Fusion of Qualitative Beliefs Based on Linguistic Labels

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***Abstract*—In order to model beliefs expressed by natural language, the qualitative belief assignment based on the linguistic labels has been presented in the Dezert-Smarandache Theory (DSmT). However, the combination of qualitative beliefs is only based on the equidistant linguistic labels and the calculation of non-equidistant labels is approximate. To solve the fusion problem of qualitative beliefs denoted by non-equidistant labels, this paper proposed the universal arithmetic operations of linguistic labels, and analyzed the fusion of qualitative beliefs based on the general labels. The new operations which were based on the linear or nonlinear mapping relationship between linguistic labels and numerical values, could be effectively used on 1-tuple or 2-tuple linguistic representation model, and could increase the precision and reliability of the combination of qualitative beliefs or mixed beliefs. The experimental results have proved the validity of universal operations.**

Keywords-linguistic label; qualitative belief; combination of beliefs; information fusion

I. INTRODUCTION

The traditional information fusion methods always focus on the fusion of quantitative information. However, we have to dispose of the qualitative information on the high application levels of information fusion. Especially with the development of science and technology, the requirement of qualitative fusion becomes more intense.

Zadeh [1][2] proposed a kind of methodology named as Computing with Words (CW), in which the fuzzy logic played the leading role. CW is an important tool to enhance the ability to analyze really vague or uncertain problems, and has become a significant concept in knowledge processing. Having done further researches for the CW, Jerry and Dongrui [3] presented the Perceptual Reasoning (PR) which was a kind of CW engine. Wellman [4] proposed the Qualitative Probability Net (QPN) which replaced numerical probabilities in the Bayesian belief network with qualitative influences. In order to measure the weights of qualitative influences, Yue and Liu [5] proposed the QPN with rough-set-based weights. Parson [6] presented a Qualitative Evidence Theory (QET), which was a qualitative Dempster-Shafer Theory (DST) in essence，but only used the un-normalized version of Dempster's rule. Brewka et al. [7] proposed the Qualitative Choice Logic (QCL), which presented some new qualitative logic operators and extended the logic theory. In 2006, Smarandache and Dezert [8] proposed the concept of qualitative belief based on linguistic labels in the Dezert-Smarandache Theory (DSmT). The linguistic label is used to describe the qualitative information represented by natural language, and can be directly reported by human itself or intelligentized system. In order to calculate the qualitative beliefs effectively, they [9] also presented the DSm Linear Algebra of Refined Labels (DSm-LARL) and the corresponding arithmetic operators. However, the DSm-LARL is based on the equidistant labels and the calculation of non-equidistant labels is approximate.

In this paper, we propose the universal operations of linguistic labels. The new qualitative operators are defined according to the mapping relationship between linguistic labels and numerical values, and can be extended to use in different linguistic models. Based on the universal operators of linguistic labels, we can effectively tackle the qualitative beliefs or the mixed beliefs.

II. DSM LINEAR ALGEBRA OF REFINED LABELS

On the basis of the linguistic labels, Smarandache and Dezert [8] presented the definition of qualitative beliefs. They also proposed the DSm-LARL [9] to calculate the linguistic labels effectively.

A. Linguistic Labels

L_n is used to represent the linguistic label usually. If L_n is a classical label, then n is an integer and $n > 0$. Let L_1, L_2, $L_3 \ldots L_m$ represent a group of linguistic labels for describing a certain characteristic of a thing. For example, we can use L_1, L_2 and L_3 to represent "short", "medium" and "tall" respectively. Then, the labels can make up of a label set $L' = \{L_1, L_2, L_3 \ldots L_m\}$. Let the symbol "<" denote the description degrees and $L_1<L_2<L_3<\ldots<L_m$. In order to work on a closed set under arithmetic operators, L' is extended with two extreme labels L_0 and L_{m+1}, then the linguistic labels set $L=\{L_0,L_1,L_2\ldots L_m,L_{m+1}\}$ form. L_0 is corresponding to the minimal value and $L_0 = L_{\min}$; L_{m+1} is corresponding to the maximal value and $L_{m+1} = L_{\max}$. If the distance between adjacent labels are the same, the linguistic labels will be equidistant and $L_{i+1} - L_i = L_{j+1} - L_j$; if the distances between adjacent labels are different, the linguistic labels will be non-equidistant and $L_{i+1} - L_i \neq L_{j+1} - L_j$ $(i \neq j)$.

In the applications, the equidistant labels L_0, L_1, $L_2 \ldots L_m$, L_{m+1} are usually mapped into the unit interval [0, 1]. Then, the set L is isomorphic with the numerical set $\{i/(m+1), i = 0, 1, 2\ldots m, m+1\}$ and $L_i = i/(m+1)$. L is discrete and limited. In order to combine qualitative beliefs precisely, L was continued and extended to the left and the right sides. The refined labels include L_Z, L_Q and L_R. In L_Z or L_Q or L_R, $L_i = i/(m+1)$, and i is an integer or a rational number or a real number respectively.

The mentioned linguistic labels above only have one linguistic term and are called as 1-tuple linguistic model. The 1-tuple labels are vague and lose information in calculation. In order to enhance the precision, Herrera and Martínez [10] proposed 2-tuple linguistic model. The 2-tuple model (L_i, α_i) includes a label center L_i and an accessorial offset α_i. L_i is the closest index label to the actual value and α_i indicates the offset between the actual value and the label center. Because of the increase of the offset, the computation of 2-tuple is more precise than that of 1-tuple label.

B. Linear Arithmetic Operators

On the basis of the refined labels, Smarandache and Dezert proved the DSm field (L_R, +, ×) and proposed the DSm-LARL theory [9]. Let a, b, ω in R, ω is a scalar and the labels $L_a=a/(m+1)$, $L_b=b/(m+1)$. m is the number of linguistic labels in label set. Because the linguistic labels are equidistant, (L_R, +, ×) is isomorphic with the set of real numbers (R, +, ×). According to the arithmetic operations of the corresponding real numbers, the operations of linguistic labels can be got as follows:

$$\begin{cases} L_a + L_b = L_{a+b} \\ L_a - L_b = L_{a-b} \\ L_a \times L_b = L_{(ab)/(m+1)} \\ L_a \div L_b = L_{(a/b)/(m+1)} \end{cases} \quad (1)$$

The proposed label operators only considered the equidistant linguistic labels. Although the same operators can be applied to calculate non-equidistant labels, the calculation is approximate. The larger the gaps of the distance between adjacent labels are, the more trustless the results are.

III. CALULATION OF GENERAL LINGUISTIC LABLES

In real world, the equidistant label is just a special case in linguistic labels and many general linguistic labels are non-equidistant. For example, the linguistic labels, which denote such different age groups as childhood, juvenile, youth, middle age and senility, are non-equidistant.

A. Mapping relationship between Linguistic labels and Numerical Values

It is effective to calculate linguistic labels by converting linguistic labels into numerical values. The reasoning of the DSm-LARL is also based on the operations of numerical values corresponding to linguistic labels, but the mapping relationship between linguistic labels and numerical values is linear. In the paper, $f(.)$ is used to denote the mapping relationship. If the real number n is the mapping value of the linguistic label L_i, then:

$$n = f(i) \text{ and } i = f^{-1}(n). \quad (2)$$

In order to ensure the precision and convenience of calculation of the labels, the following label L_i is the refined label. Because the linguistic labels in the label set are ordinal, the mapping function $f(.)$ must monotonously increase. If $f(.)$ is linear, the labels are equidistant; if $f(.)$ is non-linear, the labels are non-equidistant.

According to the mapping function $f(.)$ and (2), we can get the following conversions: the conversion of linguistic labels to numerical values

$$\Delta : L \to \mathrm{R} \text{ with } \Delta : L_i \to n = f(i), \quad (3)$$

and the conversion of numerical values to linguistic labels

$$\nabla : \mathrm{R} \to L \text{ with } \nabla : n \to L_i = L_{f^{-1}(n)}. \quad (4)$$

In order to simplify the description, n_i is used to denote the mapping value of L_i. The label set L is mapped into the interval [0, 1], and the mapping value of L_0 or L_{m+1} is 0 or 1. Then,

$$n_0 = f(0) = 0 \text{ and } n_{m+1} = f(m+1) = 1. \quad (5)$$

On the other hand, according to (3), if $i = f^{-1}(x)$, then:

$$n_{f^{-1}(x)} = f\left(f^{-1}(x)\right) = x \text{ and } f^{-1}(n_x) = f^{-1}\left(f(x)\right) = x. \quad (6)$$

The equations (5)(6) are mainly applied in the following deduction about the operational axioms of linguistic labels.

B. Universal Operators of Linguistic Labels

According to (3)(4), we got the universal operations of linguistic labels:

- Addition of linguistic labels:

$$L_a + L_b = L_b + L_a = L_{f^{-1}(n_a+n_b)} \quad (7)$$

since $\Delta : L_a + L_b \to n_a + n_b \quad L_b + L_a \to n_b + n_a$
and $\nabla : n_a + n_b \to L_{f^{-1}(n_a+n_b)}$.

- Subtraction of linguistic labels:

$$L_a - L_b = L_{f^{-1}(n_a-n_b)} \quad (8)$$

since $\Delta : L_a - L_b \to n_a - n_b$ and $\nabla : n_a - n_b \to L_{f^{-1}(n_a-n_b)}$.

- Multiplication of linguistic labels:

$$L_a \times L_b = L_b \times L_a = L_{f^{-1}(n_a \cdot n_b)} \quad (9)$$

since $\Delta : L_a \times L_b \to n_a \cdot n_b \quad L_b \times L_a \to n_b \cdot n_a$
and $\nabla : n_a \cdot n_b \to L_{f^{-1}(n_a \cdot n_b)}$.

- Division of linguistic labels:

$$L_a \div L_b = L_{f^{-1}(n_a/n_b)} \quad n_b \neq 0 \quad (10)$$

since $\Delta : L_a \div L_b \to n_a / n_b$ and $\nabla : n_a / n_b \to L_{f^{-1}(n_a/n_b)}$.

The arithmetic operations of numerical values and linguistic labels are as follows:

- Addition of numerical values and linguistic labels:

$$L_a + \omega = \omega + L_a = L_{f^{-1}(n_a+\omega)} \quad (11)$$

since $\Delta : L_a + \omega \to n_a + \omega \quad \omega + L_a \to \omega + n_a$
and $\nabla : n_a + \omega \to L_{f^{-1}(n_a+\omega)}$.

- Subtraction of numerical values and linguistic labels:

$$L_a - \omega = L_{f^{-1}(n_a-\omega)} \quad \omega - L_a = L_{f^{-1}(\omega-n_a)} \quad (12)$$

since $\Delta : L_a - \omega \to n_a - \omega \quad \omega - L_a \to \omega - n_a$
and $\nabla : n_a - \omega \to L_{f^{-1}(n_a-\omega)} \quad \omega - n_a \to L_{f^{-1}(\omega-n_a)}$.

- Multiplication of numerical values and linguistic labels:

$$\omega \cdot L_a = L_a \cdot \omega = L_{f^{-1}(\omega \cdot n_a)} \quad (13)$$

since $\Delta : \omega \cdot L_a \to \omega \cdot n_a \quad L_a \cdot \omega \to n_a \cdot \omega$
and $\nabla : \omega \cdot n_a \to L_{f^{-1}(\omega \cdot n_a)}$.

- Division of numerical values and linguistic labels:

$$L_a \div \omega = L_{f^{-1}(n_a/\omega)} \quad \omega \neq 0 \quad \omega \div L_a = L_{f^{-1}(\omega/n_a)} \quad n_a \neq 0 \quad (14)$$

since $\Delta: L_a \div \omega \to n_a/\omega \quad \omega \div L_a \to \omega/n_a$

and $\nabla: n_a/\omega \to L_{f^{-1}(n_a/\omega)} \quad \omega/n_a \to L_{f^{-1}(\omega/n_a)}$.

The above arithmetic operations are the basis of fusion calculation of qualitative beliefs denoted by linguistic labels.

C. Correlative Operations Axioms of Labels

According to (3)(4)(5)(6) and the above mentioned arithmetic operations of linguistic labels, we can prove some operational axioms of labels

- Axiom 1:

$$L_a \times (L_b \times L_c) = (L_a \times L_b) \times L_c \quad (15)$$

since $L_a \times (L_b \times L_c) = L_a \times L_{f^{-1}(n_b \cdot n_c)} = L_{f^{-1}(n_a \cdot n_b \cdot n_c)}$

and $(L_a \times L_b) \times L_c = L_{f^{-1}(n_a \cdot n_b)} \times L_c = L_{f^{-1}(n_a \cdot n_b \cdot n_c)}$.

- Axiom 2:

$$L_a \times (L_b + L_c) = (L_b + L_c) \times L_a = L_a \times L_b + L_a \times L_c \quad (16)$$

since $L_a \times (L_b + L_c) = L_a \times L_{f^{-1}(n_b + n_c)} = L_{f^{-1}(n_a \cdot n_b + n_a \cdot n_c)}$,

$(L_b + L_c) \times L_a = L_{f^{-1}(n_b + n_c)} \times L_a = L_{f^{-1}(n_a \cdot n_b + n_a \cdot n_c)}$

and $L_a \times L_b + L_a \times L_c = L_{f^{-1}(n_a \cdot n_b)} + L_{f^{-1}(n_a \cdot n_c)} = L_{f^{-1}(n_a \cdot n_b + n_a \cdot n_c)}$.

- Axiom3:

$$\omega \cdot (L_a \times L_b) = (\omega \cdot L_a) \times L_b = L_a \times (\omega \cdot L_b) \quad (17)$$

since $\omega \cdot (L_a \times L_b) = \omega \times L_{f^{-1}(n_a \cdot n_b)} = L_{f^{-1}(\omega \cdot n_a \cdot n_b)}$,

$(\omega \cdot L_a) \times L_b = L_{f^{-1}(\omega \cdot n_a)} \times L_b = L_{f^{-1}(\omega \cdot n_a \cdot n_b)}$

and $L_a \times (\omega \cdot L_b) = L_a \times L_{f^{-1}(\omega \cdot n_b)} = L_{f^{-1}(\omega \cdot n_a \cdot n_b)}$.

- Axiom 4:

$$\omega \times (L_a + L_b) = \omega \cdot L_a + \omega \cdot L_b \quad (18)$$

since $\omega \cdot (L_a + L_b) = \omega \times L_{f^{-1}(n_a + n_b)} = L_{f^{-1}(\omega \cdot n_a + \omega \cdot n_b)}$

and $\omega \cdot L_a + \omega \cdot L_b = L_{f^{-1}(\omega \cdot n_a)} + L_{f^{-1}(\omega \cdot n_b)} = L_{f^{-1}(\omega \cdot n_a + \omega \cdot n_b)}$.

- Axiom 5:

$$L_{m+1} \times L_a = 1 \cdot L_a = L_a \quad (19)$$

according to (5)(6), $L_a \times L_{m+1} = L_{f^{-1}(n_a \cdot n_{m+1})} = L_{f^{-1}(n_a \cdot 1)} = L_a$

and $1 \cdot L_a = L_{f^{-1}(1 \cdot n_a)} = L_a$.

- Axiom 6:

$$L_0 \times L_a = 0 \cdot L_a = L_0 \quad (20)$$

according to (5)(6), $L_a \times L_0 = L_{f^{-1}(n_a \cdot n_0)} = L_{f^{-1}(n_a \cdot 0)} = L_0$

and $0 \cdot L_a = L_{f^{-1}(0 \cdot n_a)} = L_0$.

The above operation axioms, which include the commutative axioms, the associative axioms and the unitary axioms, can be effectively used in the fusion calculation of qualitative beliefs denoted by linguistic labels.

D. Arithmetic Operations of 2-Tuple Labels

The 2-tuple linguistic label (L_i, α_i) defined by Herrera and Martínez [10] are formed by a label center L_i and an accessorial offset α_i. L_i is the index label and i is an integer. α_i is in the interval [-0,5,0.5] and denotes the actual offset from the actual value to L_i. In fact, we can use the 1-tuple refined labels to express the 2-tuple labels:

$$(L_i, \alpha_i) = L_{i+\alpha_i}. \quad (21)$$

Therefore, the arithmetic operations of 2-tuple linguistic labels can be carried out as follows: convert 2-tuple labels to 1-tuple refined labels firstly according to (21), then do the calculations according to the above mentioned universal operators and axioms of 1-tuple labels.

IV. Fusion of Qualitative Beliefs

A. Qualitative Belief

The original qualitative belief assignment is defined as a kind of mapping function:

$$qm(.): G^\Theta \to L. \quad (22)$$

G^Θ denotes the space of propositions and can be the power set 2^Θ corresponding to the Shafer model, or the hyper-power set D^Θ corresponding to the DSm model. The original qualitative operators only included addition and multiplication. Because of the lack of division, there is no way to define the normalization of qualitative beliefs.

In the above section, we have presented the universal operators of linguistic labels. Therefore, according to the definition of quantitative belief, the qualitative belief can be defined by:

$$\sum_{X \in G^\Theta} qm(X) = L_{\max} = L_{m+1} \text{ with } qm(\phi) = L_{\min} = L_0. \quad (23)$$

The qualitative belief is still the mapping from G^Θ to label set L, but needs to meet the normalization condition. The sum of qualitative beliefs provided by all evidence sources should be L_{m+1}. If the sun is not L_{m+1}, the provided qualitative beliefs need to be normalized, and the normalization operation can be carried out according to the universal operators of linguistic labels. It needs to highlighted that the provided beliefs are denoted by classical labels (the index value i of L_i is an integer and $i \geq 0$) but the normalized beliefs are denoted by refined labels (i is a real number).

B. Fusion Calculation of Qualitative Beliefs

Because the definition of qualitative belief is similar to that of quantitative belief, the qualitative combination rules are similar to the corresponding quantitative combination rules. The calculation of qualitative beliefs can be carried out on the basis of the operations and axioms of linguistic labels which have been mentioned in Section III.

Let the qualitative beliefs provided by two independent and equally reliable sources (m_1 and m_2), are denoted by linguistic labels. According to the corresponding quantitative combination rules, we can get some classical qualitative rules as follows:

- The average rule

$$qm_{av}(X) = qm_{12}(X) + 0.5 \sum_{X \cap Y = \phi} qm_1(X) qm_2(Y) \quad \forall X \in G^\Theta \setminus \{\phi\} \quad (24)$$

This rule averagely distributes the conflict masses to the conflict elements.

- The absorb rule

$$qm_{ab}(X) = qm_{12}(X) + p(X) \sum_{X \cap Y = \phi} qm_1(X) qm_2(Y) \quad \forall X \in G^\Theta \setminus \{\phi\} \quad (25)$$

If $qm_1(X)$ is more than $qm_2(X)$, $p(X) = 1$, otherwise, $p(X) = 0$. In the absorb rule, the conflict masses are distributed to the conflict elements whose original beliefs are more.

- The Dempster's rule

$$qm_{DS}(X) = \left(\frac{1}{L_{\max} - k_{12}}\right) \sum_{\substack{A,B\in 2^\Theta \\ A\cap B = X}} qm_1(A) qm_2(B) \quad \forall X \in 2^\Theta \setminus \{\phi\}$$

$$\text{and } k_{12} = \sum_{\substack{A,B\in 2^\Theta \\ A\cap B=\phi}} qm_1(A) qm_2(B) \tag{26}$$

In the equation, K_{12} is the total conflict between two evidence sources. The Dempster's rule [11] utilizes normalization operation to eliminate the conflict masses.

- The PCR2 (Proportional Conflict Redistribution) rule

$$qm_{PCR2}(X) = qm_{12}(X) + g(X)\frac{c_{12}(X)}{e_{12}} k_{12} \quad \forall X \in G^\Theta \setminus \{\phi\} \tag{27}$$

If X is conflict element, $g(X) = 1$, otherwise, $g(X) = 0$. $c_{12}(X)$ is the total original beliefs of X, and e_{12} is the total beliefs of each element included in global conflict. The PCR2 rule [8] redistributes the global conflict according to the proportion of the total beliefs of each conflict element.

- The PCR6 rule

$$qm_{PCR6}(X) = qm_{12}(X) + \sum_{\substack{Y\in G^\Theta\setminus\{X\} \\ X\cap Y=\phi}} \left[\frac{(qm_1(X))^2 qm_2(Y)}{qm_1(X)+qm_2(Y)} + \frac{(qm_2(X))^2 qm_1(Y)}{qm_2(X)+qm_1(Y)}\right] \tag{28}$$

The PCR6 rule [8] redistributes each local conflict mass according to the original belief proportion of the conflict elements.

In the above five rules, $qm_{12}(X) = \sum_{\substack{A,B\in G^\Theta \\ A\cap B=X}} qm_1(A) qm_2(B)$.

The calculation results of qualitative fusion are the refined labels. In order to increase the precision, we can directly use the refined labels to draw the conclusion. On the other hand, the above mentioned equations all aim at two evidence sources, and the combination rules can be extended to deal with multi-sources easily.

C. Fusion Operations of Mixed Beliefs

Because the operators and axioms of numerical values and linguistic labels have been presented in Section III, the qualitative beliefs and quantitative beliefs can be effectively combined. The combination rules of the mixed beliefs are similar to those of quantitative rules or qualitative rules. Before the fusion, quantitative beliefs and the qualitative beliefs all need to be normalized, and the 2-tuple labels should be transformed into the 1-tuple refined labels.

V. Fusion Examples and Analysis

Two fusion examples are represented in the following. One is the fusion of qualitative beliefs and the other is the fusion of mixed beliefs. In the two examples, we suppose the linguistic labels L_0, L_1, L_2, L_3, L_4, L_5 denotes "certainly not", "impossibly", "improbably", "may", "probably", "certainly" respectively. In the paper, the six labels are non-equidistant, and the mapping functions are:

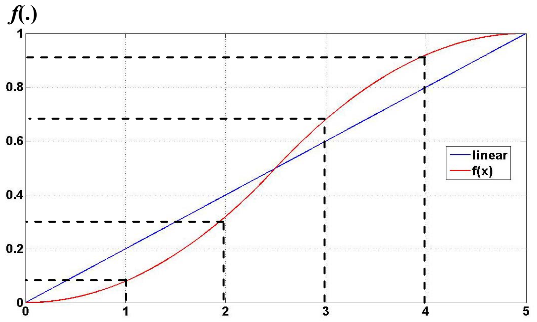

Figure 1. The mapping function.

$$f(x) = \begin{cases} 2\left(x/(m+1)\right)^2 & 0 \le x < 0.5(m+1) \\ 1 - 2\left(1 - x/(m+1)\right)^2 & 0.5(m+1) \le x \le 1 \end{cases} \tag{29}$$

$$f^{-1}(x) = \begin{cases} (m+1)\sqrt{x/2} & 0 \le x < 0.5 \\ (m+1)\left(1 - \sqrt{(1-x)/2}\right) & 0.5 \le x \le 1 \end{cases} \tag{30}$$

In the examples, $m = 4$. According to (29), n_0=0, n_1=0.08, n_2=0.32, n_3=0.68, n_4=0.92, n_5=1. As shown in Fig.1, the red line denotes the mapping function $f(x)$, and L_3–L_2 > L_2–L_1 = L_4–L_3 > L_1–L_0 = L_5–L_4. Therefore, the middle value occupies more range.

A. Example 1

Let us consider $\Theta = \{\theta_1, \theta_2\}$ in Example 1. Two independent and equally reliable evidence sources provide the following qualitative beliefs:

$qm_{1\text{-ori}}(\theta_1) = L_4$ $qm_{1\text{-ori}}(\theta_2) = L_1$ $qm_{1\text{-ori}}(\theta_1 \cup \theta_2) = L_0$

$qm_{2\text{-ori}}(\theta_1) = L_3$ $qm_{2\text{-ori}}(\theta_2) = L_1$ $qm_{2\text{-ori}}(\theta_1 \cup \theta_2) = L_1$

The Shafer model is held and $\theta_1 \cap \theta_2 = \Phi$. Then,

$\sum qm_{1-ori} = L_4 + L_1 + L_0 = L_{f^{-1}(n_4+n_1+n_0)} = L_{f^{-1}(1)} = L_5$,

$\sum qm_{2-ori} = L_3 + L_1 + L_1 = L_{f^{-1}(n_3+n_1+n_1)} = L_{f^{-1}(0.84)} = L_{3.5858} \ne L_5$.

Therefore, we need to normalize the provided qualitative beliefs. The normalized beliefs are as follows:

$qm_1(\theta_1) = L_4$ $qm_1(\theta_2) = L_1$ $qm_1(\theta_1 \cup \theta_2) = L_0$

$qm_2(\theta_1) = L_{3.4569}$ $qm_2(\theta_2) = L_{1.0910}$ $qm_2(\theta_1 \cup \theta_2) = L_{1.0910}$

Thereinto, $qm_2(\theta_1) = qm_{2-ori}(\theta_1) / \sum qm_2 = L_{f^{-1}(n_3/(n_3+n_1+n_1))} = L_{3.4569}$.

According to (24)(25)(26)(27)(28), we can get the combination results of the average rule ($qm_{av}(.)$), the absorb rule ($qm_{ab}(.)$), the Dempster's rule ($qm_{DS}(.)$), the PCR2 rule ($qm_{PCR2}(.)$) and the PCR6 rule ($qm_{PCR6}(.)$). The results are shown in Tab.I.

TABLE I. Combination Results in Example 1

qm(.)	θ_1	θ_2	$\theta_1 \cup \theta_2$
$qm_{av}(.)$	$L_{3.9305}$	$L_{1.0689}$	L_0
$qm_{ab}(.)$	$L_{4.5622}$	$L_{0.4363}$	L_0
$qm_{DS}(.)$	$L_{4.5243}$	$L_{0.4739}$	L_0
$qm_{PCR2}(.)$	$L_{4.3943}$	$L_{0.6046}$	L_0
$qm_{PCR6}(.)$	$L_{4.3941}$	$L_{0.6049}$	L_0

The calculation of the Dempster's rule includes addition, subtraction, multiplication, and division. The calculation process of $qm_{DS}(\theta_1)$ is as follows:

$$qm_{12}(\theta_1)$$
$$= qm_1(\theta_1)qm_2(\theta_1) + qm_1(\theta_1)qm_2(\theta_1 \cup \theta_2) + qm_1(\theta_1 \cup \theta_2)qm_2(\theta_1)$$
$$= L_4 \times L_{3.4569} + L_4 \times L_{1.0910} + L_0 \times L_{3.4569}$$
$$= L_{f^{-1}(f(4)\times f(3.4569))} + L_{f^{-1}(f(4)\times f(1.0910))} + L_{f^{-1}(0)} = L_{f^{-1}(0.8323)} = L_{3.5522}$$
$$k_{12} = qm_1(\theta_1)qm_2(\theta_2) + qm_1(\theta_2)qm_2(\theta_1)$$
$$= L_4 \times L_{1.0910} + L_1 \times L_{3.4569} = L_{f^{-1}(f(4)\times f(1.0910))} + L_{f^{-1}(f(1)\times f(3.4569))}$$
$$= L_{f^{-1}(0.0876)} + L_{f^{-1}(0.0648)} = L_{f^{-1}(0.1524)} = L_{1.3802}$$
$$qm_{DS}(\theta_1) = \left(1/\left(L_{\max} - k_{12}\right)\right) qm_{12}(\theta_1)$$
$$= L_{3.5522} / \left(L_5 - L_{1.3802}\right) = L_{3.5522} / L_{f^{-1}(f(5)-f(1.3802))}$$
$$= L_{3.5522} / L_{f^{-1}(0.8476)} = L_{f^{-1}(0.9819)} = L_{4.5243}$$

B. Example 2

In example 2, we consider $\Theta = \{\theta_1, \theta_2, \theta_3\}$ and hold Shafer model. θ_1, θ_2 and θ_3 respectively represent three selected projects. Three evidence sources judge the selected projects and provide the following quantitative and qualitative beliefs:

$nm_{1\text{-ori}}(\theta_1) = 0.7$ $nm_{1\text{-ori}}(\theta_2) = 0.2$ $nm_{1\text{-ori}}(\theta_3) = 0.1$

$qm_{2\text{-ori}}(\theta_1) = L_4$ $qm_{2\text{-ori}}(\theta_2) = L_1$ $qm_{2\text{-ori}}(\theta_3) = L_0$

$qm_{3\text{-o}}(\theta_1)=(L_3,+0.4)$ $qm_{3\text{-o}}(\theta_2)=(L_2,-0.2)$ $qm_{3\text{-o}}(\theta_3)=(L_0,+0.3)$

Because $\sum qm_{3-ori} = L_{3+0.4} + L_{2-0.2} + L_{0+0.3} = L_{f^{-1}(1.0616)} \neq L_5$, the provided beliefs need to be normalized, and the normalized beliefs are:

$nm_1(\theta_1) = 0.7$ $nm_1(\theta_2) = 0.2$ $nm_1(\theta_3) = 0.1$

$qm_2(\theta_1) = L_4$ $qm_2(\theta_2) = L_1$ $qm_2(\theta_3) = L_0$

$qm_3(\theta_1) = L_{3.2287}$ $qm_3(\theta_2) = L_{1.7471}$ $qm_3(\theta_3) = L_{0.2915}$

According to the above mentioned combination rules, we can combine the provided beliefs and get the fusion results of different rules. The results are shown in Tab.II.

The deduced operators and axioms in Section III are based on the mapping relationship between refined labels and real numbers. However, in the real application, the provided labels are in the interval $[L_0, L_{m+1}]$ and are usually mapped into the interval [0, 1] to simplify the operations. In other words, the definition domain and range of mapping function f(.) are both limited. The definition domain and range can be extended to the refined labels field and real number field respectively, but may be different sometimes.

TABLE II. COMBINATION RESULTS IN EXAMPLE 2

m(.)	θ_1	θ_2	θ_3
$m_{av}(.)$	$L_{3.3785}$	$L_{1.4779}$	$L_{0.6671}$
$m_{ab}(.)$	$L_{4.7257}$	$L_{0.2732}$	$L_{0.0250}$
$m_{DS}(.)$	$L_{4.6831}$	$L_{0.3169}$	L_0
$m_{PCR2}(.)$	$L_{3.8170}$	$L_{1.0821}$	$L_{0.4781}$
$m_{PCR6}(.)$	$L_{4.0169}$	$L_{0.9381}$	$L_{0.2938}$

On the other hand, the sum calculation results of some linguistic labels may surpass the limits, such as $\sum qm_{3\text{-ori}}$. However, the calculation results based on the combination rules would be within the normal range. Therefore, we can ignore the problem in the calculation of information fusion.

VI. CONCLUSION

The linguistic label is an effective method to express the natural language and is used to denote the qualitative belief in DSmT. In order to combine the qualitative beliefs, the DSm-LARL has been proposed, but can only solve the problem about the precise calculation of equidistant labels. Based on the refined linguistic label, the paper provided the universal operations of labels. The proposed operators and axioms are based on the mapping relationship between linguistic labels and numerical values, and can be used in different linguistic models. They can help not only solve the fusion problems of the qualitative beliefs denoted by equidistant or non-equidistant labels, but also increase the precision and validity of the combination of qualitative beliefs or mixed beliefs. How to assure the mapping function is the key point and the difficulty in the calculation. This paper does not further research on the mapping relationship between linguistic labels and numerical values, and the deep research of mapping function is our next step.

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