Neutrosophic Sets and Systems, Vol. 54, 2023
University of New Mexico

On Phi-Euler's Function in Refined Neutrosophic Number Theory and The Solutions of Fermat's Diophantine Equation

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Abstract:

The objective of this paper is to answer the open problem proposed about the validity of phi-Euler's theorem in the refined neutrosophic ring of integers $Z(I_1, I_2)$. This work presents an algorithm to compute the values of Euler's function on refined neutrosophic integers, and it prove that phi-Euler's theorem is still true in $Z(I_1, I_2)$.

On the other hand, we present a solution for another open question about the solutions of Fermat's Diophantine equation in refined neutrosophic ring of integers, where we determine the solutions of Fermat's Diophantine equation $X^n + Y^n = Z^n; n \geq 3$ in $Z(I_1, I_2)$.

Key Words: refined Neutrosophic integer, Neutrosophic Euler's function, Neutrosophic Fermat's equation

1. Introduction

Neutrosophy is a new generalization of fuzzy ideas by considering three membership states (truth, falsity, and indeterminacy) founded by Smarandache in 1995 [1].

In the literature [2], the indeterminacy element I was used to build some interesting extensions of algebraic rings. By adding I (with a logical property $I^2 = I$) to any ring R, we get $R(I)$ the corresponding neutrosophic ring as follows:

$R(I) = \{a + bI; a, b \in R\}$[2].

In [3], Agboola et.al, proposed the structure of refined neutrosophic rings.

As a natural development, neutrosophic number theory was studied in [4,6], where we can find neutrosophic congruencies, Diophantine equations, primes, and neutrosophic Euler's theorem.

In [5], Ibrahim et.al, proposed the basic ideas in refined neutrosophic number theory, where they defined congruencies, Pell's equation, and divisibility in $Z(I_1, I_2)$. On the other hand, an interesting open question has been asked as follows:

Define phi-Euler's function in $Z(I_1, I_2)$? Is Euler's theorem still true ?.

Through this paper, we aim to solve this problem by proving that Euler's theorem is still true in refined neutrosophic number theory.

Also, we find all possible solutions for The non-linear Fermat's Diophantine equation $X^n + Y^n = Z^n; n \geq 3$, which was proposed as an open question in [7].

For more results and findings of neutrosophic number theory and algebraic structures, see [8-15].

For definitions and basic concepts in refined neutrosophic number theory, see [5].

Main discussion :

First of all, we will give an example to explain our idea.

Example :

Let $Z(I_1, I_2)$ be the refined neutrosophic ring of integers, consider $x = (3, I_1, -I_2) \epsilon Z(I_1, I_2)$.

To compute the value of $\varphi(x)$, we have to know the number of refined neutrosophic integers:

$y = (y_0, y_1 I_1, y_2 I_2)$, with the property :

$$\begin{cases} \gcd(x, y) = (1,0,0) \\ \quad 0 < y \leq x \end{cases}$$

According to the definition of (gcd) in refined neutrosophic ring of integers, we get

$\gcd(3, y_0) = 1$, $\gcd(2, y_0 + y_2) = 1$, $\gcd(3, y_0 + y_1 + y_2) = 1$.Also, $y \leq x$ implies that:

$$\begin{cases} \quad 0 \leq y \leq 3 \\ 0 \leq y_0 + y_2 \leq 2 \\ 0 \leq y_0 + y_1 + y_2 \leq 3 \end{cases}$$

The possible values of y_0 are $\{1,2\}$. The possible values of $y_0 + y_2$ are $\{1\}$.The possible values of $y_0 + y_1 + y_2$ are $\{1,2\}$. This implies that we get the following solutions :

$$y = (1,0,0), \ y = (1, I_1, 0), \ y = (2, 0, I_2), \ y = (2, I_1, -I_2)$$

So, $\varphi(x) = 4$ which is equal to $\varphi(3) \times \varphi(2) \times \varphi(3)$. Now, we are able to study the general case.

Definition:

Let $0 < x = (x_0, x_1 I_1, x_2 I_2) \epsilon Z(I_1, I_2)$, we define Euler's function as follows:

$\varphi(x) = |\{y = (y_0, y_1 I_1, y_2 I_2) \epsilon Z(I_1, I_2) : \gcd(x, y) = (1,0,0) and\ 0 < y \leq x\}|$.

Theorem::

Let $x = (x_0, x_1 I_1, x_2 I_2)$ be any positive refined neutrosophic integer, hence: $\varphi(x) = \varphi(x_0) \times \varphi(x_0 + x_2) \times \varphi(x_0 + x_1 + x_2)$.

Proof:

Let $y = (y_0, y_1 I_1, y_2 I_2)$ be a refined neutrosophic integer with $\begin{cases} 0 \leq y \leq x \\ \gcd(x,y) = (1,0,0) \end{cases}$

We have, ($y_0 \leq x_0$, $y_0 + y_2 \leq x_0 + x_2$, $y_0 + y_1 + y_2 \leq x_0 + x_1 + x_2$) and ($\gcd(x_0, y_0) = \gcd(x_0 + x_2, y_0 + y_2) = \gcd(x_0 + x_1 + x_2, y_0 + y_1 + y_2) = (1,0,0)$. This implies that we have $\varphi(x_0)$ ways to chose y_0, $\varphi(x_0 + x_2)$ ways to chose $y_0 + y_2$ and $\varphi(x_0 + x_1 + x_2)$ ways to chose $y_0 + y_1 + y_2$. By using the essential concept in combinatory, we get $\varphi(x) = \varphi(x_0) \times \varphi(x_0 + x_2) \times \varphi(x_0 + x_1 + x_2)$.

Example:

Let $x = (4,0,2I_2)$, we have :

$\varphi(4) = 2$, $\varphi(4+2) = \varphi(6) = 2$, $\varphi(4+0+2) = \varphi(6) = 2$.

Hence $\varphi(x) = 2 \times 2 \times 2 = 8$.

We shall find the 8 refined neutrosophic integers with the property $\begin{cases} 0 \leq y \leq x \\ \gcd(x,y) = (1,0,0) \end{cases}$

Let $y = (y_0, y_1 I_1, y_2 I_2)$, we have :$\begin{cases} y_0 \leq 4, \ \gcd(y_0, 4) = 1 \Longrightarrow y_0 \in \{1,3\} \\ y_0 + y_2 \leq 6, \ \gcd(y_0 + y_2, 6) = 1 \Longrightarrow y_0 + y_2 \in \{1,5\} \\ y_0 + y_1 + y_2 \leq 6, \ \gcd(y_0 + y_1 + y_2, 6) = 1 \Longrightarrow y_0 + y_1 + y_2 \in \{1,5\} \end{cases}$

The possible solutions are:

1) $y = (1,0,0)$.

2) $y = (1, -4I_1, 4I_2)$.

3) $y = (1,0,4I_2)$.

4) $y = (1,4I_1, 0)$.

5) $y = (3,0,-2I_2)$.

6) $y=(3,4I_1,-2I_2)$.

7) $y=(1,-4I_1,2I_2)$.

8) $y=(1,0,2I_2)$.

The following theorem clarifies how to compute natural powers in $Z(I_1,I_2)$.

Theorem :

Let $x=(x_0,x_1I_1,x_2I_2)\in Z(I_1,I_2)$, let n be any positive integer, hence $x^n=(x_0^n,I_1[(x_0+x_1+x_2)^n-(x_0+x_2)^n],I_2[(x_0+x_2)^n-x_0^n])$.

Theorem:

Let $Z(I_1,I_2)$ be the refined neutrosophic ring of integers. Let $x=(x_0,x_1I_1,x_2I_2)$, $y=(y_0,y_1I_1,y_2I_2)\in Z(I_1,I_2)$ with gcd(x,y)=1, hence $x^{\varphi(y)}=1(mod\ y)$.

Proof:

According to the assumption , we have :

$$x^{\varphi(y)}=x^{\varphi(y_0)\times\varphi(y_0+y_2)\times\varphi(y_0+y_1+y_2)}=\left(x_0^{\varphi(y)},I_1[(x_0+x_1+x_2)^{\varphi(y)}-(x_0+x_2)^{\varphi(y)}],I_2[(x_0+x_2)^{\varphi(y)}-x_0^{\varphi(y)}]\right).$$

Now, let's compute the following :

$$x_0^{\varphi(y)}=[x_0^{\varphi(y_0)}]^{\varphi(y_0+y_2)\times\varphi(y_0+y_1+y_2)}\equiv 1^{\varphi(y_0+y_2)\times\varphi(y_0+y_1+y_2)}(mod\ y_0)\equiv 1(mod\ y_0)\ .$$

(That is because $\gcd(x_0,y_0)=1$)

$$(x_0+x_2)^{\varphi(y)}=[(x_0+x_2)^{\varphi(y_0+y_2)}]^{\varphi(y_0)\times\varphi(y_0+y_1+y_2)}\equiv 1(mod\ y_0+y_2).$$

(That is because $\gcd(x_0+x_2,y_0+y_2)=1$)

$$(x_0+x_1+x_2)^{\varphi(y)}=[(x_0+x_1+x_2)^{\varphi(y_0+y_1+y_2)}]^{\varphi(y_0)\times\varphi(y_0+y_2)}\equiv 1(mod\ y_0+y_1+y_2).$$

(That is because $\gcd(x_0+x_1+x_2,y_0+y_1+y_2)=1$).

We get that:

$$x_0^{\varphi(y)}\equiv 1(mod\ y_0),$$

$$x_0^{\varphi(y)}+\left[(x_0+x_2)^{\varphi(y)}-x_0^{\varphi(y)}\right]=(x_0+x_2)^{\varphi(y)}\equiv 1(\mathrm{mod}\, y_0+y_2),$$

$x_0^{\varphi(y)} + \left[(x_0 + x_2)^{\varphi(y)} - x_0^{\varphi(y)}\right] + [(x_0 + x_1 + x_2)^{\varphi(y)} - (x_0 + x_2)^{\varphi(y)} = (x_0 + x_1 + x_2)^{\varphi(y)} \equiv$

$1(\mathrm{mod}\, y_0 + y_1 + y_2)$,

Under the definition of congruencies in refined neutrosophic rings we can write:

$x^{\varphi(y)} \equiv (1,0,0)\ (mod\ y)$.

This implies that Euler's theorem is true in $Z(I_1, I_2)$.

Definition : [7]

Let R be a ring, $F = (X, Y, Z)$ be a triple, where $X, Y, Z \in R$. F is called a general Fermat's triple if and only if $X^n + Y^n = Z^n; for\ all\ \ integers\ n \geq 3$.
This is equivalent to the condition that (X, Y, Z) is a solution of Fermat's equation.

Theorem :

Let $Z(I_1, I_2)$ be the refined neutrosophic ring of integers. The Equation $X^n + Y^n = Z^n; n \geq 3$ has only 27 solutions.

Proof:

$$X^n + Y^n = Z^n \Leftrightarrow \begin{cases} x_0{}^n + y_0{}^n = z_0{}^n \ldots (1) \\ (x_0 + x_2)^n + (y_0 + y_2)^n = (z_0 + z_2)^n \ldots (2) \\ (x_0 + x_1 + x_2)^n + (y_0 + y_1 + y_2)^n = (z_0 + z_1 + z_2)^n \ldots (3) \end{cases}$$

Now, solutions of (1) is.

$$\begin{cases} x_0 = y_0 = z_0 = 0 \ldots (a) \\ x_0 = z_0 = 1, y_0 = 0 \ldots (b) \\ y_0 = z_0 = 1, x_0 = 0 \ldots (c) \end{cases}$$

solutions of (2) is.

$$\begin{cases} x_0 + x_2 = y_0 + y_2 = z_0 + z_2 = 0 \ldots (d) \\ x_0 + x_2 = z_0 + z_2 = 1, y_0 + y_2 = 0 \ldots (e) \\ y_0 + y_2 = z_0 + z_2 = 1, x_0 + x_2 = 0 \ldots (f) \end{cases}$$

solutions of (3) is.

$$\begin{cases} x_0 + x_1 + x_2 = y_0 + y_1 + y_2 = z_0 + z_1 + z_2 = 0 \ldots (g) \\ x_0 + x_1 + x_2 = z_0 + z_1 + z_2 = 1, y_0 + y_1 + y_2 = 0 \ldots (h) \\ y_0 + y_1 + y_2 = z_0 + z_1 + z_2 = 1, x_0 + x_1 + x_2 = 0 \ldots (i) \end{cases}$$

We discuss possible cases.

Case1. If $(a), (d), (g)$, then $X = Y = Z = (0,0,0)$.

Case2. If $(a), (d), (h)$, then $X = (0,1,0), Z = (0,1,0), Y = (0,0,0)$.

Case3. If $(a), (d), (i)$, then $X = (0,0,0), Z = (0, I_1, 0), Y = (0, I_1, 0)$.

Case4. If $(a), (e), (g)$, then $X = (0, -I_1, I_2), Z = (0, -I_1, I_2), Y = (0,0,0)$.

Case5. If $(a),(e),(g)$, then $X=(0,-I_1,I_2), Z=(0,0,I_2), Y=(0,0,0)$.

Case6. If $(a),(e),(h)$, then $X=(0,0,I_2), Z=(0,0,I_2), Y=(0,0,0)$.

Case7. If $(a),(f),(g)$, then $X=(0,0,0), Z=(0,-I_1,I_2), Y=(0,-I_1,I_2)$.

Case8. If $(a),(f),(h)$, then $X=(0,I_1,0), Z=(0,-I_1,I_2), Y=(0,-I_1,I_2)$.

Case9. If $(a),(f),(i)$, then $X=(0,0,0), Z=(0,0,I_2), Y=(0,0,I_2)$.

Case10. If $(b),(d),(g)$, then $X=(1,0,-I_2), Z=(1,0,-I_2), Y=(0,0,0)$.

Case11. If $(b),(d),(h)$, then $X=(1,I_1,-I_2), Z=(1,I_1,-I_2), Y=(0,0,I_2)$.

Case12. If $(a),(d),(i)$, then $X=(1,0,-I_2), Z=(1,I_1,I_2), Y=(0,I_1,0)$.

Case13. If $(b),(e),(g)$, then $X=(1,-I_1,0), Z=(1,-I_1,0), Y=(0,0,0)$.

Case14. *If* $(b),(e),(h)$, *then* $X=(1,0,0), Z=(1,0,0), Y=(0,I_1,0)$.

Case15. If $(b),(e),(i)$, then $X=(1,-I_1,0), Z=(1,0,0), Y=(0,I_1,0)$.

Case16. If $(b),(f),(g)$, then $X=(1,0,-I_2), Z=(1,-I_1,0), Y=(0,-I_1,I_2)$.

Case17. If $(b),(f),(h)$, then $X=(1,I_1,I_2), Z=(1,0,0), Y=(0,-I_1,I_2)$.

Case18. If $(b),(f),(i)$, then $X=(1,0,-I_2), Z=(1,0,0), Y=(0,0,I_2)$.

Case19. If $(c),(d),(h)$, then $X=(0,I_1,0), Z=(1,I_1,-I_2), Y=(1,0,-I_2)$.

Case20. If $(c),(d),(g)$, then $X=(0,0,0), Z=(1,0,-I_2), Y=(1,0,-I_2)$.

Case21. If $(c),(d),(i)$, then $X=(0,0,0), Z=(1,I_1,-I_2), Y=(1,I_1,-I_2)$.

Case22. If $(c),(e),(g)$, then $X=(0,-I_1,I_2), Z=(1,-I_1,0), Y=(1,0,-I_2)$.

Case23. If $(a),(e),(h)$, then $X=(0,0,-I_2), Z=(1,0,0), Y=(1,0,-I_2)$.

Case24. If $(c),(e),(i)$, then $X=(0,-I_1,I_2), Z=(1,0,0), Y=(1,I_1,I_2)$.

Case25. If $(c),(f),(g)$, then $X=(0,0,0), Z=(1,-I_1,0), Y=(1,-I_1,0)$.

Case26. If $(c),(f),(h)$, then $X=(0,I_1,0), Z=(1,0,0), Y=(1,-I_1,0)$.

Case27. If $(c),(f),(i)$, then $X=(0,0,0), Z=(1,0,0), Y=(1,0,0)$.

Conclusion

In this paper, we have defined the Euler's function in the refined neutrosophic ring of integers (I_1,I_2) , as well as, we have presented an algorithm to compute the values of this function.

Also, we have proved that Euler's famous theorem is still true in the case of refined neutrosophic number theory.

In particular, we have determined the possible solutions of Fermat's equation in the refined neutrosophic ring of integers.

As a future research direction, we aim to study the Euler's theorem in n-refined neutrosophic number theory and n-cyclic refined neutrosophic integers, as well as Fermat's equation in these rings.

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Received: December 30, 2022. **Accepted:** April 01, 2023