Neutrosophic Sets and Systems, Vol. 29, 2019
University of New Mexico

Cyclic Associative Groupoids (CA-Groupoids) and Cyclic Associative Neutrosophic Extended Triplet Groupoids (CA-NET-Groupoids)

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Abstract: Group is the basic algebraic structure describing symmetry based on associative law. In order to express more general symmetry (or variation symmetry), the concept of group is generalized in various ways, for examples, regular semigroups, generalized groups, neutrosophic extended triplet groups and AG-groupoids. In this paper, based on the law of cyclic association and the background of non-associative ring, left weakly Novikov algebra and CA-AG-groupoid, a new concept of cyclic associative groupoid (CA-groupoid) is firstly proposed, and some examples and basic properties are presented. Moreover, as a combination of neutrosophic extended triplet group (NETG) and CA-groupoid, the notion of cyclic associative neutrosophic extended triplet groupoid (CA-NET-groupoid) is introduced, some important results are obtained, particularly, a decomposition theorem of CA-NET-groupoid is proved.

Keywords: Cyclic associative groupoid (CA-groupoid); CA-AG-groupoid; neutrosophic extended triplet group (NETG); CA-NET-groupoid; Decomposition theorem

1. Introduction

For algebraic operations, the associative law is very important, and it also characterizes the symmetry of operation: since from $(ab)c = a(bc)$, turn it upside down, we have $(cb)a = c(ba)$. This is also associative, that is, symmetry. Based on associative law, the concept of group is studied as basic algebraic structure describing symmetry. In order to express more general symmetry (or variation symmetry), group is generalized in various ways, for examples, regular semigroups, generalized groups, neutrosophic extended triplet groups and AG-groupoids (see [1, 16, 17, 22-24, 32]).

In many fields (such as non-associative rings and non-associative algebras [5, 18, 20, 21]), image processing [14] and networks [7]), non-associativity has important research significance. This paper focuses on non-associative algebraic structures satisfying the following operation law:

$$x(yz) = z(xy). \qquad \text{(Cyclic associative law)}$$

As early as 1995, M. Kleinfeld studied the rings with $x(yz) = z(xy)$ in [13], this research comes from the study of Novikov rings. After then, A. Behn, I. Correa, I. R. Hentzel and D. Samanta further investigated this kind of ring and algebra in [2, 3, 19]. Moreover, Zhan and Tan [34] introduced the notion of left weakly Novikov algebra: a non-associative algebra is called left weakly Novikov if it satisfies

$$(xy)z = (zx)y. \qquad \text{(Left weakly Novikov law)}$$

Obviously, the equation above is antithetical parallelism of the cyclic associative law (turn it upside down, $y(xz) = z(yx)$, that is cyclic associative).

Not only that, cyclic associativity is also applied to the research of AG-groupoids: in 2016, M. Iqbal, I. Ahmad, M. Shah and M.I. Ali [11] proposed the notion of cyclic associative AG-groupoid (CA-AG-groupoid), some new results are obtained in [9, 10].

Since cyclic associative law is widely used in algebraic systems, so we focus on basic algebra structure endow with a binary operation satisfying cyclic associative law in this paper, call it cyclic associative groupoid (CA-groupoid). We will also study the relationships between CA-groupoids and other related algebraic structures (see [4, 8, 12, 15, 26-31]).

The rest of this paper is organized as follows: in Section 2, we give some basic concepts and properties on semigroup, AG-groupoid and neutrosophic extended triplet groupoid (NETG); in Section 3, we give the definition of CA-groupoid and some interesting examples; in Section 4, we discuss the basic properties of CA-groupoids and analyze the relationships among some related algebraic systems; specially, we prove that every CA-groupoid with a left (or right) identity element is a commutative semigroup; in Section 5, we propose the new notion of cyclic associative neutrosophic extended triplet groupoid (CA-NET-groupoid), investigate basic properties of CA-NET-groupoids, and prove the composition theorem of CA-NET-groupoids.

2. Preliminaries

In this paper, a groupoid means that an algebraic structure consisting of a non-empty set with a single binary operation acting on it.

Let $(S, \cdot)$ be a groupoid. Some concepts are defined as follows (traditionally, the dot operator is omitted without confusion):

(1) S is called left nuclear square if for any $a, b, c \in S$, $a^2(bc) = (a^2b)c$; middle nuclear square if $a(b^2c) = (ab^2)c$; right nuclear square if $a(bc^2) = (ab)c^2$. S is called nuclear square if it is left, middle, and right nuclear square.

(2) S is called a Bol*-groupoid if $(\forall a, b, c, d \in S)$ $a((bc)d) = ((ab)c)d$.

(3) S is called left alternative if for all $a, b \in S$, $(aa)b = a(ab)$; and is called right alternative if $b(aa) = (ba)a$. S is called alternative, if it is both left alternative and right alternative.

(4) S is called right commutative if for all $a, b, c \in S$, $a(bc) = a(cb)$; and is called left commutative if $(ab)c = (ba)c$. S is called bi-commutative groupoid, if it is right and left commutative.

(5) An element $a \in S$ is called idempotent if $a^2 = a$.

(7) S is called transitively commutative if $ab = ba$ and $bc = cb$ implies $ac = ca$ for all $a, b, c \in S$.

(8) S is called semigroup, if for any $a, b, c \in S$, $a(bc) = (ab)c$. A semigroup $(S, \cdot)$ is commutative, if for all $a, b \in S$, $ab = ba$. A semigroup $(S, \cdot)$ is called band, if for all $a \in S$, $a^2 = a$.

Definition 1. ([24]) Assume that $(S, \cdot)$ is a groupoid. S is called an Abel-Grassmann's groupoid (or simply AG-groupoid), if S satisfying the left invertive law:

$$\forall a, b, c \in S,\ (ab)c = (cb)a.$$

For any AG-groupoid $(S, \cdot)$, the medial law holds, that is,

$$(ab)(cd) = (ac)(bd),\ \forall a, b, c \in S.$$

Definition 2. ([10, 11]) Let $(S, \cdot)$ be an AG-groupoid. (1) S is called an AG*-groupoid, if $(ab)c = b(ac)$ for all $a, b, c \in S$. (2) S is called an AG**-groupoid, if $(\forall a, b, c \in S)$ $a(bc) = b(ac)$. (3) S is called an T^1-AG-groupoid, if $(\forall a, b, c, d \in S)$ $ab = cd \Rightarrow ba = dc$.

Definition 3. ([22, 23]) Suppose that N is a non-empty set and $\cdot$ is a binary operation on N. If for any $a \in N$, there exist $neut(a)$, $anti(a) \in N$ such that

$$neut(a) \cdot a = a \cdot neut(a) = a;$$
$$anti(a) \cdot a = a \cdot anti(a) = neut(a).$$

Then $(N, \cdot)$ is called a neutrosophic extended triplet set, $neut(a)$ is called a neutral of "a", $anti(a)$ is called an opposite of "a", and $(a, neut(a), anti(a))$ is called a neutrosophic extended triplet.

Definition 4. ([22, 23]) Assume that $(N, \cdot)$ is a neutrosophic extended triplet set. If

(1) $(N, \cdot)$ is well-defined, that is, $(\forall a, b \in N)$ $a \cdot b \in N$.

(2) $(N, \cdot)$ is associative, that is, $(\forall a, b, c \in N)$ $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

Then, $(N, \cdot)$ is called a neutrosophic extended triplet group (NETG).

Theorem 1. ([30, 32]) Suppose that $(N, \cdot)$ is a neutrosophic extended triplet group (NETG). Then $(\forall a \in N)$ $neut(a)$ is unique.

3. Cyclic Associative Groupoids (CA-Groupoids)

Definition 5. Assume that $(S, \cdot)$ is a groupoid. If

$$a\cdot(b\cdot c) = c\cdot(a\cdot b),\ \forall\ a, b, c \in S,$$

then $(S, \cdot)$ is called a cyclic associative groupoid (shortly, CA-groupoid). By convention, operator $\cdot$ can be omitted without confusion.

Example 1. Considering the regular pentagon as shown in Figure 1, the center is at the origin of the x-y plane and the bottom side is parallel to the x-axis, the vertices are labeled a, b, c, d, e.

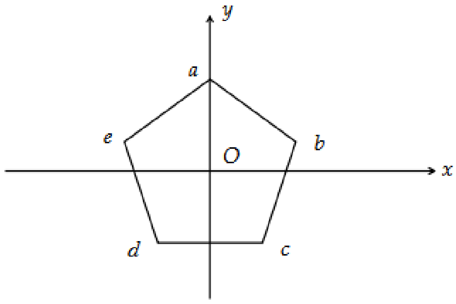

Figure 1. Regular pentagon

Denote $S=\{I, R, R^2, R^3, R^4\}$, representing some transformations of the regular pentagon, where I is 0 degrees clockwise around the center, R is 72 degrees clockwise around the center, R^2 is 144 degrees clockwise around the center, R^3 is 216 degrees clockwise around the center, R^4 is 288 degrees clockwise around the center. Define binary operation as a composition of functions in S, for arbitrary $U, V \in S$, $U \circ V$ is that the first transforming V and then transforming U. We can verity that $(S, \circ)$ is a CA-groupoid, the Cayley table can be presented as Table 1.

Table 1. The operation $\circ$ on $S = \{I, R, R^2, R^3, R^4\}$

$\circ$	$\boldsymbol{I}$	$\boldsymbol{R}$	$\boldsymbol{R^2}$	$\boldsymbol{R^3}$	$\boldsymbol{R^4}$
$\boldsymbol{I}$	I	R	R^2	R^3	R^4
$\boldsymbol{R}$	R	R^2	R^3	R^4	I
$\boldsymbol{R^2}$	R^2	R^3	R^4	I	R
$\boldsymbol{R^3}$	R^3	R^4	I	R	R^2
$\boldsymbol{R^4}$	R^4	I	R	R^2	R^3

Example 2. Suppose that Z is the set of all integer and $n \in Z$. Denote $W_n = \{a^2+nb^2 \mid a, b \in Z\}$, then $(W_n, \cdot)$ is a CA-groupoid, where $\cdot$ is the normal multiplication. In fact, for arbitrary element $w_1 = a_1^2+nb_1^2$, $w_2= a_2^2+nb_2^2$, $w_3 = a_3^2+nb_3^2 \in W_n$, we have

$$w_1\cdot(w_2\cdot w_3) = (a_1a_2a_3 - na_1b_2b_3 - nb_1a_2b_3 - nb_1b_2b_3)^2+n(a_1a_2b_3 + a_1b_2a_3 + b_1a_2a_3 - nb_1b_2b_3)^2 = w_3\cdot(w_1\cdot w_2).$$

Moreover, the result above can be extended and applied to solving binary indefinite equation, please see [6, 25]. We can obtain the following results (the proof is omitted).

Proposition 1. (1) Every commutative semigroup is a CA-groupoid. (2) Assume that $(S, \cdot)$ is a CA-groupoid. If S is commutative, then S is a commutative semigroup.

The following example shows that there exists CA-groupoid which is not a semigroup and not an AG-groupoid.

Example 3. Suppose $S = \{1, 2, 3, 4\}$, define a binary operation $\cdot$ on S in Table 2. Then, $(S, \cdot)$ is a CA-groupoid.

Table 2. The operation · on S

·	**1**	**2**	**3**	**4**
1	1	1	1	1
2	1	1	2	1
3	1	1	4	2
4	1	1	2	1

Moreover, S is not a AG-groupoid because $(4\cdot3)\cdot3 \neq (3\cdot3)\cdot4$. S isn't a semigroup because $(3\cdot4)\cdot 3 \neq 3\cdot(4\cdot3)$.

From the following example, we know that there exists CA-groupoid which is a semigroup and but it is not commutative.

Example 4. Assume $S = \{1, 2, 3, 4\}$, define a binary operation · on S by Table 3. Then, $(S, \cdot)$ is a CA-groupoid, and $(S, \cdot)$ is a semigroup, but · is not commutation because $2\cdot4 \neq 4\cdot2$.

Table 3. The operation · on S

·	**1**	**2**	**3**	**4**
1	1	1	1	1
2	1	3	1	3
3	1	1	1	1
4	1	1	1	3

Example 5. ([2]) Let A be an algebra (i.e. A be a linear space over a field F) with basis x_1, x_2, x_3, x_4, x_5, and the following nonzero products of basis elements

$$x_2\cdot x_1 = x_3,\ x_4\cdot x_2 = x_3,\ x_5\cdot x_1 = -x_3,\ x_3\cdot x_1 = x_4,\ x_3\cdot x_2 = x_5. \qquad \text{(NZP)}$$

For any $a, b \in A$, denote $a=\sum_{i=1}^{5} a_i x_i$, $b=\sum_{j=1}^{5} b_j x_j$, where $a_i, b_j \in F$ (i, j=1, 2, 3, 4, 5), then

$$a\cdot b = (\sum_{i=1}^{5} a_i x_i)\cdot(\sum_{j=1}^{5} b_j x_j) = a_2b_1x_3 + a_3b_1x_4 + a_3b_2x_5 + a_4b_2x_3 - a_5b_1x_3$$
$$= (a_2b_1 + a_4b_2 - a_5b_1)x_3 + a_3b_1x_4 + a_3b_2x_5$$

This means that $A^2=<x_3, x_4, x_5>$. Moreover, $AA^2=0$, since for any $c\in A$, $c=\sum_{k=1}^{5} c_k x_k$, where $c_k\in F$ (k=1, 2, 3, 4, 5),

$$c\cdot(a\cdot b) = (\sum_{k=1}^{5} c_k x_k)\cdot[(a_2b_1 + a_4b_2 - a_5b_1)x_3 + a_3b_1x_4 + a_3b_2x_5]$$

Note that, all nonzero products of basis elements are presented in (NZP), therefore, other products of basis elements are zero, that is, $x_1\cdot x_3 = x_1\cdot x_4 = x_1\cdot x_5 = \dots = 0$. Hence, $(A, \cdot)$ is a CA-groupoid, since it satisfies the stronger identity $a\cdot(b\cdot c) = 0 = c\cdot(a\cdot b)$, $\forall a, b, c\in A$.

Example 6. ([2]) Let $N = \{x_1, x_2, x_3, \dots\}$ a countably infinite set of indeterminates, for any element $x_i\in N$, call it is a letter. Denote P that is the set of the words in the letters x_i such that each letter occurs at most once in each word. For any word $u\in P$, if it is formed by k letters x_i, then say that u has length k, denote by length(u) = k. Obviously, length(u)≥1 for any $u\in P$. Suppose K is a field and A is the set of finite formal sums of words of P and with coefficient in K. For any $u, v\in P$, define multiplication · by:

(1) $u\cdot v = 0$, if length(v) >1, $u=v$ or v is a letter that is in the composition of u;

(2) $u\cdot v = uv$, if v is a letter that is not in the composition of u, where uv is the word obtained adding the letter v at the end of the word u.

For any $a, b\in A$, denote $a=\sum_{i=1}^{m} a_i p_i$, $b=\sum_{j=1}^{n} b_j q_j$, where $a_i, b_j\in K$, $p_i, q_j\in P$ (i=1, 2, …, m, j=1, 2, …, n), then

$$a\cdot b = (\sum_{i=1}^{m} a_i p_i)\cdot(\sum_{j=1}^{n} b_j q_j) = \sum_{t\le nm} d_t u_t$$

Where, $d_t \in K$, $u_t \in P$. By the definition of the multiplication in A, $u_t = 0$ or length(u_t)>1. Therefore, $AA^2=0$, since for any $c\in A$, $c=\sum_{s=1}^{l} c_s v_s$, where $c_s\in K$, $v_s\in P$ (s=1, 2, …, l),

$$c\cdot(a\cdot b)=(\sum_{s=1}^{l} c_s v_s)\cdot(\sum_{t\le mn} d_t u_t)=0$$

Hence, $(A, \cdot)$ is a CA-groupoid, since it satisfies the stronger identity $a\cdot(b\cdot c) = 0 = c\cdot(a\cdot b)$, $\forall a, b, c\in A$.

Example 7. Let $S = [1, 2]$ (real number interval). For any $a, b\in S$, define the multiplication $\cdot$ by

$$a\cdot b=\begin{cases} a+b-1, & \text{if} \quad a+b\le 3 \\ a+b-2, & \text{if} \quad a+b>3 \end{cases}$$

Then $(S, \cdot)$ is a CA-groupoid, since it satisfies $a\cdot(b\cdot c) = c\cdot(a\cdot b)$, $\forall a, b, c\in S$, the proof is as follows:

Case 1: $a+b+c-1\le 3$. It follows that $b+c \le a+b+c-1\le 3$ and $a+b \le a+b+c-1\le 3$. Then $a\cdot(b\cdot c) = a\cdot(b+c-1) = a+b+c-2 = c\cdot(a+b-1) = c\cdot(a\cdot b)$.

Case 2: $a+b+c-1>3$, $b+c\le 3$ and $a+b\le 3$. Then $a\cdot(b\cdot c) = a\cdot(b+c-1) = a+b+c-3 = c\cdot(a+b-1) = c\cdot(a\cdot b)$.

Case 3: $a+b+c-1>3$, $b+c\le 3$ and $a+b>3$. It follows that $a+b+c-2 \le a+3-2 = a+1 \le 3$. Then $a\cdot(b\cdot c) = a\cdot(b+c-1) = a+b+c-3 = c\cdot(a+b-2) = c\cdot(a\cdot b)$.

Case 4: $a+b+c-1>3$, $b+c>3$ and $a+b\le 3$. It follows that $a+b+c-2 \le 3+c-2 = c+1 \le 3$. Then $a\cdot(b\cdot c) = a\cdot(b+c-2) = a+b+c-3 = c\cdot(a+b-1) = c\cdot(a\cdot b)$.

Case 5: $a+b+c-1>3$, $b+c>3$ and $a+b>3$. When $a+b+c-2\le 3$, $a\cdot(b\cdot c) = a\cdot(b+c-2) = a+b+c-3 = c\cdot(a+b-2) = c\cdot(a\cdot b)$; When $a+b+c-2>3$, $a\cdot(b\cdot c) = a\cdot(b+c-2) = a+b+c-4 = c\cdot(a+b-2) = c\cdot(a\cdot b)$.

4. Some Properties of CA-Groupoids

Proposition 2. If $(S, \cdot)$ is a CA-groupoid, then, for any,

(1) $\forall a, b, c, d\in S$, $(ab)(cd) = (da)(cb)$;

(2) $\forall a, b, c, d, x, y\in S$, $(ab)((cd)(xy)) = (da)((cb)(xy))$.

Proof. Assume that $a, b, c, d, x, y\in S$, by Definition 5 we have

$$(ab)(cd) = d((ab)c) = c(d(ab)) = c(b(da)) = (da)(cb).$$

$$(ab)((cd)(xy)) = (xy)((ab)(cd)) = (xy)((da)(cb)) = (da)((cb)(xy)).\ \square$$

Theorem 2. Let $(S, \cdot)$ be a CA-groupoid.

(1) If S have a left identity element, that is, there exists $e\in S$ such that $e\cdot a=a$ for all $a\in S$, then S is a commutative semigroup.

(2) If $e\in S$ is a left identity element in S, then $e\in S$ is an identity element in S.

(3) If $e\in S$ is a right identity element in S, that is, $a\cdot e=a$ for all $a\in S$, then $e\in S$ is an identity element in S.

(4) If S have a right identity element, then S is a commutative semigroup.

Proof. (1) Suppose $a, b\in S$, $a\cdot b = a\cdot(e\cdot b) = b\cdot(a\cdot e) = e\cdot(b\cdot a) = b\cdot a$. It follows that $(S, \cdot)$ is a commutative CA-groupoid. By Proposition 1 (2) we know that $(S, \cdot)$ is a commutative semigroup.

(2) Assume that $e\in S$ is a left identity element in S, then for any $a\in S$, $a\cdot e = a\cdot(e\cdot e) = e\cdot(a\cdot e) = e\cdot(e\cdot a) = e\cdot a = a$. This means that $e\in S$ is an identity element in S.

(3) Assume that $e\in S$ is a right identity element in S, then for any $a\in S$, $e\cdot a = e\cdot(a\cdot e) = e\cdot(e\cdot a) = a\cdot(e\cdot e) = a\cdot e = a$. This means that $e\in S$ is an identity element in S.

(4) It follows from (1) and (3). $\square$

Theorem 3. Let $(S, \cdot)$ be a semigroup.

(1) When S is right commutative CA-groupoid, S is an AG-groupoid.
(2) When S is right commutative CA-groupoid, S is left commutative CA-groupoid.
(3) When S is left commutative CA-groupoid, S is right commutative CA-groupoid
(4) When S is left commutative CA-groupoid, S is an AG-groupoid.
(5) When S is left commutative AG-groupoid, S is an CA-groupoid.
(6) When S is left commutative AG-groupoid, S is right commutative AG-groupoid.
(7) When S is right commutative AG-groupoid, S is left commutative AG-groupoid.
(8) When $(S, *)$ is right commutative AG-groupoid, S is an CA-groupoid.

Proof. (1) If $(S, \cdot)$ is right commutative CA-groupoid, then, $\forall a, b, c \in S$, $(a \cdot b) \cdot c = a \cdot (b \cdot c) = c \cdot (a \cdot b) = c \cdot (b \cdot a)$. It follows that $(S, \cdot)$ is an AG-groupoid by Definition 1.

(2) If $(S, \cdot)$ is right commutative CA-groupoid, then, $\forall a, b, c \in S$, $(a \cdot b) \cdot c = a \cdot (b \cdot c) = a \cdot (c \cdot b) = b \cdot (a \cdot c) = (b \cdot a) \cdot c$. That is, $(S, \cdot)$ is left commutative CA-groupoid.

(3) Assume that $(S, \cdot)$ is left commutative CA-groupoid. Then, for any $a, b, c \in S$, $a \cdot (b \cdot c) = c \cdot (a \cdot b) = (c \cdot a) \cdot b = (a \cdot c) \cdot b = a \cdot (c \cdot b)$. This means that $(S, \cdot)$ is right commutative CA-groupoid.

(4) It follows from (1) and (3).

(5) Suppose that $(S, \cdot)$ is left commutative AG-groupoid. Then, for any $a, b, c \in S$,

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c = (b \cdot a) \cdot c = c \cdot (a \cdot b).$$

Using Definition 5, $(S, \cdot)$ is a CA-groupoid.

(6) If $(S, \cdot)$ is left commutative AG-groupoid, then, $\forall a, b, c \in S$, $a \cdot (b \cdot c) = (a \cdot b) \cdot c = (c \cdot b) \cdot a = (b \cdot c) \cdot a = (a \cdot c) \cdot b = a \cdot (c \cdot b)$. That is, $(S, \cdot)$ is right commutative AG-groupoid.

(7) If $(S, \cdot)$ is right commutative AG-groupoid, then, $\forall a, b, c \in S$,

$$(a \cdot b) \cdot c = a \cdot (b \cdot c) = a \cdot (c \cdot b) = (a \cdot c) \cdot b = (b \cdot c) \cdot a = b \cdot (c \cdot a) = b \cdot (a \cdot c) = (b \cdot a) \cdot c.$$

This means that $(S, \cdot)$ is left commutative AG-groupoid.

(8) It follows from (5) and (7). □

Example 8. Let $S = \{a, b, c, d\}$. Define the operate $\cdot$ on S in Table 4. Then, $(S, \cdot)$ is a CA-groupoid, but isn't a CA-AG-groupoid because $(b \cdot d) \cdot d \neq (d \cdot d) \cdot b$.

Table 4. The operation on S

$\cdot$	**a**	**b**	**c**	**d**
a	a	a	a	a
b	a	a	a	b
c	a	a	c	c
d	a	a	c	c

Example 9. Let $S = \{a, b, c, d, e\}$. Define the operate $\cdot$ on S in Table 5. Then, $(S, \cdot)$ is a CA-AG-groupoid, and $(S, \cdot)$ is not a semigroup, because $(a \cdot a) \cdot a \neq a \cdot (a \cdot a)$.

Table 5. The operation on S

$\cdot$	**a**	**b**	**c**	**d**	**e**
a	b	c	c	c	e
b	d	c	c	c	e
c	c	c	c	c	e
d	c	c	c	c	e
e	e	e	e	e	c

From Proposition 1, Theorem 3, Example 4, Example 8 and Example 9, we know the relationships among some algebraic systems, we can present as Figure 2.

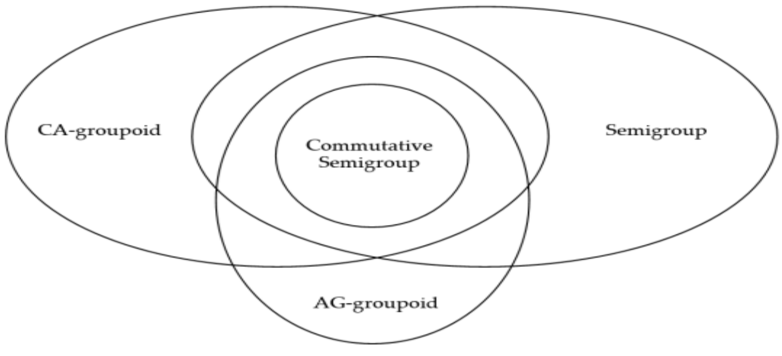

Figure 2. The relationships among some algebraic systems

Theorem 4. Let $(S, \cdot)$ be a CA-groupoid. If for all $a\in S$, $a^2=a$, then S is commutative.

Proof. Suppose that $(S, \cdot)$ is a CA-groupoid and $\forall a, b\in S$, we have

$$a\cdot b=(a\cdot a)\cdot(b\cdot b)=b\cdot((a\cdot a)\cdot b)=b\cdot(b\cdot(a\cdot a))=b\cdot(a\cdot(b\cdot a))=(b\cdot a)\cdot(b\cdot a)=b\cdot a;$$

hence S is commutative.

It follows that $(S, \cdot)$ is a commutative CA-groupoid, and it is a commutative semigroup.

Definition 6. Let $(S_1, \cdot_1)$ and $(S_2, \cdot_2)$ be two CA-groupoids, $S_1\times S_2 = \{(a, b)\mid a\in S_1, b\in S_2\}$. Define binary operation $\cdot$ on $S_1\times S_2$ as following: $(a_1, a_2)\cdot(b_1, b_2) = (a_1\cdot_1 b_1, a_2\cdot_2 b_2)$, $\forall(a_1, a_2), (b_1, b_2)\in S_1\times S_2$.

$(S_1\times S_2, \cdot)$ is called the direct product of $(S_1, \cdot_1)$ and $(S_2, \cdot_2)$, and S_1 and S_2 are called the direct factors of $S_1\times S_2$.

Theorem 5. Let $(S_1, \cdot_1)$ and $(S_2, \cdot_2)$ be two CA-groupoids. Then the direct product $(S_1\times S_2, \cdot)$ defined in Definition 7 is a CA-groupoid.

Proof. If $(a_1, a_2), (b_1, b_2), (c_1, c_2)\in S_1\times S_2$, then

$$(a_1, a_2)\cdot((b_1, b_2)\cdot(c_1, c_2)) = (a_1, a_2)\cdot(b_1\cdot_1 c_1, b_2\cdot_2 c_2) = (a_1\cdot_1(b_1\cdot_1 c_1), a_2\cdot_2(b_2\cdot_2 c_2))$$
$$= (c_1\cdot_1(a_1\cdot_1 b_1), c_2\cdot_2(a_2\cdot_2 b_2)) = (c_1, c_2)\cdot(a_1\cdot_1 b_1, a_2\cdot_2 b_2) = (c_1, c_2)\cdot((a_1, a_2)\cdot(b_1, b_2)).$$

Hence, $(S_1\times S_2, \cdot)$ is a CA-groupoid.

5. Cyclic Associative Neutrosophic Extended Triplet Groupoids (CA-NET-Groupoids)

In this section, we mainly study a class of important CA-groupoids, called CA-NET-groupoids. The research ideas are derived from regular semigroups in classical semigroup theory and the recent research results on neutrosophic extended triplet groupoids (NETGs, see [15, 22-23, 26, 30, 32-33]). After giving the basic definitions and properties, this section focuses on the structure of CA-NET-groupoids. The results show that every CA-NET-groupoid can be decomposed into disjoint union of some of its subgroups, which is actually an extension of the famous Clifford's theorem in semigroup theory.

Definition 7. Assume that $(N, \cdot)$ be a neutrosophic extended triplet set. If

(1) $(N, \cdot)$ is well-defined, that is, $(\forall a, b\in N)$ $a*b\in N$;

(2) $(N, \cdot)$ is cyclic associative, that is, $(\forall a, b, c\in N)$ $a\cdot(b\cdot c) = c\cdot(a\cdot b)$.

Then $(N, \cdot)$ is called a cyclic associative neutrosophic extended triplet groupoid (shortly, CA-NET-groupoid). A CA-NET-groupoid $(N, \cdot)$ is commutative, if $(\forall a, b\in N)$ $a\cdot b = b\cdot a$.

Theorem 6. If $(N, \cdot)$ is a CA-NET-groupoid and $a\in N$. Then the local unit element $neut(a)$ is unique in N.

Proof. Suppose that local unit element $neut(a)$ is not unique in S. Then, there exists $s, t\in\{neut(a)\}$ such that $(p, q\in N)$

$$as = sa = a \text{ and } ap=pa=s;\ at = ta = a \text{ and } aq = qa = t.$$

(1) $s = ts$. Since $s = pa = p(at) = t(pa) = ts$.

(2) $t = st$. Since $t = qa = q(as) = s(qa) = st$.

(3) $s = ss$ and $t = tt$. Since $s = pa = p(as) = s(pa) = ss$, and $t = qa = q(at) = t(qa) = tt$.

(4) $ts = st$. Since $ts = t(ts) = s(tt) = st$.

Hence s=t and $neut(a)$ is unique in N. □

From the following example, we know that $anti(a)$ may be not unique.

Example 10. Denote $N = \{1, 2, 3, 4\}$. Define the operate $\cdot$ on N in Table 6. Then, $(N, \cdot)$ is CA-NET-groupoid. Moreover, $neut(1) = 1$ and $\{anti(1)\} = \{1, 2, 3, 4\}$.

Table 6. The operation $\cdot$ on N

·	1	2	3	4
1	1	1	1	1
2	1	4	1	2
3	1	1	3	1
4	1	2	1	4

Theorem 7. If $(N, \cdot)$ be a CA-NET-groupoid, then

(1) $\forall\ a \in N$, $neut(a)neut(a) = neut(a)$;
(2) $\forall\ a \in N$, $neut(neut(a)) = neut(a)$;
(3) $\forall\ a \in N$, $\forall\ anti(neut(a)) \in \{anti(neut(a))\}$, $anti(neut(a))a = a$.

Proof. (1) By $a(anti(a)) = anti(a)a = neut(a)$, we get

$$neut(a)neut(a) = neut(a)(a(anti(a))) = anti(a)(neut(a)a) = anti(a)a = neut(a).$$

(2) For any $a \in N$, using the definition of $neut(neut(a))$ we have

$$neut(neut(a))neut(a) = neut(a)neut(neut(a)) = neut(a).$$

By the definition of $anti(neut(a))$ we have

$$anti(neut(a))neut(a) = neut(a)anti(neut(a)) = neut(neut(a)).$$

By (1) and Theorem 7, we get that $neut(neut(a)) = neut(a)$.

(3) Using Definition 5, Definition 8 and above (1), for all $a \in N$,

$$anti(neut(a))a = anti(neut(a))(neut(a)a) = a(anti(neut(a))neut(a)) = a(neut(neut(a))) = a(neut(a)) = a.$$

It follows that $anti(neut(a))a = a$. □

From the following example, $neut(anti(a))$ may be not equal to $neut(a)$.

Example 11. Denote $N = \{1, 2, 3, 4\}$. Define the operate $\cdot$ on N in Table 7. Then, $(N, \cdot)$ is CA-NET-groupoid. Moreover, $neut(1) = 1$, $neut(2) = 2$, $\{anti(1)\} = \{1, 2, 3, 4\}$. While $anti(1) = 2$, $neut(anti(a)) \neq neut(a)$, because $neut(anti(1)) = neut(2) = 2 \neq 1 = neut(1)$.

Table 7. The operation $\cdot$ on N

$\cdot$	**1**	**2**	**3**	**4**
1	1	1	1	1
2	1	2	1	4
3	1	1	3	1
4	1	4	1	2

Theorem 8. If $(N, \cdot)$ is a CA-NET-groupoid. Then
(1) $\forall\ a \in N$, $\forall\ p, q \in \{anti(a)\}$, $p(neut(a)) = q(neut(a))$;
(2) $\forall\ a \in N$, $\forall\ anti(a) \in \{anti(a)\}$, $anti(neut(a))anti(a) \in \{anti(a)\}$;
(3) $\forall\ a \in N$, $\forall\ q \in \{anti(a)\}$, $anti(q)neut(a) = a(neut(q))$;

Proof. (1) $\forall\ a \in N$, $\forall\ p, q \in \{anti(a)\}$, by the definition of neutral and opposite element, using Theorem 7, we get

$pa = ap = neut(a)$, $qa = aq = neut(a)$.

$p(neut(a)) = p(aq) = q(pa) = q(neut(a))$.

(2) $a \in N$, $\forall\ anti(a) \in \{anti(a)\}$, $\forall\ anti(neut(a)) \in \{anti(neut(a))\}$,

$a[anti(neut(a))anti(a)] = anti(a)\{a[anti(neut(a))]\} = anti(neut(a))[anti(a)a] = anti(neut(a))neut(a) = neut(neut(a)) = neut(a)$;

$[anti(neut(a))anti(a)]a = [anti(neut(a))anti(a)]\{a[neut(a)]\} = neut(a)\{[anti(neut(a))anti(a)]a\} = a\{neut(a)[anti(neut(a))anti(a)]\} = a\{anti(a)[neut(a)anti(neut(a))]\} = a[anti(a)neut(a)] = neut(a)[a(anti(a))] = neut(a)neut(a) = neut(a)$.

Thus, $anti(neut(a))anti(a) \in \{anti(a)\}$.

(3) $\forall\ a \in N$, $\forall\ q \in \{anti(a)\}$, by $aq = qa = neut(a)$ and $q(anti(q)) = anti(q)q = neut(q)$, we get

$a(neut(q)) = a[q(anti(q))] = anti(q)(aq) = anti(q)neut(a)$.

This shows that $anti(q)neut(a) = a(neut(q))$.

Proposition 3. If $(N, \cdot)$ is a CA-NET-groupoid. Then
(1) $\forall\ a, b, c \in N$, $ab = ac \Rightarrow b(neut(a)) = c(neut(a))$;
(2) $\forall\ a, b, c \in N$, $ba = ca$ *if and only if* $b(neut(a)) = c(neut(a))$.

Proof. (1) Assume $ab = ac$. For $a \in N$, by the definition of CA-NET-groupoid, $anti(a) \in N$. Multiply $anti(a)$ to the left side with $ab = ac$,

$anti(a)(ab) = anti(a)(ac)$, $b[anti(a)a] = c[anti(a)a]$, $b(neut(a)) = c(neut(a))$.

(2) Assume $ba = ca$. Then,
$anti(a)(ba) = anti(a)(ca)$, $a[anti(a)b] = a[anti(a)c]$, $b[a(anti(a))] = c[a(anti(a))]$, $b(neut(a)) = c(neut(a))$.
Conversely, suppose that $b(neut(a)) = c(neut(a))$. By Definition 5,
$a[b(neut(a))] = a[c(neut(a))]$, $neut(a)(ab) = neut(a)(ac)$, $b[neut(a)a] = c[neut(a)a]$, $ba = ca$.

Proposition 4. Suppose that $(N, \cdot)$ is a commutative CA-NET-groupoid. Then
$\forall\ a, b \in N$, $neut(a)neut(b) = neut(ab)$.

Proof. Because the local unit element of every element is unique in N, consider left hand side, $neut(a)neut(b)$. Now multiply to the left with ab,
$(ab)[neut(a)neut(b)] = neut(b)[(ab)neut(a)] = neut(a)[neut(b)(ab)] = neut(a)[b(neut(b)a)] =$
$= neut(a)\{a[b(neut(b))]\} = neut(a)(ab) = b[neut(a)a] = ba = ab$.
And multiply to the right with ab for $neut(a)neut(b)$, we can get
$[neut(a)neut(b)](ab) = b\{[neut(a)neut(b)]a\} = a\{b[neut(a)neut(b)]\} = a\{neut(b)[b(neut(a))]\} =$
$a\{neut(a)[neut(b)b]\} = a[neut(a)b] = b[a(neut(a))] = ba = ab$.
Therefore, $neut(a)neut(b) = neut(ab)$.

Definition 8. Let $(N, \cdot)$ be a CA-NET-groupoid. If $(\forall\ a, b \in N)$ $a(neut(b)) = neut(b)a$, then N is called a weak commutative CA-NET-groupoid (briefly, WC-CA-NET-groupoid).

Theorem 9. Assume that $(N, \cdot)$ is a CA-NET-groupoid. Then N is a commutative CA-NET-groupoid if and only if N is a weak commutative CA-NET-groupoid.

Proof. Suppose that N is a commutative CA-NET-groupoid. Obviously, N is a weak commutative CA-NET-groupoid. Conversely, if N is a weak commutative CA-NET-groupoid, then $(\forall\ a, b \in N)$

$$ab = a[neut(b)b] = b[a(neut(b))] = neut(b)(ba) = neut(b)[b(neut(a)a)] = neut(b)\{a[b(neut(a))]\}$$
$$= neut(b)[neut(a)(ab)] = (ab)[neut(b)neut(a)] = neut(a)[(ab)neut(b)] = neut(a)[neut(b)(ab)]$$
$$= neut(a)[b(neut(b)a)] = neut(a)\{a(b[neut(b)]) = neut(a)(ab) = b[neut(a)a] = ba.$$

Therefore, N is a commutative CA-NET-groupoid. □

Theorem 10. Suppose that $(N, \cdot)$ is a CA-NET-groupoid. Denote the set of all different neutral element in N by $E(N)$. For any $e \in E(N)$, denote $N(e) = \{a \in N \mid neut(a) = e\}$. Then
(1) $\forall\ e \in E(N)$, $N(e)$ is a subgroup of N.
(2) $\forall\ e_1, e_2 \in E(N)$, $e_1 \neq e_2 \Rightarrow N(e_1) \cap N(e_2) = \phi$.
(3) $N = \bigcup_{e \in E(N)} N(e)$.

Proof. (1) $\forall\ x \in N(e)$, $neut(x) = e$. This means that e is an identity element in $N(e)$. Moreover, by Theorem 8 (1), $ee = e$.

If $x, y \in N(e)$, then $neut(x) = neut(y) = e$. We prove that $neut(xy) = e$. In fact, by Definition 5 and Proposition 2 (1) we have

$$(xy)e = (xy)(ee) = (ex)(ey) = xy;\ e(xy) = y(ex) = x(ye) = xy.$$

On the other hand, $\forall\ anti(x) \in \{anti(x)\}$, $\forall\ anti(y) \in \{anti(y)\}$, by Proposition 2 (1),

$$(xy)[anti(x)anti(y)] = (anti(y)x)(anti(x)y) = [y(anti(y))](anti(x)x) = neut(y)neut(x) = ee = e.$$
$$[anti(x)anti(y)](xy) = [y(anti(x))][x(anti(y))] = (anti(y)y)[x(anti(x))] = neut(y)neut(x) = ee = e.$$

Thus, by the definition of neutral element and Theorem 7, we know that $neut(xy) = e$. It follows that $xy \in N(e)$, that is, $N(e)$ is closed under operation $\cdot$.

Moreover, $\forall\ x \in N(e)$, there exists $q \in N$ and $q \in \{anti(x)\}$. Using Theorem 11 (1), $q(neut(x)) \in \{anti(x)\}$; and using Theorem 11 (5), $neut(q(neut(x))) = neut(x)$. Denote $t = q(neut(x))$, then

$$t = q(neut(x)) \in \{anti(x)\}, \text{ and } neut(t) = neut(q(neut(x))) = neut(x) = e.$$

This means that there exists $t \in \{anti(x)\}$, $neut(t) = e$, that is, $t \in N(e)$.

Combing above results, we know that $(N(e), \cdot)$ is a subgroup of N.

(2) Assume that $x \in N(e_1) \cap N(e_2)$ and $e_1, e_2 \in E(N)$. Then $neut(x) = e_1$, $neut(x) = e_2$. By Theorem 7 we get $e_1 = e_2$. Therefore, $e_1 \neq e_2 \Rightarrow N(e_1) \cap N(e_2) = \phi$.

(3) $\forall\ x \in N$, there exists $neut(x) \in N$. Denote $e = neut(x)$, then $e \in E(N)$ and $x \in N(e)$. This means that $N = \bigcup_{e \in E(N)} N(e)$. □

6. Conclusions

In this paper, the concept of cyclic associative groupoid (CA-groupoid) is introduced for the first time from various backgrounds, such as non-associative rings and non-associative algebras, weak Novikov algebras and CA-AG-groupoids. The research results of this paper show that CA-groupoid, as a non-associative algebraic structure, has typical representativeness and rich connotation, and is closely related to many kinds of algebraic structures. This paper obtains many interesting conclusions. Here are some important results：

(1) Every commutative semigroup is CA-groupoid, every commutative CA-groupoid is a semigroup. (see Example 1, 2 and Proposition 1)

(2) From some non-associative and non-commutative algebras (as vector spaces over fields), we can get some CA-groupoids. (see Example 5 and 6)

(3) Every CA-groupoid with left (or right) identity element is a commutative semigroup, every left cancellative element of a CA-groupoid is right cancellative. (see Theorem 2 and 4)

(4) CA-groupoids and AG-groupoids are closely related, but they do not contain each other. (see Theorem 3 and Figure 2)

(5) For cyclic associative neutrosophic extended triplet groupoids (CA-NET- groupoids), there are some interesting properties. (see Theorem 7, 8, 9 and 11)

(6) A CA-groupoid is weak commutative if and only if it is commutative CA-NET-groupoid. (see Definition 9 and Theorem 10)

(7) Every CA-NET- groupoid is a disjoint union of its subgroup. (Decomposition Theorem of the CA-NET-groupoids, see Theorem 12)

As a direction of future research, we'll investigate regularity, cancellability and the relationships among CA-groupoids, CA-NET-groupoids and related algebraic systems.

Acknowledgments: The research was supported by National Natural Science Foundation of China (No. 61976130).

Author Contributions: Xiaohong Zhang and Zhirou Ma initiated the research and wrote the paper, Wangtao Yuan added some new properties.

Conflicts of Interest: The authors declare no conflict of interest.

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Received: June 13, 2019. Accepted: October 20, 2019