

I /Peutrosophic'Upace

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Mg{y qtf ur Group action, G-space, orbit, stabilizer, G-neutrosophic space, neutrosophic orbit, neutrosophic stabilizer.

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The Concept of a G-space came into being as a consequence of Group action on an ordinary set. Over the history of Mathematics and Algebra, theory of group action has emerged and proven to be an applicable and effective framework for the study of different kinds of structures to make connection among them. The applications of group action in different areas of science such as physics, chemistry, biology, computer science, game theory, cryptography etc has been worked out very well. The abstraction provided by group actions is a powerful one, because it allows geometrical ideas to be applied to more abstract objects. Many objects in Mathematics have natural group actions defined on them. In particular, groups can act on other groups, or even on themselves. Despite this generality, the theory of group actions contains wide-reaching theorems, such as the orbit stabilizer theorem, which can be used to prove deep results in several fields. Neutrosophy is a branch of neutrosophic philosophy which handles the origin and stages of neutralities. Neutrosophic science is a newly emerging science which has been firstly introduced by Florentin Smarandache in 1995. This is quite a general phenomenon which can be found almost everywhere in the nature. Neutrosophic approach provides a generosity to absorbing almost all the corresponding algebraic structures open heartedly. This tradition is also maintained in our work here. The combination of neutrosophy and group action gives some extra ordinary excitement while forming this new structure called G-neutrosophic space. This is a generalization of all the work of the past and some new notions are also raised due to this approach. Some new types of spaces and their core properties have been discovered here for the first time. Examples and counter examples have been illustrated wherever required. In this paper we have also coined a new term called pseudo neutrosophic spaces and a new property

called ideal property. The link of transitivity with ideal property and the corresponding results are established.

4.0.1. Theorem 1

Definition 1

Definition 3: Let Ω be a non empty set and I be a group. Let $\mu: \Omega \times I \rightarrow \Omega$ be a mapping. Then μ is called an action of I on Ω such that for all $\omega \in \Omega$ and $i, j \in I$.

- 1) $\mu(\mu(\omega, i), j) = \mu(\omega, ij)$
- 2) $\mu(\omega, 1) = \omega$, where 1 is the identity element in I .

Usually we write ω^i instead of $\mu(\omega, i)$. Therefore 1 and 2 becomes as

- 1) $\omega^{i \cdot j} = \omega^{ij}$. For all $\omega \in \Omega$ and $i, j \in I$.
- 2) $\omega^1 = \omega$.

Definition 4: Let Ω be a I -space. Let $\Omega_1 \neq \emptyset$ be a subset of Ω . Then Ω_1 is called I -subspace of Ω if $\omega^i \in \Omega_1$ for all $\omega \in \Omega_1$ and $i \in I$.

Definition 5: We say that Ω is transitive I -space if for any $\alpha, \beta \in \Omega$, there exist $i \in I$ such that $\alpha^i = \beta$.

Definition 6: Let $\alpha \in \Omega$, then α^I or αI is called I -orbit and is defined as $\alpha^I = \{\alpha^i : i \in I\}$.

A transitive I -subspace is also called an orbit.

Theorem 1: A transitive I space has only one orbit.

Definition 7: Let I be a group acting on Ω and if $\alpha \in \Omega$, we denote stabilizer of α by I_α and is define as $I_\alpha = \{i \in I : \alpha^i = \alpha\}$.

Lemma 1: Let Ω be a I -space and $\alpha \in \Omega$. Then

- 1) $I_\alpha \leq I$ and
- 2) There is one-one correspondence between the right cosets of I_α and the I -orbit α^I in Ω .

Proposition 1: If I is finite, then $|I| = |I_\alpha| \cdot |\alpha^I|$

Proposition 8: Let Ω be a I -space and $i \in I$. Then

$$|K_{\Omega}^i| = |\alpha \in \Omega : \alpha^i = \alpha|.$$

Lemma 3: Let Ω and I be finite. Then

$$|Qd_{\Omega} I| = \frac{1}{|I|} \sum_{i \in I} |K_{\Omega}^i|,$$

where $|Qd_{\Omega} I|$ is the number of orbits of I in Ω .

50 Proposition 10

Proposition 10: Let Ω be a I -space. Then $P \Omega$ is called I -neutrosophic space if $P \Omega = \langle \Omega \cup K \rangle$ which is generated by Ω and I .

Example 1: Let $\Omega = \{g, z, z^2, \{, z\{, z^2\} = S_3$ and $I = \{g, \{$. Let $\mu: \Omega \times I \rightarrow \Omega$ be an action of I on Ω defined by $\mu(\omega, i) = i \cdot \omega$, for all $\omega \in \Omega$ and $i \in I$. Then Ω be a I -space under this action. Let $P \Omega$ be the corresponding I -neutrosophic space, where

$$P \Omega = \langle \Omega \cup K \rangle = \{g, z, z^2, \{, z\{, z^2\}, K, K\{, K^2, K\{, K^2\}$$

Lemma 3: $P \Omega$ always contains Ω .

Proposition 11: Let $P \Omega$ be a neutrosophic space and $P \Omega_1$ be a subset of $P \Omega$. Then $P \Omega_1$ is called neutrosophic subspace of $P \Omega$ if $z^i \in P \Omega_1$ for all $z \in P \Omega_1$ and $i \in I$.

Example 2: In the above example 1. Let $P \Omega_1 = \{z, z\{$ and $P \Omega_2 = \{K^2, K\{$ are subsets of $P \Omega$. Then clearly $P \Omega_1$ and $P \Omega_2$ are neutrosophic subspaces of $P \Omega$.

Proposition 4: Let $P \Omega$ be a I -neutrosophic space and Ω be a I -space. Then Ω is always a neutrosophic subspace of $P \Omega$.

Proof: The proof is straightforward.

Definition 12: A neutrosophic subspace $P \Omega_1$ is called strong neutrosophic subspace or pure neutrosophic subspace if all the elements of $P \Omega_1$ are neutrosophic elements.

Example 3: In example 1, the neutrosophic subspace $P \Omega_2 = \mathbb{K}^2, \mathbb{K}^2\{$ is a strong neutrosophic subspace or pure neutrosophic subspace of $P \Omega$.

Remark 2: Every strong neutrosophic subspace or pure neutrosophic subspace is trivially neutrosophic subspace.

The converse of the above remark is not true.

Example 4: In previous example $P \Omega_1 = z, z\{$ is a neutrosophic subspace but it is not strong neutrosophic subspace or pure neutrosophic subspace of $P \Omega$.

Definition 13: Let $P \Omega$ be a I -neutrosophic space. Then $P \Omega$ is said to be transitive I -neutrosophic space if for any $z, \{ \in P \Omega$, there exists $i \in I$ such that $z^i = \{$.

Example 5: Let $\Omega = I = \mathbb{Z}_4, +$, where \mathbb{Z}_4 is a group under addition modulo 4. Let $\mu: \Omega \times I \rightarrow \Omega$ be an action of I on itself defined by $\mu(\omega, i) = \omega + i$, for all $\omega \in \Omega$ and $i \in I$. Then Ω is a I -space and $P \Omega$ be the corresponding I -neutrosophic space, where $P \Omega = \{0, 1, 2, 3, \mathbb{K}2, \mathbb{K}3, \mathbb{K}4, \mathbb{K}1 + \mathbb{K}2 + \mathbb{K}3 + \mathbb{K}1 + 2\mathbb{K}2 + 2\mathbb{K}2 + 3\mathbb{K}3 + 2\mathbb{K}3 + 3\mathbb{K}$

Then $P \Omega$ is not transitive neutrosophic space.

Theorem 5: All the I -neutrosophic spaces are intransitive I -neutrosophic spaces.

Proposition 14: Let $p \in P \Omega$, the neutrosophic orbit of p is denoted by PQ_p and is defined as $PQ_p = \{ p^i : i \in I \}$.

Equivalently neutrosophic orbit is a transitive neutrosophic subspace.

Example 8: In example 1, the neutrosophic space $P \Omega$ has 6 neutrosophic orbits which are given below

$$\begin{aligned} PQ_g &= \{ g \}, PQ_z = \{ z, z^2 \}, \\ PQ_{z^2} &= \{ z^2, z^4 \}, PQ_K = \{ K, K^2 \}, \\ PQ_{K^2} &= \{ K^2, K^4 \}. \end{aligned}$$

Definition 37: A neutrosophic orbit PQ_p is called strong neutrosophic orbit or pure neutrosophic orbit if it has only neutrosophic elements.

Example 7: In example 1,

$$\begin{aligned} PQ_K &= \{ K, K^2 \}, \\ PQ_{K^2} &= \{ K^2, K^4 \}, \\ PQ_{K^4} &= \{ K^4, K^8 \}. \end{aligned}$$

are strong neutrosophic orbits or pure neutrosophic orbits of $P \Omega$.

Theorem 9: All strong neutrosophic orbits or pure neutrosophic orbits are neutrosophic orbits.

Proof: Straightforward

To show that the converse is not true, let us check the following example.

Example 8: In example 1

$$\begin{aligned} PQ_g &= \{ g \}, \\ PQ_z &= \{ z, z^2 \}, \\ PQ_{z^2} &= \{ z^2, z^4 \}. \end{aligned}$$

are neutrosophic orbits of $P \Omega$ but they are not strong or pure neutrosophic orbits.

Proposition 16: Let I be a group acting on Ω and $z \in P \Omega$. The neutrosophic stabilizer of z is defined as $I_z = \{i \in I : z^i = z\}$.

Example 9: Let $\Omega = \{g, z, z^2, \dots, z^k\}$ and $I = \{g, z, z^2, \dots, z^k\}$. Let $\mu: \Omega \times I \rightarrow \Omega$ be an action of I on Ω defined by $\mu(\omega, i) = i \cdot \omega$, for all $\omega \in \Omega$ and $i \in I$. Then Ω is a I -space under this action. Now $P \Omega$ be the I -neutrosophic space, where

$$P \Omega = \{g, z, z^2, \dots, z^k, K, K^2, \dots, K^k\}$$

Let $z \in P \Omega$, then the neutrosophic stabilizer of z is $I_z = g$ and also let $K \in P \Omega$, so the neutrosophic stabilizer of K is $I_K = g$.

Theorem 4: Let $P \Omega$ be a neutrosophic space and $z \in P \Omega$, then

- 1) $I_z \leq I$.
- 2) There is also one-one correspondence between the right cosets of I_z and the neutrosophic orbit PQ_z .

Corollary 4: Let I is finite and $z \in P \Omega$, then $|I| = |I_z| \cdot |PQ_z|$.

Proposition 17: Let $z \in P \Omega$, then the neutrosophic stabilizer of z is called strong neutrosophic stabilizer or pure neutrosophic stabilizer if and only if z is a neutrosophic element of $P \Omega$.

Example 10: In above example (9), $I_K = g$ is a strong neutrosophic or pure neutrosophic stabilizer of neutrosophic element K , where $K \in P \Omega$.

Theorem 3: Every strong neutrosophic stabilizer or pure neutrosophic stabilizer is always a neutrosophic stabilizer

but the converse is not true.

Gzco rrg 11: Let $z \in P \Omega$, where

$$P \Omega = \{g, z, z^2, \{z, z^2\}, K, K^2, K, K^2\}$$

Then $I_z = g$ is the neutrosophic stabilizer of z but it is not strong neutrosophic stabilizer or pure neutrosophic stabilizer as z is not a neutrosophic element of $P \Omega$.

Fghplkqp'18: Let $P \Omega$ be a neutrosophic space and I be a finite group acting on Ω . For $i \in I$, $H_{z_{P \Omega}} i = \{z \in P \Omega : z^i = z\}$

Gzco rrg 12: In example 11,

$$H_{z_{P \Omega}} g = \{g, z, z^2, \{z, z^2\}, K, K^2, K, K^2\}$$

$$H_{z_{P \Omega}} i = \emptyset, \text{ where } i \neq g.$$

'Vj gqt go '8: Let $P \Omega$ be a finite neutrosophic space, then

$$|PQ_{P \Omega} I| = \frac{1}{|I|} \sum_{i \in I} |H_{z_{P \Omega}} i|.$$

Proof: The proof is same as in group action.

Gzco rrg 13: Consider example 1, only identity element of I fixes all the elements of $P \Omega$. Hence $H_{z_{P \Omega}} g = \{g, z, z^2, \{z, z^2\}, K, K^2, K, K^2\}$ and hence $|H_{z_{P \Omega}} g| = 12$.

The number of neutrosophic orbits of $P \Omega$ are given by above theorem

$$|PQ_{P \Omega} I| = \frac{1}{2} 12 = 6$$

Hence $P \Omega$ has 6 neutrosophic orbits.

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Fghplkqp'19: A neutrosophic space $P \Omega$ is called pseudo neutrosophic space which does not contain a proper set which is a I -space.

Gzco rrg 14: Let $\Omega = I = \mathbb{Z}_2$ where \mathbb{Z}_2 is a group under addition modulo 2.

Let $\mu: \Omega \times I \rightarrow \Omega$ be an action of I on Ω defined by $\mu(\omega, i) = \omega + i$, for all

$\omega \in \Omega$ and $i \in I$. Then Ω be a I -space under this action and $P \Omega$ be the I -neutrosophic space, where $P \Omega = \{0, 1, K, 1+K\}$.

Then clearly $P \Omega$ is a pseudo neutrosophic space.

Vj gqt go '32 Every pseudo neutrosophic space is a neutrosophic space but the converse is not true in general.

Gzco rrg 15: In example 1, $P \Omega$ is a neutrosophic space but it is not pseudo neutrosophic space because $\{g, f\}$, $\{z, z^2\}$ and $\{z^2, z^2\}$ are proper subsets which are I -spaces.

Fghplkqp'20: Let $P \Omega$ be a neutrosophic space and $P \Omega_1$ be a neutrosophic subspace of $P \Omega$. Then $P \Omega_1$ is called pseudo neutrosophic subspace of $P \Omega$ if $P \Omega_1$ does not contain a proper subset of Ω which is a I -subspace of Ω .

Gzco rrg 16: In example 1, $\{g, f\}$, $\{K, K\}$ etc are pseudo neutrosophic subspaces of $P \Omega$ but $\{g, f, K, K\}$ is not pseudo neutrosophic subspace of $P \Omega$ as $\{g, f\}$ is a proper I -subspace of Ω .

Vj gqt go '32 All pseudo neutrosophic subspaces are neutrosophic subspaces but the converse is not true in general.

Gzco rrg 17: In example 1, $\{g, f, K, K\}$ is a neutrosophic subspace of $P \Omega$ but it is not pseudo neutrosophic subspace of $P \Omega$.

Vj gqt go '33 A neutrosophic space $P \Omega$ has neutrosophic subspaces as well as pseudo neutrosophic subspaces.

Proof : The proof is obvious.

Vj gqt go '34 A transitive neutrosophic subspace is always a pseudo neutrosophic subspace but the converse is not true in general.

Proof: A transitive neutrosophic subspace is a neutrosophic orbit and hence neutrosophic orbit does not contain any other subspace and so pseudo neutrosophic subspace.

The converse of the above theorem does not holds in general. For instance let see the following example.

Gzco rig 18: In example 1 , K, K', K'' is a pseudo neutrosophic subspace of P, Ω but it is not transitive.

Vj gqt go '35: All transitive pseudo neutrosophic subspaces are always neutrosophic orbits.

Proof: The proof is followed from by definition.

Fghplskp'21: The pseudo property in a pseudo neutrosophic subspace is called ideal property.

Vj gqt go '36: The transitive property implies ideal property but the converse is not true.

Proof: Let us suppose that P, Ω_1 be a transitive neutrosophic subspace of P, Ω . Then by following above theorem, P, Ω_1 is pseudo neutrosophic subspace of P, Ω and hence transitivity implies ideal property.

The converse of the above theorem is not holds.

Gzco rig 19: In example 1 , K, K', K'' is a pseudo neutrosophic subspace of P, Ω but it is not transitive.

Vj gqt go '37: The ideal property and transitivity both implies to each other in neutrosophic orbits.

Proof: The proof is straightforward.

Fghplskp'22: A neutrosophic space P, Ω is called ideal space or simply if all of its proper neutrosophic subspaces are pseudo neutrosophic subspaces.

Gzco rig 20: In example 14 , the neutrosophic space $P \Omega$ is an ideal space because $0,1, K_1+K$ are only proper neutrosophic subspaces which are pseudo neutrosophic subspaces of $P \Omega$.

Vj gqt go '38<Every ideal space is trivially a neutrosophic space but the converse is not true.

For converse, we take the following example

Gzco rig 21: In example 1 , $P \Omega$ is a neutrosophic space but it is not an ideal space.

Vj gqt go '17: A neutrosophic space $P \Omega$ is an ideal space If Ω is transitive I -space.

Vj gqt go '18: Let $P \Omega$ be a neutrosophic space, then $P \Omega$ is pseudo neutrosophic space if and only if $P \Omega$ is an ideal space.

Proof: Suppose that $P \Omega$ is a pseudo neutrosophic space and hence by definition all proper neutrosophic subspaces are also pseudo neutrosophic subspaces. Thus $P \Omega$ is an ideal space.

Conversely suppose that $P \Omega$ is an ideal space and so all the proper neutrosophic subspaces are pseudo neutrosophic subspaces and hence $P \Omega$ does not contain any proper set which is I -subspace and consequently $P \Omega$ is a pseudo neutrosophic space.

Vj gqt go '3;< If the neutrosophic orbits are only the neutrosophic proper subspaces of $P \Omega$, then $P \Omega$ is an ideal space.

Proof: The proof is obvious.

Vj gqt go '42<'A neutrosophic space $P \Omega$ is an ideal space if $|PQ_p \Omega I| = 2$

'Vj gqt go '21: A neutrosophic space $P \Omega$ is ideal space if all of its proper neutrosophic subspaces are neutrosophic orbits.

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The main theme of this paper is the extension of neutrosophy to group action and G-spaces to form G-neutrosophic spaces. Our aim is to see the newly generated structures and finding their links to the old versions in a logical manner. Fortunately enough, we have found some new type of algebraic structures here, such as ideal space, Pseudo spaces. Pure parts of neutrosophy and their corresponding properties and theorems are discussed in detail with a sufficient supply of examples.

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