Computers & Industrial Engineering 124 (2018) 493–508

Contents lists available at ScienceDirect

Computers & Industrial Engineering

journal homepage: www.elsevier.com/locate/caie

New evidential reasoning rule with both weight and reliability for evidence combination

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ARTICLE INFO

Keywords:
Evidential reasoning
Evidence combination rule
Evidence discounting
Evidence weight
Evidence reliability

ABSTRACT

Two aspects of problems such as weight over-bounding and reliability-dependence cannot be well solved in the evidential reasoning (ER) approach with both weight and reliability. In order to solve the above problems, the characteristics of weight and reliability are investigated and summarized, i.e., the reliability of evidence is objective and absolute to reflect information quality, while the weight of evidence is subjective and relative to reflect information importance. Then a new discounting method is defined to generate probability masses for the evidence by assigning residual support of weight to empty set and that of reliability to power set. A new ER rule is established for recursively combining the evidence with both reliability and weight by the orthogonal sum operation and a series of theorems and corollaries are introduced and proved. Finally numerical comparison and illustrative example are provided to demonstrate the performances and the applicabilities of the proposed rule and algorithm.

1. Introduction

The evidential reasoning (ER) approach is a general approach for analyzing multiple criteria decision making (MCDM) problems under uncertainties (Yang & Singh, 1994). In the ER approach, the belief structure and the belief decision matrix are introduced to model uncertainties of various types of nature, and a generic conjunctive probabilistic reasoning process satisfying a generalized bayesian inference process is established to combine multiple pieces of evidence generated from criteria or experts. The ER approach has been widely applied to solve some practical problems in different fields such as medical quality assessment (Kong, Xu, Yang, & Ma, 2015), technical analysis in forex trading expert system (Dymova, Sevastjanov, & Kaczmarek, 2016), trauma outcome prediction (Kong et al., 2016), smart home subcontractor selection (Polat, Cetindere, Damci, & Bingol, 2016), environmentally-friendly designs selection (NG, 2016), navigational risk assessment (Zhang, Yan, Zhang, Yang, & Wang, 2016), data classification (Xu, Zheng, Yang, Xu, & Chen, 2017), and so on.

In the published literature, three versions of the ER approach are successively developed during the past decades. The first version of the ER approach is introduced to discover the link between the MCDM and the Dempster-Shafer theory of evidence (DST) (Yang & Singh, 1994) by using the original concept of ER for criteria aggregation. It employs the Dempster's combination rule and the Shafer's discounting method for criteria aggregation with the introduction of criteria weights in the probability mass assignment. Since it is incapable of separating an unassigned probability mass into the part caused by incompleteness and that caused by criterion weight, the reasoning process is approximate and the ignorance may be exaggerated in the fusion result. Meanwhile, this version approach has other drawbacks such as the compensation among criteria is unable to be reflected, random numbers or interval uncertainty are not to be handled, etc. In order to overcome the drawbacks existing in the first version, the second version of ER approach is proposed to hold a more rigorous and rational reasoning process. The second version approach is equipped with a new ER rule and information transformation techniques (Yang, 2006). It is capable of properly handling both qualitative and quantitative information, probabilistic uncertainty, incomplete information and complete/global ignorance in some assessments. It is important to note that the residual support (for weight) generalized by Shafer's discounting method is distinguished from the degree of global ignorance denoted by frame of discernment in the new ER rule. The global ignorance in a piece of evidence is considered as an intrinsic property and has no relevance to other evidence, while the residual support is considered as an extrinsic property of the evidence and it may be incurred when applying weights to discount evidence. Accordingly, if the residual support and global

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https://doi.org/10.1016/j.cie.2018.07.037
Received 28 January 2017; Received in revised form 1 June 2018; Accepted 24 July 2018
Available online 29 July 2018
0360-8352/

ignorance are confused in the basic probability assignment (BPA) calculation or the process of combining evidence as in the first version of ER approach, the combination results are bound to be unreasonable.

Obviously, there exists only one parameter such as evidence weight in the earlier two versions of ER approach. The DSmT theory puts forward the problem of combining evidence with both weight and reliability for the first time and attempts to solve it by the enhanced proportional conflict redistribution rule no.5 (PCR5) (Smarandache & Dezert, 2006). In contrast, the ER approach believes that the processing way in the PCR5 changes the specificity of evidence and it is no longer a bayesian inference process. Thus the third version of ER approach is proposed, in which the discounting mode for processing evidence weights as described in the second version is inherited (Yang & Xu, 2013). In this version, a novel concept of weighted belief distribution (WBD) is proposed and firstly extended to WBD with reliability (WBDR) to characterize evidence in complement of belief distribution (BD) introduced in the DST. It establishes a unique ER rule to combine multiple pieces of independent evidence conjunctively with both weights and reliabilities. The implementation of the orthogonal sum operation on WBDs and WBDRs leds to the establishment of a new ER rule. Since the newly established ER rule holds an important property of constituting a generic conjunctive probabilistic reasoning process or a generalizing bayesian inference process, it is quickly employed to construct a data-driven approximate causal inference model (Chen et al., 2015), and solve expert assessments' integration problem (Du & Xu, 2017; Dymova et al., 2016). Note that, the ER approach mentioned hereafter in this paper refers to the third version unless otherwise specified.

The evidence fusion problem with both weights and reliabilities frequently exists in multi-criteria group decision making (MCGDM) problems. Restricted by discipline backgrounds and professional areas, the cognitive abilities of experts are always limited. Each expert can only give decision information on one or several aspects for the decision problem, in other words, he/she can only evaluate the performance of alternatives on one or several criteria. The decision information on a specific criterion evaluated by an expert can be regarded as a piece of evidence and it is usually profiled as a BD (Du & Wang, 2017). It is reasonable to believe that the reliability of evidence depends on the cognitive ability of expert and the weight depends on the relative importance of criterion. How to scientifically fuse the decision information evaluated by experts with both parameters will strongly affect the qualities of MCGDM. We believe the ER approach with both weight and reliability is capable of holding the specificity of evidence and following a bayesian inference process, and its combination result can be well explained than other approaches. However, the ER approach may be more reasonable if it can well consider and reflect the characteristics of weight and reliability in its discounting method and combination rule. Otherwise, the derived result may remain two aspects of problems as follows. The first problem involves weight over-bounding. The ER's discounting method for weight seems to be capable of reflecting relative importance degree, but the weight over-bounding problem that the degree of constraint on the combined evidence is overly bounded than its weight in the ER rule, will potentially lead to an unreasonable combination result. The second problem relates to reliability-dependence. When there are both weights and reliabilities in the evidence discounting, each kind of parameters should play the corresponding roles in the combination process. The ER's discounting method will lose effectiveness if the evidence is the most reliable or its reliability is equal to one. Specifically, the result discounted by ER's discounting method is still the BD no matter how much is the weight, accordingly the weight works or not in the discounting heavily depends on whether the reliability is equal to one. Details on the above two problems are discussed in Section 3.2.

The weight of evidence reflects the importance degree preferred in decision maker's mind, and the reliability of evidence is used to measure the quality of information generated by a piece of evidence (Smarandache & Dezert, 2010). The former is subjective and depends on who makes the judgement when combining several pieces of evidence, but the latter is objective and is independent of who may use the evidence. In our opinion, the reason why there exist the above two problems in the ER approach is that the characteristics of weight and reliability are not well considered in the process of evidence discounting and combining. In order to overcome the above drawbacks in the ER approach and derive a rational combination result, this paper proposes a new discounting method and a new ER rule with both weight and reliability for evidence combination. To facilitate our discussion, in Section 2, we describe background knowledge related to the DST and the ER approach. In Section 3, a new discounting method and a new ER combination rule are established to fuse evidence with both weight and reliability. Numerical comparison and illustrative example are provided to demonstrate the performances and the applicabilities of the proposed approach in Sections 4 and 5. We come to the conclusion of this paper in Section 6.

2. Preliminaries

The background knowledge presented in this section deals with the interpretations of the DST and the ER approach. The DST is an approach for uncertainty reasoning. It enables us to combine evidence from different sources and arrive at a degree of belief (Dempster, 1967; Shafer, 1996). It is modeled based on a frame of discernment denoted by $\Theta = \{\theta_1, \cdots, \theta_N\}$, in which elements are mutually exclusive. The power set of Θ denoted by 2^Θ or $P(\Theta)$, contains all subsets of Θ and is expressed as $P(\Theta) = 2^\Theta = \{\varnothing, \theta_1, \cdots, \theta_N, \{\theta_1, \theta_2\}, \cdots, \{\theta_1, \theta_N\}, \cdots, \{\theta_1, \cdots, \theta_{N-1}\}, \Theta\}$. If a set is assumed to be true, then all subsets are considered to be true as well. An expert who believes that one or several sets in $P(\Theta)$ maybe true can assign belief masses to these sets. Belief mass on a singleton set is interpreted as the belief that the set in question is true. Belief mass on a non-singleton set is interpreted as the belief that one of the singleton elements it contains is true, but that the expert is uncertain about which of them is true. According to the above principles, several key concepts in the DST such as basic probability assignment function and Dempster's rule are defined as follows.

Definition 1 (*Dempster, 1967*). Suppose $\Theta = \{\theta_1, \cdots, \theta_N\}$ is a frame of discernment, if the mapping function m: $2^\Theta \to [0, 1]$ could fulfill

$$\begin{cases} m(\theta) = 0 & \theta = \varnothing \\ m(\theta) \geqslant 0, \ \sum\limits_{\theta \subseteq \Theta} m(\theta) = 1 & \theta \neq \varnothing \end{cases} \tag{1}$$

then $m(\cdot)$ is called basic probability assignment (BPA) function of Θ. If $m(\theta) > 0$, θ is named as a focal element.

Definition 2 (*Dempster, 1967*). Suppose the BPA functions of two pieces of evidence are m_1 and m_2 on Θ, $\oplus$ is the orthogonal sum operator, then the combined evidence with Dempster's rule from m_1 and m_2 for $\theta \neq \varnothing$ can be defined as:

$$m_{e(2)}(\theta) = [m_1 \oplus m_2](\theta) = \frac{1}{1-k} \sum\nolimits_{B \cap C = \theta, B, C \subseteq \Theta} m_1(B) m_2(C) \tag{2}$$

where $k = \sum_{B \cap C = \varnothing, B, C \subseteq \Theta} m_1(B) m_2(C)$ is conflict factor and equal to the amount of conflict among these pieces of evidence.

The Dempster's rule is associative for the reason that it is based on an orthogonal sum operation, so as to the third or more pieces of evidence can be combined with the combined joint mass by Eq. (2) recursively. It is necessary to point out that the counter-intuitive problem may be occurred when the Dempster's rule is used to combine evidence under the situation $k \to 1$. A plethora of modified methods has been developed to solve the counter-intuitive problem, in which modifying the initial belief function to better represent original information is a widely accepted way. The BPA function is usually regarded as the probability mass that has already taken into account the weight of evidence, so the degree of support for a proposition is proportional to both the weight of evidence and the belief degree (Yang & Xu, 2013).

Definition 3 (*Yang and Xu, 2013*). Suppose $(\theta, p_{\theta,i})$ shows that the evidence e_i points to proposition θ to a belief degree $p_{\theta,i}$, then the profiled expression

$$b_i = \left\{(\theta, p_{\theta,i}),\ \forall \theta \subseteq \Theta,\ \sum_{\theta\subseteq\Theta} p_{\theta,i} = 1\right\} \tag{3}$$

is called the belief distribution (BD) of e_i.

Definition 4 (*Shafer, 1996*). Suppose the BD of evidence e_i is b_i as in Eq. (3), λ_i is the weight of evidence e_i used to discount b_i, then the Shafer's discounting method can be defined to generate BPA for the evidence e_i as follows:

$$m_{\theta,i} = m_i(\theta) = \begin{cases} 0 & \theta = \varnothing \\ \lambda_i p_i(\theta) & \theta \subset \Theta \\ \lambda_i p_i(\theta) + (1-\lambda_i) & \theta = \Theta \end{cases} \tag{4}$$

Shafer's discounting method may change the specificity of the original evidence in that global ignorance is introduced to a BD even when the evidence points to a proposition precisely and unambiguously. In order to solve such a problem, the ER's discounting method with weight is introduced as in Eq. (5), and its discounting result m_i is referred to as weighted belief distribution (WBD). Besides, if there are both weight and reliability in the evidence combination, the WBD can be further discounted by reliability as in Eq. (6) and referred to as weighted belief distribution with reliability (WBDR). The basic probability masses of evidence, whether the WBDs discounted by Eq. (5) or the WBDRs discounted by Eq. (6), are capable of being combined by Eqs. (7a)–(7d). Note that, the combined degrees of belief may consist of local ignorance or global ignorance. In order to distribute the combined result on each element of frame of discernment, the pignistic probability is usually employed.

Definition 5 (*Yang and Xu, 2013*). Suppose the BD of evidence e_i is b_i as in Eq. (3), $w_i (0 \leqslant w_i \leqslant 1)$ is the weight to discount e_i, $P(\Theta)$ is the power set of Θ, then the ER's discounting method with weight is defined to generate the WBD for evidence e_i as follows:

$$m_{\theta,i} = m_i(\theta) = \begin{cases} 0 & \theta = \varnothing \\ w_i p_{\theta,i} & \theta \subseteq \Theta \\ 1-w_i & \theta = P(\Theta) \end{cases} \tag{5}$$

Definition 6 (*Yang and Xu, 2013*). Suppose the BD of evidence e_i is b_i as in Eq. (3), with w_i and r_i $(0 \leqslant w_i, r_i \leqslant 1)$ as its weight and reliability, $P(\Theta)$ is the power set of Θ, then the ER's discounting method with both weight and reliability is defined to generate the WBDR for evidence e_i as follows:

$$m_{\theta,i} = m_i(\theta) = \begin{cases} 0 & \theta = \varnothing \\ \widetilde{w}_i p_{\theta,i} & \theta \subseteq \Theta \\ 1-\widetilde{w}_i & \theta = P(\Theta) \end{cases} \tag{6}$$

where $c_{rw,i} = 1/(1 + w_i - r_i)$ is a normalization factor determined by $\sum_{\theta\subseteq\Theta} m_{\theta,i} + m_{P(\Theta),i} = 1$, $\widetilde{w}_i = c_{rw,i} w_i$ is called as the new weight or the adjusted weight, $1-\widetilde{w}_i = c_{rw,i}(1-r_i)$.

Definition 7 (*Yang and Xu, 2013*). Suppose I pieces of independent evidence $e_1, \cdots, e_I$ are asked to combine, each is discounted by Eq. (5) or Eq. (6), $e(i)$ denotes the combination of the first i pieces of evidence, $m_{\theta,e(i)}$ is the probability mass to which $e(i)$ jointly supports proposition θ with $m_{\theta,e(1)} = m_{\theta,1}$ and $m_{P(\Theta),e(1)} = m_{P(\Theta),1}$, then the combined evidence with ER's rule from the first i pieces of evidence can be defined as:

$$m_{\theta,e(i)} = [m_1 \oplus \cdots \oplus m_i](\theta) = \begin{cases} 0 & \theta = \varnothing \\ \dfrac{\widetilde{m}_{\theta,e(i)}}{\sum_{\vartheta\subseteq\Theta} \widetilde{m}_{\vartheta,e(i)} + \widetilde{m}_{P(\Theta),e(i)}} & \theta \neq \varnothing \end{cases} \tag{7a}$$

$$\widetilde{m}_{\theta,e(i)} = (m_{P(\Theta),i} m_{\theta,e(i-1)} + m_{P(\Theta),e(i-1)} m_{\theta,i}) + \sum_{B\cap C=\theta, B, C\subseteq\Theta} m_{B,e(i-1)} m_{C,i},\ \theta \subseteq \Theta \tag{7b}$$

$$\widetilde{m}_{P(\Theta),e(i)} = m_{P(\Theta),i} m_{P(\Theta),e(i-1)} \tag{7c}$$

where $0 \leqslant m_{\theta,e(i)}, m_{P(\Theta),e(i)} \leqslant 1$ and $\sum_{\theta\subseteq\Theta} m_{\theta,e(i)} + m_{P(\Theta),e(i)} = 1$ for $i = 1, \cdots, I$ recursively. Note that, if the evidence only consists of weight, then the probability mass is generated by Eq. (5), otherwise the evidence consists of both weight and reliability, then the probability mass is generated by Eq. (6). When the pieces of evidence are all combined at the end of the recursive process, the combined degree of belief to which I pieces of independent evidence jointly support proposition θ is given by

$$p_\theta = p_{\theta,e(I)} = \begin{cases} 0 & \theta = \varnothing \\ \dfrac{\widetilde{m}_{\theta,e(I)}}{\sum_{\vartheta\subseteq\Theta} \widetilde{m}_{\vartheta,e(I)}} & \theta \subseteq \Theta \end{cases} \tag{7d}$$

with $0 \leqslant p_\theta \leqslant 1$, $\forall \theta \subseteq \Theta$, $\sum_{\theta\subseteq\Theta} p_\theta = 1$.

Definition 8 (*Smets, 2005*). Suppose the frame of discernment is $\Theta = \{\theta_1, \cdots, \theta_N\}$, the combined degree of belief is p_θ, $\sum_\theta p_\theta = 1$, $\theta \subseteq \Theta$. The belief function and the plausibility function corresponding to θ_n are $Bel(\theta_n) = \sum_{\theta\subseteq\theta_n} p_\theta$ and $Pl(\theta_n) = \sum_{\theta_n\cap\theta\neq\varnothing} p_\theta$, set a relevant factor $\varepsilon = \left[1-\sum_{\theta_n\subseteq\Theta} Bel(\theta_n)\right] / \sum_{\theta_n\subseteq\Theta} Pl(\theta_n)$. Then the pignistic probability is

$$\gamma(\theta_n) = Bel(\theta_n) + \varepsilon \cdot Pl(\theta_n), \quad n = 1, \cdots, N. \tag{8}$$

3. The proposed method

3.1. Reliability and weight

Reliability is an important concept in various fields (Fu, Yang, & Yang, 2015), such as engineering (Sriramdas, Chaturvedi, & Gargama, 2014), industry (Gonzalez-Gonzalez, Cantu-Sifuentes, Praga-Alejo, Flores-Hermosillo, & Zuniga-Salazar, 2014), transportation (Gaonkar, Xie, & Fu, 2013), computer networks (Lin & Yeng, 2013), wireless networks (Chen & Lyu, 2005), software (Yacoub, Cukic, & Ammar, 2004), etc. In these domains, reliability is assessed to improve system performance or safety. Recently, reliability is introduced into behavior evaluation field for assessing the proficiency of specialists, in which human reliability analysis or expert reliability is becoming an important topic since human behavior can significantly influence system performance and safety (Akyuz & Celik, 2016; Ribeiro, Sousa, Duarte, & e Melo, 2016). In information fusion field, reliability is defined as an ability of evidence source to provide correct assessment/solution for the given problem, and the reliability of an evidence source should be estimated by statistics or other techniques (Smarandache & Dezert, 2010). Its reasonable range lies in $[0, 1]$ with 0 and 1 respectively standing for not reliable at all and the most reliable. For example, if an evidence source totally generalized or received 100 records, in which 90 records are correct, then the reliability of this evidence source is $90/100 = 0.9$.

Weight is a basic concept in the MCDM field with the meaning that the subjectively relative importance degree of a criterion than another with respect to a given problem (Fu & Wang, 2015). It is usually determined by some computing methods such as analytic hierarchy process (AHP) (Saaty, 2003), analytic network process (ANP) (Saaty, 2007), delphi (Stebler, Schuepbach-Regula, Braam, & Falzon, 2015), etc. In the same way, the weight of an evidence source is a relative importance degree of an evidence source than another and it should be determined by a fusion system designer, expert, or decision maker. Its reasonable range usually lies in $[0,1]$ with 0 and 1 respectively standing for not important at all and the most important. The sum of weights is frequently equal to 1 but this requirement is unnecessary. For example, a decision maker is asked to give the weights for three pieces of evidence with respect to a problem, he/she does pair-wised comparisons and gives a judgement matrix as $[1, 2, 2;1/2, 1, 1;1/2, 1, 1]$, and finally the weights can be determined as [0.5, 0.25, 0.25] according to the AHP method (Saaty, 2003). In particular, the weights of three pieces of evidence may be changed when the given problem or the decision maker is changed.

Consequently, the reliability and the weight of a piece of evidence are not the same thing and there is at least a significant difference between them, i.e., the reliability of evidence is objective and absolute to reflect information quality, while the weight of evidence is subjective and relative to reflect information importance.

3.2. New discounting method with both weight and reliability

There are two aspects of problems such as weight over-bounding and reliability-dependence in the ER approach.

The weight over-bounding problem is described as that the degree of constraint on the combined evidence is overly bounded than its weight, resulting in the combination results to be unreasonable. Suppose there are only two pieces of evidence to be combined and the sum of their weights is equal to 1. In this case, Eq. (7b) is reduced to $\widetilde{m}_{\theta,e(2)} = [(1-w_1)m_{\theta,2} + (1-w_2)m_{\theta,1}] + \sum_{B\cap C=\theta,B,C\subseteq\Theta} m_{B,1}m_{C,2}$ with $m_{\theta,i} = w_i p_{\theta,i}$ for $\forall\ \theta \subseteq \Theta$, in which $[(1-w_1)m_{\theta,2} + (1-w_2)m_{\theta,1}]$ is called the bounded sum of individual support (BSIS) and $\sum_{B\cap C=\theta,B,C\subseteq\Theta} m_{B,1}m_{C,2}$ is called the orthogonal sum of collective support (OSCS). Specially, the BSIS is explained to set a bound on the role that the combined evidence can play in its individual support for a proposition θ. We believe such an explanation has a certain degree of rationality, but the role played by the combined evidence may be inaccurately bounded. Taking $m_{\theta,i} = w_i p_{\theta,i}$ and $w_1 + w_2 = 1$ into the BSIS, we get $[(1-w_1)m_{\theta,2} + (1-w_2)m_{\theta,1}] = [(1-w_1)w_2 p_{\theta,2} + (1-w_2)w_1 p_{\theta,1}] = (w_2)^2 p_{\theta,2} + (w_1)^2 p_{\theta,1}$. From this equation, we know the bound for the combined evidence is $(w_i)^2$ which should be equal to the weight w_i for $i = 1, 2$, thus the weight of evidence is overly bounded than it should be.

Example 1. Suppose $\Theta = \{\theta_1, \theta_2, \theta_3\}$, the BDs of evidence e_1 and e_2 are $b_1 = \{(\theta_1, 0.40), (\theta_3, 0.60)\}$ and $b_2 = \{(\theta_1, 0.70), (\theta_2, 0.30)\}$, their weights are $w_1 = 0.60$ and $w_2 = 0.40$. Use Eq. (7b) in the ER's rule to combine e_1 with e_2, we have $\widetilde{m}_{\theta_2,e(2)} = [(1-w_2)m_{\theta_2,1} + (1-w_1)m_{\theta_2,2}] + \sum_{B\cap C=\theta_2} m_{B,1}m_{C,2} = (1-w_1)m_{\theta_2,2} = (1-w_1)w_2 p_{\theta_2,2} = (1-0.60) \times 0.40 \times 0.30 = (0.40)^2 \times 0.30 = 0.048$. Obviously, the BD of e_2 is bounded by 0.4 for two times and there exists weight over-bounding problem in this example.

Much attention should be paid to the weight over-bounding problem which also exists in the situation of non-normalized weights. In Example 1, if weights are not required to be normalized, there is $1-w_2 \neq w_1$. As such, there is $(1-w_1) \times w_2 \times p_{\theta_2,2} \neq w_2^2 \times p_{\theta_2,2}$. With respect to the element θ_2, its belief distribution $p_{\theta_2,2}$ has been correctly bounded by its weight w_2, does it make sense to redundantly bounded by $(1-w_1)$? Obviously, the belief distribution $p_{\theta_2,2}$ of the element θ_2 should be bounded by either its weight w_2 or the remaining weight of the other evidence $1-w_1$, but it should not be simultaneously bounded by w_2 and $1-w_1$ (i.e., $w_2 \times (1-w_1)$). Thus the weight over-bounding problem also exists in the situation of $w_1 + w_2 \neq 1$ for the reason that the evidence is overly bounded than it should be. In particular, if the weight as mentioned above is replaced with the adjusted weight as in Eq. (6), then it is easy to find that the weight over-bounding problem also exists in the ER's rule with both reliability and weight for the similar reason, especially in the situation of $w_1 = r_1$, $w_2 = r_2$, and $w_1 + w_2 = 1$.

Example 2. Also suppose $\Theta = \{\theta_1, \theta_2, \theta_3\}$, the BDs of evidence e_1 and e_2 are $b_1 = \{(\theta_1, 0.40), (\theta_3, 0.60)\}$ and $b_2 = \{(\theta_1, 0.70), (\theta_2, 0.30)\}$, their weights and reliabilities are $w_1 = r_1 = 0.60$ and $w_2 = r_2 = 0.40$. Use Eq. (6) to discount e_1 and e_2, we have $\widetilde{w}_i = c_{rw,i}w_i = w_i/(1 + w_i-r_i) = w_i$ for $i = 1, 2$. Use Eq. (7b) in the ER's rule to combine e_1 with e_2, we have $\widetilde{m}_{\theta_2,e(2)} = [(1-\widetilde{w}_2)m_{\theta_2,1} + (1-\widetilde{w}_1)m_{\theta_2,2}] + \sum_{B\cap C=\theta_2} m_{B,1}m_{C,2} = (1-\widetilde{w}_1)m_{\theta_2,2} = (1-\widetilde{w}_1)\widetilde{w}_2 p_{\theta_2,2} = (1-w_1)w_2 p_{\theta_2,2} = (1-0.60) \times 0.40 \times 0.30 = (0.40)^2 \times 0.30 = 0.048$. Similar to Example 1, the BD of e_2 is bounded by 0.4 for two times and the weight over-bounding problem appears.

The reliability-dependence problem is described as that the ER's discounting method will lose effectiveness as long as the reliability $r_i = 1$, in other words, if the reliability r_i is equal to one, then the result discounted by the ER is still the BD no matter how much is the weight. Taking $r_i = 1$ into Eq. (6), we have $c_{rw,i} = 1/(1 + w_i-r_i) = 1/w_i$, so we further derive $m_{\theta,i} = c_{rw,i}w_i p_{\theta,i} = (1/w_i)w_i p_{\theta,i} = p_{\theta,i}$ for $\forall\ \theta \subseteq \Theta$, $m_{P(\Theta),i} = c_{rw,i}(1-r_i) = 1/w_i \times 0 = 0$, and $m_{\emptyset,i} = 0$. Obviously, there is $m_{\theta,i} = p_{\theta,i}$ for $\forall\ \theta \subseteq \Theta$. As a result, whether the weight works or not in the discounting heavily depends on whether the reliability is equal to one or not.

Example 3. Suppose $p_{\theta,i} = \{(\theta_1, 0.4), (\theta_2, 0.6)\}$, $w_i = 0.6$, $r_i = 1.0$. According to Eq. (6), there is $c_{rw,i} = 1/(1 + w_i-r_i) = 1/w_i = 5/3$, $m_{\theta_1,i} = c_{rw,i}w_i p_{\theta_1,i} = 5/3 \times 0.6 \times 0.4 = 0.4$, $m_{\theta_2,i} = 5/3 \times 0.6 \times 0.6 = 0.6$, $m_{P(\Theta),i} = c_{rw,i}(1-r_i) = 5/3 \times (1-1) = 0$, $m_{\emptyset,i} = 0$. So there is $m_{\theta,i} = \{(\theta_1, 0.4), (\theta_2, 0.6)\} = p_{\theta,i}$.

Some experts may argue that if one is sure that evidence e_i is fully reliable, and he/she still believes this piece of evidence fully when receiving other pieces of evidence with reliability less than 1. If not, e_i is incapable of saying to be fully reliable. One does that without making a tradeoff among pieces of evidence by considering weights. That is, fully reliable evidence can dominate other pieces of evidence with reliabilities less than 1. In our opinion, reliability and weight are two different concepts and each plays its corresponding role in evidence discounting. In a MCGDM context, let each criterion be judged by an expert on the performance of an alternative, so each expert can be regarded as an evidence source, and the judgment information given by an expert can be regarded as a piece of evidence. Obviously, the reliability of evidence depends on the cognitive ability of the expert, and the weight of the evidence depends on the relative importance of the criterion with respect to the decision problem. It is unreasonable to say that the influence of a criterion on the decision problem (represented by weight) is not to be considered just because an expert gives the completely correct information on only one criterion (unless the criterion is absolutely important). Therefore, only when the evidence is fully reliable and fully important, the BD before and after the discounting should remain the same, otherwise the two kinds of parameters should be both reflected in the evidence discounting.

Example 4. Suppose an alternative is evaluated by two criteria c_1 and c_2, their weights are $w_1 = 0.0001$ and $w_2 = 0.9999$, expert e_1 and e_2 are asked to participate in decision making, their reliabilities are $r_1 = r_2 = 1.0$, the frame of discernment is $\Theta = \{\theta_1, \theta_2\} = \{Good, Bad\}$. Also suppose the alternative performance on c_1 given by expert e_1 is $b_1 = \{(Good, 0.9999), (Bad, 0.0001)\}$, and that on c_2 given by expert e_2 is $b_2 = \{(Good, 0.0001), (Bad, 0.9999)\}$. $r_1 = r_2 = 1.0$ shows that b_1 and b_2 are both absolutely correct. From b_1 and b_2 we know the alternative performance on c_1 is approximate to *Good* and that on c_2 is approximate to *Bad*. Because of $w_1 = 0.0001$ and $w_2 = 0.9999$, it is logical and reasonable to infer that the comprehensive performance of alternative on the two criteria should heavily depend on c_2 and should be approximate to *Bad*. Now we use the ER approach to solve this decision problem. Since there is the reliability-dependence problem in the ER's discounting method with both weight and reliability as in Eq. (6), the discounted probability mass for the two pieces of evidence are unchanged. Then taking b_1 and b_2 into the ER's rule as in Eqs. (7a)–(7d), the eventually combined result is determined as $p(Good) = p(Bad) = 0.5$. Such a result means that the comprehensive performance of alternative on the two criteria has 50% probability to

Good and 50% probability to *Bad*. Obviously, it is conflicting with the intuitional result. The root cause of the above errors is that there exists reliability-dependence problem in the ER's discounting method with both weight and reliability. As a result, we believe that the weight and the reliability should be both reflected in the evidence discounting whether or not the reliability is equal to 1.

Definition 9. Suppose the BD of evidence e_i is b_i as in Eq. (3), with w_i and r_i ($0 \leqslant w_i, r_i \leqslant 1$) as its weight and reliability, α is a coefficient for subjective fusion and β is a coefficient for objective fusion with $\alpha + \beta = 1$, $\alpha, \beta > 0$. Then the new discounting method with both weight and reliability is defined to generate probability mass for the evidence as follows:

$$m_{\theta,i} = m_i(\theta) = \begin{cases} \alpha w_i & \theta = P(\varnothing) \\ \beta r_i p_{\theta,i} & \theta \subseteq \Theta \\ \alpha(1-w_i) + \beta(1-r_i) & \theta = P(\Theta) \end{cases} \tag{9}$$

where $P(\Theta)$ is the power set of Θ, $P(\varnothing)$ is an empty set generated by weight discounting and it is used to distinguish with the empty set generated by orthogonality of focus element.

In order to overcome the mentioned drawbacks in the ER approach, we introduce a new discounting method with both reliability and weight as in Eq. (9). The reliability and the weight are two kinds of different concepts with the former objectively and absolutely reflecting information quality and the latter subjectively and relatively reflecting information importance. The new discounting method takes reliability r_i to discount the BD $p_{\theta,i}$ and assigns the residual support of reliability $(1-r_i)$ to the power set $P(\Theta)$. It means that the residual support of reliability also participates in combining with other evidence, and it is completely consistent with the characteristics of reliability for the reason that the reliability is the ability of an evidence source to provide correct assessment/solution with respect to the given problem as mentioned in Section 3.1. The above principles can be directly explained with the following example.

Example 5. Suppose $\Theta = \{\theta_1, \theta_2\}$, the BD given by evidence source e_i is $b_i = \{(\theta_1, 0.4), (\theta_2, 0.6)\}$, the reliability is $r_i = 0.8$, $\alpha = 0$ and $\beta = 1$. Without loss of generality, we assume the evidence source e_i totally generalized or received 100 records. In this case, $b_i = \{(\theta_1, 0.4), (\theta_2, 0.6)\}$ means that $100 \times 0.4 = 40$ records show θ_1 to be true and $100 \times 0.6 = 60$ records show θ_2 to be true. According to the characteristics of reliability, $r_i = 0.8$ means that the evidence source e_i has the ability to provide 80% correct information so as to there is 80% information in b_i is correct. It is easily inferred that there are $r_i \times 40 = 32$ records that show θ_1 to be true must be correct, and $r_i \times 60 = 48$ records that show θ_2 to be true must be correct. How to deal with the residual records $(100-r_i \times 40-r_i \times 60 = 20)$? Since we are not sure these residual records may show which subsets in the power set to be true, it is reasonable to assign the 20 records to the power set. The above processing principle can effectively describe the discounting thoughts by the reliability in Definition 9.

Although it is meaningful to take the reliability to discount the BD as shown in Example 5, we believe that such a processing mode is unsuitable for the weight. The weight is different from the reliability and it just reflects information importance subjectively and relatively. The main question is how to deal with different weights of evidence sources in the fusion process in such a way that a clear distinction is made/preserved between reliability and weight? Our preliminary investigations show that the discounting idea which defines the weight discounting with respect to the empty set is a nice way (Smets, 1993; Smarandache & Dezert, 2010). As a result, the new discounting method assigns the weight w_i to the empty set $P(\varnothing)$ as well as assigns the residual support of weight $(1-w_i)$ to the power set $P(\Theta)$. The interest of this new discounting is not only to preserve the specificity of the evidence since all BDs of focal elements are unchanged, but also to set a bound on the role that the combined evidence can play in its individual support for a proposition θ. The basic probability mass for $P(\varnothing)$ in the eventually fusion result will be reassigned to focal elements. We shall use the positive mass of the empty set just as an intermediate/preliminary step of the fusion process. Working with positive mass of belief on the empty set is not new and has been introduced in Smets' transferable belief model (Smets, 1993), and Smarandache's DSmT theory (Smarandache & Dezert, 2010). Above introductions will be discussed deeply in next section.

The BD discounted by the reliability is to improve information qualities from the objective and absolute perspectives, while that discounted by the weight is to reflect information importances from the subjective and relative perspectives. A pair of parameters α and β, such that $\alpha, \beta > 0$ and $\alpha + \beta = 1$, is used to balance the relative importance relationship between the two kinds of perspectives. Taking α and β into the discounting with both reliability and weight, we derive the new discounting method as in Eq. (9). How to determine α and β rationally in practice will be discussed later. No matter what value α and β take, the new discounting method must satisfy the property that the sum of all parts is equal to 1 (see Theorem 1).

Theorem 1. *Suppose the BD of evidence e_i is b_i as in Eq. (3), with w_i and r_i ($0 \leqslant w_i, r_i \leqslant 1$) as its weight and reliability, $\alpha + \beta = 1$, $\alpha, \beta > 0$, $m_{\theta,i}$ is the result discounted by the new discounting method with both weight and reliability as in Eq. (9). Then there must be* $\sum_{\theta \subseteq \Theta} m_{\theta,i} + m_{P(\varnothing),i} + m_{P(\Theta),i} = 1$.

Proof. See Appendix A.1. □

Differentiating with the ER's discounting method as in Eq. (6), the new discounting method presented in this paper is a symmetry form of weight and reliability which is benefit to discount the evidence by weight and reliability simultaneously in terms of the features of two concepts. Moreover the reliability-dependence problem does not exist in the new discounting method. Taking $r_i = 1$ into Eq. (9), we get $m_{\theta,i} = \{\alpha w_i, \theta = P(\varnothing); \beta p_{\theta,i}, \theta \subseteq \Theta; \alpha(1-w_i), \theta = P(\Theta)\}$, it is obvious to find that $m_{\theta,i} \neq p_{\theta,i}$. Note that, here $m_{P(\Theta),i} = \alpha(1-w_i)$ means that the residual support of reliability is zero and there is only the residual support of weight left to participate in combination.

3.3. New ER combination rule

Suppose the BD of evidence e_i is b_i as in Eq. (3), its weight and reliability are w_i and r_i, $0 \leqslant w_i, r_i \leqslant 1$, $m_{\theta,i}$ is the probability mass of e_i discounted by taking b_i, w_i and r_i into Eq. (9). Now two pieces of independent evidence e_1 and e_2 with $m_{\theta,1}$ and $m_{\theta,2}$ are asked to combine. In order to hold a generalized bayesian inference process, the orthogonal sum operation is utilized to make combination for $m_{\theta,1}$ and $m_{\theta,2}$, and the conjunctive probabilistic reasoning process for two pieces of evidence is shown as in Fig. 1. In Fig. 1, $\widetilde{m}_{\theta,e(2)}$ and $\widetilde{m}_{P(\Theta),e(2)}$ are the initially orthogonal results of $m_{\theta,1}$ and $m_{\theta,2}$ without extracting the probability mass of empty set, $\widetilde{m}_{\varnothing,e(2)} = \sum_{B \cap C = \varnothing, B, C \subseteq \Theta} m_{B,1} m_{C,2}$ is the probability mass of empty set generated by orthogonality of focal elements. Note that, $\widetilde{m}_{\varnothing,e(2)}$ is frequently called the conflict factor. Similar to the DST and the ER approach, the conflict factor is reassigned into other parts and thus we get the eventually joint probability masses for $\forall\, \theta \subseteq \Theta$, $P(\Theta)$ and $P(\varnothing)$ as in Eqs. (10a)–(10c).

$$m_{\theta,e(2)} = \frac{1}{1-k}\left[\sum_{B \cap C = \theta, B, C \subseteq \Theta} m_{B,1} m_{C,2} + m_{P(\Theta),1} m_{\theta,2} + m_{P(\Theta),2} m_{\theta,1}\right],\ \theta \subseteq \Theta \tag{10a}$$

$$m_{P(\Theta),e(2)} = \frac{1}{1-k} m_{P(\Theta),1} m_{P(\Theta),2} \tag{10b}$$

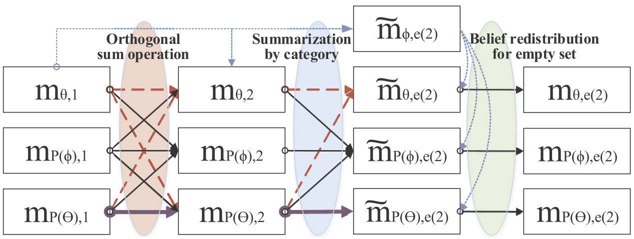

Fig. 1. The conjunctive probabilistic reasoning process.

$$m_{P(\emptyset),e(2)} = \frac{1}{1-k}\left[m_{P(\emptyset),1}\left(m_{P(\emptyset),2} + \sum_{\theta\subseteq\Theta} m_{\theta,2} + m_{P(\Theta),2}\right) + m_{P(\emptyset),2}\left(\sum_{\theta\subseteq\Theta} m_{\theta,1} + m_{P(\Theta),1}\right)\right] \tag{10c}$$

where $k = \sum_{B\cap C=\emptyset} m_{B,1}m_{C,2}$ is the conflict factor.

From Eq. (7d), we know that the combined degree of belief is computed by normalizing the probability masses of $\theta \subseteq \Theta$ in the ER approach. Such a principle is employed to compute the combined degree of belief in this work. There is an important property that $\sum_{\theta\subseteq\Theta} m_{\theta,e(2)} + m_{P(\Theta),e(2)} + m_{P(\emptyset),e(2)} = 1$ in the combined probability mass as shown in Theorem 2, so as to the combined degree of belief denoted by $p_{\theta,e(2)}$ is actually generated by reassigning $m_{P(\Theta),e(2)}$ and $m_{P(\emptyset),e(2)}$ back to all focal elements of Θ as in Eq. (11), and it is capable of being determined by Eqs. (12a) and (12b) as shown in Theorem 3.

$$p_{\theta,e(2)} = \frac{m_{\theta,e(2)}}{1-m_{P(\Theta),e(2)}-m_{P(\emptyset),e(2)}}, \quad \theta \subseteq \Theta \tag{11}$$

Theorem 2. *Suppose the probability masses of two pieces of independent evidence e_1 and e_2 are $m_{\theta,1}$ and $m_{\theta,2}$ discounted by Eq. (9), and the probability mass to which both e_1 and e_2 jointly support proposition θ is $m_{\theta,e(2)}$ as shown in Eqs. (10a)–(10c). Then there must be $\sum_{\theta\subseteq\Theta} m_{\theta,e(2)} + m_{P(\Theta),e(2)} + m_{P(\emptyset),e(2)} = 1$.*

Proof. See Appendix A.2. □

Theorem 3. *Suppose the probability masses of two pieces of independent evidence e_1 and e_2 are $m_{\theta,1}$ and $m_{\theta,2}$ discounted by Eq. (9). Then the combined degree of belief denoted by $p_{\theta,e(2)}$ to which both e_1 and e_2 jointly support proposition θ is given as follows:*

$$p_{\theta,e(2)} = \begin{cases} 0 & \theta = \emptyset \\ \dfrac{\widetilde{m}_{\theta,e(2)}}{\sum_{B\subseteq\Theta} \widetilde{m}_{B,e(2)}} & \theta \subseteq \Theta \end{cases} \tag{12a}$$

$$\widetilde{m}_{\theta,e(2)} = \sum_{B\cap C=\theta,B,C\subseteq\Theta} m_{B,1}m_{C,2} + \beta[(1-r_1)m_{\theta,2} + (1-r_2)m_{\theta,1}] + \alpha[(1-w_1)m_{\theta,2} + (1-w_2)m_{\theta,1}] \tag{12b}$$

where $0 \leqslant p_{\theta,e(2)} \leqslant 1$ *for* $\forall\ \theta \subseteq \Theta$, $\sum_{\theta\subseteq\Theta} p_{\theta,e(2)} = 1,\ \ \alpha,\beta > 0,\ \ \alpha + \beta = 1.$

Proof. See Appendix A.3. □

As shown in Eq. (12b), Theorem 3 reinforces the notion that the combined degree of belief to which two pieces of independent evidence jointly support a proposition consists of three parts: $\sum_{B\cap C=\theta,B,C\subseteq\Theta} m_{B,1}m_{C,2}$, $\beta[(1-r_1)m_{\theta,2} + (1-r_2)m_{\theta,1}]$, and $\alpha[(1-w_1)m_{\theta,2} + (1-w_2)m_{\theta,1}]$. We name the first part as orthogonal sum of collective support (OSCS), the second part as reliability-bounded sum of individual support (RBSIS), and the third part as weight-bounded sum of individual support (WBSIS). Eq. (12b) illustrates that if two pieces of evidence each play a limited role bounded by the reliability and the weight, in addition to their collective support, the individual supports from any evidence not only bounded by reliability but also bounded by weight should be counted as part of the combined support in general. Obviously, the OSCS and the WBSIS have the similar forms with Eq. (7b), but both of them differ from that in the ER approach in that they are discounted by Eq. (9) and do not exist the over-bounded problem and the reliability-dependence problem.

Example 6. Suppose $\Theta = \{\theta_1, \theta_2, \theta_3\}$, the BDs of evidence e_1 and e_2 are $b_1 = \{(\theta_1, 0.40), (\theta_3, 0.60)\}$ and $b_2 = \{(\theta_1, 0.70), (\theta_2, 0.30)\}$, their weights are $w_1 = 0.60$ and $w_2 = 0.40$, their reliabilities are $r_1 = r_2 = 1.0$, $\alpha = \beta = 0.5$. Using Eq. (12b) to combine e_1 with e_2, we have $\widetilde{m}_{\theta_2,e(2)} = \sum_{B\cap C=\theta_2} m_{B,1}m_{C,2} + \beta[(1-r_1)m_{\theta_2,2} + (1-r_2)m_{\theta_2,1}] + \alpha[(1-w_1)m_{\theta_2,2} + (1-w_2)m_{\theta_2,1}] = \alpha[(1-w_1)m_{\theta_2,2}] = \alpha w_2\beta r_2 b_{\theta_2,2} = 0.5 \times (1-0.60) \times 0.5 \times 0.30 = 0.03$. In this computing process, it is obvious to see that the BD of e_2 is bounded by the weight (which is $w_2 = 1-w_1$ in this case) for just one time rather than two times in the ER's rule (see Example 1). Thus there is no weight over-bounding problems in Eq. (12b). Similarly, we can derive $\widetilde{m}_{\theta_1,e(2)} = 0.2$ and $\widetilde{m}_{\theta_3,e(2)} = 0.009$. Using Eq. (12a) to compute the combined degree of belief, and we have $p_{\theta_1,e(2)} = 0.6250$, $p_{\theta_2,e(2)} = 0.0937$, $p_{\theta_3,e(2)} = 0.2813$.

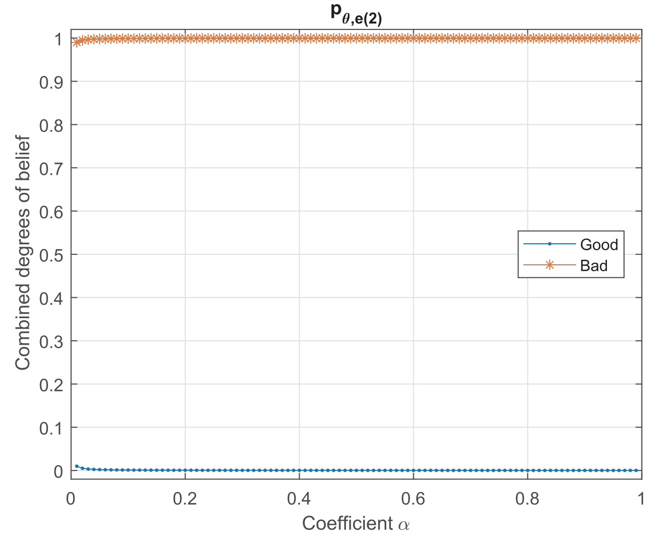

Fig. 2. The combined degrees of belief.

Example 7. Suppose an alternative is evaluated by two criteria c_1 and c_2, the alternative performance on c_1 is evaluated by expert e_1 and that on c_2 is evaluated by expert e_2, the frame of discernment is $\Theta = \{\theta_1, \theta_2\} = \{Good, Bad\}$, and the parameters such as BDs given by experts, weights, reliabilities are the same as in Example 4. Let the coefficient α be valued from 0.01 to 0.99 with the step 0.01, we use the proposed combination rule as in Eqs. (12a) and (12b) to solve this problem. The combination results are illustrated in Fig. 2. Fig. 2 shows that the combined degree of belief is changed as α is defined to different values, but the combined degrees of belief for *Good* are always approximate to 0 and that for *Bad* are always approximate to 1. It is consistent with the intuitive result that the comprehensive performance of alternative on the two criteria should heavily depend on the performance on c_2 which should be approximate to *Bad*. Obviously, the reliability-dependence problem does not exist in the proposed combination rule.

From Theorem 3 we know that if the evidence e_1 is not the most reliable or not the most important then the fusion result will consist of all focal elements of the evidence e_2, and if the two pieces of evidence are both not the most reliable or the most important then the fusion result will consist of all their focal elements. Such a conclusion is consistent with the practice, i.e., if e_1 does not have the ability to completely deny e_2 (the reliability problem), or e_1 is not fully trusted by the decision maker (the weight problem), then it is reasonable and logical to assign the propositions (focal elements) of e_2 in the final fusion result.

If there are more than two pieces of evidence to be combined, Eqs.

(10a)–(10c) can be repeated to combine the third piece of evidence with the previously-combined assessment $m_{\theta,e(2)}$ for $\theta \subseteq \Theta$, $\theta = m_{P(\Theta),e(2)}$, $\theta = m_{P(\emptyset),e(2)}$, and so on until all pieces of evidence are combined recursively. Theorem 4 is established to calculate the combined probability masses for the first i pieces of evidence. Suppose there are I pieces of evidence to be combined, the recursive fusion as in Eqs. (13a)–(13d) needs to be applied for $I-1$ times. As mentioned before, the finally combined degree of belief, denoted by $p_{\theta,e(I)}$, should be generated by reassigning $m_{P(\Theta),e(I)}$ and $m_{P(\emptyset),e(I)}$ back to all focal elements of Θ as in Eq. (11). Theorem 5 is established to calculate the finally combined degree of belief after all the pieces of evidence are combined, and it is applied only one time at the end of the recursive process. Such a recursive combination process can be summarized as in Algorithm 1.

Theorem 4. *Suppose the BD of evidence e_i is profiled by b_i as in Eq. (3), $m_{\theta,i}$ is the probability mass of e_i discounted by Eq. (9), $i = 1, \cdots, I$, $e(i-1)$ denotes the combination of the first $i-1$ pieces of evidence and $m_{\theta,e(i-1)}$ is the probability mass to which $e(i-1)$ jointly supports proposition θ, with $m_{\theta,e(1)} = m_{\theta,1}$. Then the combined probability mass denoted by $m_{\theta,e(i)}$ to which $e(i-1)$ and e_i jointly support proposition θ can be computed by the recursive combination rule as:*

$$m_{\theta,e(i)} = [m_1 \oplus \cdots \oplus m_i] = \frac{\widetilde{m}_{\theta,e(i)}}{\sum_{B\subseteq\Theta} \widetilde{m}_{B,e(i)} + \widetilde{m}_{P(\Theta),e(i)} + \widetilde{m}_{P(\emptyset),e(i)}}, \quad \forall\, \theta \tag{13a}$$

$$\widetilde{m}_{\theta,e(i)} = \sum_{B\cap C=\theta,B,C\subseteq\Theta} m_{B,e(i-1)} m_{C,i} + [\alpha(1-w_i) + \beta(1-r_i)] m_{\theta,e(i-1)} + m_{P(\Theta),e(i-1)} m_{\theta,i} \tag{13b}$$

$$\widetilde{m}_{P(\Theta),e(i)} = [\alpha(1-w_i) + \beta(1-r_i)] m_{P(\Theta),e(i-1)} \tag{13c}$$

$$\widetilde{m}_{P(\emptyset),e(i)} = \alpha w_i + m_{P(\emptyset),e(i-1)} - \alpha w_i m_{P(\emptyset),e(i-1)} \tag{13d}$$

where $0 \leqslant m_{\theta,e(i)} \leqslant 1$ *for* $\forall\, \theta \subseteq \Theta, \theta = P(\Theta), \theta = P(\emptyset)$ *and* $\sum_{\theta\subseteq\Theta} m_{\theta,e(i)} + m_{P(\Theta),e(i)} + m_{P(\emptyset),e(i)} = 1$, $i = 1, \cdots, I$.

Proof. See Appendix A.4. □

Theorem 5. *Suppose I pieces of independent evidence are all combined by Theorem 4, then the combined degree of belief denoted by $p_{\theta,e(I)}$ to which I pieces of independent evidence jointly support proposition θ is given as follows:*

$$p_\theta = p_{\theta,e(I)} = \begin{cases} 0 & \theta = \emptyset \\ \frac{\widetilde{m}_{\theta,e(I)}}{\sum_{B\subseteq\Theta} \widetilde{m}_{B,e(I)}} & \theta \subseteq \Theta \end{cases} \tag{14}$$

where $\widetilde{m}_{\theta,e(I)}$ is given by Eq. (13b) for $i = I$, $0 \leqslant p_\theta \leqslant 1$, $\forall\, \theta \subseteq \Theta$ and $\sum_{\theta\subseteq\Theta} p_\theta = 1$.

Proof. See Appendix A.5. □

Algorithm 1. Combination algorithm by the new ER approach

Input: The BDs of evidence ($b_i = \{(\theta, p_{\theta,i}), \forall\, \theta \subseteq \Theta, \sum_{\theta\subseteq\Theta} p_{\theta,i} = 1\}$, $i = 1, \cdots, I$), the weights of evidence (w_i, $i = 1, \cdots, I$), the reliabilities of evidence (r_i, $i = 1, \cdots, I$), the pair of fusion coefficients (α and β).
Output: The combined degree of belief for I pieces of evidence ($p_{\theta,e(I)}$).
Begin
%Generate the probability masses by discounting evidence with both weights and reliabilities as in Eq. (9)
For $i = 1$ to I
  If $\theta \subseteq \Theta$
    %Generate the probability masses for $\theta \subseteq \Theta$
    Then $m_{\theta,i} = \beta r_i p_{\theta,i}$
    %Generate the probability masses for $\theta = P(\emptyset)$ and $\theta = P(\Theta)$
    Else $m_{P(\emptyset),i} = \alpha w_i$, $m_{P(\Theta),i} = \alpha(1-w_i) + \beta(1-r_i)$
  ElseIf
EndFor
%Initialize the first i pieces of evidence $e(i)$
$e(1) = e_1$, $m_{\theta,e(1)} = m_{\theta,1}$
%Combine probability masses recursively by the new ER rule as in Eqs. (13a)–(13d)
For $i = 2$ to I
  If $\theta \subseteq \Theta$
    %Combine probability masses without normalization for $\theta \subseteq \Theta$ by Eq. (13b)
    Then
$\widetilde{m}_{\theta,e(i)} = \sum_{B\cap C=\theta,B,C\subseteq\Theta} m_{B,e(i-1)} m_{C,i} + [\alpha(1-w_i) + \beta(1-r_i)] m_{\theta,e(i-1)} + m_{P(\Theta),e(i-1)} m_{\theta,i}$
    %Combine probability masses without normalization for $\theta = P(\emptyset)$ and $\theta = P(\Theta)$ by Eqs. (13c) and (13d)
    Else $\widetilde{m}_{P(\emptyset),e(i)} = \alpha w_i + m_{P(\emptyset),e(i-1)} - \alpha w_i m_{P(\emptyset),e(i-1)}$,
$\widetilde{m}_{P(\Theta),e(i)} = [\alpha(1-w_i) + \beta(1-r_i)] m_{P(\Theta),e(i-1)}$
  EndIf
  %Normalize the combined probability masses by Eq. (13a)
  For $\theta \subseteq \Theta$, $\theta = P(\emptyset)$, and $\theta = P(\Theta)$
    $m_{\theta,e(i)} = \frac{\widetilde{m}_{\theta,e(i)}}{\sum_{B\subseteq\Theta} \widetilde{m}_{\theta,e(i)} + \widetilde{m}_{P(\Theta),e(i)} + \widetilde{m}_{P(\emptyset),e(i)}}$
  EndFor
EndFor
%Compute the combined degree of belief by Eq. (14)
For $\theta \subseteq \Theta$
  $p_\theta = p_{\theta,e(I)} = \frac{\widetilde{m}_{\theta,e(I)}}{\sum_{B\subseteq\Theta} \widetilde{m}_{B,e(I)}}$
EndFor
End

3.4. Discussion of fusion coefficients

From Eqs. (12b) and (13b) we know that the eventually combined degree of belief is influenced by the pair of fusion coefficients, i.e., the coefficient for subjective fusion α and the coefficient for objective fusion β with $\alpha + \beta = 1$, $\alpha, \beta > 0$. How to rationally determine the two coefficients is important to guarantee the effectiveness of fusion results. In the simple context, the two coefficients can be determined in terms of the practical requirements by decision makers. For solving the decision problem, if the objective fusion result is more valuable, β should be valued bigger than α, else if the subjective fusion result is more valuable, α should be valued bigger than β, else set $\alpha = \beta = 0.5$.

In the complex context, the decision makers may feel difficult to determine the two coefficients, so the following principles are suggested. Suppose there are four sets of evidence E_1, E_2, E_3, E_4 and each set consists of several pieces of evidence. From the perspective of overall information qualities, E_1 is high and E_2 is low. From the perspective of overall importance degrees for solving the given problem, E_3 is high and E_4 is low. All pieces of evidence in each set are combined to determine the eventually combined degree of belief (result). The following assumptions are logical and rational to be existing in the minds of decision makers.

- The combined result of E_1 is more valuable than that of E_2 ($E_1 \succ E_2$), since the overall information quality of E_1 is better than E_2.
- The combined result of E_3 is more valuable than that of E_4 ($E_3 \succ E_4$), since the overall importance of E_3 is higher than E_4 to the given

problem.

- When the combined result of E_1 is compared with that of E_3, the decision makers may consider the two are both valuable ($E_1 = E_3$), since the two aspects are both important in solving the given decision problems.

From the above assumptions it can be easily inferred that:

- When the combined result of E_1 is compared with that of E_4, the decision makers may consider the former is more valuable than the latter ($E_1 = E_3$ and $E_3 \succ E_4$ so there is $E_1 \succ E_4$).
- When the combined result of E_3 is compared with that of E_2, the decision makers may consider the former is more valuable than the latter ($E_1 \succ E_2$ and $E_1 = E_3$ so there is $E_3 \succ E_2$).

As mentioned in Section 3.1, the information quality can be reflected by reliabilities and the information importance can be reflected by weights. It is reasonable to believe that the values of the two coefficients are closely related to the overall performance of the reliabilities and the weights. According to the proposed assumptions and inferences, we establish the methods for determining fusion coefficients as in Eqs. (15a) and (15b).

Definition 10. Suppose the weight and the reliability of evidence e_i are w_i and r_i respectively, $0 \leqslant w_i, r_i \leqslant 1$, $i = 1, \cdots, I$. The coefficient for subjective fusion α and the coefficient for objective fusion β can be computed as follows.

$$\alpha = \frac{\sum_i [w_i/\max(w_i|\ \forall\ i)\ ]}{\sum_i [w_i/\max(w_i|\ \forall\ i)\ ] + \sum_i [(r_i)^2/\max(r_i|\ \forall\ i)\ ]} \tag{15a}$$

$$\beta = \frac{\sum_i [(r_i)^2/\max(r_i|\ \forall\ i)\ ]}{\sum_i [w_i/\max(w_i|\ \forall\ i)\ ] + \sum_i [(r_i)^2/\max(r_i|\ \forall\ i)\ ]} \tag{15b}$$

In Eqs. (15a) and (15b), $\sum_i [w_i/\max(w_i|\ \forall\ i)]$ is introduced to reflect the overall performance of weights. If $w_i = w_{i'}$ for $i, i' = 1, \cdots, I$, $i \neq i'$, there is $\sum_i [w_i/\max(w_i|\ \forall\ i)] = I$ with the meaning that all pieces of evidence are all important for solving the given problem. If $w_i = \max(w_{i'}|\ \forall\ i)$ and $w_{i'} = 0$ for $i' \neq i$, there is $\sum_i [w_i/\max(w_i|\ \forall\ i)] = 1$ with the meaning that only one piece of evidence is important for solving the given problem. Obviously, the former illustrates that the subjective fusion based on weights is very important and necessary, while the latter illustrates that it is not very important and unnecessary since the subjective fusion result is equal to the belief distribution of evidence with the maximum weight. With the similar reasons, $\sum_i [(r_i)/\max(r_i|\ \forall\ i)]$ is introduced to reflect the overall performance of reliabilities. Note that, the sum of weights is frequently equal to 1 but this requirement is unnecessary, in other words, the weight of evidence is subjective and relative to reflect information importance. Differentiating with the weight, the reliability of evidence is objective and absolute to reflect information quality. In order to describe the objective characteristics (the greater the reliabilities of all evidence, the more important the objective fusion), $\sum_i [(r_i)^2/\max(r_i|\ \forall\ i)]$ is established by multiplying $\sum_i [(r_i)/\max(r_i|\ \forall\ i)]$ with the reliability r_i. Accordingly, Eqs. (15a) and (15b) are constructed to reflect the relative importance degrees respectively for the subjective fusion and the objective fusion.

Theorem 6. *Suppose the two fusion coefficients are determined by Eqs. (15a) and (15b), then there must be $\alpha + \beta = 1$ and (1) if all pieces of evidence are the most reliable and the same important, then $\alpha = \beta = 0.5$; (2) if all pieces of evidence are the most reliable but their importance degrees are different, then $0.5 < \beta < 1$; (3) if all pieces of evidence are the same important but their reliabilities are different, then $0.5 < \alpha < 1$.*

Proof. See Appendix A.6. □

The two coefficients can be valued by decision makers according to the practical requirements for the decision problem, or determined by the proposed methods as in Eqs. (15a) and (15b). When the fusion coefficients are close to the maximum or the minimum values, the following useful corollaries can be obtained.

Corollary 1. *If the coefficient for subjective fusion $\alpha \to 0$, then the combined degree of belief $p_{\theta,e(2)}$ in Theorem 3 is able to be approximately computed by $\tilde{n}_{\theta,e(2)}$ as follows:*

$$p_{\theta,e(2)} = \begin{cases} 0 & \theta = \varnothing \\ \frac{\tilde{n}_{\theta,e(2)}}{\sum_{B\subseteq\Theta} \tilde{n}_{B,e(2)}} & \theta \subseteq \Theta \end{cases} \tag{16a}$$

$$\tilde{n}_{\theta,e(2)} = r_1 r_2 \sum_{B\cap C=\theta, B,C\subseteq\Theta} p_{B,1} p_{C,2} + [(1-r_1) r_2 p_{\theta,2} + (1-r_2) r_1 p_{\theta,1}] \tag{16b}$$

where $0 \leqslant p_{\theta,e(2)} \leqslant 1$ for $\forall\ \theta \subseteq \Theta$, $\sum_{\theta\subseteq\Theta} p_{\theta,e(2)} = 1$.

Proof. See Appendix A.7. □

Corollary 2. *If the coefficient for objective fusion $\beta \to 0$, then the combined degree of belief $p_{\theta,e(2)}$ as in Eq. (16a) in Corollary 1 is able to be approximately computed by $\tilde{n}_{\theta,e(2)}$ as follows:*

$$\tilde{n}_{\theta,e(2)} = (1-w_1) r_2 p_{\theta,2} + (1-w_2) r_1 p_{\theta,1} \tag{16c}$$

Proof. See Appendix A.8. □

Corollary 3. *If the coefficient for subjective fusion $\alpha \to 0$, then the orthogonal sum of the first i piece of evidence which is used to derive the combined degree of belief $p_{\theta,e(I)}$ in Theorem 5 is able to be recursively computed by:*

$$n_{\theta,e(i)} = [n_1 \oplus \cdots \oplus n_i] = \frac{\tilde{n}_{\theta,e(i)}}{\sum_{B\subseteq\Theta} \tilde{n}_{B,e(i)} + \tilde{n}_{P(\Theta),e(i)}}, \quad \theta \subseteq \Theta,\ \theta = P(\Theta) \tag{17a}$$

$$\tilde{n}_{\theta,e(i)} = \sum_{B\cap C=\theta, B,C\subseteq\Theta} n_{B,e(i-1)} n_{C,i} + [(1-r_i) n_{\theta,e(i-1)} + n_{P(\Theta),e(i-1)} n_{\theta,i}] \tag{17b}$$

$$\tilde{n}_{P(\Theta),e(i)} = (1-r_i) n_{P(\Theta),e(i-1)} \tag{17c}$$

with $n_{\theta,i} = r_i p_{\theta,i}$ *for* $\theta \subseteq \Theta$, $n_{P(\Theta),i} = 1-r_i$, $i = 1, \cdots, I$ *and* $n_{\theta,e(1)} = n_{\theta,1}$ *for* $\theta \subseteq \Theta$, $\theta = P(\Theta)$.

Proof. See Appendix A.9. □

Corollary 4. *If the coefficient for objective fusion $\beta \to 0$, then the orthogonal sum of the first i pieces of evidence which is used to derive the combined degree of belief $p_{\theta,e(I)}$ in Theorem 5 is able to be recursively computed by:*

$$n_{\theta,e(i)} = \tilde{n}_{\theta,e(i)} = (1-w_i) n_{\theta,e(i-1)} + n_{P(\Theta),e(i-1)} n_{\theta,i} \tag{18a}$$

$$n_{P(\Theta),e(i)} = \tilde{n}_{P(\Theta),e(i)} = (1-w_i) n_{P(\Theta),e(i-1)} \tag{18b}$$

$$n_{P(\varnothing),e(i)} = \tilde{n}_{P(\varnothing),e(i)} = n_{P(\varnothing),e(i-1)} + w_i - w_i n_{P(\varnothing),e(i-1)} \tag{18c}$$

with $n_{\theta,i} = r_i p_{\theta,i}$ *for* $\theta \subseteq \Theta$, $n_{P(\Theta),i} = 1-w_i$, $n_{P(\varnothing),i} = w_i$, $i = 1, \cdots, I$ *and* $n_{\theta,e(1)} = n_{\theta,1}$ *for* $\theta \subseteq \Theta$, $\theta = P(\varnothing)$, $\theta = P(\Theta)$.

Proof. See Appendix A.10. □

From Corollary 1–4, it can be inferred that the eventually fusion result combined by the new ER approach only consists of an objective fusion result when the coefficient for subjective fusion $\alpha \to 0$, while the eventually fusion result only consists of a subjective fusion result when the coefficient for objective fusion $\beta \to 0$. Above four Corollaries can effectively simplify the complexities of the recursive operation in the extremely subjective ($\beta \to 0$) circumstance and extremely objective ($\alpha \to 0$) circumstance. Meanwhile, we can also find that some features of the extremely subjective fusion and the extremely objective fusion as follows. From Corollaries 1 and 3, we know the weights do not appear in the recursive fusion process, so it is able to find that the extremely objective fusion result is independent of weights. From Corollaries 2 and 4, we know that the reliabilities appear in the recursive fusion process with the role correcting the evidences from the perspective of information quality, so it is able to find that the extremely subjective fusion result is dependent of reliabilities. Note that, Eqs. (17a)–(17c) in Corollary 3 and Eqs. (7a)–(7c) in the ER's rule are very similar in appearance. But there is an essential difference between them, i.e., Eqs. (17a)–(17c) are the objective fusion result by fusing with reliability, and Eqs. (7a)–(7c) seem to be the subjective fusion result by fusing with weight in which there exists a weight over-bounding problem and does not well reflect the characteristics of weight as mentioned in Section 3.1.

4. Numerical comparisons

From the literature, there are only two kinds of approaches to solve the evidence fusion problems with both reliability and weight, i.e., the ER approach and the proportional conflict redistribution rules no 5(PCR5) method. The DST is an important tool for uncertainty reasoning, and it initially uses the Shafer's discounting to modify belief functions by integrating weights or reliabilities with belief functions. Although the DST does not distinguish the differences between weights and reliabilities, it is favorable to give expressions to the results of two methods. In this section, these three kinds of approaches are compared with the proposed approach in this paper by using the same example as examined in literature (Smarandache & Dezert, 2010). Suppose $\Theta = \{A, B, C\}$ with A, B and C mutually exclusive and collectively exhaustive, and three pieces of independent evidence e_1, e_2 and e_3 are represented by three BDs as shown in Table 1. Since the relative importance for the subjective fusion with weight and the objective fusion with reliability is not considered in the DST, the ER and the PCR5, this example assumes $\alpha = \beta = 0.5$ which means the two kinds of fusion are equally valuable.

4.1. Comparison with the DST

The DST can only solve such an evidence combination problem that all pieces of evidence are absolutely reliable or absolutely important. When not all of the evidence are absolutely reliable or absolutely important, the DST usually uses the Shafer's discounting method to integrate the weight or the reliability with the BDs of evidence, but the difference between the weight and the reliability is not distinguished at all (Dempster, 1967; Shafer, 1996). In order to compare the DST with the new ER approach presented in this paper, we assume $r_1 = r_2 = r_3 = 1$ and $w_1 = w_2 = 1$, $w_3 = 0.4$ in this example.

Taking the data of Table 1 directly into the Dempster's rule as in Eq. (2) and the new ER rule as in Theorems 4 and 5, we can recursively determine the fusion result of three pieces of evidence as shown in Tables 2 and 3. In Table 2, $\widetilde{m}_{DST,\theta,e(2)}$ and $\widetilde{m}_{DST,\theta,e(3)}$ are the DST's results of the first two pieces of evidence and all three pieces of evidence without normalization by conflict factor, $m_{DST,\theta,e(2)}$ and $m_{DST,\theta,e(3)}$ are normalization results of $\widetilde{m}_{DST,\theta,e(2)}$ and $\widetilde{m}_{DST,\theta,e(3)}$. Note that, the third piece of evidence needs to be discounted by Shafer's discounting method before combining it with the combined result of the first two pieces of evidence. In Table 3, $m_{\theta,e(2)}$ and $m_{\theta,e(3)}$ are the new ER's jointly probability masses of the first two pieces of evidence and all three pieces of evidence generated by Theorem 4, $p_{\theta,e(2)}$ and $p_{\theta,e(3)}$ are the combined degrees of belief by taking $m_{\theta,e(2)}$ and $m_{\theta,e(3)}$ into Theorem 5.

Table 1
The BDs of three pieces of independent evidence.

	A	B	C	{A, B}	{A, C}	{B, C}	{A, B, C}
e_1	0.8000	–	–	0.1000	0.1000	–	–
e_2	0.4000	0.3000	–	0.2000	–	0.1000	–
e_3	0.1000	0.3000	0.5000	–	–	–	0.1000

Table 2
Combination result of the DST.

	∅	A	B	C	{A, B}	{A, C}	{B, C}	{A, B, C}
$\widetilde{m}_{DST,\theta,e(2)}$	0.3500	0.5800	0.0400	0.0100	0.0200	–	–	–
$m_{DST,\theta,e(2)}$		0.8923	0.0615	0.0154	0.0308	–	–	–
$\widetilde{m}_{DST,,\theta,e(3)}$	0.5508	0.3952	0.0328	0.0084	0.0128	–	–	–
$m_{DST,,\theta,e(3)}$		0.8798	0.0730	0.0187	0.0285	–	–	–

Comparing the 3rd row of Table 2 and the 3rd row of Table 3, it is not difficult to find that there is no difference between the DST and the new ER on the fusion results of the first two pieces of evidence. This illustrates the two methods are equivalent when combining the absolutely reliable and absolutely weighting evidence. However, the fusion results of the two approaches are not the same when combining the combined results of the first two pieces of evidence with the third one, as shown in the 5th row of Table 2 and the 5th row of Table 3. The third piece of evidence is discounted by Shafer's discounting method as in Eq. (4), and there is $m_{\Theta,3} = \lambda_3 p_3(\Theta) + (1-\lambda_3)$ for the global ignorance. The difference between the weight and the reliability is not distinguished in the DST, so as to there is $\lambda_3 = w_3$, and further we have $m_{\Theta,3} = \lambda_3 p_3(\Theta) + (1-\lambda_3) = 0.4 \times 0.1 + (1-0.4) = 0.04 + 0.6 = 0.64$. In this process, the residual result of weight is assigned to the global ignorance and it leads to the specificity of the original evidence to be changed for the reason that $m_{\Theta,i}/m_{\Theta,j} \neq p_{\Theta,i}/p_{\Theta,j}$ in the DST. Thus the combination result of the DST is imprecise and unreasonable. Differentiating from the DST, the new ER is able to hold the specificity of the original evidence in its discounting method with $r_3 = 1$ and $w_3 = 0.4$ as in Eq. (9), in which the characteristic of weight can be well reflected, and its result is more precise and reasonable than the DST. It is interesting to find that, if we discount the third piece of evidence with $w_3 = 1$ and $r_3 = 0.4$ by Eq. (9), the fusion result of new ER is as the same as the DST (see $p'_{\theta,e(3)}$ in the 7th row of Table 3). The reason is the residual support of reliability is assigned to the power set $P(\Theta)$ and the belief degree for $\theta \subseteq \Theta$ is also discounted by $r_i p_{\theta,i}$ in the new ER, whose discounting result is similar to the DST's. This not only coincides with the characteristics of reliability but also holds the specificity of the original evidence. In other words, the DST with Shafer's discounting method may be only suitable for combination with reliability but not for that with weight.

4.2. Comparison with the ER

In order to make an effective comparison for the ER and the new ER, this example assumes $r_1 = r_2 = 1$, $r_3 = 0.6$, $w_1 = 0.7$, $w_2 = 0.4$, $w_3 = 0.8$. The fusion result generated by using the ER is as shown in Table 4. The support for each proposition from each piece of evidence is given as the probability mass generating by Eq. (6), as shown in rows 2–4 of Table 4. $m_{ER,\theta,e(2)}$ and $m_{ER,\theta,e(3)}$ in the 5th and 7th row of Table 4 generated by Eqs. (7a)–(7d) are the jointly probability masses, and $p_{ER,\theta,e(2)}$ and $p_{ER,\theta,e(3)}$ in the 6th and 8th row of Table 4 generated by Eq. (7d) are the combined degree of belief to which three pieces of independent evidence with both weight and reliability.

The fusion result generated by using the new ER is shown in Table 5. The support for each proposition from each piece of evidence is given as the probability mass generated by Eq. (9), as shown in rows 2–4 of Table 5. $m_{\theta,e(2)}$ and $m_{\theta,e(3)}$ in the 5th and 7th row of Table 5 generated by Theorem 4 are the jointly probability mass, and $p_{\theta,e(2)}$ and $p_{\theta,e(3)}$ in the 6th and 8th row of Table 5 generated by Theorem 5 are the combined degree of belief to which three pieces of independent evidence with both weight and reliability.

From the rows 2–3 of Table 4, it is not difficult to find that the

Table 3
Combination result of the new ER.

	$P(\varnothing)$	A	B	C	{A, B}	{A, C}	{B, C}	{A, B, C}	$P(\Theta)$
$m_{\theta,e(2)}$	0.0304	0.8651	0.0597	0.0149	0.0298	–	–	–	–
$p_{\theta,e(2)}$	–	0.8923	0.0615	0.0154	0.0308	–	–	–	–
$m_{\theta,e(3)}$	0.9209	0.0685	0.0068	0.0018	0.0021	–	–	–	0.0000
$p_{\theta,e(3)}$	–	0.8662	0.0855	0.0223	0.0260	–	–	–	–
$m'_{\theta,e(3)}$	0.9367	0.0557	0.0046	0.0012	0.0018	–	–	–	0.0000
$p'_{\theta,e(3)}$	–	0.8798	0.0730	0.0187	0.0285	–	–	–	–

discounted result for the first two pieces of evidence is the same as the original BD, or say there is no discounting for them. Such a discounted result is correct in the case that reliability and weight of evidence are both equal to 1, but is clearly unreasonable in the case that reliability is equal to 1 but weight is 0.7. This situation illustrates that there is a reliability-dependence problem in the ER.

From $\alpha = 0.5$ we know that the subjective fusion with weight and objective fusion with reliability are equally valuable in this example. From Eq. (12b) we further know that the eventually fusion result should consist of all focal elements of the two combined pieces of evidence especially when each weight of the two pieces of evidence is smaller than 1 ($w_1 = 0.7$, $w_2 = 0.4$). Comparing the 6th row of Table 4 with the 6th row of Table 5, we know that two focal elements {A, C} and {B, C} appear in the combination result of the new ER, while the both do not appear in the ER. Obviously, the focal element {A, C} of e_1 and {B, C} of e_2 are lost in the fusion result of the first two pieces of evidence by the ER. Furthermore, comparing the 8th row with the 6th row of Table 4, we know that the focal elements of the eventual fusion result derived by combining the fusion result of the first two pieces of evidence with the third one, are unchanged. This illustrates that the focal element {A, B, C} is also lost in the eventual result by the ER. Thus it can be inferred that the reliability-dependence problem makes the ER lose the ability to reflect the impact of weight or subjective fusion in the fusing process, as a result directly leading to the loss of the focal elements in this example. On the contrary, the new ER is capable of taking into account the differences between the reliability and the weight and well balance the relationship between the subjective fusion and the objective fusion, so that all the focal elements of three pieces of evidence are kept in the eventual fusion result.

Besides, the ER also has the weight over-bounding problem and the order-dependent problem. These two problems are illustrated by hereinbefore examples, so we do not illustrate them again. In particular, if we let $r_1 = 0$ and other parameters remain unchanged, the ER and the new ER are respectively employed to make combination for three pieces of evidence, and we can obtain the eventual fusion results as shown in Tables 6 and 7. For the first two pieces of evidence, the fusion result of the new ER is the same as the BD of e_2 in Table 1 completely, while the ER's is not the same. Which result is more reasonable? We know r_1 is given by 0 means that all the evidence information in e_1 is false. Obviously, if a fusion method cannot eliminate the impact of false evidence, its fusion result is bound to be irrational. Fortunately, the new ER is capable of excluding the impact of the false evidence such as e_1 because its fusion result is the same as the BD of e_2. Therefore, the result of the new ER is more reasonable than the ER's.

4.3. Comparison with the PCR5

The fusion result generated by the PCR5 is shown in Table 8 with $r_1 = 0.8$, $r_2 = 0.5$, $r_3 = 0.2$, $w_1 = 0.9$, $w_2 = 0.3$, $w_3 = 0.6$, which has been originally provided in the paper by Smarandache and Dezert (2010). In Table 8, $m_{ri,wi}(\theta)$ is a reliability-importance discounting which is performed for e_i by Shafer's reliability discounting method and followed by the PCR5 importance discounting method, $i = 1, 2, 3$. The fusion result for $m_{r1,w1}(\theta)$, $m_{r2,w2}(\theta)$ and $m_{r3,w3}(\theta)$ is $\widetilde{m}_{PCR5\varnothing,r,w}(\theta)$ as shown in the 7th row of Table 8. Similarly, $m_{wi,ri}(\theta)$ is an importance-reliability discounting which reverses the order of two discounting methods, $i = 1, 2, 3$, and their combination result is $\widetilde{m}_{PCR5\varnothing,w,r}(\theta)$ as shown in the 8th row of Table 8. $\widetilde{m}_{PCR5}(\theta)$ in the 9th row of Table 8 is the arithmetic mean value of $\widetilde{m}_{PCR5\varnothing,r,w}(\theta)$ and $\widetilde{m}_{PCR5\varnothing,w,r}(\theta)$. More details on the PCR5 can be found in Smarandache and Dezert (2010).

We also use the weights and reliabilities as mentioned above in the PCR5 to combine three pieces of evidence by the new ER (here also let $\alpha = \beta = 0.5$) and the eventual fusion result is shown in Table 9. The support for each proposition from each piece of evidence is given as probability masses generated by Eq. (9) and as shown in rows 2–4 of Table 9. $m_{\theta,e(2)}$ and $m_{\theta,e(3)}$ in the 5th and the 7th row of Table 9 generated by Theorem 4 are the jointly probability masses, and $p_{\theta,e(2)}$ and $p_{\theta,e(3)}$ in the 6th and the 8th of Table 9 generated by Theorem 5 are the combined degrees of belief to which three pieces of independent evidence with both weight and reliability.

Comparing the discounted results in rows 2–7 of Table 8 with those in rows 2–4 of Table 1, it is able to find that the specificity of the original evidence is changed in the PCR5, i.e., $m_{\theta,i}/m_{\theta,j} = p_{\theta,i}/p_{\theta,j}$ is true for all $\theta \subset \Theta$ but not for $\theta = \Theta$. For example, e_2 does not contain any global ignorance with $p_{\{A,B,C\},2} = 0$ as shown in the 3rd row and last column of Table 1, but after discounting there is $m_{r2,w2}(\Theta) = 0.1500$ or $m_{w2,r2}(\Theta) = 0.5000$ as shown in the 3rd or the 6th row and last column of Table 8. The specificity of the original evidence is also changed for e_1 and e_3. Especially for e_3, the specificity change leads to a mix of global ignorance and residual support of weight or reliability. However, such a problem of changing the specificity of the original evidence does not occur in new ER, i.e., $m_{\theta,i}/m_{\theta,j} = p_{\theta,i}/p_{\theta,j}$ for $\forall\ \theta \subseteq \Theta$. Therefore, from the perspective of evidence discounting, the new ER is more reasonable than the PCR5.

Table 4
Combination result of the ER.

	A	B	C	{A, B}	{A, C}	{B, C}	{A, B, C}	$P(\Theta)$
$m_{ER,\theta,1}$	0.8000	–	–	0.1000	0.1000	–	–	–
$m_{ER,\theta,2}$	0.4000	0.3000	–	0.2000	–	0.1000	–	–
$m_{ER,\theta,3}$	0.0667	0.2000	0.3333	–	–	–	0.0667	0.3333
$m_{ER,\theta,e(2)}$	0.8923	0.0615	0.0154	0.0308	–	–	–	0.0000
$p_{ER,\theta,e(2)}$	0.8923	0.0615	0.0154	0.0308	–	–	–	–
$m_{ER,\theta,e(3)}$	0.8626	0.0888	0.0233	0.0254	–	–	–	0.0000
$p_{ER,\theta,e(3)}$	0.8626	0.0888	0.0233	0.0254	–	–	–	–

Table 5
Combination results of the new ER.

	$P(\varnothing)$	A	B	C	{A, B}	{A, C}	{B, C}	{A, B, C}	$P(\Theta)$
$m_{\theta,1}$	0.3500	0.4000	–	–	0.0500	0.0500	–	–	0.1500
$m_{\theta,2}$	0.2000	0.2000	0.1500	–	0.1000	–	0.0500	–	0.3000
$m_{\theta,3}$	0.4000	0.0300	0.0900	0.1500	–	–	–	0.0300	0.3000
$m_{\theta,e(2)}$	0.5260	0.3233	0.0356	0.0027	0.0384	0.0164	0.0082	–	0.0493
$p_{\theta,e(2)}$	–	0.7613	0.0839	0.0065	0.0903	0.0387	0.0194	–	
$m_{\theta,e(3)}$	0.7880	0.1316	0.0260	0.0137	0.0139	0.0060	0.0030	0.0016	0.0163
$p_{\theta,e(3)}$	–	0.6722	0.1327	0.0698	0.0712	0.0305	0.0153	0.0083	–

Table 6
Combination result of the ER with $r_1 = 0$.

	A	B	C	{A, B}	{A, C}	{B, C}	{A, B, C}	$P(\Theta)$
$m_{\theta,1}$	0.3294	–	–	0.0412	0.0412	–	–	0.5882
$m_{\theta,2}$	0.4000	0.3000	–	0.2000	–	0.1000	–	–
$m_{\theta,e(2)}$	0.5540	0.2254	0.0048	0.1471	–	0.0687	–	–
$p_{\theta,e(2)}$	0.5540	0.2254	0.0048	0.1471	–	0.0687	–	–

Table 7
Combination result of the new ER with $r_1 = 0$.

	$P(\varnothing)$	A	B	C	{A, B}	{A, C}	{B, C}	{A, B, C}	$P(\Theta)$
$m_{\theta,1}$	0.3500	–	–	–	–	–	–	–	0.6500
$m_{\theta,2}$	0.2000	0.2000	0.1500	–	0.1000	–	0.0500	–	0.3000
$m_{\theta,e(2)}$	0.4800	0.1300	0.0975	–	0.0650	–	0.0325	–	0.1950
$p_{\theta,e(2)}$	–	0.4000	0.3000	–	0.2000	–	0.1000	–	–

Let's examine the original and the generated data for {A, C}, {B, C} and {A, B, C} as adopt in literature (Yang & Xu, 2013). In the original data, each of them is only given a belief degree of 0.1 in e_1, e_2, and e_3 respectively, or $p_{\{A,C\},1} = 0.1$, $p_{\{B,C\},2} = 0.1$ and $p_{\{A,B,C\},2} = 0.1$ as shown in Table 1. It seems reasonable to expect that none of them should be given a larger degree of belief than 0.1 in the eventual fusion result after the three pieces of evidence are combined. In the result as shown in the last row of Table 8, there is $\widetilde{m}_{PCR5}(\{A, B, C\}) = 0.3506$, which is much larger than 0.1 even their sum is 0.3, inconsistent with the expectation. Besides, the uncertainty degree of combination result which can be reflected by the interval between the lower bound ($Bel(\theta) = \sum_{\vartheta\subseteq\theta} m(\vartheta)$) and the upper bound ($Pl(\theta) = \sum_{\vartheta\cap\theta\neq\varnothing} m(\vartheta)$). The uncertainty degree of the combination result in PCR5 is much larger than that in the new ER. For example, the lower bound and the upper bound of {A} is respectively 0.5334 and 0.9538(0.5334 + 0.0388 + 0.0310 + 0.3506) in the PCR5, while that is 0.7395 and 0.9000(0.7395 + 0.0912 + 0.0653 + 0.0040) in the new ER. The interval of the former is 0.9538–0.5334 = 0.4204, which is more than the interval of the latter with 0.9000–0.7395 = 0.1605. From the perspective of uncertainty degree of the combination result, the new ER is superior to the PCR5.

5. Illustrative example

The second round of National Marine Functional Zoning (2011–2020) has been launched by China for several years. It has played an important role in the development of marine economy and the construction of marine ecological civilization. In order to further raise the marine management level and rationally allocate marine resources, a scientific method should be established for evaluating implementation performance of a specified marine functional zoning (MFZ). The new ER method proposed in this paper is employed to solve such a MFZ evaluation problem as follows.

Suppose the government is responsible for the evaluation work and the criteria listed in Fig. 3 are utilized as the evaluation index system for MFZ. Each first class index in Fig. 3 is respectively regarded as a criterion, so the criteria can be denoted by c_i, $i = 1, \cdots, 6$. The government selects six experts $\{e_i | i = 1, \cdots, 6\}$ from different fields to participate in the evaluation. Expert e_i is responsible for giving the assessment information on criterion c_i in the overall view of all the second class indices included in c_i. Taking e_1 for example, he/she should investigate the overall implementation performance of the MFZ on such three aspects as functional area adjustment, reasonable marine demand, management enforcement, and then gives the assessment information on executive force (c_1). The assessment is made on five grades such as Excellent (E), Good (G), Average (A), Poor (P), and Worst (W), and thus the frame of discernment is constructed as $\Theta = \{\theta_1, \cdots, \theta_5\} = \{W, P, A, G, E\}$. The decision information given by expert e_i is profiled by b_i as in Eq. (3), which is listed in Eq. (19).

$$\begin{cases} b_1 = \{E, 0.2; A, 0.5; (G, P), 0.3\} \\ b_2 = \{(A, G), 0.7; (P, E), 0.3\} \\ b_3 = \{P, 0.5; (A, G), 0.5\} \\ b_4 = \{G, 0.7; (P, W), 0.3\} \\ b_5 = \{E, 0.6; A, 0.4\} \\ b_6 = \{G, 0.5; A, 0.4; (P, W), 0.1\} \end{cases} \tag{19}$$

The weights of criteria are given by the government, which are supposed to be $w_1 = w_2 = w_3 = 0.6$, $w_4 = w_5 = w_6 = 0.8$. The reliabilities

Table 8
Combination result of the PCR5.

	∅	A	B	C	{A, B}	{A, C}	{B, C}	{A, B, C}
$m_{r1,w1}(\theta)$	0.1000	0.5760	–	–	0.0720	0.0720	–	0.1800
$m_{r2,w2}(\theta)$	0.7000	0.0600	0.0450	–	0.0300	–	0.0150	0.1500
$m_{r3,w3}(\theta)$	0.4000	0.0120	0.0360	0.0600	–	–	–	0.4920
$m_{w1,r1}(\theta)$	0.0800	0.5760	–	–	0.0720	0.0720	–	0.2000
$m_{w2,r2}(\theta)$	0.3500	0.0600	0.0450	–	0.0300	–	0.0150	0.5000
$m_{w3,r3}(\theta)$	0.0800	0.0120	0.0360	0.0600	–	–	–	0.8120
$\widetilde{m}_{PCR5\varnothing,r,w}(\theta)$	–	0.5741	0.0254	0.0182	0.0311	0.0233	0.0032	0.3247
$\widetilde{m}_{PCR5\varnothing,w,r}(\theta)$	–	0.4927	0.0254	0.0182	0.0311	0.0233	0.0032	0.3765
$\widetilde{m}_{PCR5}(\theta)$	–	0.5334	0.0249	0.0182	0.0388	0.0310	0.0032	0.3506

Table 9
Combination result of the new ER.

	$P(\varnothing)$	A	B	C	{A, B}	{A, C}	{B, C}	{A, B, C}	$P(\Theta)$
$m_{\theta,1}$	0.4500	0.3200	–	–	0.0400	0.0400	–	–	0.1500
$m_{\theta,2}$	0.1500	0.1000	0.0750	–	0.0500	–	0.0250	–	0.6000
$m_{\theta,3}$	0.3000	0.0100	0.0300	0.0500	–	–	–	0.0100	0.6000
$m_{\theta,e(2)}$	0.5518	0.2746	0.0158	0.0010	0.0347	0.0249	0.0039	–	0.0933
$p_{\theta,e(2)}$	–	0.7737	0.0445	0.0029	0.0978	0.0701	0.0109	–	–
$m_{\theta,e(3)}$	0.7042	0.1763	0.0144	0.0070	0.0217	0.0156	0.0024	0.0010	0.0574
$p_{\theta,e(3)}$	–	0.7395	0.0606	0.0292	0.0912	0.0653	0.0102	0.0040	–

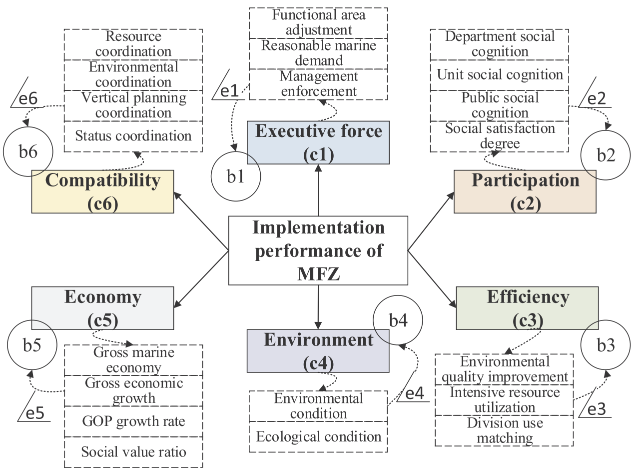

Fig. 3. The evaluation index system.

of experts are estimated by statistics and are supposed to be $r_1 = r_2 = 1.0$, $r_3 = r_4 = 0.8$, $r_5 = r_6 = 0.6$. If the b_i is regarded as a piece of evidence, with the weight w_i derived from criteria and the reliability r_i derived from experts, $i = 1, \cdots, 6$, then the evaluation problem on the MFZ is a MCGDM problem and can be solved by fusing all pieces of evidence.

Because the coefficients α and β are not given by the government, so they are determined by the suggested method as in Section 3.4. Taking the weights and the reliabilities into Eqs. (15a) and (15b), we have $\alpha = 0.5676$ and $\beta = 0.4324$. We take the BDs of evidence as listed in Eq. (19), weights, reliabilities, and the pair of coefficients into the combination algorithm by the new ER approach (Algorithm 1), and the combined degree of belief can be gradually obtained as follows.

The probability masses for six pieces of evidence are generated by discounting with both weights and reliabilities as in Eq. (9), and the discounted results are listed as in Eq. (20). Taking the first piece of evidence for example, w_1, r_1 and b_1 are substituted into Eq. (9), we have $m_{E,1} = \beta r_1 p_{E,1} = 0.4324 \times 1.0 \times 0.2 = 0.0865$, $m_{A,1} = \beta r_1 p_{A,1} = 0.4324 \times 1.0 \times 0.5 = 0.2162$, $m_{(P,G),1} = \beta r_1 p_{(P,G),1} = 0.4324 \times 1.0 \times 0.3 = 0.1297$, $m_{P(\varnothing),1} = \alpha w_1 = 0.5676 \times 0.6 = 0.3406$, $m_{P(\Theta),1} = \alpha(1-w_1) + \beta(1-r_1) = 0.5676 \times (1-0.6) + 0.4324 \times (1-1) = 0.2270$.

$$\begin{cases} m_1(\theta) = \{E,\ 0.0865; A,\ 0.2162; (P,\ G),\ 0.1297; P(\varnothing),\ 0.3406; P(\Theta),\ 0.2270\} \\ m_2(\theta) = \{(A,\ G),\ 0.3973; (P,\ E),\ 0.1703; P(\varnothing),\ 0.3406; P(\Theta),\ 0.0918\} \\ m_3(\theta) = \{P,\ 0.1730; (A,\ G),\ 0.1730; P(\varnothing),\ 0.3406; P(\Theta),\ 0.3135\} \\ m_4(\theta) = \{G,\ 0.2421; (W,\ P),\ 0.1038; P(\varnothing),\ 0.4541; P(\Theta),\ 0.2000\} \\ m_5(\theta) = \{E,\ 0.1557; A,\ 0.1038; P(\varnothing),\ 0.4541; P(\Theta),\ 0.2865\} \\ m_6(\theta) = \{G,\ 0.1297; A,\ 0.1038; (W,\ P),\ 0.0259; P(\varnothing),\ 0.4541; P(\Theta),\ 0.2865\} \end{cases} \tag{20}$$

Combining the probability masses of e_1 and e_2 by the new ER approach, we have $\widetilde{m}_{E,e(2)} = \sum_{B\cap C=E,B,C\subseteq\Theta} m_{B,e(1)} m_{C,2} + [\alpha(1-w_2) + \beta(1-r_2)] m_{E,e(1)} + m_{P(\Theta),e(1)} m_{E,2} = m_{E,e(1)} m_{(P,E),2} + [\alpha(1-w_2) + \beta(1-r_2)] m_{E,e(1)} = 0.0865 \times 0.1297 + [0.5676 \times (1-0.6) + 0.4324 \times (1-1)] \times 0.0865 \doteq 0.0344$; $\widetilde{m}_{A,e(2)} = m_{A,e(1)} m_{(A,G),2} + [\alpha(1-w_2) + \beta(1-r_2)] m_{A,e(1)} = 0.2162 \times 0.3027 + [0.5676 \times (1-0.6) + 0.4324 \times (1-1)] \times 0.2162 \doteq 0.1145$; $\widetilde{m}_{G,e(2)} = m_{(P,G),e(1)} m_{(A,G),2} = 0.1297 \times 0.3027 \doteq 0.0393$; $\widetilde{m}_{P,e(2)} = m_{(P,G),e(1)} m_{(P,E),2} = 0.1297 \times 0.1297 \doteq 0.0168$; $\widetilde{m}_{(P,G),e(2)} = [\alpha(1-w_2) + \beta(1-r_2)] m_{(P,G),e(1)} = [0.5676 \times (1-0.6) + 0.4324 \times (1-1)] \times 0.1297 \doteq 0.0294$; $\widetilde{m}_{(A,G),e(2)} = m_{P(\Theta),e(1)} m_{(A,G),2} = 0.2270 \times 0.3027 \doteq 0.0687$; $\widetilde{m}_{(P,E),e(2)} = m_{P(\Theta),e(1)} m_{(P,E),2} = 0.2270 \times 0.1297 \doteq 0.0294$; $\widetilde{m}_{P(\varnothing),e(2)} = \alpha w_2 + m_{P(\varnothing),e(1)} - \alpha w_2 m_{P(\varnothing),e(1)} = 0.5676 \times 0.6 + 0.3406 - 0.5676 \times 0.6 \times 0.3406 \doteq 0.5652$; $\widetilde{m}_{P(\Theta),e(2)} = [\alpha(1-w_2) + \beta(1-r_2)] m_{P(\Theta),e(1)} = [0.5676 \times (1-0.6) + 0.4324 \times (1-1)] \times 0.2270 \doteq 0.0515$. Then we use the equation $m_{\theta,e(i)} = \frac{\widetilde{m}_{\theta,e(i)}}{\sum_{B\subseteq\Theta} \widetilde{m}_{\theta,e(i)} + \widetilde{m}_{P(\Theta),e(i)} + \widetilde{m}_{P(\varnothing),e(i)}}$ to deal with the above probability masses for $\forall\, \theta \subseteq \Theta$, $\theta = P(\varnothing)$ and $\theta = P(\Theta)$. We obtain that $m_{E,e(2)} \doteq 0.0326$, $m_{A,e(2)} \doteq 0.1211$, $m_{G,e(2)} \doteq 0.0415$, $m_{P,e(2)} \doteq 0.0178$, $m_{(P,G),e(2)} \doteq 0.0311$, $m_{(A,G),e(2)} \doteq 0.0727$, $m_{(P,E),e(2)} \doteq 0.0311$, $m_{P(\varnothing),e(2)} \doteq 0.5976$, $m_{P(\Theta),e(2)} \doteq 0.0545$.

It is similar to the combination process of $m_{\theta,e(2)}$, we can obtain $m_{\theta,e(3)}$, $m_{\theta,e(4)}$, $m_{\theta,e(5)}$ and $m_{\theta,e(6)}$, for $\forall\, \theta \subseteq \Theta$, $\theta = P(\varnothing)$ and $\theta = P(\Theta)$. Taking $m_{\theta,e(6)}$ into Eq. (14), the combined degree of belief is able to be computed and the result is as follows. $p_{G,e(6)} = 0.4225$, $p_{A,e(6)} = 0.2851$, $p_{P,e(6)} = 0.1065$, $p_{E,e(6)} = 0.0534$, $p_{(W,P),e(6)} = 0.0197$, $p_{(A,G),e(6)} = 0.0787$, $p_{(P,G),e(6)} = 0.0171$, $p_{(P,E),e(6)} = 0.0171$. In order to distribute the combined result on each grade of frame of discernment, the pignistic probability is computed by taking the combined degree of belief into Eq. (8) and we have $\gamma(W) = 0.0023$, $\gamma(P) = 0.1252$, $\gamma(A) = 0.3277$, $\gamma(G) = 0.4832$, $\gamma(E) = 0.0616$.

The eventually combined result shows that the overall implementation performance of the MFZ by six experts/criteria has 23% probability to be *Worst*, 12.52% probability to be *Poor*, 32.77% probability to be *Average*, 48.32% probability to be *Good*, and 6.16% probability to be *Excellent*. According to the principle of maximum membership, the assessment grade *Good* is the final evaluation result. It is easy to find that the new ER approach can be solved recursively and can be programmed as shown in Algorithm 1, thus the proposed approach in this work is valid and applicable.

6. Conclusions

The ER approach with both weight and reliability has two aspects of problems such as weight over-bounding and reliability-dependence. The reason why there exists above two aspects of problems in the ER approach is that the characteristics of weight and reliability are not well considered in the process of evidence discounting and combining. In this paper, we investigate the characteristics of the weight and the reliability, and find that the reliability of evidence is objective and absolute to reflect information quality, while the weight of evidence is subjective and relative to reflect information importance. A new discounting method with both weight and reliability is defined to generate probability masses for the evidence by assigning the residual support of weight to the empty set and that of reliability to the power set. On the basis of the new discounting method, we use the orthogonal sum operation to establish a new ER combination rule with both reliability and weight for recursively combining the evidence. The new ER combination rule consists of the subjective fusion with weight and the objective fusion with reliability which are integrated by a pair of fusion coefficients, and a series of theorems and corollaries are introduced and proved. Numerical comparisons are introduced to compare the new ER approach with the DST, the ER, and the PCR5, and illustrate the superiority of the new ER approach. An illustrative example is provided to demonstrate the applicabilities of the proposed combination rules and algorithm. The new ER combination rule cannot only maintain the specificity of the evidence but also solve the problems such as weight over-bounding and reliability-dependence.

We have to point out that: (1) The weights sometimes can be determined by the objective methods, in which the weight-determination thoughts should comply with some subjective principles given by decision-makers. For examples, the subjective principle may be that the weights should lessen the separation between each alternative and the ideal one as much as possible (Ma, Fan, & Huang, 1999); the weights should reflect the disorder degrees of data denoted by information entropy, the greater the entropy, the lower the weight (Zhou, Lin, Deng, Li, & Liu, 2016); the weights should be favorable for each decision making unit as much as possible (Pendharkar, 2018). In other words, weight-determination thoughts should be subjectively goal-oriented according to the needs of decision makers, based on which the weights may be determined by the objective data. From this viewpoint, the weight of evidence also can be regarded to be subjective in the objective methods. (2) This work only studies how to scientifically fuse the evidence with both weights and reliabilities from the static perspective, however, sometimes contentious meetings of what might be described as BOGSATs (Bunch of Guys/Gals Sitting Around A Table) may happen. Further studying the evidence combination in dynamic situation would be a good research direction in the future.

Acknowledgement

This research was supported by the National Natural Science Foundation of China (NSFC) under Grant No. 71462022 and 61773123, the Fundamental Research Funds for the Central Universities under Grant No. 201762026, the Special Funds of Taishan Scholars Project of Shandong Province under Grant No. tsqn20171205, and Humanities and Social Science Foundation of Ministry of Education under Grant No. 17YJC790120. The authors would like to thank four anonymous referees and the Editor-in-Chief for their valuable comments and suggestions that help to improve the quality of the paper to its current standard.

Appendix A

A.1. Proof of Theorem 1

Proof. Taking the expression of $m_{\theta,i}, m_{P(\varnothing),i}$ and $m_{P(\Theta),i}$ of Eq. (9) into $\sum_{\theta\subseteq\Theta} m_{\theta,i}+m_{\varnothing,i}+m_{P(\theta),i}$, we have $\sum_{\theta\subseteq\Theta} m_{\theta,i}+m_{P(\varnothing),i}+m_{P(\Theta),i}=\sum_{\theta\subseteq\Theta}\beta r_i p_{\theta,i}+\alpha w_i+\alpha(1-w_i)+\beta(1-r_i)=\beta r_i\sum_{\theta\subseteq\Theta}p_{\theta,i}+\alpha w_i+\alpha-\alpha w_i+\beta-\beta r_i$. Since $\sum_{\theta\subseteq\Theta}p_{\theta,i}=1$ and $\alpha+\beta=1$, so as to $\sum_{\theta\subseteq\Theta} m_{\theta,i}+m_{P(\varnothing),i}+m_{P(\Theta),i}=\beta r_i+\alpha+\beta-\beta r_i=\alpha+\beta=1$. □

A.2. Proof of Theorem 2

Proof. From Theorem 1, we know $\sum_{\theta\subseteq\Theta} m_{\theta,i}+m_{P(\Theta),i}+m_{P(\varnothing),i}=1$, $i=1,2$.We make an orthogonal sum operation for $m_{\theta,1}$ and $m_{\theta,2}$, and get four parts as following: $\widetilde{m}_{\theta,e(2)}=\sum_{B\cap C=\theta,B,C\subseteq\Theta} m_{B,1}m_{C,2}+m_{P(\Theta),1}m_{\theta,2}+m_{P(\Theta),2}m_{\theta,1}$ for $\forall\,\theta\subseteq\Theta$, $\widetilde{m}_{P(\Theta),e(2)}=m_{P(\Theta),1}m_{P(\Theta),2}$ for $\theta=P(\Theta)$, $\widetilde{m}_{P(\varnothing),e(2)}=m_{P(\varnothing),1}(m_{P(\varnothing),2}+\sum_{\theta\subseteq\Theta} m_{\theta,2}+m_{P(\Theta),2})+m_{P(\varnothing),2}(\sum_{\theta\subseteq\Theta} m_{\theta,1}+m_{P(\Theta),1})$ for $\theta=P(\varnothing)$, $\widetilde{m}_{\varnothing,e(2)}=\sum_{B\cap C=\varnothing,B,C\subseteq\Theta} m_{B,1}m_{C,2}$ for $\theta=\varnothing$. If there are two parts with the sum of each part being equal to 1, then the orthogonal sum for the two parts must be equal to 1.Thus there is $\sum_{\theta\subseteq\Theta}\widetilde{m}_{\theta,e(2)}+\widetilde{m}_{P(\Theta),e(2)}+\widetilde{m}_{P(\varnothing),e(2)}+\widetilde{m}_{\varnothing,e(2)}=1$ (Yang & Xu, 2013).Let $k=\widetilde{m}_{\varnothing,e(2)}=\sum_{B\cap C=\varnothing,B,C\subseteq\Theta} m_{B,1}m_{C,2}$, $m_{\theta,e(2)}=\widetilde{m}_{\theta,e(2)}/(1-k)$ for $\theta\subseteq\Theta,\theta=P(\Theta)$ and $\theta=P(\varnothing)$. Because $\sum_{\theta\subseteq\Theta}\widetilde{m}_{\theta,e(2)}+\widetilde{m}_{P(\Theta),e(2)}+\widetilde{m}_{P(\varnothing),e(2)}=1-\widetilde{m}_{\varnothing,e(2)}=1-k$, there is $\sum_{\theta\subseteq\Theta} m_{\theta,e(2)}+m_{P(\Theta),e(2)}+m_{P(\varnothing),e(2)}=[\sum_{\theta\subseteq\Theta}\widetilde{m}_{\theta,e(2)}+\widetilde{m}_{P(\Theta),e(2)}+\widetilde{m}_{P(\varnothing),e(2)}]/(1-k)=(1-k)/(1-k)=1$. □

A.3. Proof of Theorem 3

Proof. From Eq. (3), we get the discounted results on $P(\Theta)$ of e_1 and e_2 are $m_{P(\Theta),1}=\alpha(1-w_1)+\beta(1-r_1)$ and $m_{P(\Theta),2}=\alpha(1-w_2)+\beta(1-r_2)$. Let the numerator in Eq. (10a) be $\widetilde{m}_{\theta,e(2)}=\sum_{B\cap C=\theta,B,C\subseteq\Theta} m_{B,1}m_{C,2}+m_{P(\Theta),1}m_{\theta,2}+m_{P(\Theta),2}m_{\theta,1}$, and take the expressions of $m_{P(\Theta),1}$ and $m_{P(\Theta),2}$ into $\widetilde{m}_{\theta,e(2)}$. We get $\widetilde{m}_{\theta,e(2)}=\sum_{B\cap C=\theta,B,C\subseteq\Theta} m_{B,1}m_{C,2}+m_{P(\Theta),1}m_{\theta,2}+m_{P(\Theta),2}m_{\theta,1}=\sum_{B\cap C=\theta,B,C\subseteq\Theta} m_{B,1}m_{C,2}+[\alpha(1-w_1)+\beta(1-r_1)]m_{\theta,2}+[\alpha(1-w_2)+\beta(1-r_2)]m_{\theta,1}$ $=\sum_{B\cap C=\theta,B,C\subseteq\Theta} m_{B,1}m_{C,2}+[\beta(1-r_1)m_{\theta,2}+\beta(1-r_2)m_{\theta,1}]+[\alpha(1-w_1)m_{\theta,2}+\alpha(1-w_2)m_{\theta,1}]$. Since $\sum_{\theta\subseteq\Theta} m_{\theta,i}+m_{P(\varnothing),i}+m_{P(\Theta),i}=1$ from Theorem 1, $i=1,2$, and $\sum_{\theta\subseteq\Theta} m_{\theta,e(2)}+m_{P(\Theta),e(2)}+m_{P(\varnothing),e(2)}=1$ from Theorem 2, as a result, there exists $\sum_{\theta\subseteq\Theta} m_{\theta,e(2)}=1-m_{P(\Theta),e(2)}-m_{P(\varnothing),e(2)}$. Besides, take Eq. (12b) into Eq. (10a), we can easily get $m_{\theta,e(2)}=\widetilde{m}_{\theta,e(2)}/(1-k)$. At last, take the expressions of $\sum_{\theta\subseteq\Theta} m_{\theta,e(2)}$ and $m_{\theta,e(2)}$ into Eq. (10), we can get $p_{\theta,e(2)}=\frac{m_{\theta,e(2)}}{1-m_{P(\Theta),e(2)}-m_{P(\varnothing),e(2)}}=\frac{m_{\theta,e(2)}}{\sum_{\theta\subseteq\Theta} m_{\theta,e(2)}}=\frac{\widetilde{m}_{\theta,e(2)}/(1-k)}{\sum_{\theta\subseteq\Theta}\widetilde{m}_{\theta,e(2)}/(1-k)}=\frac{\widetilde{m}_{\theta,e(2)}}{\sum_{B\subseteq\Theta}\widetilde{m}_{B,e(2)}}$, $\sum_{\theta\subseteq\Theta}p_{\theta,e(2)}=\sum_{\theta\subseteq\Theta}\frac{\widetilde{m}_{\theta,e(2)}}{\sum_{B\subseteq\Theta}\widetilde{m}_{B,e(2)}}=\frac{\sum_{\theta\subseteq\Theta}\widetilde{m}_{\theta,e(2)}}{\sum_{B\subseteq\Theta}\widetilde{m}_{B,e(2)}}=1$. Because $p_{\theta,e(2)}\geqslant 0$ and $\sum_{\theta\subseteq\Theta}p_{\theta,e(2)}=1$, we can get $0\leqslant p_{\theta,e(2)}\leqslant 1$, $\forall\,\theta\subseteq\Theta$. □

A.4. Proof of Theorem 4

Proof. For $i=2$, since $m_{\theta,e(1)}=m_{\theta,1}$ for $\theta\subseteq\Theta,\theta=P(\Theta)$ and $\theta=P(\varnothing)$, Eq.(13b) becomes Eq.(12b), Eq.(13c) becomes $\widetilde{m}_{P(\Theta),e(2)}=m_{P(\Theta),e(1)}m_{P(\Theta),2}=[\alpha(1-w_1)+\beta(1-r_1)][\alpha(1-w_2)+\beta(1-r_2)]$, and (13d) becomes $\widetilde{m}_{P(\varnothing),e(2)}=\alpha w_2+m_{P(\varnothing),e(1)}-\alpha w_2 m_{P(\varnothing),e(1)}=\alpha w_2+m_{P(\varnothing),1}-\alpha w_2 m_{P(\varnothing),1}=\alpha w_2+\alpha w_1-\alpha^2 w_1 w_2$. They are as the same as that in the proof of Theorem 2. Besides, Eq.(13a) is true with $0\leqslant m_{\theta,e(2)}\leqslant 1$ for $\forall\,\theta\subseteq\Theta,\theta=P(\Theta),\theta=P(\varnothing)$, $\sum_{\theta\subseteq\Theta} m_{\theta,e(2)}+m_{P(\Theta),e(2)}+m_{P(\varnothing),e(2)}=1$ also has been proved in Theorem 2. Suppose for $i=i-1$, Eqs.(13a)–(13d) are true, that is $m_{\theta,e(i-1)}=[m_1\oplus\cdots\oplus m_{i-1}](\theta)$, with $0\leqslant m_{\theta,e(i-1)}\leqslant 1$ for $\forall\,\theta\subseteq\Theta,\theta=P(\Theta),\theta=P(\varnothing)$ and$\sum_{\theta\subseteq\Theta} m_{\theta,e(i-1)}+m_{P(\Theta),e(i-1)}+m_{P(\varnothing),e(i-1)}=1$. For $i=i$, since the orthogonal sum operation is independent of the order, we have $m_{\theta,e(i)}=[m_1\oplus\cdots\oplus m_{i-1}](\theta)\oplus m_{\theta,i}=[(m_1\oplus\cdots\oplus m_{i-1})\oplus m_i](\theta)$. The above equation means that combining i pieces of evidence is equal to combining the first $i-1$ pieces with the i^{th} piece. The orthogonal sum of $m_{\theta,e(i-1)}$ and $m_{\theta,i}$ without normalization leads to $\widetilde{m}_{\theta,e(i)} = \sum_{B\cap C=\theta,B,C\subseteq\Theta} m_{B,e(i-1)}m_{C,i} + [m_{P(\Theta),i}m_{\theta,e(i-1)} + m_{P(\Theta),e(i-1)}m_{\theta,i}] = \sum_{B\cap C=\theta,B,C\subseteq\Theta} m_{B,e(i-1)}m_{C,i} + [\alpha(1-w_i) + \beta(1-r_i)]m_{\theta,e(i-1)} + m_{P(\Theta),e(i-1)}m_{\theta,i}$, $\widetilde{m}_{P(\Theta),e(i)} = m_{P(\Theta),i}m_{P(\Theta),e(i-1)} = [\alpha(1-w_i) + \beta(1-r_i)]m_{P(\Theta),e(i-1)}$. Since $\sum_{\theta\subseteq\Theta} m_{\theta,e(i-1)} + m_{P(\Theta),e(i-1)} + m_{P(\varnothing),e(i-1)} = 1$ and $\sum_{\theta\subseteq\Theta} m_{\theta,i} + m_{P(\Theta),i} + m_{P(\varnothing),i} = 1$, there is $\sum_{\theta\subseteq\Theta} m_{\theta,i} + m_{P(\Theta),i} = 1-m_{P(\varnothing),i}$. Thus $\widetilde{m}_{P(\varnothing),e(i)} = m_{P(\varnothing),i}(\sum_{\theta\subseteq\Theta} m_{\theta,e(i-1)} + m_{P(\Theta),e(i-1)} + m_{P(\varnothing),e(i-1)}) + m_{P(\varnothing),e(i-1)}(\sum_{\theta\subseteq\Theta} m_{\theta,i} + m_{P(\Theta),i}) = m_{P(\varnothing),i} + m_{P(\varnothing),e(i-1)}(1-m_{P(\varnothing),i}) = m_{P(\varnothing),i} + m_{P(\varnothing),e(i-1)}-m_{P(\varnothing),e(i-1)}m_{P(\varnothing),i} = \alpha w_i + m_{P(\varnothing),e(i-1)}-\alpha w_i m_{P(\varnothing),e(i-1)}$, $\widetilde{m}_{\varnothing,e(i)} = \sum_{B\cap C=\varnothing,B,C\subseteq\Theta} m_{B,e(i-1)}m_{C,i} = k$. There must be $\sum_{\theta\subseteq\Theta} \widetilde{m}_{\theta,e(i)} + \widetilde{m}_{P(\Theta),e(i)} + \widetilde{m}_{P(\varnothing),e(i)} + \widetilde{m}_{\varnothing,e(i)} = 1$, and $\sum_{\theta\subseteq\Theta} \widetilde{m}_{\theta,e(i)} + \widetilde{m}_{P(\Theta),e(i)} + \widetilde{m}_{P(\varnothing),e(i)} = 1-k$. Conflict factor k is the probability mass of empty set and its influence should be eliminated by normalizing to keep characteristics of other expressions unchanged as mentioned above. Thus let $m_{\theta,e(i)} = \widetilde{m}_{\theta,e(i)}/(1-k)$ for $\forall\ \theta \subseteq \Theta$, $\theta = P(\Theta)$, $\theta = P(\varnothing)$, we get $m_{\theta,e(i)} = \widetilde{m}_{\theta,e(i)}/(1-k) = \widetilde{m}_{\theta,e(i)}/[\sum_{\theta\subseteq\Theta} \widetilde{m}_{\theta,e(i)} + \widetilde{m}_{P(\Theta),e(i)} + \widetilde{m}_{P(\varnothing),e(i)}]$ for $\forall\ \theta \subseteq \Theta$, $\theta = P(\Theta)$, $\theta = P(\varnothing)$; $\sum_{\theta\subseteq\Theta} m_{\theta,e(i)} + m_{P(\Theta),e(i)} + m_{P(\varnothing),e(i)} = [\sum_{\theta\subseteq\Theta} \widetilde{m}_{\theta,e(i)} + \widetilde{m}_{P(\Theta),e(i)} + \widetilde{m}_{P(\varnothing),e(i)}]/(1-k) = (1-k)/(1-k) = 1$. Since $\widetilde{m}_{\theta,e(i)}$ is non-negative so as to $m_{\theta,e(i)} \geqslant 0$ for $\forall\ \theta \subseteq \Theta$, $\theta = P(\Theta)$, $\theta = P(\varnothing)$. Besides $\sum_{\theta\subseteq\Theta} m_{\theta,e(i)} + m_{P(\Theta),e(i)} + m_{P(\varnothing),e(i)} = 1$, thus there must be $0 \leqslant m_{\theta,e(i)} \leqslant 1$ for $\forall\ \theta \subseteq \Theta$, $\theta = P(\Theta)$, $\theta = P(\varnothing)$. □

A.5. Proof of Theorem 5

Proof. Similar to the proof of Theorem 3, it is straightforward to be proved. □

A.6. Proof of Theorem 6

Proof. Taking Eqs. (15a) and (15b) into $\alpha + \beta$, we have $\alpha + \beta = 1$ directly.

Case 1: When all pieces of evidence are the most reliable and the same important, there are $r_i = 1$ $(i = 1, \cdots, I)$ and $w_i = w_{i'}$ $(i, i' = 1, \cdots, I)$. Taking the above two parameters into Eqs. (15a) and (15b), we have $\alpha = \beta = 0.5$.

Case 2: When all pieces of evidence are the most reliable but their importance degrees are different, there are $r_i = 1$ $(\forall\ i)$ and $\exists\ w_{i'} < \max(w_i|\ i \neq i')$. We have

$$\begin{cases} \sum_i \ [(r_i)^2/\max(r_i|\ \forall\ i)] = I \\ \sum_i \ [w_i/\max(w_i|\ \forall\ i)\ ] = w_{i'}/\max(w_i|i \neq i') + \sum_{i\neq i'} \ [w_i/\max(w_i|i \neq i')] \end{cases}$$

If let $\sum_i [w_i/\max(w_i|\ \forall\ i)] = \mu$, then $0 < \mu < I$. Taking the above result into Eq. (15b), there is $\alpha = \mu/(\mu + I) = 1-I/(\mu + I)$. Since $0 < \mu < I$, there exist the following relationships, i.e., $I < (\mu + I) < 2I$, $1/(2I) < 1/(\mu + I) < 1/I$, $I/(2I) < I/(\mu + I) < I/I$, $1/2 < I/(\mu + I) < 1$. From $\alpha = 1-I/(\mu + I)$, we have $0 < \alpha < 0.5$. From $\alpha + \beta = 1$, we have $0.5 < \beta < 1$.

Case 3: If all pieces of evidence are the same important but their reliabilities are different, there are $w_i = 1/I$ $(\forall\ i)$ and $\exists\ r_{i'} < \max(r_i|i \neq i')$, so

$$\begin{cases} \sum_i \ [w_i/\max(w_i|\ \forall\ i)] = I \\ \sum_i \ [(r_i)^2/\max(r_i|\ \forall\ i)] = (r_{i'})^2/\max(r_i|i \neq i') + \sum_{i\neq i'} \ [(r_i)^2/\max(r_i|i \neq i')] \end{cases}$$

If let $\sum_i [(r_i)^2/\max(r_i|\ \forall\ i)] = \eta$, then there must be $0 < \eta < I$. Taking the above result into Eq. (15b), there is $\beta = \eta/(\eta + I) = 1-I/(\eta + I)$. Since $0 < \eta < I$, there exist the following relationships, i.e., $I < (\eta + I) < 2I$, $1/(2I) < 1/(\eta + I) < 1/I$, $I/(2I) < I/(\eta + I) < I/I$, $1/2 < I/(\eta + I) < 1$. From $\beta = 1-I/(\eta + I)$, we have $0 < \beta < 0.5$. From $\alpha + \beta = 1$, we have $0.5 < \alpha < 1$. □

A.7. Proof of Corollary 1

Proof. From Eqs. (9) and (12b), we get

$\widetilde{m}_{\theta,e(2)} = \sum_{B\cap C=\theta,B,C\subseteq\Theta} m_{B,1}m_{C,2} + [\beta(1-r_1)m_{\theta,2} + \beta(1-r_2)m_{\theta,1}] + [\alpha(1-w_1)m_{\theta,2} + \alpha(1-w_2)m_{\theta,1}] = \sum_{B\cap C=\theta,B,C\subseteq\Theta} \beta r_1 p_{B,1}\beta r_2 p_{C,2} + [\beta(1-r_1)\beta r_2 p_{\theta,2} + \beta(1-r_2)\beta r_1 p_{\theta,1}] + [\alpha(1-w_1)\beta r_2 p_{\theta,2} + \alpha(1-w_2)\beta r_1 p_{\theta,1}] = \beta\{\beta\sum_{B\cap C=\theta,B,C\subseteq\Theta} r_1 p_{B,1} r_2 p_{C,2} + [\beta(1-r_1)r_2 p_{\theta,2} + \beta(1-r_2)r_1 p_{\theta,1}] + [\alpha(1-w_1)r_2 p_{\theta,2} + \alpha(1-w_2)r_1 p_{\theta,1}]\}$. Let $\widetilde{N}_{\theta,e(2)} = \beta\sum_{B\cap C=\theta,B,C\subseteq\Theta} r_1 p_{B,1} r_2 p_{C,2} + [\beta(1-r_1)r_2 p_{\theta,2} + \beta(1-r_2)r_1 p_{\theta,1}] + [\alpha(1-w_1)r_2 p_{\theta,2} + \alpha(1-w_2)r_1 p_{\theta,1}]$, so $\widetilde{m}_{\theta,e(2)} = \beta\widetilde{N}_{\theta,e(2)}$. Since $\alpha + \beta = 1$ and $\alpha, \beta > 0$, there is $\beta \to 1$ when $\alpha \to 0$. $\lim_{\alpha\to0,\beta\to1}\widetilde{N}_{\theta,e(2)} = \lim_{\alpha\to0,\beta\to1}\{\beta\sum_{B\cap C=\theta,B,C\subseteq\Theta} r_1 p_{B,1} r_2 p_{C,2} + [\beta(1-r_1)r_2 p_{\theta,2} + \beta(1-r_2)r_1 p_{\theta,1}] + [\alpha(1-w_1)r_2 p_{\theta,2} + \alpha(1-w_2)r_1 p_{\theta,1}]\} = (\lim_{\beta\to1}\beta)\sum_{B\cap C=\theta,B,C\subseteq\Theta} r_1 p_{B,1} r_2 p_{C,2} + (\lim_{\beta\to1}\beta)[(1-r_1)r_2 p_{\theta,2} + (1-r_2)r_1 p_{\theta,1}] + (\lim_{\alpha\to0}\alpha)[(1-w_1)r_2 p_{\theta,2} + (1-w_2)r_1 p_{\theta,1}] = r_1 r_2\sum_{B\cap C=\theta,B,C\subseteq\Theta} p_{B,1}p_{C,2} + [(1-r_1)r_2 p_{\theta,2} + (1-r_2)r_1 p_{\theta,1}]$. Let $\widetilde{n}_{\theta,e(2)} = r_1 r_2\sum_{B\cap C=\theta,B,C\subseteq\Theta} p_{B,1}p_{C,2} + [(1-r_1)r_2 p_{\theta,2} + (1-r_2)r_1 p_{\theta,1}]$. Thus when the coefficient for subjective fusion $\alpha \to 0$, there is $p_{\theta,e(2)} = \lim_{\alpha\to0,\beta\to1}\frac{\widetilde{m}_{\theta,e(2)}}{\sum_{B\subseteq\Theta}\widetilde{m}_{B,e(2)}} = \lim_{\alpha\to0,\beta\to1}\frac{\beta\widetilde{N}_{\theta,e(2)}}{\sum_{\theta\subseteq\Theta}\beta\widetilde{N}_{\theta,e(2)}} = \lim_{\alpha\to0,\beta\to1}\frac{\widetilde{N}_{\theta,e(2)}}{\sum_{\theta\subseteq\Theta}\widetilde{N}_{\theta,e(2)}} = \frac{\lim_{\alpha\to0,\beta\to1}\widetilde{N}_{\theta,e(2)}}{\sum_{\theta\subseteq\Theta}(\lim_{\alpha\to0,\beta\to1}\widetilde{N}_{\theta,e(2)})} = \frac{\widetilde{n}_{\theta,e(2)}}{\sum_{B\subseteq\Theta}\widetilde{n}_{B,e(2)}}$.□

A.8. Proof of Corollary 2

Proof. As shown in the proof of Corollary 1, there is $\widetilde{m}_{\theta,e(2)} = \beta\widetilde{N}_{\theta,e(2)}$,$\widetilde{N}_{\theta,e(2)} = \beta\sum_{B\cap C=\theta,B,C\subseteq\Theta} r_1 p_{B,1} r_2 p_{C,2} + [\beta(1-r_1)r_2 p_{\theta,2} + \beta(1-r_2)r_1 p_{\theta,1}] + [\alpha(1-w_1)r_2 p_{\theta,2} + \alpha(1-w_2)r_1 p_{\theta,1}]$. Since $\alpha + \beta = 1$ and $\alpha, \beta > 0$, there is $\alpha \to 1$ when $\beta \to 0$. $\lim_{\alpha\to1,\beta\to0}\widetilde{N}_{\theta,e(2)} = \lim_{\alpha\to1,\beta\to0}\{\beta\sum_{B\cap C=\theta,B,C\subseteq\Theta} r_1 p_{B,1} r_2 p_{C,2} + [\beta(1-r_1)r_2 p_{\theta,2} + \beta(1-r_2)r_1 p_{\theta,1}] + [\alpha(1-w_1)r_2 p_{\theta,2} + \alpha(1-w_2)r_1 p_{\theta,1}]\} = (\lim_{\beta\to0}\beta)\sum_{B\cap C=\theta,B,C\subseteq\Theta} r_1 p_{B,1} r_2 p_{C,2} + (\lim_{\beta\to0}\beta)[(1-r_1)r_2 p_{\theta,2} + (1-r_2)r_1 p_{\theta,1}] + (\lim_{\alpha\to1}\alpha)[(1-w_1)r_2 p_{\theta,2} + (1-w_2)r_1 p_{\theta,1}] = (1-w_1)r_2 p_{\theta,2} + (1-w_2)r_1 p_{\theta,1}$. Let $\widetilde{n}_{\theta,e(2)} = (1-w_1)r_2 p_{\theta,2} + (1-w_2)r_1 p_{\theta,1}$. Thus when the coefficient for subjective fusion $\beta \to 0$, there is $p_{\theta,e(2)} = \lim_{\alpha\to1,\beta\to0}\frac{\widetilde{m}_{\theta,e(2)}}{\sum_{B\subseteq\Theta}\widetilde{m}_{B,e(2)}} = \lim_{\alpha\to1,\beta\to0}\frac{\beta\widetilde{N}_{\theta,e(2)}}{\sum_{\theta\subseteq\Theta}\beta\widetilde{N}_{\theta,e(2)}} = \lim_{\alpha\to1,\beta\to0}\frac{\widetilde{N}_{\theta,e(2)}}{\sum_{\theta\subseteq\Theta}\widetilde{N}_{\theta,e(2)}} = \frac{\lim_{\alpha\to1,\beta\to0}\widetilde{N}_{\theta,e(2)}}{\sum_{\theta\subseteq\Theta}(\lim_{\alpha\to1,\beta\to0}\widetilde{N}_{\theta,e(2)})} = \frac{\widetilde{n}_{\theta,e(2)}}{\sum_{B\subseteq\Theta}\widetilde{n}_{B,e(2)}}$. □

A.9. Proof of Corollary 3

Proof. Since $\alpha + \beta = 1$ and $\alpha, \beta > 0$, there is $\alpha \to 0$ when $\beta \to 1$. For $i = 2$, take $n_{\theta,e(1)} = n_{\theta,1} = r_1 p_{\theta,1}$, $n_{P(\Theta),e(1)} = n_{P(\Theta),1} = 1 - r_1$, and $n_{\theta,2} = r_2 p_{\theta,2}$ into Eq. (15b), we have $\tilde{n}_{\theta,e(2)} = \sum_{B\cap C=\theta,B,C\subseteq\Theta} n_{B,e(1)} n_{C,2} + [(1-r_2) n_{\theta,e(1)} + n_{P(\Theta),e(1)} n_{\theta,2}] = \sum_{B\cap C=\theta,B,C\subseteq\Theta} r_1 p_{B,1} r_2 p_{C,2} + [(1-r_2) r_1 p_{\theta,1} + (1-r_1) r_2 p_{\theta,2}]$. This is consistent with Eq. (16b) in Corollary 1. From Eq. (13c), when $\alpha \to 0$ and $\beta \to 1$ there is $\widetilde{m}_{P(\Theta),e(2)} = \lim_{\alpha\to0,\beta\to1}[\alpha(1-w_2) + \beta(1-r_2)] m_{P(\Theta),e(1)} = \lim_{\alpha\to0,\beta\to1}[\alpha(1-w_2) + \beta(1-r_2)] m_{P(\Theta),1} = \lim_{\alpha\to0,\beta\to1}[\alpha(1-w_2) + \beta(1-r_2)][\alpha(1-w_1) + \beta(1-r_1)] = (1-r_1)(1-r_2)$. This is consistent with Eq. (17c) as $\tilde{n}_{P(\Theta),e(2)} = (1-r_2) n_{P(\Theta),e(1)} = (1-r_2) n_{P(\Theta),1} = (1-r_1)(1-r_2)$. From Eq. (13d), when $\alpha \to 0$ and $\beta \to 1$ there is $\widetilde{m}_{P(\varnothing),e(2)} = \lim_{\alpha\to0,\beta\to1}[\alpha w_2 + m_{P(\varnothing),e(1)} - \alpha w_2 m_{P(\varnothing),e(1)}] = \lim_{\alpha\to0,\beta\to1}[\alpha w_2 + \alpha w_1 - \alpha^2 w_1 w_2] = 0$. So here we get $m_{\theta,e(2)} = \lim_{\alpha\to0,\beta\to1} \frac{\widetilde{m}_{\theta,e(2)}}{\sum_{B\subseteq\Theta} \widetilde{m}_{\theta,e(2)} + \widetilde{m}_{P(\Theta),e(2)} + \widetilde{m}_{P(\varnothing),e(2)}} = \frac{\tilde{n}_{\theta,e(2)}}{\sum_{B\subseteq\Theta} \tilde{n}_{\theta,e(2)} + \tilde{n}_{P(\Theta),e(2)}}$. Without loss of generality, $\lim_{\alpha\to0,\beta\to1} \widetilde{m}_{P(\varnothing),e(2)} = 0$ is omitted in this corollary. Suppose for $i = i$, Eqs. (17a)–(17c) are true when $\alpha \to 0$ and $\beta \to 1$. As presented in Corollary 1, Eq. (17b) is true in this situation means $\lim_{\alpha\to0,\beta\to1} \widetilde{m}_{\theta,e(i)} = \tilde{n}_{\theta,e(i)}$ for $\theta \subseteq \Theta$ and $\lim_{\alpha\to0,\beta\to1} p_{\theta,e(i)} = \lim_{\alpha\to0,\beta\to1} \frac{\widetilde{m}_{\theta,e(i)}}{\sum_{B\subseteq\Theta} \widetilde{m}_{\theta,e(i)}} = \frac{\tilde{n}_{\theta,e(i)}}{\sum_{B\subseteq\Theta} \tilde{n}_{B,e(i)}}$. Eq. (17c) is true means $\lim_{\alpha\to0,\beta\to1} \widetilde{m}_{P(\Theta),e(i)} = \tilde{n}_{P(\Theta),e(i)}$, Eq. (17a) is true means $\lim_{\alpha\to0,\beta\to1} \widetilde{m}_{P(\varnothing),e(i)} = \tilde{n}_{P(\varnothing),e(i)} = 0$. So we get $\lim_{\alpha\to0,\beta\to1}[\sum_{\theta\subseteq\Theta} \widetilde{m}_{\theta,e(i)} + \widetilde{m}_{P(\Theta),e(i)} + \widetilde{m}_{P(\varnothing),e(i)}] = \sum_{\theta\subseteq\Theta} \tilde{n}_{\theta,e(i)} + \tilde{n}_{P(\Theta),e(i)}$. For $i = i+1$, when $\alpha \to 0$ and $\beta \to 1$ there is $\widetilde{m}_{\theta,e(i+1)} = \lim_{\alpha\to0,\beta\to1}\{\sum_{B\cap C=\theta,B,C\subseteq\Theta} m_{B,e(i)} m_{C,i+1} + [m_{P(\Theta),i+1} m_{\theta,e(i)} + m_{P(\Theta),e(i)} m_{\theta,i+1}]\} = \lim_{\alpha\to0,\beta\to1}\left\{\sum_{B\cap C=\theta,B,C\subseteq\Theta} \frac{\widetilde{m}_{B,e(i)} \beta r_{i+1} p_{C,i+1}}{\sum_{\theta\subseteq\Theta} \widetilde{m}_{\theta,e(i)} + \widetilde{m}_{P(\Theta),e(i)} + \widetilde{m}_{P(\varnothing),e(i)}} + \left[\frac{(1-r_{i+1}) \widetilde{m}_{\theta,e(i)}}{\sum_{\theta\subseteq\Theta} \widetilde{m}_{\theta,e(i)} + \widetilde{m}_{P(\Theta),e(i)} + \widetilde{m}_{P(\varnothing),e(i)}} + \frac{\widetilde{m}_{P(\Theta),e(i)} \beta r_{i+1} p_{\theta,i+1}}{\sum_{\theta\subseteq\Theta} \widetilde{m}_{\theta,e(i)} + \widetilde{m}_{P(\Theta),e(i)} + \widetilde{m}_{P(\varnothing),e(i)}}\right]\right\} = \sum_{B\cap C=\theta,B,C\subseteq\Theta} \frac{\tilde{n}_{B,e(i)} r_{i+1} p_{C,i+1}}{\sum_{\theta\subseteq\Theta} \tilde{n}_{\theta,e(i)} + \tilde{n}_{P(\Theta),e(i)}} + \frac{(1-r_{i+1}) \tilde{n}_{\theta,e(i)}}{\sum_{\theta\subseteq\Theta} \tilde{n}_{\theta,e(i)} + \tilde{n}_{P(\Theta),e(i)}} + \frac{\tilde{n}_{P(\Theta),e(i)} r_{i+1} p_{\theta,i+1}}{\sum_{\theta\subseteq\Theta} \tilde{n}_{\theta,e(i)} + \tilde{n}_{P(\Theta),e(i)}}$. From Eq. (17a) and $n_{\theta,i+1} = r_{i+1} p_{\theta,i+1}$ for $\theta \subseteq \Theta$, $n_{P(\Theta),i+1} = 1 - r_{i+1}$, here we have $\widetilde{m}_{\theta,e(i+1)} = \sum_{B\cap C=\theta,B,C\subseteq\Theta} \frac{\tilde{n}_{B,e(i)} (r_{i+1} p_{C,i+1})}{\sum_{\theta\subseteq\Theta} \tilde{n}_{\theta,e(i)} + \tilde{n}_{P(\Theta),e(i)}} + \frac{\tilde{n}_{\theta,e(i)} (1-r_{i+1})}{\sum_{\theta\subseteq\Theta} \tilde{n}_{\theta,e(i)} + \tilde{n}_{P(\Theta),e(i)}} + \frac{\tilde{n}_{P(\Theta),e(i)} (r_{i+1} p_{\theta,i+1})}{\sum_{\theta\subseteq\Theta} \tilde{n}_{\theta,e(i)} + \tilde{n}_{P(\Theta),e(i)}} = \sum_{B\cap C=\theta,B,C\subseteq\Theta} n_{B,e(i)} n_{C,i+1} + [n_{P(\Theta),i+1} n_{\theta,e(i)} + n_{P(\Theta),e(i)} n_{\theta,i+1}] = \tilde{n}_{\theta,e(i+1)}$, $\widetilde{m}_{P(\Theta),e(i+1)} = lim_{\alpha\to0,\beta\to1}(m_{P(\Theta),i+1} m_{P(\Theta),e(i)}) = lim_{\alpha\to0,\beta\to1} \frac{[\alpha(1-w_{i+1}) + \beta(1-r_{i+1})] \widetilde{m}_{P(\Theta),e(i)}}{\sum_{\theta\subseteq\Theta} \widetilde{m}_{\theta,e(i)} + \widetilde{m}_{P(\Theta),e(i)} + \widetilde{m}_{P(\varnothing),e(i)}} = (1-r_{i+1}) \frac{\tilde{n}_{P(\Theta),e(i)}}{\sum_{\theta\subseteq\Theta} \tilde{n}_{\theta,e(i)} + \tilde{n}_{P(\Theta),e(i)}} = (1-r_{i+1}) n_{P(\Theta),e(i)} = \tilde{n}_{P(\Theta),e(i+1)}$. Since $lim_{\alpha\to0} \alpha = 0$ and $\lim_{\alpha\to0,\beta\to1} \widetilde{m}_{P(\varnothing),e(i)} = \tilde{n}_{P(\varnothing),e(i)} = 0$, we have $\widetilde{m}_{P(\varnothing),e(i+1)} = lim_{\alpha\to0,\beta\to1}[\alpha w_{i+1} + \frac{\widetilde{m}_{P(\varnothing),e(i)}}{\sum_{\theta\subseteq\Theta} \widetilde{m}_{\theta,e(i)} + \widetilde{m}_{P(\Theta),e(i)} + \widetilde{m}_{P(\varnothing),e(i)}} - \frac{\alpha w_{i+1} \widetilde{m}_{P(\varnothing),e(i)}}{\sum_{\theta\subseteq\Theta} \widetilde{m}_{\theta,e(i)} + \widetilde{m}_{P(\Theta),e(i)} + \widetilde{m}_{P(\varnothing),e(i)}}] = 0$. Thus when $\alpha \to 0$ and $\beta \to 1$ we have $m_{\theta,e(i+1)} = \lim_{\alpha\to0,\beta\to1} \frac{\widetilde{m}_{\theta,e(i+1)}}{\sum_{B\subseteq\Theta} \widetilde{m}_{\theta,e(i+1)} + \widetilde{m}_{P(\Theta),e(i+1)} + \widetilde{m}_{P(\varnothing),e(i+1)}} = \frac{\tilde{n}_{\theta,e(i+1)}}{\sum_{B\subseteq\Theta} \tilde{n}_{\theta,e(i+1)} + \tilde{n}_{P(\Theta),e(i+1)}} = n_{\theta,e(i+1)}$. □

A.10. Proof of Corollary 4

Proof. Since $\alpha + \beta = 1$ and $\alpha, \beta > 0$, there is $\beta \to 0$ when $\alpha \to 1$. For $i = 2$, take $n_{\theta,e(1)} = n_{\theta,1} = r_1 p_{\theta,1}$, $n_{P(\Theta),e(1)} = n_{P(\Theta),1} = 1 - w_1$, and $n_{\theta,2} = r_2 p_{\theta,2}$ into Eq. (16b), we have $\tilde{n}_{\theta,e(2)} = (1-w_2) n_{\theta,e(1)} + n_{P(\Theta),e(1)} n_{\theta,2} = (1-w_2) n_{\theta,1} + n_{P(\Theta),1} n_{\theta,2} = (1-w_1) r_2 p_{\theta,2} + (1-w_2) r_1 p_{\theta,1}$. This is consistent with Eq. (16c) in Corollary 2. From Eq. (13c), when $\alpha \to 1$ and $\beta \to 0$ there is $\widetilde{m}_{P(\Theta),e(2)} = \lim_{\alpha\to1,\beta\to0}[\alpha(1-w_2) + \beta(1-r_2)] m_{P(\Theta),e(1)} = \lim_{\alpha\to1,\beta\to0}[\alpha(1-w_2) + \beta(1-r_2)] m_{P(\Theta),1} = \lim_{\alpha\to1,\beta\to0}[\alpha(1-w_2) + \beta(1-r_2)][\alpha(1-w_1) + \beta(1-r_1)] = (1-w_1)(1-w_2)$. This is consistent with Eq. (18b) as $\tilde{n}_{P(\Theta),e(2)} = (1-w_2) n_{P(\Theta),e(1)} = (1-w_2) n_{P(\Theta),1} = (1-w_2)(1-w_1)$. From Eq. (13d), when $\alpha \to 1$ and $\beta \to 0$ there is $\widetilde{m}_{P(\varnothing),e(2)} = \lim_{\alpha\to1,\beta\to0}[\alpha w_2 + m_{P(\varnothing),e(1)} - \alpha w_2 m_{P(\varnothing),e(1)}] = \lim_{\alpha\to1,\beta\to0}[\alpha w_2 + \alpha w_1 - \alpha^2 w_1 w_2] = w_1 + w_2 - w_1 w_2$. This is consistent with Eq. (18c) as $\tilde{n}_{P(\varnothing),e(2)} = n_{P(\varnothing),e(1)} + w_i - w_i n_{P(\varnothing),e(1)} = w_1 + w_2 - w_1 w_2$. Since $\lim_{\alpha\to1,\beta\to0} \widetilde{m}_{\varnothing,e(2)} = \lim_{\alpha\to1,\beta\to0}(\sum_{B\cap C=\varnothing,B,C\subseteq\Theta} m_{B,1} m_{C,2}) = \lim_{\alpha\to1,\beta\to0}(\sum_{B\cap C=\varnothing,B,C\subseteq\Theta} \beta r_1 p_{B,1} \beta r_1 p_{C,2}) = 0$, there is $\lim_{\alpha\to1,\beta\to0}(\sum_{\theta\subseteq\Theta} \widetilde{m}_{\theta,e(2)} + \widetilde{m}_{P(\Theta),e(2)} + \widetilde{m}_{P(\varnothing),e(2)} + \widetilde{m}_{\varnothing,e(2)}) = \sum_{\theta\subseteq\Theta} \tilde{n}_{\theta,e(2)} + \tilde{n}_{P(\Theta),e(2)} + \tilde{n}_{P(\varnothing),e(2)} + \tilde{n}_{\varnothing,e(2)} = \sum_{\theta\subseteq\Theta} \tilde{n}_{\theta,e(2)} + \tilde{n}_{P(\Theta),e(2)} + \tilde{n}_{P(\varnothing),e(2)} = 1$. so $n_{\theta,e(2)} = \lim_{\alpha\to1,\beta\to0} \frac{\widetilde{m}_{\theta,e(2)}}{\sum_{\theta\subseteq\Theta} \widetilde{m}_{\theta,e(2)} + \widetilde{m}_{P(\Theta),e(2)} + \widetilde{m}_{P(\varnothing),e(2)}} = \frac{\tilde{n}_{\theta,e(2)}}{\sum_{B\subseteq\Theta} \tilde{n}_{\theta,e(2)} + \tilde{n}_{P(\Theta),e(2)} + \tilde{n}_{P(\varnothing),e(2)}} = \tilde{n}_{\theta,e(2)}$ for $\theta \subseteq \Theta$, $P(\varnothing)$, $P(\Theta)$. From Eqs. (3), (13a) and (13b), we know $\widetilde{m}_{\theta,e(i)}$ is a function of β, so also let $\widetilde{m}_{\theta,e(i)} = \beta \widetilde{N}_{\theta,e(i)}$ as in Corollary 2. Suppose for $i = i$, Eqs. (18a)–(18c) are true when $\alpha \to 1$ and $\beta \to 0$. As presented in Corollary 2, Eq. (18a) is true means $\lim_{\alpha\to1,\beta\to0} \widetilde{N}_{\theta,e(i)} = \tilde{n}_{\theta,e(i)}$ and $\lim_{\alpha\to1,\beta\to0} p_{\theta,e(i)} = \lim_{\alpha\to1,\beta\to0} \frac{\widetilde{m}_{\theta,e(i)}}{\sum_{B\subseteq\Theta} \widetilde{m}_{B,e(i)}} = \lim_{\alpha\to1,\beta\to0} \frac{\beta \widetilde{N}_{\theta,e(i)}}{\sum_{B\subseteq\Theta} \beta \widetilde{N}_{B,e(i)}} = \frac{\lim_{\alpha\to1,\beta\to0} \widetilde{N}_{\theta,e(i)}}{\sum_{B\subseteq\Theta}(\lim_{\alpha\to1,\beta\to0} \widetilde{N}_{B,e(i)})} = \frac{\tilde{n}_{\theta,e(i)}}{\sum_{B\subseteq\Theta} \tilde{n}_{B,e(i)}}$. Eqs. (16b)–(16c) are true means $\lim_{\alpha\to1,\beta\to0} \widetilde{m}_{P(\varnothing),e(i)} = \tilde{n}_{P(\varnothing),e(i)}$, $\lim_{\alpha\to1,\beta\to0} \widetilde{m}_{P(\Theta),e(i)} = \tilde{n}_{P(\Theta),e(i)}$. Since $\lim_{\alpha\to1,\beta\to0} \widetilde{m}_{\varnothing,e(i)} = \lim_{\alpha\to1,\beta\to0}(\sum_{B\cap C=\varnothing,B,C\subseteq\Theta} \widetilde{m}_{\varnothing,e(i-1)} m_{C,i}) = \lim_{\alpha\to1,\beta\to0}(\sum_{B\cap C=\varnothing,B,C\subseteq\Theta} \beta \widetilde{N}_{B,e(i-1)} \beta r_i p_{C,i}) = 0$, there is $\lim_{\alpha\to1,\beta\to0}(\sum_{B\subseteq\Theta} \widetilde{m}_{\theta,e(i)} + \widetilde{m}_{P(\Theta),e(i)} + \widetilde{m}_{P(\varnothing),e(i)} + \widetilde{m}_{\varnothing,e(i)}) = \lim_{\alpha\to1,\beta\to0}(\sum_{B\subseteq\Theta} \widetilde{m}_{\theta,e(i)} + \widetilde{m}_{P(\Theta),e(i)} + \widetilde{m}_{P(\varnothing),e(i)}) = 1$. For $i = i+1$, Since $\widetilde{m}_{\theta,e(i)} = \beta \widetilde{N}_{\theta,e(i)}$ for $\theta \subseteq \Theta$ and $m_{\theta,e(i)} = \frac{\widetilde{m}_{\theta,e(i)}}{\sum_{B\subseteq\Theta} \widetilde{m}_{\theta,e(i)} + \widetilde{m}_{P(\Theta),e(i)} + \widetilde{m}_{P(\varnothing),e(i)}}$ for $\theta \subseteq \Theta$, $P(\Theta)$, $P(\varnothing)$, there is $\widetilde{m}_{\theta,e(i+1)} = \sum_{B\cap C=\theta,B,C\subseteq\Theta} m_{B,e(i)} m_{C,i+1} + [\alpha(1-w_{i+1}) + \beta(1-r_{i+1})] m_{\theta,e(i)} + m_{P(\Theta),e(i)} m_{\theta,i+1} = \sum_{B\cap C=\theta,B,C\subseteq\Theta} \frac{\beta \widetilde{N}_{B,e(i)} \beta r_{i+1} p_{C,i+1}}{\sum_{\theta\subseteq\Theta} \widetilde{m}_{\theta,e(i)} + \widetilde{m}_{P(\varnothing),e(i)} + \widetilde{m}_{P(\Theta),e(i)}} + \frac{[\alpha(1-w_{i+1}) + \beta(1-r_{i+1})] \beta \widetilde{N}_{\theta,e(i)}}{\sum_{\theta\subseteq\Theta} \widetilde{m}_{\theta,e(i)} + \widetilde{m}_{P(\varnothing),e(i)} + \widetilde{m}_{P(\Theta),e(i)}} + \frac{\widetilde{m}_{P(\Theta),e(i)} \beta r_{i+1} p_{\theta,i+1}}{\sum_{\theta\subseteq\Theta} \widetilde{m}_{\theta,e(i)} + \widetilde{m}_{P(\varnothing),e(i)} + \widetilde{m}_{P(\Theta),e(i)}} = \beta\left\{\sum_{B\cap C=\theta,B,C\subseteq\Theta} \frac{\widetilde{N}_{B,e(i)} \beta r_{i+1} p_{C,i+1}}{\sum_{\theta\subseteq\Theta} \widetilde{m}_{\theta,e(i)} + \widetilde{m}_{P(\varnothing),e(i)} + \widetilde{m}_{P(\Theta),e(i)}} + \frac{[\alpha(1-w_{i+1}) + \beta(1-r_{i+1})] \widetilde{N}_{\theta,e(i)}}{\sum_{\theta\subseteq\Theta} \widetilde{m}_{\theta,e(i)} + \widetilde{m}_{P(\varnothing),e(i)} + \widetilde{m}_{P(\Theta),e(i)}} + \frac{\widetilde{m}_{P(\Theta),e(i)} r_{i+1} p_{\theta,i+1}}{\sum_{\theta\subseteq\Theta} \widetilde{m}_{\theta,e(i)} + \widetilde{m}_{P(\varnothing),e(i)} + \widetilde{m}_{P(\Theta),e(i)}}\right\}$. Let $\widetilde{N}_{\theta,e(i+1)} = \sum_{B\cap C=\theta,B,C\subseteq\Theta} \frac{\widetilde{N}_{B,e(i)} \beta r_{i+1} p_{C,i+1}}{\sum_{\theta\subseteq\Theta} \widetilde{m}_{\theta,e(i)} + \widetilde{m}_{P(\varnothing),e(i)} + \widetilde{m}_{P(\Theta),e(i)}} + \frac{[\alpha(1-w_{i+1}) + \beta(1-r_{i+1})] \widetilde{N}_{\theta,e(i)}}{\sum_{\theta\subseteq\Theta} \widetilde{m}_{\theta,e(i)} + \widetilde{m}_{P(\varnothing),e(i)} + \widetilde{m}_{P(\Theta),e(i)}} + \frac{\widetilde{m}_{P(\Theta),e(i)} r_{i+1} p_{\theta,i+1}}{\sum_{\theta\subseteq\Theta} \widetilde{m}_{\theta,e(i)} + \widetilde{m}_{P(\varnothing),e(i)} + \widetilde{m}_{P(\Theta),e(i)}}$, so $\widetilde{m}_{\theta,e(i+1)} = \beta \widetilde{N}_{\theta,e(i+1)}$. Since $\lim_{\alpha\to1,\beta\to0}(\sum_{B\subseteq\Theta} \widetilde{m}_{\theta,e(i)} + \widetilde{m}_{P(\Theta),e(i)} + \widetilde{m}_{P(\varnothing),e(i)}) = 1$, $n_{\theta,e(i)} = \tilde{n}_{\theta,e(i)}$, and $n_{P(\Theta),e(i)} = \tilde{n}_{P(\Theta),e(i)}$, we have $\lim_{\alpha\to1,\beta\to0} \widetilde{N}_{\theta,e(i+1)} = \lim_{\alpha\to1,\beta\to0}\{\sum_{B\cap C=\theta,B,C\subseteq\Theta} \widetilde{N}_{B,e(i)} \beta r_{i+1} p_{C,i+1} + [\alpha(1-w_{i+1}) + \beta(1-r_{i+1})] \widetilde{N}_{\theta,e(i)} + \widetilde{m}_{P(\Theta),e(i)} r_{i+1} p_{\theta,i+1}\} = (1-w_{i+1}) \lim_{\alpha\to1,\beta\to0} \widetilde{N}_{\theta,e(i)} + (\lim_{\alpha\to1,\beta\to0} \widetilde{m}_{P(\Theta),e(i)}) r_{i+1} p_{\theta,i+1} = (1-w_{i+1}) \tilde{n}_{\theta,e(i)} + \tilde{n}_{P(\Theta),e(i)} r_{i+1} p_{\theta,i+1} = (1-w_{i+1}) n_{\theta,e(i)} + n_{P(\Theta),e(i)} r_{i+1} p_{\theta,i+1} = (1-w_{i+1}) n_{\theta,e(i)} + n_{P(\Theta),e(i)} n_{\theta,i+1} = n_{\theta,e(i+1)}$. When $\alpha \to 1$ and $\beta \to 0$ there is $p_{\theta,e(i+1)} = \lim_{\alpha\to1,\beta\to0} \frac{\widetilde{m}_{\theta,e(i+1)}}{\sum_{B\subseteq\Theta} \widetilde{m}_{\theta,e(i+1)}} = \lim_{\alpha\to1,\beta\to0} \frac{\beta \widetilde{N}_{\theta,e(i+1)}}{\sum_{B\subseteq\Theta} \beta \widetilde{N}_{B,e(i+1)}} = \lim_{\alpha\to1,\beta\to0} \frac{\widetilde{N}_{\theta,e(i+1)}}{\sum_{B\subseteq\Theta} \widetilde{N}_{B,e(i+1)}} = \frac{n_{\theta,e(i+1)}}{\sum_{B\subseteq\Theta} n_{\theta,e(i+1)}}$. Since $\widetilde{m}_{P(\Theta),e(i+1)} = m_{P(\Theta),e(i)} m_{P(\Theta),i+1} = m_{P(\Theta),e(i)}[\alpha(1-w_i) + \beta(1-r_i)] = \frac{\widetilde{m}_{P(\Theta),e(i)}}{\sum_{B\subseteq\Theta} \widetilde{m}_{\theta,e(i)} + \widetilde{m}_{P(\Theta),e(i)} + \widetilde{m}_{P(\varnothing),e(i)}}[\alpha(1-w_i) + \beta(1-r_i)]$ $\widetilde{m}_{P(\varnothing),e(i+1)} = m_{P(\varnothing),e(i)} + m_{P(\varnothing),i+1} - m_{P(\varnothing),e(i)} m_{P(\varnothing),i+1} = \frac{\widetilde{m}_{P(\varnothing),e(i)}}{\sum_{B\subseteq\Theta} \widetilde{m}_{\theta,e(i)} + \widetilde{m}_{P(\Theta),e(i)} + \widetilde{m}_{P(\varnothing),e(i)}} + \alpha w_i - \frac{\alpha w_i \widetilde{m}_{P(\varnothing),e(i)}}{\sum_{B\subseteq\Theta} \widetilde{m}_{\theta,e(i)} + \widetilde{m}_{P(\Theta),e(i)} + \widetilde{m}_{P(\varnothing),e(i)}}$. So when $\alpha \to 1$ and $\beta \to 0$ there is $\widetilde{m}_{P(\Theta),e(i+1)} = \lim_{\alpha\to1,\beta\to0} \widetilde{m}_{P(\Theta),e(i)} \lim_{\alpha\to1,\beta\to0}\{[\alpha(1-w_{i+1}) + \beta(1-r_{i+1})]\} = (1-w_i) \tilde{n}_{P(\Theta),e(i)} = (1-w_{i+1}) n_{P(\Theta),e(i)} = n_{P(\Theta),e(i+1)}$, $\widetilde{m}_{P(\varnothing),e(i+1)} = \lim_{\alpha\to1,\beta\to0} \widetilde{m}_{P(\varnothing),e(i)} + w_{i+1} - w_{i+1} \lim_{\alpha\to1,\beta\to0} \widetilde{m}_{P(\varnothing),e(i)} = n_{P(\varnothing),e(i)} + w_{i+1} - w_{i+1} n_{P(\varnothing),e(i)} = n_{P(\varnothing),e(i+1)}$. □

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