

A novel decision probability transformation method based on belief interval

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ABSTRACT

In Dempster–Shafer evidence theory, the basic probability assignment (BPA) can effectively represent and process uncertain information. How to transform the BPA of uncertain information into a decision probability remains a problem to be solved. In the light of this issue, we develop a novel decision probability transformation method to realize the transition from the belief decision to the probability decision in the framework of Dempster–Shafer evidence theory. The newly proposed method considers the transformation of BPA with multi-subset focal elements from the perspective of the belief interval, and applies the continuous interval argument ordered weighted average operator to quantify the data information contained in the belief interval for each singleton. Afterward, we present an approach to calculate the support degree of the singleton based on quantitative data information. According to the support degree of the singleton, the BPA of multi-subset focal elements is allocated reasonably. Furthermore, we introduce the concepts of probabilistic information content in this paper, which is utilized to evaluate the performance of the decision probability transformation method. Eventually, a few numerical examples and a practical application are given to demonstrate the rationality and accuracy of our proposed method.

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1. Introduction

To combine the different types of information provided by different information sources, some new and interesting mathematical theories are gradually formed. These include rough sets [1–3], Dempster–Shafer (DS) evidence theory [4–6], D number [7–9], Z number [10–12], and fuzzy sets theory [13–15], etc. In these theories, the DS evidence theory has attracted great attention. DS evidence theory is a combinatorial uncertainty and imprecise reasoning theory proposed and developed by Dempster [16] and Shafer [17]. This theory, also known as belief function theory, has been successfully utilized in different areas [18–22] and has become the mainstream theory of information fusion [23–25]. In management, DS evidence theory can provide a new way to solve the problems such as lot-sizing decisions in supply chains and inventory model [26–31], and evaluation of the sustainable transportation system [32,33]. The interested authors can employ the DS evidence theory for performance measurement of inventory and supply chain systems [34–38]. Moreover, DS evidence theory can effectively handle uncertain and imprecise information to achieve decision support [39,40].

In the DS evidence theory, the basic probability assignment (BPA) of multi-subset focal elements can express uncertain information directly. Due to the uncertainty, it is difficult to make a

decision directly based on the BPA. How to make a reasonable decision through the BPA is an issue that must be solved. A simple and effective approach is to transform the BPA into a decision probability [41] to deal with the uncertain information contained in multi-subset focal elements. The BPA of multi-subset focal elements can reflect the support degree for the singleton, which can be quantified by decision probability transformation. If we can put forward a reasonable method to transform the BPA into a decision probability, and make use of the mature probabilistic decision model, it is of great significance for the system to get a correct decision analysis. Many researchers have investigated the evidence theory model, and proposed many decision probability transformation methods [42–44]. Smets [45] first presented a transferable belief model, and applying the Pignistic probability transformation method to transform the BPA of multi-subset focal elements into a decision probability. The Pignistic probability transformation method has aroused great interest and broadly utilized in various fields [46–48]. Unfortunately, the Pignistic probability transformation method assigns the BPA of multi-subset focal elements to the singleton by employing average allocation, which is too conservative and sometimes fails to yield reasonable decision probability. Sudano [49] proposed several transformation methods in the framework of DS evidence theory for approximating any quantitative BPA by the subjective probability. Pan and Deng [50] presented a decision probability transformation method by combining the ordered

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weighted averaging with entropy difference. However, the transformation method cannot make reasonable allocation when a certain singleton is not contained in the multi-subset focal elements. From a geometric interpretation of Dempster's combination rule, Cuzzolin [51] defined a new decision probability transformation method in the framework of DS evidence theory. This method assigns the BPA of multi-subset focal elements in terms of the uncertainty proportion of the singleton. However, when there is no intersection between singleton and multi-subset focal elements, the proposed method will get unreasonable assignment results. Dezert and Smarandache [52,53] put forward a novel generalized Pignistic probability transformation method in the framework of Dezert–Smarandache Theory (DSmT). This approach considers both the belief values and the cardinality of elements in the process of the proportional redistribution. Since the transformation performance of the method varies with parameter ε , however, it is difficult to determine the value of parameter ε in the practical application.

To overcome the defects of these methods, we develop a novel decision probability transformation method. The new decision probability transformation method considers the transformation of BPA for multi-subset focal elements from the perspective of the belief interval. The data information contained in the belief interval of the singleton can reflect the degree of support for the singleton by other multi-subset focal elements. In our work, we use the continuous interval argument ordered weighted average (C-OWA) operator to quantify the data information about the belief interval of each singleton. Then, we propose a method to calculate the support degree of the singleton based on quantitative data information. According to the support degree of the singleton, the BPA of multi-subset focal elements is allocated reasonably. Besides, we also introduce the concepts of probabilistic information content (PIC) in our work. Afterward, the PIC criterion is applied to evaluate the performance of the decision probability transformation method. The new decision probability transformation method can overcome the defects of the traditional methods, and obtain more reasonable and precise decision probability. Finally, several numerical examples and a practical application are given to illustrate the rationality and accuracy of our method.

This article is composed as follows. Some basics knowledge about DS evidence theory is briefly depicted in Section 2. Section 3 reviews some traditional decision probability transformation methods. A novel decision probability transformation method based on the belief interval is presented, and some properties of the new decision probability transformation method are proved in Section 4. Section 5 introduces the related concepts of probabilistic information content. Section 6 gives a few numerical examples to demonstrate the rationality and accuracy of our proposed method. A practical application of our proposed method is described in Section 7. Section 8 summarizes and analyzes this paper.

2. Basics of Dempster–Shafer evidence theory

DS evidence theory can handle the uncertain information caused by the randomness and fuzziness of the objective information. Compared with the traditional uncertainty reasoning method, DS evidence theory can effectively distinguish unknown and uncertain information, decrease the redundancy of information, and improve the certainty of decision-making. The proposal of DS evidence theory provides a practical mathematical tool for the field of decision analysis and artificial intelligence, which can realize the representation and combination of uncertain information, and has great advantages in modeling uncertain information [54–56]. In this chapter, some basics knowledge of DS evidence theory is briefly introduced.

The theory of belief function is constructed on the complete set Θ composed of finite mutually exclusive elements. We call $\Theta = \{F_1, F_2, \dots, F_N\}$ the frame of discernment (FOD), which contains all possible results of the considered problem. The set of propositions formed by all subsets in Θ is called the power set, denoted as 2^Θ . The power set 2^Θ of Θ is described as:

$$2^\Theta = \{\emptyset, F_1, F_2, \dots, F_N, \{F_1, F_2\}, \dots, \{F_1, F_2, F_3\}, \dots, \Theta\} \quad (1)$$

where \emptyset denotes the empty set.

Under the Shafer's model, the basic probability assignment (BPA) of any proposition C on the FOD Θ is described as a mapping of its power set 2^Θ from 0 to 1, namely, $m : 2^\Theta \rightarrow [0, 1]$, which meets as follows:

$$\sum_{C \in 2^\Theta} m(C) = 1, \quad m(\emptyset) = 0 \quad (2)$$

$m(C)$ is the basic probability assignment of the proposition C , which indicates the support degree of the evidence m to the proposition C . $\forall C \subseteq \Theta$, if $m(C) > 0$, C is a focal element of the evidence m .

In the FOD Θ , Shafer defines the belief function and plausibility function of $C \subseteq \Theta$ as follows:

$$Bel(C) = \sum_{D \in 2^\Theta, D \subseteq C} m(D) \quad (3)$$

$$Pl(C) = \sum_{D \in 2^\Theta, D \cap C \neq \emptyset} m(D) = 1 - Bel(\bar{C}) \quad (4)$$

$Bel(C)$ characterizes the total degree of belief in proposition C , and constitutes the lower bound of the probability distribution of proposition C . $Pl(C)$ denotes the degree to which proposition C is not opposed, and constitutes the upper bound of the probability distribution of proposition C .

From the above definition, we know that $Bel(C) \leq Pl(C)$. $[Bel(C), Pl(C)]$ is the belief interval of the proposition C . The longer the length of the belief interval, the greater the uncertainty of the corresponding proposition. DS evidence theory describes the uncertain information through the belief interval, which can solve the problem that traditional probability theory cannot effectively deal with the uncertain information.

In the DS evidence theory, Dempster's combination rule is utilized to combine the multiple sources of evidence. Dempster's combination rule is defined as follows.

$$m(C) = \begin{cases} 0 & C = \emptyset \\ \frac{1}{1-k} \sum_{C_i \cap C_j = C} m_1(C_i)m_2(C_j) & C \neq \emptyset \end{cases} \quad (5)$$

$$k = \sum_{C_i \cap C_j = \emptyset} m_1(C_i)m_2(C_j)$$

where k is the conflict coefficient.

3. Decision probability transformation method

In the DS evidence theory, the BPA can express and process uncertain information effectively. When using BPA to describe the uncertain information of multi-element propositions, it is difficult for us to get accurate decision results through the BPA. How to make an effective decision through the BPA is still a problem to be solved. All along, many researchers have studied the probability transformation methods. A simple and intuitive idea is to transform the BPA into a probability function for decision-making. In this section, we briefly introduce several common probability transformation methods.

(1) Pignistic probability transformation

To solve the decision-making problem under uncertainty situation, Smets [45] presented the Pignistic probability transformation method in the transferable belief model, denoted as *BetP*. In the FOD Θ , the definition of the Pignistic probability transformation method is described below.

$$BetP(C) = \sum_{D \in 2^\Theta, D \neq \emptyset} \frac{|C \cap D|}{|D|} \frac{m(D)}{1 - m(\emptyset)} \quad (6)$$

where 2^Θ is the power set of the FOD Θ , $|D|$ is the cardinality of set D .

The Pignistic probability transformation method does not take full advantage of the known information, and directly transforms the BPA of multi-element propositions into the single element proposition by means of equal distribution. This transformation method is conservative, which is not beneficial to making a correct decisions.

(2) Sudano's probability transformation method

In the framework of DS evidence theory, Sudano [49] proposed several decision probability transformation methods by using belief function and plausibility function. These methods apply the mapping proportional to the plausibility(PrPl), the mapping proportional to the belief(PrBel), the mapping proportional to the normalized plausibility(PrNPl), and the mapping proportional to all plausibilities(PraPl), respectively. The definitions of these approaches are depicted below:

The *PrPl* transformation method is defined as follows:

$$PrPl(C) = Pl(C) \cdot \sum_{D \in 2^\Theta, C \subseteq D} \frac{1}{\sum_{\substack{D_i \in 2^\Theta, |D_i|=1 \\ \cup_i D_i = D}} Pl(D_i)} m(D) \quad (7)$$

The *PrBel* transformation method is shown below:

$$PrBel(C) = Bel(C) \cdot \sum_{D \in 2^\Theta, C \subseteq D} \frac{1}{\sum_{\substack{D_i \in 2^\Theta, |D_i|=1 \\ \cup_i D_i = D}} Bel(D_i)} m(D) \quad (8)$$

The *PrNPl* transformation method is denoted as follows:

$$PrNPl(C) = \frac{Pl(C)}{\sum_{D \in 2^\Theta} Pl(D)} \quad (9)$$

The *PraPl* transformation method is described as follows:

$$PraPl(C) = Bel(C) + \varepsilon \cdot Pl(C) \quad (10)$$

$$\varepsilon = \frac{1 - \sum_{C \in 2^\Theta} Bel(C)}{\sum_{C \in 2^\Theta} Pl(C)}$$

(3) Cuzzolin's probability transformation method

From the geometric significance of Dempster's combination rule, Cuzzolin [51] presented a new probability transformation method in the framework of DS evidence theory. This method processes the BPA of multi-element propositions through the uncertainty proportion, and proportionally repartition the non-specific mass to the singleton. The definition of Cuzzolin's probability(CuzzP) transformation method is described as follows:

In the FOD $\Theta = \{F_1, F_2, \dots, F_n\}$, Cuzzolin's probability transformation method is defined as follows:

$$CuzzP(F_i) = m(F_i) + \frac{\Delta(F_i)}{\sum_{j=1}^n \Delta(F_j)} \times TNSM \quad (11)$$

where $\Delta(F_i) = Pl(F_i) - m(F_i)$ and

$$TNSM = 1 - \sum_{j=1}^n m(F_j) = \sum_{F \in 2^\Theta, |F|>1} m(F) \quad (12)$$

However, Cuzzolin's probability transformation method has some limitations. When $F_i \in \Theta, F_i \cap F = \emptyset$, the uncertain information contained in the *TNSM* will also be assigned to the

singleton F_i . This distribution is not reasonable and does not make sense in our point of view. Furthermore, it is easy to see from the definition of *CuzzP* that when the belief value of the evidence m is a probability distribution, there is no definition of *CuzzP* in the mathematical sense. Because in this case all $\Delta(*)$ equals 0, and we will get 0/0 indetermination, which is meaningless from a mathematical point of view.

(4) *DSmP* probability transformation method

In the theoretical framework of *DSmT* [52], Dezert and Smarandache [53] proposed a novel generalized Pignistic probability transformation method, represented as *DSmP*, which is defined as follows.

Let Θ be the FOD in the given model, the definition of *DSmP* is described as follows:

$$DSmP(C) = \sum_{D \in C^\Theta} \frac{\sum_{B \subseteq C \cap D, |B|=1} m(B) + \varepsilon \cdot |C \cap D|}{\sum_{B \subseteq D, |B|=1} m(B) + \varepsilon \cdot |D|} \cdot m(D) \quad (13)$$

where $\varepsilon \geq 0$ is a adjustment parameter. C^Θ is the corresponding hyper power set, which includes all the integrity restrains of the *DSmT* theoretical framework.

This method uses the parameter ε to combine the Pignistic probability transformation method with the proportional belief transformation method. However, parameter ε is not easily determined in the practical application.

4. A new decision probability transformation method

In the DS evidence theory, the BPA can effectively represent the uncertain information of multi-subset focal elements. Due to the uncertain of the focal element, it is sometimes impossible to make a decision through the BPA in practical application. Therefore, the BPA needs to be transformed into a decision probability. The mapping from the belief function to the probability domain is a controversial problem. How to transform the BPA into a decision probability accurately is still a hot topic. In our work, a novel decision probability transformation approach is presented to consider the transformation of BPA from the perspective of the belief interval. The continuous interval argument ordered weighted average(C-OWA) operator is utilized to quantify the data information about the belief interval of each singleton, and then using the preference degree of the singleton to modify the quantitative data information, so as to obtain the support degree of the singleton. According to the support degree of the singleton, the BPA of multi-subset focal elements is reasonably distributed.

4.1. A new measure for the singleton support degree

In the DS evidence theory, the data information contained in the belief interval can reflect the support degree of the multi-subset focal elements to the singleton. To quantify the data information contained in the belief interval, the C-OWA operator is introduced in our work. According to the belief interval of the singleton, the support degree of the singleton is calculated by using the C-OWA operator. We first review some basics knowledge of the interval number and C-OWA operator.

4.1.1. Some properties of interval number

In this chapter, we first introduce some properties of interval numbers.

Assuming that $F = [f^L, f^U] = \{x | f^L \leq x \leq f^U\}$, then F is called the nonnegative interval number [57]. Note that, we only discuss the nonnegative interval numbers in this article.

Suppose $F_1 = [f_1^L, f_1^U]$ and $F_2 = [f_2^L, f_2^U]$ are two interval numbers, and $l_1 = f_1^U - f_1^L$ and $l_2 = f_2^U - f_2^L$ are the length of

two intervals, respectively. Then the possibility degree of $F_1 \geq F_2$ is defined as follows [57]:

$$p(F_1 \geq F_2) = \max \left\{ 1 - \max \left(\frac{f_2^U - f_1^L}{l_1 + l_2}, 0 \right), 0 \right\} \quad (14)$$

Similarly, the possibility degree of $F_2 \geq F_1$ can be defined as follows:

$$p(F_2 \geq F_1) = \max \left\{ 1 - \max \left(\frac{f_1^U - f_2^L}{l_1 + l_2}, 0 \right), 0 \right\} \quad (15)$$

As can be seen from the above definition:

1. $0 \leq p(F_1 \geq F_2) \leq 1, \quad 0 \leq p(F_2 \geq F_1) \leq 1$
2. $p(F_1 \geq F_2) + p(F_2 \geq F_1) = 1$
3. $p(F_1 \geq F_1) = p(F_2 \geq F_2) = \frac{1}{2}$

For convenience, we set $p_{ij} = p(F_i \geq F_j)(i, j = 1, 2, \dots, n)$. Assuming that there are n interval numbers, we use Eq. (14) to calculate the possibility degree between each interval number F_i and all interval number F_j . Then construct the possibility degree matrix $P = (p_{ij})_{n \times n}$. Where $p_{ij} \geq 0, p_{ij} + p_{ji} = 1, p_{ii} = 0.5$.

4.1.2. C-OWA operator

In the process of handling continuous interval data, Yager [58, 59] firstly proposed the C-OWA operator for processing the continuous data information, and applying the C-OWA operator to integrate the continuous interval data. The definition of C-OWA operator is described as follows:

For any interval number $[a, b]$, if satisfied

$$f_s([a, b]) = \int_0^1 \frac{ds(x)}{dx} (b - x(b - a)) dx \quad (16)$$

where function $s(x)$ satisfies the following properties:

1. $s(0) = 0$
2. $s(1) = 1$
3. if $x > y$, then $s(x) \geq s(y)$

Then f is called the C-OWA operator, $s(x)$ is the basic unit-interval monotonic (BUM).

The C-OWA operator has the following properties:

For any BUM function $s(x)$, there exists

$$a \leq f_s([a, b]) \leq b$$

If get $s(x) = x^r (r \geq 0)$, then the C-OWA operator can be expressed as follows:

$$f_s([a, b]) = \frac{b + ar}{r + 1} \quad (17)$$

The C-OWA operator can quantify the interval data information. Based on this, the C-OWA operator is utilized to quantify the data information contained in the belief interval.

4.1.3. Singleton support degree measurement based on C-OWA operator

The interval number can effectively express the uncertain information, and the belief interval can represent the uncertainty of the given evidence in DS evidence theory. In terms of uncertainty representation, interval numbers and belief intervals have the same form. Therefore, we can generalize the properties of interval numbers to belief intervals, and use these properties to handle the information contained in the belief intervals. According to the characteristics of belief interval data, the C-OWA operator is applied to quantify the belief interval data information of the singleton in our work. Then, calculating the preference degree of the singleton based on the related properties of the interval number. Afterward, using the preference degree of the singleton

to modify the quantized data information of the belief interval, so as to obtain the support degree of the singleton. The process of measuring the support degree of the singleton is described as follows.

The C-OWA operator is utilized to integrate the belief interval data to get the quantized information about the belief interval. The definition of the belief interval data quantization method based on the C-OWA operator is depicted as follows:

Let $[Bel(A_i), Pl(A_i)]$ be the belief interval of the focal element A_i . Using the C-OWA operator to quantify the belief interval data information of the focal element, the definition is described as follows:

$$f_s([Bel(A_i), Pl(A_i)]) = \int_0^1 \frac{ds(x)}{dx} (Pl(A_i) - x(Pl(A_i) - Bel(A_i))) dx \quad (18)$$

In this paper, the BUM function is taken as $s(x) = x^r$, then

$$f_s([Bel(A_i), Pl(A_i)]) = \frac{Pl(A_i) + Bel(A_i) \cdot r}{r + 1} \quad (19)$$

Hence, the quantization information of the belief interval data for the focal element A_i is expressed as

$$\beta_i = \frac{Pl(A_i) + Bel(A_i) \cdot r}{r + 1} \quad (20)$$

β_i represents the quantitative data information of the focal element A_i . In order to calculate the quantitative data information of the singleton more effectively, we take $r = 2^{|\Theta|}$.

Since the belief interval contains uncertainty information, it is necessary to evaluate the support degree for the belief interval of the singleton to obtain more accurate quantitative data information about the belief interval. The calculation method for the support degree of the singleton is given below.

In the given FOD $\Theta = \{A_1, A_2, \dots, A_n\}$, suppose the belief interval of the focal element A_i is $[Bel(A_i), Pl(A_i)]$, ($i = 1, 2, \dots, n$), and the belief interval of the focal element A_j is $[Bel(A_j), Pl(A_j)]$, ($j = 1, 2, \dots, n$), the possibility degree $p(A_i \geq A_j)$ is calculated by using Eq. (14). For the evidence containing n singletons, the possibility degree matrix P between the belief intervals of n singletons is constructed by utilizing the properties of interval numbers, as shown below.

$$P = \begin{bmatrix} 0.5 & p_{12} & \cdots & p_{1n} \\ p_{21} & 0.5 & \cdots & p_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ p_{n1} & p_{n2} & \cdots & 0.5 \end{bmatrix} \quad (21)$$

Based on the possibility degree matrix P , the preference degree p of the singleton A_i is defined as follows.

$$p(A_i) = \frac{1}{n-1} \sum_{j=1, j \neq i}^n p_{ij} \quad i = 1, 2, \dots, n \quad (22)$$

Using the preference degree to modify the quantized belief interval data information, we can obtain the support degree of the singleton A_i . The support degree of the singleton A_i is defined as follows:

$$Sup(A_i) = p(A_i) \times \beta_i \quad (23)$$

The support degree of the singleton can reflect the support degree of other multi-subset focal elements to the singleton. The higher the support degree of the singleton, the more belief values should be allocated to that focal element. Based on this idea, we develop a novel decision probability transformation method.

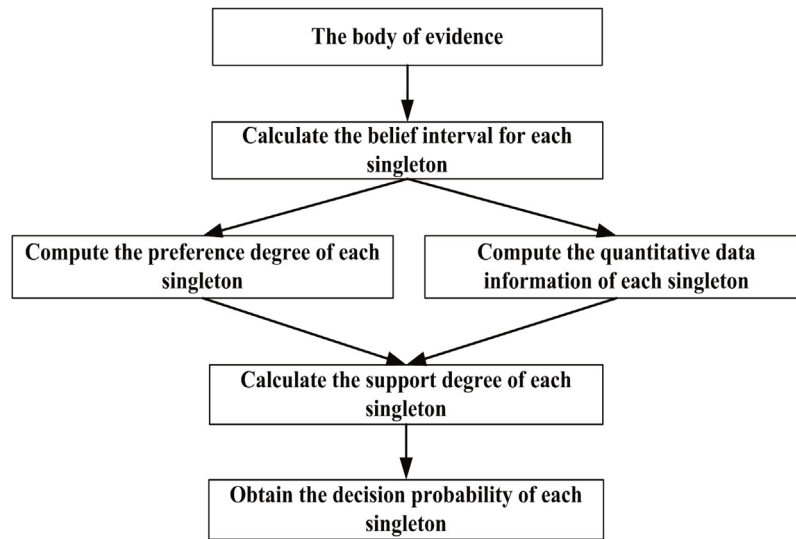


Fig. 1. The implementation process of our proposed method.

4.2. A new decision probability transformation based on the support degree of the singleton

In the DS evidence theory, the belief interval of the singleton is composed of the belief function and plausibility function of the proposition. Therefore, the belief interval of the singleton can contain all the data information of the singleton. Based on this idea, we define the new decision probability transformation method as follows:

Assuming that B is the multi-subset focal elements in the FOD $\Theta = \{A_1, A_2, \dots, A_n\}$, the new decision probability transformation method is depicted as follows:

$$ITP(A_i) = m(A_i) + \sum_{A_j \subset B \in 2^\Theta} \varepsilon_{A_j} \cdot m(B) \tag{24}$$

$$\varepsilon_{A_i} = \frac{Sup(A_i)}{\sum_{j=1, A_j \subset B \in 2^\Theta} Sup(A_j)}$$

where $Sup(A_i)$ is the support degree of the singleton A_i , $|B|$ is the cardinality of set B . The new decision probability transformation method is denoted as ITP . The implementation process of our proposed method is presented in Fig. 1.

ITP can realize the transition from BPA to decision probability, which is convenient for decision making in an uncertain environment, and can obtain reasonable decision probability.

Daniel defines the decision probability transformation function $PT(*)$ in Ref. [44]. According to Daniel's definition, the rationality of the new decision probability transformation method is verified.

Theorem 1. $PT(*)$ needs to satisfy the consistency of the upper and lower boundaries, namely $Bel(A_i) \leq PT(A_i) \leq Pl(A_i)$

Proof. $ITP(A_i) = m(A_i) + \sum_{A_j \subset B \in 2^\Theta} \varepsilon_{A_j} \cdot m(B)$
 Because $Sup(A_i) \geq 0$
 Then $\varepsilon_{A_i} \geq 0$
 Therefore $ITP(A_i) \geq m(A_i) = Bel(A_i)$
 Due to $\varepsilon_{A_i} \leq 1$
 Thus
 $ITP(A_i) = m(A_i) + \sum_{A_j \subset B \in 2^\Theta} \varepsilon_{A_j} \cdot m(B) \leq m(A_i) + \sum_{A_j \subset B \in 2^\Theta} m(B) = Pl(A_i)$
 Hence $Bel(A_i) \leq ITP(A_i) \leq Pl(A_i)$

Theorem 2. For the permutation function $R : \Theta \rightarrow \Theta^*$ in the FOD Θ , $PT(*)$ requires to be satisfied $PT^*(R(A_i)) = PT(A_i)$. That is, the decision probability of the focal element after the action of permutation function remains unchanged.

Proof. Because R is the permutation function of $\Theta \rightarrow \Theta^*$
 Therefore, we can obtain $R(A_i) = A_i$
 Thus, the BPA for $R(A_i)$ is $m^*(R(A_i)) = m(A_i)$
 Hence

$$\begin{aligned} ITP^*(R(A_i)) &= m^*(R(A_i)) + \sum_{A_j \subset B \in 2^\Theta} \varepsilon_{A_j} \cdot m(B) \\ &= m(A_i) + \sum_{A_j \subset B \in 2^\Theta} \varepsilon_{A_j} \cdot m(B) = ITP(A_i) \end{aligned}$$

As can be seen from the above proof that the new decision probability transformation method meets the definition of decision probability transformation function given by Daniel, and is a reasonable decision probability transformation method. We give an example to illustrate the calculation process of the ITP .

Example 1. Let m be a BPA defined on the FOD $\Theta = \{x_1, x_2\}$, and the BPA is described below:

$$m(x_1) = 0.2 \quad m(x_2) = 0.1 \quad m(\{x_1, x_2\}) = 0.7$$

Applying the ITP to calculate the probability distribution of each singleton.

The belief interval of the singleton x_1 is $[Bel(x_1), Pl(x_1)] = [0.2, 0.9]$

The belief interval of the singleton x_2 is $[Bel(x_2), Pl(x_2)] = [0.1, 0.8]$

According to Eq. (20), we can get the quantitative data information of each singleton as follows:

$$\beta_1 = 0.34 \quad \beta_2 = 0.24$$

Compute the preference degree of each singleton by using Eqs. (14) and (22), the results are as follows:

$$p(x_1) = 0.5714 \quad p(x_2) = 0.4286$$

According to Eq. (23), the support degree of each singleton is achieved, which is shown as follows:

$$Sup(x_1) = 0.1943 \quad Sup(x_2) = 0.1029$$

Using Eq. (24) to compute the decision probability of each singleton.

$$ITP(x_1) = 0.2 + \frac{Sup(x_1)}{Sup(x_1) + Sup(x_2)} \times m(\{x_1, x_2\}) = 0.6576$$

$$ITP(x_2) = 0.1 + \frac{Sup(x_2)}{Sup(x_1) + Sup(x_2)} \times m(\{x_1, x_2\}) = 0.3424$$

In Example 1, since the belief value of the focal element x_1 is greater than that of the focal element x_2 , the support degree of the focal element x_1 for multi-subset focal elements $\{x_1, x_2\}$ is greater than that of the focal element x_2 . Focal element $\{x_1, x_2\}$ should assign more belief values to the focal element x_1 . According to the calculation results, we find that the results obtained by the *ITP* approach accord with intuitive judgment.

4.3. Advantages of *ITP*

Applying the Pignistic probability transformation method to calculate the probability distribution of each focal element in Example 1. We can obtain the result as follows:

$$Bet(x_1) = 0.55 \quad Bet(x_2) = 0.45$$

By comparison, it can be seen that our proposed method is more reasonable to allocate the belief value of multi-subset focal elements. *ITP* takes into account the belief function and plausibility function of the singleton, which can accurately redistribute the BPA of multi-subset focal elements to the singleton, and the decision result is more reasonable, low computational complexity, and convenient decision-making in an uncertain environment. Moreover, *ITP* overcomes the shortcomings of the traditional methods and can achieve reasonable results in different situations. In Section 6, we further illustrate the rational and effectiveness of the *ITP* through several examples.

5. Evaluation criteria for decision probability transformation method

In the process of decision probability transformation, we hope to contain more information content in the decision probability distribution after transformation, to facilitate making the correct decision. For measuring the information content in the transformed decision probability, we introduce the concepts of probabilistic information content (PIC) criterion in this paper. PIC criterion is utilized to quantify the information content of the transformed decision probability. Afterward, the PIC criterion is employed to evaluate the performance of each transformation method in our analysis. The larger the PIC value is, the more information content is included in the transformed decision probability, and the more accurate decision is made. Accordingly, the performance of the transformation method is stronger. To compare the performance of various transformation methods, we adopt the PIC criterion as an index to evaluate the performance of the transformation method. In Section 6, we give a detailed discussion. We first review the content for Shannon entropy.

5.1. Shannon Entropy

Entropy is a measure of uncertainty. For the first time, Shannon [60] introduced the concepts of entropy into the field of information theory, and defined the expression form for entropy in discrete finite sets.

Assuming that a probability distribution on random variable $X = \{x_1, x_2, \dots, x_n\}$ is $P = \{p(x_1), p(x_2), \dots, p(x_n)\}$, then the Shannon entropy of the random variable X is defined as follows:

$$H(X) = - \sum_{i=1}^n p(x_i) \log_b(p(x_i)) \quad (25)$$

In this paper, we take $b = 2$. Information entropy can measure the uncertainty for the random variable X . When the random variable X is uniformly distributed, i.e. $p = 1/n$, at this time, the random variable X has the maximum entropy. The maximum entropy is described as follows:

$$H_{max}(X) = - \sum_{i=1}^n \frac{1}{n} \log_2 \left(\frac{1}{n} \right) = \log_2(n) \quad (26)$$

The greater the information entropy is, the higher the degree of uncertainty for the random variable X is, at this time, it is not conducive to making a decision.

5.2. Probabilistic information content

In Ref. [61], Sudano proposed to use the probabilistic information content (PIC) criterion to evaluate the performance of various decision probability transformation methods. PIC criterion is an important evaluation index in the decision-making system. In the discrete probability distribution, the PIC criterion is defined as follows.

Suppose a probability distribution on random variable $X = \{x_1, x_2, \dots, x_n\}$ is $P = \{p(x_1), p(x_2), \dots, p(x_n)\}$, then the PIC criterion for the random variable X is defined as follows [61]:

$$PIC(X) = 1 + \frac{1}{H_{max}(X)} \left(\sum_{i=1}^n p(x_i) \log_2(p(x_i)) \right) \quad (27)$$

The PIC criterion is the normalized Shannon entropy dual, which can effectively quantify the probabilistic information content for the random variable X . The value range of the PIC criterion is [0,1]. A PIC value of one indicates that the information content in the probability distribution is completely determined, and there is no interference of uncertain information when making decisions. A PIC value of zero means no information can be utilized to make the right decision. In our work, we evaluate the performance of the decision probability transformation method by calculating the PIC criterion of the transformed decision probability.

6. Experiments and comparisons

In this chapter, the following examples are from [53], which are given to verify the rationality and accuracy of the *ITP* and compared it to the existing probability transformation methods. The PIC criterion is used to evaluate the performances of various probability transformation methods.

Example 2. Suppose there is a FOD $\Theta = \{a, b\}$ in the Shafer's model. m is a BPA defined on Θ , and the BPA is depicted as follows:

$$m(a) = 0.3 \quad m(b) = 0.1 \quad m(\{a, b\}) = 0.6$$

Sudano's probability transformation, Pignistic probability transformation, Cuzzolin's probability transformation, DSMP method, and *ITP* are employed to calculate the decision probability of the evidence m , respectively. The PIC values of each probability transformation method are calculated separately to evaluate the performance of different probability transformation methods. The results of the decision probability are presented in Table 1, and the evaluation results of each method are shown in Fig. 2.

Since the parameter ε in the DSMP method is difficult to determine, we take $\varepsilon = 0.5$ in this article. As shown in Table 1, among all decision probability transformation methods, the decision probability obtained by focal element a is the largest, which

Table 1
Comparison results in example 2.

	<i>a</i>	<i>b</i>	PIC(*)
<i>PrNPI</i>	0.5625	0.4375	0.0113
<i>BetP</i>	0.6	0.4	0.0291
<i>CuzzP</i>	0.6	0.4	0.0291
<i>PraPI</i>	0.6375	0.3625	0.0553
<i>PrPI</i>	0.6375	0.3625	0.0553
<i>DSmP</i>	0.6429	0.3571	0.0598
<i>PrBel</i>	0.75	0.25	0.1887
<i>ITP</i>	0.7754	0.2246	0.2315

Table 2
Comparison results in example 3.

	<i>a</i>	<i>b</i>	<i>c</i>	PIC(*)
<i>PrNPI</i>	0.1112	0.4444	0.4444	0.1216
<i>BetP</i>	0.2	0.4	0.4	0.0398
<i>PrPI</i>	0.2	0.4	0.4	0.0398
<i>PrBel</i>	0.2	NaN ^a	NaN	NaN
<i>CuzzP</i>	0.2	0.4	0.4	0.0398
<i>DSmP</i>	0.2	0.4	0.4	0.0398
<i>ITP</i>	0.2	0.4	0.4	0.0398
<i>PraPI</i>	0.2890	0.3555	0.3555	0.0041

^aRepresents not a number. Replace with 0 when drawing.

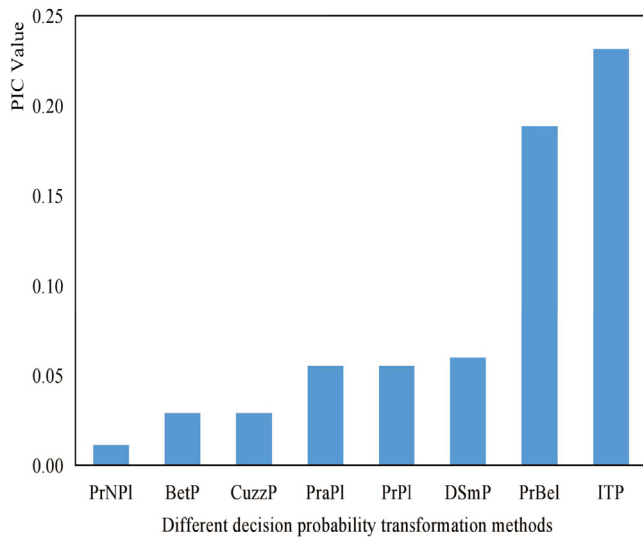


Fig. 2. The PIC values for each decision probability transformation method in Example 2.

is consistent with the intuitive analysis. This shows that our proposed method is reasonable. We note that the decision probability of the focal element *a* obtained by our proposed method is greater than that of other methods. Moreover, Fig. 2 shows that the PIC value obtained with ITP is greater than other probability transformation methods. This indicates that the transformed probability distribution generated by our proposed method can accurately reflect the information of the BPA, and effectively carry out the decision probability transformation. In this example, although the *PrBel* method can get relatively ideal results, there are some limitations of the *PrBel* method, which will be illustrated in the following examples.

Example 3. Let us consider the FOD $\Theta = \{a, b, c\}$ with Shafer's model, a BPA defined on Θ is described as $m(a) = 0.2$ $m(\{b, c\}) = 0.8$. Several decision probability transformation methods mentioned in this article are used to compute the decision probability of the evidence *m*, the results are described in Table 2, and the PIC values of different methods are presented in Fig. 3.

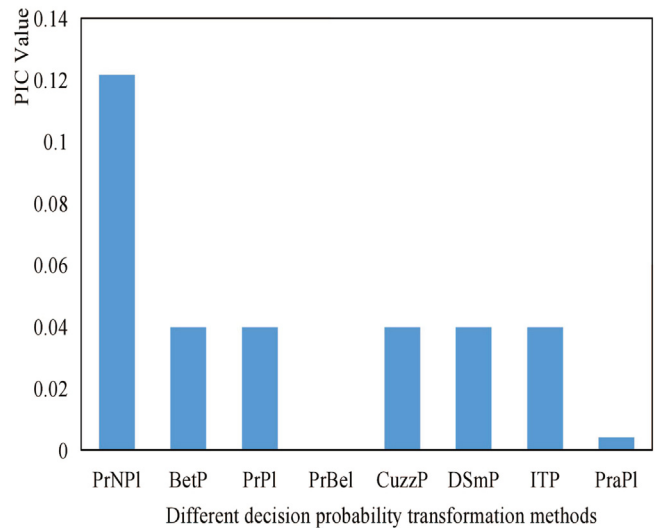


Fig. 3. The PIC values for each decision probability transformation method in Example 3.

In this example, the multi-subset focal elements do not contain the information of the singleton *a*, hence the belief value of the focal element *a* is unchanged when assigning the BPA of multi-subset focal elements $\{b, c\}$. Since the belief values of focal elements *b* and *c* are vacuous, the BPA of multi-subset focal elements $\{b, c\}$ cannot be reasonably allocated according to the existing information. The most conservative allocation is equal distribution. As can be seen from Table 2, in the results obtained by the *PrNPI* transformation method, the decision probability of the focal element *a* decreases ($0.1112 < 0.2$), which is unreasonable. The decision probability of the singleton should be greater than or equal to the belief value of that singleton. In the process of decision probability transformation, the decrease of belief value for the singleton signifies that there has information loss in the process of transformation, which is abnormal. In Fig. 3, although the *PrNPI* method has the largest PIC value, its probability transformation results are not satisfactory. *PrBel* method cannot effectively distribute the BPA of multi-subset focal elements $\{b, c\}$, because the decision probabilities of focal elements *b* and *c* are mathematically undefined. We can note that when the belief value of the singleton is vacuous, the *PrBel* method cannot reasonably allocate the BPA of multi-subset focal elements containing this singleton. Because of this limitation, the *PrBel* method cannot be extended to practical applications. It is interesting to note that the decision probability of the focal element *a* is increased in the results of the *PraPI* method, which is also obviously contradictory to the intuitive analysis. Such results are invalid. Because in multi-subset focal elements, there is no information about the focal element *a*. One sees that ITP coincides with other traditional methods and achieves reasonable results in this special case. This is an excellent illustration of the correctness of our theoretical work.

Example 4. In the given FOD $\Theta = \{a, b\}$, one of the BPA definition in Θ is described below:

$$m(a) = 0.4 \quad m(\{a, b\}) = 0.6$$

Sudano's probability transformation, Pignistic probability transformation, Cuzzolin's probability transformation, DSmP method, and ITP are utilized to calculate the decision probability of the evidence *m*, respectively. The results of decision probability and their corresponding PIC values are displayed in Table 3 and Fig. 4.

Table 3
Comparison results in example 4.

	<i>a</i>	<i>b</i>	PIC(*)
<i>PrNPI</i>	0.6250	0.3750	0.0456
<i>BetP</i>	0.7	0.3	0.1187
<i>CuzzP</i>	0.7	0.3	0.1187
<i>PrPI</i>	0.7750	0.2250	0.2308
<i>PraPI</i>	0.7750	0.2250	0.2308
<i>DSmP</i>	0.7857	0.2143	0.2504
<i>ITP</i>	0.9735	0.0265	0.8235
<i>PrBel</i>	1	NaN	NaN

Table 4
Comparison results in example 5.

	<i>a</i>	<i>b</i>	<i>c</i>	PIC(*)
<i>PrNPI</i>	0.3043	0.3913	0.3044	0.0067
<i>PrPI</i>	0.3093	0.4377	0.2530	0.0240
<i>BetP</i>	0.3333	0.3833	0.2834	0.0068
<i>CuzzP</i>	0.3455	0.3681	0.2864	0.0050
<i>DSmP</i>	0.3591	0.3659	0.2750	0.0073
<i>PraPI</i>	0.3739	0.3522	0.2739	0.0077
<i>ITP</i>	0.4140	0.3885	0.1975	0.0418
<i>PrBel</i>	0.7	NaN	NaN	NaN

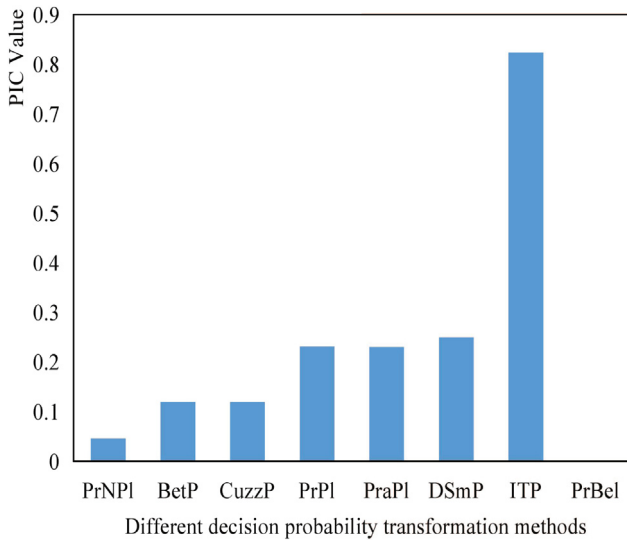


Fig. 4. The PIC values for each decision probability transformation method in Example 4.

As shown in Table 3, *PrBel* method allocated all BPA of multi-subset focal elements $\{a, b\}$ to the focal element *a*, which is obviously inconsistent with intuitive judgment. Since the multi-subset focal elements $\{a, b\}$ support the focal element *b* to some extent, some belief values should be allocated to the focal element *b*. The results allocated by the *PrBel* method are too optimistic to be conducive for decision-making. Our proposed method and other traditional methods can get reasonable results. We note that the decision probability of the singleton *a* obtained by *ITP* is greater than other traditional methods. Moreover, *ITP* gives here the best results in terms of the PIC values with compare to all other methods, which is shown in Fig. 4. This indicates that our proposed method can extract more information from the pre-transformation BPA, and make the decision probability transformation accurately.

Example 5. Suppose there is an evidence *m* in the FOD $\Theta = \{a, b, c\}$, whose BPA is described as follows:

$$m(a) = 0.1 \quad m(\{a, b\}) = 0.2 \quad m(\{b, c\}) = 0.3 \quad m(\{a, b, c\}) = 0.4$$

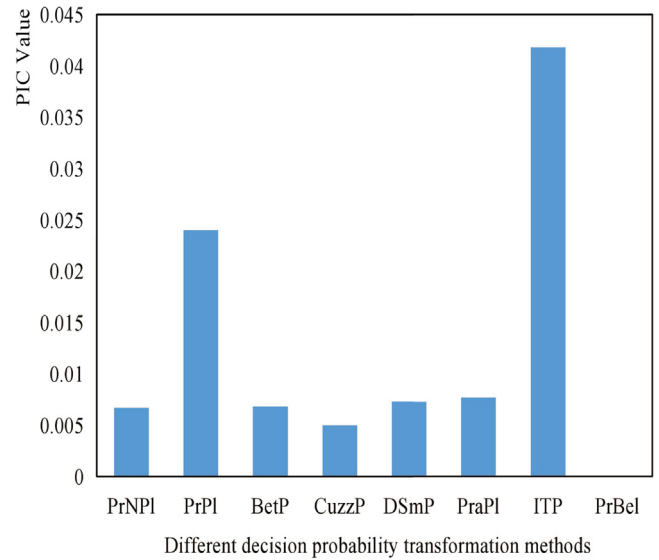


Fig. 5. The PIC values for each decision probability transformation method in Example 5.

Sudano's probability transformation, Pignistic probability transformation, Cuzzolin's probability transformation, DSmP method, and *ITP* are used to calculate the decision probability of the evidence *m*, respectively. The results of the decision probability are given in Table 4, and the values of the corresponding PIC are presented in Fig. 5.

In the DS evidence theory, multi-subset focal elements are a manifestation of the support for the singleton. As can be seen from the BPA of the evidence *m*, the belief value of the focal element *b* is vacuous, while the belief value of the singleton *a* is not vacuous, therefore, the support degree of multi-subset focal elements on singleton *a* should be greater than that on singleton *b*. Hence, the transformed decision probability should be that the singleton *a* is greater than the singleton *b*. As shown in Table 4, both the *PraPI* method and the *ITP* method can obtain reasonable transformation results. However, the decision probability of the singleton *a* obtained by the *ITP* method is greater than that of the *PraPI* method. It is shown that the probability transformation results of the *ITP* method are more accurate. *PrBel* method cannot achieve the decision probability of focal elements *b* and *c* through the existing information, and the multi-subset focal elements containing the singleton *a* distribute all the BPA to the singleton *a*, which is not fair nor intuitive. The decision probability of the singleton *b* obtained by other methods is the largest, which is different from the intuitive judgment. Fig. 5 shows that the *ITP* method has the largest PIC value, which indicates that the *ITP* method has the optimal probability transformation performance. Compared with other methods, the decision probability of getting focal element *a* in our proposed method is the largest, which is more conducive to making the correct decision. When making a decision probability transformation, our proposed method is more accurate and reasonable.

By analyzing the above examples, we note that the *ITP* method can be applied to any model, and the resulting decision probability is more reasonable and accurate than the traditional method. In some special evidence models, *ITP* method can overcome the shortcomings of the traditional methods. Furthermore, *ITP* method can precisely redistribute the BPA of multi-subset focal elements based on the belief value of the singleton. Ergo, *ITP* method is more efficient and applicable in decision probability transformation.

Table 5
The BPAs of each sensor data.

	T_1	T_2	T_3	$\{T_1, T_2\}$	$\{T_1, T_3\}$	$\{T_2, T_3\}$	\emptyset
m_1	0.38	0.15	0.15	0.15	0.07	0.07	0.03
m_2	0.30	0.40	0	0.30	0	0	0
m_3	0.28	0.42	0	0.30	0	0	0

7. Practical application

The transformation for the basic probability assignment to the decision probability distribution is a very common apply in many fields, such as pattern recognition [62,63], multi-criteria decision-making [64,65], etc. In these practical applications, probability transformation is helpful for people to make reasonable decisions. In this section, we present a new conflicting evidence fusion method based on the *ITP* approach, and apply it to the practical problem of target recognition to further validate the effectiveness and rationality of our proposed method.

7.1. A new method for fusing the conflict evidence

According to the *ITP* method, we present a novel fusion method for the conflict evidence. The main steps of the proposed method are described below.

- Step 1: Apply the evidence theory to model the data for each sensor to obtain the BPA of each sensor.
- Step 2: Use the *ITP* method to calculate the transformation probability of each evidence.
- Step 3: Suppose there are k pieces of evidence on the frame of discernment $F = \{F_1, F_2, \dots, F_n\}$. Using Eq. (28) to calculate the inconsistency between the transformation probabilities of each evidence, denoted as $Dis(m_i, m_j) (i \neq j)$, which is described below.

$$Dis(m_i, m_j) = \sqrt{1 - \left[\frac{\sum_{s=1}^n ITP_{m_i}(F_s) \cdot ITP_{m_j}(F_s)}{\|ITP_{m_i}\| \cdot \|ITP_{m_j}\|} \right]^2} \quad i, j = 1, 2, \dots, k \quad (28)$$

- Step 4: According to the inconsistency information, the similarity between each evidence is calculated by employing equation (29), denoted as $Sim(m_i, m_j) (i \neq j)$, which is depicted as follows.

$$Sim(m_i, m_j) = 1 - Dis(m_i, m_j) \quad i, j = 1, 2, \dots, k \quad (29)$$

- Step 5: According to the similarity between each evidence, the credibility $C(m_i)$ of each evidence is computed by:

$$C(m_i) = \sum_{j=1, j \neq i}^k Sim(m_i, m_j) \quad i = 1, 2, \dots, k \quad (30)$$

- Step 6: On the basis of the credibility of the evidence, the weight of the evidence is defined as follows.

$$\omega(m_i) = \frac{C(m_i)}{\sum_{i=1}^k C(m_i)} \quad (31)$$

- Step 7: On account of the weight of each evidence, the original evidence is modified to obtain the weighted average evidence $AVE(m)$, which is described as follows.

$$AVE(m) = \sum_{i=1}^k \omega(m_i) \times m_i \quad (32)$$

- Step 8: The weighted average evidence is fused $k - 1$ times by using Dempster's combination rule to generate the ultimate results.

Table 6
The decision probability distribution of each evidence.

	T_1	T_2	T_3
m_1	0.6004	0.2157	0.1839
m_2	0.3833	0.6167	0
m_3	0.3403	0.6597	0

7.2. Target recognition problem

Assuming that there is a multi-sensor target recognition problem [66], the actual target type needs to be identified from three known targets. The frame of discernment composed of the three kinds of targets is $T = \{T_1, T_2, T_3\}$. Three different types of sensors are employed to detect the target. Each sensor provides data information about the target type on the same FOD. The output readings of each sensor are modeled as BPAs, which are presented in Table 5.

In Table 5, evidence m_1 has the highest support degree for the target T_1 , while m_2 and m_3 have the maximum belief value for the target T_2 . Hence, we know that there is a high conflict between m_1 and other two bodies of evidence, which will generate a negative impact on the fusion process. Therefore, Dempster's combination rule cannot be directly utilized to fuse these evidence. According to the intuitive judgment, the result should have the greatest support degree for the target T_2 .

Using the *ITP* method to calculate the transformation probability of each evidence, and the results are shown in Table 6.

The new conflict evidence fusion method is applied to solve the problem of target recognition, and the results are displayed in Table 7.

From Table 7, it can be seen that the DS approach [16], Martin's method [67], and Jiang's method [68] almost get the same decision result. They all have the maximum belief support for the target T_1 , which contradicts the intuitive judgment. By contrast, our proposed approach has the greatest support degree for the target T_2 , which indicates that the results generated by our proposed method are more rational. This simple example shows that our proposed method is more reasonable and effective than other methods.

7.3. Managerial implications

How to transform the BPA into a decision probability accurately is an issue to be solved. The main intention of this paper is to provide an efficient approach to realize the transformation of BPA to decision probability to assist decision-makers in making decisions. In the DS evidence theory, BPA can effectively express and process uncertain information. However, it is difficult to make decisions directly through the BPA. Therefore, we need to transform the BPA into a decision probability. Some existing methods cannot make full use of the known information, and there is information loss in the transformation process. Moreover, we found that in some special evidence models, the existing methods will get unreasonable results. *ITP* method can overcome these problems. By using *ITP* method to transform the BPA, we can obtain precise decision probability. Furthermore, *ITP* method makes full use of the known information of the focal elements

Table 7
Combination results for different fusion methods.

	T_1	T_2	T_3	$\{T_1, T_2\}$	$\{T_1, T_3\}$	$\{T_2, T_3\}$	Θ	Target
DS	0.502	0.458	0	0.040	0	0	0	T_1
Martin	0.491	0.462	0	0.047	0	0	0	T_1
Jiang	0.452	0.438	0.005	0.092	0.002	0.002	0.009	T_1
Our results	0.416	0.538	0.001	0.045	0	0	0	T_2

and considers the preference relation between the focal elements, which can improve the accuracy of decision probability transformation. In different evidence models, *ITP* method can generate accurate decision results. In practical decision-making problems, *ITP* method is more reasonable and effective.

8. Conclusion

When using BPA to describe the uncertain information of multi-elements proposition, we cannot give the accurate decision probability. How to transfer the BPA into a decision probability precisely is a problem that must be solved. In decision applications, it is necessary to use a better decision probability transformation method to assign the BPA of multi-elements proposition. In view of the deficiencies of existing decision probability transformation methods, we developed a new decision probability transformation method based on the belief interval, called *ITP*. *ITP* method considers the transformation of BPA from the perspective of the belief interval. First, applying the C-OWA operator to quantify the data information about the belief interval of the singleton. Then, we propose a method to compute the support degree of the singleton based on quantitative data information. Ultimately, according to the support degree of the singleton, the BPA of multi-subset focal elements is allocated reasonably. We also introduce the concepts of probabilistic information content in our work, which is utilized to evaluate the performance of the decision probability transformation method. Furthermore, *ITP* method coincides with the *BetP* method when all belief values of singletons included in ignorance are zero. We have clearly shown through numerical examples that the *ITP* method can accurately and reasonably transform the BPA into a decision probability, and has the larger PIC values than other traditional methods. In future work, we will further research the rationality of the *ITP* method, and extend the method to practical engineering applications.

CRedit authorship contribution statement

Zhan Deng: Designed research, Performed research, Wrote this paper. **Jianyu Wang:** Designed research, Performed research, Wrote this paper.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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