Comparative Study of Contradiction Measures in the Theory of Belief Functions

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***Abstract*—Uncertainty measures in the theory of belief functions are important for the uncertainty representation and reasoning. Many measures of uncertainty in the theory of belief functions have been introduced. The degree of discord (or conflict) inside a body of evidence is an important index for measuring uncertainty degree. Recently, distance of evidence is used to define a contradiction measure for quantifying the degree of discord inside a body of evidence. The contradiction measure is actually the weighted summation of the distance values between a given basic belief assignment (bba) and the categorical bba's defined on each focal element of the given bba redefined in this paper. It has normalized value and can well characterize the self-discord incorporated in bodies of evidence. We propose here, some numerical examples with comparisons among different uncertainty measures are provided, together with related analyses, to show the rationality of the proposed contradiction measure.**

***Index Terms*—Evidence theory, uncertainty measure, belief function, discord, conflict.**

I. Introduction

Dempster-Shafer evidence theory [1], also known as theory of belief functions, is one of the important uncertainty reasoning tools. It has been widely used in many applications. Evidence theory can be seen as a generalization of probability theory, where the additivity axiom is excluded. In probability theory, Shannon entropy [2] is often used for quantifying uncertainty while in the framework of evidence theory, there also need the uncertainty measure for quantifying the degree of uncertainty incorporated in a body of evidence (BOE).

In uncertainty theories, we can consider two types of uncertainty including discord (or conflict) and non-specificity, hence ambiguity [3]. There have emerged several types of uncertainty measures in the theory of belief functions. They are either the generalization of Shannon entropy and other types of uncertainty measures in probability theory or are established based on the conflict obtained by using some combination rule. For example, non-specificity [4] proposed by Dubois and Prade is a generalization of Hartley entropy [5]; aggregate uncertainty (AU) measure [6] and ambiguity measure (AM) [3] can be regarded as the generalized forms of Shannon entropy. In Martin's work [7], [8], the auto-conflict measure was proposed based on the conjunctive combination rule. There are also lots of other types of uncertainty measures in the theory of belief functions (See details in [3], [9], [11]). All the available uncertainty measures characterize the uncertainty either from one aspect (*e.g.* non-specificity and discord) or as a whole, *i.e.* the total uncertainty (*e.g.*, AM and AU).

Like in [7], [11], we attempt to break the traditional ways to establish uncertainty measure in the theory of belief functions. That is, we do not generalize the uncertainty measures in probability theory or use combination rule to obtain the uncertainty measures in theory of belief functions. In this paper we modify the contradiction measure proposed in [11] to characterize the internal conflict (or discord) degree of the uncertainty in bba's. For a bba with L focal elements, based on each focal element, a categorical bba (a bba with a unique focal element) can be obtained. Thus there are totally L categorical bba's. We calculate Jousselme's distance of evidence [10] between the original given bba and each categorical bba then we can obtained L values of distance. By using the masses of the given bba to generate the weights and executing weighted summation of the corresponding L distance values, the contradiction can be obtained. To make the contradiction measure be normalized, the normalization factor is designed and added. Some simulation results are provided to verify the correctness of the normalization factor. This contradiction measure can well characterize the conflict incorporated in a BOE, *i.e.* the self-conflict or internal conflict. Some numerical examples with comparisons among different uncertainty measures in the theory of belief functions are also provided to show the rationality of the proposed contradiction measure. It should be noted that this work is based on our previous paper [11]. The idea of constructing contradiction measure based on distance of evidence is first preliminarily proposed in that paper, where there exist some errors in the definition -corrected here- and related analyses are far from enough.

II. Basics in the theory of belief functions

A. Basic concepts in the theory of belief functions

In Dempster-Shafer evidence theory [1], The elements in the frame of discernment (FOD) (denoted by Θ) are mutually exclusive and exhaustive. Suppose that 2^Θ denotes the powerset of FOD and define the function $m : 2^\Theta \to [0, 1]$ as the basic belief assignment (bba) satisfying:

$$\sum_{A \subseteq \Theta} m(A) = 1,\ m(\emptyset) = 0 \tag{1}$$

A bba is also called a mass function. Belief function (Bel) and plausibility function (Pl) are defined below, respectively:

$$Bel(A) = \sum_{B \subseteq A} m(B) \tag{2}$$

$$pl(A) = \sum_{A \cap B \neq \emptyset} m(B) \tag{3}$$

Suppose there are two bba's: m_1, m_2 over the FOD Θ with focal elements $A_1, \dots, A_k$ and $B_1, \dots, B_l$, respectively. If $k = \sum_{A_i \cap B_j = \emptyset} m_1(A_i)m_2(B_j) < 1$, $m : 2^\Theta \to [0,1]$ denoted by

$$m(A) = \begin{cases} 0,\ A = \emptyset \\ \dfrac{\sum\limits_{A_i \cap B_j = A} m_1(A_i)m_2(B_j)}{1 - \sum\limits_{A_i \cap B_j = \emptyset} m_1(A_i)m_2(B_j)}, A \neq \emptyset \end{cases} \tag{4}$$

is a bba. The rule defined in Eq. (4) is called Dempster's rule of combination. In Dempster's rule of combination,

$$K = 1 - \sum_{A_i \cap B_j = \emptyset} m_1(A_i)m_2(B_j) \tag{5}$$

is used to represent the conflict between two BOEs. In recent research [12], both K and distance of evidence are used to construct a two tuple to represent the conflict between BOEs.

B. Uncertainty measures in the theory of belief functions

In the theory of belief functions, a BOE hides two types of uncertainty: non-specificity [4] and discord, hence ambiguity [3]. The available related definitions on degree of uncertainty in the theory of belief functions are briefly introduced below.

1) Auto-conflict

A n-order auto-conflict measure was proposed in [7] based on non-normalized conjunctive combination rule [13].

$$a_n = \left(\overset{n}{\underset{i=1}{\oplus}} m \right)(\emptyset) \tag{6}$$

The conjunctive combination rule $\oplus$ is defined as

$$m_{Conj}(C) = \sum_{A \cap B = C} m_1(A)m_2(B) := (m_1 \oplus m_2)(C) \tag{7}$$

When $n = 2$, the auto-conflict equals to K in Dempster's rule of combination.

2) Non-specificity

$$N(m) = \sum_{A \subseteq \Theta} m(A) \log_2 |A| \tag{8}$$

Non-specificity can be seen as weighted sum of the Hartley measure for different focal elements.

3) Confusion

Höhle proposed the measure of confusion [14] by using bba and belief function in spirit of entropy as follows.

$$Confusion(m) = -\sum_{A \in \Theta} m(A) \log_2(Bel(A)) \tag{9}$$

4) Dissonance

Yager proposed the measure of Dissonance [14] by using bba and plausibility function in spirit of entropy as follows.

$$Dissonance(m) = -\sum_{A \in \Theta} m(A) \log_2(Pl(A)) \tag{10}$$

5) Aggregate Uncertainty measure (AU)

There have emerged several definitions aiming to represent the total uncertainty in the theory of belief functions. The most representational one is a kind of generalized Shannon entropy [2], *i.e.* the aggregated uncertainty (AU) [6].

Let Bel be a belief measure on the FOD Θ. The AU associated with Bel is measured by:

$$\text{AU}(Bel) = \max_{\mathcal{P}_{Bel}} [-\sum_{\theta \in \Theta} p_\theta \log_2 p_\theta] \tag{11}$$

where the maximum is taken over all probability distributions that are consistent with the given belief function. $\mathcal{P}_{Bel}$ consists of all probability distributions $\langle p_\theta | \theta \in \Theta \rangle$ satisfying:

$$\begin{cases} p_\theta \in [0,1], \forall \theta \in \Theta \\ \sum_{\theta \in \Theta} p_\theta = 1 \\ Bel(A) \leq \sum_{\theta \in A} p_\theta \leq 1 - Bel(\bar{A}),\ \forall A \subseteq \Theta \end{cases} \tag{12}$$

As illustrated in Eq. (11) and Eq. (12), in the definition of AU, the calculation of AU is an optimization problem and bba's (or belief functions) are used to establish the constraints of the optimization problem. It is also called the "upper entropy". AU is an aggregated total uncertainty (ATU) measure, which can capture both non-specificity and discord.

AU satisfies all the requirements for uncertainty measure [9], which include probability consistency, set consistency, value range, sub-additivity and additivity for the joint BPA in Cartesian space. However, AU has the following shortcomings [3]: high computing complexity, high insensitivity to the changes of evidence, etc.

6) Ambiguity Measure (AM)

Jousselme *et al* [3] proposed AM (ambiguity measure) aiming to describe the non-specificity and discord in the theory of belief functions. Let $\Theta = \{\theta_1, \theta_2, \dots, \theta_n\}$ be a FOD. Let m be a bba defined on Θ. Define

$$\text{AM}(m) = -\sum_{\theta \in \Theta} \text{BetP}_m(\theta) \log_2(\text{BetP}_m(\theta)) \tag{13}$$

where $\text{BetP}_m(\theta) = \sum_{\theta \in B, B \subseteq \Theta} m(B)/|B|$ is the pignistic probability distribution proposed by Smets [16]. Jousselme *et al* [3] declared that the ambiguity measure satisfies the requirements of uncertainty measure and at the same time it overcomes the defects of AU, but in fact AM does not satisfy the sub-additivity which has been pointed out by Klir [17]. Moreover in the work of Abellan [9], AM has been proved to be logically non-monotonic under some circumstances.

There are also other existing uncertainty measures in the theory of belief functions, see details in related reference [3].

III. Contradiction Measure Based on Distance of Evidence

As we can see in the previous section, all the available uncertainty measures in the theory of belief functions are direct or indirect generalization of entropy defined in probability theory or are defined by using some combination rule. Hence in [11], we break such ways in spirit of entropy in probability theory. Distance of evidence is used to construct the uncertainty degree, which is called contradiction and shown below.

$$Contr_m(m) = \sum_{X \in \mathcal{X}} m(X) \cdot d(m, m_X) \tag{14}$$

where $\mathcal{X}$ represents the set of all the focal elements of $m(\cdot)$. But it should be noted that the definition in Eq. (14) is not a normalized value. We should obtain a normalized definition for the convenience of use.

The maximum contradiction measure for $m(\cdot)$ defined on $\Theta = \{\theta_1, \theta_2, ..., \theta_n\}$ occurs when $m(\cdot)$ has a uniform distribution:

$$m(\{\theta_1\}) = m(\{\theta_2\}) = \cdots = m(\{\theta_n\}) = \frac{1}{n}$$

It depends on the cardinality of Θ and the distance used.

For $|\Theta| = n$, we use Jousselme's distance, we get max $Contr_m = \sqrt{\frac{n-1}{2n}}$.

Proof:

$$Contr_m = n \cdot \frac{1}{n} \cdot d(m, m_{\theta_i}) = d(m, m_{\theta_i})$$

i.e.: where

$$\begin{cases} m_{\theta_i}(\{\theta_i\}) = 1, \\ m_{\theta_i}(\{\theta_j\}) = 0, j \neq i, j = 1, ..., n \end{cases}$$

But the distance between m and m_{θ_i} is the same,

$$d(m, m_{\theta_i}) = \sqrt{(m - m_{\theta_i})^T \mathbf{Jac}(m - m_{\theta_i})}$$

$$= \sqrt{0.5 \left[\tfrac{n-1}{n}, -\tfrac{1}{n}, \cdots, -\tfrac{1}{n}\right] \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \frac{n-1}{n} \\ -\frac{1}{n} \\ \vdots \\ -\frac{1}{n} \end{bmatrix}}$$

$$= \sqrt{0.5 \left[\tfrac{n-1}{n}, -\tfrac{1}{n}, \cdots, -\tfrac{1}{n}\right] \begin{bmatrix} \frac{n-1}{n} \\ -\frac{1}{n} \\ \vdots \\ -\frac{1}{n} \end{bmatrix}}$$

$$= \sqrt{0.5 \left[\left(\tfrac{n-1}{n}\right)^2 + (n-1) \cdot \tfrac{1}{n^2}\right]}$$

$$= \sqrt{0.5 \tfrac{(n-1)^2 + n - 1}{n^2}} = \sqrt{0.5 \tfrac{n^2 - n}{n^2}} = \sqrt{\tfrac{n-1}{2n}}$$

Therefore, in this paper, we use the normalized factor $\sqrt{\frac{n-1}{2n}}$ and then the correct normalized contradiction measure is defined below:

$$Contr_m(m) = \sqrt{\frac{2n}{n-1}} \cdot \sum_{X \in \mathcal{X}} m(X) \cdot d(m, m_X) \tag{15}$$

To further verify the correctness of the normalization factor, we design the experiments as follows.

Randomly generate 500 bba's and calculate their corresponding contradiction values based on Eq. (15). The method to randomly generate bba's is as follows [18].

Input: Θ : Frame of discernment;
N_{max}: Maximum number of focal elements
Output: Bel: Belief function (under the form of a bba, m)
Generate the power set of Θ $\mathcal{P}(\Theta)$;
Generate a random permutation of $\mathcal{P}(\Theta) \to \mathcal{R}(\Theta)$;
Generate a integer between 1 and $N_{max} \to k$;
FOReach First k elements of $\mathcal{R}(\Theta)$ do
Generate a value within $[0, 1] \to m_i, i = 1, ..., k$;
END
Normalize the vector $m = [m_1, ..., m_k] \to m'$;
$m(A_k) = m_k$;
Algorithm 1: Random generation of bba

Based on the above algorithm, the bba's generated have random number of focal elements. We set the cardinality of FOD to be 3 and 4, respectively in each experiment. Thus we totally do two experiments and the experimental results are illustrated in Fig.1.

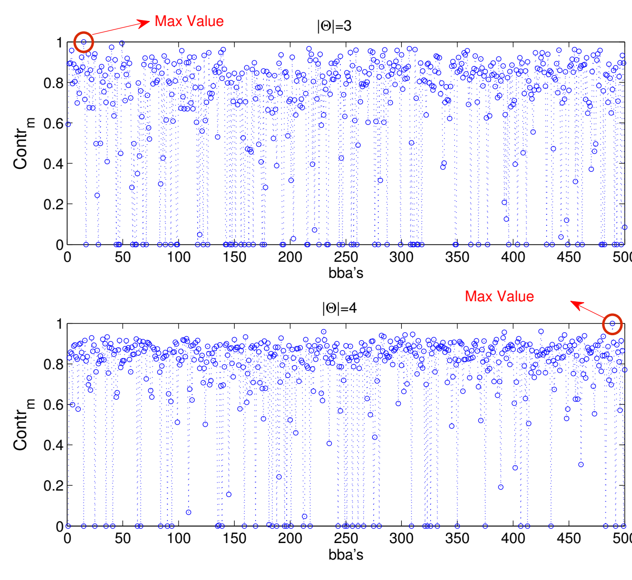

Fig. 1. Values of contradiction $Contr_m$

As shown in Fig.1, when $|\Theta| = 3$, the max value (one) is obtained at the 15th bba, which is:

$$m(\{\theta_1\}) = m(\{\theta_2\}) = m(\{\theta_3\}) = 1/3.$$

When $|\Theta| = 4$, the max value (one) is obtained at the 489th bba, which is:

$$m(\{\theta_1\}) = m(\{\theta_2\}) = m(\{\theta_3\}) = m(\{\theta_4\}) = 1/4.$$

From the proof and the experiments above, it can be seen that the selection of normalized factor is correct.

IV. EXAMPLES

A. Example 1

In this experiment, we use the bba's with focal elements of singletons and the total set. Suppose that the FOD is $\Theta = \{\theta_1, \theta_2, ..., \theta_5\}$. The initial bba is

$$m(\{\theta_1\}) = m(\{\theta_2\}) = m(\{\theta_3\}) = m(\{\theta_4\}) = m(\{\theta_5\}) = 0; \quad m(\Theta) = 1$$

Then at each step, the mass of $m(\Theta)$ decreases by $\Delta = 0.05$, and the mass of each $m(\{\theta_i\})$ increase by $\Delta/5 = 0.01$, where $i = 1, ..., 5$. After 20 steps, $m(\Theta)$ will become zero and $m(\{\theta_1\}) = m(\{\theta_2\}) = m(\{\theta_3\}) = m(\{\theta_4\}) = m(\{\theta_5\}) = 0.2$. Then the experiment will finish.

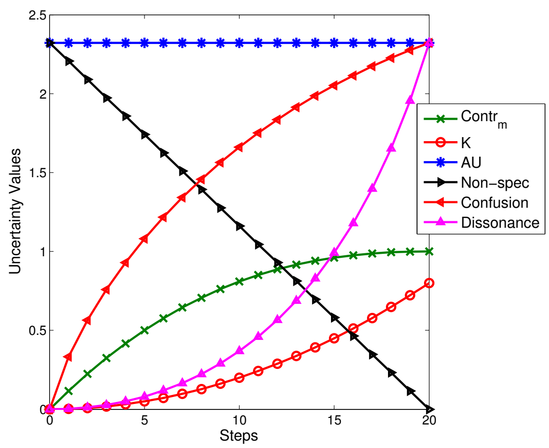

Fig. 2. Comparisons among different uncertainty measures in Example 1

As we can see in Fig. 2, although AU is deemed as a total uncertainty measure, we cannot detect the change of bba in each step based on AU.

The values of non-specificity decrease with the increase of masses of singletons.

For contradiction, K, dissonance and confusion, their values all increase with the increase of masses of singletons. Contradiction increases faster than K in the first half of all the steps and then it increases slower than K in the second half. Confusion increases faster than dissonance in the first half of all the steps and it increases slower than dissonance in the second half. The change trends of contradiction and confusion are more rational. Because at the first half of all the steps, the relative changes of the masses of singletons increase more significantly than the relative changes in the second half.

The value of contradiction belongs to $[0, 1]$ and it reaches its maximum value at the final step, *i.e.*:

When $m(\{\theta_1\}) = m(\{\theta_2\}) = m(\{\theta_3\}) = m(\{\theta_4\}) = m(\{\theta_5\}) = 0.2$, $Contr_m = 1$

B. Example 2

In this experiment, we use the bba's with focal elements of singletons and the total set. Suppose that the FOD is $\Theta = \{\theta_1, \theta_2, ..., \theta_5\}$. The initial bba is

$$m(\{\theta_1\}) = m(\{\theta_2\}) = m(\{\theta_3\}) = m(\{\theta_4\}) = m(\{\theta_5\}) = 0; \quad m(\Theta) = 1$$

Then at each step, the mass of $m(\Theta)$ decreases by $\Delta = 0.05$, and the mass of one singleton $m(\{\theta_1\})$ increase by $\Delta = 0.05$ at each step. After 20 steps, $m(\Theta)$ will become zero and the experiment will finish.

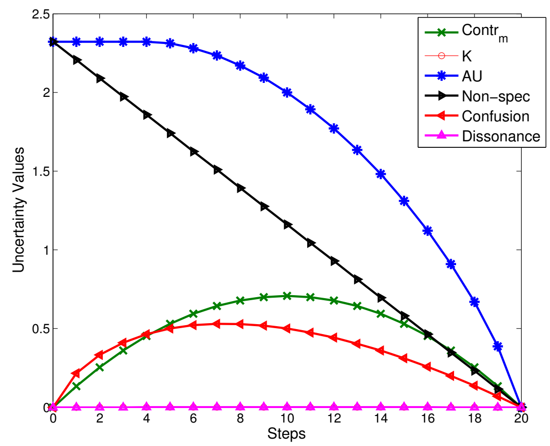

Fig. 3. Comparisons among different uncertainty measures in Example 2

As we can see in Fig. 3, with the increase of $m(\{\theta_1\})$ and the decrease of $m(\Theta)$ in each step, the AU and non-specificity decrease.

Although for the original bba, the non-specificity is highest, the conflict inside should be the least. So AU can not characterize the discord part of the uncertainty incorporated in the BOE.

K and Dissonance cannot detect the change of bba.

The value of the proposed contradiction increases at first and reaches the max value when the bba becomes

$$m(\{\theta_1\}) = 0.5, m(\Theta) = 0.5$$

Then with the increase of $m(\{\theta_1\})$ and the decrease of $m(\Theta)$ in following steps, the value of the proposed contradiction decrease and it reach zero when $m(\{\theta_1\}) = 1$, which is the clearest case.

If we consider the two focal elements $\{\theta_1\}$ and Θ are different in the power-set of Θ, when their values are equal the uncertainty reaches the max value. This should be more rational.

Confusion has the similar change trend compared to that of our proposed contradiction measure. But the maximum value of confusion does not occur at the middle.

C. Example 3

In this experiment, we use the bba's with focal elements of the same cardinality. Suppose that the FOD is $\Theta = \{\theta_1, \theta_2, ..., \theta_5\}$.

The initial bba is

$$m(\{\theta_1, \theta_2\}) = m(\{\theta_1, \theta_3\}) = m(\{\theta_1, \theta_4\})$$
$$= m(\{\theta_2, \theta_3\}) = m(\{\theta_2, \theta_4\}) = 0; m(\{\theta_3, \theta_4\}) = 1$$

Then at each step, the mass of $m(\{\theta_3, \theta_4\})$ decreases by $\Delta = 0.05$, and the masses of all the other focal elements increase by $\Delta = 0.05/5 = 0.01$ at each step. After 16 steps, masses of all the focal elements become equal.

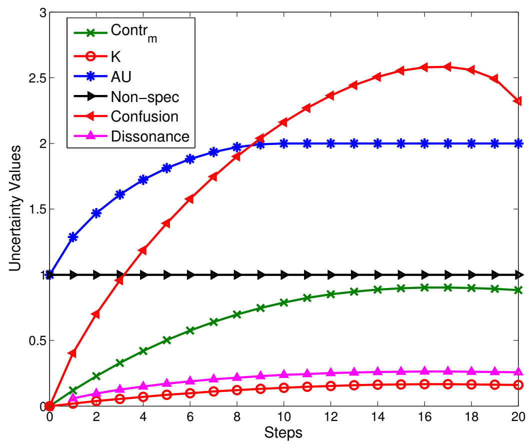

Fig. 4. Comparisons among different uncertainty measures in Example 3

As we can see in Fig. 4, Non-specificity can not detect the change of bba. This is because Non-specificity mainly concerns the cardinality of focal elements.

AU can detect the change of bba, but after step 10, the values of AU are the same with the change of bba in following steps. Thus AU is not sensitive to the change of bba.

With the change of bba in each step, K and Dissonance change very little. Thus here K and dissonance are not so sensitive to the change of bba.

For contradiction proposed and confusion, they can detect the change of bba well.

D. Example 4

Suppose that the FOD is $\Theta = \{\theta_1, \theta_2\}$. The initial bba is

$$m(\{\theta_1\}) = a, \quad m(\{\theta_2\}) = b,$$

$$m(\{\theta_1, \theta_1\}) = 1 - a - b.$$

Suppose that $a, b \in [0, 0.5]$, we calculate the values of all the uncertainty measures according to the change of a and b

As we can see in Fig. 5, with the change of a and b, AU are always the same.

All the other measures can detect the change of a and b.

We can see that the value of the proposed contradiction varies relatively uniformly when compared with other meaures. Thus the contradiction is not too sensitive and at the same time not too insensitive to the change of bba.

The value range belongs to [0,1], which is good characteristic for being a measure for quantifying the degree of discord.

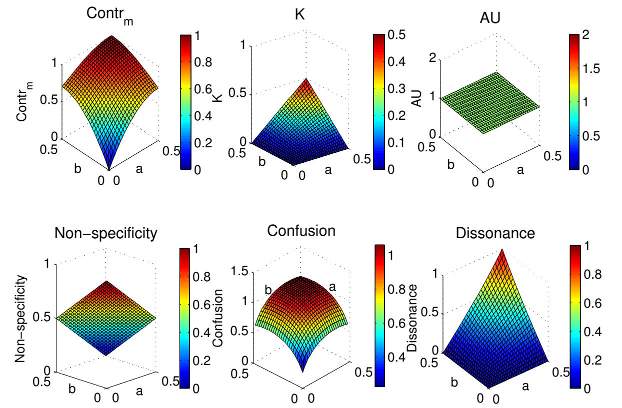

Fig. 5. Comparisons among different uncertainty measures in Example 4

V. Further Analysis

In definition of $Contr_m$ in Eq. (15), the distance used is Jousselme's distance. In our work, we have also tried other types of distances in the theory of belief functions to construct the contradiction, which include

1) Betting commitment distance (Pignistic probability distance)

$$d_T(m_1, m_2) = \max_{A \subseteq \Theta} \{|\text{BetP}_1(A) - \text{BetP}_2(A)|\} \tag{16}$$

where BetP represents the pignistic probability of corresponding bba.

2) Cuzzonlin distance

$$d_{Cuzz}(m_1, m_2) = \sqrt{(m_1, m_2)^T \mathbf{Inc}\mathbf{Inc}^T (m_1, m_2)} \tag{17}$$

where $\mathbf{Inc}$ is

$$\begin{cases} \mathbf{Inc}(A, B) = 1, if A \subseteq B \\ \quad 0, \text{others} \end{cases} \tag{18}$$

3) Conflict distance

$$d_K((m_1, m_2)) = m_1^T (\mathbf{I} - \mathbf{Inc}) m_2 \tag{19}$$

4) Bhattacharyya distance

$$d_B(m_1, m_2) = (1 - \sqrt{m_1}^T \mathbf{I} \sqrt{m_2})^p \tag{20}$$

We do following experiments to compare the different contradiction measures defined on the different distance definitions above. When we use d_{Cuzz} and d_K to construct normalized contradiction measures, the normalization factor should be $(n - 1)/n$.

A. Example 5

Suppose that the FOD is $\Theta = \{\theta_1, \theta_2, \theta_3\}$.

The initial bba is

$$m(\{\theta_1\}) = m(\{\theta_2\}) = m(\{\theta_3\}) = 0; m(\Theta) = 1$$

Then at each step, the mass of $m(\Theta)$ decreases by $\Delta = 0.05$, and the mass of each $m(\{\theta_i\})$ increase by $\Delta/3 = 0.05/3$,where $i = 1, 2, 3$.

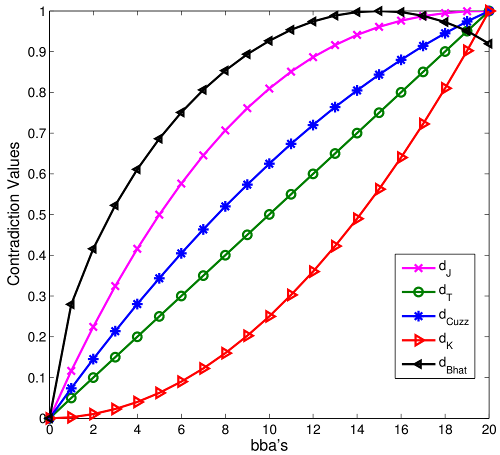

Fig. 6. Comparisons among different contradiction measures based on different distance measures - Example 5

B. Example 6

Suppose that the FOD is $\Theta = \{\theta_1, \theta_2, \theta_3\}$.
The initial bba is

$$m(\{\theta_1\}) = m(\{\theta_2\}) = m(\{\theta_3\}) = 0; m(\Theta) = 1$$

Then at each step, the mass of $m(\Theta)$ decreases by $\Delta = 0.05$, and the mass of $m(\{\theta_1\})$ increase by $\Delta = 0.05$. In the final step, the bba obtained is

$$\begin{array}{l} m(\{\theta_1\}) = 1 \\ m(\{\theta_2\}) = m(\{\theta_3\}) = m(\Theta) = 0; \end{array}$$

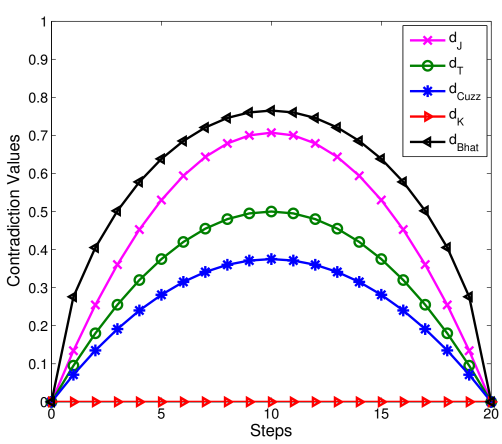

Fig. 7. Comparisons among different contradiction measures based on different distance measures - Example 5

As we can see in Example 5 and 6, all the contradiction measures obtained based on different distance definitions can well characterize the degree of discord inside BOEs. Till now, only Jousselme's distance is a strict distance metric, so we suggest to use Jousselme's distance.

VI. Conclusion

In this paper, we propose a new normalization of a measure called contradiction to characterize the degree of discord or conflict inside a body of evidence. This contradiction measure is distance-based and it can well describe the discord part of the uncertainty in the theory of belief functions. Some numerical examples are provided to support the rationality of the proposed contradiction measure.

In our work, we have also preliminarily tried other types of distance in evidence theory to construct the contradiction measure. In our future work, we will further analyze the contradiction defined on different distance measures. Contradiction measure can represent the qualities of different information sources to some extent. Thus we will also try to use the contradiction measure in applications based on the evaluation of bba's, for example, the weights determination in weighted evidence combination.

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