EMAN.M.EL-NAKEEB[1], H. ELGHAWALBY[2], A.A.SALAMA[3], S.A.EL-HAFEEZ[4]

1,2 Port Said University, Faculty of Engineering, Physics and Engineering Mathematics Department, Egypt.
E-mails: emanmarzouk1991@gmail.com, hewayda2011@eng.psu.edu.eg
3,4 Port Said University, Faculty of Science, Department of Mathematics and Computer Science, Egypt.
E-mails: drsalama44@gmail.com, samyabdelhafeez@yahoo.com

Foundation for Neutrosophic Mathematical Morphology

Abstract
The aim of this paper is to introduce a new approach to Mathematical Morphology based on neutrosophic set theory. Basic definitions for neutrosophic morphological operations are extracted and a study of its algebraic properties is presented. In our work we demonstrate that neutrosophic morphological operations inherit properties and restrictions of Fuzzy Mathematical Morphology.

Keywords
Crisp sets operations, fuzzy sets, neutrosophic sets, mathematical morphology, fuzzy mathematical morphology.

1. Introduction
Established in 1964, Mathematical Morphology was firstly introduced by Georges Matheron and Jean Serra, as a branch of image processing [12]. As morphology is the study of shapes, Mathematical Morphology mostly deals with the mathematical theory of describing shapes using set theory. In image processing, the basic morphological operators dilation, erosion, opening and closing form the fundamentals of this theory [12]. A morphological operator transforms an image into another image, using some structuring element which can be chosen by the user. Mathematical Morphology stands somewhat apart from traditional linear image processing, since the basic operations of morphology are non-linear in nature, and thus make use of a totally different type of algebra than the linear algebra. At first, the theory was purely based on set theory and operators which defined for binary cases only. Later on the theory was extended to the grayscale images as the theory of lattices was introduced, hence, a representation theory for image processing was given [7]. As a scientific branch, Mathematical Morphology expanded worldwide during the 1990's. It is also during that period, different models based on fuzzy set theory were introduced [3, 4]. Today, Mathematical Morphology remains a challenging research field [6, 7].

In 1995, Samarandache initiated the theory of neutrosophic set as new mathematical tool for handling problems involving imprecise indeterminacy, and inconsistent data [14]. Later on, several researchers such as Bhowmik and Pal [2], and Salama [11], studied the concept of neutrosophic set. Neutrosophy introduces a new concept which represents indeterminacy with respect to some event, which can solve certain problems that cannot be solved by fuzzy logic.

This work is devoted for introducing the neutrosophic concepts to Mathematical Morphology. The rest of the paper is structured as follows: In §2, we introduce the fundamental definitions from the Mathematical Morphology whereas, the concepts of Fuzzy Morphology are introduced in §3. The basic definitions for Neutrosophic Morphological operations are extracted and a study of its algebraic properties is presented in §4.

2. Mathematical Morphology [5]

Basically, Mathematical Morphology describes an image's regions in the form of sets. Where the image is considered to be the universe with values are pixels in the image, hence, standard set notations can be used to describe image operations [7]. The essential idea, is to explore an image with a simple, pre-defined shape, drawing conclusions on how this shape fits or misses the shapes in the image [12]. This simple pre-defined shape is called the "structuring element", and it is usually small relative to the image.

In the case of digital images, a simple binary structuring elements like a cross or a square is used. The structuring elements can be placed at any pixel in the image, nevertheless, the rotation is not allowed. In this process, some reference pixel whose position defines where the structuring element is to be placed. The choice of this reference pixel is often arbitrary.

2.1. Binary Morphology

In binary morphology, an image is viewed as a subset of an Euclidean space R^n or the integer grid Z^n, for some dimension *n*. The structuring element is a binary image (i.e., a subset of the space or the grid). In this section we briefly review the basic morphological operations, the dilation, the erosion, the opening and the closing.

2.1.1. Binary Dilation: (Minkowski addition)

Dilation is one of the basic operations in Mathematical Morphology, which originally developed for binary images [15]. The dilation operation uses a structuring element for exploring and expanding the shapes contained in the input image. In binary morphology, dilation is a shift-invariant (translation invariant) operator, strongly related to the Minkowski addition.

For any Euclidean space E and a binary image A in E, the dilation of A by some structuring element B is defined by: $A\oplus B = \bigcup_{b\in B} A_b$ where A_b is the translate of the set A along the vector *b*, i.e., A_b=$\{a+b \in E | a \in A, b \in B\}$.

The dilation is commutative, and may also be given by: $A\oplus B = B\oplus A = \bigcup_{a\in A} B_a$.

An interpretation of the dilation of A by B can be understood as, if we put a copy of B at each pixel in A and union all of the copies, then we get $A\oplus B$.

The dilation can also be obtained by: $A\oplus B = \{b \in E \,|(-B) \cap A \neq \emptyset\}$, where (–B) denotes the reflection of B, that is, $-B = \{x \in E | -x \in B\}$.

Where the reflection satisfies the following property: $-(A\oplus B) = (-A)\oplus(-B)$

2.1.2. Binary Erosion: (Minkowski subtraction)

Strongly related to the Minkowski subtraction, the erosion of the binary image A by the structuring element B is defined by: $A \ominus B = \bigcap_{b \in B} A_{-b}$.

Unlike dilation, erosion is not commutative, much like how addition is commutative while subtraction is not [8, 15]. An interpretation for the erosion of A by B can be understood as, if we again put a copy of B at each pixel in A, this time we count only those copies whose translated structuring elements lie entirely in A; hence $A \ominus B$ is all pixels in A that these copies were translated to. The erosion of A by B is also may be given by the expression: $\mathrm{A} \ominus \mathrm{B} = \{\mathrm{p} \in \mathrm{E} \,|\, \mathrm{B_p} \subseteq \mathrm{A}\}$, where B_p is the translation of B by the vector p, i.e.,

$B_p = \{b + p \in E \,|\, b \in B\}$, $\forall\, p \in E$.

2.1.3. Binary Opening [15]

The opening of A by B is obtained by the erosion of A by B, followed by dilation of the resulting image by B: $A \circ B = (A \ominus B) \oplus B$.

The opening is also given by $A \circ B = \bigcup_{B_x \subseteq A} B_x$, which means that, an opening can be consider to be the union of all translated copies of the structuring element that can fit inside the object. Generally, openings can be used to remove small objects and connections between objects.

2.1.4. Binary Closing [6]

The closing of A by B is obtained by the dilation of A by B, followed by erosion of the resulting structure by B: $\mathrm{A} \bullet \mathrm{B} = (A \oplus B) \ominus B$.

The closing can also be obtained by $\mathrm{A} \bullet \mathrm{B} = (\mathrm{A^c} \circ (-\mathrm{B}))^{\mathrm{c}}$, where $\mathrm{A^c}$ denotes the complement of A relative to E (that is, $\mathrm{A^c} = \{a \in E \,|\, a \notin A\}$). Whereas opening removes all pixels where the structuring element won't fit inside the image foreground, closing fills in all places where the structuring element will not fit in the image background, that is opening removes small objects, while closing removes small holes.

2.2. Properties of the Basic Binary Operations

Here are some properties of the basic binary morphological operations (dilation, erosion, opening and closing[8]). We define the power set of X, denoted by P(X), to be the set of all crisp subset of X.

For all A, B, C $\in P(X)$, the following properties hold:

- $A \oplus B = B \oplus A$,
- $A \subseteq B \Longrightarrow (A \oplus C) \subseteq (B \oplus C)$,
- $A \subseteq (A \oplus B)$,
- $(A \oplus B) \oplus C = A \oplus (B \oplus C)$, and $(A \ominus B) \ominus C = A \ominus (B \ominus C)$,
- Erosion and dilation satisfy the duality that is:

  $\mathrm{A} \oplus \mathrm{B} = (\mathrm{A^c} \ominus (-\mathrm{B}))^{\mathrm{c}}$, and $\mathrm{A} \ominus \mathrm{B} = (\mathrm{A^c} \oplus (-\mathrm{B}))^{\mathrm{c}}$,
- $A \subseteq B \Longrightarrow (A \circ C) \subseteq (B \circ C)$,

- $A \circ B \subseteq A$,
- Opening and closing satisfy the duality that is:

$$A \bullet B = (A^c \circ (-B))^c, \text{ and } A \circ B = \left(A^c \bullet (-B)\right)^c.$$

3. Fuzzy Mathematical Morphology

When operations are expressed in algebraic or logical terms, one powerful approach leading to good properties consists of formally replacing the classical symbols in the equations by their fuzzy equivalent. This framework led to an infinity of fuzzy Mathematical Morphologies, which are constructed in families with specific properties described in [3, 13].

3.1. Fuzzy Set

Since introduced by Zadeh [16], fuzzy sets have received a great deal of interest [17]. For an ordinary set, a given element either belongs or does not belong to the set, whereas for a fuzzy set the membership of an element is determined by the value of a given membership function, which assigns to each element a degree of membership ranging between zero and one.

3.1.1. Definition [16]

Let X be a fixed set. A fuzzy set A of X is an object having the form $A = \langle \mu_A \rangle$, where the function $\mu_A : X \to [0,1]$ defines the degree of membership of the element $x \in X$ to the set A. The set of all fuzzy subset of X is denoted by $\mathcal{F}(X)$.

The fuzzy empty set in X is denoted by $0_f = \langle \underline{0} \rangle$, $where$ $\underline{0} : X \longrightarrow [0,1]$ and $\underline{0}(x) = 0$, $\forall x \in X$. Moreover, the fuzzy universe set in X is denoted by $1_f = \langle \underline{1} \rangle$, $where$ $\underline{1} : X \longrightarrow [0,1]$ and $\underline{1}(x) = 1$, $\forall x \in X$.

3.2. Fuzzy Mathematical Operations [4]

The fuzziness concept was introduced to the morphology by defining the degree to which the structuring element fits into the image. The operations of dilation and erosion of a fuzzy image by a fuzzy structuring element having a bounded support, are defined in terms of their membership functions.

3.2.1. Fuzzy Dilation [4]

Let us consider the notion of dilation within the original formulation of Mathematical Morphology in Euclidean space E. For any two n-dimensional gray-scale images, A and B, the fuzzy dilation, $A \oplus B = \langle \mu_{A \oplus B} \rangle$, of A by the structuring element B is an n-dimensional gray-scale image, that is: $\mu_{A \oplus B} : Z^2 \longrightarrow [0,1]$, and

$$\mu_{A \oplus B}(v) = \sup_{u \in Z^2} \min[\, \mu_A(v+u),\ \mu_B(u)]$$

Where $u, v \in Z^2$ are the spatial co-ordinates of pixels in the image and the structuring element; while μ_A , μ_B are the membership functions of the image and the structuring element, respectively.

3.2.2. Fuzzy Erosion [4]

For any two n-dimensional gray-scale image, A and B, the fuzzy erosion $A \ominus B = \langle \mu_{A \ominus B} \rangle$ of A by the structuring element B is an n-dimensional gray-scale image, that is:

$\mu_{A\ominus B}: Z^2 \longrightarrow [0,1]$, and

$$\mu_{A\ominus B}(v) = \inf_{u\in Z^2} max[\mu_A(v+u), 1-\mu_B(u)]$$

where u, $v \in Z^2$are the spatial co-ordinates of pixels in the image and the structuring element; while μ_A, μ_B are the membership functions of the image and the structuring element respectively.

3.2.3. Fuzzy Closing and Fuzzy Opening [3]

In a similar way the two fuzzy operations for closing and opening for any two n-dimensional gray-scale images, A and B, are defined as follows:

$$\mu_{A\bullet B}(v) = \inf_{u\in Z^2} max\left(\sup_{w\in Z^2} min\ \left(\mu_A(v-u+w), \mu_B(u)\right), 1-\mu_B(u)\right)$$

$$\mu_{A\circ B}(v) = \sup_{u\in Z^2} min\left(\inf_{w\in Z^2} max\ \left(\mu_A(v-u+w), \mu_B(u)\right), 1-\mu_B(u)\right)$$

where u, v, w $\in Z^2$ are the spatial co-ordinates of pixels in the image and the structuring element; while μ_A, μ_B are the membership functions of the image and the structuring element respectively.

3.3. Properties of the Basic Operations

Here are some properties of the basic fuzzy morphological operations (dilation, erosion, opening and closing [4]). We define the power set of X, denoted by $\mathcal{F}(Z^2)$, to be the set of all fuzzy subset of X,

For all $A, B, C \in \mathcal{F}(Z^2)$ the following properties hold:

i. Monotonicity (increasing in both argument)

$A \subseteq B \Longrightarrow A \oplus C \subseteq B \oplus C$

$A \subseteq B \Longrightarrow C \oplus A \subseteq C \oplus B$

ii. Monotonicity (increasing in the first and decreasing in the argument)

$A \subseteq B \Longrightarrow A \ominus C \subseteq B \ominus C$

$A \subseteq B \Longrightarrow C \ominus A \supseteq C \ominus B$

iii. Monotonicity (increasing in the first argument)

$A \subseteq B \Longrightarrow A \bullet C \subseteq B \bullet C$

iv. Monotonicity (increasing in the first argument)

$A \subseteq B \Longrightarrow A \circ C \subseteq B \circ C$

For any family $(A_i|i \in I)\ in \mathcal{F}(\mathrm{Z}^2)\ and\ B \in \mathcal{F}(\mathrm{Z}^2)$,

i. $\bigcap_{i\in I} \mathrm{A}_i \oplus \mathrm{B} \subseteq \bigcap_{i\in I} (\mathrm{A}_i \oplus \mathrm{B})$ and $\mathrm{B} \oplus \bigcap_{i\in \mathrm{I}} \mathrm{A}_i \subseteq \bigcap_{i\in I} (\mathrm{B} \oplus \mathrm{A}_i)$

ii. $\bigcap_{i\in I} \mathrm{A}_i \ominus \mathrm{B} \subseteq \bigcap_{i\in I} (\mathrm{A}_i \ominus \mathrm{B})$ and $\mathrm{B} \ominus \bigcap_{i\in \mathrm{I}} \mathrm{A}_i \supseteq \bigcap_{i\in I} (\mathrm{B} \ominus \mathrm{A}_i)$

iii. $\bigcap_{i\in I} \mathrm{A}_i \bullet \mathrm{B} \subseteq \bigcap_{i\in I} (\mathrm{A}_i \bullet \mathrm{B})$

iv. $\bigcap_{i\in I} \mathrm{A}_i \circ \mathrm{B} \subseteq \bigcap_{i\in I} (\mathrm{A}_i \circ \mathrm{B})$

For any family $(A_i|i \in I)\ in \mathcal{F}(\mathrm{Z}^2)\ and\ B \in \mathcal{F}(\mathrm{Z}^2)$,

i. $\bigcup_{i\in I} A_i \oplus B \supseteq \bigcup_{i\in I} (A_i \oplus B)$ and $B \oplus \bigcup_{i\in I} A_i \supseteq \bigcup_{i\in I} (B \oplus A_i)$

ii. $\bigcup_{i\in I} A_i \ominus B \supseteq \bigcap_{i\in I} (A_i \ominus B)$ and $B \ominus \bigcap_{i\in I} A_i \subseteq \bigcap_{i\in I} (B \ominus A_i)$

iii. $\bigcup_{i\in I} A_i \bullet B \supseteq \bigcap_{i\in I} (A_i \bullet B)$

iv. $\bigcup_{i\in I} A_i \circ B \supseteq \bigcap_{i\in I} (A_i \circ B)$.

4. Neutrosophic Approach to Mathematical Morphology

Smarandache [14] introduced the neutrosophic components (T, I, F) which represent the membership, indeterminacy, and non-membership values respectively, $T, I, F : X \to]^-0,1^+[$ where $]^-0,1^+[$ is non-standard unit interval. Let $\varepsilon > 0$ be some infinitesimal number, hence, $1^+ = 1+\varepsilon$ and $^-0 = 0-\varepsilon$.

4.1. Neutrosophic Sets [1]

We denote the set of all neutrosophic subset of X by $\mathcal{N}(X)$. In [1, 14], the authors gave different definition for the concept of the neutrosophic sets. For more convenience we are choosing the following definitions to follow up our work for neutrosophic morphology. In the following definitions, we consider a space E and two neutrosophic subsets of X; $A, B \in \mathcal{N}(X)$.

4.1.1. Definition [11, 14]

A neutrosophic set A on the universe of discourse X is defined as:

$A = \langle T_A, I_A, F_A \rangle$, where $T_A, I_A, F_A : X \to [0,1]$.

4.1.2. Definition [11]

The complement of a neutrosophic set A is denoted by A^c and is defined as:

$A^c = \langle T_{A^c}, I_{A^c}, F_{A^c} \rangle$, where $T_{A^c}, I_{A^c}, F_{A^c} : X \to [0,1]$ and for all x in X.

$$T_{A^c}(x) = 1 - T_A(x), \quad I_{A^c}(x) = 1 - I_A(x) \quad \text{and} \quad F_{A^c}(x) = 1 - F_A(x)$$

The neutrosophic empty Set of X is the triple, $0_{\mathcal{N}} = \langle \underline{0}\ ,\ \underline{0}\ ,\ \underline{1} \rangle$, where

$$\underline{1}(x) = 1 \ and\ \underline{0}(x) = 0, \quad \forall x \in X.$$

The neutrosophic universe set of X is the triple, $1_{\mathcal{N}} = \langle \underline{1}\ ,\ \underline{1}\ ,\ \underline{0} \rangle$, where

$$\underline{1}(x) = 1 \ and\ \underline{0}(x) = 0 \quad \forall x \in X.$$

4.2. Neutrosophic Mathematical Morphology

In this section we introduce the concept of neutrosophic morphology based on the fact that the basic morphological operators make use of fuzzy set operators, or equivalently, crisp logical operators. Hence, such expressions can easily be extended using the context of neutrosophic sets.

4.2.1. Definition

The reflection of the structuring element B mirrored in its origin is defined as:

- $-B = \langle -T_B, -I_B, -F_B \rangle$, where

  $-T_B(u) = T_B(-u), \qquad -I_B(u) = I_B(-u) \quad and \quad -F_B(u) = F_B(-u)$

- For every p in E, Translation of A by $p \in Z^2$ is $A_p = \langle T_{A_p}, I_{A_p}, F_{A_p} \rangle$, Where $T_{A_p}(u) = T_{A_p}(u+p)$, $I_{A_p}(u) = I_{A_p}(u+p)$ and $F_{A_p}(u) = F_{A_p}(u+p)$

Most morphological operations on neutrosophic can be obtained by combining neutrosophic set theoretical operations with two basic operations, dilation and erosion.

4.3 Neutrosophic Morphological Operations

The neutrosophy concept is introduced to morphology by a triple degree to which the structuring element fits into the image in the three levels of trueness, indeterminacy, and falseness. The operations of neutrosophic erosion, dilation, opening and closing of the neutrosophic image by neutrosophic structuring element, are defined in terms of their membership, indeterminacy and non-membership functions; which is defined for the first time as far as we know.

4.3.1. The Operation of Dilation

The process of the structuring element B on the image A and moving it across the image in a way like convolution is defined as dilation operation. The two main inputs for the dilation operator [7] are the image which is to be dilated and a set of coordinate points known as a structuring element which may be considered as a kernel. The exact effect of the dilation on the input image is determined by this structuring element [6].

4.3.1.1. Definition: (Neutrosophic Dilation)

let A and B are two neutrosophic sets; then the neutrosophic dilation is given as $\left(\mathrm{A} \,\widetilde{\oplus}\, \mathrm{B}\right) = \left\langle \mathrm{T}_{\mathrm{A}\widetilde{\oplus}\mathrm{B}}, \mathrm{I}_{\mathrm{A}\widetilde{\oplus}\mathrm{B}}, \mathrm{F}_{\mathrm{A}\widetilde{\oplus}\mathrm{B}} \right\rangle$, where for each u and v $\in Z^2$.

$$T_{A\widetilde{\oplus}B}(v) = \sup_{u\in Z^n} min(T_A(v+u), T_B(u))$$

$$I_{A\widetilde{\oplus}B}(v) = \sup_{u\in Z^n} min(I_A(v+u), I_B(u))$$

$$F_{A\widetilde{\oplus}B}(v) = \inf_{u\in Z^n} max(1-F_A(v+u), 1-F_B(u))$$

4.3.2. The Operation of Erosion

The erosion process is as same as dilation, but the pixels are converted to 'white', not 'black'. The two main inputs for the erosion operator [12], are the image which is to be eroded and

a structuring element. The exact effect of the erosion on the input image is determined by this structuring element. The following steps are the mathematical definition of erosion for gray-scale images.

4.3.2.1. Definition: (Neutrosophic Erosion)

let A and B are two neutrosophic sets , then the neutrosophic erosion is given

$\left(\mathrm{A} \,\widetilde{\ominus}\, \mathrm{B}\right) = \left\langle \mathrm{T}_{\mathrm{A}\widetilde{\ominus}\mathrm{B}}, \mathrm{I}_{\mathrm{A}\widetilde{\ominus}\mathrm{B}}, \mathrm{F}_{A\widetilde{\ominus}B} \right\rangle$; where for each u and v $\in Z^2$

$$T_{A\widetilde{\ominus}B}(v) = \inf_{u\in Z^n} max(T_A(v+u), 1-T_B(u))$$

$$I_{A\widetilde{\ominus}B}(v) = \inf_{u\in Z^n} max(I_A(v+u), 1-I_B(u))$$

$$F_{A\widetilde{\ominus}B}(v) = \sup_{u\in Z^n} min\left(1-F_A(v+u), F_B(u)\right)$$

4.3.3. The Operation of Opening and Closing

The combination of the two main operations, dilation and erosion, can produce more complex sequences. Opening and closing are the most useful of these for morphological filtering [8]. An opening operation is defined as erosion followed by a dilation using the same structuring element for both operations. The basic two inputs for opening operator are an image to be opened, and a structuring element. Gray-level opening consists simply of gray-level erosion followed by gray-level dilation. The morphological opening $\circ$ and closing $\bullet$ are defined by:

$$A \,\tilde{\circ}\, B \;=\; (A \,\widetilde{\ominus}\, B) \,\widetilde{\oplus}\, B$$

$$\mathrm{A} \,\tilde{\bullet}\, \mathrm{B} \;=\; (A \,\widetilde{\oplus}\, B) \,\widetilde{\ominus}\, B$$

From a granularity perspective, opening and closing provide coarser descriptions of the set A. The opening describes A as closely as possible using not the individual pixels but by fitting (possibly overlapping) copies of E within A. The closing describes the complement of A by fitting copies of E^* outside A. The actual set is always contained within these two extremes: $A \,\tilde{\circ}\, B \subseteq A \subseteq A \,\tilde{\bullet}\, B$ and the informal notion of fitting copies of E, or of E^*, within a set is made precise in these equations:

The operator $\mathcal{N}(E) \to \mathcal{N}(E) : A \to A \,\tilde{\circ}\, B$ is called the opening by B; it is the composition of the erosion $\ominus$, followed by the dilation $\oplus$. On the other hand, the operator $\mathcal{N}(E) \to \mathcal{N}(E) : A \to A \,\tilde{\bullet}\, B$ is called the closing.

To understand what *e.g.,* a closing operation does: imagine the closing applied to a set; the dilation will expand object boundaries, which will be partly undone by the following erosion. Small, (*i.e.*, smaller than the structuring element) holes and thin tubelike structures in the interior or at the boundaries of objects will be filled up by the dilation, and not reconstructed by the erosion, inasmuch as these structures no longer have a boundary for the erosion to act upon. In this sense the term 'closing' is a well-chosen one, as the operation removes holes and thin cavities. In the same sense the opening opens up holes that are near (with respect to the size of the structuring element) a boundary, and removes small object protuberances.

4.3.3.1. Neutrosophic Opening

let A and B are two neutrosophic sets it's defined as the flowing:

$(A \,\tilde{\circ}\, B) = \langle T_{A\tilde{\circ}B}, I_{A\tilde{\circ}B}, F_{A\tilde{\circ}B} \rangle$, where u, v, w$\in Z^2$

$$T_{A\tilde{\circ}B}(v) = \sup_{u\in Z^n} \min \left[\inf_{z\in R^n} \max\big(T_A(v-u+w), 1-T_B(w)\big), T_B(u) \right]$$

$$I_{A\tilde{\circ}B}(v) = \sup_{u\in Z^n} \min \left[\inf_{z\in R^n} \max\big(I_A(v-u+w), 1-I_B(w)\big), I_B(u) \right]$$

$$F_{A\tilde{\circ}B}(v) = \inf_{u\in Z^n} \max \left[\sup_{z\in R^n} \min\big(1-F_A(v-u+w), F_B(w)\big), 1-F_B(u) \right]$$

4.3.3.2. Neutrosophic Closing

let A and B be two neutrosophic sets it's defined as the flowing:

$(A \,\tilde{\bullet}\, B) = \langle T_{A\tilde{\bullet}B}, I_{A\tilde{\bullet}B}, F_{A\tilde{\bullet}B} \rangle$, where u, v, w$\in Z^2$

$$T_{A \tilde{\bullet} B}(v) = \inf_{u \in Z^2} \max \left[\sup_{w \in Z^2} \min\big(T_A(v-u+w), T_B(w)\big), 1 - T_B(u) \right]$$

$$I_{A \tilde{\bullet} B}(v) = \inf_{u \in Z^2} \max \left[\sup_{w \in Z^2} \min\big(I_A(v-u+w), I_B(w)\big), 1 - I_B(u) \right]$$

$$F_{A \tilde{\bullet} B}(v) = \sup_{u \in Z^2} \min \left[\inf_{w \in Z^2} \max\big(1 - F_A(v-u+w), 1 - F_B(w)\big), F_B(u) \right]$$

4.4. Algebraic Properties in Neutrosophic

The algebraic properties for Neutrosophic Mathematical Morphology erosion and dilation, as well as for neutrosophic opening and closing operations are now considered.

4.4.1. Proposition Duality Theorem of Dilation

let A and B be two neutrosophic sets. neutrosophic erosion and dilation are dual operations i.e. $(\mathrm{A}^c \,\widetilde{\oplus}\, \mathrm{B})^{\mathrm{c}} = \langle \mathrm{T}_{(\mathrm{A}^c \widetilde{\oplus} \mathrm{B})^{\mathrm{c}}}, \mathrm{I}_{(\mathrm{A}^c \widetilde{\oplus} \mathrm{B})^{\mathrm{c}}}, \mathrm{F}_{(\mathrm{A}^c \widetilde{\oplus} \mathrm{B})^{\mathrm{c}}} \rangle$; where for each u, v $\in Z^2$

$$\begin{aligned}
T_{(\mathrm{A}^c \widetilde{\oplus} B)^c}(v) &= 1 - T_{(\mathrm{A}^c \widetilde{\oplus} B)}(v) \\
&= 1 - \sup_{u \in Z^2} \min(T_{A^c}(v+u), T_B(u)) = \inf_{u \in Z^2} [1 - \min(T_{A^c}(v+u), T_B(u))] \\
&= \inf_{u \in Z^2} [\max(1 - \mathrm{T}_{\mathrm{A}^c}(v+u), 1 - T_B(u))] \\
&= \inf_{u \in Z^2} [\max(T_A(v+u), 1 - T_B(u))] \quad = T_{A \widetilde{\ominus} B}(v)
\end{aligned}$$

$$\begin{aligned}
I_{(\mathrm{A}^c \widetilde{\oplus} B)^c}(v) &= 1 - I_{(\mathrm{A}^c \widetilde{\oplus} B)}(v) \quad = 1 - \sup_{x \in R^n} \min(\mathrm{A}^c(v+u), I_B(u)) \\
&= \inf_{u \in Z^2} [1 - \min(\mathrm{I}_{\mathrm{A}^c}(v+u), I_B(x))] \\
&= \inf_{u \in Z^2} [\max(1 - \mathrm{I}_{\mathrm{A}^c}(v+u), 1 - I_B(u))] \\
&= \inf_{u \in Z^2} [\max(I_A(v+u), 1 - I_B(u))] \quad = I_{A \widetilde{\ominus} B}(v)
\end{aligned}$$

$$\begin{aligned}
F_{(\mathrm{A}^c \widetilde{\oplus} B)^c}(v) &= 1 - F_{(\mathrm{A}^c \widetilde{\oplus} B)}(v) \\
&= 1 - \inf_{x \in R^n} \max(1 - \mathrm{F}_{\mathrm{A}^c}(v+u), 1 - F_B(u)) \\
&= \sup_{u \in Z^2} [1 - \max(1 - \mathrm{F}_{\mathrm{A}^c}(v+u), 1 - F_B(u))] \\
&= \sup_{u \in Z^2} [\min(1 - \mathrm{F}_A(v+u), F_B(u))] \quad = F_{A \widetilde{\ominus} B}(v)
\end{aligned}$$

$$\langle \mathrm{T}_{(\mathrm{A}^c \widetilde{\oplus} \mathrm{B})^{\mathrm{c}}}, \mathrm{I}_{(\mathrm{A}^c \widetilde{\oplus} \mathrm{B})^{\mathrm{c}}}, \mathrm{F}_{(\mathrm{A}^c \widetilde{\oplus} \mathrm{B})^{\mathrm{c}}} \rangle = \langle T_{A \widetilde{\ominus} B}, I_{A \widetilde{\ominus} B}, F_{A \widetilde{\ominus} B} \rangle.$$

4.4.2. Proposition the Duality Theorem Closing

let A and B be two neutrosophic sets, neutrosophic opening and neutrosophic closing are also dual operation i.e.

$(\mathrm{A}^c \,\tilde{\bullet}\, \mathrm{B})^{\mathrm{c}} = \langle \mathrm{T}_{(\mathrm{A}^c \tilde{\bullet} \mathrm{B})^{\mathrm{c}}}, \mathrm{I}_{(\mathrm{A}^c \tilde{\bullet} \mathrm{B})^{\mathrm{c}}}, \mathrm{F}_{(\mathrm{A}^c \tilde{\bullet} \mathrm{B})^{\mathrm{c}}} \rangle$, where for all $x \in X$

$\mathrm{T}_{(\mathrm{A}^c \tilde{\bullet} \mathrm{B})^{\mathrm{c}}}(v) = 1 - T_{\mathrm{A}^c \tilde{\bullet} B}(v)$

$$T_{(A^c \tilde{\bullet} B)^c}(v) = 1 - \inf_{u \in Z^2} \max\left[\sup_{z \in R^n} \min\left(T_{A^c}(v-u+w), T_{B(w)}\right), 1 - T_B(u)\right]$$

$$= \sup_{u \in Z^2} \min\left[1 - \sup_{z \in R^n} \min\left(T_{A^c}(v-u+w), T_{B(w)}\right), T_B(u)\right]$$

$$= \sup_{u \in Z^2} \min\left[\inf_{z \in R^n} \max\left(1 - T_{A^c}(v-u+w), 1 - T_{B(w)}\right), T_B(u)\right]$$

$$= \sup_{u \in Z^2} \min\left[\inf_{z \in R^n} \max\left(T_{A^c}(v-u+w), 1 - T_{B(w)}\right), T_B(u)\right] \quad = T_{A \tilde{\circ} B}(v)$$

$$\mathrm{I}_{(\mathrm{A}^c \tilde{\bullet} \mathrm{B})^c}(v) = 1 - I_{\mathrm{A}^c \tilde{\bullet} B}(v)$$

$$I_{(A^c \tilde{\bullet} B)^c}(v) = 1 - \inf_{u \in Z^2} \max\left[\sup_{z \in R^n} \min\left(I_{A^c}(v-u+w), I_{B(w)}\right), 1 - I_B(u)\right]$$

$$= \sup_{u \in Z^2} \min\left[1 - \sup_{z \in R^n} \min\left(I_{A^c}(v-u+w), I_{B(w)}\right),\ I_B(u)\right]$$

$$= \sup_{u \in Z^2} \min\left[\inf_{z \in R^n} \max\left(1 - I_{A^c}(v-u+w), 1 - I_{B(w)}\right),\ I_B(u)\right]$$

$$= \sup_{u \in Z^2} \min\left[\inf_{z \in R^n} \max\left(I_A(v-u+w), 1 - I_{B(w)}\right),\ I_B(u)\right] \quad = I_{A \tilde{\circ} B}(v)$$

$$\mathrm{F}_{(\mathrm{A}^c \tilde{\bullet} \mathrm{B})^c}(\mathrm{v}) = 1 - \mathrm{F}_{\mathrm{A}^c \tilde{\bullet} \mathrm{B}}$$

$$F_{(A^c \tilde{\bullet} B)^c}(v) = 1 - \sup_{u \in Z^2} \min\left[\inf_{z \in R^n} \max\left(1 - F_A(v-u+w), 1 - F_{B(w)}\right), F_B(u)\right]$$

$$F_{(A^c \tilde{\bullet} B)^c}(v) = \inf_{u \in Z^2} \max\left[1 - \inf_{z \in R^n} \max\left(1 - F_A(v-u+w), 1 - F_{B(w)}\right), F_B(u)\right]$$

$$F_{(A^c \tilde{\bullet} B)^c}(v) = \inf_{u \in Z^2} \max\left[\sup_{z \in R^n} \min\left(1 - F_A(v-u+w), F_{B(w)}\right), 1 - F_B(u)\right]$$

$$= \mathrm{F}_{A \tilde{\circ} B}(v)$$

$$\left\langle \mathrm{T}_{(\mathrm{A}^c \tilde{\bullet} \mathrm{B})^c}, \mathrm{I}_{(\mathrm{A}^c \tilde{\bullet} \mathrm{B})^c}, \mathrm{F}_{(\mathrm{A}^c \tilde{\bullet} \mathrm{B})^c} \right\rangle = \langle \mathrm{T}_{A \tilde{\circ} B}, \mathrm{I}_{A \tilde{\circ} B}, F_{A \tilde{\circ} B} \rangle.$$

.

Lemma 1: for any $A \in \mathcal{N}(X)$, and the neutrosophic universal set $1_{\mathcal{N}}$, we have that $\mathrm{A} \widetilde{\oplus} 1_{\mathcal{N}} \subseteq A$, $\mathrm{A} \widetilde{\oplus} 1_{\mathcal{N}} = \left\langle T_{\mathrm{A}\widetilde{\oplus}1_{\mathcal{N}}}, I_{\mathrm{A}\widetilde{\oplus}1_{\mathcal{N}}}, F_{A\widetilde{\oplus}1_{\mathcal{N}}} \right\rangle$

Proof:

$$T_{A\widetilde{\oplus}1_{\mathcal{N}}}(v) = \sup_{u \in Z^2} \min(T_A(v+u), 1) \qquad = \sup_{u \in Z^2}\left(T_A(y+x)\right) = T_A(v)$$

$$I_{A\widetilde{\oplus}1_{\mathcal{N}}}(v) = \sup_{u \in Z^2} \min(I_A(v+u), 1) \qquad = \sup_{u \in Z^2}\left(I_A(y+x)\right) = I_A(v)$$

$$F_{A\widetilde{\oplus}1_{\mathcal{N}}}(v) = \inf_{u \in Z^2} \max(1 - F_A(v+u), 1 - 0) \qquad = \underline{1}(v)$$

$$\langle \mathrm{T_A}, \mathrm{I_A}, \underline{1} \rangle \subseteq \langle \mathrm{T_A}, \mathrm{I_A}, \mathrm{F_A} \rangle = \mathrm{A}$$

Lemma 2: for any $A \in \mathcal{N}(X)$, and the neutrosophic empty set $0_{\mathcal{N}}$, we have that

$\mathbf{A} \widetilde{\oplus}\, 0_{\mathcal{N}} \subseteq A^c$, $\quad \mathrm{A} \widetilde{\oplus}\, 0_{\mathcal{N}} = \langle T_{\mathrm{A}\widetilde{\oplus}\, 0_{\mathcal{N}}}, I_{\mathrm{A}\widetilde{\oplus}\, 0_{\mathcal{N}}}, F_{A\widetilde{\oplus} 0_{\mathcal{N}}} \rangle$

Proof:

$$T_{A\widetilde{\oplus}0_{\mathcal{N}}}(v) = \sup_{u\in Z^2} min(T_A(v+u), 0) \qquad = \underline{0}(v)$$

$$I_{A\widetilde{\oplus}0_{\mathcal{N}}}(v) = \sup_{u\in Z^2} min(I_A(v+u), 0) \qquad = \underline{0}(v)$$

$$F_{A\widetilde{\oplus}0_{\mathcal{N}}}(v) = \inf_{u\in Z^2} max(1 - F_A(v+u), 1-1) \quad = \inf_{u\in Z^2}\left(1 - F_A(v+u)\right) = F_{A^c}(v)$$

$$\langle \underline{0}, \underline{0}, \mathrm{F}_{\mathrm{A}^c} \rangle \subseteq \langle \mathrm{T}_{\mathrm{A}^c}, \mathrm{I}_{\mathrm{A}^c}, \mathrm{F}_{\mathrm{A}^c} \rangle = \mathrm{A}^c$$

4.5. Properties of the Neutrosophic Morphological Operations

In this section, we investigate the basic properties of the neutrosophic morphological operation (dilation, erosion, opening and closing), which we defined in §4.

4.5.1. Properties of the Neutrosophic Dilation

Proposition 1

The neutrosophic dilation satisfies the following properties: $\forall\ \mathrm{A}, \mathrm{B} \in \mathcal{N}(\mathrm{Z}^2)$

- ***i.*** ***Commutativity:*** $\mathrm{A}\widetilde{\oplus}\mathrm{B} = \mathrm{B} \widetilde{\oplus} \mathrm{A}$
- ***ii.*** ***Associativity:*** $\left(\mathrm{A} \widetilde{\oplus} \mathrm{B}\right) \widetilde{\oplus} \mathrm{C} = \mathrm{A} \widetilde{\oplus} \left(\mathrm{B} \widetilde{\oplus} \mathrm{C}\right)$.
- iii. ***Monotonicity: (***increasing in both arguments):
  - a) $\mathrm{A} \subseteq \mathrm{B} \Longrightarrow \langle \mathrm{T}_{\mathrm{A}\widetilde{\oplus}\mathrm{C}}, I_{\mathrm{A}\widetilde{\oplus}\mathrm{C}}, F_{\mathrm{A}\widetilde{\oplus}\mathrm{C}} \rangle \subseteq \langle \mathrm{T}_{\mathrm{B}\widetilde{\oplus}\mathrm{C}}, I_{\mathrm{B}\widetilde{\oplus}\mathrm{C}}, F_{\mathrm{B}\widetilde{\oplus}\mathrm{C}} \rangle$

    $\mathrm{T}_{\mathrm{A}\widetilde{\oplus}\mathrm{C}} \subseteq \mathrm{T}_{\mathrm{B}\widetilde{\oplus}\mathrm{C}}$, $\mathrm{I}_{\mathrm{A}\widetilde{\oplus}\mathrm{C}} \subseteq \mathrm{I}_{\mathrm{B}\widetilde{\oplus}\mathrm{C}}$ *and* $\mathrm{F}_{\mathrm{A}\widetilde{\oplus}\mathrm{C}} \supseteq \mathrm{F}_{\mathrm{B}\widetilde{\oplus}\mathrm{C}}$
  - b) $\mathrm{A} \subseteq \mathrm{B} \Longrightarrow \langle \mathrm{T}_{\mathrm{C}\widetilde{\oplus}\mathrm{A}}, I_{\mathrm{C}\widetilde{\oplus}\mathrm{A}}, F_{\mathrm{C}\widetilde{\oplus}\mathrm{A}} \rangle \subseteq \langle \mathrm{T}_{\mathrm{C}\widetilde{\oplus}\mathrm{B}}, I_{\mathrm{C}\widetilde{\oplus}\mathrm{B}}, F_{\mathrm{C}\widetilde{\oplus}\mathrm{B}} \rangle$

    $\mathrm{T}_{\mathrm{C}\widetilde{\oplus}\mathrm{A}} \subseteq \mathrm{T}_{\mathrm{C}\widetilde{\oplus}\mathrm{B}}$, $\mathrm{I}_{\mathrm{C}\widetilde{\oplus}\mathrm{A}} \subseteq \mathrm{I}_{\mathrm{C}\widetilde{\oplus}\mathrm{B}}$ *and* $\mathrm{F}_{\mathrm{C}\widetilde{\oplus}\mathrm{A}} \supseteq \mathrm{F}_{\mathrm{C}\widetilde{\oplus}\mathrm{B}}$

Proof*:*

i), ii), iii) Obvious.

Proposition2: for any family $(A_i | i \in I)$ in $\mathcal{N}(\mathrm{Z}^2)$ and $B \in \mathcal{N}(\mathrm{Z}^2)$

a) $\langle \mathrm{T}_{\bigcap_{\mathrm{i}\in\mathrm{I}} \mathrm{A_i}\widetilde{\oplus}\mathrm{B}}, \mathrm{I}_{\bigcap_{\mathrm{i}\in\mathrm{I}} \mathrm{A_i}\widetilde{\oplus}\mathrm{B}}, \mathrm{F}_{\bigcap_{\mathrm{i}\in\mathrm{I}} \mathrm{A_i}\widetilde{\oplus}\mathrm{B}} \rangle \subseteq \langle \mathrm{T}_{\bigcap_{\mathrm{i}\in\mathrm{I}}(\mathrm{A}_i\widetilde{\oplus}\mathrm{B})}, \mathrm{I}_{\bigcap_{\mathrm{i}\in\mathrm{I}}(\mathrm{A}_i\widetilde{\oplus}\mathrm{B})}, \mathrm{F}_{\bigcap_{\mathrm{i}\in\mathrm{I}}(\mathrm{A}_i\widetilde{\oplus}\mathrm{B})} \rangle$

$T_{\bigcap_{i\in I} \mathrm{A}_i\widetilde{\oplus}\mathrm{B}} \subseteq T_{\bigcap_{i\in I}(\mathrm{A}_i\widetilde{\oplus}\mathrm{B})}$, $I_{\bigcap_{i\in I} \mathrm{A}_i\widetilde{\oplus}\mathrm{B}} \subseteq I_{\bigcap_{i\in I}(\mathrm{A}_i\widetilde{\oplus}\mathrm{B})}$ and $F_{\bigcap_{i\in I} \mathrm{A}_i\widetilde{\oplus}\mathrm{B}} \supseteq F_{\bigcup_{i\in I}(\mathrm{A}_i\widetilde{\oplus}\mathrm{B})}$

b) $\langle \mathrm{T}_{\mathrm{B}\widetilde{\oplus}\bigcap_{\mathrm{i}\in\mathrm{I}} \mathrm{A}_i}, \mathrm{I}_{\mathrm{B}\widetilde{\oplus}\bigcap_{\mathrm{i}\in\mathrm{I}} \mathrm{A}_i}, \mathrm{F}_{\mathrm{B}\widetilde{\oplus}\bigcap_{\mathrm{i}\in\mathrm{I}} \mathrm{A}_i} \rangle \subseteq \langle \mathrm{T}_{\bigcap_{\mathrm{i}\in\mathrm{I}}(\mathrm{B}\widetilde{\oplus}\mathrm{A}_i)}, \mathrm{I}_{\mathrm{T}_{\bigcap_{\mathrm{i}\in\mathrm{I}}(\mathrm{B}\widetilde{\oplus}\mathrm{A}_i)}}, \mathrm{F}_{\mathrm{T}_{\bigcap_{\mathrm{i}\in\mathrm{I}}(\mathrm{B}\widetilde{\oplus}\mathrm{A}_i)}} \rangle$

$T_{B\widetilde{\oplus}\bigcap_{i\in I}\mathrm{A}_i} \subseteq T_{\bigcap_{i\in I}(\mathrm{B}\widetilde{\oplus}\mathrm{A}_i)}$, $I_{B\widetilde{\oplus}\bigcap_{i\in I}\mathrm{A}_i} \subseteq I_{\bigcap_{i\in I}(\mathrm{B}\widetilde{\oplus}\mathrm{A}_i)}$ and $F_{B\widetilde{\oplus}\bigcap_{i\in I}\mathrm{A}_i} \supseteq F_{\bigcup_{i\in I}(\mathrm{B}\widetilde{\oplus}\mathrm{A}_i)}$

Proof: a)

$$\langle T_{\bigcap_{i\in I} A_i\widetilde{\oplus}B}, I_{\bigcap_{i\in I} A_i\widetilde{\oplus}B}, F_{\bigcap_{i\in I} A_i\widetilde{\oplus}B} \rangle \subseteq \langle T_{\bigcap_{i\in I}(A_i\widetilde{\oplus}B)}, I_{\bigcap_{i\in I}(A_i\widetilde{\oplus}B)}, F_{\bigcap_{i\in I}(A_i\widetilde{\oplus}B)} \rangle$$

$$T_{\underset{i\in I}{\cap} A_i \widetilde{\oplus} B}(v) = \sup_{u\in Z^n} \min\left(T_{\underset{i\in I}{\cap} A_i}(v+u), T_B(u)\right) \quad = \sup_{u\in Z^n} \min\left(\inf_{i\in I}\ T_{A_i}(v+u), T_B(u)\right)$$

$$= \sup_{u\in Z^n} \inf_{i\in I}\left(\min T_{\boldsymbol{A_i}}(v+u), T_B(u)\right) \quad \leq \inf_{\boldsymbol{i\in I}} \sup_{u\in Z^n}\left(\min T_{\boldsymbol{A_i}}(v+u), T_B(u)\right)$$

$$\leq \underset{i\in I}{\cap} T_{(A_i \widetilde{\oplus} B)}(v) \qquad \leq T_{\underset{i\in I}{\cap}(A_i \widetilde{\oplus} B)}(v)$$

$$I_{\underset{i\in I}{\cap} A_i \widetilde{\oplus} B}(v) = \sup_{u\in Z^n} \min\left(I_{\underset{i\in I}{\cap} \boldsymbol{A_i}}(v+u), I_B(u)\right)$$

$$= \sup_{u\in Z^n} \min\left(\inf_{i\in I}\ I_{\boldsymbol{A_i}}(v+u), I_B(u)\right) \qquad = \sup_{u\in Z^n} \inf_{i\in I}\left(\min\ I_{\boldsymbol{A_i}}(v+u), I_B(u)\right)$$

$$\leq \underset{\boldsymbol{i\in I}}{\cap} \sup_{u\in Z^n}\left(\min\ I_{\boldsymbol{A_i}}(v+u), I_B(u)\right) \qquad \leq \underset{i\in I}{\cap} I_{(A_i \widetilde{\oplus} B)}(v)$$

$$\leq I_{\underset{i\in I}{\cap}(A_i \widetilde{\oplus} B)}(v)$$

$$F_{\underset{i\in I}{\cap} \boldsymbol{A_i} \widetilde{\oplus} \boldsymbol{B}}(v) = \inf_{u\in Z^n} \max\left(1 - F_{\underset{i\in I}{\cap} \boldsymbol{A_i}}(v+u), 1 - F_B(u)\right)$$

$$= \inf_{u\in Z^n} \max\left(1 - \inf_{i\in I} F_{\boldsymbol{A_i}}(v+u), 1 - F_B(u)\right)$$

$$= \inf_{u\in Z^n} \max\left(\sup_{i\in I}\left(1 - F_{\boldsymbol{A_i}}(v+u)\right), 1 - F_B(u)\right)$$

$$= \inf_{u\in Z^n} \sup_{i\in I}\left(\max F_{\boldsymbol{A_i}}(v+u), 1 - F_B(u)\right)$$

$$\geq \sup_{i\in I} \inf_{u\in Z^n}\left(\max F_{\boldsymbol{A_i}}(v+u), 1 - F_B(u)\right)$$

$$\geq \underset{i\in I}{\cup} \inf_{u\in Z^n}\left(\max F_{A_i}(v+u), 1 - F_B(u)\right) \qquad \geq F_{\underset{i\in I}{\cup}(A_i \widetilde{\oplus} B)}(v)$$

b) The proof is similar to a).

Proposition 3: for any family $(A_i | i \in I)$ in $\mathcal{N}(\mathrm{Z}^2)$ and $B \in \mathcal{N}(\mathrm{Z}^2)$

a) $\langle T_{\underset{i\in I}{\cup} \mathrm{A}_i \widetilde{\oplus} \mathrm{B}}, I_{\underset{i\in I}{\cup} \mathrm{A}_i \widetilde{\oplus} \mathrm{B}}, F_{\underset{i\in I}{\cup} \mathrm{A}_i \widetilde{\oplus} \mathrm{B}}\rangle \supseteq \langle T_{\underset{i\in I}{\cup}(\mathrm{A}_i \widetilde{\oplus} \mathrm{B})}, I_{\underset{i\in I}{\cup}(\mathrm{A}_i \widetilde{\oplus} \mathrm{B})}, F_{\underset{i\in I}{\cup}(\mathrm{A}_i \widetilde{\oplus} \mathrm{B})}\rangle$

$T_{\underset{i\in I}{\cup} \mathrm{A}_i \widetilde{\oplus} \mathrm{B}} \supseteq T_{\underset{i\in I}{\cup}(\mathrm{A}_i \widetilde{\oplus} \mathrm{B})}$, $I_{\underset{i\in I}{\cup} \mathrm{A}_i \widetilde{\oplus} \mathrm{B}} \supseteq I_{\underset{i\in I}{\cup}(\mathrm{A}_i \widetilde{\oplus} \mathrm{B})}$ and $F_{\underset{i\in I}{\cup} \mathrm{A}_i \widetilde{\oplus} \mathrm{B}} \subseteq F_{\underset{i\in I}{\cap}(\mathrm{A}_i \widetilde{\oplus} \mathrm{B})}$

b) $\langle T_{B \widetilde{\oplus} \underset{i\in I}{\cup} \mathrm{A}_i}, I_{B \widetilde{\oplus} \underset{i\in I}{\cup} \mathrm{A}_i}, F_{B \widetilde{\oplus} \underset{i\in I}{\cup} \mathrm{A}_i}\rangle \supseteq \langle T_{\underset{i\in I}{\cup}(\mathrm{A}_i \widetilde{\oplus} \mathrm{B})}, I_{\underset{i\in I}{\cup}(\mathrm{A}_i \widetilde{\oplus} \mathrm{B})}, F_{\underset{i\in I}{\cup}(\mathrm{A}_i \widetilde{\oplus} \mathrm{B})}\rangle$

$T_{B \widetilde{\oplus} \underset{i\in I}{\cup} \mathrm{A}_i} \supseteq T_{\underset{i\in I}{\cup}(\mathrm{A}_i \widetilde{\oplus} \mathrm{B})}$, $I_{B \widetilde{\oplus} \underset{i\in I}{\cup} \mathrm{A}_i} \supseteq I_{\underset{i\in I}{\cup}(\mathrm{A}_i \widetilde{\oplus} \mathrm{B})}$ *and* $F_{B \widetilde{\oplus} \underset{i\in I}{\cup} \mathrm{A}_i} \subseteq F_{\underset{i\in I}{\cap}(\mathrm{A}_i \widetilde{\oplus} \mathrm{B})}$

Proof: b)

$$\langle T_{B \widetilde{\oplus} \underset{i\in I}{\cup} A_i}, I_{B \widetilde{\oplus} \underset{i\in I}{\cup} A_i}, F_{B \widetilde{\oplus} \underset{i\in I}{\cup} A_i}\rangle \supseteq \langle T_{\underset{i\in I}{\cup}(B \widetilde{\oplus} A_i)}, I_{\underset{i\in I}{\cup}(B \widetilde{\oplus} A_i)}, F_{\underset{i\in I}{\cup}(B \widetilde{\oplus} A_i)}\rangle$$

$$T_{B\widetilde{\oplus}\underset{i\in I}{\cup}A_i}(v) = \sup_{u\in Z^n} \min\left(T_B(v+u), T_{\underset{i\in I}{\cup}A_i}(u)\right) \quad = \sup_{u\in Z^n} \min\left(T_{\boldsymbol{B}}(v+u), \sup_{i\in I} T_{\boldsymbol{A_i}}(u)\right)$$

$$\geq \sup_{u\in Z^n}\left(\sup_{i\in I}\ \min T_{\boldsymbol{B}}(v+u), T_{\boldsymbol{A_i}}(u)\right) \quad \geq \bigcup_{\boldsymbol{i\in I}}\left(\sup_{u\in Z^n}\ \min T_{\boldsymbol{B}}(v+u), T_{\boldsymbol{A_i}}(u)\right)$$

$$\geq \bigcup_{i\in I} T_{(B\widetilde{\oplus}\boldsymbol{A_i})}(v+u) \quad \geq T_{\underset{i\in I}{\cup}(B\widetilde{\oplus}\boldsymbol{A_i})}(v)$$

$$I_{B\widetilde{\oplus}\underset{i\in I}{\cup}A_i}(v) = \sup_{u\in Z^2} \min\left(I_{\boldsymbol{B}}(v+u), I_{\underset{i\in I}{\cup}\boldsymbol{A_i}}(u)\right) \quad = \sup_{u\in Z^2} \min\left(I_{\boldsymbol{B}}(v+u), \sup_{i\in I} I_{\boldsymbol{A_i}}(u)\right)$$

$$\geq \sup_{u\in Z^2}\left(\sup_{i\in I} \min\ I_{\boldsymbol{A_i}}(v+u), I_{\boldsymbol{A_i}}(u)\right) \quad \geq \bigcup_{\boldsymbol{i\in I}}\left(\sup_{u\in Z^2} \min I_{\boldsymbol{A_i}}(v+u), I_{\boldsymbol{A_i}}(u)\right)$$

$$\geq \bigcup_{i\in I} I_{(B\widetilde{\oplus}A_i)}(v) \quad \geq I_{\underset{i\in I}{\cup}(B\widetilde{\oplus}A_i)}(v)$$

$$F_{\underset{i\in I}{\cup}\boldsymbol{A_i}\widetilde{\oplus}A_i}(v) = \inf_{u\in Z^n} \max\left(1 - F_{\boldsymbol{B}}(v+u), 1 - F_{\underset{i\in I}{\cup}A_i}(u)\right)$$

$$= \inf_{u\in Z^n} \max\left(1 - F_{\boldsymbol{B}}(v+u), 1 - \sup_{i\in I} F_{A_i}(u)\right)$$

$$= \inf_{u\in Z^n} \max\left(1 - F_{\boldsymbol{B}}(v+u), \inf_{i\in I}\left(1 - F_{A_i}(u)\right)\right)$$

$$\leq \inf_{u\in Z^n}\left(\inf_{i\in I} \max\left(1 - F_{\boldsymbol{B}}(v+u), 1 - F_{A_i}(u)\right)\right)$$

$$\leq \inf_{i\in I} \inf_{u\in Z^n}\left(\max\left(1 - F_{\boldsymbol{B}}(v+u), 1 - F_{A_i}(u)\right)\right)$$

$$\leq \bigcap_{i\in I} \inf_{u\in Z^n} \max\left(1 - F_B(v+u), 1 - F_{A_i}(u)\right) \quad \leq F_{\underset{i\in I}{\cap}(B\widetilde{\oplus}A_i)}(v)$$

a) The proof is similar to b).

4.5.2. Proposition (properties of the neutrosophic erosion)***:***

Proposition 1:

The neutrosophic erosion satisfies the monotonicity, $\forall \mathrm{A}, \mathrm{B}, \mathrm{C} \in \mathcal{N}(\mathrm{Z}^2)$.

a) $\mathrm{A} \subseteq \mathrm{B} \Longrightarrow \langle \mathrm{T}_{\mathrm{A}\widetilde{\ominus}\mathrm{C}}, I_{\mathrm{A}\widetilde{\ominus}\mathrm{C}}, F_{\mathrm{A}\widetilde{\ominus}\mathrm{C}}\rangle \subseteq \langle \mathrm{T}_{\mathrm{B}\widetilde{\ominus}\mathrm{C}}, I_{\mathrm{B}\widetilde{\ominus}\mathrm{C}}, F_{\mathrm{B}\widetilde{\ominus}\mathrm{C}}\rangle$

$$\mathrm{T}_{\mathbf{A}\widetilde{\ominus}\mathbf{C}} \subseteq \mathrm{T}_{\mathbf{B}\widetilde{\ominus}\mathbf{C}}\ ,\ \ \mathrm{I}_{\mathbf{A}\widetilde{\ominus}\mathbf{C}} \subseteq \mathrm{I}_{\mathbf{B}\widetilde{\ominus}\mathbf{C}}\ \ and\ \ \mathrm{F}_{\mathbf{A}\widetilde{\ominus}\mathbf{C}} \supseteq \mathrm{F}_{\mathbf{B}\widetilde{\ominus}\mathbf{C}}$$

b) $\mathrm{A} \subseteq \mathrm{B} \Longrightarrow \langle \mathrm{T}_{\mathrm{C}\widetilde{\ominus}\mathrm{A}}, I_{\mathrm{C}\widetilde{\ominus}\mathrm{A}}, F_{\mathrm{C}\widetilde{\ominus}\mathrm{A}}\rangle \supseteq \langle \mathrm{T}_{\mathrm{C}\widetilde{\ominus}\mathrm{B}}, I_{\mathrm{C}\widetilde{\ominus}\mathrm{B}}, F_{\mathrm{C}\widetilde{\ominus}\mathrm{B}}\rangle$

$$\mathrm{T}_{\mathbf{C}\widetilde{\ominus}\mathbf{A}} \supseteq \mathrm{T}_{\mathbf{C}\widetilde{\ominus}\mathbf{B}}\ ,\ \ \mathrm{I}_{\mathbf{C}\widetilde{\ominus}\mathbf{A}} \supseteq \mathrm{I}_{\mathbf{C}\widetilde{\ominus}\mathbf{B}}\ \ and\ \ \mathrm{F}_{\mathbf{C}\widetilde{\ominus}\mathbf{A}} \subseteq \mathrm{F}_{\mathbf{C}\widetilde{\ominus}\mathbf{B}}$$

Note that: dislike the dilation operator, the erosion does not satisfy commutativity and associativity.

Proposition 2:

for any family $(A_i | i \in I)\ in\ \mathcal{N}(\mathrm{Z}^2) and\ B \in \mathcal{N}(\mathrm{Z}^2)$

a) $\langle \mathrm{T}_{\underset{i\in I}{\cap}\mathrm{A_i}\widetilde{\ominus}\mathrm{B}}, \mathrm{I}_{\underset{i\in I}{\cap}\mathrm{A_i}\widetilde{\ominus}\mathrm{B}}, \mathrm{F}_{\underset{i\in I}{\cap}\mathrm{A_i}\widetilde{\ominus}\mathrm{B}}\rangle \subseteq \langle \mathrm{T}_{\underset{i\in I}{\cap}(\mathrm{A_i}\widetilde{\ominus}\mathrm{B})}, \mathrm{I}_{\underset{i\in I}{\cap}(\mathrm{A_i}\widetilde{\ominus}\mathrm{B})}, \mathrm{F}_{\underset{i\in I}{\cap}(\mathrm{A_i}\widetilde{\ominus}\mathrm{B})}\rangle$

$$T_{\underset{i\in I}{\cap}\mathbf{A_i}\widetilde{\ominus}\mathbf{B}} \subseteq T_{\underset{i\in I}{\cap}(\mathbf{A_i}\widetilde{\ominus}\mathbf{B})},\ I_{\underset{i\in I}{\cap}\mathbf{A_i}\widetilde{\ominus}\mathbf{B}} \subseteq I_{\underset{i\in I}{\cap}(\mathbf{A_i}\widetilde{\ominus}\mathbf{B})} \quad \text{and} F_{\underset{i\in I}{\cap}\mathbf{A_i}\widetilde{\ominus}\mathbf{B}} \supseteq F_{\underset{i\in I}{\cup}(\mathbf{A_i}\widetilde{\ominus}\mathbf{B})}$$

b) $\langle T_{B\widetilde{\ominus}\underset{i\in I}{\cap}A_i}, I_{B\widetilde{\ominus}\underset{i\in I}{\cap}A_i} F_{B\widetilde{\ominus}\underset{i\in I}{\cap}A_i}\rangle \supseteq \langle T_{\underset{i\in I}{\cap}(B\widetilde{\ominus}A_i)}, I_{\underset{i\in I}{\cap}(B\widetilde{\ominus}A_i)}, F_{\underset{i\in I}{\cap}(B\widetilde{\ominus}A_i)}\rangle$

$$T_{B\widetilde{\oplus}\underset{i\in I}{\cap}A_i} \supseteq T_{\underset{i\in I}{\cup}(B\widetilde{\oplus}A_i)},\ I_{B\widetilde{\oplus}\underset{i\in I}{\cap}A_i} \supseteq I_{\underset{i\in I}{\cup}(B\widetilde{\oplus}A_i)} \ \text{and} F_{B\widetilde{\oplus}\underset{i\in I}{\cap}A_i} \subseteq F_{\underset{i\in I}{\cap}(B\widetilde{\oplus}A_i)}$$

Proof: a)

$$\langle T_{\underset{i\in I}{\cap}A_i\widetilde{\ominus}B}, I_{\underset{i\in I}{\cap}A_i\widetilde{\ominus}B}, F_{\underset{i\in I}{\cap}A_i\widetilde{\ominus}B}\rangle \subseteq \langle T_{\underset{i\in I}{\cap}(A_i\widetilde{\ominus}B)}, I_{\underset{i\in I}{\cap}(A_i\widetilde{\ominus}B)}, F_{\underset{i\in I}{\cap}(A_i\widetilde{\ominus}B)}\rangle$$

$$T_{\underset{i\in I}{\cap}A_i\widetilde{\ominus}B}(v) = \inf_{u\in Z^n} \max\left(T_{\underset{i\in I}{\cap}A_i}(v+u), 1-T_B(u)\right)$$

$$= \inf_{u\in Z^n} \max\left(\inf_{i\in I} T_{\boldsymbol{A_i}}(v+u), 1-T_B(u)\right)$$

$$\le \inf_{u\in Z^n}\inf_{i\in I}\left(\max T_{\boldsymbol{A_i}}(v+u), 1-T_B(u)\right)$$

$$\le \underset{\boldsymbol{i\in I}}{\cap} \inf_{u\in Z^n}\left(\max T_{\boldsymbol{A_i}}(v+u), 1-T_B(u)\right) \qquad \le \underset{i\in I}{\cap} T_{(A_i\widetilde{\ominus}B)}(v)$$

$$I_{\underset{i\in I}{\cap}A_i\widetilde{\ominus}B}(v) = \inf_{u\in Z^n} \max\left(I_{\underset{i\in I}{\cap}A_i}(v+u), 1-I_B(u)\right)$$

$$= \inf_{u\in Z^n} \max\left(\min_{i\in I} I_{\boldsymbol{A_i}}(v+u), 1-I_B(u)\right)$$

$$\le \inf_{u\in Z^n}\min_{i\in I}\left(\max I_{\boldsymbol{A_i}}(v+u), 1- I_B(u)\right)$$

$$\le \underset{\boldsymbol{i\in I}}{\cap} \inf_{u\in Z^n}\left(\max I_{\boldsymbol{A_i}}(v+u), 1-I_B(u)\right) \qquad \le \underset{i\in I}{\cap} I_{(A_i\widetilde{\ominus}B)}(v)$$

$$F_{\underset{i\in I}{\cap}\boldsymbol{A_i}\widetilde{\ominus}\boldsymbol{B}}(v) = \sup_{u\in Z^n} \min\left(1-F_{\underset{i\in I}{\cap}\boldsymbol{A_i}}(v+u), F_B(u)\right)$$

$$= \sup_{u\in Z^n} \min\left(1-\inf_{i\in I} F_{\boldsymbol{A_i}}(v+u), F_B(u)\right)$$

$$= \sup_{u\in Z^n} \min\left(\sup_{i\in I}\left(1- F_{\boldsymbol{A_i}}(v+u)\right), F_B(u)\right)$$

$$\ge \sup_{u\in Z^n}\sup_{i\in I}\left(\min 1- F_{\boldsymbol{A_i}}(v+u), F_B(u)\right)$$

$$\ge \underset{i\in I}{\cup}\sup_{u\in Z^n}\left(\min 1- F_{\boldsymbol{A_i}\widetilde{\ominus}\boldsymbol{B}}(v+u), F_B(u)\right) \qquad \ge F_{\underset{i\in I}{\cup}(A_i\widetilde{\ominus}B)}(v)$$

b) The proof is similar to a).

Proposition 3: for any family $(A_i|i\in I)$ *in* $\mathcal{N}(\mathrm{Z}^2)$ *and* $B\in\mathcal{N}(\mathrm{Z}^2)$

a) $\langle T_{\underset{i\in I}{\cup}A_i\widetilde{\ominus}B}, I_{\underset{i\in I}{\cup}A_i\widetilde{\ominus}B}, F_{\underset{i\in I}{\cup}A_i\widetilde{\ominus}B}\rangle \supseteq \langle T_{\underset{i\in I}{\cup}(A_i\widetilde{\ominus}B)}, I_{\underset{i\in I}{\cup}(A_i\widetilde{\ominus}B)}, F_{\underset{i\in I}{\cup}(A_i\widetilde{\ominus}B)}\rangle$

$$T_{\underset{i\in I}{\cup}\boldsymbol{A_i}\widetilde{\ominus}\mathbf{B}} \supseteq T_{\underset{i\in I}{\cup}(\boldsymbol{A_i}\widetilde{\ominus}\mathbf{B})},\ I_{\underset{i\in I}{\cup}\boldsymbol{A_i}\widetilde{\ominus}\mathbf{B}} \supseteq I_{\underset{i\in I}{\cup}(\boldsymbol{A_i}\widetilde{\ominus}\mathbf{B})} \text{ and } F_{\underset{i\in I}{\cup}\boldsymbol{A_i}\widetilde{\ominus}\mathbf{B}} \subseteq F_{\underset{i\in I}{\cap}(\boldsymbol{A_i}\widetilde{\ominus}\mathbf{B})}$$

b) $\langle T_{B\widetilde{\ominus}\underset{i\in I}{\cup}A_i}, I_{B\widetilde{\ominus}\underset{i\in I}{\cup}A_i}, F_{B\widetilde{\ominus}\underset{i\in I}{\cup}A_i}\rangle \subseteq \langle T_{\underset{i\in I}{\cup}(B\widetilde{\ominus}A_i)}, I_{\underset{i\in I}{\cup}(B\widetilde{\ominus}A_i)}, F_{\underset{i\in I}{\cup}(B\widetilde{\ominus}A_i)}\rangle$

$$T_{B\widetilde{\ominus}\underset{i\in I}{\cup}A_i} \subseteq T_{\underset{i\in I}{\cap}(\boldsymbol{B}\widetilde{\ominus}\boldsymbol{A_i})},\ I_{B\widetilde{\ominus}\underset{i\in I}{\cup}A_i} \subseteq I_{\underset{i\in I}{\cap}(\boldsymbol{B}\widetilde{\ominus}\boldsymbol{A_i})} \ \textit{and}\ F_{B\widetilde{\ominus}\underset{i\in I}{\cup}A_i} \supseteq F_{\underset{\boldsymbol{i\in I}}{\cup}(\boldsymbol{B}\widetilde{\ominus}\boldsymbol{A_i})}$$

Proof: a)

$$\langle T_{\underset{i\in I}{\cup} A_i\widetilde{\ominus}B}, I_{\underset{i\in I}{\cup} A_i\widetilde{\ominus}B}, F_{\underset{i\in I}{\cup} A_i\widetilde{\ominus}B}\rangle \supseteq \langle T_{\underset{i\in I}{\cup}(A_i\widetilde{\ominus}B)}, I_{\underset{i\in I}{\cup}(A_i\widetilde{\ominus}B)}, F_{\underset{i\in I}{\cup}(A_i\widetilde{\ominus}B)}\rangle$$

$$T_{\underset{i\in I}{\cup} A_i\widetilde{\ominus}B}(v) = \inf_{u\in Z^n} \max\left(T_{\underset{i\in I}{\cup} A_i}(v+u), T_B(u)\right)$$

$$= \inf_{u\in Z^n} \max\left(\sup_{i\in I} T_{\boldsymbol{A_i}}(v+u), T_B(u)\right) \quad = \inf_{u\in Z^n} \sup_{i\in I}\left(\max T_{\boldsymbol{A_i}}(v+u), T_B(u)\right)$$

$$\geq \underset{\boldsymbol{i\in I}}{\cup} \inf_{u\in Z^n}\left(\max T_{\boldsymbol{A_i}}(v+u), T_B(u)\right) \qquad \geq \underset{i\in I}{\cup} T_{(A_i\widetilde{\ominus}B)}(v)$$

$$\geq T_{\underset{i\in I}{\cup}(A_i\widetilde{\ominus}B)}(v)$$

$$I_{\underset{i\in I}{\cup} A_i\widetilde{\ominus}B}(v) = \inf_{u\in Z^n} \max\left(I_{\underset{i\in I}{\cup} A_i}(v+u), I_B(u)\right)$$

$$= \inf_{u\in Z^n} \max\left(\sup_{i\in I} I_{\boldsymbol{A_i}}(v+u), I_B(u)\right) \qquad = \inf_{u\in Z^n} \sup_{i\in I}\left(\max I_{\boldsymbol{A_i}}(v+u), I_B(u)\right)$$

$$\geq \underset{\boldsymbol{i\in I}}{\cup} \inf_{u\in Z^n}\left(\max I_{\boldsymbol{A_i}}(v+u), I_B(u)\right) \qquad \geq \underset{i\in I}{\cup} I_{(A_i\widetilde{\ominus}B)}(v)$$

$$\geq I_{\underset{i\in I}{\cup}(A_i\widetilde{\ominus}B)}(v)$$

$$F_{\underset{i\in I}{\cup} \boldsymbol{A_i}\widetilde{\ominus}\boldsymbol{B}}(v) = \sup_{u\in Z^n} \min\left(1 - F_{\underset{i\in I}{\cup} \boldsymbol{A_i}}(v+u), F_B(u)\right)$$

$$= \sup_{u\in Z^n} \min\left(1 - \sup_{i\in I} F_{\boldsymbol{A_i}}(v+u), F_B(u)\right)$$

$$= \sup_{u\in Z^n} \min\left(\inf_{i\in I}\left(1 - F_{\boldsymbol{A_i}}(v+u)\right), F_B(u)\right)$$

$$= \sup_{u\in Z^n} \inf_{i\in I}\left(\min 1 - F_{\boldsymbol{A_i}}(v+u), F_B(u)\right)$$

$$\leq \inf_{i\in I} \sup_{u\in Z^n}\left(\min 1 - F_{\boldsymbol{A_i}}(v+u), F_B(u)\right)$$

$$\leq \underset{i\in I}{\cap} \sup_{u\in Z^n}\left(\min 1 - F_{A_I\widetilde{\ominus}B}(v+u), F_B(u)\right) \quad \leq F_{\underset{i\in I}{\cap}(A_i\widetilde{\ominus}B)}(v)$$

b) The proof is similar to a).

4.5.3. Proposition (properties of the neutrosophic closing)***:***

The neutrosophic closing satisfies the following properties

Proposition 1: The neutrosophic closing satisfies:

Monotonicity, $\forall\ A, B, C\ \in \mathcal{N}(Z^2)$

$$A \subseteq B \Longrightarrow \langle T_{A\tilde{\bullet}C}, I_{A\tilde{\bullet}C}, F_{A\tilde{\bullet}C}\rangle \subseteq \langle T_{B\tilde{\bullet}C}, I_{B\tilde{\bullet}C}, F_{B\tilde{\bullet}C}\rangle$$

$$T_{A\tilde{\bullet}C} \subseteq T_{B\tilde{\bullet}C} \quad , \quad I_{A\tilde{\bullet}C} \subseteq I_{B\tilde{\bullet}C} \quad \textit{and} \quad F_{A\tilde{\bullet}C} \supseteq F_{B\tilde{\bullet}C}$$

Proposition 2: For any family $(A_i | i \in I)$ *in* $\mathcal{N}(Z^2)$ *and* $B \in \mathcal{N}(Z^2)$

$$\langle T_{\underset{i\in I}{\cap} A_i\tilde{\bullet}B}, I_{\underset{i\in I}{\cap} A_i\tilde{\bullet}B}, F_{\underset{i\in I}{\cap} A_i\tilde{\bullet}B}\rangle \subseteq \langle T_{\underset{i\in I}{\cap}(A_i\tilde{\bullet}B)}, I_{\underset{i\in I}{\cap}(A_i\tilde{\bullet}B)}, F_{\underset{i\in I}{\cap}(A_i\tilde{\bullet}B)}\rangle$$

$$T_{\underset{i\in I}{\cap} A_i \tilde{\bullet} B} \subseteq T_{\underset{i\in I}{\cap}(A_i \tilde{\bullet} B)},\ I_{\underset{i\in I}{\cap} A_i \tilde{\bullet} B} \subseteq I_{\underset{i\in I}{\cap}(A_i \tilde{\bullet} B)}\ \ and\ \ F_{\underset{i\in I}{\cap} A_i \tilde{\bullet} B} \supseteq F_{\underset{i\in I}{\cup}(A_i \tilde{\bullet} B)}$$

Proposition 3: For any family $(A_i|i \in I)\ in\ \mathcal{N}(Z^2)\ \ and\ \ B \in \mathcal{N}(Z^2)$

$$\langle T_{\underset{i\in I}{\cup} A_i \tilde{\bullet} B}, I_{\underset{i\in I}{\cup} A_i \tilde{\bullet} B}, F_{\underset{i\in I}{\cup} A_i \tilde{\bullet} B}\rangle \supseteq \langle T_{\underset{i\in I}{\cup}(A_i \tilde{\bullet} B)}, I_{\underset{i\in I}{\cup}(A_i \tilde{\bullet} B)}, F_{\underset{i\in I}{\cup}(A_i \tilde{\bullet} B)}\rangle$$

$$T_{\underset{i\in I}{\cup} A_i \tilde{\bullet} B} \supseteq T_{\underset{i\in I}{\cup}(A_i \tilde{\bullet} B)},\ I_{\underset{i\in I}{\cup} A_i \tilde{\bullet} B} \supseteq I_{\underset{i\in I}{\cup}(A_i \tilde{\bullet} B)}\ \ and\ F_{\underset{i\in I}{\cup} A_i \tilde{\bullet} B} \subseteq F_{\underset{i\in I}{\cap}(A_i \tilde{\bullet} B)}$$

Proof: The proof is similar to the procedure used in propositions §4.5.1 and §4.5.2.

4.5.4. Proposition (properties of the neutrosophic opening)**:**

The neutrosophic opening satisfies the following properties

Proposition 1: The neutrosophic opening satisfies:

Monotonicity: $\forall\ \ A, B, C \in \mathcal{N}(Z^2)$

$$A \subseteq B \Longrightarrow \langle T_{A\,\tilde{\circ}\,C}, I_{A\,\tilde{\circ}\,C}, F_{A\,\tilde{\circ}\,C}\rangle \subseteq \langle T_{B\,\tilde{\circ}\,C}, I_{B\,\tilde{\circ}\,C}, F_{B\,\tilde{\circ}\,C}\rangle$$

$$T_{\mathbf{A}\tilde{\circ}\,\mathbf{C}} \subseteq T_{\mathbf{B}\tilde{\circ}\,\mathbf{C}}\ \ ,\ \ I_{\mathbf{A}\tilde{\circ}\,\mathbf{C}} \subseteq I_{\mathbf{B}\tilde{\circ}\,\mathbf{C}}\ \ and\ \ F_{\mathbf{A}\tilde{\circ}\,\mathbf{C}} \supseteq F_{\mathbf{B}\tilde{\circ}\,\mathbf{C}}$$

Proposition 2: For any family $(A_i|i \in I)\ in\ \mathcal{N}(Z^2)\ and\ B \in \mathcal{N}(Z^2)$

$$\langle T_{\underset{i\in I}{\cap} A_i \tilde{\circ} B}, I_{\underset{i\in I}{\cap} A_i \tilde{\circ} B}, F_{\underset{i\in I}{\cap} A_i \tilde{\circ} B}\rangle \subseteq \langle T_{\underset{i\in I}{\cap}(A_i \tilde{\circ} B)}, I_{\underset{i\in I}{\cap}(A_i \tilde{\circ} B)}, F_{\underset{i\in I}{\cap}(A_i \tilde{\circ} B)}\rangle$$

$$T_{\underset{i\in I}{\cap} A_i \tilde{\circ} B} \subseteq T_{\underset{i\in I}{\cap}(A_i \tilde{\circ} B)},\ I_{\underset{i\in I}{\cap} A_i \tilde{\circ} B} \subseteq I_{\underset{i\in I}{\cap}(A_i \tilde{\circ} B)}\ \text{ and }\ F_{\underset{i\in I}{\cap} A_i \tilde{\circ} B} \supseteq F_{\underset{i\in I}{\cup}(A_i \tilde{\circ} B)}$$

Proposition 3: For any family $(A_i|i \in I)\ in\ \mathcal{N}(Z^2)\ and\ B \in \mathcal{N}(Z^2)$

$$\langle T_{\underset{i\in I}{\cup} A_i \tilde{\circ} B}, I_{\underset{i\in I}{\cup} A_i \tilde{\circ} B}, F_{\underset{i\in I}{\cup} A_i \tilde{\circ} B}\rangle \supseteq \langle T_{\underset{i\in I}{\cup}(A_i \tilde{\circ} B)}, I_{\underset{i\in I}{\cup}(A_i \tilde{\circ} B)}, F_{\underset{i\in I}{\cup}(A_i \tilde{\circ} B)}\rangle$$

$$T_{\underset{i\in I}{\cup} A_i \tilde{\circ} B} \supseteq T_{\underset{i\in I}{\cup}(A_i \tilde{\circ} B)},\ I_{\underset{i\in I}{\cup} A_i \tilde{\circ} B} \supseteq I_{\underset{i\in I}{\cup}(A_i \tilde{\circ} B)}\ \text{and}\ F_{\underset{i\in I}{\cup} A_i \tilde{\circ} B} \subseteq F_{\underset{i\in I}{\cap}(A_i \tilde{\circ} B)}$$

Proof The proof is similar to the procedure used in propositions §4.5.1 and §4.5.2.

5. Conclusion

In this paper, our aim was to establish a foundation for what we called, Neutrosophic Mathematical Morphology. It is a new approach to Mathematical Morphology based on neutrosophic set theory. Several basic definitions for Neutrosophic Morphological operations were extracted and a study of its algebraic properties was presented. In addition, we were able to prove that Neutrosophic Morphological operations inherit properties and restrictions of fuzzy Mathematical Morphology. In future, we plane to apply the introduced concepts in Image Processing. For instance, Image Smoothing, Enhancement and Retrieval, as well as in medical imaging.

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