Extension of Crisp Functions on Neutrosophic Sets

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Abstract. In this paper, we generalize the definition of Neutrosophic sets and present a method for extending crisp functions on Neutrosophic sets and study some properties of such extended functions.

1 Introduction


2 Preliminaries

Definition 2.1. [26] Let X be a nonempty set. A fuzzy set A of X is a mapping A : X → [0, 1], that is,

A = \{ (x, \mu_A(x)) : \mu_A(x) is the grade of membership of x in A, x ∈ X \}.

The set of all the fuzzy sets on X is denoted by \( F(X) \).

Definition 2.2. [8] Let X be a nonempty ordinary set, L a complete lattice. An L-fuzzy set on X is a mapping A : X → L, that is the family of all the L-fuzzy sets on X is just \( L^X \) consisting of all the mappings from X to L.

Definition 2.3. [1] An Intuitionistic Fuzzy Set on X is a set

A = \{ (x, \mu_A(x), \nu_A(x)) : x ∈ X \},

where \( \mu_A(x) \in [0, 1] \) denotes the membership degree and \( \nu_A(x) \in [0, 1] \) denotes the non-membership degree of x in A and

\[ \mu_A(x) + \nu_A(x) \leq 1, \forall x \in X. \]

The neutrosophic set (NS) was introduced by F. Smarandache [22] who introduced the degree of indeterminacy (i) as independent component in his manuscripts that was published in 1998.

Multi-fuzzy sets [12, 13, 16] was proposed in 2009 by Sabu Sebastian as an extension of fuzzy sets [8, 26] in terms of multi membership functions. In this paper we generalize the definition of neutrosophic sets and introduce extension of crisp functions on neutrosophic sets.

Definition 2.4. [22] A Neutrosophic Set on X is a set

\[ A = \{ (x, T_A(x), I_A(x), F_A(x)) : x \in X \}, \]

where \( T_A(x) \in [0, 1] \) denotes the truth membership degree, \( I_A(x) \in [0, 1] \) denotes the indeterminacy membership degree and \( F_A(x) \in [0, 1] \) denotes the falsity membership degree of x in A respectively and

\[ 0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3, \forall x \in X. \]

For single valued neutrosophic logic \( (T, I, F) \), the sum of the components is: 0 ≤ T + I + F ≤ 3 when all three components are independent; 0 ≤ T + I + F ≤ 2 when two components are dependent, while the third one is independent from them; 0 ≤ T + I + F ≤ 1 when all three components are dependent.

Definition 2.5. [12, 13, 16] Let X be a nonempty set, J be an indexing set and \{ L_j : j ∈ J \} a family of partially ordered sets. A multi-fuzzy set A in X is a set :

\[ A = \{ (x, (\mu_j(x)))_{j \in J} : x \in X, \mu_j \in L_j, j \in J \}. \]
The indexing set \( J \) may be uncountable. The function \( \mu_A = (\mu_j)_{j \in J} \) is called the membership function of the multi-fuzzy set \( A \) and \( \prod_{j \in J} L_j \) is called the value domain.

If \( J = \{1, 2, ..., n\} \) or the set of all natural numbers, then the membership function \( \mu_A = (\mu_1, \mu_2, ...) \) is a sequence.

In particular, if the sequence of the membership function having precisely \( n \)-terms and \( L_j = [0, 1] \), for \( J = \{1, 2, ..., n\} \), then \( n \) is called the dimension and \( M^\text{FS}(X) \) denotes the set of all multi-fuzzy sets in \( X \).

Properties of multi-fuzzy sets, relations on multi-fuzzy sets and multi-fuzzy extensions of crisp functions are depend on the order relations defined in the membership functions. Most of the results in the initial papers [12, 13, 15, 16, 18] are based on product order in the membership functions. The paper [21] discussed other order relations like dictionary order, reverse dictionary order on their membership functions.

Let \( \{L_j : j \in J\} \) be a family of partially ordered sets, and \( A = \{(x, (\mu_j(x)))_{j \in J} : x \in X, \mu_j \in L_j^X, j \in J\} \) and \( B = \{(x, (\nu_j(x)))_{j \in J} : x \in X, \nu_j \in L_j^X, j \in J\} \) be multi-fuzzy sets in a nonempty set \( X \). Note that, if the order relation in their membership functions are either product order, dictionary order or reverse dictionary order[16, 21], then:

- \( A = B \) if and only if \( \mu_j(x) = \nu_j(x), \forall x \in X \) and for all \( j \in J \).
- \( A \sqcup B = \{(x, (\mu_j(x) \lor \nu_j(x)))_{j \in J} : x \in X\} \) and \( A \sqcap B = \{(x, (\mu_j(x) \land \nu_j(x)))_{j \in J} : x \in X\} \), where \( \lor \) and \( \land \) are the supremum and infimum defined in \( L_j \) with partial order relation \( \leq_j \). Set inclusion defined as follows:
  - In product order, \( A \subseteq B \) if and only if \( \mu_j(x) < \nu_j(x), \forall x \in X \) and for all \( j \in J \).
  - In dictionary order, \( A \subseteq B \) if and only if \( \mu_1(x) < \nu_1(x) \) or if \( \mu_1(x) = \nu_1(x) \) and \( \mu_2(x) < \nu_2(x), \forall x \in X \).

**Definition 2.6.** Let \( L \) be a lattice. A mapping \( \prime : L \to L \) is called an order reversing involution [25], if for all \( a, b \in L \):

1. \( a \leq b \Rightarrow b' \leq a' \);
2. \( (a')' = a \).

**Definition 2.7.** [23] Let \( \prime : M \to M \) and \( \prime : L \to L \) be order reversing involutions. A mapping \( h : M \to L \) is called an order homomorphism, if it satisfies the conditions:

1. \( h(0_M) = 0_L \);
2. \( h(\vee a_i) = \vee (h(a_i)) \);
3. \( h^{-1}(b') = (h^{-1}(b))' \),

where \( h^{-1} : L \to M \) is defined by, for every \( b \in L \),
\[
h^{-1}(b) = \lor \{a \in M : h(a) \leq b\}.
\]

Generalized Zadeh extension of crisp functions [24] have prime importance in the study of fuzzy mappings. Sabu Sebastian [16, 13]generalized this concept as multi-fuzzy extension of crisp functions and it is useful to map a multi-fuzzy set into another multi-fuzzy set. In the case of a crisp function, there exists infinitely many multi-fuzzy extensions, even though the domain and range of multi-fuzzy extensions are same.

**Definition 2.8.** [16] Let \( f : X \to Y \) and \( h : \prod M_i \to \prod L_j \) be a functions. The multi-fuzzy extension of \( f \) and the inverse of the extension are \( f : \prod M_i^X \to \prod L_j^Y \) and \( f^{-1} : \prod L_j^Y \to \prod M_i^X \) defined by
\[
f(A)(y) = \bigvee_{y = f(x)} h(A(x)), \quad A \in \prod M_i^X, \quad y \in Y
\]
and
\[
f^{-1}(B)(x) = h^{-1}(B(f(x))), \quad B \in \prod L_j^Y, \quad x \in X;
\]
where \( h^{-1} \) is the upper adjoint [23] of \( h \). The function \( h : \prod M_i \to \prod L_j \) is called the bridge function of the multi-fuzzy extension of \( f \).
Remark 2.9. In particular, the multi-fuzzy extension of a crisp function \( f : X \rightarrow Y \) based on the bridge function \( h : F^k \rightarrow F^n \) can be written as \( f : \text{M}^k\text{FS}(X) \rightarrow \text{M}^n\text{FS}(Y) \) and \( f^{-1} : \text{M}^n\text{FS}(Y) \rightarrow \text{M}^k\text{FS}(X) \), where
\[
f(A)(y) = \sup_{y = f(x)} h(A(x)), \ A \in \text{M}^k\text{FS}(X), \ y \in Y
\]
and
\[
f^{-1}(B)(x) = h^{-1}(B(f(x))), \ B \in \text{M}^n\text{FS}(Y), \ x \in X.
\]
In the following section \( \prod M_i = \prod L_j = F^3 \).

Remark 2.10. There exists infinitely many bridge functions. Lattice homomorphism, order homomorphism, lattice valued fuzzy lattices and strong L-fuzzy lattices are examples of bridge functions.

Definition 2.11. [10] A function \( t : [0, 1] \times [0, 1] \rightarrow [0, 1] \) is a \( t \)-norm if \( \forall a, b, c \in [0, 1] ; (1) t(a, 1) = a ; (2) t(a, b) = t(b, a) ; (3) t(a, t(b, c)) = t(t(a, b), c) ; (4) b \leq c \) implies \( t(a, b) \leq t(a, c) \).

Similarly, a \( t \)-conorm (\( s \)-norm) is a commutative, associative and non-decreasing mapping \( s : [0, 1] \times [0, 1] \rightarrow [0, 1] \) that satisfies the boundary condition:
\[
s(a, 0) = a, \text{ for all } a \in [0, 1].
\]

Definition 2.12. [9] A function \( c : [0, 1] \rightarrow [0, 1] \) is called a complement (fuzzy) operation, if it satisfies the following conditions:
(1) \( c(0) = 1 \) and \( c(1) = 0 \),
(2) for all \( a, b \in [0, 1] \), if \( a \leq b \), then \( c(a) \geq c(b) \).

Definition 2.13. [9] A \( t \)-norm \( t \) and a \( t \)-conorm \( s \) are dual with respect to a fuzzy complement operation \( c \) if and only if
\[
c(t(a, b)) = s(c(a), c(b))
\]
and
\[
c(s(a, b)) = t(c(a), c(b)),
\]
for all \( a, b \in [0, 1] \).

Definition 2.14. [9] Let \( n \) be an integer greater than or equal to 2. A function \( m : [0, 1]^n \rightarrow [0, 1] \) is said to be an aggregation operation for fuzzy sets, if it satisfies the following conditions:
1. \( m \) is continuous;
2. \( m \) is monotonic increasing in all its arguments;
3. \( m(0, 0, ..., 0) = 0 \);
4. \( m(1, 1, ..., 1) = 1 \).

3 Neutrosophic Sets

In this section, we generalize the definition of neutrosophic sets on \([0, 1]\). Throughout the following sections \( X \) is the universe of discourse and \( A \in \text{M}^3\text{FS}(X) \) means \( A \) is a multi-fuzzy sets of dimension 3 with value domain \( \mathcal{F}^3 \), where \( \mathcal{F}^3 = [0, 1] \times [0, 1] \times [0, 1] \). That is, \( A \in (\mathcal{F}^3)^X \).

Definition 3.1. Let \( X \) be a nonempty crisp set and \( 0 \leq \alpha \leq 3 \). A multi-fuzzy set \( A \in \text{M}^3\text{FS}(X) \) is called a neutrosophic set of order \( \alpha \), if
\[
A = \{ (x, T_A(x), I_A(x), F_A(x)) : x \in X, 0 \leq T_A(x) + I_A(x) + F_A(x) \leq \alpha \}.
\]

Definition 3.2. Let \( A, B \) be neutrosophic sets in \( X \) of order 3 and let \( t, s, m, c \) be the \( t \)-norm, \( s \)-norm, aggregation operation and complement operation respectively. Then the union, intersection and complement are given by

1. \( A \cup B = \{ (x, s(T_A(x), T_B(x)), m(I_A(x), I_B(x)), t(F_A(x), F_B(x))) : x \in X \} \);
2. \( A \cap B = \{ (x, t(T_A(x), T_B(x)), m(I_A(x), I_B(x)), s(F_A(x), F_B(x))) : x \in X \} \);
3. \( A^c = \{ (x, c(T_A(x)), c(I_A(x)), c(F_A(x))) : x \in X \} \).
4 Extension of crisp functions on neutrosophic set using order homomorphism as bridge function

Theorem 4.1. If an order homomorphism \( h : F^3 \rightarrow F^3 \) is the bridge function for the multi-fuzzy extension of a crisp function \( f : X \rightarrow Y \), then for every \( k \in K \) neutrosophic sets \( A_k \) in \( X \) and \( B_k \) in \( Y \) of order 3;

1. \( A_1 \subseteq A_2 \) implies \( f(A_1) \subseteq f(A_2) \);
2. \( f(\cup A_k) = \cup f(A_k) \);
3. \( f(\cap A_k) \subseteq \cap f(A_k) \);
4. \( B_1 \subseteq B_2 \) implies \( f^{-1}(B_1) \subseteq f^{-1}(B_2) \);
5. \( f^{-1}(\cup B_k) = \cup f^{-1}(B_k) \);
6. \( f^{-1}(\cap B_k) = \cap f^{-1}(B_k) \);
7. \( (f^{-1}(B))' = f^{-1}(B')' \);
8. \( A \subseteq f^{-1}(f(A)) \);
9. \( f(f^{-1}(B)) \subseteq B \).

Proof.
1. \( A_1 \subseteq A_2 \) implies \( A_1(x) \leq A_2(x), \forall x \in X \) and implies
   \( h(A_1(x)) \leq h(A_2(x)), \forall x \in X \).
   Hence
   \[
   \forall \{h(A_1(x)) : x \in X, \ y = f(x)\} \leq \forall \{h(A_2(x)) : x \in X, \ y = f(x)\},
   \]
   \[
   \forall y \in Y. \text{That is, } f(A_1) \subseteq f(A_2).
   \]
2. For every \( y \in Y \),
   \[
   f(\cup A_k)(y) = \forall \{h(\cup A_k(x)) : x \in X, \ y = f(x)\} = \forall \{h(\vee A_k(x)) : x \in X, \ y = f(x)\} = \forall \{h(A_k(x)) : x \in X, \ y = f(x)\},
   \]
   \[
   \forall k \in K \cup \{h(A_k(x)) : x \in X, \ y = f(x)\} \text{,}
   \]
   thus \( f(\cup A_k) = \cup f(A_k) \).
3. For every \( y \in Y \),
   \[
   f(\cap A_k)(y) = \forall \{h(\cap A_k(x)) : x \in X, \ y = f(x)\} = \forall \{h(A_k(x)) : x \in X, \ y = f(x)\},
   \]
   \[
   \forall k \in K \cup \{h(A_k(x)) : x \in X, \ y = f(x)\},
   \]
   thus \( f(\cap A_k) = \cap f(A_k) \).
4. \( f(\cap A_k)(y) \leq \forall \{h(A_k(x)) : x \in X, \ y = f(x)\}, \forall k \in K \).
   Hence
   \[
   f(\cap A_k)(y) = \forall \{h(A_k(x)) : x \in X, \ y = f(x)\}, \forall k \in K, \forall y \in Y.
   \]
   \[
   f^{-1}(B_1(x)) \leq h^{-1}(B_1(f(x))) \leq h^{-1}(B_2(f(x))) = f^{-1}(B_2)(x), \forall x \in X.
   \]
   Therefore, \( f^{-1}(B_1) \subseteq f^{-1}(B_2) \).
5. For every \( x \in X \), we have
   \[
   f^{-1}(\cup B_k)(x) = h^{-1}(\cup B_k(f(x))) = h^{-1}(\sup_{k \in K} B_k(f(x))) = \sup_{k \in K} h^{-1}(B_k(f(x))) = f^{-1}(B_k)(x).
   \]
   \[
   f^{-1}(\cup B_k)(x) = (\cup f^{-1}(B_k))(x).
   \]
   Hence \( f^{-1}(\cup B_k) = \cup f^{-1}(B_k) \).
6. For every \( x \in X \),
   \[
   f^{-1}(\cap B_k)(x) = h^{-1}(\cap B_k(f(x))) = h^{-1}(\inf_{k \in K} B_k(f(x))) = \inf_{k \in K} h^{-1}(B_k(f(x))) = f^{-1}(B_k)(x).
   \]
   Hence \( f^{-1}(\cap B_k) = \cap f^{-1}(B_k) \).
7. For every \( x \in X \),
   \[
   f^{-1}(B')(x) = h^{-1}(B'(f(x))) = h^{-1}(B(f(x)))' = (f^{-1}(B))'(x), \text{since } f^{-1}(B)(x) = h^{-1}(B(f(x))).
   \]
   That is, \( f^{-1}(B') = (f^{-1}(B))' \).
8. For every \( x_0 \in X \),
   \[
   A(x_0) \leq \forall \{A(x) : x \in X, \ x \in f^{-1}(f(x_0))\} \leq h^{-1}(\forall \{A(x) : x \in X, \ x \in f^{-1}(f(x_0))\}) = h^{-1}(\forall \{A(x) : x \in X, \ x \in f^{-1}(f(x_0))\}) = h^{-1}(f(A)(f(x_0))) = f^{-1}(f(A)(x_0)).
   \]
9. For every \( y \in Y \),
   \[
   f(f^{-1}(B))(y) = \sup_{y=f(x)} h^{-1}(B(f(x))) = \sup_{y=f(x)} h^{-1}(B(f(x))))
   \]
Proposition 4.2. If an order homomorphism $h : f^3 \rightarrow f^3$ is the bridge function for the extension of a crisp function $f : X \rightarrow Y$, then for any $k \in K$ neutrosophic sets $A_k$ in $X$ and $B$ in $Y$:

1. $f(0_X) = 0_Y$;
2. $f(\cup A_k) = \cup f(A_k)$; and
3. $(f^{-1}(B))^\prime = f^{-1}(B^\prime)$,

that is, the extension map $f$ is an order homomorphism.

Acknowledgement

The authors are very grateful to referees for their constructive comments and suggestions.

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Received: August 4, 2017. Accepted: August 24, 2017.