Smarandache Multi-Space Theory(III)

-Map geometries and pseudo-plane geometries

Linfan Mao

Academy of Mathematics and System Sciences
Chinese Academy of Sciences, Beijing 100080
maolinfan@163.com

Abstract. A Smarandache multi-space is a union of n different spaces equipped with some different structures for an integer $n \geq 2$, which can be both used for discrete or connected spaces, particularly for geometries and spacetimes in theoretical physics. This monograph concentrates on characterizing various multi-spaces including three parts altogether. The first part is on *algebraic multi-spaces with structures*, such as those of multi-groups, multi-rings, multi-vector spaces, multi-metric spaces, multi-operation systems and multi-manifolds, also multi-voltage graphs, multi-embedding of a graph in an n-manifold,$\cdots$, etc.. The second discusses *Smarandache geometries*, including those of map geometries, planar map geometries and pseudo-plane geometries, in which the *Finsler geometry*, particularly the *Riemann geometry* appears as a special case of these Smarandache geometries. The third part of this book considers the *applications of multi-spaces to theoretical physics*, including the relativity theory, the M-theory and the cosmology. Multi-space models for p-branes and cosmos are constructed and some questions in cosmology are clarified by multi-spaces. The first two parts are relative independence for reading and in each part open problems are included for further research of interested readers.

Key words: Smarandache geometries, map geometries, planar map geometries, pseudo-plane geometries, Finsler geometries.

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3. Map geometries

As a kind of multi-metric spaces, Smarandache geometries were introduced by Smarandache in [86] and investigated by many mathematicians. These geometries are related with the *Euclid geometry*, the *Lobachevshy-Bolyai-Gauss geometry* and the *Riemann geometry*, also related with *relativity theory* and *parallel universes* (see [56], [35] – [36], [38] and [77] – [78] for details). As a generalization of Smarandache manifolds of dimension 2, *Map geometries* were introduced in [55], [57] and [62], which can be also seen as a realization of Smarandache geometries on surfaces or Smarandache geometries on maps.

§3.1 Smarandache Geometries

3.1.1. What are lost in classical mathematics?

As we known, mathematics is a powerful tool of sciences for its unity and neatness, without any shade of mankind. On the other hand, it is also a kind of aesthetics deep down in one's mind. There is a famous proverb says that *only the beautiful things can be handed down to today*, which is also true for the mathematics.

Here, the terms *unity* and *neatness* are relative and local, maybe also have various conditions. For obtaining a good result, many unimportant matters are abandoned in the research process. Whether are those matters still unimportant in another time? It is not true. That is why we need to think a queer question: *what are lost in the classical mathematics?*

For example, a compact surface is topological equivalent to a polygon with even number of edges by identifying each pairs of edges along its a given direction ([68], [92]). If label each pair of edges by a letter $e, e \in \mathcal{E}$, a surface S is also identified to a cyclic permutation such that each edge $e, e \in \mathcal{E}$ just appears two times in S, one is e and another is e^{-1} (orientable) or e (non-orientable). Let $a, b, c, \cdots$ denote letters in $\mathcal{E}$ and $A, B, C, \cdots$ the sections of successive letters in a linear order on a surface S (or a string of letters on S). Then, an orientable surface can be represented by

$$S = (\cdots, A, a, B, a^{-1}, C, \cdots),$$

where$a \in \mathcal{E}$ and A, B, C denote strings of letter. Three elementary transformations are defined as follows:

(O_1) $(A, a, a^{-1}, B) \Leftrightarrow (A, B)$;

(O_2) (i) $(A, a, b, B, b^{-1}, a^{-1}) \Leftrightarrow (A, c, B, c^{-1})$;
(ii) $(A, a, b, B, a, b) \Leftrightarrow (A, c, B, c)$;

(O_3) $\quad (i) \quad (A, a, B, C, a^{-1}, D) \Leftrightarrow (B, a, A, D, a^{-1}, C);$
$\quad\quad (ii) \quad (A, a, B, C, a, D) \Leftrightarrow (B, a, A, C^{-1}, a, D^{-1}).$

If a surface S_0 can be obtained by these elementary transformations O_1-O_3 from a surface S, it is said that S is *elementary equivalent* with S_0, denoted by $S \sim_{El} S_0$.

We have known the following formulae from [43]:

$(i) \quad (A, a, B, b, C, a^{-1}, D, b^{-1}, E) \sim_{El} (A, D, C, B, E, a, b, a^{-1}, b^{-1});$
$(ii) \quad (A, c, B, c) \sim_{El} (A, B^{-1}, C, c, c);$
$(iii) \quad (A, c, c, a, b, a^{-1}, b^{-1}) \sim_{El} (A, c, c, a, a, b, b).$

Then we can get the classification theorem of compact surfaces as follows([68]):

Any compact surface is homeomorphic to one of the following standard surfaces:
(P_0) The sphere: aa^{-1}*;*
(P_n) The connected sum of $n, n \geq 1$*, tori:*

$$a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1};$$

(Q_n) The connected sum of $n, n \geq 1$*, projective planes:*

$$a_1 a_1 a_2 a_2 \cdots a_n a_n.$$

As we have discussed in Chapter 2, a combinatorial map is just a kind of decomposition of a surface. Notice that all the standard surfaces are one face map underlying an one vertex graph, i.e., a bouquet B_n with $n \geq 1$. By a combinatorial view, *a combinatorial map is nothing but a surface*. This assertion is needed clarifying. For example, let us see the left graph Π_4 in Fig. 3.1, which is a tetrahedron.

Fig.3.1

Whether can we say $\prod_4$ is a sphere? Certainly NOT. Since any point u on a sphere has a neighborhood $N(u)$ homeomorphic to an open disc, thereby all angles incident with the point 1 must be 120° degree on a sphere. But in Π_4, those are only 60° degree. For making them same in a topological sense, i.e., homeomorphism, we must blow up the Π_4 and make it become a sphere. This physical processing is shown in the Fig.3.1. Whence, for getting the classification theorem of compact surfaces, we lose the *angle,area, volume,distance,curvature*,$\cdots$, etc, which are also lost in combinatorial maps.

By a geometrical view, *Klein Erlanger Program* says that *any geometry is nothing but find invariants under a transformation group of this geometry*. This is essentially the group action idea and widely used in mathematics today. Surveying topics appearing in publications for combinatorial maps, we know the following problems are applications of *Klein Erlanger Program*:

(i) *to determine isomorphism maps or rooted maps;*
(ii) *to determine equivalent embeddings of a graph;*
(iii) *to determine an embedding whether exists or not;*
(iv) *to enumerate maps or rooted maps on a surface;*
(v) *to enumerate embeddings of a graph on a surface;*
(vi) $\cdots$, etc.

All the problems are extensively investigated by researches in the last century and papers related those problems are still frequently appearing in journals today. Then,

what are their importance to classical mathematics?

and

what are their contributions to sciences?

Today, we have found that combinatorial maps can contribute an underlying frame for applying mathematics to sciences, i.e., through by map geometries or by graphs in spaces.

3.1.2. **Smarandache geometries**

Smarandache geometries were proposed by Smarandache in [86] which are generalization of classical geometries, i.e., these *Euclid, Lobachevshy-Bolyai-Gauss* and *Riemann geometries* may be united altogether in a same space, by some Smarandache geometries. These last geometries can be either partially Euclidean and partially Non-Euclidean, or Non-Euclidean. Smarandache geometries are also connected with the *Relativity Theory* because they include Riemann geometry in a subspace and with the *Parallel Universes* because they combine separate spaces into one space too. For a detail illustration, we need to consider classical geometries first.

As we known, the axiom system of an *Euclid geometry* is in the following:

(A1) *there is a straight line between any two points.*
(A2) *a finite straight line can produce a infinite straight line continuously.*
(A3) *any point and a distance can describe a circle.*
(A4) *all right angles are equal to one another.*
(A5) *if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, then the two straight lines, if produced*

indefinitely, meet on that side on which are the angles less than the two right angles.

The axiom (A5) can be also replaced by:

(A5') *given a line and a point exterior this line, there is one line parallel to this line.*

The *Lobachevshy-Bolyai-Gauss geometry*, also called *hyperbolic geometry*, is a geometry with axioms $(A1) - (A4)$ and the following axiom $(L5)$:

(L5) *there are infinitely many lines parallel to a given line passing through an exterior point.*

The *Riemann geometry*, also called *elliptic geometry*, is a geometry with axioms $(A1) - (A4)$ and the following axiom $(R5)$:

there is no parallel to a given line passing through an exterior point.

By a thought of anti-mathematics: *not in a nihilistic way, but in a positive one, i.e., banish the old concepts by some new ones: their opposites*, Smarandache introduced thse *paradoxist geometry, non-geometry, counter-projective geometry* and *anti-geometry* in [86] by contradicts axioms $(A1) - (A5)$ in an Euclid geometry.

Paradoxist geometry

In this geometry, its axioms consist of $(A1) - (A4)$ and one of the following as the axiom $(P5)$:

(i) *there are at least a straight line and a point exterior to it in this space for which any line that passes through the point intersect the initial line.*

(ii) *there are at least a straight line and a point exterior to it in this space for which only one line passes through the point and does not intersect the initial line.*

(iii) *there are at least a straight line and a point exterior to it in this space for which only a finite number of lines* $l_1, l_2, \cdots, l_k, k \geq 2$ *pass through the point and do not intersect the initial line.*

(iv) *there are at least a straight line and a point exterior to it in this space for which an infinite number of lines pass through the point (but not all of them) and do not intersect the initial line.*

(v) *there are at least a straight line and a point exterior to it in this space for which any line that passes through the point and does not intersect the initial line.*

Non-Geometry

The non-geometry is a geometry by denial some axioms of $(A1) - (A5)$, such as:

$(A1^-)$ *It is not always possible to draw a line from an arbitrary point to another arbitrary point.*

$(A2^-)$ *It is not always possible to extend by continuity a finite line to an infinite line.*

$(A3^-)$ *It is not always possible to draw a circle from an arbitrary point and of an arbitrary interval.*

$(A4^-)$ *not all the right angles are congruent.*

$(A5^-)$ *if a line, cutting two other lines, forms the interior angles of the same side of it strictly less than two right angle, then not always the two lines extended towards infinite cut each other in the side where the angles are strictly less than two right angle.*

Counter-Projective geometry

Denoted by P the point set, L the line set and R a relation included in $P \times L$. A counter-projective geometry is a geometry with these counter-axioms $(C_1) - (C_3)$:

$(C1)$ *there exist: either at least two lines, or no line, that contains two given distinct points.*

$(C2)$ *let p_1, p_2, p_3 be three non-collinear points, and q_1, q_2 two distinct points. Suppose that $\{p_1.q_1, p_3\}$ and $\{p_2, q_2, p_3\}$ are collinear triples. Then the line containing p_1, p_2 and the line containing q_1, q_2 do not intersect.*

$(C3)$ *every line contains at most two distinct points.*

Anti-Geometry

A geometry by denial some axioms of the Hilbert's 21 axioms of Euclidean geometry. As shown in [38], there are at least $2^{21} - 1$ anti-geometries.

In general, Smarandache geometries are defined as follows.

Definition 3.1.1 *An axiom is said to be Smarandachely denied if the axiom behaves in at least two different ways within the same space, i.e., validated and invalided, or only invalided but in multiple distinct ways.*

*A Smarandache geometry is a geometry which has at least one Smarandachely denied axiom(*1969*).*

In a Smarandache geometries, points, lines, planes, spaces, triangles, $\cdots$, etc are called s-points, s-lines, s-planes, s-spaces, s-triangles, $\cdots$, respectively in order to distinguish them from classical geometries. An example of Smarandache geometries in the classical geometrical sense is in the following.

Example 3.1.1 Let us consider an Euclidean plane $\mathbf{R}^2$ and three non-collinear points A, B and C. Define s-points as all usual Euclidean points on $\mathbf{R}^2$ and s-lines any Euclidean line that passes through one and only one of points A, B and C. Then this geometry is a Smarandache geometry because two axioms are Smarandachely denied comparing with an Euclid geometry:

(i) The axiom (A5) that through a point exterior to a given line there is only one parallel passing through it is now replaced by two statements: *one parallel*, and *no parallel*. Let L be an s-line passes through C and is parallel in the euclidean sense to AB. Notice that through any s-point not lying on AB there is one s-line parallel to L and through any other s-point lying on AB there is no s-lines parallel to L such as those shown in Fig.3.2(a).

Fig.3.2

(ii) The axiom that through any two distinct points there exist one line passing through them is now replaced by; *one s-line*, and *no s-line*. Notice that through any two distinct s-points D, E collinear with one of A, B and C, there is one s-line passing through them and through any two distinct s-points F, G lying on AB or non-collinear with one of A, B and C, there is no s-line passing through them such as those shown in Fig.3.2(b).

3.1.3. **Smarandache manifolds**

Generally, a *Smarandache manifold* is an n-dimensional manifold that support a Smarandache geometry. For $n = 2$, a nice model for Smarandache geometries called *s-manifolds* was found by Iseri in [35][36], which is defined as follows:

An s-manifold is any collection $\mathcal{C}(T, n)$ *of these equilateral triangular disks* $T_i, 1 \leq i \leq n$ *satisfying the following conditions:*

(i) *each edge* e *is the identification of at most two edges* e_i, e_j *in two distinct triangular disks* $T_i, T_j, 1 \leq i, j \leq n$ *and* $i \neq j$*;*

(ii) *each vertex* v *is the identification of one vertex in each of five, six or seven distinct triangular disks.*

The vertices are classified by the number of the disks around them. A vertex around five, six or seven triangular disks is called an *elliptic vertex*, an *euclidean vertex* or a *hyperbolic vertex*, respectively.

In a plane, an elliptic vertex O, an euclidean vertex P and a hyperbolic vertex Q and an s-line L_1, L_2 or L_3 passes through points O, P or Q are shown in Fig.3.3(a), (b), (c), respectively.

Fig.3.3

Smarandache paradoxist geometries and non-geometries can be realized by s-manifolds, but other Smarandache geometries can be only partly realized by this kind of manifolds. Readers are inferred to Iseri's book [35] for those geometries.

An s-manifold is called closed if each edge is shared exactly by two triangular disks. An elementary classification for closed s-manifolds by triangulation were introduced in [56]. They are classified into 7 classes. Each of those classes is defined in the following.

Classical Type:

(1) $\Delta_1 = \{5 - \textit{regular triangular maps}\}$ (*elliptic*);
(2) $\Delta_2 = \{6 - \textit{regular triangular maps}\}$(*euclidean*);
(3) $\Delta_3 = \{7 - \textit{regular triangular maps}\}$(*hyperbolic*).

Smarandache Type:

(4) $\Delta_4 = \{\textit{triangular maps with vertex valency } 5 \textit{ and } 6\}$ (*euclid-elliptic*);
(5) $\Delta_5 = \{\textit{triangular maps with vertex valency } 5 \textit{ and } 7\}$ (*elliptic-hyperbolic*);
(6) $\Delta_6 = \{\textit{triangular maps with vertex valency } 6 \textit{ and } 7\}$ (*euclid-hyperbolic*);
(7) $\Delta_7 = \{\textit{triangular maps with vertex valency } 5, 6 \textit{ and } 7\}$ (*mixed*).

It is proved in [56] that $|\Delta_1| = 2$, $|\Delta_5| \geq 2$ and $|\Delta_i|, i = 2, 3, 4, 6, 7$ are infinite. Iseri proposed a question in [35]: *Do the other closed* 2*-manifolds correspond to s-manifolds with only hyperbolic vertices?* Since there are infinite Hurwitz maps, i.e., $|\Delta_3|$ is infinite, the answer is affirmative.

§3.2 Map Geometries without Boundary

A combinatorial map can be also used to construct new models for Smarandache geometries. By a geometrical view, these models are generalizations of Isier's model for Smarandache geometries. For a given map on a locally orientable surface, map geometries without boundary are defined in the following definition.

Definition 3.2.1 *For a combinatorial map M with each vertex valency≥ 3, associates a real number $\mu(u), 0 < \mu(u) < \frac{4\pi}{\rho_M(u)}$, to each vertex $u, u \in V(M)$. Call (M, μ) a map geometry without boundary, $\mu(u)$ an angle factor of the vertex u and* *orientablle or non-orientable if M is orientable or not.*

The realization for vertices $u, v, w \in V(M)$ in a space $\mathbf{R}^3$ is shown in Fig.3.4, where $\rho_M(u)\mu(u) < 2\pi$ for the vertex u, $\rho_M(v)\mu(v) = 2\pi$ for the vertex v and $\rho_M(w)\mu(w) > 2\pi$ for the vertex w, respectively.

$\rho_M(u)\mu(u) < 2\pi$ $\quad$ $\rho_M(u)\mu(u) = 2\pi$ $\quad$ $\rho_M(u)\mu(u) > 2\pi$

Fig.3.4

As we have pointed out in Section 3.1, this kind of realization is not a surface, but it is homeomorphic to a locally orientable surface by a view of topological equivalence. Similar to s-manifolds, we also classify points in a map geometry (M, μ) without boundary into *elliptic points*, *euclidean points* and *hyperbolic points*, defined in the next definition.

Definition 3.2.2 *A point u in a map geometry (M, μ) is said to be elliptic, euclidean or hyperbolic if $\rho_M(u)\mu(u) < 2\pi$, $\rho_M(u)\mu(u) = 2\pi$ or $\rho_M(u)\mu(u) > 2\pi$.*

Then we get the following results.

Theorem 3.2.1 *Let M be a map with $\forall u \in V(M), \rho_M(u) \geq 3$. Then for $\forall u \in V(M)$, there is a map geometry (M, μ) without boundary such that u is elliptic, euclidean or hyperbolic.*

Proof Since $\rho_M(u) \geq 3$, we can choose an angle factor $\mu(u)$ such that $\mu(u)\rho_M(u) < 2\pi$, $\mu(u)\rho_M(u) = 2\pi$ or $\mu(u)\rho_M(u) > 2\pi$. Notice that

$$0 < \frac{2\pi}{\rho_M(u)} < \frac{4\pi}{\rho_M(u)}.$$

Thereby we can always choose $\mu(u)$ satisfying that $0 < \mu(u) < \frac{4\pi}{\rho_M(u)}$. ♮

Theorem 3.2.2 *Let M be a map of order≥ 3 and $\forall u \in V(M), \rho_M(u) \geq 3$. Then there exists a map geometry (M, μ) without boundary in which elliptic, euclidean and hyperbolic points appear simultaneously.*

Proof According to Theorem 3.2.1, we can always choose an angle factor μ such that a vertex $u, u \in V(M)$ to be elliptic, or euclidean, or hyperbolic. Since $|V(M)| \geq 3$, we can even choose the angle factor μ such that any two different vertices $v, w \in V(M) \setminus \{u\}$ to be elliptic, or euclidean, or hyperbolic as we wish. Then the map geometry (M, μ) makes the assertion hold. ♮

A *geodesic* in a manifold is a curve as straight as possible. Applying conceptions such as angles and straight lines in an Euclid geometry, we define m-lines and m-points in a map geometry in the next definition.

Definition 3.2.3 *Let (M, μ) be a map geometry without boundary and let $S(M)$ be the locally orientable surface represented by a plane polygon on which M is embedded. A point P on $S(M)$ is called an m-point. A line L on $S(M)$ is called an m-line if it is straight in each face of M and each angle on L has measure $\frac{\rho_M(v)\mu(v)}{2}$ when it passes through a vertex v on M.*

Two examples for m-lines on the torus are shown in the Fig.3.5(a) and (b), where $M = M(B_2)$, $\mu(u) = \frac{\pi}{2}$ for the vertex u in (a) and

$$\mu(u) = \frac{135 - \arctan(2)}{360}\pi$$

for the vertex u in (b), i.e., u is euclidean in (a) but elliptic in (b). Notice that in (b), the m-line L_2 is self-intersected.

Fig.3.5

If an m-line passes through an elliptic point or a hyperbolic point u, it must has an angle $\frac{\mu(u)\rho_M(u)}{2}$ with the entering line, not 180° which are explained in Fig.3.6.

a $= \frac{\mu(u)\rho_M(u)}{2} < \pi$, a $= \frac{\mu(u)\rho_M(u)}{2} > \pi$

Fig.3.6

In an Euclid geometry, a right angle is an angle with measure $\frac{\pi}{2}$, half of a straight angle and parallel lines are straight lines never intersecting. They are very important research objects. Many theorems characterize properties of them in classical Euclid geometry.

In a map geometry, we can also define a straight angle, a right angle and parallel m-lines by Definition 3.2.2. Now a *straight angle* is an angle with measure π for points not being vertices of M and $\frac{\rho_M(u)\mu(u)}{2}$ for $\forall u \in V(M)$. A *right angle* is an angle with a half measure of a straight angle. Two m-lines are said *parallel* if they are never intersecting. The following result asserts that map geometries without boundary are paradoxist geometries.

Theorem 3.2.3 *For a map M on a locally orientable surface with $|M| \geq 3$ and $\rho_M(u) \geq 3$ for $\forall u \in V(M)$, there exists an angle factor μ such that (M, μ) is a Smarandache geometry by denial the axiom (A5) with these axioms (A5),(L5) and (R5).*

Proof According to Theorem 3.2.1, we know that there exists an angle factor μ such that there are elliptic vertices, euclidean vertices and hyperbolic vertices in (M, μ) simultaneously. The proof is divided into three cases according to M is planar, orientable or non-orientable. Not loss of generality, we assume that an angle is measured along a clockwise direction, i.e., as these cases in Fig.3.6 for an m-line passing through an elliptic point or a hyperbolic point.

Case 1. M **is a planar map**

Notice that for a given line L not intersection with the map M and a point u in (M, μ), if u is an euclidean point, then there is one and only one line passing through u not intersecting with L, and if u is an elliptic point, then there are infinite lines passing through u not intersecting with L, but if u is a hyperbolic point, then each line passing through u will intersect with L. See also in Fig.3.7, where the planar graph is a complete graph K_4 and points $1, 2$ are elliptic, the point 3 is euclidean but the point 4 is hyperbolic. Then all m-lines in the field A do not intersect with L and each m-line passing through the point 4 will intersect with the line L. Therefore, (M, μ) is a Smarandache geometry by denial the axiom (A5) with these axioms (A5), (L5) and (R5).

Fig.3.7

Case 2. M **is an orientable map**

According to the classification theorem of compact surfaces, We only need to prove this result for a torus. Notice that m-lines on a torus has the following property (see [82] for details):

If the slope ς of an m-line L is a rational number, then L is a closed line on the torus. Otherwise, L is infinite, and moreover L passes arbitrarily close to every point on the torus.

Whence, if L_1 is an m-line on a torus with an irrational slope not passing through an elliptic or a hyperbolic point, then for any point u exterior to L_1, if u is an euclidean point, then there is only one m-line passing through u not intersecting with L_1, and if u is elliptic or hyperbolic, any m-line passing through u will intersect with L_1.

Now let L_2 be an m-line on the torus with an rational slope not passing through an elliptic or a hyperbolic point, such as the m-line L_2 in Fig.3.8, v is an euclidean point. If u is an euclidean point, then each m-line L passing through u with rational slope in the area A will not intersect with L_2 but each m-line passing through u with irrational slope in the area A will intersect with L_2.

Fig.3.8

Therefore, (M, μ) is a Smarandache geometry by denial the axiom (A5) with axioms (A5),(L5) and (R5) in this case.

Case 3. M **is a non-orientable map**

Similar to the Case 2, we only need to prove this result for the projective plane. An m-line in a projective plane is shown in Fig.3.9(a), (b) or (c), in where case (a) is an m-line passing through an euclidean point, (b) passing through an elliptic point and (c) passing through an hyperbolic point.

Fig.3.9

Now let L be an m-line passing through the center in the circle. Then if u is an euclidean point, there is only one m-line passing through u such as the case (a) in Fig.3.10. If v is an elliptic point then there is an m-line passing through it and intersecting with L such as the case (b) in Fig.3.10. We assume the point 1 is a point such that there exists an m-line passing through 1 and 0, then any m-line in the shade of Fig.3.10(b) passing through v will intersect with L.

Fig.3.10

If w is an euclidean point and there is an m-line passing through it not intersecting with L such as the case (c) in Fig.3.10, then any m-line in the shade of Fig.3.10(c) passing through w will not intersect with L. Since the position of the vertices of a map M on a projective plane can be choose as our wish, we know (M, μ) is a Smarandache geometry by denial the axiom (A5) with axioms (A5),(L5) and (R5).

Combining these discussions of Cases $1, 2$ and 3, the proof is complete. ♮.

Similar to Iseri's s-manifolds, among map geometries without boundary there are non-geometries, anti-geometries and counter-projective geometries, $\cdots$, etc..

Theorem 3.2.4 *There are non-geometries in map geometries without boundary.*

Proof We prove there are map geometries without boundary satisfying axioms $(A_1^-) - (A_5^-)$. Let (M, μ) be such a map geometry with elliptic or hyperbolic points.

(i) Assume u is an eulicdean point and v is an elliptic or hyperbolic point on (M, μ). Let L be an m-line passing through points u and v in an Euclid plane. Choose a point w in L after but nearly enough to v when we travel on L from u to

v. Then there does not exist a line from u to w in the map geometry (M,μ) since v is an elliptic or hyperbolic point. So the axiom (A_1^-) is true in (M,μ).

(ii) In a map geometry (M,μ), an m-line maybe closed such as we have illustrated in the proof of Theorem 3.2.3. Choose any two points A, B on a closed m-line L in a map geometry. Then the m-line between A and B can not continuously extend to indefinite in (M,μ). Whence the axiom (A_2^-) is true in (M,μ).

(iii) An m-circle in a map geometry is defined to be a set of continuous points in which all points have a given distance to a given point. Let C be a m-circle in an Euclid plane. Choose an elliptic or a hyperbolic point A on C which enables us to get a map geometry (M,μ). Then C has a gap in A by definition of an elliptic or hyperbolic point. So the axiom (A_3^-) is true in a map geometry without boundary.

(iv) By the definition of a right angle, we know that a right angle on an elliptic point can not equal to a right angle on a hyperbolic point. So the axiom (A_4^-) is held in a map geometry with elliptic or hyperbolic points.

(v) The axiom (A_5^-) is true by Theorem 3.2.3.

Combining these discussions of (i)-(v), we know that there are non-geometries in map geometries. This completes the proof. ♮

The *Hilbert's axiom system* for an Euclid plane geometry consists five group axioms stated in the following, where we denote each group by a capital *Roman* numeral.

I. Incidence

$I-1$. *For every two points A and B, there exists a line L that contains each of the points A and B.*
$I-2$. *For every two points A and B, there exists no more than one line that contains each of the points A and B.*
$I-3$. *There are at least two points on a line. There are at least three points not on a line.*

II. Betweenness

$II-1$. *If a point B lies between points A and C, then the points A, B and C are distinct points of a line, and B also lies between C and A.*
$II-2$. *For two points A and C, there always exists at least one point B on the line AC such that C lies between A and B.*
$II-3$. *Of any three points on a line, there exists no more than one that lies between the other two.*
$II-4$. *Let A, B and C be three points that do not lie on a line, and let L be a line which does not meet any of the points A, B and C. If the line L passes through a point of the segment AB, it also passes through a point of the segment AC, or through a point of the segment BC.*

III. Congruence

$III-1$. If A_1 and B_1 are two points on a line L_1, and A_2 is a point on a line L_2 then it is always possible to find a point B_2 on a given side of the line L_2 through A_2 such that the segment A_1B_1 is congruent to the segment A_2B_2.
$III-2$. If a segment A_1B_1 and a segment A_2B_2 are congruent to the segment AB, then the segment A_1B_1 is also congruent to the segment A_2B_2.
$III-3$. On the line L, let AB and BC be two segments which except for B have no point in common. Furthermore, on the same or on another line L_1, let A_1B_1 and B_1C_1 be two segments, which except for B_1 also have no point in common. In that case, if AB is congruent to A_1B_1 and BC is congruent to B_1C_1, then AC is congruent to A_1C_1.
$III-4$. Every angle can be copied on a given side of a given ray in a uniquely determined way.
$III-5$ If for two triangles ABC and $A_1B_1C_1$, AB is congruent to A_1B_1, AC is congruent to A_1C_1 and $\angle BAC$ is congruent to $\angle B_1A_1C_1$, then $\angle ABC$ is congruent to $\angle A_1B_1C_1$.

IV. **Parallels**

$IV-1$. There is at most one line passes through a point P exterior a line L that is parallel to L.

V. **Continuity**

$V-1$(Archimedes) *Let AB and CD be two line segments with $|AB| \geq |CD|$. Then there is an integer m such that*

$$m|CD| \leq |AB| \leq (m+1)|CD|.$$

$V-2$(Cantor) Let $A_1B_1, A_2B_2, \cdots, A_nB_n, \cdots$ be a segment sequence on a line L. If

$$A_1B_1 \supseteq A_2B_2 \supseteq \cdots \supseteq A_nB_n \supseteq \cdots,$$

then there exists a common point X on each line segment A_nB_n for any integer $n, n \geq 1$.

Smarandache defined an anti-geometries by denial some axioms of Hilbert axiom system for an Euclid geometry. Similar to the discussion in the reference [35], We obtain the following result for anti-geometries in map geometries without boundary.

Theorem 3.2.5 *Unless axioms $I-3$, $II-3$, $III-2$, $V-1$ and $V-2$, an anti-geometry can be gotten from map geometries without boundary by denial other axioms in Hilbert axiom system.*

Proof The axiom $I-1$ has been denied in the proof of Theorem 3.2.4. Since there maybe exists more than one line passing through two points A and B in a map geometry with elliptic or hyperbolic points u such as those shown in Fig.3.11. So the axiom $II-2$ can be Smarandachely denied.

Fig.3.11

Notice that an m-line maybe has self-intersection points in a map geometry without boundary. So the axiom $II-1$ can be denied. By the proof of Theorem 3.2.4, we know that for two points A and B, an m-line passing through A and B may not exist. Whence, the axiom $II-2$ can be denied. For the axiom $II-4$, see Fig.3.12, in where v is a non-euclidean point such that $\rho_M(v)\mu(v) \geq 2(\pi + \angle ACB)$ in a map geometry.

Fig.3.12

So $II-4$ can be also denied. Notice that an m-line maybe has self-intersection points. There are maybe more than one m-lines passing through two given points A, B. Therefore, the axioms $III-1$ and $III-3$ are deniable. For denial the axiom $III-4$, since an elliptic point u can be measured at most by a number $\frac{\rho_M(u)\mu(u)}{2} < \pi$, i.e., there is a limitation for an elliptic point u. Whence, an angle with measure bigger than $\frac{\rho_M(u)\mu(u)}{2}$ can not be copied on an elliptic point on a given ray.

Because there are maybe more than one m-lines passing through two given points A and B in a map geometry without boundary, the axiom $III-5$ can be Smarandachely denied in general such as those shown in Fig.3.13(a) and (b) where u is an elliptic point.

Fig.3.13

For the parallel axiom $IV-1$, it has been denied by the proofs of Theorems 3.2.3 and 3.2.4.

Notice that axioms $I-3$, $II-3$ $III-2$, $V-1$ and $V-2$ can not be denied in a map geometry without boundary. This completes the proof. $\natural$

For counter-projective geometries, we have a result as in the following.

Theorem 3.2.6 *Unless the axiom* $(C3)$, *a counter-projective geometry can be gotten from map geometries without boundary by denial axioms* $(C1)$ *and* $(C2)$.

Proof Notice that axioms $(C1)$ and $(C2)$ have been denied in the proof of Theorem 3.2.5. Since a map is embedded on a locally orientable surface, every m-line in a map geometry without boundary may contains infinite points. Therefore the axiom $(C3)$ can not be Smarandachely denied. $\natural$

§3.3 Map Geometries with Boundary

A *Poincaré's model* for a hyperbolic geometry is an upper half-plane in which lines are upper half-circles with center on the x-axis or upper straight lines perpendicular to the x-axis such as those shown in Fig.3.14.

Fig.3.14

If we think that all infinite points are the same, then a Poincaré's model for a hyperbolic geometry is turned to a *Klein model* for a hyperbolic geometry which uses a boundary circle and lines are straight line segment in this circle, such as those shown in Fig.3.15.

Fig.3.15

By a combinatorial map view, a Klein's model is nothing but a one face map geometry. This fact hints us to introduce map geometries with boundary, which is defined in the next definition.

Definition 3.3.1 *For a map geometry* (M,μ) *without boundary and faces* $f_1, f_2, \cdots, f_l \in F(M), 1 \leq l \leq \phi(M) - 1$, *if* $S(M) \setminus \{f_1, f_2, \cdots, f_l\}$ *is connected, then call* $(M,\mu)^{-l} = (S(M) \setminus \{f_1, f_2, \cdots, f_l\}, \mu)$ *a map geometry with boundary* $f_1, f_2, \cdots, f_l$ *and orientable or not if* (M,μ) *is orientable or not, where* $S(M)$ *denotes the locally orientable surface on which* M *is embedded.*

The m-points and m-lines in a map geometry $(M,\mu)^{-l}$ are defined as same as Definition 3.2.3 by adding an m-line terminated at the boundary of this map geometry. Two m^--lines on the torus and projective plane are shown in these Fig.3.16 and Fig.3.17, where the shade field denotes the boundary.

Fig.3.16

Fig.3.17

All map geometries with boundary are also Smarandache geometries which is convince by a result in the following.

Theorem 3.3.1 *For a map M on a locally orientable surface with order≥ 3, vertex valency≥ 3 and a face $f \in F(M)$, there is an angle factor μ such that $(M,\mu)^{-1}$ is a Smarandache geometry by denial the axiom (A5) with these axioms (A5),(L5) and (R5).*

Proof Similar to the proof of Theorem 3.2.3, we consider a map M being a planar map, an orientable map on a torus or a non-orientable map on a projective plane, respectively. We can get the assertion. In fact, by applying the property that m-lines in a map geometry with boundary are terminated at the boundary, we can get an more simpler proof for this theorem. ♮

Notice that in a one face map geometry $(M,\mu)^{-1}$ with boundary is just a Klein's model for hyperbolic geometry if we choose all points being euclidean.

Similar to map geometries without boundary, we can also get non-geometries, anti-geometries and counter-projective geometries from map geometries with boundary.

Theorem 3.3.2 *There are non-geometries in map geometries with boundary.*

Proof The proof is similar to the proof of Theorem 3.2.4 for map geometries without boundary. Each of axioms $(A_1^-)-(A_5^-)$ is hold, for example, cases $(a)-(e)$ in Fig.3.18,

Fig.3.18

in where there are no an m-line from points A to B in (a), the line AB can not be continuously extended to indefinite in (b), the circle has gap in (c), a right angle at an euclidean point v is not equal to a right angle at an elliptic point u in (d) and there are infinite m-lines passing through a point P not intersecting with the m-line L in (e). Whence, there are non-geometries in map geometries with boundary. $\natural$

Theorem 3.3.3 *Unless axioms $I-3$, $II-3$ $III-2$, $V-1$ and $V-2$ in the Hilbert's axiom system for an Euclid geometry, an anti-geometry can be gotten from map geometries with boundary by denial other axioms in this axiom system.*

Theorem 3.3.4 *Unless the axiom $(C3)$, a counter-projective geometry can be gotten from map geometries with boundary by denial axioms $(C1)$ and $(C2)$.*

Proof The proofs of Theorems 3.3.3 and 3.3.4 are similar to the proofs of Theorems 3.2.5 and 3.2.6. The reader is required to complete their proof. $\natural$

§3.4 The Enumeration of Map Geometries

For classifying map geometries, the following definition is needed.

Definition 3.4.1 *Two map geometries (M_1, μ_1) and (M_2, μ_2) or $(M_1, \mu_1)^{-l}$ and $(M_2, \mu_2)^{-l}$ are said to be equivalent each other if there is a bijection $\theta : M_1 \to M_2$ such that for $\forall u \in V(M)$, $\theta(u)$ is euclidean, elliptic or hyperbolic if and only if u is euclidean, elliptic or hyperbolic.*

A relation for the numbers of unrooted maps with map geometries is in the following result.

Theorem 3.4.1 *Let $\mathcal{M}$ be a set of non-isomorphic maps of order n and with m faces. Then the number of map geometries without boundary is $3^n|\mathcal{M}|$ and the number of map geometries with one face being its boundary is $3^n m|\mathcal{M}|$.*

Proof By the definition of equivalent map geometries, for a given map $M \in \mathcal{M}$, there are 3^n map geometries without boundary and $3^n m$ map geometries with one face being its boundary by Theorem 3.3.1. Whence, we get $3^n|\mathcal{M}|$ map geometries without boundary and $3^n m|\mathcal{M}|$ map geometries with one face being its boundary from $\mathcal{M}$. $\natural$.

We get an enumeration result for non-equivalent map geometries without boundary as follows.

Theorem 3.4.2 *The numbers $n^O(\Gamma, g)$ and $n^N(\Gamma, g)$ of non-equivalent orientable and non-orientable map geometries without boundary underlying a simple graph Γ by denial the axiom (A5) by (A5), (L5) or (R5) are*

$$n^O(\Gamma, g) = \frac{3^{|\Gamma|} \prod\limits_{v \in V(\Gamma)} (\rho(v)-1)!}{2|\mathrm{Aut}\Gamma|},$$

and

$$n^N(\Gamma,g)=\frac{(2^{\beta(\Gamma)}-1)3^{|\Gamma|}\prod\limits_{v\in V(\Gamma)}(\rho(v)-1)!}{2|\mathrm{Aut}\Gamma|},$$

where $\beta(\Gamma)=\varepsilon(\Gamma)-\nu(\Gamma)+1$ *is the Betti number of the graph* Γ.

Proof Denote the set of non-isomorphic maps underlying the graph Γ on locally orientable surfaces by $\mathcal{M}(\Gamma)$ and the set of embeddings of the graph Γ on locally orientable surfaces by $\mathcal{E}(\Gamma)$. For a map $M, M\in\mathcal{M}(\Gamma)$, there are $\frac{3^{|M|}}{|\mathrm{Aut}M|}$ different map geometries without boundary by choice the angle factor μ on a vertex u such that u is euclidean, elliptic or hyperbolic. From permutation groups, we know that

$$|\mathrm{Aut}\Gamma\times\langle\alpha\rangle\,|=|(\mathrm{Aut}\Gamma)_M||M^{\mathrm{Aut}\Gamma\times\langle\alpha\rangle}|=|\mathrm{Aut}M||M^{\mathrm{Aut}\Gamma\times\langle\alpha\rangle}|.$$

Therefore, we get that

$$\begin{aligned}
n^O(\Gamma,g) &= \sum_{M\in\mathcal{M}(\Gamma)}\frac{3^{|M|}}{|\mathrm{Aut}M|}\\
&= \frac{3^{|\Gamma|}}{|\mathrm{Aut}\Gamma\times\langle\alpha\rangle\,|}\sum_{M\in\mathcal{M}(\Gamma)}\frac{|\mathrm{Aut}\Gamma\times\langle\alpha\rangle\,|}{|\mathrm{Aut}M|}\\
&= \frac{3^{|\Gamma|}}{|\mathrm{Aut}\Gamma\times\langle\alpha\rangle\,|}\sum_{M\in\mathcal{M}(\Gamma)}|M^{\mathrm{Aut}\Gamma\times\langle\alpha\rangle}|\\
&= \frac{3^{|\Gamma|}}{|\mathrm{Aut}\Gamma\times\langle\alpha\rangle\,|}|\mathcal{E}^O(\Gamma)|\\
&= \frac{3^{|\Gamma|}\prod\limits_{v\in V(\Gamma)}(\rho(v)-1)!}{2|\mathrm{Aut}\Gamma|}.
\end{aligned}$$

Similarly, we can also get that

$$\begin{aligned}
n^N(\Gamma,g) &= \frac{3^{|\Gamma|}}{|\mathrm{Aut}\Gamma\times\langle\alpha\rangle\,|}|\mathcal{E}^N(\Gamma)|\\
&= \frac{(2^{\beta(\Gamma)}-1)3^{|\Gamma|}\prod\limits_{v\in V(\Gamma)}(\rho(v)-1)!}{2|\mathrm{Aut}\Gamma|}.
\end{aligned}$$

This completes the proof. ♮

For classifying map geometries with boundary, we get a result as in the following.

Theorem 3.4.3 *The numbers* $n^O(\Gamma,-g)$, $n^N(\Gamma,-g)$ *of non-equivalent orientable, non-orientable map geometries with one face being its boundary underlying a simple graph* Γ *by denial the axiom (A5) by (A5), (L5) or (R5) are respective*

$$n^O(\Gamma,-g)=\frac{3^{|\Gamma|}}{2|\mathrm{Aut}\Gamma|}[(\beta(\Gamma)+1)\prod_{v\in V(\Gamma)}(\rho(v)-1)!-\frac{2d(g[\Gamma](x))}{dx}|_{x=1}]$$

and

$$n^N(\Gamma,-g)=\frac{(2^{\beta(\Gamma)}-1)3^{|\Gamma|}}{2|\mathrm{Aut}\Gamma|}[(\beta(\Gamma)+1)\prod_{v\in V(\Gamma)}(\rho(v)-1)!-\frac{2d(g[\Gamma](x))}{dx}|_{x=1}],$$

where $g[\Gamma](x)$ *is the genus polynomial of the graph* Γ, *i.e.,* $g[\Gamma](x)=\sum\limits_{k=\gamma(\Gamma)}^{\gamma_m(\Gamma)} g_k[\Gamma]x^k$ *with* $g_k[\Gamma]$ *being the number of embeddings of* Γ *on the orientable surface of genus* k.

Proof Notice that $\nu(M)-\varepsilon(M)+\phi(M)=2-2g(M)$ for an orientable map M by the Euler-Poincar*é* formula. Similar to the proof of Theorem 3.4.2 with the same meaning for $\mathcal{M}(\Gamma)$, we know that

$$\begin{aligned}
n^O(\Gamma,-g) &= \sum_{M\in\mathcal{M}(\Gamma)}\frac{\phi(M)3^{|M|}}{|\mathrm{Aut}M|}\\
&= \sum_{M\in\mathcal{M}(\Gamma)}\frac{(2+\varepsilon(\Gamma)-\nu(\Gamma)-2g(M))3^{|M|}}{|\mathrm{Aut}M|}\\
&= \sum_{M\in\mathcal{M}(\Gamma)}\frac{(2+\varepsilon(\Gamma)-\nu(\Gamma))3^{|M|}}{|\mathrm{Aut}M|}-\sum_{M\in\mathcal{M}(\Gamma)}\frac{2g(M)3^{|M|}}{|\mathrm{Aut}M|}\\
&= \frac{(2+\varepsilon(\Gamma)-\nu(\Gamma))3^{|M|}}{|\mathrm{Aut}\Gamma\times\langle\alpha\rangle\,|}\sum_{M\in\mathcal{M}(\Gamma)}\frac{|\mathrm{Aut}\Gamma\times\langle\alpha\rangle\,|}{|\mathrm{Aut}M|}\\
&- \frac{2\times 3^{|\Gamma|}}{|\mathrm{Aut}\Gamma\times\langle\alpha\rangle\,|}\sum_{M\in\mathcal{M}(\Gamma)}\frac{g(M)|\mathrm{Aut}\Gamma\times\langle\alpha\rangle\,|}{|\mathrm{Aut}M|}\\
&= \frac{(\beta(\Gamma)+1)3^{|M|}}{|\mathrm{Aut}\Gamma\times\langle\alpha\rangle\,|}\sum_{M\in\mathcal{M}}(\Gamma)|M^{\mathrm{Aut}\Gamma\times\langle\alpha\rangle}|\\
&- \frac{3^{|\Gamma|}}{|\mathrm{Aut}\Gamma|}\sum_{M\in\mathcal{M}(\Gamma)}g(M)|M^{\mathrm{Aut}\Gamma\times\langle\alpha\rangle}|\\
&= \frac{(\beta(\Gamma)+1)3^{|\Gamma|}}{2|\mathrm{Aut}\Gamma|}\prod_{v\in V(\Gamma)}(\rho(v)-1)!-\frac{3^{|\Gamma|}}{|\mathrm{Aut}\Gamma|}\sum_{k=\gamma(\Gamma)}^{\gamma_m(\Gamma)}kg_k[\Gamma]\\
&= \frac{3^{|\Gamma|}}{2|\mathrm{Aut}\Gamma|}[(\beta(\Gamma)+1)\prod_{v\in V(\Gamma)}(\rho(v)-1)!-\frac{2d(g[\Gamma](x))}{dx}|_{x=1}].
\end{aligned}$$

by Theorem 3.4.1.

Notice that $n^L(\Gamma,-g) = n^O(\Gamma,-g)+n^N(\Gamma,-g)$ and the number of re-embeddings an orientable map M on surfaces is $2^{\beta(M)}$ (see also [56] for details). We know that

$$
\begin{aligned}
n^L(\Gamma,-g) &= \sum_{M\in\mathcal{M}(\Gamma)} \frac{2^{\beta(M)}\times 3^{|M|}\phi(M)}{|\mathrm{Aut}M|} \\
&= 2^{\beta(M)} n^O(\Gamma,-g).
\end{aligned}
$$

Whence, we get that

$$
\begin{aligned}
n^N(\Gamma,-g) &= (2^{\beta(M)}-1)n^O(\Gamma,-g) \\
&= \frac{(2^{\beta(M)}-1)3^{|\Gamma|}}{2|\mathrm{Aut}\Gamma|}[(\beta(\Gamma)+1)\prod_{v\in V(\Gamma)}(\rho(v)-1)! - \frac{2d(g[\Gamma](x))}{dx}|_{x=1}].
\end{aligned}
$$

This completes the proof. ♮

§3.5 Remarks and Open Problems

3.5.1 A complete Hilbert axiom system for an Euclid geometry contains axioms $I-i, 1\leq i\leq 8$; $II-j, 1\leq j\leq 4$; $III-k, 1\leq k\leq 5$; $IV-1$ and $V-l, 1\leq l\leq 2$, which can be also applied to the geometry of space. Unless $I-i, 4\leq i\leq 8$, other axioms are presented in Section 3.2. Each of axioms $I-i, 4\leq i\leq 8$ is described in the following.

$I-4$ *For three non-collinear points A, B and C, there is one and only one plane passing through them.*

$I-5$ *Each plane has at least one point.*

$I-6$ *If two points A and B of a line L are in a plane $\sum$, then every point of L is in the plane $\sum$.*

$I-7$ *If two planes $\sum_1$ and $\sum_2$ have a common point A, then they have another common point B.*

$I-8$ *There are at least four points not in one plane.*

By the Hilbert's axiom system, the following result for parallel planes can be obtained.

(**T**) *Passing through a given point A exterior to a given plane $\sum$ there is one and only one plane parallel to $\sum$.*

This result seems like the Euclid's fifth axiom. Similar to the Smarandache's notion, we present problems by denial this theorem for the geometry of space as follows.

Problem 3.5.1 *Construct a geometry of space by denial the parallel theorem of planes with*

(T_1^-) *there are at least a plane* $\sum$ *and a point* A *exterior to the plane* $\sum$ *such that no parallel plane to* $\sum$ *passing through the point* A.

(T_2^-) *there are at least a plane* $\sum$ *and a point* A *exterior to the plane* $\sum$ *such that there are finite parallel planes to* $\sum$ *passing through the point* A.

(T_3^-) *there are at least a plane* $\sum$ *and a point* A *exterior to the plane* $\sum$ *such that there are infinite parallel planes to* $\sum$ *passing through the point* A.

Problem 3.5.2 *Similar to the Iseri's idea define an elliptic, euclidean, or hyperbolic point or plane in* $\mathbf{R}^3$ *and apply these Plato polyhedrons to construct Smarandache geometries of a space* $\mathbf{R}^3$.

Problem 3.5.3 *Similar to map geometries define graph in a space geometries and apply graphs in* $\mathbf{R}^3$ *to construct Smarandache geometries of a space* $\mathbf{R}^3$.

Problem 3.5.4 *For an integer* $n, n \geq 4$, *define Smarandache geometries in* $\mathbf{R}^n$ *by denial some axioms for an Euclid geometry in* $\mathbf{R}^n$ *and construct them.*

3.5.2 The terminology *map geometry* was first appeared in [55] which enables us to find non-homogenous spaces from already known homogenous spaces and is also a typical example for application combinatorial maps to metric geometries. Among them there are many problems not solved yet until today. Here we would like to describe some of them.

Problem 3.5.5 *For a given graph* G, *determine non-equivalent map geometries with an underlying graph* G, *particularly, for graphs* K_n, $K(m,n), m, n \geq 4$ *and enumerate them.*

Problem 3.5.6 *For a given locally orientable surface* S, *determine non-equivalent map geometries on* S, *such as a sphere, a torus or a projective plane,* $\cdots$ *and enumerate them.*

Problem 3.5.7 *Find characteristics for equivalent map geometries or establish new ways for classifying map geometries.*

Problem 3.5.8 *Whether can we rebuilt an intrinsic geometry on surfaces, such as a sphere, a torus or a projective plane,* $\cdots$, *by map geometries?*

4. **Planar map geometries**

Fundamental elements in an Euclid geometry are those of points, lines, polygons and circles. For a map geometry, the situation is more complex since a point maybe an elliptic, euclidean or a hyperbolic point, a polygon maybe a line, $\cdots$, etc.. This chapter concentrates on discussing fundamental elements and measures such as angle, area, curvature, $\cdots$, etc., also parallel bundles in planar map geometries, which can be seen as a first step for comprehending map geometries on surfaces. All materials of this chapter will be used in Chapters 5-6 for establishing relations of an integral curve with a differential equation system in a pseudo-plane geometry and continuous phenomena with discrete phenomena

§4.1 **Points in a Planar Map Geometry**

Points in a map geometry are classified into three classes: *elliptic, euclidean* and *hyperbolic*. There are only finite non-euclidean points considered in Chapter 3 because we had only defined an elliptic, euclidean or a hyperbolic point on vertices of a map. In a planar map geometry, we can present an even more delicate consideration for euclidean or non-euclidean points and find infinite non-euclidean points in a plane.

Let (M, μ) be a planar map geometry on a plane $\sum$. Choose vertices $u, v \in V(M)$. A mapping is called an *angle function between u and v* if there is a smooth monotone mapping $f : \sum \to \sum$ such that $f(u) = \frac{\rho_M(u)\mu(u)}{2}$ and $f(v) = \frac{\rho_M(v)\mu(v)}{2}$. Not loss of generality, we can assume that each edge in a planar map geometry is an angle function. Then we know a result as in the following.

Theorem 4.1.1 *A planar map geometry (M, μ) has infinite non-euclidean points if and only if there is an edge $e = (u, v) \in E(M)$ such that $\rho_M(u)\mu(u) \neq \rho_M(v)\mu(v)$, or $\rho_M(u)\mu(u)$ is a constant but $\neq 2\pi$ for $\forall u \in V(M)$, or a loop $(u, u) \in E(M)$ attaching a non-euclidean point u.*

Proof If there is an edge $e = (u, v) \in E(M)$ such that $\rho_M(u)\mu(u) \neq \rho_M(v)\mu(v)$, then at least one of vertices u and v in (M, μ) is non-euclidean. Not loss of generality, we assume the vertex u is non-euclidean.

If u and v are elliptic or u is elliptic but v is euclidean, then by the definition of angle functions, the edge (u, v) is correspondent with an angle function $f : \sum \to \sum$ such that $f(u) = \frac{\rho_M(u)\mu(u)}{2}$ and $f(v) = \frac{\rho_M(v)\mu(v)}{2}$, each points is non-euclidean in $(u, v)\setminus\{v\}$. If u is elliptic but v is hyperbolic, i.e., $\rho_M(u)\mu(u) < 2\pi$ and $\rho_M(v)\mu(v) > 2\pi$, since f is smooth and monotone on (u, v), there is one and only one point x^* in (u, v) such that $f(x^*) = \pi$. Thereby there are infinite non-euclidean points on (u, v).

Similar discussion can be gotten for the cases that u and v are both hyperbolic, or u is hyperbolic but v is euclidean, or u is hyperbolic but v is elliptic.

If $\rho_M(u)\mu(u)$ is a constant but $\neq 2\pi$ for $\forall u \in V(M)$, then each point on an edges is a non-euclidean point. Thereby there are infinite non-euclidean points in (M,μ).

Now if there is a loop $(u,u) \in E(M)$ and u is non-eucliean, then by definition, each point v on the loop (u,u) satisfying that $f(v) >$ or $< \pi$ according to $\rho_M(u)\mu(u) > \pi$ or $< \pi$. Therefore there are also infinite non-euclidean points on the loop (u,u).

On the other hand, if there are no an edge $e = (u,v) \in E(M)$ such that $\rho_M(u)\mu(u) \neq \rho_M(v)\mu(v)$, i.e., $\rho_M(u)\mu(u) = \rho_M(v)\mu(v)$ for $\forall (u,v) \in E(M)$, or there are no vertices $u \in V(M)$ such that $\rho_M(u)\mu(u)$ is a constant but $\neq 2\pi$ for $\forall$, or there are no loops $(u,u) \in E(M)$ with a non-eucliean point u, then all angle functions on these edges of M are an constant π. Therefore there are no non-euclidean points in the map geometry (M,μ). This completes the proof. ♮

For euclidean points in a planar map geometry (M,μ), we get the following result.

Theorem 4.1.2 *For a planar map geometry (M,μ) on a plane $\sum$,*

(i) every point in $\sum \setminus E(M)$ is an euclidean point;

(ii) there are infinite euclidean points on M if and only if there exists an edge $(u,v) \in E(M)$ ($u = v$ or $u \neq v$) such that u and v are both euclidean.

Proof By the definition of angle functions, we know that every point is euclidean if it is not on M. So the assertion (i) is true.

According to the proof of Theorem 4.1.1, there are only finite euclidean points unless there is an edge $(u,v) \in E(M)$ with $\rho_M(u)\mu(u) = \rho_M(v)\mu(v) = 2\pi$. In this case, there are infinite euclidean points on the edge (u,v). Thereby the assertion (ii) is also holds. ♮

According to Theorems 4.1.1 and 4.1.2, we classify edges in a planar map geometry (M,μ) into six classes as follows.

C_E^1 **(euclidean-elliptic edges)**: *edges $(u,v) \in E(M)$ with $\rho_M(u)\mu(u) = 2\pi$ but $\rho_M(v)\mu(v) < 2\pi$.*

C_E^2 **(euclidean-euclidean edges)**: *edges $(u,v) \in E(M)$ with $\rho_M(u)\mu(u) = 2\pi$ and $\rho_M(v)\mu(v) = 2\pi$.*

C_E^3 **(euclidean-hyperbolic edges)**: *edges $(u,v) \in E(M)$ with $\rho_M(u)\mu(u) = 2\pi$ but $\rho_M(v)\mu(v) > 2\pi$.*

C_E^4 **(elliptic-elliptic edges)**: *edges $(u,v) \in E(M)$ with $\rho_M(u)\mu(u) < 2\pi$ and $\rho_M(v)\mu(v) < 2\pi$.*

C_E^5 **(elliptic-hyperbolic edges)**: *edges $(u,v) \in E(M)$ with $\rho_M(u)\mu(u) < 2\pi$ but $\rho_M(v)\mu(v) > 2\pi$.*

C_E^6 **(hyperbolic-hyperbolic edges)**: *edges $(u,v) \in E(M)$ with $\rho_M(u)\mu(u) > 2\pi$ and $\rho_M(v)\mu(v) > 2\pi$.*

In Fig.4.1$(a)-(f)$, these m-lines passing through an edge in one of classes of C_E^1-C_E^6 are shown, where u is elliptic and v is eucildean in (a), u and v are both euclidean in (b), u is eucildean but v is hyperbolic in (c), u and v are both elliptic in (d), u is elliptic but v is hyperbolic in (e) and u and v are both hyperbolic in (f), respectively.

Fig.4.1

Denote by $V_{el}(M), V_{eu}(M)$ and $V_{hy}(M)$ the respective sets of elliptic, euclidean and hyperbolic points in $V(M)$ in a planar map geometry (M, μ). Then we get a result as in the following.

Theorem 4.1.3 *Let (M, μ) be a planar map geometry. Then*

$$\sum_{u \in V_{el}(M)} \rho_M(u) + \sum_{v \in V_{eu}(M)} \rho_M(v) + \sum_{w \in V_{hy}(M)} \rho_M(w) = 2 \sum_{i=1}^{6} |C_E^i|$$

and

$$|V_{el}(M)| + |V_{eu}(M)| + |V_{hy}(M)| + \phi(M) = \sum_{i=1}^{6} |C_E^i| + 2.$$

where $\phi(M)$ denotes the number of faces of a map M.

Proof Notice that

$$|V(M)| = |V_{el}(M)| + |V_{eu}(M)| + |V_{hy}(M)| \text{ and } |E(M)| = \sum_{i=1}^{6} |C_E^i|$$

for a planar map geometry (M, μ). By two well-known results

$$\sum_{v \in V(M)} \rho_M(v) = 2|E(M)| \quad \text{and} \quad |V(M)| - |E(M)| + \phi(M) = 2$$

for a planar map M, we know that

$$\sum_{u\in V_{el}(M)} \rho_M(u) + \sum_{v\in V_{eu}(M)} \rho_M(v) + \sum_{w\in V_{hy}(M)} \rho_M(w) = 2\sum_{i=1}^{6}|C_E^i|$$

and

$$|V_{el}(M)| + |V_{eu}(M)| + |V_{hy}(M)| + \phi(M) = \sum_{i=1}^{6}|C_E^i| + 2. \quad \natural$$

§4.2 Lines in a Planar Map Geometry

The situation of m-lines in a planar map geometry (M,μ) is more complex. Here an m-line maybe open or closed, with or without self-intersections in a plane. We discuss all of these m-lines and their behaviors in this section, .

4.2.1. Lines in a planar map geometry

As we have seen in Chapter 3, m-lines in a planar map geometry (M,μ) can be classified into three classes.

C_L^1**(opened lines without self-intersections)**: *m-lines in (M,μ) have an infinite number of continuous m-points without self-intersections and endpoints and may be extended indefinitely in both directions.*

C_L^2**(opened lines with self-intersections)**: *m-lines in (M,μ) have an infinite number of continuous m-points and self-intersections but without endpoints and may be extended indefinitely in both directions.*

C_L^3**(closed lines)**: *m-lines in (M,μ) have an infinite number of continuous m-points and will come back to the initial point as we travel along any one of these m-lines starting at an initial point.*

By this classification, a straight line in an Euclid plane is nothing but an opened m-line without non-euclidean points. Certainly, m-lines in a planar map geometry (M,μ) maybe contain non-euclidean points. In Fig.4.2, these m-lines shown in (a), (b) and (c) are opened m-line without self-intersections, opened m-line with a self-intersection and closed m-line with A, B, C, D and E non-euclidean points, respectively.

Fig.4.2

Notice that a closed m-line in a planar map geometry maybe also has self-intersections. A closed m-line is said to be *simply closed* if it has no self-intersections, such as the m-line in Fig.4.2(c). For simply closed m-lines, we know the following result.

Theorem 4.2.1 *Let (M, μ) be a planar map geometry. An m-line L in (M, μ) passing through n non-euclidean points $x_1, x_2, \cdots, x_n$ is simply closed if and only if*

$$\sum_{i=1}^{n} f(x_i) = (n-2)\pi,$$

where $f(x_i)$ denotes the angle function value at an m-point $x_i, 1 \le i \le n$.

Proof By results in an Euclid geometry of plane, we know that the angle sum of an n-polygon is $(n-2)\pi$. In a planar map geometry (M, μ), a simply closed m-line L passing through n non-euclidean points $x_1, x_2, \cdots, x_n$ is nothing but an n-polygon with vertices $x_1, x_2, \cdots, x_n$. Whence, we get that

$$\sum_{i=1}^{n} f(x_i) = (n-2)\pi.$$

Now if a simply m-line L passing through n non-euclidean points $x_1, x_2, \cdots, x_n$ with

$$\sum_{i=1}^{n} f(x_i) = (n-2)\pi$$

held, then L is nothing but an n-polygon with vertices $x_1, x_2, \cdots, x_n$. Therefore, L is simply closed. ♮

By applying Theorem 4.2.1, we can also find conditions for an opened m-line with or without self-intersections.

Theorem 4.2.2 *Let (M, μ) be a planar map geometry. An m-line L in (M, μ) passing through n non-euclidean points $x_1, x_2, \cdots, x_n$ is opened without self-intersections if and only if m-line segments $x_i x_{i+1}, 1 \le i \le n-1$ are not intersect two by two and*

$$\sum_{i=1}^{n} f(x_i) \ge (n-1)\pi.$$

Proof By the Euclid's fifth postulate for a plane geometry, two straight lines will meet on the side on which the angles less than two right angles if we extend them to indefinitely. Now for an m-line L in a planar map geometry (M, μ), if it is opened without self-intersections, then for any integer $i, 1 \le i \le n-1$, m-line segments $x_i x_{i+1}$ will not intersect two by two and the m-line L will also not intersect before it enters x_1 or leaves x_n.

Fig.4.3

Now look at Fig.4.3, in where line segment x_1x_n is an added auxiliary m-line segment. We know that

$$\angle 1 + \angle 2 = f(x_1) \text{ and } \angle 3 + \angle 4 = f(x_n).$$

According to Theorem 4.2.1 and the Euclid's fifth postulate, we know that

$$\angle 2 + \angle 4 + \sum_{i=2}^{n-1} f(x_i) = (n-2)\pi,$$

$$\angle 1 + \angle 3 \geq \pi$$

Therefore, we get that

$$\sum_{i=1}^{n} f(x_i) = (n-2)\pi + \angle 1 + \angle 3 \geq (n-1)\pi. \qquad \natural$$

For opened m-lines with self-intersections, we know a result as in the following.

Theorem 4.2.3 Let (M,μ) be a planar map geometry. An m-line L in (M,μ) passing through n non-euclidean points $x_1, x_2, \cdots, x_n$ is opened only with l self-intersections if and only if there exist integers i_j and s_{i_j}, $1 \leq j \leq l$ with $1 \leq i_j, s_{i,j} \leq n$ and $i_j \neq i_t$ if $t \neq j$ such that

$$(s_{i_j} - 2)\pi \ < \ \sum_{h=1}^{s_{i_j}} f(x_{i_j+h}) \ < \ (s_{i_j} - 1)\pi.$$

Proof If an m-line L passing through m-points $x_{t+1}, x_{t+2}, \cdots, x_{t+s_t}$ only has one self-intersection point, let us look at Fig.4.4 in where $x_{t+1}x_{t+s_t}$ is an added auxiliary m-line segment.

Fig.4.4

We know that

$$\angle 1 + \angle 2 = f(x_{t+1}) \text{ and } \angle 3 + \angle 4 = f(x_{t+s_t}).$$

Similar to the proof of Theorem 4.2.2, by Theorem 4.2.1 and the Euclid's fifth postulate, we know that

$$\angle 2 + \angle 4 + \sum_{j=2}^{s_t-1} f(x_{t+j}) = (s_t - 2)\pi$$

and

$$\angle 1 + \angle 3 \ < \ \pi.$$

Whence, we get that

$$(s_t - 2)\pi \ < \ \sum_{j=1}^{s_t} f(x_{t+j}) \ < \ (s_t - 1)\pi.$$

Therefore, if L is opened only with l self-intersection points, we can find integers i_j and $s_{i_j}, 1 \leq j \leq l$ with $1 \leq i_j, s_{i,j} \leq n$ and $i_j \neq i_t$ if $t \neq j$ such that L passing through $x_{i_j+1}, x_{i_j+2}, \cdots, x_{i_j+s_j}$ only has one self-intersection point. By the previous discussion, we know that

$$(s_{i_j} - 2)\pi \ < \ \sum_{h=1}^{s_{i_j}} f(x_{i_j+h}) \ < \ (s_{i_j} - 1)\pi.$$

This completes the proof. ♮

Notice that all m-lines considered in this section are consisted by line segments or rays in an Euclid plane geometry. If the length of each line segment tends to zero, then we get a curve at the limitation in the usually sense. Whence, an m-line in a planar map geometry can be also seen as a discretization for plane curves and also has relation with differential equations. Readers interested in those materials can see in Chapter 5 for more details.

4.2.2. **Curvature of an m-line**

The curvature at a point of a curve C is a measure of how quickly the tangent vector changes direction with respect to the length of arc, such as those of the Gauss curvature, the Riemann curvature, $\cdots$, etc.. In Fig.4.5 we present a smooth curve and the changing of tangent vectors.

Fig.4.5

To measure the changing of vector v_1 to v_2, a simpler way is by the changing of the angle between vectors v_1 and v_2. If a curve $C = f(s)$ is smooth, then the changing rate of the angle between two tangent vector with respect to the length of arc, i.e., $\frac{df}{ds}$ is continuous. For example, as we known in the differential geometry, the Gauss curvature at every point of a circle $x^2 + y^2 = r^2$ of radius r is $\frac{1}{r}$. Whence, the changing of the angle from vectors v_1 to v_2 is

$$\int_A^B \frac{1}{r} ds = \frac{1}{r} |\widehat{AB}| = \frac{1}{r} r\theta = \theta.$$

By results in an Euclid plane geometry, we know that θ is also the angle between vectors v_1 and v_2. As we illustrated in Subsection 4.2.1, an m-line in a planar map geometry is consisted by line segments or rays. Therefore, the changing rate of the angle between two tangent vector with respect to the length of arc is not continuous. Similar to the definition of the set curvature in the reference [1], we present a discrete definition for the curvature of m-lines as follows.

Definition 4.2.1 *Let L be an m-line in a planar map geometry (M, μ) with the set W of non-euclidean points. The curvature $\omega(L)$ of L is defined by*

$$\omega(L) = \sum_{p \in W} (\pi - \varpi(p)),$$

where $\varpi(p) = f(p)$ if p is on an edge (u, v) in map M on a plane $\sum$ with an angle function $f : \sum \to \sum$.

In the classical differential geometry, the *Gauss mapping* and the *Gauss curvature* on surfaces are defined as follows:

Let $\mathcal{S} \subset R^3$ be a surface with an orientation $\mathbf{N}$. The mapping $N : \mathcal{S} \to S^2$ takes its value in the unit sphere

$$S^2 = \{(x, y, z) \in R^3 | x^2 + y^2 + z^2 = 1\}$$

along the orientation $\mathbf{N}$. The map $N : \mathcal{S} \to S^2$, thus defined, is called a Gauss mapping and the determinant of $K(p) = d\mathbf{N}_p$ a Gauss curvature.

We know that for a point $p \in \mathcal{S}$ such that the Gaussian curvature $K(p) \neq 0$ and a connected neighborhood V of p with K does not change sign,

$$K(p) = \lim_{A \to 0} \frac{N(A)}{A},$$

where A is the area of a region $B \subset V$ and $N(A)$ is the area of the image of B by the Gauss mapping $N : \mathcal{S} \to S^2$.

The well-known *Gauss-Bonnet theorem* for a compact surface says that

$$\int\int_{\mathcal{S}} K d\sigma = 2\pi\chi(S),$$

for any orientable compact surface S.

For a simply closed m-line, we also have a result similar to the Gauss-Bonnet theorem, which can be also seen as a discrete Gauss-Bonnet theorem on a plane.

Theorem 4.2.4 *Let L be a simply closed m-line passing through n non-euclidean points $x_1, x_2, \cdots, x_n$ in a planar map geometry (M, μ). Then*

$$\omega(L) = 2\pi.$$

Proof According to Theorem 4.2.1, we know that

$$\sum_{i=1}^{n} f(x_i) = (n-2)\pi,$$

where $f(x_i)$ denotes the angle function value at an m-point $x_i, 1 \leq i \leq n$. Whence, by Definition 4.2.1 we know that

$$\begin{aligned}
\omega(L) &= \sum_{i=1}^{n}(\pi - f(x_i)) \\
&= \pi n - \sum_{i=1}^{n} f(x_i) \\
&= \pi n - (n-2)\pi = 2\pi. \qquad \natural
\end{aligned}$$

Similarly, we get a result for the sum of curvatures on the planar map M in a planar geometry (M, μ).

Theorem 4.2.6 *Let (M, μ) be a planar map geometry. Then the sum $\omega(M)$ of curvatures on edges in a map M is*

$$\omega(M) = 2\pi s(M),$$

where $s(M)$ denotes the sum of length of edges in M.

Proof Notice that the sum $\omega(u,v)$ of curvatures on an edge (u,v) of M is

$$\omega(u,v)=\int\limits_{v}^{u}(\pi-f(s))ds=\pi|\widehat{(u,v)}|-\int\limits_{v}^{u}f(s)ds.$$

Since M is a planar map, each of its edges appears just two times with an opposite direction. Whence, we get that

$$\begin{aligned}\omega(M) &= \sum_{(u,v)\in E(M)}\omega(u,v)+\sum_{(v,u)\in E(M)}\omega(v,u)\\ &= \pi\sum_{(u,v)\in E(M)}(|\widehat{(u,v)}|+|\widehat{(v,u)}|)-(\int\limits_{v}^{u}f(s)ds+\int\limits_{u}^{v}f(s)ds)\\ &= 2\pi s(M)\end{aligned}$$ ♮

Notice that if we assume $s(M)=1$, then Theorem 4.2.6 turns to the Gauss-bonnet theorem for a sphere. Similarly, if we consider general map geometry on an orientable surface, similar results can be also obtained such as those materials in Problem 4.7.8 and Conjecture 4.7.1 in the final section of this chapter.

§4.3 Polygons in a Planar Map Geometry

4.3.1. Existence

In an Euclid plane geometry, we have encountered triangles, quadrilaterals, $\cdots$, and generally, n-polygons, i.e., these graphs on a plane with n straight line segments not on the same line connected with one after another. There are no 1 and 2-polygons in an Euclid plane geometry since every point is euclidean. The definition of n-polygons in a planar map geometry (M,μ) is similar to that of an Euclid plane geometry.

Definition 4.3.1 *An n-polygon in a planar map geometry (M,μ) is defined to be a graph on (M,μ) with n m-line segments two by two without self-intersections and connected with one after another.*

Although their definition is similar, the situation is more complex in a planar map geometry (M,μ). We have found a necessary and sufficient condition for 1-polygon in Theorem 4.2.1, i.e., 1-polygons maybe exist in a planar map geometry. In general, we can find n-polygons in a planar map geometry for any integer $n, n\geq 1$.

Examples of polygon in a planar map geometry (M,μ) are shown in Fig.4.6, in where (a) is a 1-polygon with u,v,w and t being non-euclidean points, (b) is a 2-polygon with vertices A,B and non-euclidean points u,v, (c) is a triangle with vertices A,B,C and a non-euclidean point u and (d) is a quadrilateral with vertices A,B,C and D.

Fig.4.6

Theorem 4.3.1 *There exists a* 1*-polygon in a planar map geometry* (M,μ) *if and only if there are non-euclidean points* $u_1, u_2, \cdots, u_l$ *with* $l \geq 3$ *such that*

$$\sum_{i=1}^{l} f(u_i) = (l-2)\pi,$$

where $f(u_i)$ *denotes the angle function value at the point* u_i*,* $1 \leq i \leq l$*.*

Proof According to Theorem 4.2.1, an m-line passing through l non-euclidean points $u_1, u_2, \cdots, u_l$ is simply closed if and only if

$$\sum_{i=1}^{l} f(u_i) = (l-2)\pi,$$

i.e., 1-polygon exists in (M,μ) if and only if there are non-euclidean points $u_1, u_2, \cdots, u_l$ with the above formula hold.

Whence, we only need to prove $l \geq 3$. Since there are no 1-polygons or 2-polygons in an Euclid plane geometry, we must have $l \geq 3$ by the Hilbert's axiom $I-2$. In fact, for $l = 3$ we can really find a planar map geometry (M,μ) with a 1-polygon passing through three non-euclidean points u, v and w. Look at Fig.4.7,

Fig.4.7

in where the angle function values are $f(u) = f(v) = f(w) = \frac{2}{3}\pi$ at u, v and w. ♮

Similarly, for 2-polygons we get the following result.

Theorem 4.3.2 *There are* 2*-polygons in a planar map geometry* (M,μ) *only if there are at least one non-euclidean point in* (M,μ)*.*

Proof In fact, if there is a non-euclidean point u in (M, μ), then each straight line enter u will turn an angle $\theta = \pi - \frac{f(u)}{2}$ or $\frac{f(u)}{2} - \pi$ from the initial straight line dependent on that u is elliptic or hyperbolic. Therefore, we can get a 2-polygon in (M, μ) by choice a straight line AB passing through euclidean points in (M, μ), such as the graph shown in Fig.4.8.

Fig.4.8

This completes the proof. ♮

For the existence of n-polygons with $n \geq 3$, we have a general result as in the following.

Theorem 4.3.3 *For any integer $n, n \geq 3$, there are n-polygons in a planar map geometry (M, μ).*

Proof Since in an Euclid plane geometry, there are n-polygons for any integer $n, n \geq 3$. Therefore, there are also n-polygons in a planar map geometry (M, μ) for any integer $n, n \geq 3$. ♮

4.3.2. **Sum of internal angles**

For the sum of the internal angles in an n-polygon, we have the following result.

Theorem 4.3.4 *Let $\prod$ be an n-polygon in a map geometry with its edges passing through non-euclidean points $x_1, x_2, \cdots, x_l$. Then the sum of internal angles in $\prod$ is*

$$(n + l - 2)\pi - \sum_{i=1}^{l} f(x_i),$$

where $f(x_i)$ denotes the value of the angle function f at the point $x_i, 1 \leq i \leq l$.

Proof Denote by U, V the sets of elliptic points and hyperbolic points in $x_1, x_2, \cdots, x_l$ and $|U| = p, |V| = q$, respectively. If an m-line segment passes through an elliptic point u, add an auxiliary line segment AB in the plane as shown in Fig.4.9(1). Then we get that

$$\angle \mathrm{a} = \angle 1 + \angle 2 = \pi - f(u).$$

If an m-line passes through a hyperbolic point v, also add an auxiliary line segment AB in the plane as that shown in Fig.4.9(2). Then we get that

$$\text{angle } b = \text{angle3} + \text{angle4} = f(v) - \pi.$$

Fig.4.9

Since the sum of internal angles of an n-polygon in a plane is $(n-2)\pi$ whenever it is a convex or concave polygon, we know that the sum of the internal angles in $\prod$ is

$$(n-2)\pi + \sum_{x\in U}(\pi - f(x)) - \sum_{y\in V}(f(y)-\pi)$$
$$= (n+p+q-2)\pi - \sum_{i=1}^{l} f(x_i)$$
$$= (n+l-2)\pi - \sum_{i=1}^{l} f(x_i).$$

This completes the proof. ♮

A triangle is called *euclidean, elliptic* or *hyperbolic* if its edges only pass through one kind of euclidean, elliptic or hyperbolic points. As a consequence of Theorem 4.3.4, we get the sum of the internal angles of a triangle in a map geometry which is consistent with these already known results .

Corollary 4.3.1 *Let $\triangle$ be a triangle in a planar map geometry (M,μ). Then*

(i) the sum of its internal angles is equal to π if $\triangle$ is euclidean;

(ii) the sum of its internal angles is less than π if $\triangle$ is elliptic;

(iii) the sum of its internal angles is more than π if $\triangle$ is hyperbolic.

Proof Notice that the sum of internal angles of a triangle is

$$\pi + \sum_{i=1}^{l}(\pi - f(x_i))$$

if it passes through non-euclidean points $x_1, x_2, \cdots, x_l$. By definition, if these $x_i, 1 \leq i \leq l$ are one kind of euclidean, elliptic, or hyperbolic, then we have that $f(x_i) = \pi$, or $f(x_i) < \pi$, or $f(x_i) > \pi$ for any integer $i, 1 \leq i \leq l$. Whence, the sum of internal angles of an euclidean, elliptic or hyperbolic triangle is equal to, or lees than, or more than π. $\natural$

4.3.3. **Area of a polygon**

As it is well-known, calculation for the area $A(\triangle)$ of a triangle $\triangle$ with two sides a, b and the value of their include angle θ or three sides a, b and c in an Euclid plane is simple. Formulae for its area are

$$A(\triangle) = \frac{1}{2}ab\sin\theta \text{ or } A(\triangle) = \sqrt{s(s-a)(s-b)(s-c)},$$

where $s = \frac{1}{2}(a+b+c)$. But in a planar map geometry, calculation for the area of a triangle is complex since each of its edge maybe contains non-euclidean points. Where, we only present a programming for calculation the area of a triangle in a planar map geometry.

STEP 1 *Divide a triangle into triangles in an Euclid plane such that no edges contain non-euclidean points unless their endpoints;*

STEP 2 *Calculate the area of each triangle;*

STEP 3 *Sum up all of areas of these triangles to get the area of the given triangle in a planar map geometry.*

The simplest cases for triangle is the cases with only one non-euclidean point such as those shown in Fig.4.10(1) and (2) with an elliptic point u or with a hyperbolic point v.

Fig.4.10

Add an auxiliary line segment AB in Fig.4.10. Then by formulae in the plane trigonometry, we know that

$$A(\triangle ABC) = \sqrt{s_1(s_1-a)(s_1-b)(s_1-t)} + \sqrt{s_2(s_2-c)(s_2-d)(s_2-t)}$$

for case (1) in Fig.4.10 and

$$A(\triangle ABC) = \sqrt{s_1(s_1-a)(s_1-b)(s_1-t)} - \sqrt{s_2(s_2-c)(s_2-d)(s_2-t)}$$

for case (2) in Fig.4.10, where

$$t=\sqrt{c^2+d^2-2cd\cos\frac{f(x)}{2}}$$

with $x=u$ or v and

$$s_1=\frac{1}{2}(a+b+t),\quad s_2=\frac{1}{2}(c+d+t).$$

Generally, let $\triangle ABC$ be a triangle with its edge AB passing through p elliptic or p hyperbolic points $x_1,x_2,\cdots,x_p$ simultaneously, as those shown in Fig.4.11(1) and (2).

Fig.4.11

Where $|AC|=a$, $|BC|=b$ and $|Ax_1|=c_1$, $|x_1x_2|=c_2,\cdots,|x_{p-1}x_p|=c_p$ and $|x_pB|=c_{p+1}$. Adding auxiliary line segments $Ax_2,Ax_3,\cdots,Ax_p,AB$ in Fig.4.11, then we can find its area by the programming STEP 1 to STEP 3. By formulae in the plane trigonometry, we get that

$$|Ax_2|=\sqrt{c_1^2+c_2^2-2c_1c_2\cos\frac{f(x_1)}{2}},$$

$$\angle Ax_2x_1=\cos^{-1}\frac{c_1^2-c_2^2-|Ax_1|^2}{2c_2|Ax_2|},$$

$$\angle Ax_2x_3=\frac{f(x_2)}{2}-\angle Ax_2x_1\ \text{ or }\ 2\pi-\frac{f(x_2)}{2}-\angle Ax_2x_1,$$

$$|Ax_3|=\sqrt{|Ax_2|^2+c_3^2-2|Ax_2|c_3\cos(\frac{f(x_2)}{2}-\angle Ax_2x_3)},$$

$$\angle Ax_3x_2=\cos^{-1}\frac{|Ax_2|^2-c_3^2-|Ax_3|^2}{2c_3|Ax_3|},$$

$$\angle Ax_2x_3=\frac{f(x_3)}{2}-\angle Ax_3x_2\ \text{ or }\ 2\pi-\frac{f(x_3)}{2}-\angle Ax_3x_2,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

and generally, we get that

$$|AB| = \sqrt{|Ax_p|^2 + c_{p+1}^2 - 2|Ax_p|c_{p+1}\cos\angle Ax_pB}.$$

Then the area of the triangle $\triangle ABC$ is

$$\begin{aligned} A(\triangle ABC) &= \sqrt{s_p(s_p-a)(s_p-b)(s_p-|AB|)} \\ &+ \sum_{i=1}^{p}\sqrt{s_i(s_i-|Ax_i|)(s_i-c_{i+1})(s_i-|Ax_{i+1}|)} \end{aligned}$$

for case (1) in Fig.4.11 and

$$\begin{aligned} A(\triangle ABC) &= \sqrt{s_p(s_p-a)(s_p-b)(s_p-|AB|)} \\ &- \sum_{i=1}^{p}\sqrt{s_i(s_i-|Ax_i|)(s_i-c_{i+1})(s_i-|Ax_{i+1}|)} \end{aligned}$$

for case (2) in Fig.4.11, where for any integer $i, 1 \leq i \leq p-1$,

$$s_i = \frac{1}{2}(|Ax_i| + c_{i+1} + |Ax_{i+1}|)$$

and

$$s_p = \frac{1}{2}(a + b + |AB|).$$

Certainly, this programming can be also applied to calculate the area of an n-polygon in a planar map geometry in general.

§4.4 Circles in a Planar Map Geometry

The length of an m-line segment in a planar map geometry is defined in the following definition.

Definition 4.4.1 *The length $|AB|$ of an m-line segment AB consisted by k straight line segments $AC_1, C_1C_2, C_2C_3, \cdots, C_{k-1}B$ in a planar map geometry (M,μ) is defined by*

$$|AB| = |AC_1| + |C_1C_2| + |C_2C_3| + \cdots + |C_{k-1}B|.$$

As that shown in Chapter 3, there are not always exist a circle with any center and a given radius in a planar map geometry in the sense of the Euclid's definition. Since we have introduced angle function on a planar map geometry, we can likewise the Euclid's definition to define an m-circle in a planar map geometry in the next definition.

Definition 4.4.2 *A closed curve C without self-intersection in a planar map geometry (M,μ) is called an m-circle if there exists an m-point O in (M,μ) and a real number r such that $|OP| = r$ for each m-point P on C.*

Two Examples for m-circles in a planar map geometry (M,μ) are shown in Fig.4.12(1) and (2). The m-circle in Fig.4.12(1) is a circle in the Euclid's sense, but (2) is not. Notice that in Fig.4.12(2), m-points u and v are elliptic and the length $|OQ| = |Ou| + |uQ| = r$ for an m-point Q on the m-circle C, which seems likely an ellipse but it is not. The m-circle C in Fig.4.12(2) also implied that m-circles are more complex than those in an Euclid plane geometry.

Fig.4.12

We have a necessary and sufficient condition for the existence of an m-circle in a planar map geometry.

Theorem 4.4.1 *Let (M,μ) be a planar map geometry on a plane $\sum$ and O an m-point on (M,μ). For a real number r, there is an m-circle of radius r with center O if and only if O is in a non-outer face of M or O is in the outer face of M but for any $\epsilon, r > \epsilon > 0$, the initial and final intersection points of a circle of radius ϵ with M in an Euclid plane $\sum$ are euclidean points.*

Proof If there is a solitary non-euclidean point A with $|OA| < r$, then by those materials in Chapter 3, there are no m-circles in (M,μ) of radius r with center O.

Now if O is in the outer face of M but there exists a number $\epsilon, r > \epsilon > 0$ such that one of the initial and final intersection points of a circle of radius ϵ with M on $\sum$ is non-euclidean point, then points with distance r to O in (M,μ) at least has a gap in a circle with an Euclid sense. See Fig.4.13 for details, in where u is a non-euclidean point and the shade field denotes the map M. Therefore there are no m-circles in (M,μ) of radius r with center O.

Fig.4.13

Now if O in the outer face of M but for any $\epsilon, r > \epsilon > 0$, the initial and final intersection points of a circle of radius ϵ with M on $\sum$ are euclidean points or O is in a non-outer face of M, then by the definition of angle functions, we know that all points with distance r to O is a closed smooth curve on $\sum$, for example, see Fig.4.14(1) and (2).

Fig.4.14

Whence it is an m-circle. ♮

We construct a polar axis OX with center O in a planar map geometry as that in an Euclid geometry. Then each m-point A has a coordinate (ρ, θ), where ρ is the length of the m-line segment OA and θ is the angle between OX and the straight line segment of OA containing the point A. We get an equation for an m-circle of radius r which has the same form as that in the analytic geometry of plane.

Theorem 4.4.2 *In a planar geometry (M, μ) with a polar axis OX of center O, the equation of each m-circle of radius r with center O is*

$$\rho = r.$$

Proof By the definition of an m-circle C of radius r, every m-point on C has a distance r to its center O. Whence, its equation is $\rho = r$ in a planar map geometry with a polar axis OX of center O. ♮

§4.5 Line Bundles in a Planar Map Geometry

The behaviors of m-line bundles is need to clarify from a geometrical sense. Among those m-line bundles the most important is parallel bundles defined in the next definition, which is also motivated by the Euclid's fifth postulate discussed in the reference [54] first.

Definition 4.5.1 *A family $\mathcal{L}$ of infinite m-lines not intersecting each other in a planar geometry (M, μ) is called a parallel bundle.*

In Fig.4.15, we present all cases of parallel bundles passing through an edge in planar geometries, where, (a) is the case with the same type points u, v and $\rho_M(u)\mu(u) = \rho_M(v)\mu(v) = 2\pi$, (b) and (c) are the same type cases with $\rho_M(u)\mu(u) > \rho_M(v)\mu(v)$ or $\rho_M(u)\mu(u) = \rho_M(v)\mu(v) > 2\pi$ or $< 2\pi$ and (d) is the case with an elliptic point u but a hyperbolic point v.

Fig.4.15

Here, we assume the angle at the intersection point is in clockwise, that is, a line passing through an elliptic point will bend up and passing through a hyperbolic point will bend down, such as those cases (b),(c) in the Fig.4.15. Generally, we define a *sign function $sign(f)$ of an angle function f* as follows.

Definition 4.5.2 *For a vector $\overrightarrow{O}$ on the Euclid plane called an orientation, a sign function $sign(f)$ of an angle function f at an m-point u is defined by*

$$sign(f)(u) = \begin{cases} 1, & \text{if } u \text{ is elliptic,} \\ 0, & \text{if } u \text{ is euclidean,} \\ -1, & \text{if } u \text{ is hyperbolic.} \end{cases}$$

We classify parallel bundles in planar map geometries along an orientation $\overrightarrow{O}$ in this section.

4.5.1. **A condition for parallel bundles**

We investigate the behaviors of parallel bundles in a planar map geometry (M, μ). Denote by $f(x)$ the angle function value at an intersection m-point of an m-line L with an edge (u, v) of M and a distance x to u on (u, v) as shown in Fig.4.15(a). Then we get an elementary result as in the following.

Theorem 4.5.1 *A family $\mathcal{L}$ of parallel m-lines passing through an edge (u, v) is a parallel bundle if and only if*

$$\left.\frac{df}{dx}\right|_{+} \geq 0.$$

Proof If $\mathcal{L}$ is a parallel bundle, then any two m-lines L_1, L_2 will not intersect after them passing through the edge uv. Therefore, if θ_1, θ_2 are the angles of L_1, L_2 at the intersection m-points of L_1, L_2 with (u, v) and L_2 is far from u than L_1, then we know $\theta_2 \geq \theta_1$. Thereby we know that

$$f(x + \Delta x) - f(x) \geq 0$$

for any point with distance x from u and $\Delta x > 0$. Therefore, we get that

$$\left.\frac{df}{dx}\right|_{+} = \lim_{\Delta x \to +0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \geq 0.$$

As that shown in the Fig.4.15.

Now if $\left.\frac{df}{dx}\right|_{+} \geq 0$, then $f(y) \geq f(x)$ if $y \geq x$. Since $\mathcal{L}$ is a family of parallel m-lines before meeting uv, any two m-lines in $\mathcal{L}$ will not intersect each other after them passing through (u, v). Therefore, $\mathcal{L}$ is a parallel bundle. ♮

A general condition for a family of parallel m-lines passing through a cut of a planar map being a parallel bundle is the following.

Theorem 4.5.2 *Let (M, μ) be a planar map geometry, $C = \{(u_1, v_1), (u_2, v_2), \cdots, (u_l, v_l)\}$ a cut of the map M with order $(u_1, v_1), (u_2, v_2), \cdots, (u_l, v_l)$ from the left to the right, $l \geq 1$ and the angle functions on them are $f_1, f_2, \cdots, f_l$ (also seeing Fig.4.16), respectively.*

Fig.4.16

Then a family $\mathcal{L}$ of parallel m-lines passing through C is a parallel bundle if and only if for any $x, x \geq 0$,

$$sign(f_1)(x) f'_{1+}(x) \geq 0$$

$$sign(f_1)(x)f'_{1+}(x) + sign(f_2)(x)f'_{2+}(x) \geq 0$$
$$sign(f_1)(x)f'_{1+}(x) + sign(f_2)(x)f'_{2+}(x) + sign(f_3)(x)f'_{3+}(x) \geq 0$$
$$\cdots\cdots\cdots\cdots$$
$$sign(f_1)(x)f'_{1+}(x) + sign(f_2)(x)f'_{2+}(x) + \cdots + sign(f_l)(x)f'_{l+}(x) \geq 0.$$

Proof According to Theorem 4.5.1, we know that m-lines will not intersect after them passing through (u_1, v_1) and (u_2, v_2) if and only if for $\forall \Delta x > 0$ and $x \geq 0$,

$$sign(f_2)(x)f_2(x + \Delta x) + sign(f_1)(x)f'_{1+}(x)\Delta x \geq sign(f_2)(x)f_2(x),$$

seeing Fig.4.17 for an explanation.

Fig.4.17

That is,

$$sign(f_1)(x)f'_{1+}(x) + sign(f_2)(x)f'_{2+}(x) \geq 0.$$

Similarly, m-lines will not intersect after them passing through $(u_1, v_1), (u_2, v_2)$ and (u_3, v_3) if and only if for $\forall \Delta x > 0$ and $x \geq 0$,

$$sign(f_3)(x)f_3(x + \Delta x) + sign(f_2)(x)f'_{2+}(x)\Delta x$$
$$+ sign(f_1)(x)f'_{1+}(x)\Delta x \geq sign(f_3)(x)f_3(x).$$

That is,

$$sign(f_1)(x)f'_{1+}(x) + sign(f_2)(x)f'_{2+}(x) + sign(f_3)(x)f'_{3+}(x) \geq 0.$$

Generally, m-lines will not intersect after them passing through $(u_1, v_1), (u_2, v_2), \cdots, (u_{l-1}, v_{l-1})$ and (u_l, v_l) if and only if for $\forall \Delta x > 0$ and $x \geq 0$,

$$sign(f_l)(x)f_l(x + \Delta x) + sign(f_{l-1})(x)f'_{l-1+}(x)\Delta x +$$
$$\cdots + sign(f_1)(x)f'_{1+}(x)\Delta x \geq sign(f_l)(x)f_l(x).$$

Whence, we get that

$$sign(f_1)(x)f'_{1+}(x) + sign(f_2)(x)f'_{2+}(x) + \cdots + sign(f_l)(x)f'_{l+}(x) \geq 0.$$

Therefore, a family $\mathcal{L}$ of parallel m-lines passing through C is a parallel bundle if and only if for any $x, x \geq 0$, we have that

$$\begin{aligned} sign(f_1)(x)f'_{1+}(x) &\geq 0 \\ sign(f_1)(x)f'_{1+}(x) + sign(f_2)(x)f'_{2+}(x) &\geq 0 \\ sign(f_1)(x)f'_{1+}(x) + sign(f_2)(x)f'_{2+}(x) + sign(f_1)(x)f'_{3+}(x) &\geq 0 \\ \cdots\cdots\cdots\cdots \\ sign(f_1)(x)f'_{1+}(x) + sign(f_2)(x)f'_{2+}(x) + \cdots + sign(f_1)(x)f'_{l+}(x) &\geq 0. \end{aligned}$$

This completes the proof. $\natural$.

Corollary 4.5.1 *Let (M, μ) be a planar map geometry, $C = \{(u_1, v_1), (u_2, v_2), \cdots, (u_l, v_l)\}$ a cut of the map M with order $(u_1, v_1), (u_2, v_2), \cdots, (u_l, v_l)$ from the left to the right, $l \geq 1$ and the angle functions on them are $f_1, f_2, \cdots, f_l$, respectively. Then a family $\mathcal{L}$ of parallel lines passing through C is still parallel lines after them leaving C if and only if for any $x, x \geq 0$,*

$$\begin{aligned} sign(f_1)(x)f'_{1+}(x) &\geq 0 \\ sign(f_1)(x)f'_{1+}(x) + sign(f_2)(x)f'_{2+}(x) &\geq 0 \\ sign(f_1)(x)f'_{1+}(x) + sign(f_2)(x)f'_{2+}(x) + sign(f_1)(x)f'_{3+}(x) &\geq 0 \\ \cdots\cdots\cdots\cdots \\ sign(f_1)(x)f'_{1+}(x) + sign(f_2)(x)f'_{2+}(x) + \cdots + sign(f_1)(x)f'_{l-1+}(x) &\geq 0. \end{aligned}$$

and

$$sign(f_1)(x)f'_{1+}(x) + sign(f_2)(x)f'_{2+}(x) + \cdots + sign(f_1)(x)f'_{l+}(x) = 0.$$

Proof According to Theorem 4.5.2, we know the condition is a necessary and sufficient condition for $\mathcal{L}$ being a parallel bundle. Now since lines in $\mathcal{L}$ are parallel lines after them leaving C if and only if for any $x \geq 0$ and $\Delta x \geq 0$, there must be that

$$sign(f_l)f_l(x+\Delta x)+sign(f_{l-1})f'_{l-1+}(x)\Delta x+\cdots+sign(f_1)f'_{1+}(x)\Delta x = sign(f_l)f_l(x).$$

Therefore, we get that

$$sign(f_1)(x)f'_{1+}(x) + sign(f_2)(x)f'_{2+}(x) + \cdots + sign(f_1)(x)f'_{l+}(x) = 0. \qquad \natural$$

When do some parallel m-lines parallel the initial parallel lines after them passing through a cut C in a planar map geometry? The answer is in the next result.

Theorem 4.5.3 *Let (M, μ) be a planar map geometry, $C = \{(u_1, v_1), (u_2, v_2), \cdots, (u_l, v_l)\}$ a cut of the map M with order $(u_1, v_1), (u_2, v_2), \cdots, (u_l, v_l)$ from the left to the right, $l \geq 1$ and the angle functions on them are $f_1, f_2, \cdots, f_l$, respectively. Then the parallel m-lines parallel the initial parallel lines after them passing through C if and only if for $\forall x \geq 0$,*

$$\begin{aligned}
sign(f_1)(x)f'_{1+}(x) &\geq 0 \\
sign(f_1)(x)f'_{1+}(x) + sign(f_2)(x)f'_{2+}(x) &\geq 0 \\
sign(f_1)(x)f'_{1+}(x) + sign(f_2)(x)f'_{2+}(x) + sign(f_1)(x)f'_{3+}(x) &\geq 0 \\
\cdots\cdots\cdots\cdots \\
sign(f_1)(x)f'_{1+}(x) + sign(f_2)(x)f'_{2+}(x) + \cdots + sign(f_1)(x)f'_{l-1+}(x) &\geq 0.
\end{aligned}$$

and

$$sign(f_1)f_1(x) + sign(f_2)f_2(x) + \cdots + sign(f_1)(x)f_l(x) = l\pi.$$

Proof According to Theorem 4.5.2 and Corollary 4.5.1, we know that these parallel m-lines satisfying conditions of this theorem is a parallel bundle.

We calculate the angle $\alpha(i, x)$ of an m-line L passing through an edge $u_iv_i, 1 \leq i \leq l$ with the line before it meeting C at the intersection of L with the edge (u_i, v_i), where x is the distance of the intersection point to u_1 on (u_1, v_1), see also Fig.4.18. By definition, we know the angle $\alpha(1, x) = sign(f_1)f(x)$ and $\alpha(2, x) = sign(f_2)f_2(x) - (\pi - sign(f_1)f_1(x)) = sign(f_1)f_1(x) + sign(f_2)f_2(x) - \pi$.

Now if $\alpha(i, x) = sign(f_1)f_1(x) + sign(f_2)f_2(x) + \cdots + sign(f_i)f_i(x) - (i-1)\pi$, then we know that $\alpha(i+1, x) = sign(f_{i+1})f_{i+1}(x) - (\pi - \alpha(i, x)) = sign(f_{i+1})f_{i+1}(x) + \alpha(i, x) - \pi$ similar to the case $i = 2$. Thereby we get that

$$\alpha(i+1, x) = sign(f_1)f_1(x) + sign(f_2)f_2(x) + \cdots + sign(f_{i+1})f_{i+1}(x) - i\pi.$$

Notice that an m-line L parallel the initial parallel line after it passing through C if and only if $\alpha(l, x) = \pi$, i.e.,

$$sign(f_1)f_1(x) + sign(f_2)f_2(x) + \cdots + sign(f_l)f_l(x) = l\pi.$$

This completes the proof. $\natural$

4.5.2. **Linear conditions and combinatorial realization for parallel bundles**

For the simplicity, we can assume even that the function $f(x)$ is linear and denoted it by $f_l(x)$. We calculate $f_l(x)$ in the first.

Theorem 4.5.4 *The angle function $f_l(x)$ of an m-line L passing through an edge (u,v) at a point with distance x to u is*

$$f_l(x) = (1 - \frac{x}{d(u,v)})\frac{\rho(u)\mu(v)}{2} + \frac{x}{d(u,v)}\frac{\rho(v)\mu(v)}{2},$$

where, $d(u,v)$ is the length of the edge (u,v).

Proof Since $f_l(x)$ is linear, we know that $f_l(x)$ satisfies the following equation.

$$\frac{f_l(x) - \frac{\rho(u)\mu(u)}{2}}{\frac{\rho(v)\mu(v)}{2} - \frac{\rho(u)\mu(u)}{2}} = \frac{x}{d(u,v)},$$

Calculation shows that

$$f_l(x) = (1 - \frac{x}{d(u,v)})\frac{\rho(u)\mu(v)}{2} + \frac{x}{d(u,v)}\frac{\rho(v)\mu(v)}{2}. \quad \natural$$

Corollary 4.5.2 *Under the linear assumption, a family $\mathcal{L}$ of parallel m-lines passing through an edge (u,v) is a parallel bundle if and only if*

$$\frac{\rho(u)}{\rho(v)} \leq \frac{\mu(v)}{\mu(u)}.$$

Proof According to Theorem 4.5.1, a family of parallel m-lines passing through an edge (u,v) is a parallel bundle if and only if $f'(x) \geq 0$ for $\forall x, x \geq 0$, i.e.,

$$\frac{\rho(v)\mu(v)}{2d(u,v)} - \frac{\rho(u)\mu(u)}{2d(u,v)} \geq 0.$$

Therefore, a family $\mathcal{L}$ of parallel m-lines passing through an edge (u,v) is a parallel bundle if and only if

$$\rho(v)\mu(v) \geq \rho(u)\mu(u).$$

Whence,

$$\frac{\rho(u)}{\rho(v)} \leq \frac{\mu(v)}{\mu(u)}. \quad \natural$$

For a family of parallel m-lines passing through a cut, we get the following condition.

Theorem 4.5.5 *Let (M,μ) be a planar map geometry, $C=\{(u_1,v_1),(u_2,v_2),\cdots,(u_l,v_l)\}$ a cut of the map M with order $(u_1,v_1),(u_2,v_2),\cdots,(u_l,v_l)$ from the left to the right, $l\geq 1$. Then under the linear assumption, a family L of parallel m-lines passing through C is a parallel bundle if and only if the angle factor μ satisfies the following linear inequality system*

$$\rho(v_1)\mu(v_1)\geq\rho(u_1)\mu(u_1)$$

$$\frac{\rho(v_1)\mu(v_1)}{d(u_1,v_1)}+\frac{\rho(v_2)\mu(v_2)}{d(u_2,v_2)}\geq\frac{\rho(u_1)\mu(u_1)}{d(u_1,v_1)}+\frac{\rho(u_2)\mu(u_2)}{d(u_2,v_2)}$$

$$\cdots\cdots\cdots\cdots$$

$$\begin{aligned}\frac{\rho(v_1)\mu(v_1)}{d(u_1,v_1)} &+ \frac{\rho(v_2)\mu(v_2)}{d(u_2,v_2)}+\cdots+\frac{\rho(v_l)\mu(v_l)}{d(u_l,v_l)}\\ &\geq \frac{\rho(u_1)\mu(u_1)}{d(u_1,v_1)}+\frac{\rho(u_2)\mu(u_2)}{d(u_2,v_2)}+\cdots+\frac{\rho(u_l)\mu(u_l)}{d(u_l,v_l)}.\end{aligned}$$

Proof Under the linear assumption, for any integer $i, i\geq 1$ we know that

$$f'_{i+}(x)=\frac{\rho(v_i)\mu(v_i)-\rho(u_i)\mu(u_i)}{2d(u_i,v_i)}$$

by Theorem 4.5.4. Thereby according to Theorem 4.5.2, we get that a family L of parallel m-lines passing through C is a parallel bundle if and only if the angle factor μ satisfies the following linear inequality system

$$\rho(v_1)\mu(v_1)\geq\rho(u_1)\mu(u_1)$$

$$\frac{\rho(v_1)\mu(v_1)}{d(u_1,v_1)}+\frac{\rho(v_2)\mu(v_2)}{d(u_2,v_2)}\geq\frac{\rho(u_1)\mu(u_1)}{d(u_1,v_1)}+\frac{\rho(u_2)\mu(u_2)}{d(u_2,v_2)}$$

$$\cdots\cdots\cdots\cdots$$

$$\begin{aligned}\frac{\rho(v_1)\mu(v_1)}{d(u_1,v_1)} &+ \frac{\rho(v_2)\mu(v_2)}{d(u_2,v_2)}+\cdots+\frac{\rho(v_l)\mu(v_l)}{d(u_l,v_l)}\\ &\geq \frac{\rho(u_1)\mu(u_1)}{d(u_1,v_1)}+\frac{\rho(u_2)\mu(u_2)}{d(u_2,v_2)}+\cdots+\frac{\rho(u_l)\mu(u_l)}{d(u_l,v_l)}.\end{aligned}$$

This completes the proof. ♮

For planar maps underlying a regular graph, we have an interesting consequence for parallel bundles in the following.

Corollary 4.5.3 *Let (M,μ) be a planar map geometry with M underlying a regular graph, $C=\{(u_1,v_1),(u_2,v_2),\cdots,(u_l,v_l)\}$ a cut of the map M with order $(u_1,v_1),(u_2,v_2),\cdots,(u_l,v_l)$ from the left to the right, $l\geq 1$. Then under the linear assumption, a family L of parallel lines passing through C is a parallel bundle if and only if the angle factor μ satisfies the following linear inequality system.*

$$\mu(v_1)\geq\mu(u_1)$$

$$\frac{\mu(v_1)}{d(u_1,v_1)}+\frac{\mu(v_2)}{d(u_2,v_2)}\geq\frac{\mu(u_1)}{d(u_1,v_1)}+\frac{\mu(u_2)}{d(u_2,v_2)}$$

$$\cdots\cdots\cdots\cdots$$

$$\frac{\mu(v_1)}{d(u_1,v_1)}+\frac{\mu(v_2)}{d(u_2,v_2)}+\cdots+\frac{\mu(v_l)}{d(u_l,v_l)}\geq\frac{\mu(u_1)}{d(u_1,v_1)}+\frac{\mu(u_2)}{d(u_2,v_2)}+\cdots+\frac{\mu(u_l)}{d(u_l,v_l)}$$

and particularly, if assume that all the lengths of edges in C are the same, then

$$\begin{array}{rcl}\mu(v_1) & \geq & \mu(u_1)\\ \mu(v_1)+\mu(v_2) & \geq & \mu(u_1)+\mu(u_2)\\ \cdots\cdots & \cdots & \cdots\cdots\\ \mu(v_1)+\mu(v_2)+\cdots+\mu(v_l) & \geq & \mu(u_1)+\mu(u_2)+\cdots+\mu(u_l).\end{array}$$

Certainly, by choice different angle factors, we can also get combinatorial conditions for the existence of parallel bundles under the linear assumption.

Theorem 4.5.6 *Let (M,μ) be a planar map geometry, $C=\{(u_1,v_1),(u_2,v_2),\cdots,(u_l,v_l)\}$ a cut of the map M with order $(u_1,v_1),(u_2,v_2),\cdots,(u_l,v_l)$ from the left to the right, $l\geq 1$. If*

$$\frac{\rho(u_i)}{\rho(v_i)}\leq\frac{\mu(v_i)}{\mu(u_i)}$$

for any integer $i,i\geq 1$, then a family L of parallel m-lines passing through C is a parallel bundle under the linear assumption.

Proof Under the linear assumption we know that

$$f'_{i+}(x)=\frac{\rho(v_i)\mu(v_i)-\rho(u_i)\mu(u_i)}{2d(u_i,v_i)}$$

for any integer $i,i\geq 1$ by Theorem 4.5.4. Thereby $f'_{i+}(x)\geq 0$ for $i=1,2,\cdots,l$. We get that

$$\begin{aligned} f_1'(x) &\geq 0 \\ f_{1+}'(x) + f_{2+}'(x) &\geq 0 \\ f_{1+}'(x) + f_{2+}'(x) + f_{3+}'(x) &\geq 0 \\ &\cdots\cdots\cdots\cdots \\ f_{1+}'(x) + f_{2+}'(x) + \cdots + f_{l+}'(x) &\geq 0. \end{aligned}$$

By Theorem 4.5.2 we know that a family L of parallel m-lines passing through C is still a parallel bundle. ♮

§4.6 Examples of Planar Map Geometries

By choice different planar maps and define angle factors on their vertices, we can get various planar map geometries. In this section, we present some concrete examples for planar map geometries.

Example 4.6.1 *A complete planar map* K_4

We take a complete map K_4 embedded on a plane $\sum$ with vertices u, v, w and t and angle factors

$$\mu(u) = \frac{\pi}{2}, \quad \mu(v) = \mu(w) = \pi \text{ and } \mu(t) = \frac{2\pi}{3},$$

such as shown in Fig.4.18 where each number on the side of a vertex denotes $\rho_M(x)\mu(x)$ for $x = u, v, w$ and t.

Fig.4.18

We assume the linear assumption is holds in this planar map geometry (M, μ). Then we get a classifications for m-points in (M, μ) as follows.

$$V_{el} = \{\text{points in } (\text{uA} \setminus \{\text{A}\}) \bigcup (\text{uB} \setminus \{\text{B}\}) \bigcup (\text{ut} \setminus \{\text{t}\})\},$$

where A and B are euclidean points on (u, w) and (u, v), respectively.

$$V_{eu} = \{A, B, t\} \bigcup (P \setminus E(K_4))$$

and

$$V_{hy} = \{\text{points in } (\text{wA} \setminus \{\text{A}\}) \bigcup (\text{wt} \setminus \{\text{t}\}) \bigcup \text{wv} \bigcup (\text{tv} \setminus \{\text{t}\}) \bigcup (\text{vB} \setminus \{\text{B}\})\}.$$

Edges in K_4 are classified into $(u,t) \in C_E^1, (t,w),(t,v) \in C_E^3$, $(u,w),(u,v) \in C_E^5$ and $(w,u) \in C_E^6$.

Various m-lines in this planar map geometry are shown in Fig.4.19.

Fig.4.19

There are no 1-polygons in this planar map geometry. One 2-polygon and various triangles are shown in Fig.4.20.

Fig.4.20

Example 4.6.2 *A wheel planar map* $W_{1.4}$

We take a wheel $W_{1.4}$ embedded on a plane $\sum$ with vertices O and u, v, w, t and angle factors

$$\mu(O) = \frac{\pi}{2}, \text{ and } \mu(u) = \mu(v) = \mu(w) = \mu(t) = \frac{4\pi}{3},$$

such as shown in Fig.4.21.

Fig.4.21

There are no elliptic points in this planar map geometries. Euclidean and hyperbolic points V_{eu}, V_{hy} are as follows.

$$V_{eu} = P\bigcup\backslash(E(W_{1.4}) \setminus \{O\})$$

and

$$V_{hy} = E(W_{1.4}) \setminus \{O\}.$$

Edges are classified into $(O,u),(O,v),(O,w),(O,t) \in C_E^3$ and $(u,v),(v,w),(w,t),(t,u) \in C_E^6$. Various m-lines and one 1-polygon are shown in Fig.4.22 where each m-line will turn to its opposite direction after it meeting $W_{1.4}$ such as those m-lines L_1, L_2 and L_4, L_5 in Fig.4.22.

Fig.4.22

Example 4.6.3 *A parallel bundle in a planar map geometry*

We choose a planar ladder and define its angle factor as shown in Fig.4.23 where each number on the side of a vertex u denotes the number $\rho_M(u)\mu(u)$. Then we find a parallel bundle $\{L_i; 1 \leq i \leq 6\}$ as those shown in Fig.4.23.

Fig.4.23

§4.7 Remarks and Open Problems

4.7.1. Unless the Einstein's relativity theory, nearly all other branches of physics use an Euclid space as their spacetime model. This has their own reason, also due to one's sight because the moving of an object is more likely as it is described by an Euclid geometry. As a generalization of an Euclid geometry of plane by the Smarandache's notion, planar map geometries were introduced in the references [54] and [62]. The same research can be also done for an Euclid geometry of a space $\mathbf{R}^3$ and open problems are selected in the following.

Problem 4.7.1 *Establish Smarandache geometries of a space* $\mathbf{R}^3$ *and classify their fundamental elements, such as points, lines, polyhedrons,* $\cdots$*, etc..*

Problem 4.7.2 *Determine various surfaces in a Smarandache geometry of a space* $\mathbf{R}^3$*, such as a sphere, a surface of cylinder, circular cone, a torus, a double torus and a projective plane, a Klein bottle,* $\cdots$*, also determine various convex polyhedrons such as a tetrahedron, a pentahedron, a hexahedron,* $\cdots$*, etc..*

Problem 4.7.3 *Define the conception of volume and find formulae for volumes of convex polyhedrons in a Smarandache geometry of a space* $\mathbf{R}^3$*, such as a tetrahedron, a pentahedron or a hexahedron,* $\cdots$*, etc..*

Problem 4.7.4 *Apply Smarandache geometries of a space* $\mathbf{R}^3$ *to find knots and characterize them.*

4.7.2. As those proved in Chapter 3, we can also research these map geometries on a locally orientable surfaces and find its fundamental elements in a surface, such as a sphere, a torus, a double torus, $\cdots$ and a projective plane, a Klein bottle, $\cdots$, i.e., to establish an intrinsic geometry on a surface. For this target, open problems for surfaces with small genus should be solved in the first.

Problem 4.7.5 *Establish an intrinsic geometry by map geometries on a sphere or a torus and find its fundamental elements.*

Problem 4.7.6 *Establish an intrinsic geometry on a projective or a Klein bottle and find its fundamental elements.*

Problem 4.7.7 *Define various measures of map geometries on a locally orientable surface S and apply them to characterize the surface S.*

Problem 4.7.8 *Define the conception of curvature for a map geometry (M, μ) on a locally orientable surface and calculate the sum $\omega(M)$ of curvatures on all edges in M.*

Conjecture 4.7.1 $\omega(M) = 2\pi\chi(M)s(M)$, *where $s(M)$ denotes the sum of length of edges in M.*

5. **Pseudo-Plane geometries**

The essential idea in planar map geometries is to associate each point in a planar map with an angle factor which turns flatness of a plane to tortuous as we have seen in Chapter 4. When the order of a planar map tends to infinite and its diameter of each face tends to zero (such planar maps exist, for example, triangulations of a plane), we get a tortuous plane at the limiting point, i.e., a plane equipped with a vector and straight lines maybe not exist. We concentrate on discussing these pseudo-planes in this chapter. A relation for integral curves with differential equations is established, which enables us to find good behaviors of plane curves.

§5.1 **Pseudo-Planes**

In the classical analytic geometry of plane, each point is correspondent with the Descartes coordinate (x, y), where x and y are real numbers which ensures the flatness of a plane. Motivated by the ideas in Chapters 3 and 4, we find a new kind of planes, called *pseudo-planes* which distort the flatness of a plane and can be applied to classical mathematics.

Definition 5.1.1 *Let $\sum$ be an Euclid plane. For $\forall u \in \sum$, if there is a continuous mapping $\omega : u \to \omega(u)$ where $\omega(u) \in \mathbf{R}^n$ for an integer $n, n \geq 1$ such that for any chosen number $\epsilon > 0$, there exists a number $\delta > 0$ and a point $v \in \sum$, $\|u - v\| < \delta$ such that $\|\omega(u) - \omega(v)\| < \epsilon$, then $\sum$ is called a pseudo-plane, denoted by $(\sum, \omega)$, where $\|u - v\|$ denotes the norm between points u and v in $\sum$.*

An explanation for Definition 5.1.1 is shown in Fig.5.1, in where $n = 1$ and $\omega(u)$ is an angle function $\forall u \in \sum$.

Fig.5.1

We can also explain $\omega(u)$, $u \in \mathcal{P}$ to be the coordinate z in $u = (x, y, z) \in \mathbf{R}^3$ by taking also $n = 1$. Thereby a pseudo-plane can be also seen as a projection of an Euclid space $\mathbf{R}^{n+2}$ on an Euclid plane. This fact implies that some characteristic of the geometry of space may reflected by a pseudo-plane.

We only discuss the case of $n = 1$ and explain $\omega(u), u \in \sum$ being a periodic function in this chapter, i.e., for any integer k, $4k\pi + \omega(u) \equiv \omega(u)(mod\ 4\pi)$. Not loss of generality, we assume that $0 < \omega(u) \leq 4\pi$ for $\forall u \in \sum$. Similar to map geometries, points in a pseudo-plane are classified into three classes, i.e., *elliptic points* V_{el}, *euclidean points* V_{eu} and *hyperbolic points* V_{hy}, defined by

$$V_{el} = \{u \in \sum |\omega(u) \ < 2\pi\},$$

$$V_{eu} = \{v \in \sum |\omega(v) \ = 2\pi\}$$

and

$$V_{hy} = \{w \in \sum |\omega(w) \ > 2\pi\}.$$

We define a sign function $sign(v)$ on a point of a pseudo-plane $(\sum, \omega)$

$$sign(v) = \begin{cases} 1, & \text{if } v \text{ is elliptic,} \\ 0, & \text{if } v \text{ is euclidean,} \\ -1, & \text{if } v \text{ is hyperbolic.} \end{cases}$$

Then we get a result as in the following.

Theorem 5.1.1 *There is a straight line segment AB in a pseudo-plane $(\sum, \omega)$ if and only if for $\forall u \in AB$, $\omega(u) = 2\pi$, i.e., every point on AB is euclidean.*

Proof Since $\omega(u)$ is an angle function for $\forall u \in \sum$, we know that AB is a straight line segment if and only if for $\forall u \in AB$,

$$\frac{\omega(u)}{2} = \pi,$$

i.e., $\omega(u) = 2\pi$, u is an euclidean point. ♮

Theorem 5.1.1 implies that not every pseudo-plane has straight line segments.

Corollary 5.1.1 *If there are only finite euclidean points in a pseudo-plane $(\sum, \omega)$, then there are no straight line segments in $(\sum, \omega)$.*

Corollary 5.1.2 *There are not always exist a straight line between two given points u and v in a pseudo-plane $(\mathcal{P}, \omega)$.*

By the intermediate value theorem in calculus, we know the following result for points in a pseudo-plane.

Theorem 5.1.2 *In a pseudo-plane $(\sum, \omega)$, if $V_{el} \neq \emptyset$ and $V_{hy} \neq \emptyset$, then*

$$V_{eu} \neq \emptyset.$$

Proof By these assumptions, we can choose points $u \in V_{el}$ and $v \in V_{hy}$. Consider points on line segment uv in an Euclid plane $\sum$. Since $\omega(u) < 2\pi$ and $\omega(v) > 2\pi$, there exists at least a point $w, w \in uv$ such that $\omega(w) = 2\pi$, i.e., $w \in V_{eu}$ by the intermediate value theorem in calculus. Whence, $V_{eu} \neq \emptyset$. $\natural$

Corollary 5.1.3 *In a pseudo-plane* $(\sum, \omega)$, *if* $V_{eu} = \emptyset$, *then every point of* $(\sum, \omega)$ *is elliptic or every point of* $\sum$ *is hyperbolic.*

According to Corollary 5.1.3, pseudo-planes can be classified into four classes as follows.

C_P^1(euclidean): *pseudo-planes whose each point is euclidean.*

C_P^2(elliptic): *pseudo-planes whose each point is elliptic.*

C_P^3(hyperbolic): *pseudo-planes whose each point is hyperbolic.*

C_P^4(Smarandache's): *pseudo-planes in which there are euclidean, elliptic and hyperbolic points simultaneously.*

For the existence of an algebraic curve C in a pseudo-plane $(\sum, \omega)$, we get a criteria as in the following.

Theorem 5.1.3 *There is an algebraic curve* $F(x, y) = 0$ *passing through* (x_0, y_0) *in a domain* D *of a pseudo-plane* $(\sum, \omega)$ *with Descartes coordinate system if and only if* $F(x_0, y_0) = 0$ *and for* $\forall (x, y) \in D$,

$$(\pi - \frac{\omega(x,y)}{2})(1 + (\frac{dy}{dx})^2) = sign(x, y).$$

Proof By the definition of pseudo-planes in the case of that ω being an angle function and the geometrical meaning of the differential value of a function at a point, we know that an algebraic curve $F(x, y) = 0$ exists in a domain D of $(\sum, \omega)$ if and only if

$$(\pi - \frac{\omega(x,y)}{2}) = sign(x, y)\frac{d(\arctan(\frac{dy}{dx}))}{dx},$$

for $\forall (x, y) \in D$, i.e.,

$$(\pi - \frac{\omega(x,y)}{2}) = \frac{sign(x, y)}{1 + (\frac{dy}{dx})^2},$$

such as shown in Fig.5.2, where $\theta = \pi - \angle 2 + \angle 1$, $\lim\limits_{\triangle x \to 0} \theta = \omega(x, y)$ and (x, y) is an elliptic point.

Fig.5.2

Therefore we get that

$$(\pi - \frac{\omega(x,y)}{2})(1+(\frac{dy}{dx})^2) = sign(x,y). \qquad \natural$$

A plane curve C is called *elliptic* or *hyperbolic* if $sign(x,y) = 1$ or -1 for each point (x,y) on C. We know a result for the existence of an elliptic or a hyperbolic curve in a pseudo-plane.

Corollary 5.1.4 *An elliptic curve $F(x,y) = 0$ exists in a pseudo-plane $(\sum,\omega)$ with the Descartes coordinate system passing through (x_0,y_0) if and only if there is a domain $D \subset \sum$ such that $F(x_0,y_0) = 0$ and for $\forall(x,y) \in D$,*

$$(\pi - \frac{\omega(x,y)}{2})(1+(\frac{dy}{dx})^2) = 1$$

and there exists a hyperbolic curve $H(x,y) = 0$ in a pseudo-plane $(\sum,\omega)$ with the Descartes coordinate system passing through (x_0,y_0) if and only if there is a domain $U \subset \sum$ such that for $H(x_0,y_0) = 0$ and $\forall(x,y) \in U$,

$$(\pi - \frac{\omega(x,y)}{2})(1+(\frac{dy}{dx})^2) = -1.$$

Now construct a polar axis (ρ,θ) in a pseudo-plane $(\sum,\omega)$. Then we get a result as in the following.

Theorem 5.1.4 *There is an algebraic curve $f(\rho,\theta) = 0$ passing through (ρ_0,θ_0) in a domain F of a pseudo-plane $(\sum,\omega)$ with a polar coordinate system if and only if $f(\rho_0,\theta_0) = 0$ and for $\forall(\rho,\theta) \in F$,*

$$\pi - \frac{\omega(\rho,\theta)}{2} = sign(\rho,\theta)\frac{d\theta}{d\rho}.$$

Proof Similar to the proof of Theorem 5.1.3, we know that $\lim\limits_{\triangle x \to 0} \theta = \omega(x,y)$ and $\theta = \pi - \angle 2 + \angle 1$ if (ρ,θ) is elliptic, or $\theta = \pi - \angle 1 + \angle 2$ if (ρ,θ) is hyperbolic in Fig.5.2. Whence, we get that

$$\pi - \frac{\omega(\rho,\theta)}{2} = sign(\rho,\theta)\frac{d\theta}{d\rho}.$$

Corollary 5.1.5 *An elliptic curve $F(\rho,\theta) = 0$ exists in a pseudo-plane $(\sum,\omega)$ with a polar coordinate system passing through (ρ_0,θ_0) if and only if there is a domain $F \subset \sum$ such that $F(\rho_0,\theta_0) = 0$ and for $\forall(\rho,\theta) \in F$,*

$$\pi - \frac{\omega(\rho,\theta)}{2} = \frac{d\theta}{d\rho}$$

and there exists a hyperbolic curve $h(x,y) = 0$ in a pseudo-plane $(\sum,\omega)$ with a polar coordinate system passing through (ρ_0,θ_0) if and only if there is a domain $U \subset \sum$ such that $h(\rho_0,\theta_0) = 0$ and for $\forall(\rho,\theta) \in U$,

$$\pi - \frac{\omega(\rho,\theta)}{2} = -\frac{d\theta}{d\rho}.$$

Now we discuss a kind of expressions in an Euclid plane $\mathbf{R}^2$ for points in $\mathbf{R}^3$ and its characteristics.

Definition 5.1.2 *For a point $P = (x,y,z) \in \mathbf{R}^3$ with center O, let ϑ be the angle of vector $\overrightarrow{OP}$ with the plane XOY. Then define an angle function $\omega : (x,y) \to 2(\pi-\vartheta)$, i.e., the presentation of a point (x,y,z) in $\mathbf{R}^3$ is a point (x,y) with $\omega(x,y) = 2(\pi - \angle(\overrightarrow{OP}, XOY))$ in a pseudo-plane $(\sum,\omega)$.*

An explanation for Definition 5.2.1 is shown in Fig.5.3 where θ is an angle between the vector $\overrightarrow{OP}$ and plane XOY.

Fig.5.3

Theorem 5.1.5 *Let $(\sum,\omega)$ be a pseudo-plane and $P = (x,y,z)$ a point in $\mathbf{R}^3$. Then the point (x,y) is elliptic, euclidean or hyperbolic if and only if $z > 0$, $z = 0$ or $z < 0$.*

Proof By Definition 5.1.2, we know that $\omega(x,y) > 2\pi$, $= 2\pi$ or $< 2\pi$ if and only if $\theta > 0$,$= 0$ or < 0 since $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Those conditions are equivalent to $z > 0$, $= 0$ or < 0. ♮

The following result reveals the shape of points with a constant angle function value in a pseudo-plane $(\sum, \omega)$.

Theorem 5.1.6 *For a constant $\eta, 0 < \eta \leq 4\pi$, all points (x, y, z) in $\mathbf{R}^3$ with $\omega(x, y) = \eta$ consist an infinite circular cone with vertex O and an angle $\pi - \frac{\eta}{2}$ between its generatrix and the plane XOY.*

Proof Notice that $\omega(x_1, y_1) = \omega(x_2, y_2)$ for two points A, B in $\mathbf{R}^3$ with $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$ if and only if

$$\angle(\overrightarrow{OA}, XOY) = \angle(\overrightarrow{OB}, XOY) = \pi - \frac{\eta}{2},$$

that is, points A and B is on a circular cone with vertex O and an angle $\pi - \frac{\eta}{2}$ between $\overrightarrow{OA}$ or $\overrightarrow{OB}$ and the plane XOY. Since $z \to +\infty$, we get an infinite circular cone in $\mathbf{R}^3$ with vertex O and an angle $\pi - \frac{\eta}{2}$ between its generatrix and the plane XOY. ♮

§5.2 Integral Curves

An integral curve in an Euclid plane is defined by the next definition.

Definition 5.2.1 If the solution of a differential equation

$$\frac{dy}{dx} = f(x, y)$$

with an initial condition $y(x_0) = y_0$ exists, then all points (x, y) consisted by their solutions of this initial problem on an Euclid plane $\sum$ is called an integral curve.

By the ordinary differential equation theory, a well-known result for the unique solution of an ordinary differential equation is stated in the following. See also the reference [3] for details.

If the following conditions hold:

(i) $f(x, y)$ is continuous in a field F:

$$F : x_0 - a \leq x \leq x_0 + a, \quad y_0 - b \leq y \leq y_0 + b$$

(ii) there exist a constant ς such that for $\forall (x, y), (x, \overline{y}) \in F$,

$$|f(x, y) - f(x, \overline{y})| \leq \varsigma |y - \overline{y}|,$$

then there is an unique solution

$$y = \varphi(x), \quad \varphi(x_0) = y_0$$

for the differential equation

$$\frac{dy}{dx} = f(x, y)$$

with an initial condition $y(x_0) = y_0$ *in the interval* $[x_0 - h_0, x_0 + h_0]$*, where* $h_0 = \min(a, \frac{b}{M})$*,* $M = \max_{(x,y)\in R} |f(x, y)|$.

The conditions in this theorem are complex and can not be applied conveniently. As we have seen in Section 5.1 of this chapter, a pseudo-plane $(\sum, \omega)$ is related with differential equations in an Euclid plane $\sum$. Whence, by a geometrical view, to find an integral curve in a pseudo-plane $(\sum, \omega)$ is equivalent to solve an initial problem for an ordinary differential equation. Thereby we concentrate on to find integral curves in a pseudo-plane in this section.

According to Theorem 5.1.3, we get the following result.

Theorem 5.2.1 *A curve* C*,*

$$C = \{(x, y(x)) | \frac{dy}{dx} = f(x, y), y(x_0) = y_0\}$$

exists in a pseudo-plane $(\sum, \omega)$ *if and only if there is an interval* $I = [x_0 - h, x_0 + h]$ *and an angle function* $\omega : \sum \to \mathbf{R}$ *such that*

$$\omega(x, y(x)) = 2(\pi - \frac{sign(x, y(x))}{1 + f^2(x, y)})$$

for $\forall x \in I$ *with*

$$\omega(x_0, y(x_0)) = 2(\pi - \frac{sign(x, y(x))}{1 + f^2(x_0, y(x_0))}).$$

Proof According to Theorem 5.1.3, a curve passing through the point $(x_0, y(x_0))$ in a pseudo-plane $(\sum, \omega)$ if and only if $y(x_0) = y_0$ and for $\forall x \in I$,

$$(\pi - \frac{\omega(x, y(x))}{2})(1 + (\frac{dy}{dx})^2) = sign(x, y(x)).$$

Solving $\omega(x, y(x))$ from this equation, we get that

$$\omega(x, y(x)) = 2(\pi - \frac{sign(x, y(x))}{1 + (\frac{dy}{dx})^2}) = 2(\pi - \frac{sign(x, y(x))}{1 + f^2(x, y)}). \qquad \natural$$

Now we consider curves with an constant angle function value at each of its point.

Theorem 5.2.2 *Let* $(\sum, \omega)$ *be a pseudo-plane. Then for a constant* $0 < \theta \le 4\pi$*,*

(i) *a curve* C *passing through a point* (x_0, y_0) *and* $\omega(x, y) = \eta$ *for* $\forall (x, y) \in C$ *is closed without self-intersections on* $(\sum, \omega)$ *if and only if there exists a real number* s *such that*

$$s\eta = 2(s-2)\pi.$$

(ii) a curve C passing through a point (x_0, y_0) with $\omega(x,y) = \theta$ for $\forall(x,y) \in C$ is a circle on $(\sum, \omega)$ if and only if

$$\eta = 2\pi - \frac{2}{r},$$

where $r = \sqrt{x_0^2 + y_0^2}$, i.e., C is a projection of a section circle passing through a point (x_0, y_0) on the plane XOY.

Proof Similar to Theorem 4.3.1, we know that a curve C passing through a point (x_0, y_0) in a pseudo-plane $(\sum, \omega)$ is closed if and only if

$$\int_0^s (\pi - \frac{\omega(s)}{2})ds = 2\pi.$$

Now since $\omega(x,y) = \eta$ is constant for $\forall(x,y) \in C$, we get that

$$\int_0^s (\pi - \frac{\omega(s)}{2})ds = s(\pi - \frac{\eta}{2}).$$

Whence, we get that

$$s(\pi - \frac{\eta}{2}) = 2\pi,$$

i.e.,

$$s\eta = 2(s-2)\pi.$$

Now if C is a circle passing through a point (x_0, y_0) with $\omega(x,y) = \theta$ for $\forall(x,y) \in C$, then by the Euclid plane geometry we know that $s = 2\pi r$, where $r = \sqrt{x_0^2 + y_0^2}$. Therefore, there must be that

$$\eta = 2\pi - \frac{2}{r}.$$

This completes the proof. ♮

Two spiral curves without self-intersections are shown in Fig.5.4, in where (a) is an input but (b) an output curve.

Fig.5.4

We call the curve in Fig.5.4(a) an *elliptic in-spiral* and Fig.5.4(b) an *elliptic out-spiral*, correspondent to the right hand rule. In a polar coordinate system (ρ, θ), a spiral curve has equation

$$\rho = ce^{\theta t},$$

where c, t are real numbers and $c > 0$. If $t < 0$, then the curve is an in-spiral as the curve shown in Fig.5.4(a). If $t > 0$, then the curve is an out-spiral as shown in Fig.5.4(b).

For the case $t = 0$, we get a circle $\rho = c$ (or $x^2 + y^2 = c^2$ in the Descartes coordinate system).

Now in a pseudo-plane, we can easily find conditions for in-spiral or out-spiral curves. That is the following theorem.

Theorem 5.2.3 *Let $(\sum, \omega)$ be a pseudo-plane and let η, ζ be constants. Then an elliptic in-spiral curve C with $\omega(x, y) = \eta$ for $\forall(x, y) \in C$ exists in $(\sum, \omega)$ if and only if there exist numbers $s_1 > s_2 > \cdots > s_l > \cdots$, $s_i > 0$ for $i \geq 1$ such that*

$$s_i\eta < 2(s_i - 2i)\pi$$

for any integer $i, i \geq 1$ and an elliptic out-spiral curve C with $\omega(x, y) = \zeta$ for $\forall(x, y) \in C$ exists in $(\sum, \omega)$ if and only if there exist numbers $s_1 > s_2 > \cdots > s_l > \cdots$, $s_i > 0$ for $i \geq 1$ such that

$$s_i\zeta > 2(s_i - 2i)\pi$$

for any integer $i, i \geq 1$.

Proof Let L be an m-line like an elliptic in-spiral shown in Fig.5.5, in where x_1, $x_2, \cdots, x_n$ are non-euclidean points and x_1x_6 is an auxiliary line segment.

Fig.5.5

Then we know that

$$\sum_{i=1}^{6}(\pi - f(x_1)) < 2\pi,$$

$$\sum_{i=1}^{12}(\pi - f(x_1)) < 4\pi,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots.$$

Similarly from any initial point O to a point P far s to O on C, the sum of lost angles at P is

$$\int_0^s (\pi - \frac{\eta}{2})ds = (\pi - \frac{\eta}{2})s.$$

Whence, the curve C is an elliptic in-spiral if and only if there exist numbers $s_1 > s_2 > \cdots > s_l > \cdots$, $s_i > 0$ for $i \geq 1$ such that

$$(\pi - \frac{\eta}{2})s_1 < 2\pi,$$

$$(\pi - \frac{\eta}{2})s_2 < 4\pi,$$

$$(\pi - \frac{\eta}{2})s_3 < 6\pi,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$(\pi - \frac{\eta}{2})s_l < 2l\pi.$$

Therefore, we get that

$$s_i\eta < 2(s_i - 2i)\pi$$

for any integer $i, i \geq 1$.

Similarly, consider an m-line like an elliptic out-spiral with x_1, $x_2, \cdots$, x_n non-euclidean points. We can also find that C is an elliptic out-spiral if and only if there exist numbers $s_1 > s_2 > \cdots > s_l > \cdots$, $s_i > 0$ for $i \geq 1$ such that

$$(\pi - \frac{\zeta}{2})s_1 > 2\pi,$$

$$(\pi - \frac{\zeta}{2})s_2 > 4\pi,$$

$$(\pi - \frac{\zeta}{2})s_3 > 6\pi,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$(\pi - \frac{\zeta}{2})s_l > 2l\pi.$$

Whence, we get that

$$s_i\eta < 2(s_i - 2i)\pi.$$

for any integer $i, i \geq 1$. $\natural$

Similar to elliptic in or out-spirals, we can also define a *hyperbolic in-spiral* or *hyperbolic out-spiral* correspondent to the left hand rule, which are mirrors of curves in Fig.5.4. We get the following result for a hyperbolic in or out-spiral in a pseudo-plane.

Theorem 5.2.4 *Let $(\sum, \omega)$ be a pseudo-plane and let η, ζ be constants. Then a hyperbolic in-spiral curve C with $\omega(x, y) = \eta$ for $\forall(x, y) \in C$ exists in $(\sum, \omega)$ if and only if there exist numbers $s_1 > s_2 > \cdots > s_l > \cdots$, $s_i > 0$ for $i \geq 1$ such that*

$$s_i\eta > 2(s_i - 2i)\pi$$

for any integer $i, i \geq 1$ and a hyperbolic out-spiral curve C with $\omega(x, y) = \zeta$ for $\forall(x, y) \in C$ exists in $(\sum, \omega)$ if and only if there exist numbers $s_1 > s_2 > \cdots > s_l > \cdots$, $s_i > 0$ for $i \geq 1$ such that

$$s_i\zeta < 2(s_i - 2i)\pi$$

for any integer $i, i \geq 1$.

Proof The proof for (i) and (ii) is similar to the proof of Theorem 5.2.3. $\natural$

§5.3 Stability of a Differential Equation

For an ordinary differential equation system

$$\frac{dx}{dt} = P(x,y),$$
$$\frac{dy}{dt} = Q(x,y), \quad (A^*)$$

where t is a time parameter, the Euclid plane XOY with the Descartes coordinate system is called its a *phase plane* and the orbit $(x(t), y(t))$ of its a solution $x = x(t), y = y(t)$ is called an *orbit curve.* If there exists a point (x_0, y_0) on XOY such that

$$P(x_0, y_0) = Q(x_0, y_0) = 0,$$

then there is an obit curve which is only a point (x_0, y_0) on XOY. The point (x_0, y_0) is called a *singular point of* (A^*). Singular points of an ordinary differential equation are classified into four classes: *knot, saddle, focal* and *central points.* Each of these classes are introduced in the following.

Class 1. **Knots**

A *knot* O of a differential equation is shown in Fig.5.6 where (a) denotes that O is stable but (b) is unstable.

Fig.5.6

A *critical knot* O of a differential equation is shown in Fig.5.7 where (a) denotes that O is stable but (b) is unstable.

Fig.5.7

A *degenerate knot* O of a differential equation is shown in Fig.5.8 where (a) denotes that O is stable but (b) is unstable.

Fig.5.8

Class 2. **Saddle points**

A *saddle point* O of a differential equation is shown in Fig.5.9.

Fig.5.9

Class 3. **Focal points**

A *focal point* O of a differential equation is shown in Fig.5.10 where (a) denotes that O is stable but (b) is unstable.

Fig.5.10

Class 4. **Central points**

A *central point* O of a differential equation is shown in Fig.5.11, which is just the center of a circle.

Fig.5.11

In a pseudo-plane $(\sum, \omega)$, not all kinds of singular points exist. We get a result for singular points in a pseudo-plane as in the following.

Theorem 5.3.1 *There are no saddle points and stable knots in a pseudo-plane plane* $(\sum, \omega)$.

Proof On a saddle point or a stable knot O, there are two rays to O, seeing Fig.5.6(a) and Fig.5.10 for details. Notice that if this kind of orbit curves in Fig.5.6(a) or Fig.5.10 appears, then there must be that

$$\omega(O) = 4\pi.$$

Now according to Theorem 5.1.1, every point u on those two rays should be euclidean, i.e., $\omega(u) = 2\pi$, unless the point O. But then ω is not continuous at the point O, which contradicts Definition 5.1.1. ♮

If an ordinary differential equation system (A^*) has a closed orbit curve C but all other orbit curves are not closed in a neighborhood of C nearly enough to C and those orbits curve tend to C when $t \to +\infty$ or $t \to -\infty$, then C is called a *limiting ring of* (A^*) and *stable* or *unstable* if $t \to +\infty$ or $t \to -\infty$.

Theorem 5.3.2 *For two constants* ρ_0, θ_0, $\rho_0 > 0$ *and* $\theta_0 \neq 0$, *there is a pseudo-plane* $(\sum, \omega)$ *with*

$$\omega(\rho, \theta) = 2(\pi - \frac{\rho_0}{\theta_0 \rho})$$

or

$$\omega(\rho, \theta) = 2(\pi + \frac{\rho_0}{\theta_0 \rho})$$

such that

$$\rho = \rho_0$$

is a limiting ring in $(\sum, \omega)$.

Proof Notice that for two given constants ρ_0, θ_0, $\rho_0 > 0$ and $\theta_0 \neq 0$, the equation

$$\rho(t) = \rho_0 e^{\theta_0 \theta(t)}$$

has a stable or unstable limiting ring

$$\rho = \rho_0$$

if $\theta(t) \to 0$ when $t \to +\infty$ or $t \to -\infty$. Whence, we know that

$$\theta(t) = \frac{1}{\theta_0} \ln \frac{\rho_0}{\rho(t)}.$$

Therefore,

$$\frac{d\theta}{d\rho} = \frac{\rho_0}{\theta_0 \rho(t)}.$$

According to Theorem 5.1.4, we get that

$$\omega(\rho, \theta) = 2(\pi - sign(\rho, \theta)\frac{d\theta}{d\rho}),$$

for any point $(\rho, \theta) \in \sum$, i.e.,

$$\omega(\rho, \theta) = 2(\pi - \frac{\rho_0}{\theta_0 \rho})$$

or

$$\omega(\rho, \theta) = 2(\pi + \frac{\rho_0}{\theta_0 \rho}). \quad \natural$$

A general pseudo-space is discussed in the next section which enables us to know the Finsler geometry is a particular case of Smnarandache geometries.

§5.4 Remarks and Open Problems

Definition 5.1.1 can be generalized as follows, which enables us to enlarge our fields of mathematics for further research.

Definition 5.4.1 *Let U and W be two metric spaces with metric ρ, $W \subseteq U$. For $\forall u \in U$, if there is a continuous mapping $\omega : u \to \omega(u)$, where $\omega(u) \in \mathbf{R}^n$ for an integer $n, n \geq 1$ such that for any number $\epsilon > 0$, there exists a number $\delta > 0$ and a point $v \in W$, $\rho(u - v) < \delta$ such that $\rho(\omega(u) - \omega(v)) < \epsilon$, then U is called a metric pseudo-space if $U = W$ or a bounded metric pseudo-space if there is a number $N > 0$ such that $\forall w \in W$, $\rho(w) \leq N$, denoted by (U, ω) or (U^-, ω), respectively.*

By choice different metric spaces U and W in this definition, we can get various metric pseudo-spaces. For the case $n = 1$, we can also explain $\omega(u)$ being an angle function with $0 < \omega(u) \leq 4\pi$, i.e.,

$$\omega(u) = \begin{cases} \omega(u)(mod 4\pi), & \text{if u} \in \text{W}, \\ 2\pi, & \text{if u} \in \text{U} \setminus \text{W} \end{cases} \qquad (*)$$

and get some interesting metric pseudo-spaces.

5.4.1. **Bounded pseudo-plane geometries** Let C be a closed curve in an Euclid plane $\sum$ without self-intersections. Then C divides $\sum$ into two domains. One of them is finite. Denote by D_{fin} the finite one. Call C a boundary of D_{fin}. Now let $U = \sum$ and $W = D_{fin}$ in Definition 5.4.1 for the case of $n = 1$. For example, choose C be a 6-polygon such as shown in Fig.5.12.

Fig.5.12

Then we get a geometry $(\sum^{-}, \omega)$ partially euclidean and partially non-euclidean.

Problem 5.4.1 *Similar to Theorem* 4.5.2*, find conditions for parallel bundles on* $(\sum^{-}, \omega)$.

Problem 5.4.2 *Find conditions for existing an algebraic curve* $F(x, y) = 0$ *on* $(\sum^{-}, \omega)$.

Problem 5.4.3 *Find conditions for existing an integer curve* C *on* $(\sum^{-}, \omega)$.

5.4.2. **Pseudo-Space geometries** For any integer $m, m \geq 3$ and a point $\overline{u} \in \mathbf{R}^m$. Choose $U = W = \mathbf{R}^m$ in Definition 5.4.1 for the case of $n = 1$ and $\omega(\overline{u})$ an angle function. Then we get a pseudo-space geometry $(\mathbf{R}^m, \omega)$ on $\mathbf{R}^m$.

Problem 5.4.4 *Find conditions for existing an algebraic surface* $F(x_1, x_2, \cdots, x_m) = 0$ *in* $(\mathbf{R}^m, \omega)$*, particularly, for an algebraic surface* $F(x_1, x_2, x_3) = 0$ *existing in* $(\mathbf{R}^3, \omega)$.

Problem 5.4.5 *Find conditions for existing an integer surface in* $(\mathbf{R}^m, \omega)$.

If we take $U = \mathbf{R}^m$ and W a bounded convex point set of $\mathbf{R}^m$ in Definition 5.4.1. Then we get a bounded pseudo-space $(\mathbf{R}^{m-}, \omega)$, which is partially euclidean and partially non-euclidean.

Problem 5.4.6 *For a bounded pseudo-space* $(\mathbf{R}^{m-}, \omega)$*, solve Problems* 5.4.4 *and* 5.4.5 *again.*

5.4.3. **Pseudo-Surface geometries** For a locally orientable surface S and $\forall u \in S$, we choose $U = W = S$ in Definition 5.4.1 for $n = 1$ and $\omega(u)$ an angle function. Then we get a pseudo-surface geometry (S, ω) on the surface S.

Problem 5.4.7 *Characterize curves on a surface S by choice angle function ω. Whether can we classify automorphisms on S by applying pseudo-surface geometries (S, ω)?*

Notice that Thurston had classified automorphisms of a surface $S, \chi(S) \leq 0$ into three classes in [86]: *reducible, periodic* or *pseudo-Anosov*.

If we take $U = S$ and W a bounded simply connected domain of S in Definition 5.4.1. Then we get a bounded pseudo-surface (S^-, ω).

Problem 5.4.8 *For a bounded pseudo-surface* (S^-, ω)*, solve Problem* 5.4.7.

5.4.4. **Pseudo-Manifold geometries** For an m-manifold M^m and $\forall u \in M^m$, choose $U = W = M^m$ in Definition 5.4.1 for $n = 1$ and $\omega(u)$ a smooth function. Then we get a pseudo-manifold geometry (M^m, ω) on the m-manifold M^m. This geometry includes the *Finsler geometry*, i.e., equipped each m-manifold with a Minkowski norm defined in the following ([13], [39]).

A *Minkowski norm* on M^m is a function $F : M^m \to [0, +\infty)$ such that

(*i*) F is smooth on $M^m \setminus \{0\}$;

(*ii*) F is 1-homogeneous, i.e., $F(\lambda\overline{u}) = \lambda F(\overline{u})$ for $\overline{u} \in M^m$ and $\lambda > 0$;

(*iii*) for $\forall y \in M^m \setminus \{0\}$, the symmetric bilinear form $g_y : M^m \times M^m \to R$ with

$$g_y(\overline{u}, \overline{v}) = \frac{1}{2} \frac{\partial^2 F^2(y + s\overline{u} + t\overline{v})}{\partial s \partial t}|_{t=s=0}$$

.

is positive definite.

Then a *Finsler manifold* is a manifold M^m and a function $F : TM^m \to [0, +\infty)$ such that

(*i*) F is smooth on $TM^m \setminus \{0\} = \bigcup\{T_{\overline{x}}M^m \setminus \{0\} : \overline{x} \in M^m\}$;

(*ii*) $F|_{T_{\overline{x}}M^m} \to [0, +\infty)$ is a Minkowski norm for $\forall \overline{x} \in M^m$.

As a special case of pseudo-manifold geometries, we choose $\omega(\overline{x}) = F(\overline{x})$ for $\overline{x} \in M^m$, then (M^m, ω) is a *Finsler manifold*, particularly, if $\omega(\overline{x}) = g_{\overline{x}}(y, y) = F^2(x, y)$, then (M^m, ω) is a *Riemann manifold*. Thereby, *Smarandache geometries, particularly pseudo-manifold geometries include the Finsler geometry.*

Open problems for pseudo-manifold geometries are presented in the following.

Problem 5.4.9 *Characterize these pseudo-manifold geometries (M^m, ω) without boundary and apply them to classical mathematics and to classical mechanics.*

Similarly, if we take $U = M^m$ and W a bounded submanifold of M^m in Definition 5.4.1. Then we get a bounded pseudo-manifold (M^{m-}, ω).

Problem 5.4.10 *Characterize these pseudo-manifold geometries* (M^{m-}, ω) *with boundary and apply them to classical mathematics and to classical mechanics, particularly, to hamiltonian mechanics.*