DOI :10.13338/j.issn.1006-8341.2007.04.012

第 20 卷第 4 期 纺 织 高 校 基 础 科 学 学 报 Vol.20, No.4
2007 年 12 月 BASIC SCIENCES JOURNAL OF TEXTILE UNIVERSITIES Dec., 2007

Article ID: 1006-8341(2007)04-0361-03

On the divisor function $\sigma_\alpha(n)$ involving the F. Smarandache simple function

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Abstract: The asymptotic properties of $\sigma_\alpha(p(n))$ are studied by using the analytic method in this paper, and two interesting asymptotic formulas for it are given, where $\sigma_\alpha(n)$ is the divisor function.

Key words: F. Smarandache simple function; mean value; asymptotic formula

CLC number: O 156.4 **Document code**: A

1 Introduction and theorems

For any positive integer n, the F. Smarandache function $S(n)$ is defined as the smallest $m\in\mathbf{N}_+$, such that $n|m!$. Reference [1] studied the relations between the functions $S_1(n)$ and $\varphi(n)$, where $S_1(n)$ is defined as follows: $S_1(n)=\max\{\alpha_i p_i\}$ if $n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$, where $n>1$. For a fixed prime p, the Smarandache simple function $S_p(n)$ is defined as the smallest $m\in\mathbf{N}_+$, where $p^n\mid m!$. In reference [2], Jozsef Sandor introduced the additive analogue of the Smarandache simple function $p(x)$ as follows:

$$p(x)=\min\{m\in\mathbf{N}_+: p^x\leqslant m!\},$$

and

$$p_*(x)=\max\{m\in\mathbf{N}_+: m!\leqslant p^x\},$$

which is denfied on a subset of real numbers. It is obvious that $p(x)=m$, if $(m-1)!<p^x\leqslant m!$ for $x\geqslant 1$. About the properties of $p(x)$, many scholars showed great interests in it[2-4]. In reference [4], Liu Hua studied the mean value properties of $d(p(x))$ and proved the following asymptotic formula:

$$\sum_{n\leqslant x}d(p(n))=x(\ln x-\ln\ln x)+O(x\ln p),$$

where $d(x)$ is the Dirichlet divisor function.

The main purpose of this paper is to study the asymptotic properties of the mean value of $\sigma_\alpha(p(x))$, where $\sigma_\alpha(n)$ is the divisor function, and give some interesting asymptotic formulas for it. That is, we shall prove the following:

Theorem 1 For any real number $x\geqslant 1$, we have

$$\sum_{n\leqslant x}\sigma_\alpha(p(n))=\begin{cases}\dfrac{\zeta(\alpha+1)}{\alpha+1}\dfrac{x^{\alpha+1}\ln^\alpha p}{\ln^{\alpha+1}x}\left[\ln\left(\dfrac{x\ln p}{\ln x}\right)-\dfrac{1}{\alpha+1}\right]+O\left(\dfrac{x^\alpha}{\ln^{\alpha-1}x}\right), & \text{if } \alpha>1,\\ \dfrac{\pi^2}{12}\dfrac{x^2\ln p}{\ln^2 x}\left[\ln\left(\dfrac{x\ln p}{\ln x}\right)-\dfrac{1}{2}\right]+O(x\ln x), & \text{if } \alpha=1,\\ \dfrac{\zeta(\alpha+1)}{\alpha+1}\dfrac{x^{\alpha+1}\ln^\alpha p}{\ln^{\alpha+1}x}\left[\ln\left(\dfrac{x\ln p}{\ln x}\right)-\dfrac{1}{\alpha+1}\right]+O\left(\dfrac{x}{\ln x}\right), & \text{if } 0<\alpha<1,\end{cases}$$

Received Date: 2007-05-27

Funding item: Supported by NSFC(10671155)

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where $\zeta(s)$ is the Riemann zeta-function.

Theorem 2 For any real number $x\geqslant 1$, we have

$$\sum_{n\leqslant x}\sigma_\alpha(p_*(n))=\begin{cases}\frac{\zeta(\alpha+1)}{\alpha+1}\frac{x^{\alpha+1}\ln^{\alpha}p}{\ln^{\alpha+1}x}\left(\ln\left(\frac{x\ln p}{\ln x}\right)-\frac{1}{\alpha+1}\right)+O\left(\frac{x^{\alpha}}{\ln^{\alpha-1}x}\right), & \text{if } \alpha>1,\\ \frac{\pi^2}{12}\frac{x^2\ln p}{\ln^2 x}\left(\ln\left(\frac{x\ln p}{\ln x}\right)-\frac{1}{2}\right)+O(x\ln x), & \text{if } \alpha=1,\\ \frac{\zeta(\alpha+1)}{\alpha+1}\frac{x^{\alpha+1}\ln^{\alpha}p}{\ln^{\alpha+1}x}\left(\ln\left(\frac{x\ln p}{\ln x}\right)-\frac{1}{\alpha+1}\right)+O\left(\frac{x}{\ln x}\right), & \text{if } 0<\alpha<1.\end{cases}$$

2 Proof of the Theorems

2.1 Lemmas

In this section, we will complete the proof of the theorems. Firstly, we need the following two lemmas.

Lemma 1 For any real number $x\geqslant 1$, we have the asymptotic formula

$$\sum_{n\leqslant x}\sigma_1(n)=\frac{\pi^2}{12}x^2+O(x\ln x).$$

Lemma 2 For any real number $x\geqslant 1$ and $\alpha>0$, $\alpha\neq 1$, we have the asymptotic formula

$$\sum_{n\leqslant x}\sigma_\alpha(n)=\frac{\zeta(\alpha+1)}{\alpha+1}x^{\alpha+1}+O(x^{\beta}),$$

where $\beta=\max\{1,\alpha\}$.

The proof of Lemma 1 and Lemma 2 can be found in reference [5].

2.2 Proof of the Theorems

Now we use these lemmas to prove the preceding theorems.

If $\alpha=1$, from the definitions of $p(n)$ and $\sigma_1(n)$, we know that

$$\sum_{n\leqslant x}\sigma_1(p(n))=\sum_{n\leqslant x}\sum_{\frac{\ln(m-1)!}{\ln p}<n\leqslant\frac{\ln(m!)}{\ln p}}\sigma_1(m).$$

Since $p(n)=m$, when $n\in\left(\frac{\ln(m-1)!}{\ln p},\frac{\ln m!}{\ln p}\right]$, and $n\leqslant x$, so the biggest number in the interval $\left(\frac{\ln(m-1)!}{\ln p},\frac{\ln m!}{\ln p}\right]$ is less than or equal to x. That is, $\frac{\ln m!}{\ln x}\leqslant x$, then we get $\ln m!\leqslant x\ln p$. Applying the Euler's summation formula[6], we obtain the main term of $\ln m!$ is $m\ln m$ and $m\ln m\leqslant x\ln p$.

If $m\geqslant x^{1/2}\ln p/\ln x$, then $\ln m$ is asymptotic to $\ln x$, we get $m\leqslant x\ln p/\ln x$.

From Lemma 1 and the Abel's identity[5], we have

$$\sum_{n\leqslant x}\sigma_1(p(n))=\sum_{n\leqslant x}\sum_{\frac{\ln(m-1)!}{\ln p}<n\leqslant\frac{\ln(m!)}{\ln p}}\sigma_1(m)=\sum_{\frac{x^{1/2}\ln p}{\ln x}<m\leqslant\frac{x\ln p}{\ln x}}\sigma_1(m)\frac{\ln m}{\ln p}+O\left(\frac{x\ln^2 p}{\ln x}\right)=$$

$$\frac{1}{\ln p}\sum_{\frac{x^{1/2}\ln p}{\ln x}<m\leqslant\frac{x\ln p}{\ln x}}\sigma_1(m)\ln m+O\left(\frac{x\ln^2 p}{\ln x}\right)=$$

$$\frac{1}{\ln p}\ln\left(\frac{x\ln p}{\ln x}\right)\sum_{m\leqslant\frac{x\ln p}{\ln x}}\sigma_1(m)-\frac{1}{\ln p}\ln\left(\frac{x^{1/2}\ln p}{\ln x}\right)\sum_{m\leqslant\frac{x^{1/2}\ln p}{\ln x}}\sigma_1(m)-$$

$$\frac{1}{\ln p}\int_{x^{1/2}\ln p/\ln x}^{x\ln p/\ln x}\frac{(\zeta(2)/2)t^2+O(t\ln t)}{t}\mathrm{d}t+O\left(\frac{x\ln^2 p}{\ln x}\right)=$$

$$\frac{1}{\ln p}\left(\frac{\zeta(2)}{2}\frac{x^2\ln^2 p}{\ln^2 x}\ln\left(\frac{x\ln p}{\ln x}\right)+O(x\ln x)-\frac{\zeta(2)}{4}\frac{x^2\ln^2 p}{\ln^2 x}\right)+O\frac{x\ln^2 p}{\ln x}=$$

$$\frac{\pi^2}{12}\frac{x^2\ln p}{\ln^2 x}\left(\ln\left(\frac{x\ln p}{\ln x}\right)-\frac{1}{2}\right)+O(x\ln x).$$

If $\alpha>1$

$$\sum_{n\leqslant x}\sigma_\alpha(p(n))=\sum_{n\leqslant x}\sum_{\frac{\ln(m-1)!}{\ln p}<n\leqslant\frac{\ln(m!)}{\ln p}}\sigma_\alpha(m)=\frac{1}{\ln p}\sum_{\frac{x^{1/2}\ln p}{\ln x}<m\leqslant\frac{x\ln p}{\ln x}}\sigma_\alpha(m)\ln m+O\left(\frac{x^{(\alpha+1)/2}}{\ln^\alpha x}\right).$$

From Lemma 2 we know that $\sum_{n\leqslant x}\sigma_\alpha(n)=\frac{\zeta(\alpha+1)}{\alpha+1}x^{\alpha+1}+O(x^\alpha)$.

If $\alpha>1$, by using the Euler's summation formula and the Abel's identity we obtain that

$$\sum_{n\leqslant x}\sigma_\alpha(p(n))=\sum_{n\leqslant x}\sum_{\frac{\ln(m-1)!}{\ln p}<n\leqslant\frac{\ln(m!)}{\ln p}}\sigma_\alpha(m)=\frac{1}{\ln p}\sum_{\frac{x^{1/2}\ln p}{\ln x}<m\leqslant\frac{x\ln p}{\ln x}}\sigma_\alpha(m)\ln m+O\left(\frac{x^{(\alpha+1)/2}}{\ln^\alpha x}\right)=$$

$$\frac{1}{\ln p}\ln\left(\frac{x\ln p}{\ln x}\right)\left(\frac{\zeta(\alpha+1)}{\alpha+1}\left(\frac{x\ln p}{\ln x}\right)^{\alpha+1}+O\left(\frac{x\ln p}{\ln x}\right)^{\alpha}\right)-$$

$$\frac{1}{\ln p}\ln\left(\frac{x^{1/2}p}{\ln x}\right)\left(\frac{\zeta(\alpha+1)}{\alpha+1}\left(\frac{x^{1/2}\ln p}{\ln x}\right)^{\alpha+1}+O\left(\frac{x^{1/2}\ln p}{\ln x}\right)^{\alpha}\right)-$$

$$\int_{x^{1/2}\ln p/\ln x}^{x\ln p/\ln x}\frac{(\zeta(\alpha+1)/(\alpha+1))t^{\alpha+1}+O(t^\alpha)}{t}\mathrm{d}t+O\left(\frac{x^{(\alpha+1)/2}}{\ln^\alpha x}\right)=$$

$$\frac{\zeta(\alpha+1)}{\alpha+1}\frac{x^{\alpha+1}\ln^\alpha p}{\ln^\alpha x}\ln\left(\frac{x\ln p}{\ln x}\right)+O\left(\frac{x^\alpha\ln^\alpha p}{\ln^{\alpha-1}x}\right)-$$

$$\frac{\zeta(\alpha+1)}{\alpha+1}\frac{x^{(\alpha+1)/2}\ln^\alpha p}{\ln^{\alpha+1}x}\ln\left(\frac{x^{1/2}\ln p}{\ln x}\right)-\frac{1}{\ln p}\frac{\zeta(\alpha+1)}{(\alpha+1)^2}\left(\frac{x\ln p}{\ln x}\right)^{\alpha+1}-$$

$$\frac{\zeta(\alpha+1)}{(\alpha+1)^2}\frac{1}{\ln p}\left(\frac{x^{1/2}\ln p}{\ln x}\right)^{\alpha+1}+O\left(\frac{x^\alpha}{\ln^\alpha x}\right)=$$

$$\frac{\zeta(\alpha+1)}{\alpha+1}\frac{x^{\alpha+1}\ln^\alpha p}{\ln^{\alpha+1}x}\left(\ln\left(\frac{x\ln p}{\ln x}\right)-\frac{1}{\alpha+1}\right)+O\left(\frac{x^\alpha}{\ln^{\alpha-1}x}\right).$$

If $\alpha<1$, using the same method, we can obtain the result easily. This complete the proof of Theorem 1.

By using the same way as in the proof of Theorem 1, we can deduce Theorem 2.

Acknowledgments: The author express her gratitude to her supervisor Professor Zhang Wenpeng for his very helpful and detailed instructions.

References:

[1] 冀永强. 数论函数及其方程[J]. 纺织高校基础科学学报, 2006, 19(1): 5-6.

[2] SANDOR Jozsef. On additive analogues of certain arithmetic function[J]. Smarandache Notion Journal, 2004, 14: 128-132.

[3] LE Mao-hua. Some problems concerning the Smarandache square complementary function[J]. Smarandache Notion Journal, 2004, 14: 220-222.

[4] LIU Hua. On the F. Smarandache simple function[J]. Scientia Magna, 2006, 2: 98-100.

[5] APOSTOL T M. Introduction to analytic number theory[M]. New York: Spring-Verlag, 1976.

[6] ZHU Min-hui. The additive analogue of Smarandache simple function[C]// Reseach on Smarandache problem in number theory, USA: Hexis, 2004: 39-40.

关于包含 F. Smarandache 简单函数的除数函数 $\sigma_\alpha(n)$

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摘要: 用解析的方法研究了 $\sigma_\alpha(p(n))$ 的渐近性质, 并给出了关于 $\sigma_\alpha(p(n))$ 的两 个渐近公式.

关键词: F. Smarandache 简单函数; 均值; 渐近公式

编辑、校对: 黄燕萍