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On The 3-Refined Neutrosophic Analytical Structures and Number Theoretical Concepts

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Abstract:

n-refined neutrosophic structures are considered as generalizations of classical structures, and neutrosophic structures.

The main goal of this paper, is to study several structures generated by using 3-refined neutrosophic numbers, where we find the mathematical formulas of 3-refined neutrosophic real functions. Also, the inner products over 3-refined neutrosophic vector spaces and orthogonal properties. In addition, we present the foundations of 3-refined neutrosophic number theory, especially division, congruencies, and some related equations.

Keywords: 3-refined neutrosophic real function, 3-refined neutrosophic inner product, 3-refined neutrosophic vector space, 3-refined neutrosophic number theory

Introduction and basic concepts

The concept of neutrosophic structures plays an important role in the theory of algebraic structures and analysis. Many concepts and structures were defined previously, such as neutrosophic vector spaces, neutrosophic matrices, and algebraic rings [1-3, 5-7, 14-16].

Laterally, refined neutrosophic structures were defined to generalize the neutrosophic structures, where refined neutrosophic rings, modules, and other structures were presented [4, 8-11, 24-28].

The concept of n-refined neutrosophic structure is considered as a generalization of refined structure [12, 29]. For each value of the integer n, we get a generalized structure.

This work will study some of 3-refined neutrosophic structures, where we present the formulas of 3-refined neutrosophic real functions, 3-refined neutrosophic inner products defined over 3-refined neutrosophic vector spaces, and 3-refined number theoretical concepts.

First, we recall some basic concepts.

Definition:

Let (R,+,×) be a ring, $R(I)=\{a+bI\ ;\ a,b\in R\}$ is called the neutrosophic ring where I is a neutrosophic element with condition $I^2=I$.

Definition:

Let (R,+,×) be a ring, (R(I_1,I_2) , + ,×) is called a refined neutrosophic ring generated by R , I_1, I_2.

Definition:

Let (R,+,×) be a ring and $I_k; 1\le k\le n$ be n indeterminacies. We define R_n(I)=$\{a_0+a_1I+\cdots+a_nI_n\ ;\ a_i\in R\}$ to be n-refined neutrosophic ring.

Addition and multiplication on R_n(I) are defined as:

$\sum_{i=0}^{n} x_iI_i+\sum_{i=0}^{n} y_iI_i=\sum_{i=0}^{n}(x_i+y_i)I_i$,$\sum_{i=0}^{n} x_iI_i\times\sum_{i=0}^{n} y_iI_i=\sum_{i,j=0}^{n}(x_i\times y_j)I_iI_j$.

Where × is the multiplication defined on the ring R.

For n=3, we get the 3-refined neutrosophic ring.

Main Discussion

Definition.

Let $R_3(I)$ be the 3-refined neutrosophic ring of reals, $f: R_3(I)\to R_3(I); f=f(X); X\in R_3(I)$. f is called 3-refined neutrosophic real function with one variable.

Theorem.

$R_3(I)\cong R^4$.

Proof.

We define $g: R_3(I)\to R^4; g(a+bI_1+cI_2+dI_3)=(a,a+b+c+d,a+c+d,a+d)$.

It is clear that g is well defined function.

$ker(g) = \{0\}$, thus g is injective.

$Im(g) = R^4$, thus g is surjective, so that g is one-to-one.

Now, let $A = a_0 + a_1 I_1 + a_2 I_2 + a_3 I_3, B = b_0 + b_1 I_1 + b_2 I_2 + b_3 I_3$,

$A + B = (a_0 + b_0) + (a_1 + b_1)I_1 + (a_2 + b_2)I_2 + (a_3 + b_3)I_3$

$g(A + B) = g(A) + g(B)$.

$$\begin{aligned} A.B = a_0 b_0 &+ [(a_0 + a_1 + a_2 + a_3)(b_0 + b_1 + b_2 + b_3) - (a_0 + a_2 + a_3)(b_0 + b_2 + b_3)]I_1 \\ &+ [(a_0 + a_2 + a_3)(b_0 + b_2 + b_3) - (a_0 + a_3)(b_0 + b_3)]I_2 \\ &+ [(a_0 + a_3)(b_0 + b_3) - a_0 b_0]I_3 \end{aligned}$$

$g(A.B) = g(A).g(B)$, hence g is a ring isomorphism.

Remark.

Let $f: R_3(I) \to R_3(I)$ be a 3-refined neutrosophic real function with one variable, then f can be represented by four classical real functions by taking the direct isomorphic image $g\big(f(X)\big)$.

Example.

Take $f(X) = (1 + I_1)X^2 + (2 - I_2 - I_3)X + 1 + 2I_1 + I_2 + I_3$, f can be represented as follows:

$$g\big(f(X)\big) = g(1 + I_1)\big(g(X)\big)^2 + g(2 - I_2 - I_3)g(X) + g(1 + 2I_1 + I_2 + I_3)$$

$$\begin{aligned} g\big(f(X)\big) = &(1,2,1,1)({x_0}^2, (x_0 + x_1 + x_2 + x_3)^2, (x_0 + x_2 + x_3)^2, (x_0 + x_3)^2) \\ &+ (2,0,0,1)(x_0, x_0 + x_1 + x_2 + x_3, x_0 + x_2 + x_3, x_0 + x_3) + (1,5,3,2) \end{aligned}$$

$$\begin{aligned} g\big(f(X)\big) = &({x_0}^2 + 2x_0 + 1, 2(x_0 + x_1 + x_2 + x_3)^2 + 5, (x_0 + x_2 + x_3)^2 + 3, (x_0 + x_3)^2 \\ &+ (x_0 + x_3) + 2) \end{aligned}$$

The four classical functions that represent (f) are:

$f_1: R \to R;\ f_1(x_0) = {x_0}^2 + 2x_0 + 1$

$f_2: R \to R;\ f_2(x_0 + x_1 + x_2 + x_3) = 2(x_0 + x_1 + x_2 + x_3)^2 + 5$

$f_3: R \to R;\ f_3(x_0 + x_2 + x_3) = (x_0 + x_2 + x_3)^2 + 3$

$f_4: R \to R;\ f_4(x_0 + x_3) = (x_0 + x_3)^2 + (x_0 + x_3) + 2$

Theorem.

Let $g: R_3(I) \to R^4$ be the isomorphism defined above, then:

$g^{-1}: R^4 \to R_3(I); g^{-1}(a, b, c, d) = a + (b - c)I_1 + (c - d)I_2 + (d - a)I_3$.

The proof is easy.

Remark.

To find the formula of a 3-refined neutrosophic real function $f: R_3(I) \to R_3(I)$, we went compute:

$g^{-1}\left(g\big(f(X)\big)\right)$.

Example.

For the function $f(X) = (1 + I_1)X^2 + (2 - I_2 - I_3)X + 1 + 2I_1 + I_2 + I_3$, we compute:

$$g^{-1}\left(g\big(f(X)\big)\right) = ({x_0}^2 + 2x_0 + 1) + [2(x_0 + x_1 + x_2 + x_3)^2 + 5 - (x_0 + x_2 + x_3)^2 - 3]I_1$$
$$+ [(x_0 + x_2 + x_3)^2 + 3 - (x_0 + x_3)^2 - (x_0 + x_3) - 2]I_2$$
$$+ [(x_0 + x_3)^2 + (x_0 + x_3) + 2 - {x_0}^2 - 2x_0 - 1]I_3$$
$$= {x_0}^2 + 2x_0 + 1 + [2(x_0 + x_1 + x_2 + x_3)^2 - (x_0 + x_2 + x_3)^2 + 2]I_1$$
$$+ [(x_0 + x_2 + x_3)^2 - (x_0 + x_3)^2 - (x_0 + x_3) + 1]I_2$$
$$+ [(x_0 + x_3)^2 + (x_0 + x_3) - {x_0}^2 - 2x_0 + 1]I_3$$

Definition.

Let $f: R_3(I) \to R_3(I)$ be a 3-refined neutrosophic real function, and $g\big(f(X)\big) = (f_1, f_2, f_3, f_4)$, with $f_i: R \to R; 1 \le i \le 4$, we say:

a). f is differentiable if and only if f_i are differentiable.

b). f is integrable if and only if f_i are integrable.

We mean by differentiable/integrable on all R not only for sub-domains $]a, b[\subseteq R$.

Example on famous functions.

1. $f: R_3(I) \to R_3(I), f(X) = sin(X)$.

It's formula is $f(X) = g^{-1}\left(g\big(f(X)\big)\right) = sin(x_0) + [sin(x_0 + x_1 + x_2 + x_3) - sin(x_0 + x_2 + x_3)]I_1 + [sin(x_0 + x_2 + x_3) - sin(x_0 + x_3)]I_2 + [sin(x_0 + x_3) - sin(x_0)]I_3$.

2. $f(X) = cos(X) = g^{-1}\left(g\big(f(X)\big)\right) = cos(x_0) + [cos(x_0 + x_1 + x_2 + x_3) - cos(x_0 + x_2 + x_3)]I_1 + [cos(x_0 + x_2 + x_3) - cos(x_0 + x_3)]I_2 + [cos(x_0 + x_3) - cos(x_0)]I_3$.
3. $f(X) = tan(X) = tan(x_0) + [tan(x_0 + x_1 + x_2 + x_3) - tan(x_0 + x_2 + x_3)]I_1 + [tan(x_0 + x_2 + x_3) - tan(x_0 + x_3)]I_2 + [tan(x_0 + x_3) - tan(x_0)]I_3$
4. $f(X) = cot(X) = cot(x_0) + [cot(x_0 + x_1 + x_2 + x_3) - cot(x_0 + x_2 + x_3)]I_1 + [cot(x_0 + x_2 + x_3) - cot(x_0 + x_3)]I_2 + [cot(x_0 + x_3) - cot(x_0)]I_3$
5. $f(X) = e^X = e^{x_0} + [e^{x_0 + x_1 + x_2 + x_3} - e^{x_0 + x_2 + x_3}]I_1 + [e^{x_0 + x_2 + x_3} - e^{x_0 + x_3}]I_2 + [e^{x_0 + x_3} - e^{x_0}]I_3$

6. $f(X) = ln(X) = ln(x_0) + [ln(x_0 + x_1 + x_2 + x_3) - ln(x_0 + x_2 + x_3)]I_1 + [ln(x_0 + x_2 + x_3) - ln(x_0 + x_3)]I_2 + [ln(x_0 + x_3) - ln(x_0)]I_3$, with $X > 0$.
7. $f(X) = X^n = x_0{}^n + [(x_0 + x_1 + x_2 + x_3)^n - (x_0 + x_2 + x_3)^n]I_1 + [(x_0 + x_2 + x_3)^n - (x_0 + x_3)^n]I_2 + [(x_0 + x_3)^n - x_0{}^n]I_3; n \in N.$

Definition.

Let V be vector space over R, the 3-refined neutrosophic vector space is defined as follows:

$$V_3(I) = V + VI_1 + VI_2 + VI_3 = \{x + yI_1 + zI_2 + tI_3\ ; x, y, z, t \in R\}$$

Remark.

Addition on $V_3(I)$ is defined:

$$(x_0 + y_0I_1 + z_0I_2 + t_0I_3) + (x_1 + y_1I_1 + z_1I_2 + t_1I_3) = (x_0 + x_1) + (y_0 + y_1)I_1 + (z_0 + z_1)I_2 + (t_0 + t_1)I_3$$

Where $x_i, y_i, z_i, t_i \in V\ ; 0 \le i \le 1$.

Multiplication on $V_3(I)$ is defined:

$$(a + bI_1 + cI_2 + dI_3).(x + yI_1 + zI_2 + tI_3) = (a.x) + (ay + bx + by + bz + bt + cy + dy)I_1 + (az + cx + cz + ct + dz)I_2 + (at + dx + dt)I_3$$

Where $a, b, c, d \in R\ , x, y, z, t \in V$

Remark.

$(V_3(I), +, .)$ Is a module over $R_3(I)$.

Definition.

Let $f: V_3(I) \times V_3(I) \to R_3(I)$ be a well defined mapping, we call f a 3-refined neutrosophic real inner product if and only if the following conditions hold:

1). $f(X, X) \ge 0\ ;\ \forall X \in V_3(I)$.

2). $f(X, X) = 0 \Leftrightarrow X = 0$.

3). $f(X, Y) = f(Y, X)\ ;\ \forall X, Y \in V_3(I)$.

4). $f(\alpha X + \beta Y, Z) = \alpha f(X, Z) + \beta f(X, Z)\ ;\ \forall X, Y, Z \in V_3(I)\ , \alpha, \beta \in R_3(I)$.

Theorem.

Let $g: V \times V \to R$be an inner production on V, then for $X = x_0 + x_1I_1 + x_2I_2 + x_3I_3, Y = y_0 + y_1I_1 + y_2I_2 + y_3I_3 \in V_3(I)$, the mapping $f: V_3(I) \times V_3(I) \to R_3(I)$such that:

$$f(X,Y) = g(x_0, y_0) + \big(g(x_0 + x_1 + x_2 + x_3, y_0 + y_1 + y_2 + y_3) - g(x_0 + x_2 + x_3, y_0 + y_2 + y_3)\big)I_1 + \big(g(x_0 + x_2 + x_3, y_0 + y_2 + y_3) - g(x_0 + x_3, y_0 + y_3)\big)I_2 + \big(g(x_0 + x_3, y_0 + y_3) - g(x_0, y_0)\big)I_3$$

Is a 3-refined neutrosophic inner product.

Proof.

$$f(X,X) = g(x_0, x_0) + \big(g(x_0 + x_1 + x_2 + x_3, x_0 + x_1 + x_2 + x_3) - g(x_0 + x_2 + x_3, x_0 + x_2 + x_3)\big)I_1 + \big(g(x_0 + x_2 + x_3, x_0 + x_2 + x_3) - g(x_0 + x_3, x_0 + x_3)\big)I_2 + \big(g(x_0 + x_3, x_0 + x_3) - g(x_0, x_0)\big)I_3$$

$$= \|x_0\|^2 + (\|x_0 + x_1 + x_2 + x_3\|^2 - \|x_0 + x_2 + x_3\|^2)I_1 + (\|x_0 + x_2 + x_3\|^2 - \|x_0 + x_3\|^2)I_2 + (\|x_0 + x_3\|^2 - \|x_0\|^2)I_3 \geq 0$$

According to the concept of partial ordering on $R_3(I)$.

$f(X,X) = 0 \Leftrightarrow \|x_0\|^2 = \|x_0 + x_1 + x_2 + x_3\|^2 = \|x_0 + x_2 + x_3\|^2 = \|x_0 + x_3\|^2 = 0$

Thus $x_0 = x_1 = x_2 = x_3 = 0$and $X = 0$.

It is clear that $f(X,Y) = f(Y,X)$.

Now, let $A = a_0 + a_1I_1 + a_2I_2 + a_3I_3, B = b_0 + b_1I_1 + b_2I_2 + b_3I_3 \in R_3(I)$ and $Z = z_0 + z_1I_1 + z_2I_2 + z_3I_3 \in V_3(I)$, we have:

$AX + BY = (a_0x_0 + b_0y_0) + \big((a_0 + a_1 + a_2 + a_3)(x_0 + x_1 + x_2 + x_3) - (a_0 + a_2 + a_3)(x_0 + x_2 + x_3) + (b_0 + b_1 + b_2 + b_3)(y_0 + y_1 + y_2 + y_3) - (b_0 + b_2 + b_3)(y_0 + y_2 + y_3)\big)I_1 + \big((a_0 + a_2 + a_3)(x_0 + x_2 + x_3) - (a_0 + a_3)(x_0 + x_3) + (b_0 + b_2 + b_3)(y_0 + y_2 + y_3) - (b_0 + b_3)(y_0 + y_3)\big)I_2 + \big((a_0 + a_3)(x_0 + x_3) - a_0x_0 + (b_0 + b_3)(y_0 + y_3) - b_0y_0\big)I_3$

$$f(AX+BY,Z) = g(a_0x_0+b_0y_0,z_0)$$
$$+\Big(g\big((a_0+a_1+a_2+a_3)(x_0+x_1+x_2+x_3)$$
$$+(b_0+b_1+b_2+b_3)(y_0+y_1+y_2+y_3),z_0+z_1+z_2+z_3\big)$$
$$-g\big((a_0+a_2+a_3)(x_0+x_2+x_3)+(b_0+b_2+b_3)(y_0+y_2+y_3),z_0+z_2$$
$$+z_3\big)\Big)I_1$$
$$+\Big(g\big((a_0+a_2+a_3)(x_0+x_2+x_3)+(b_0+b_2+b_3)(y_0+y_2+y_3),z_0+z_2$$
$$+z_3\big)-g\big((a_0+a_3)(x_0+x_3)+(b_0+b_3)(y_0+y_3),z_0+z_3\big)\Big)I_2$$
$$+\Big(g\big((a_0+a_3)(x_0+x_3)+(b_0+b_3)(y_0+y_3),z_0+z_3\big)$$
$$-g(a_0x_0+b_0y_0,z_0)\Big)I_3$$
$$=(a_0+a_1I_1+a_2I_2+a_3I_3)f(X,Z)+(b_0+b_1I_1+b_2I_2+b_3I_3)f(X,Z)$$

Theorem.

Let $f: V_3(I)\times V_3(I)\to R_3(I)$ be a 3-refined neutrosophic real inner product, then $g: V\times V\to R$ such that:

$g(x,y)=f(x+0I_1+0I_2+0I_3,y+0I_1+0I_2+0I_3)$ is a classical inner product on V.

The proof is clear.

Definition.

Let $X=x_0+x_1I_1+x_2I_2+x_3I_3\in V_3(I)$ and $f: V_3(I)\times V_3(I)\to R_3(I)$ be a 3-refined neutrosophic real inner product, then:

1. If $Y=y_0+y_1I_1+y_2I_2+y_3I_3\in V_3(I)$, then $X\perp Y$ if and only if $f(X,Y)=0$.
2. $\|X\|^2=f(X,X)$.

Theorem.

Let f be a 3-refined neutrosophic real inner product on $V_3(I)$ and $X=x_0+x_1I_1+x_2I_2+x_3I_3, Y=y_0+y_1I_1+y_2I_2+y_3I_3\in V_3(I)$, then:

1). $X\perp Y$ if and only if $\begin{cases} x_0\perp y_0, x_0+x_3\perp y_0+y_3 \\ x_0+x_2+x_3\perp y_0+y_2+y_3 \\ x_0+x_1+x_2+x_3\perp y_0+y_1+y_2+y_3\end{cases}$

2). $\|X\|=\|x_0\|+(\|x_0+x_1+x_2+x_3\|-\|x_0+x_2+x_3\|)I_1+(\|x_0+x_2+x_3\|-\|x_0+x_3\|)I_2+(\|x_0+x_3\|-\|x_0\|)I_3$

Proof.

1). $X\perp Y\Leftrightarrow f(X,Y)=0\Leftrightarrow g(x_0,y_0)=g(x_0+x_3,y_0+y_3)=g(x_0+x_1+x_2+x_3,y_0+y_1+y_2+y_3)=g(x_0+x_2+x_3,y_0+y_2+y_3)=0$, hence the proof holds.

2). We put $T=\|x_0\|+(\|x_0+x_1+x_2+x_3\|-\|x_0+x_2+x_3\|)I_1+(\|x_0+x_2+x_3\|-\|x_0+x_3\|)I_2+(\|x_0+x_3\|-\|x_0\|)I_3$

We compute $T^2=\|x_0\|^2+(\|x_0+x_1+x_2+x_3\|^2-\|x_0+x_2+x_3\|^2)I_1+(\|x_0+x_2+x_3\|^2-\|x_0+x_3\|^2)I_2+(\|x_0+x_3\|^2-\|x_0\|^2)I_3=f(X,X)=\|X\|^2$, thus $T=\|X\|$.

Example.

Let $X=3+2I_1-I_2-I_3,\ x_0=3, x_1=2, x_2=-1, x_3=-1$, then:

$\|X\|=|3|+(|3|-|1|)I_1+(|1|-|2|)I_2+(|2|-|3|)I_3=3+2I_1-I_2-I_3$

Example.

Let $V=R^2,\ V_3(I)={R_3}^2(I)$, take $X=(1,1)+(2,1)I_1+(3,-1)I_2+(-1,4)I_3$.

$x_0=(1,1), \|x_0\|=\sqrt{2}$, $x_0+x_3=(0,5), \|x_0+x_3\|=5$, $x_0+x_2+x_3=(3,4), \|x_0+x_2+x_3\|=5,\ +x_1+x_2x_0+x_3=(5,5), \|x_0+x_1+x_2+x_3\|=5\sqrt{2}$

$\|X\|=\sqrt{2}+(5\sqrt{2}-5)I_1+(5-5)I_2+(5-\sqrt{2})I_3=\sqrt{2}+(5\sqrt{2}-5)I_1+(5-\sqrt{2})I_3$

Remark.

$\forall X,Y\in V_3(I)$, then: $\|X\|\geq 0,\ \|X+Y\|\leq\|X\|+\|Y\|$.

Theorem.

Let $X,Y\in V_3(I)$, then $|f(X,Y)|\leq\|X\|.\|Y\|$.

Proof.

$$\begin{aligned}f(X,Y)=g(x_0,y_0)&\\ &+\big(g(x_0+x_1+x_2+x_3,y_0+y_1+y_2+y_3)-g(x_0+x_2+x_3,y_0+y_2+y_3)\big)I_1\\ &+\big(g(x_0+x_2+x_3,y_0+y_2+y_3)-g(x_0+x_3,y_0+y_3)\big)I_2\\ &+\big(g(x_0+x_3,y_0+y_3)-g(x_0,y_0)\big)I_3\end{aligned}$$

According to Cauchy-Shwartz inequality on the space V, we have:

$|g(x_0,y_0)|\leq\|x_0\|.\|y_0\|,\ |g(x_0+x_2+x_3,y_0+y_2+y_3)|\leq\|x_0+x_2+x_3\|.\|y_0+y_2+y_3\|$

$|g(x_0+x_3,y_0+y_3)|\leq\|x_0+x_3\|.\|y_0+y_3\|$

$|g(x_0+x_1+x_2+x_3,y_0+y_1+y_2+y_3)|\leq\|x_0+x_1+x_2+x_3\|.\|y_0+y_1+y_2+y_3\|$

Thus, $|f(X,Y)|\leq\|X\|.\|Y\|$, according to the definition of partial order relation.

Example.

Take $V_3(I)={R_3}^2(I), X=(1,1)+(1,0)I_1+(0,1)I_2, Y=(2,0)+(0,3)I_1+(1,0)I_2+(0,1)I_3$

$x_0=(1,1), y_0=(2,0), g(x_0,y_0)=2, \|x_0\|=\sqrt{2}, \|y_0\|=2$, $x_0+x_3=(1,1), \|x_0+x_3\|=\sqrt{2}, y_0+y_3=(2,1), \|y_0+y_3\|=\sqrt{5}, g(x_0+x_3,y_0+y_3)=3$, $x_0+x_2+x_3=(1,2), \|x_0+x_2+x_3\|=\sqrt{3}, y_0+y_2+y_3=(3,1), \|y_0+y_2+y_3\|=\sqrt{10},$, $x_0+x_1+x_2+x_3=$

$(2,2), \|x_0 + x_1 + x_2 + x_3\| = 2\sqrt{2}, y_0 + y_1 + y_2 + y_3 = (3,4), \|y_0 + y_1 + y_2 + y_3\| = 5$,

$g(x_0 + x_2 + x_3, y_0 + y_2 + y_3) = 5,\ g(x_0 + x_1 + x_2 + x_3, y_0 + y_1 + y_2 + y_3) = 14.$

$f(X,Y) = 2 + (14-5)I_1 + (5-3)I_2 + (3-2)I_3 = 2 + 9I_1 + 2I_2 + I_3$

$|f(X,Y)| = 2 + 9I_1 + 2I_2 + I_3$

$\|X\| = \sqrt{2} + (2\sqrt{2} - \sqrt{5})I_1 + (\sqrt{5} - \sqrt{2})I_2 + (\sqrt{2} - \sqrt{2})I_3$

$= \sqrt{2} + (2\sqrt{2} - \sqrt{5})I_1 + (\sqrt{5} - \sqrt{2})I_2$

$\|X\| = 2 + (5 - \sqrt{10})I_1 + (\sqrt{10} - \sqrt{5})I_2 + (\sqrt{5} - \sqrt{2})I_3$

$\|X\|.\|Y\| = 2\sqrt{2} + (10\sqrt{2} - 5\sqrt{2})I_1 + (5\sqrt{2} - \sqrt{10})I_2 + (\sqrt{10} - 2\sqrt{2})I_3$

$= 2\sqrt{2} + 5\sqrt{2}I_1 + (5\sqrt{2} - \sqrt{10})I_2 + (\sqrt{10} - 2\sqrt{2})I_3$

On the other hand, we have:

$2 \le 2\sqrt{2}, 2 + 1 = 3 \le \sqrt{10}, 5 \le 5\sqrt{2}, 14 \le 10\sqrt{2}$, hence $|f(X,Y)| \le \|X\|.\|Y\|$

The Foundations 3-Refined Number Theory

Definition.

Let $Z_3(I) = \{a + bI_1 + cI_2 + dI_3; a,b,c,d \in Z\}$ be a set. It is called the ring of 3-refined neutrosophic integers if $I_i.I_j = I_{min(i,j)},\ I_i{}^2 = I_i; 1 \le i \le 3$.

It is a special case of the n-refined neutrosophic ring with $n = 3$.

Definition.

Let $X = x_0 + x_1I_1 + x_2I_2 + x_3I_3, Y = y_0 + y_1I_1 + y_2I_2 + y_3I_3, Z = z_0 + z_1I_1 + z_2I_2 + z_3I_3 \in Z_3(I)$, we define:

1). $X \setminus Y$ if there exists $T = t_0 + t_1I_1 + t_2I_2 + t_3I_3 \in Z_3(I)$ such that $T.X = Y$.

2). $X \equiv Y(mod\ Z)$ if and only if $Z \setminus X - Y$.

3). $X \ge Y$ if and only if $\begin{cases} x_0 \ge y_0 \\ x_0 + x_1 + x_2 + x_3 \ge y_0 + y_1 + y_2 + y_3 \\ x_0 + x_2 + x_3 \ge y_0 + y_2 + y_3 \\ x_0 + x_3 \ge y_0 + y_3 \end{cases}$

Theorem.

Let X, Y, Z be the previous 3-refined neutrosophic integers, then:

1. $X \setminus Y$ if and only if $x_0 \setminus y_0, x_0 + x_1 + x_2 + x_3 \setminus y_0 + y_1 + y_2 + y_3, x_0 + x_2 + x_3 \setminus y_0 + y_2 + y_3, x_0 + x_3 \setminus y_0 + y_3$.
2. If $X \setminus Y$, then $X \le Y$.

3. $X \equiv Y(mod\, Z)$ if and only if

$$\begin{cases} x_0 \equiv y_0(mod\, z_0) \\ x_0 + x_1 + x_2 + x_3 \equiv y_0 + y_1 + y_2 + y_3(mod\, z_0 + z_1 + z_2 + z_3) \\ x_0 + x_2 + x_3 \equiv y_0 + y_2 + y_3(mod\, z_0 + z_2 + z_3) \\ x_0 + x_3 \equiv y_0 + y_3(mod\, z_0 + z_3) \end{cases}$$

Proof.

1. Assume that $X \setminus Y$, this is true if and only if there exists $T = t_0 + t_1 I_1 + t_2 I_2 + t_3 I_3 \in Z_3(I)$ such that $Y = X$.T.

We have:

$$X.T = (x_0 + x_1 I_1 + x_2 I_2 + x_3 I_3)(t_0 + t_1 I_1 + t_2 I_2 + t_3 I_3)$$
$$= x_0 t_0 + (x_0 t_1 + x_1 t_0 + x_1 t_1 + x_1 t_2 + x_1 t_3 + x_2 t_1 + x_3 t_1) I_1$$
$$+ (x_0 t_2 + x_2 t_0 + x_2 t_2 + x_3 t_2) I_2 + (x_0 t_3 + x_3 t_0 + x_3 t_3) I_3$$
$$= y_0 + y_1 I_1 + y_2 I_2 + y_3 I_3$$

Thus:

$$\begin{cases} y_0 = x_0 t_0 \ldots (1) \\ y_1 = x_0 t_1 + x_1 t_0 + x_1 t_1 + x_1 t_2 + x_1 t_3 + x_2 t_1 + x_3 t_1 \ldots (2) \\ y_2 = x_0 t_2 + x_2 t_0 + x_2 t_2 + x_3 t_2 \ldots (3) \\ y_3 = x_0 t_3 + x_3 t_0 + x_3 t_3 \ldots (4) \end{cases}$$

We add (1) to (4), (1) to (2) to (4), (1) to (2) to (3) to (4).

$$\begin{cases} y_0 = x_0 t_0 \\ y_0 + y_3 = (x_0 + x_3)(t_0 + t_3) \\ y_0 + y_2 + y_3 = (x_0 + x_2 + x_3)(t_0 + t_2 + t_3) \\ y_0 + y_1 + y_2 + y_3 = (x_0 + x_1 + x_2 + x_3)(t_0 + t_1 + t_2 + t_3) \end{cases}$$

Thus, the proof of (1) is complete.

2. If $X \setminus Y$,then $x_0 \setminus y_0$, so $x_0 \leq y_0$.

Also:

$$\begin{cases} x_0 + x_3 \setminus y_0 + y_3, \text{so } x_0 + x_3 \leq y_0 + y_3 \\ x_0 + x_2 + x_3 \setminus y_0 + y_2 + y_3, \text{so } x_0 + x_2 + x_3 \leq y_0 + y_2 + y_3 \\ x_0 + x_1 + x_2 + x_3 \setminus y_0 + y_1 + y_2 + y_3, \text{so } x_0 + x_1 + x_2 + x_3 \leq y_0 + y_1 + y_2 + y_3 \end{cases}$$

Thus $X \leq Y$.

3. $X \equiv Y(mod\, Z)$ if and only if $Z \setminus X - Y$, thus:

$$\begin{cases} z_0 \setminus x_0 - y_0 \\ z_0 + z_3 \setminus (x_0 + x_3) - (y_0 + y_3) \\ z_0 + z_2 + z_3 \setminus (x_0 + x_2 + x_3) - (y_0 + y_2 + y_3) \\ z_0 + z_1 + z_2 + z_3 \setminus (x_0 + x_1 + x_2 + x_3) - (y_0 + y_1 + y_2 + y_3) \end{cases}$$

Thus $x_0 \equiv y_0(mod\, z_0), x_0 + x_3 \equiv y_0 + y_3(mod\, z_0 + z_3), x_0 + x_2 + x_3 \equiv y_0 + y_2 + y_3(mod\, z_0 + z_2 + z_3), x_0 + x_1 + x_2 + x_3 \equiv y_0 + y_1 + y_2 + y_3(mod\, z_0 + z_1 + z_2 + z_3)$.

Example.

Take $X = 3 + 2I_1 + I_2 - I_3, Y = 3 + 4I_1 + 2I_2 + I_3$, we have $X \setminus Y$ that is because:

$3 \setminus 3, 3 - 1 = 2 \setminus 3 + 1 = 4, 3 + 1 - 1 = 3 \setminus 3 + 2 + 1 = 6, 3 + 2 + 1 - 1 \setminus 3 + 4 + 2 + 1 = 10$.

Example.

Take $X = 7 + 3I_1 + I_2 + 5I_3, Y = 4 + I_1 + I_2 + I_3, Z = 3 + 2I_1 + I_3$, we have $7 \equiv 4(mod\ 3)$, $7 + 5 = 12 \equiv 4 + 1(mod\ 3 + 4)$, $7 + 1 + 5 = 13 \equiv 4 + 1 + 1(mod\ 3 + 0 + 4)$, $7 + 3 + 1 + 5 = 16 \equiv 4 + 1 + 1 + 1\ (mod 9)$ thus, $X \equiv Y(mod\ Z)$.

Theorem.

The relation $(\leq)$ is a partial order relation.

Proof.

$X \leq Y$ clearly.

If $X \leq Y$ and $Y \leq Z$, then $x_0 \leq y_0 \leq z_0$, $x_0 + x_3 \leq y_0 + y_3 \leq z_0 + z_3$, $x_0 + x_2 + x_3 \leq y_0 + y_2 + y_3 \leq z_0 + z_2 + z_3, x_0 + x_1 + x_2 + x_3 \leq y_0 + y_1 + y_2 + y_3 \leq z_0 + z_1 + z_2 + z_3$.

Thus $X \leq Z$.

If $X \leq Y$ and $Y \leq X$, then $x_0 = y_0$, $x_0 + x_3 = y_0 + y_3$, $x_0 + x_2 + x_3 = y_0 + y_2 + y_3$, $x_0 + x_1 + x_2 + x_3 = y_0 + y_1 + y_2 + y_3$, thus $X = Y$.

Theorem.

Let $X = x_0 + x_1I_1 + x_2I_2 + x_3I_3, Y = y_0 + y_1I_1 + y_2I_2 + y_3I_3, Z = z_0 + z_1I_1 + z_2I_2 + z_3I_3, T = t_0 + t_1I_1 + t_2I_2 + t_3I_3, S = s_0 + s_1I_1 + s_2I_2 + s_3I_3 \in Z_3(I)$, then:

1). If $X \equiv Y(mod\ Z)$, $T \equiv S(mod\ Z)$,then $X + T \equiv Y + S(mod\ Z)$ and $X - T \equiv Y - S(mod\ Z), X.T \equiv Y.S(mod\ Z)$.

2). $X^n = {x_0}^n + [(x_0 + x_1 + x_2 + x_3)^n - (x_0 + x_2 + x_3)^n]I_1 + [(x_0 + x_2 + x_3)^n - (x_0 + x_3)^n]I_2 + [(x_0 + x_3)^n - {x_0}^n]I_3\ ; n \in N$.

3). $X^n \equiv Y^n(mod\ Z^n)\ ; n \in N$.

Proof.

1). $X + T = (x_0 + t_0) + (x_1 + t_1)I_1 + (x_2 + t_2)I_2 + (x_3 + t_3)I_3$.

$Y + S = (y_0 + s_0) + (y_1 + s_1)I_1 + (y_2 + s_2)I_2 + (y_3 + s_3)I_3$.

Since $z_0 \setminus x_0 - y_0$, $z_0 \setminus t_0 - s_0$, then $z_0 \setminus (x_0 + t_0) - (y_0 + s_0)$ and $x_0 + t_0 \equiv y_0 + s_0(\ mod\ z_0)$.

$z_0 + z_3 \setminus (x_0 + x_3) - (y_0 + y_3), z_0 \setminus (t_0 + t_3) - (s_0 + s_3)$, then:

$z_0 + z_3 \setminus (x_0 + x_3 + t_0 + t_3) - (y_0 + y_3 + s_0 + s_3)$, thus:

$(x_0 + x_3) + (t_0 + t_3) \equiv (y_0 + y_3) + (s_0 + s_3)(\, mod\, z_0 + z_3)$

By a similar discussion, we get:

$$z_0 + z_2 + z_3 \setminus (x_0 + x_2 + x_3 + t_0 + t_2 + t_3) - (y_0 + y_2 + y_3 + s_0 + s_2 + s_3)$$

$$z_0 + z_1 + z_2 + z_3 \setminus (x_0 + x_1 + x_2 + x_3 + t_0 + t_1 + t_2 + t_3) - (y_0 + y_1 + y_2 + y_3 + s_0 + s_1 + s_2 + s_3)$$

So that $X + T \equiv Y + S(mod\, Z)$.

It is easy check that $X - T \equiv Y - S(mod\, Z), X.T \equiv Y.S(mod\, Z)$.

2). For $n = 1$ it is true clearly.

Assume that it is true for $n = k$, we must prove it for $n = k + 1$.

$$X^{k+1} = X.X^k = [x_0 + x_1I_1 + x_2I_2 + x_3I_3][{x_0}^n + [(x_0 + x_1 + x_2 + x_3)^n - (x_0 + x_2 + x_3)^n]I_1 + [(x_0 + x_2 + x_3)^n - (x_0 + x_3)^n]I_2 + [(x_0 + x_3)^n - {x_0}^n]I_3]$$

$$= {x_0}^{n+1} + [x_0(x_0 + x_1 + x_2 + x_3)^n - x_0(x_0 + x_2 + x_3)^n + x_1(x_0 + x_1 + x_2 + x_3)^n - x_1(x_0 + x_2 + x_3)^n + x_1{x_0}^n + x_1(x_0 + x_2 + x_3)^n - x_1(x_0 + x_3)^n + x_1(x_0 + x_3)^n - x_1{x_0}^n + x_2(x_0 + x_1 + x_2 + x_3)^n - x_2(x_0 + x_2 + x_3)^n + x_3(x_0 + x_1 + x_2 + x_3)^n - x_3(x_0 + x_2 + x_3)^n]I_1 + [x_0(x_0 + x_2 + x_3)^n - x_0(x_0 + x_3)^n + x_2{x_0}^n + x_2(x_0 + x_2 + x_3)^n - x_2(x_0 + x_3)^n + x_2(x_0 + x_3)^n - x_2{x_0}^n + x_3(x_0 + x_2 + x_3)^n - x_3(x_0 + x_3)^n]I_2 + [x_0(x_0 + x_3)^n - {x_0}^{n+1} + x_3{x_0}^n + x_3(x_0 + x_3)^n - x_3{x_0}^n]I_3$$

$$= {x_0}^{n+1} + [(x_0 + x_1 + x_2 + x_3)^{n+1} - (x_0 + x_2 + x_3)^{n+1}]I_1 + [(x_0 + x_2 + x_3)^{n+1} - (x_0 + x_3)^{n+1}]I_2 + [(x_0 + x_3)^{n+1} - {x_0}^{n+1}]I_3$$

This implies that is true by induction.

3). It holds directly from (1) and (2).

Example.

Take $X = 1 + 2I_1 - I_2 + I_3, n = 2$, then:

$X^2 = 1 + [(3)^2 - 1]I_1 + [1 - (2)^2]I_2 + [(2)^2 - 1]I_3 = 1 + 8I_1 - 3I_2 + 3I_3$

Theorem.

Let $X, Y \in Z_3(I)$, then $gcd(X, Y) = gcd(x_0, y_0) + [gcd(x_0 + x_1 + x_2 + x_3, y_0 + y_1 + y_2 + y_3) - gcd(x_0 + x_2 + x_3, y_0 + y_2 + y_3)]I_1 + [gcd(x_0 + x_2 + x_3, y_0 + y_2 + y_3) - gcd(x_0 + x_3, y_0 + y_3)]I_2 + [gcd(x_0 + x_3, y_0 + y_3) - gcd(x_0, y_0)]I_3$

Example.

Take $X = 4 + 3I_1 + 5I_2 - I_3, Y = 7 + I_1 + I_2 + 3I_3$.

$gcd(4,7) = 1, gcd(11,13) = 1, gcd(8,11) = 1, gcd(3,10) = 1$, thus $gcd(X,Y) = 1$.

Remark.

X, Y are called coprime (relatively prime) if and only if $gcd(X,Y) = 1$, which is equivalent to:

$gcd(x_0, y_0) = gcd(x_0 + x_3, y_0 + y_3) = gcd(x_0 + x_2 + x_3, y_0 + y_2 + y_3) = gcd(x_0 + x_1 + x_2 + x_3, y_0 + y_1 + y_2 + y_3) = 1$.

Definition.

Let $X = x_0 + x_1 I_1 + x_2 I_2 + x_3 I_3 \in Z_3(I)$, with $X > 0$, we define:

$$\varphi_s(X) = \varphi(x_0) + [\varphi(x_0 + x_1 + x_2 + x_3) - \varphi(x_0 + x_2 + x_3)]I_1 + [\varphi(x_0 + x_2 + x_3) - \varphi(x_0 + x_3)]I_2 + [\varphi(x_0 + x_3) - \varphi(x_0)]I_3$$

where φ is the ordinary Euler's function, φ_s is called the special 3-refined neutrosophic Euler's function.

Example.

Take $X = 3 + I_1 + I_2 + I_3 > 0\,; x_0 = 3, x_1 = 1, x_2 = x_3 = 1$.

$\varphi(x_0) = 2, \varphi(x_0 + x_1 + x_2 + x_3) = 2, \varphi(x_0 + x_2 + x_3) = 4, \varphi(x_0 + x_3) = 2$.

Thus

$\varphi_s(X) = 2 + [2 - 4]I_1 + [4 - 2]I_2 + [2 - 2]I_3 = 2 - 2I_1 + 2I_2 + 0I_3$.

It is clear that $\varphi_s(X) > 0;\ \forall X > 0$.

Theorem.

Let $A = a_0 + a_1 I_1 + a_2 I_2 + a_3 I_3, M = m_0 + m_1 I_1 + m_2 I_2 + m_3 I_3 \in Z_3(I)$, such that:

$A > 0, M > 0$ and $gcd(A, M) = 1$, then:

1). $A^{\varphi_s(M)} \equiv 1\ (mod\ M)$.

2). $A^{-1}\ (mod\ M) \equiv {a_0}^{-1}\ (mod m_0) + [(a_0 + a_1 + a_2 + a_3)^{-1}(mod\ m_0 + m_1 + m_2 + m_3) - (a_0 + a_2 + a_3)^{-1}(mod\ m_0 + m_2 + m_3)]I_1 + [(a_0 + a_2 + a_3)^{-1}(mod\ m_0 + m_2 + m_3) - (a_0 + a_3)^{-1}(mod\ m_0 + m_3)]I_2 + [(a_0 + a_3)^{-1}(mod\ m_0 + m_3) - {a_0}^{-1}\ (mod m_0)]I_3$.

Proof.

1). $A^{\varphi_s(M)} = {a_0}^{\varphi(m_0)} + \left[(a_0 + a_1 + a_2 + a_3)^{\varphi(m_0 + m_1 + m_2 + m_3)} - (a_0 + a_2 + a_3)^{\varphi(m_0 + m_2 + m_3)}\right]I_1 + \left[(a_0 + a_2 + a_3)^{\varphi(m_0 + m_2 + m_3)} - (a_0 + a_3)^{\varphi(m_0 + m_3)}\right]I_2 + \left[(a_0 + a_3)^{\varphi(m_0 + m_3)} - {a_0}^{\varphi(m_0)}\right]I_3 \equiv 1\ (mod\ M)$.

2). It holds directly by computing the product AA^{-1}.

3-refined Diophantine equations:

Definition.

Let $A = a_0 + a_1I_1 + a_2I_2 + a_3I_3, B = b_0 + b_1I_1 + b_2I_2 + b_3I_3, C = c_0 + c_1I_1 + c_2I_2 + c_3I_3, X = x_0 + x_1I_1 + x_2I_2 + x_3I_3, Y = y_0 + y_1I_1 + y_2I_2 + y_3I_3$, where $a_i, b_i, c_i, x_i, y_i \in Z_3(I)$.

We define the 3-refined neutrosophic linear Diophantine equation with two variables as follows:

$AX + BY = C$.

Example.

Consider the following 3-refined neutrosophic linear Diophantine equation:

$(3 + 2I_1 + I_2 + I_3)X + (2 + 4I_2)Y = 3 + 9I_1 - 7I_3$

Theorem.

Let $AX + BY = C$ be a 3-refined neutrosophic linear Diophantine equation, then it is equivalent to:

$$\begin{cases} a_0x_0 + b_0y_0 = c_0 \\ (a_0 + a_3)(x_0 + x_3) + (b_0 + b_3)(y_0 + y_3) = c_0 + c_3 \\ (a_0 + a_2 + a_3)(x_0 + x_2 + x_3) + (b_0 + b_2 + b_3)(y_0 + y_2 + y_3) = c_0 + c_2 + c_3 \\ (a_0 + a_1 + a_2 + a_3)(x_0 + x_1 + x_2 + x_3) + (b_0 + b_1 + b_2 + b_3)(y_0 + y_1 + y_2 + y_3) = c_0 + c_1 + c_2 + c_3 \end{cases}$$

Proof.

We compute $AX = a_0x_0 + [(a_0 + a_1 + a_2 + a_3)(x_0 + x_1 + x_2 + x_3) - (a_0 + a_2 + a_3)(x_0 + x_2 + x_3)]I_1 + [(a_0 + a_2 + a_3)(x_0 + x_2 + x_3) - (a_0 + a_3)(x_0 + x_3)]I_2 + [(a_0 + a_3)(x_0 + x_3) - a_0x_0]I_3$

On the other hand, we have:

$$\begin{aligned} BY = b_0y_0 &+ [(b_0 + b_1 + b_2 + b_3)(y_0 + y_1 + y_2 + y_3) - (b_0 + b_2 + b_3)(y_0 + y_2 + y_3)]I_1 \\ &+ [(b_0 + b_2 + b_3)(y_0 + y_2 + y_3) - (b_0 + b_3)(y_0 + y_3)]I_2 \\ &+ [(b_0 + b_3)(y_0 + y_3) - b_0y_0]I_3 \end{aligned}$$

The equation $AX + BY = C$ equivalents:

$$\begin{cases} a_0x_0 + b_0y_0 = c_0 \\ (a_0 + a_3)(x_0 + x_3) + (b_0 + b_3)(y_0 + y_3) = c_0 + c_3 \\ (a_0 + a_2 + a_3)(x_0 + x_2 + x_3) + (b_0 + b_2 + b_3)(y_0 + y_2 + y_3) = c_0 + c_2 + c_3 \\ (a_0 + a_1 + a_2 + a_3)(x_0 + x_1 + x_2 + x_3) + (b_0 + b_1 + b_2 + b_3)(y_0 + y_1 + y_2 + y_3) = c_0 + c_1 + c_2 + c_3 \end{cases}$$

Example.

Find a solution of the equation:

$(3 + 2I_1 + I_2 + I_3)X + (2 + 4I_2)Y = 3 + 9I_1 - 7I_3$

We have $a_0 = 3, a_1 = 2, a_2 = 1, a_3 = 1, b_0 = 2, b_1 = 0, b_2 = 4, b_3 = 0, c_0 = 3, c_1 = 9, c_2 = 0, c_3 = -7$

The equivalent system is:

$$\begin{cases} 3x_0 + 2y_0 = 3 \dots (1) \\ 4(x_0 + x_3) + 2(y_0 + y_3) = -4 \dots (2) \\ 5(x_0 + x_2 + x_3) + 7(y_0 + y_2 + y_3) = -4 \dots (3) \\ 7(x_0 + x_1 + x_2 + x_3) + 6(y_0 + y_1 + y_2 + y_3) = 5 \dots (4) \end{cases}$$

The equation (1) has a solution $x_0 = 1, y_0 = 0$.

The equation (2) has a solution $x_0 + x_3 = -1, y_0 + y_3 = 0$, thus $x_3 = -2, y_3 = 0$.

The equation (3) has a solution $x_0 + x_2 + x_3 = 9, y_0 + y_2 + y_3 = -7$, thus $x_2 = 10, y_2 = -7$.

The equation (4) has a solution $x_0 + x_1 + x_2 + x_3 = 5, y_0 + y_1 + y_2 + y_3 = 5$, thus $x_1 = -4, y_1 = 12$.

This means that $X = 1 - 4I_1 + 10I_2 - 2I_3, Y = 12I_1 - 7I_2$.

Future research directions and suggestions

3-refined neutrosophic number as generalizations of classical real numbers and integers, may have a great impact on many areas of scientific knowledge.

In the following, we suggest many possible applications of 3-refined neutrosophic real numbers.

1-) How can we build a crypto-system from 3-refined neutrosophic integers which generalize RSA algorithm. [19]

2-) How can we build a crypto-system from 3-refined neutrosophic integers which generalize El-Gamal algorithm. [20-21]

3-) How can we solve 3-refined neutrosophic differential equations, and integral equations.

4-) How can we define Hillbert and Banach 3-refined neutrosophic spaces, and do classical functional inequalities still true in this case.

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