 0.0395095, $En(x)$ =0.910089.

For using different weights, optimal solution of GPSPD are given below (Table-4):

Weights	$Er_1(x^*)$	$Er_2(x^*)$	$Er_3(x^*)$	$Vr_1(x^*)$	$Vr_2(x^*)$	$Vr_3(x^*)$	$En(x^*)$	Type
$w_{e1}= w_{e2} = w_{e3} = w_{v1} = w_{v2} = w_{v3} = w=1/7$	0.1453	0.13958	0.11195	0.03788	0.02877	0.03951	0.91009	I
$w_{e1}= w_{e2} = w_{e3} =$ 0.04, $w_{v1} = w_{v2} = w_{v3}$ =0.14, w =0.46	0.14475	0.13728	0.11248	0.03702	0.02948	0.03856	0.91493	II
$w_{e1}=w_{e2}=w_{e3}$ =0.12, $w_{v1}=w_{v3}$=0.1, w_{v2} =0.03 w=0.41	0.14538	0.13992	0.11187	0.03801	0.02867	0.03965	0.9092	III
$w_{e1}= w_{e2}$= 0.15 w_{e3} = 0.06, $w_{v1} = w_{v2} = w_{v3}$ =0.09, w =0.37	0.14552	0.1405	0.11174	0.03823	0.02851	0.0399	0.90759	IV

Table-4: Optimal solutions of GPSPD.

Here, results have been presented with the different weights to the objectives. Types-I, II, III and IV give, respectively, the results with equal importance of the objectives, more importance of the expected returns, more importance of the anyone expected return say $Er_3(x^*)$ and more importance of the any one risk say $Vr_3(x^*)$.
We also consider the condition if we do not consider falsity and indeterminacy membership functions in objective function. We see that the result remains same except the value of α (truth membership function).

7. Conclusion:

In this paper, we consider a general application of portfolio selection problem in fuzzy environment. We first consider a multi-objective Portfolio Selection model and then we added another entropy objective function and next we generalized the model. Neutrosophic optimization technique is used to solve the problems. We also take different weights on objective functions. The models are illustrated with numerical examples. The method presented in the paper is quite general and can be applied to other areas of Operation Research and Engineering Sciences.

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