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A Limit Involving the F Smarandache Square Complementary Number

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Abstract: The main purpose of this paper is using the analytic methods to study a limit problem involving the F Smarandache square complementary number $Ssc(n)$, and obtain its limit value.
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§1. Introduction and Results

For any positive integer n, we call $Ssc(n)$ as the square complementary number of n, if $Ssc(n)$ is the smallest positive integer such that $nSsc(n)$ is a perfect square. That is,

$$Ssc(n) = \min\{m : \ mn = u^2, \ u \in N\}.$$

For example, $Ssc(1) = 1$, $Ssc(2) = 2$, $Ssc(3) = 3$, $Ssc(4) = 1$, $Ssc(5) = 5$, $Ssc(6) = 6$, $Ssc(7) = 7$, $Ssc(8) = 2\cdots$. In reference [1], Professor F.Smarandache asked us to study the properties of the sequence $\{Ssc(n)\}$. About this problem, some authors had studied it, and obtained some interesting results. For example, Liu Hongyan and Gou Su[2] used the elementary method to study the mean value properties of $Ssc(n)$ and $\frac{1}{Ssc(n)}$, and obtained two interesting asymptotic formulas. Zhang Hongli and Wang Yang[3] studied the mean value of $\tau(Ssc(n))$, and obtained an asymptotic formula by the analytic method. Yi Yuan[4] studied the mean value $\sum\limits_{n\le x} d(n + Ssc(n))$, and proved that

$$\sum_{n\le x} d(n + Ssc(n)) = \frac{3}{4\pi^2} x \ln^2 x + A_1 x \ln x + A_2 x + O\left(x^{\frac{3}{4}+\epsilon}\right),$$

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where A_1 and A_2 are computable constants, ε is any fixed positive number.

On the other hand, Professor Felice Russo[5] asked us to calculate the limit $\lim_{n\to\infty}\frac{Ssc(n)}{\theta(n)}$, where $\theta(n)=\sum_{n\le k}\ln(Ssc(n))$. In fact, this problem very simple, we can easily prove that $\lim_{n\to\infty}\frac{Ssc(n)}{\theta(n)}=0$.

The main purpose of this paper is to study another limit problem involving square complementary number, and obtain an interesting limit theorem. That is, we shall prove the following conclusion:

Theorem Let $d(n)$ denotes the Dirichlet divisor function, then we have the limit formula

$$\lim_{k\to\infty}\frac{\sum\limits_{n\le k} d(Ssc(n))}{\sum\limits_{n\le k}\ln(Ssc(n))}=\frac{6}{\pi^2}\prod_p\left(1-\frac{1}{(p+1)^2}\right),$$

where $\prod\limits_p$ denotes the product over all primes p.

§2. Proof of the Theorem

In this section, we shall complete the proof of theorem. First we need the following two lemmas:

Lemma 1 Let $Ssc(n)$ denotes the square complement number of n, then for any integer $k>1$, we have the asymptotic formula:

$$\sum_{n\le k} d(Ssc(n))=\frac{h(1)}{\zeta(2)}k\ln k+\frac{h(1)}{\zeta(2)}\left(2r-1+\frac{2\zeta'(2)}{\zeta(2)}\right)k+O(\sqrt{k}\ln^3 k),$$

where $d(n)$ is the Dirichlet divisor function, $\zeta(s)$ denotes the Riemann zeta-function, $h(1)=\prod\limits_p\left(1-\frac{1}{(p+1)^2}\right)$, $\prod\limits_p$ denotes the product over all primes p, γ is the Euler's constant.

Proof See reference [3].

Lemma 2 For any real number $k>1$, we have the asymptotic formula:

$$\sum_{n\le k}\ln(Ssc(n))=k\ln k-\frac{\zeta(2)+2}{\zeta(2)}k+O(\sqrt{k}\ln^2 k).$$

Proof By the definition of the square complement and the properties of Möbius function we can get

$$\sum_{n\le k}\ln(Ssc(n))=\sum_{m^2l\le k}\ln(Ssc(m^2l))|\mu(l)|$$

$$= \sum_{m^2 l \le k} \ln l \sum_{d^2 | l} \mu(d)$$
$$= \sum_{m^2 d^2 h \le k} \mu(d) \ln(d^2 h)$$
$$= 2 \sum_{m^2 d^2 h \le k} \mu(d) \ln(d) + \sum_{m^2 d^2 h \le k} \mu(d) \ln(h). \quad (1)$$

Noting that:

$$\sum_{n \le x} \frac{1}{n^2} = \zeta(2) + O\left(\frac{1}{x}\right),$$
$$\sum_{n \le x} \frac{\mu(n)}{n^2} = \frac{1}{\zeta(2)} + O\left(\frac{1}{x}\right)$$

and

$$F(m) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^m} = \frac{1}{\zeta(m)},$$

we have

$$F'(m) = -\sum_{n=1}^{\infty} \frac{\mu(n) \ln n}{n^m} = -\frac{\zeta'(m)}{\zeta^2(m)}.$$

From this formula we may immediately get

$$\sum_{n=1}^{\infty} \frac{\mu(n) \ln n}{n^m} = \frac{\zeta'(m)}{\zeta^2(m)}.$$

From the above we have

$$\sum_{m^2 d^2 h \le k} \mu(d) \ln d \quad (2)$$
$$= \sum_{d^2 \le k} \mu(d) \ln d \sum_{m \le \frac{\sqrt{k}}{d}} \sum_{h \le \frac{k}{m^2 d^2}} 1$$
$$= \sum_{d^2 \le k} \mu(d) \ln d \sum_{m \le \frac{\sqrt{k}}{d}} \left(\frac{k}{m^2 d^2} + O(1)\right)$$
$$= k \sum_{d \le \sqrt{k}} \frac{\mu(d) \ln d}{d^2} \left(\zeta(2) + O\left(\frac{d}{\sqrt{k}}\right)\right) + O\left(\sqrt{k} \sum_{d \le \sqrt{k}} \frac{|\mu(d)| \ln d}{d}\right)$$
$$= \zeta(2) k \frac{\zeta'(2)}{\zeta^2(2)} + O\left(\sqrt{k} \ln^2 k\right)$$
$$= \frac{\zeta'(2)}{\zeta(2)} k + O\left(\sqrt{k} \ln^2 k\right). \quad (3)$$

From (1) and Euler's summation formula[6] we have

$$\sum_{m^2 d^2 h \le k} \mu(d) \ln h$$
$$= \sum_{md \le \sqrt{k}} \mu(d) \sum_{h \le \frac{k}{m^2 d^2}} \ln h$$

$$= \sum_{md\le\sqrt{k}} \mu(d)\left(\frac{k}{m^2d^2}\ln\frac{k}{m^2d^2} - \frac{k}{m^2d^2} + \int_1^{\frac{k}{m^2d^2}} (t-[t])\frac{1}{t}\mathrm{d}t + \ln\frac{k}{m^2d^2}([x]-x)\right)$$

$$= k\ln k \sum_{md\le\sqrt{k}} \frac{\mu(d)}{m^2d^2} - k \sum_{md\le\sqrt{k}} \frac{\mu(d)\ln m^2d^2}{m^2d^2} - k \sum_{md\le\sqrt{k}} \frac{\mu(d)}{m^2d^2}$$

$$+O\left(\sum_{md\le\sqrt{k}} |\mu(d)|\ln\frac{k}{m^2d^2}\right). \tag{4}$$

$$\sum_{md\le\sqrt{k}} \frac{\mu(d)}{m^2d^2}$$
$$= \sum_{m\le\sqrt{k}} \frac{1}{m^2} \sum_{d\le\frac{\sqrt{k}}{m}} \frac{\mu(d)}{d^2}$$
$$= \sum_{m\le\sqrt{k}} \frac{1}{m^2}\left(\frac{1}{\zeta(2)} + O\left(\frac{m}{\sqrt{k}}\right)\right)$$
$$= \frac{1}{\zeta(2)}\left(\zeta(2) + O\left(\frac{1}{\sqrt{k}}\right)\right) + O\left(\frac{\ln k}{\sqrt{k}}\right)$$
$$= 1 + O\left(\frac{\ln k}{\sqrt{k}}\right). \tag{5}$$

Use the same method, we also have

$$\sum_{md\le\sqrt{k}} \frac{\mu(d)\ln m^2}{m^2d^2}$$
$$= 2\sum_{m\le\sqrt{k}} \frac{\ln m}{m^2} \sum_{d\le\frac{\sqrt{k}}{m}} \frac{\mu(d)}{d^2}$$
$$= \frac{2}{\zeta(2)} \sum_{m\le\sqrt{k}} \frac{\ln m}{m^2} + O\left(\frac{2}{\sqrt{k}} \sum_{m\le\sqrt{k}} \frac{\ln m}{m}\right)$$
$$= \frac{2}{\zeta(2)} + O\left(\frac{\ln^2 k}{\sqrt{k}}\right). \tag{6}$$

$$\sum_{md\le\sqrt{k}} \frac{\mu(d)\ln d^2}{m^2d^2}$$
$$= 2\sum_{d\le\sqrt{k}} \frac{\mu(d)\ln d}{d^2} \sum_{m\le\frac{\sqrt{k}}{d}} \frac{1}{m^2}$$
$$= 2\zeta(2)\left(\sum_{n=1}^{\infty} \frac{\mu(d)\ln d}{d^2} + O\left(\frac{1}{k}\right)\right) + O\left(\frac{1}{\sqrt{k}} \sum_{d\le\sqrt{k}} \frac{|\mu(d)|\ln d}{d}\right)$$
$$= 2\frac{\zeta'(2)}{\zeta(2)} + O\left(\frac{\ln^2 k}{\sqrt{k}}\right). \tag{7}$$

Combining (4), (5) and (6) we can easily get the asymptotic formula:

$$\sum_{m^2 d^2 h \leq k} \mu(d) \ln(h) = k \ln k - \frac{2 + 2\zeta'(2) + \zeta(2)}{\zeta(2)} k + O(\sqrt{k} \ln^2 k). \quad (8)$$

Combining (1) and (7) we have

$$\sum_{n \leq k} \ln(Ssc(n)) = k \ln k - \frac{\zeta(2) + 2}{\zeta(2)} k + O(\sqrt{k} \ln^2 k).$$

This proves Lemma 2.

Now we can easily complete the proof of theorem. In fact, from Lemma 1 and Lemma 2 we have

$$\lim_{k \to \infty} \frac{\sum\limits_{n \leq k} d(Ssc(n))}{\sum\limits_{n \leq k} \ln(Ssc(n))} = \frac{\frac{h(1)}{\zeta(2)} k \ln k + \frac{h(1)}{\zeta(2)} \left(2r - 1 + \frac{2\zeta'(2)}{\zeta(2)}\right) k + O(\sqrt{k} \ln^3 k)}{k \ln k - \frac{\zeta(2)+2}{\zeta(2)} k + O(\sqrt{k} \ln^2 k)} = \frac{h(1)}{\zeta(2)}.$$

Note that $\zeta(2) = \frac{\pi^2}{6}$ and $h(1) = \prod\limits_{p} \left(1 - \frac{1}{(p+1)^2}\right)$, from the above we have

$$\lim_{k \to \infty} \frac{\sum\limits_{n \leq k} d(Ssc(n))}{\sum\limits_{n \leq k} \ln(Ssc(n))} = \frac{6}{\pi^2} \prod_{p} \left(1 - \frac{1}{(p+1)^2}\right).$$

This completes the proof of theorem.

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