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Fuzzy TOPSIS: A General View

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Abstract

The aim of this survey paper is to offer a general view of the developments of fuzzy TOPSIS methods. We begin with a literature review an we explore different fuzzy models that have been applied to the decision making field. Finally, we present some applications of fuzzy TOPSIS.

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Multi-criteria decision making, fuzzy sets, fuzzy MCDM, fuzzy numbers, linguistic variable, intuitionistic fuzzy sets, neutrosophic set, hesitant fuzzy set, fuzzy AHP, fuzzy TOPSIS

1. Introduction and literature review

The problems of Multi-Criteria Decision Making (MCDM) appear and are intensely applied in many domains, such as Economics, Social Sciences, Medical Sciences etc. Sometimes, MCDM problems are mentioned as Multiple-Criteria Decision Analysis (MCDA) or Multi-Attribute Decision-Making (MADM) (see [22, 27, 47, 60]). In spite of their diversity, the MCDM have as common characteristic multiple objectives and multiple criteria which usually are in conflict with each other. The decision makers have to select, assess or rank these alternatives according to the weights of the criteria. In the last decades the MCDM techniques have become an important branch of operations research (see [23, 46, 65]).

In many real-world situations, the problems of decision making are subjected to some constraints, objectives and consequences that are not accurately known. After Bellman and Zadeh [8] introduced for the first time fuzzy sets within MCDM, many researchers have been preoccupied by decision making in fuzzy environments. The fusion between MCDM and fuzzy set theory has led to a new decision theory, known today as fuzzy multi-criteria decision making (FMCDM), where we have decision-maker models that can deal with incomplete and uncertain knowledge and information. The most important thing is that, when we want to assess, judge or decide we usually use a natural language in which the words do not have a clear, definite meaning. As a result, we need fuzzy numbers to express linguistic variables, to describe the subjective judgement of a decision maker in a quantitative manner. Fuzzy numbers (FN) most often used are triangular FN, trapezoidal FN and Gaussian FN.

We highlight that the concept of linguistic variable introduced by Zadeh in 1975 (see [61]) allows computation with words instead of numbers and thus linguistic terms defined by fuzzy sets are intensely used in problems of decision theory for modeling uncertain information. There are very good monographs (see for instance [17]) and surveys papers [1, 12, 25, 33, 38] on FMCDM. Recently, some new methods have been explored [3, 53, 67].

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After Atanassov [4] introduced the concept of intuitionistic fuzzy sets, where each element is characterized by a membership function, as in fuzzy sets, as well as by a non-membership function, the interest in the study of the problems of decision making theory with the help of intuitionistic fuzzy sets (see [11, 26, 29, 31, 32, 62]) has increased.

As a generalization of the concept of the classic set, fuzzy set, intuitionistic fuzzy set etc., Smarandache [42] firstly proposed the concept of neutrosophic set. In paper [49] there are proposed set-theoretic operators on an instance of neutrosophic set called interval neutrosophic set. Recently, neutrosophic sets have been applied in MCDM (see [36, 56, 57, 58, 63]).

Torra and Narakawa [45] and Torra [44] introduced the concept of hesitant fuzzy set, which undergoes a much more flexible approach for decision makers when they provide their decisions. Therefore, hesitant fuzzy sets have become useful in MCDM problems [37, 39, 48, 52].

The aim of this survey paper is to offer a general view of the developments of fuzzy TOPSIS methods. We begin with a literature review an we explore different fuzzy models that have been applied to the decision making field. Finally, we present some applications of fuzzy TOPSIS.

2. Basic concepts and definitions

Definition 2.1. *[21] A fuzzy number (FN) is a fuzzy set in $\mathbb{R}$, namely a mapping $x : \mathbb{R} \to [0, 1]$, with the following properties:*

1. *x is convex, i.e. $x(t) \geq \min\{x(s), x(r)\}$, for $s \leq t \leq r$;*
2. *x is normal, i.e. $(\exists)t_0 \in \mathbb{R} \ : \ x(t_0) = 1$;*
3. *x is upper semicontinuous, i.e.*

$$(\forall)t \in \mathbb{R}, (\forall)\alpha \in (0, 1] \ : \ x(t) < \alpha, (\exists)\delta > 0 \text{ such that } |s - t| < \delta \Rightarrow x(s) < \alpha \ . \tag{1}$$

Remark 2.2. *Among the various types of FNs, triangular FNs and trapezoidal FNs are the most popular. A triangular FN is defined by its membership function*

$$x(t) = \begin{cases} 0 & \text{if} \quad t < a \\ \frac{t-a}{b-a} & \text{if} \quad a \leq t < b \\ \frac{c-t}{c-b} & \text{if} \quad b \leq t < c \\ 0 & \text{if} \quad t > c \end{cases} \ , \text{where } a \leq b \leq c \ , \tag{2}$$

and it is denoted $\tilde{x} = (a, b, c)$. A trapezoidal FN is defined by its membership function

$$x(t) = \begin{cases} 0 & \text{if} \quad t < a \\ \frac{t-a}{b-a} & \text{if} \quad a \leq t \leq b \\ 1 & \text{if} \quad b < t < c \\ \frac{d-t}{d-c} & \text{if} \quad c \leq t \leq d \\ 0 & \text{if} \quad t > d \end{cases} \ , \text{where } a \leq b \leq c \leq d \ , \tag{3}$$

and it can be expressed as $\tilde{x} = (a, b, c, d)$.

Remark 2.3. *[14, 22, 24] Let $\tilde{x} = (a_1, b_1, c_1), \tilde{y} = (a_2, b_2, c_2)$ be two non negative triangular FNs and $\alpha \in \mathbb{R}_+$. According to the extension principle, the arithmetic operations are defined as follows:*

1. $\tilde{x} + \tilde{y} = (a_1 + a_2, b_1 + b_2, c_1 + c_2)$;
2. $\tilde{x} - \tilde{y} = (a_1 - c_2, b_1 - b_2, c_1 - a_2)$;
3. $\alpha\tilde{x} = (\alpha a_1, \alpha b_1, \alpha c_1)$;
4. $\tilde{x}^{-1} \cong (1/c_1, 1/b_1, 1/a_1)$;
5. $\tilde{x} \times \tilde{y} \cong (a_1 a_2, b_1 b_2, c_1 c_2)$;
6. $\tilde{x}/\tilde{y} \cong (a_1/c_2, b_1/b_2, c_1/a_2)$.

We note that the results of $(4) - (6)$ are not triangular FNs, but they can be approximated by triangular FNs.

Remark 2.4. *[17, 28, 59] Let $\tilde{x} = (a_1, b_1, c_1, d_1), \tilde{y} = (a_2, b_2, c_2, d_2)$ be two non negative trapezoidal FNs and $\alpha \in \mathbb{R}_+$. The arithmetic operations are defined as follows:*

1. $\tilde{x} + \tilde{y} = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2)$;
2. $\tilde{x} - \tilde{y} = (a_1 - d_2, b_1 - c_2, c_1 - b_2, d_1 - a_2)$;
3. $\alpha\tilde{x} = (\alpha a_1, \alpha b_1, \alpha c_1, \alpha d_1)$;
4. $\tilde{x}^{-1} \cong (1/d_1, 1/c_1, 1/b_1, 1/a_1)$;

5. $\tilde{x} \times \tilde{y} \cong (a_1 a_2, b_1 b_2, c_1 c_2, d_1 d_2)$;
6. $\tilde{x} / \tilde{y} \cong (a_1/d_2, b_1/c_2, c_1/b_2, d_1/a_2)$.

We mention that the results of (4) – (6) *are not trapezoidal FNs, but they can be approximated by trapezoidal FNs.*

Definition 2.5. *[4] An intuitionistic fuzzy set (IFS) A in X is given by*

$$A = \{< x, \mu_A(x), \nu_A(x) > : x \in X\}, \tag{4}$$

where $\mu_A, \nu_A : X \to [0,1]$ *such that* $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ *and represent the degree of membership and the degree of non-membership of an element x to A.*

Definition 2.6. *[5] Let* $\mathcal{D}([0,1])$ *be the set of all closed subinterval of* $[0,1]$. *An interval valued intuitionistic fuzzy set (IVIFS) in X is given by*

$$\tilde{A} = \{< x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x) > : x \in X\}, \tag{5}$$

where $\mu_A, \nu_A : X \to \mathcal{D}([0,1])$ *such that* $0 \leq \sup_{x \in X} \mu_{\tilde{A}}(x) + \sup_{x \in X} \nu_{\tilde{A}}(x) \leq 1$.

Remark 2.7. *If we denote* $\mu_{\tilde{A}}(x) = [a,b], \nu_{\tilde{A}}(x) = [c,d]$, *then* $\tilde{A}$ *can be written* $\tilde{A} = ([a,b],[c,d])$. *Let* $\tilde{A}_1 = ([a_1,b_1],[c_1,d_1])$, $\tilde{A}_2 = ([a_2,b_2],[c_2,d_2])$ *and* $\alpha \in \mathbb{R}_+$. *The arithmetic operations are defined by Xu [54]:*

1. $\tilde{A}_1 + \tilde{A}_2 = ([a_1 + a_2 - a_1 a_2, b_1 + b_2 - b_1 b_2], [c_1 c_2, d_1 d_2])$;
2. $\alpha \tilde{A}_1 = ([1 - (1 - a_1)^\alpha, 1 - (1 - b_1)^\alpha], [c_1^\alpha, d_1^\alpha])$;
3. $\tilde{A}_1 \times \tilde{A}_2 = ([a_1 a_2, b_1 b_2], [c_1 + c_2 - c_1 c_2, d_1 + d_2 - d_1 d_2])$.

3. Fuzzy MCDM problem formulation

A MCDM problem with *m alternatives* $\{A_1, A_2, \cdots, A_m\}$ which should be assessed by applying *n criteria* (or *attributes*) $\{C_1, C_2, \cdots, C_n\}$ can be expressed by the *decision matrix*

$$X = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{bmatrix}, \tag{6}$$

where x_{ij} is a numeric data which represents the value of the i^{th} alternative with respect to the j^{th} criterion.

The *importance* (or *weight*) of the criterion C_j to the decision is denoted by w_j. Let w be the vector

$$w = [w_1, w_2, \cdots, w_n]. \tag{7}$$

Generally, the weights are determined on a subjective basis by a single decision maker or by a group of experts.

Some remarks must be made:

Remark 3.1. *In fuzzy MCDM, in order to assign the importance degree to the criteria, it can be used an empirical method described in [55], where an equivalence between the importance of an attribute and a triangular FN is presented.*

Table 1. Triangular FNs for the importance of criteria

Rank	Attribute grade	Triangular FN
Very low	1	(0.00, 0.10, 0.30)
Low	2	(0.10, 0.30, 0.50)
Medium	3	(0.30, 0.50, 0.75)
High	4	(0.50, 0.75, 0.90)
Very high	5	(0.75, 0.90, 1.00)

Similarly, alternatives can be evaluated by linguistic terms which can be represented by triangular FNs [15].

Remark 3.2. *If the performance ratings of alternatives on qualitative criteria is expressed by linguistic term, these linguistic terms can be represented by trapezoidal FNs or IVIFS as in the following tables:*

Remark 3.3. *It is often difficult for decision makers to assign a precise value to an alternative for the criteria considered. In this situation the fuzzy MCDM problem can be expressed by the decision matrix*

$$\tilde{X} = \begin{bmatrix} \tilde{x}_{11} & \tilde{x}_{12} & \cdots & \tilde{x}_{1n} \\ \tilde{x}_{21} & \tilde{x}_{22} & \cdots & \tilde{x}_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \tilde{x}_{m1} & \tilde{x}_{m2} & \cdots & \tilde{x}_{mn} \end{bmatrix}, \tag{8}$$

where $\tilde{x}_{ij}$ *are fuzzy value (triangular FN, trapezoidal FN, IFS, IVIFS, trapezoidal hesitant fuzzy element etc.)*

Table 2. Linguistic terms for alternatives ratings

Linguistic terms for alternatives ratings	Triangular FN
Very good	(9,10,10)
Good	(7,9,10)
Medium	(3,5,7)
Poor	(1,3,5)
Very poor	(1,1,3)

Table 3. Linguistic values of trapezoidal FNs for linguistic terms

Linguistic term	Trapezoidal FN
Very low	(0.00, 0.00, 0.00, 0.10)
Low	(0.10, 0.20, 0.25, 0.30)
Medium low	(0.30, 0.40, 0.45, 0.50)
Medium	(0.50, 0.60, 0.65, 0.70)
Medium high	(0.70, 0.80, 0.85, 0.90)
High	(0.90, 0.95, 1.00, 1.00)
Very high	(1.00, 1.00, 1.00, 1.00)

4. Fuzzy AHP

There exists a number of methods in order to obtain criteria weights [64, 66], but the AHP (Analytic Hierarchy Process) developed by Saaty [40] is the most used. Buckley [9] incorporated the fuzzy theory into AHP and obtained in this way fuzzy AHP. We note that a new method for finding fuzzy weights, based on a direct fuzzification of method proposed by Saaty is presented in paper [10]. The procedure of fuzzy AHP (see for instance [22, 51] etc.) is:

Step 1: *Construct fuzzy pairwise comparison matrices.*

Each decision maker assigns linguistic term represented by triangular FN to the pairwise comparison among all criteria. Let $\tilde{P} = [\tilde{a}_{ij}]$ be a $n \times n$ matrix, where $\tilde{a}_{ij}$ is the importance of criterion C_i with respect to criterion C_j, according to the fuzzy preference scale shown in the next table:

We note that

$$\tilde{P} = \begin{bmatrix} (1,1,1) & \tilde{a}_{12} & \cdots & \tilde{a}_{1n} \\ \tilde{a}_{21} & (1,1,1) & \cdots & \tilde{a}_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \tilde{a}_{n1} & \tilde{a}_{n2} & \cdots & (1,1,1) \end{bmatrix} = \begin{bmatrix} (1,1,1) & \tilde{a}_{12} & \cdots & \tilde{a}_{1n} \\ (1,1,1)/\tilde{a}_{12} & (1,1,1) & \cdots & \tilde{a}_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ (1,1,1)/\tilde{a}_{1n} & (1,1,1)/\tilde{a}_{2n} & \cdots & (1,1,1) \end{bmatrix}. \quad (9)$$

Step 2: *Compute the fuzzy weights by normalization.*

The fuzzy weight of criterion C_i, denoted $\tilde{w}_i$, is obtained by

$$\tilde{w}_i = \tilde{r}_i \times (\tilde{r}_1 + \tilde{r}_2 + \cdots + \tilde{r}_n)^{-1}, \quad (10)$$

where $\tilde{r}_i = [\tilde{a}_{i1} \times \tilde{a}_{i2} \times \cdots \times \tilde{a}_{in}]^{1/n}$.

5. Fuzzy TOPSIS

Technique for Order Performance by Similarity to Ideal Solution (TOPSIS) was proposed by Hwang and Yoon [27] and it is the most known technique for solving MCDM problems. This method is based on the concept that the chosen alternative should have the shortest distance to Positive Ideal Solution (PIS) (the solution which minimizes the cost criteria and maximizes the benefit criteria) and the farthest distance to Negative Ideal Solution (NIS).

Chen [15] extended TOPSIS with triangular FNs. Chen introduced a vertex method to calculate the distance between two triangular FNs. If $\tilde{x} = (a_1, b_1, c_1), \tilde{y} = (a_2, b_2, c_2)$ are two triangular FNs then

$$d(\tilde{x}, \tilde{y}) := \sqrt{\frac{1}{3}\left[(a_1 - a_2)^2 + (b_1 - b_2)^2 + (c_1 - c_2)^2\right]}. \quad (11)$$

The procedure of fuzzy TOPSIS (see for instance [6, 15] etc.) is:

Step 1. *Assignment rating to the criteria and to the alternatives.*

We assume that we have a decision group with K members. The fuzzy rating of the k^{th} decision maker about alternative A_i w.r.t. criterion C_j is denoted $\tilde{x}_{ij}^k = (a_{ij}^k, b_{ij}^k, c_{ij}^k)$ and the weight of criterion C_j is denoted $\tilde{w}_j^k = (w_{j1}^k, w_{j2}^k, w_{j3}^k)$.

Step 2. *Compute the aggregated fuzzy ratings for alternatives and the aggregated fuzzy weights for criteria.*

Table 4. Linguistic values of IVIFS for linguistic terms

Linguistic term	IVIFS
Very low	([0.00,0.10], [0.85,0.90])
Low	([0.10,0.15], [0.75,0.80])
Medium low	([0.25,0.30], [0.60,0.65])
Medium	([0.35,0.40], [0.45,0.50])
Medium high	([0.50,0.60], [0.30,0.35])
High	([0.65,0.70], [0.15,0.20])
Very high	([0.75,0.80], [0.10,0.15])
Very very high	([0.90,0.95], [0.01,0.02])
Extremely high	([1.00,1.00], [0.00,0.00])

Table 5. Fuzzy preference scale

Linguistic value	Triangular FN ($\tilde{a}_{ij}$)
Absolutely important	(7,9,9)
Very strongly extreme important	(6,8,9)
Very strongly important	(5,7,9)
Strongly important	(4,6,8)
Moderately strong important	(3,5,7)
Moderate important	(2,4,6)
Weakly important	(1,3,5)
Equally moderate important	(1,2,4)
Equally important	(1,1,3)

The aggregated fuzzy rating $\tilde{x}_{ij} = (a_{ij}, b_{ij}, c_{ij})$ of i^{th} alternative w.r.t. j^{th} criterion is obtained as follows:

$$a_{ij} = \min_k \{a^k_{ij}\}\ ,\ b_{ij} = \frac{1}{K}\sum_{k=1}^{K} b^k_{ij}\ ,\ c_{ij} = \max_k \{c^k_{ij}\}\ . \tag{12}$$

The aggregated fuzzy weight $\tilde{w}_j = (w_{j1}, w_{j2}, w_{j3})$ for the criterion C_j are calculated by formulas:

$$w_{j1} = \min_k \{w^k_{j1}\}\ ,\ w_{j2} = \frac{1}{K}\sum_{k=1}^{K} w^k_{j2}\ ,\ w_{j3} = \max_k \{w^k_{j3}\}\ . \tag{13}$$

Step 3. *Compute the normalized fuzzy decision matrix.*
The normalized fuzzy decision matrix is $\tilde{R} = [\tilde{r}_{ij}]$, where

$$\tilde{r}_{ij} = \left(\frac{a_{ij}}{c^*_j}, \frac{b_{ij}}{c^*_j}, \frac{c_{ij}}{c^*_j}\right) \text{ and } c^*_j = \max_i \{c_{ij}\} \text{ (benefit criteria)} \tag{14}$$

or

$$\tilde{r}_{ij} = \left(\frac{a^-_j}{c_{ij}}, \frac{a^-_j}{b_{ij}}, \frac{a^-_j}{a_{ij}}\right) \text{ and } c^-_j = \min_i \{a_{ij}\} \text{ (cost criteria)}\ . \tag{15}$$

Step 4. *Compute the weighted normalized fuzzy decision matrix.*
The weighted normalized fuzzy decision matrix is $\tilde{V} = (\tilde{v}_{ij})$, where $\tilde{v}_{ij} = \tilde{r}_{ij} \times w_j$.
Step 5. *Compute the Fuzzy Positive Ideal Solution (FPIS) and Fuzzy Negative Ideal Solution (FNIS).*
The FPIS and FNIS are calculated as follows:

$$A^* = (\tilde{v}^*_1, \tilde{v}^*_2, \cdots, \tilde{v}^*_n),\ \text{where } \tilde{v}^*_j = \max_i \{v_{ij3}\}\ ; \tag{16}$$

$$A^- = (\tilde{v}^-_1, \tilde{v}^-_2, \cdots, \tilde{v}^-_n),\ \text{where } \tilde{v}^-_j = \min_i \{v_{ij1}\}\ . \tag{17}$$

Step 6. *Compute the distance from each alternative to the FPIS and to the FNIS.*
Let

$$d^*_i = \sum_{j=1}^{n} d(\tilde{v}_{ij}, \tilde{v}^*_j)\ ,\ d^-_i = \sum_{j=1}^{n} d(\tilde{v}_{ij}, \tilde{v}^-_j) \tag{18}$$

be the distance from each alternative A_i to the FPIS and to the FNIS, respectively.
Step 7. *Compute the closeness coefficient CC_i for each alternative.*
For each alternative A_i we calculate the closeness coefficient CC_i as follows:

$$CC_i = \frac{d^-_i}{d^-_i + d^*_i}\ . \tag{19}$$

Step 8. *Rank the alternatives.*
The alternative with highest closeness coefficient represents the best alternative.

Remark 5.1. *If for the evaluation ratings of alternatives and the weights of criteria there are used linguistic values represented by trapezoidal FNs, then the fuzzy TOPSIS approach is slightly modified (see [16]):*
If $\tilde{x} = (a_1, b_1, c_1, d_1), \tilde{y} = (a_2, b_2, c_2, d_2)$ *are two trapezoidal FNs then*

$$d(\tilde{x}, \tilde{y}) := \sqrt{\frac{1}{4}\left[(a_1 - a_2)^2 + (b_1 - b_2)^2 + (c_1 - c_2)^2 + (d_1 - d_2)^2\right]}\,. \tag{20}$$

1. *The aggregated fuzzy ratings* $\tilde{x}_{ij} = (a_{ij}, b_{ij}, c_{ij}, d_{ij})$ *are defined as:*

$$a_{ij} = \min_k\{a_{ij}^k\}\ ,\ b_{ij} = \frac{1}{K}\sum_{k=1}^{K} b_{ij}^k\ ,\ c_{ij} = \frac{1}{K}\sum_{k=1}^{K} c_{ij}^k\ , d_{ij} = \max_k\{d_{ij}^k\}\ , \tag{21}$$

where $\tilde{x}_{ij}^k = (a_{ij}^k, b_{ij}^k, c_{ij}^k, d_{ij}^k)$ *are the fuzzy ratings of* k^{th} *decision maker.*
Let $\tilde{w}_j^k = (w_{j1}^k, w_{j2}^k, w_{j3}^k, w_{j4}^k)$ *be the importance weight of* k^{th} *decision maker. The aggregated fuzzy weight* $\tilde{w}_j = (w_{j1}, w_{j2}, w_{j3}, w_{j4})$ *for the criterion* C_j *can be calculated as:*

$$w_{j1} = \min_k\{w_{j1}^k\}\ ,\ w_{j2} = \frac{1}{K}\sum_{k=1}^{K} w_{j2}^k\ ,\ w_{j3} = \frac{1}{K}\sum_{k=1}^{K} w_{j3}^k\ ,\ w_{j4} = \max_k\{w_{j4}^k\}\ . \tag{22}$$

2. *The normalized fuzzy decision matrix is* $\tilde{R} = [\tilde{r}_{ij}]$*, where*

$$\tilde{r}_{ij} = \left(\frac{a_{ij}}{d_j^*}, \frac{b_{ij}}{d_j^*}, \frac{c_{ij}}{d_j^*}, \frac{d_{ij}}{d_j^*}\right) \text{ and } d_j^* = \max_i\{d_{ij}\} \text{ for } j \in \text{Benefit criteria} \tag{23}$$

$$\tilde{r}_{ij} = \left(\frac{a_j^-}{d_{ij}}, \frac{a_j^-}{c_{ij}}, \frac{a_j^-}{b_{ij}}, \frac{a_j^-}{a_{ij}}\right) \text{ and } a_j^- = \min_i\{a_{ij}\} \text{ for } j \in \text{Cost criteria}\ . \tag{24}$$

3.

$$A^* = (\tilde{v}_1^*, \tilde{v}_2^*, \cdots, \tilde{v}_n^*),\ \text{where } \tilde{v}_j^* = \max_i\{v_{ij4}\}\ ; \tag{25}$$

$$A^- = (\tilde{v}_1^-, \tilde{v}_2^-, \cdots, \tilde{v}_n^-),\ \text{where } \tilde{v}_j^- = \min_i\{v_{ij1}\}\ . \tag{26}$$

Remark 5.2. *In 2008, Chen and Tsao [19] extended TOPSIS method based on interval-valued fuzzy sets. In 2010, Chen and Lee [18] presented a fuzzy TOPSIS technique based on interval type-2 fuzzy sets.*

Remark 5.3. *In 2010, Li developed in paper [30] a methodology that is based on TOPSIS to solve MCDM problems with both ratings of alternatives w.r.t. criteria and weights of criteria are expressed in interval-valued intuitionistic fuzzy sets. In this case the decision matrix is* $\tilde{X} = [\tilde{x}_{ij}]$*, where* $\tilde{x}_{ij} = ([a_{ij}, b_{ij}], [c_{ij}, d_{ij}])$ *is interpreted as follows:*
- the interval $[a_{ij}, b_{ij}]$ *represents the performance rating of the alternative* A_i *w.r.t. the criterion* C_j*, namely the degree that the alternative* A_i *satisfy the criterion* C_j *may take any value between* a_{ij} *and* b_{ij}*;*
- the interval $[c_{ij}, d_{ij}]$ *represents the degree that the alternative* A_i *does not satisfies the criterion* C_j*, which means that the non-membership degree of alternative* A_i *w.r.t. criterion* C_j *may take value between* c_{ij} *and* d_{ij}*.*

The weight of criterion C_j *is denoted* $\tilde{w}_j = ([a_j, b_j], [c_j, d_j])$*. The interval* $[a_j, b_j]$ *means that the membership degree of criterion* C_j *may take any value between* a_j *and* b_j*. The interval* $[c_j, d_j]$ *shows the non-membership degree of criterion* C_j*.*

For each alternative A_i *the closeness coefficient* CC_i *is defined as follows:*

$$CC_i((\mu_{ij}), (\nu_{ij}), (w_j), (\rho_j)) = \frac{\sqrt{\sum_{j=1}^{n}\{(w_j\mu_{ij})^2 + [\rho_j(1-\nu_{ij})]^2\}}}{\sqrt{\sum_{j=1}^{n}\{(w_j\mu_{ij})^2 + [\rho_j(1-\nu_{ij})]^2\}} + \sqrt{\sum_{j=1}^{n}\{[w_j(1-\mu_{ij})]^2 + (\rho_j\nu_{ij})^2\}}}\ , \tag{27}$$

where (μ_{ij}) *and* (ν_{ij}) *are* $m \times n$ *matrices with elements* $\mu_{ij} \in [a_{ij}, b_{ij}]$ *and* $\nu_{ij} \in [c_{ij}, d_{ij}]$ *and* (w_j) *and* (ρ_j) *are n-dimensional vectors with elements* $w_j \in [a_j, b_j]$ *and* $\rho_j \in [c_j, d_j]$*.*

We note that the values of CC_i *are closed and bounded subinterval of* $[0, 1]$.

Remark 5.4. *In 2011, another TOPSIS method to solve MCDM problems in interval-valued intuitionistic fuzzy environment is proposed in paper [35].*

Step 1. *Let* $\tilde{X}^{(k)} = [\tilde{x}^k_{ij}]$ *be the interval-valued intuitionistic fuzzy (IVIF) decision matrix of decision-maker* $D_k(k = \overline{1,K})$*, where* $\tilde{x}^k_{ij} = ([a^k_{ij}, b^k_{ij}], [c^k_{ij}, d^k_{ij}])$.

Step 2. *Aggregate all individual IVIF decision matrix* $\tilde{X}^{(k)}$ *into a collective IVIF decision matrix* $\tilde{X} = [\tilde{x}_{ij}]$ *using IIFHG operator.*

Step 3. *Compute the weighted IVIF decision matrix.*

Step 4. *Compute the interval-valued intuitionistic positive ideal solution (IVIPIS) and interval-valued intuitionistic negative ideal solution (IVINIS).*

Step 5. *Compute the distance from each alternative to the IVIPIS and to the IVINIS using Hamming distance or Euclidean distance.*

Step 6. *Compute the closeness coefficient* CC_i *for each alternative as follows:*

$$CC_i = \frac{S_i^-}{S_i^- + S_i^*} \,. \tag{28}$$

Step 7. *Rank the alternatives.*

The alternative with highest closeness coefficient represents the best alternative.

6. Some applications

First of all we must note that there are very good surveys concerning fuzzy MCDM applications (see for example [1, 2, 7, 33, 34]. We mention in this paper some fuzzy TOPSIS applications:

6.1. Location problem

A fuzzy TOPSIS approach for selecting plant location is firstly proposed by Chu in 2002, in paper [20], where the rating of alternatives and the weights of criteria are assessed in linguistic terms represented by triangular FN. In paper [6] fuzzy TOPSIS method is also used . A logistic company is interested in implementing a new urban distribution center and there are three alternatives (A_1, A_2, A_3). Firstly a committee of three decision-makers is formed. The criteria are: accessibility (C1), security (C2), connectivity to multimodal transport (C3), costs (C4), environmental impact (C5), proximity to customers (C6), proximity to suppliers (C7), resource availability (C8), conformance to sustainable freight regulations (C9), possibility of expansion (C10), quality of service (C11). Similar problems are considered by various authors [41, 50].

6.2. Supplier selection

In paper [16] a fuzzy TOPSIS approach based on trapezoidal FNs is used to solve the supplier-selection problem. Five benefit criteria are considered: profitability of supplier, relationship closeness, technological capability, conformance quality, conflict resolution.

6.3. Sustainable and renewable energy

In paper [13], a fuzzy TOPSIS methodology is used to compare different heat transfer fluids. An important step, in problem formulation, is choosing the criteria. Ten criteria are selected both technical-economic and environmental. Three of them are qualitative and expressed in linguistic terms: state of knowledge of innovative technology, environmental risk and safety freezing point.

A fuzzy TOPSIS method is applied in paper [43] for ranking renewable energy supply systems in Turkey. There are five criteria with positive impact: value of CO_2 emission (environmental), job creation (social), efficiency, installed capacity, amount of energy produced (technical) and four criteria with negative impact: investment cost, operation and maintenance cost, payback period (economic), land use (environmental).

Conclusion

Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) for solving problems in decision making issues is a very popular method. In this paper we presented a general overview about the development of fuzzy TOPSIS methods. Finally we have mentioned several works that presents some applications of fuzzy TOPSIS, such as: location problem, supplier selection and, sustainable and renewable energy. We believe that this survey will be a support for future research in this field.

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