Chin. Quart. J. of Math.
2008, 23 (3): 437—441

On the Solutions of an Equation Involving the Smarandache Power Function $SP(n)$

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Abstract: For any positive integer n, the famous Smarandache power function $SP(n)$ is defined as the smallest positive integer m such that $n|m^m$, where m and n have the same prime divisors. The main purpose of this paper is using the elementary methods to study the positive integer solutions of an equation involving the Smarandache power function $SP(n)$ and obtain some interesting results. At the same time, we give an open problem about the related equation.

Key words: Smarandache power function; equation; positive integer solutions

2000 MR Subject Classification: 11B83

CLC number: O156.4 **Document code**: A

Article ID: 1002–0462 (2008) 03–0437–05

§1. Introduction and Results

For any positive integer n, we define the Smarandache power function $SP(n)$ as the smallest positive integer m such that $n|m^m$, where n and m have the same prime divisors. That is,

$$SP(n) = \min\left\{m:\ n|m^m,\ m \in \mathrm{N},\ \prod_{p|n} p = \prod_{p|m} p\right\}.$$

If n runs through all natural numbers, then we can get the Smarandache power function sequence $\{SP(n)\}$: 1, 2, 3, 2, 5, 6, 7, 4, 3, 10, 11, 6, 13, 14, 15, 4, 17, 6, 19, 10, $\cdots$.

In reference[1], professor Smarandache asked us to study the properties of the sequence $\{SP(n)\}$. From the definition of $SP(n)$ we can easily get the following conclusions:

Received date: 2007-05-10

Foundation item: Supported by the Natural Science Foundation of China(10671155)

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If $n = p^\alpha$, where p be a prime, then we have

$$SP(n) = \begin{cases} p, & \text{if } 1 \le \alpha \le p; \\ p^2, & \text{if } p+1 \le \alpha \le 2p^2; \\ p^3, & \text{if } 2p^2+1 \le \alpha \le 3p^3; \\ \cdots & \\ p^\alpha, & \text{if } (\alpha-1)p^\alpha + 1 \le \alpha \le \alpha p^\alpha. \end{cases}$$

Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ denotes the factorization of n into prime powers.

If $\alpha_i \le p_i$ for all $\alpha_i (i = 1, 2, \cdots, r)$, then we have $SP(n) = U(n)$, where $U(n) = \prod_{p|n} p$, $\prod_{p|n}$ denotes the product over all different prime divisors of n. It is clear that $SP(n)$ is not a multiplicative function. For example, $SP(8) = 4$, $SP(3) = 3$, $SP(24) = 6 \ne SP(3) \times SP(8)$. But for almost m and n with $(m, n) = 1$, we have $SP(mn) = SP(m) \cdot SP(n)$. In reference[2], doctor XU Zhe-feng had studied the mean value properties of $SP(n)$, and obtained some sharper asymptotic formulas, one of them as follows:

$$\sum_{n \le x} SP(n) = \frac{1}{2} x^2 \prod_p \left(1 - \frac{1}{p(p+1)}\right) + O\left(x^{\frac{3}{2}+\epsilon}\right),$$

where ϵ denotes any fixed positive number, and $\prod_p$ denotes the product over all primes.

In this paper, we shall use the elementary methods to study the positive integer solutions of an equation involving the Smarandache power function $SP(n)$, and prove the following conclusion:

Theorem For any positive integer m and $k > 1$, the equation

$$SP(n_1) + SP(n_2) + \cdots + SP(n_k) = m \cdot SP(n_1 + n_2 + \cdots + n_k), \tag{1.1}$$

has infinite positive integer solutions $(n_1, n_2, \cdots, n_k)$.

§2. Proof of the Theorem

In this section, we shall complete the proof of our theorem. First we need the following two important Lemmas.

Lemma 1 There exists an absolutely constant $c_1 > 0$ such that every odd number $N \ge c_1$ can be represented as a sum of three odd primes.

This Lemma is called the famous Three Primes Theorem. Its proof can be found in reference [3] and [4].

Lemma 1 can also be extended as follows: There exists an absolutely constant $c_1 > 0$ such that every odd number $N_k \ge c_1$ can be represented as a sum of $2k + 1$ odd primes.

Lemma 2 There exists an absolutely constant $c_1 > 0$ such that every large even integer $N \geq c_1$ can be represented a sum of a prime and an almost prime having at most two prime factors.

This is the famous Chen's Theorem. Its proof can also be found in reference[3].

Now we use these two Lemmas to prove our Theorem. If m and k are odd numbers, then $k \geq 3$. Let $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ be the factorization of m into prime powers, then for prime P large enough, from Lemma 1 we know that there exist primes q_1, q_2, $\cdots$, q_s satisfying the equation:

$$p_1^{\alpha_1+1} p_2^{\alpha_2+1} \cdots p_s^{\alpha_s+1} P = q_1 + q_2 + \cdots + q_k. \tag{2.1}$$

Then taking $n_i = q_i (i = 1,\ 2,\ \cdots,\ k)$ in equation (1), from the properties of $SP(n)$ and equation (2) we may immediately deduce that

$$\begin{aligned}
& SP(q_1) + SP(q_2) + \cdots + SP(q_k) \\
= & q_1 + q_2 + \cdots + q_k = p_1^{\alpha_1+1} p_2^{\alpha_2+1} \cdots p_s^{\alpha_s+1} P \\
= & p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s} \cdot p_1 p_2 \cdots p_s P = m \cdot p_1 p_2 \cdots p_s P \\
= & m \cdot SP(p_1^{\alpha_1+1} p_2^{\alpha_2+1} \cdots p_s^{\alpha_s+1} P) \\
= & m \cdot SP(q_1 + q_2 + \cdots + q_k).
\end{aligned}$$

That is to say, our theorem is correct if m and k are odd numbers.

If m be an odd number and k be an even number, then we discuss it in two cases:

Case (a) $k = 2$. We still let $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the factorization of m into prime powers, then for prime P large enough, from Lemma 2 we know that

$$2p_1^{\alpha_1+1} p_2^{\alpha_2+1} \cdots p_s^{\alpha_s+1} P = q_1 + q_2$$

or

$$2p_1^{\alpha_1+1} p_2^{\alpha_2+1} \cdots p_s^{\alpha_s+1} P = q_1 + \bar{q}_2 \bar{q}_3.$$

where q_1, $\bar{q}_2$ and $\bar{q}_3$ are primes. In any case, we still have

$$\begin{aligned}
& SP(q_1) + SP(q_2) = q_1 + q_2 \\
= & 2p_1^{\alpha_1+1} p_2^{\alpha_2+1} \cdots p_s^{\alpha_s+1} P \\
= & m \cdot 2p_1 p_2 \cdots p_s P \\
= & m \cdot SP(2p_1 p_2 \cdots p_s P) \\
= & m \cdot SP(2p_1^{\alpha_1+1} p_2^{\alpha_2+1} \cdots p_s^{\alpha_s+1} P) \\
= & m \cdot SP(q_1 + q_2)
\end{aligned}$$

or

$$\begin{aligned}
& SP(q_1) + SP(\bar{q}_2 \bar{q}_3) = q_1 + \bar{q}_2 \bar{q}_3 \\
= & 2p_1^{\alpha_1+1} p_2^{\alpha_2+1} \cdots p_s^{\alpha_s+1} P
\end{aligned}$$

$$\begin{aligned}
&= m \cdot 2p_1p_2\cdots p_sP \\
&= m \cdot SP(2p_1p_2\cdots p_sP) \\
&= m \cdot SP(2p_1^{\alpha_1+1}p_2^{\alpha_2+1}\cdots p_s^{\alpha_s+1}P) \\
&= m \cdot SP(q_1+\bar{q}_2\bar{q}_3).
\end{aligned}$$

Case (b) $k=2k_1$, $k_1\geq 2$. This time from Lemma 1 we have

$$p_1^{\alpha_1+1}p_2^{\alpha_2+1}\cdots p_k^{\alpha_k+1}P=q_1+q_2+\cdots+q_{k-1}+2.$$

Using the same method of the above we can prove that the theorem is also correct, see reference [5].

Next, we'll discuss the equation (1) in which m be an even number. We still let $m=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ be the factorization of m into prime powers, and discuss the equation (1) in the following three cases:

(I) If $k=2$, then from Lemma 2 we know that, $p_1^{\alpha_1+1}p_2^{\alpha_2+1}\cdots p_k^{\alpha_k+1}P$ is a sum of a prime and an almost prime having at most two prime factors. That is,

$$p_1^{\alpha_1+1}p_2^{\alpha_2+1}\cdots p_k^{\alpha_k+1}P=p_1'+q_1'$$

or

$$p_1^{\alpha_1+1}p_2^{\alpha_2+1}\cdots p_k^{\alpha_k+1}P=p_1'+\bar{q}_1'\bar{q}_2',$$

where P be a prime large enough, and p_1', $\bar{q}_i'(i=1,2)$ are primes. Using the same method of the above we also get that the left hand side of the equation (1) is equal to its right hand side.

(II) If $k=2k_1(k_1>1)$, then we have

$$p_1^{\alpha_1+1}p_2^{\alpha_2+1}\cdots p_k^{\alpha_k+1}P=q_1+q_2+\cdots+q_{k-1}+3,$$

where P denotes a prime large enough, $q_i(i=1,2,\cdots,k-1)$ are primes.

Hence,

$$p_1^{\alpha_1+1}p_2^{\alpha_2+1}\cdots p_k^{\alpha_k+1}P-3=q_1+q_2+\cdots+q_{k-1},$$

which satisfy Lemma 1, so (1) is also holds.

(III) If $k=2k_1+1(k_1>1)$, then we have

$$p_1^{\alpha_1+1}p_2^{\alpha_2+1}\cdots p_k^{\alpha_k+1}P=q_1+q_2+\cdots+q_{k-1}+2$$

and

$$p_1^{\alpha_1+1}p_2^{\alpha_2+1}\cdots p_k^{\alpha_k+1}P-2=q_1+q_2+\cdots+q_{k-1}.$$

Since $k-1$ is an even number, so this case is the same as in (II). Thus (1) is also holds.

Since P is a prime large enough, hence for any $m\in Z^+$, and $k>1$, the equation (1) has infinite positive integer solutions $(n_1,\ n_2,\ \cdots,\ n_k)$. This completes the proof of our Theorem.

§3. An Open Problem

If we put the number m in the right hand side of the equation (1), how about the positive integer solutions of the equation:

$$m \cdot (SP(n_1) + SP(n_2) + \cdots + SP(n_k)) = SP(n_1 + n_2 + \cdots + n_k). \tag{3.1}$$

This is an open problem.

We guess that (3) also has infinite positive integer solutions $(n_1,\ n_2,\ \cdots,\ n_k)$.

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