On Algebraic Multi-Vector Spaces

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Abstract: A Smarandache multi-space is a union of n spaces $A_1, A_2, \cdots, A_n$ with some additional conditions holding. Combining Smarandache multi-spaces with linear vector spaces in classical linear algebra, the conception of multi-vector spaces is introduced. Some characteristics of a multi-vector space are obtained in this paper.

Key words: vector, multi-space, multi-vector space, ideal subspace chain.

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1. Introduction

The notion of multi-spaces is introduced by Smarandache in [6] under his idea of hybrid mathematics: *combining different fields into a unifying field*([7]), which is defined as follows.

Definition 1.1 *For any integer $i, 1 \leq i \leq n$ let A_i be a set with ensemble of law L_i, and the intersection of k sets $A_{i_1}, A_{i_2}, \cdots, A_{i_k}$ of them constrains the law $I(A_{i_1}, A_{i_2}, \cdots, A_{i_k})$. Then the union of A_i, $1 \leq i \leq n$*

$$\widetilde{A} = \bigcup_{i=1}^{n} A_i$$

is called a multi-space.

As we known, a *vector space* or *linear space* consists of the following:

(i) a field F of scalars;

(ii) a set V of objects, called vectors;

(iii) an operation, called vector addition, which associates with each pair of vectors $\mathbf{a}, \mathbf{b}$ in V a vector $\mathbf{a} + \mathbf{b}$ in V, called the sum of $\mathbf{a}$ and $\mathbf{b}$, in such a way that

(1) addition is commutative, $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$;

(2) addition is associative, $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$;

(3) there is a unique vector $\mathbf{0}$ in V, called the zero vector, such that $\mathbf{a} + \mathbf{0} = \mathbf{a}$ for all $\mathbf{a}$ in V;

(4) for each vector $\mathbf{a}$ in Vthere is a unique vector $-\mathbf{a}$ in V such that $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$;

(iv) an operation $\cdot$, called scalar multiplication, which associates with each scalar k in F and a vector $\mathbf{a}$ in V a vector $k \cdot \mathbf{a}$ in V, called the product of k with $\mathbf{a}$, in such a way that

(1) $1 \cdot \mathbf{a} = \mathbf{a}$ for every $\mathbf{a}$ in V;

(2) $(k_1k_2)\cdot\mathbf{a} = k_1(k_2\cdot\mathbf{a})$;
(3) $k\cdot(\mathbf{a}+\mathbf{b}) = k\cdot\mathbf{a} + k\cdot\mathbf{b}$;
(4) $(k_1+k_2)\cdot\mathbf{a} = k_1\cdot\mathbf{a} + k_2\cdot\mathbf{a}$.
We say that V is a *vector space over the field F, denoted by $(V\ ;+,\cdot)$.*

By combining Smarandache multi-spaces with linear spaces, a new kind of algebraic structure called multi-vector space is found, which is defined in the following.

Definition 1.2 *Let $\widetilde{V} = \bigcup\limits_{i=1}^{k} V_i$ be a complete multi-space with binary operation set $O(\widetilde{V}) = \{(\dot{+}_i, \cdot_i) \mid 1 \le i \le m\}$ and $\widetilde{F} = \bigcup\limits_{i=1}^{k} F_i$ a multi-filed space with double binary operation set $O(\widetilde{F}) = \{(+_i, \times_i) \mid 1 \le i \le k\}$. If for any integers $i, j,\ 1 \le i, j \le k$ and $\forall\mathbf{a}, \mathbf{b}, \mathbf{c} \in \widetilde{V}$, $k_1, k_2 \in \widetilde{F}$,*

(i) $(V_i; \dot{+}_i, \cdot_i)$ is a vector space on F_i with vector additive $\dot{+}_i$ and scalar multiplication $\cdot_i$;

(ii) $(\mathbf{a}\dot{+}_i\mathbf{b})\dot{+}_j\mathbf{c} = \mathbf{a}\dot{+}_i(\mathbf{b}\dot{+}_j\mathbf{c})$;

(iii) $(k_1 +_i k_2)\cdot_j \mathbf{a} = k_1 +_i (k_2 \cdot_j \mathbf{a})$;

if all those operation results exist, then $\widetilde{V}$ is called a multi-vector space on the multi-filed space $\widetilde{F}$ with a binary operation set $O(\widetilde{V})$, denoted by $(\widetilde{V}; \widetilde{F})$.

For subsets $\widetilde{V}_1 \subset \widetilde{V}$ and $\widetilde{F}_1 \subset \widetilde{F}$, if $(\widetilde{V}_1; \widetilde{F}_1)$ is also a multi-vector space, then call $(\widetilde{V}_1; \widetilde{F}_1)$ a multi-vector subspace of $(\widetilde{V}; \widetilde{F})$.

The subject of this paper is to find some characteristics of a multi-vector space. For terminology and notation not defined here can be seen in [1], [3] for linear algebraic terminologies and in [2], [4] − [11] for multi-spaces and logics.

2. Characteristics of a multi-vector space

First, we have the following result for multi-vector subspace of a multi-vector space.

Theorem 2.1 *For a multi-vector space $(\widetilde{V}; \widetilde{F})$, $\widetilde{V}_1 \subset \widetilde{V}$ and $\widetilde{F}_1 \subset \widetilde{F}$, $(\widetilde{V}_1; \widetilde{F}_1)$ is a multi-vector subspace of $(\widetilde{V}; \widetilde{F})$ if and only if for any vector additive $\dot{+}$, scalar multiplication $\cdot$ in $(\widetilde{V}_1; \widetilde{F}_1)$ and $\forall\mathbf{a}, \mathbf{b} \in \widetilde{V}$, $\forall\alpha \in \widetilde{F}$,*

$$\alpha\cdot\mathbf{a}\dot{+}\mathbf{b} \in \widetilde{V}_1$$

if their operation result exist.

Proof Denote by $\widetilde{V} = \bigcup\limits_{i=1}^{k} V_i, \widetilde{F} = \bigcup\limits_{i=1}^{k} F_i$. Notice that $\widetilde{V}_1 = \bigcup\limits_{i=1}^{k} (\widetilde{V}_1 \bigcap V_i)$. By definition, we know that $(\widetilde{V}_1; \widetilde{F}_1)$ is a multi-vector subspace of $(\widetilde{V}; \widetilde{F})$ if and only if for any integer $i, 1 \le i \le k$, $(\widetilde{V}_1 \bigcap V_i; \dot{+}_i, \cdot_i)$ is a vector subspace of $(V_i, \dot{+}_i, \cdot_i)$ and $\widetilde{F}_1$ is a multi-filed subspace of $\widetilde{F}$ or $\widetilde{V}_1 \bigcap V_i = \emptyset$.

According to the criterion for linear subspaces of a linear space ([3]), we know that for any integer $i, 1 \leq i \leq k$, $(\widetilde{V_1} \bigcap V_i; \dot{+}_i, \cdot_i)$ is a vector subspace of $(V_i, \dot{+}_i, \cdot_i)$ if and only if for $\forall \mathbf{a}, \mathbf{b} \in \widetilde{V_1} \bigcap V_i$, $\alpha \in F_i$,

$$\alpha \cdot_i \mathbf{a} \dot{+}_i \mathbf{b} \in \widetilde{V_1} \bigcap V_i.$$

That is, for any vector additive $\dot{+}$, scalar multiplication $\cdot$ in $(\widetilde{V_1}; \widetilde{F_1})$ and $\forall \mathbf{a}, \mathbf{b} \in \widetilde{V}$, $\forall \alpha \in \widetilde{F}$, if $\alpha \cdot \mathbf{a} \dot{+} \mathbf{b}$ exists, then $\alpha \cdot \mathbf{a} \dot{+} \mathbf{b} \in \widetilde{V_1}$. ♮

Corollary 2.1 *Let $(\widetilde{U}; \widetilde{F_1}), (\widetilde{W}; \widetilde{F_2})$ be two multi-vector subspaces of a multi-vector space $(\widetilde{V}; \widetilde{F})$. Then $(\widetilde{U} \bigcap \widetilde{W}; \widetilde{F_1} \bigcap \widetilde{F_2})$ is a multi-vector space.*

For a multi-vector space $(\widetilde{V}; \widetilde{F})$, vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n \in \widetilde{V}$, if there are scalars $\alpha_1, \alpha_2, \cdots, \alpha_n \in \widetilde{F}$ such that

$$\alpha_1 \cdot_1 \mathbf{a}_1 \dot{+}_1 \alpha_2 \cdot_2 \mathbf{a}_2 \dot{+}_2 \cdots \dot{+}_{n-1} \alpha_n \cdot_n \mathbf{a}_n = \mathbf{0},$$

where $\mathbf{0} \in \widetilde{V}$ is an unit under an operation $+$ in $\widetilde{V}$ and $\dot{+}_i, \cdot_i \in O(\widetilde{V})$, then the vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$ are said to be *linearly dependent*. Otherwise, $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$ to be *linearly independent*.

Notice that in a multi-vector space, there are two cases for linearly independent vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$:

(i) for any scalars $\alpha_1, \alpha_2, \cdots, \alpha_n \in \widetilde{F}$, if

$$\alpha_1 \cdot_1 \mathbf{a}_1 \dot{+}_1 \alpha_2 \cdot_2 \mathbf{a}_2 \dot{+}_2 \cdots \dot{+}_{n-1} \alpha_n \cdot_n \mathbf{a}_n = \mathbf{0},$$

where $\mathbf{0}$ is a unit of $\widetilde{V}$ under an operation $+$ in $O(\widetilde{V})$, then $\alpha_1 = 0_{+_1}, \alpha_2 = 0_{+_2}, \cdots, \alpha_n = 0_{+_n}$, where $0_{+_i}, 1 \leq i \leq n$ are the units under the operation $+_i$ in $\widetilde{F}$.

(ii) the operation result of $\alpha_1 \cdot_1 \mathbf{a}_1 \dot{+}_1 \alpha_2 \cdot_2 \mathbf{a}_2 \dot{+}_2 \cdots \dot{+}_{n-1} \alpha_n \cdot_n \mathbf{a}_n$ does not exist.

Now for a subset $\widehat{S} \subset \widetilde{V}$, define its *linearly spanning set* $\left\langle \widehat{S} \right\rangle$ to be

$$\left\langle \widehat{S} \right\rangle = \{ \ \mathbf{a} \mid \mathbf{a} = \alpha_1 \cdot_1 \mathbf{a}_1 \dot{+}_1 \alpha_2 \cdot_2 \mathbf{a}_2 \dot{+}_2 \cdots \in \widetilde{V}, \mathbf{a}_i \in \widehat{S}, \alpha_i \in \widetilde{F}, i \geq 1\}.$$

For a multi-vector space $(\widetilde{V}; \widetilde{F})$, if there exists a subset $\widehat{S}, \widehat{S} \subset \widetilde{V}$ such that $\widetilde{V} = \left\langle \widehat{S} \right\rangle$, then we say $\widehat{S}$ is a *linearly spanning set* of the multi-vector space $\widetilde{V}$. If the vectors in a linearly spanning set $\widehat{S}$ of the multi-vector space $\widetilde{V}$ are linearly independent, then $\widehat{S}$ is said to be a *basis* of $\widetilde{V}$.

Theorem 2.2 *Any multi-vector space $(\widetilde{V}; \widetilde{F})$ has a basis.*

Proof Assume $\widetilde{V} = \bigcup\limits_{i=1}^{k} V_i, \widetilde{F} = \bigcup\limits_{i=1}^{k} F_i$ and the basis of the vector space $(V_i; \dot{+}_i, \cdot_i)$ is $\Delta_i = \{\mathbf{a}_{i1}, \mathbf{a}_{i2}, \cdots, \mathbf{a}_{in_i}\}$, $1 \leq i \leq k$. Define

$$\widehat{\Delta} = \bigcup_{i=1}^{k} \Delta_i.$$

Then $\widehat{\Delta}$ is a linearly spanning set for $\widetilde{V}$ by definition.

If vectors in $\widehat{\Delta}$ are linearly independent, then $\widehat{\Delta}$ is a basis of $\widetilde{V}$. Otherwise, choose a vector $\mathbf{b}_1 \in \widehat{\Delta}$ and define $\widehat{\Delta}_1 = \widehat{\Delta} \setminus \{\mathbf{b}_1\}$.

If we have obtained the set $\widehat{\Delta}_s, s \geq 1$ and it is not a basis, choose a vector $\mathbf{b}_{s+1} \in \widehat{\Delta}_s$ and define $\widehat{\Delta}_{s+1} = \widehat{\Delta}_s \setminus \{\mathbf{b}_{s+1}\}$.

If the vectors in $\widehat{\Delta}_{s+1}$ are linearly independent, then $\widehat{\Delta}_{s+1}$ is a basis of $\widetilde{V}$. Otherwise, we can define the set $\widehat{\Delta}_{s+2}$. Continue this process. Notice that for any integer $i, 1 \leq i \leq k$, the vectors in Δ_i are linearly independent. Therefore, we can finally get a basis of $\widetilde{V}$. ♮

Now we consider the finite-dimensional multi-vector space. A multi-vector space $\widetilde{V}$ is *finite-dimensional* if it has a finite basis. By Theorem 2.2, if for any integer $i, 1 \leq i \leq k$, the vector space $(V_i; +_i, \cdot_i)$ is finite-dimensional, then $(\widetilde{V}; \widetilde{F})$ is finite-dimensional. On the other hand, if there is an integer $i_0, 1 \leq i_0 \leq k$, such that the vector space $(V_{i_0}; +_{i_0}, \cdot_{i_0})$ is infinite-dimensional, then $(\widetilde{V}; \widetilde{F})$ is infinite-dimensional. This enables us to get the following corollary.

Corollary 2.2 *Let $(\widetilde{V}; \widetilde{F})$ be a multi-vector space with $\widetilde{V} = \bigcup_{i=1}^{k} V_i, \widetilde{F} = \bigcup_{i=1}^{k} F_i$. Then $(\widetilde{V}; \widetilde{F})$ is finite-dimensional if and only if for any integer $i, 1 \leq i \leq k$, $(V_i; +_i, \cdot_i)$ is finite-dimensional.*

Theorem 2.3 *For a finite-dimensional multi-vector space $(\widetilde{V}; \widetilde{F})$, any two bases have the same number of vectors.*

Proof Let $\widetilde{V} = \bigcup_{i=1}^{k} V_i$ and $\widetilde{F} = \bigcup_{i=1}^{k} F_i$. The proof is by the induction on k. For $k = 1$, the assertion is true by Theorem 4 of Chapter 2 in [3].

For the case of $k = 2$, notice that by a result in linearly vector space theory (see also [3]), for two subspaces W_1, W_2 of a finite-dimensional vector space, if the basis of $W_1 \bigcap W_2$ is $\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_t\}$, then the basis of $W_1 \bigcup W_2$ is

$$\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_t, \mathbf{b}_{t+1}, \mathbf{b}_{t+2}, \cdots, \mathbf{b}_{dimW_1}, \mathbf{c}_{t+1}, \mathbf{c}_{t+2}, \cdots, \mathbf{c}_{dimW_2}\},$$

where, $\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_t, \mathbf{b}_{t+1}, \mathbf{b}_{t+2}, \cdots, \mathbf{b}_{dimW_1}\}$ is a basis of W_1 and $\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_t, \mathbf{c}_{t+1}, \mathbf{c}_{t+2}, \cdots, \mathbf{c}_{dimW_2}\}$ a basis of W_2.

Whence, if $\widetilde{V} = W_1 \bigcup W_2$ and $\widetilde{F} = F_1 \bigcup F_2$, then the basis of $\widetilde{V}$ is also

$$\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_t, \mathbf{b}_{t+1}, \mathbf{b}_{t+2}, \cdots, \mathbf{b}_{dimW_1}, \mathbf{c}_{t+1}, \mathbf{c}_{t+2}, \cdots, \mathbf{c}_{dimW_2}\}.$$

Assume the assertion is true for $k = l, l \geq 2$. Now we consider the case of $k = l + 1$. In this case, since

$$\widetilde{V} = (\bigcup_{i=1}^{l} V_i)\bigcup V_{l+1},\ \ \widetilde{F} = (\bigcup_{i=1}^{l} F_i)\bigcup F_{l+1},$$

by the induction assumption, we know that any two bases of the multi-vector space $(\bigcup_{i=1}^{l} V_i; \bigcup_{i=1}^{l} F_i)$ have the same number p of vectors. If the basis of $(\bigcup_{i=1}^{l} V_i)\bigcap V_{l+1}$ is $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$, then the basis of $\widetilde{V}$ is

$$\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n, \mathbf{f}_{n+1}, \mathbf{f}_{n+2}, \cdots, \mathbf{f}_p, \mathbf{g}_{n+1}, \mathbf{g}_{n+2}, \cdots, \mathbf{g}_{dimV_{l+1}}\},$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n, \mathbf{f}_{n+1}, \mathbf{f}_{n+2}, \cdots, \mathbf{f}_p\}$ is a basis of $(\bigcup_{i=1}^{l} V_i; \bigcup_{i=1}^{l} F_i)$ and $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n, \mathbf{g}_{n+1}, \mathbf{g}_{n+2}, \cdots, \mathbf{g}_{dimV_{l+1}}\}$ a basis of V_{l+1}. Whence, the number of vectors in a basis of $\widetilde{V}$ is $p + dimV_{l+1} - n$ for the case $n = l + 1$.

Therefore, by the induction principle, we know the assertion is true for any integer k. ♮

The number of a finite-dimensional multi-vector space $\widetilde{V}$ is called its *dimension*, denoted by $dim\widetilde{V}$.

Theorem 2.4(*dimensional formula*) *For a multi-vector space $(\widetilde{V}; \widetilde{F})$ with $\widetilde{V} = \bigcup_{i=1}^{k} V_i$ and $\widetilde{F} = \bigcup_{i=1}^{k} F_i$, the dimension $dim\widetilde{V}$ of $\widetilde{V}$ is*

$$dim\widetilde{V} = \sum_{i=1}^{k}(-1)^{i-1} \sum_{\{i1,i2,\cdots,ii\}\subset\{1,2,\cdots,k\}} dim(V_{i1}\bigcap V_{i2}\bigcap\cdots\bigcap V_{ii}).$$

Proof The proof is by induction on k. For $k = 1$, the formula is the trivial case of $dim\widetilde{V} = dimV_1$. for $k = 2$, the formula is

$$dim\widetilde{V} = dimV_1 + dimV_2 - dim(V_1\bigcap dimV_2),$$

which is true by Theorem 6 of Chapter 2 in [3].

Now assume the formula is true for $k = n$. Consider the case of $k = n + 1$. According to the proof of Theorem 2.15, we know that

$$\begin{aligned} dim\widetilde{V} &= dim(\bigcup_{i=1}^{n} V_i) + dimV_{n+1} - dim((\bigcup_{i=1}^{n} V_i)\bigcap V_{n+1}) \\ &= dim(\bigcup_{i=1}^{n} V_i) + dimV_{n+1} - dim(\bigcup_{i=1}^{n}(V_i\bigcap V_{n+1})) \\ &= dimV_{n+1} + \sum_{i=1}^{n}(-1)^{i-1} \sum_{\{i1,i2,\cdots,ii\}\subset\{1,2,\cdots,n\}} dim(V_{i1}\bigcap V_{i2}\bigcap\cdots\bigcap V_{ii}) \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{i=1}^{n}(-1)^{i-1}\sum_{\{i1,i2,\cdots,ii\}\subset\{1,2,\cdots,n\}} dim(V_{i1}\bigcap V_{i2}\bigcap\cdots\bigcap V_{ii}\bigcap V_{n+1}) \\
&= \sum_{i=1}^{n}(-1)^{i-1}\sum_{\{i1,i2,\cdots,ii\}\subset\{1,2,\cdots,k\}} dim(V_{i1}\bigcap V_{i2}\bigcap\cdots\bigcap V_{ii}).
\end{aligned}
$$

By the induction principle, we know this formula is true for any integer k. ♮

From Theorem 2.4, we get the following additive formula for any two multi-vector spaces.

Corollary 2.3(*additive formula*) *For any two multi-vector spaces* $\widetilde{V}_1, \widetilde{V}_2$,

$$dim(\widetilde{V}_1\bigcup\widetilde{V}_2) = dim\widetilde{V}_1 + dim\widetilde{V}_2 - dim(\widetilde{V}_1\bigcap\widetilde{V}_2).$$

3. **Open problems for a multi-ring space**

Notice that Theorem 2.3 has told us there is a similar linear theory for multi-vector spaces, but the situation is more complex. Here, we present some open problems for further research.

Problem 3.1 *Similar to linear spaces, define linear transformations on multi-vector spaces. Can we establish a new matrix theory for linear transformations?*

Problem 3.2 *Whether a multi-vector space must be a linear space?*

Conjecture A *There are non-linear multi-vector spaces in multi-vector spaces.*

Based on Conjecture A, there is a fundamental problem for multi-vector spaces.

Problem 3.3 *Can we apply multi-vector spaces to non-linear spaces?*

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